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ENERGY DECAY TO TIMOSHENKO SYSTEM WITH INDEFINITE DAMPING

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Abstract. We consider the classical Timoshenko system for vibrations of thin rods. The system has an indefinite damping mechanism, i.e. it has a damping function $a = a(x)$ possibly changing sign, present only in the equation for the vertical displacement. We shall prove that exponential stability depends on conditions regarding of the indefinite damping function a and a nice relationship between the coefficient of the system. Finally, we give some numerical result to verify our analytical results.

1. Introduction

In this work we consider the Timoshenko system which models the transverse vibration of a thin rod of length L by taking into account the shear forces given by

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0,\infty)\times(0,L), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in} \quad (0,\infty)\times(0,L). \tag{1.2}$$

Here t denotes the time variable, x is the distance until the beam's centerline in equilibrium, the function $\varphi = \varphi(t,x)$ denotes the vertical displacement of the beam's centerline and the function $\psi = \psi(t,x)$ denotes the rotation of the vertical fibers in the beam. Moreover, the coefficients ρ_1, ρ_2, b and k denote positive constants and they depend on the density of the mass material, the area of the cross-section, the second moment of the cross-section area, the Young's model, the modulus of rigidity and the shear factor.

The system (1.1)-(1.2) is conservative. So, if we want to search about asymptotic behavior we must add a damping term. In this direction, the main types of dissipative mechanisms considered are frictional, thermal, viscoelastic and their combinations.

Recently, researches have shown that the exponential stability of the Timoshenko system is achieved regardless of any specific relations between the coefficients when there are a dissipative mechanism in both equations. We refer the reader to, e.g. [14, 8, 20, 22] and the reference therein.

However, if we consider only one damping term the scenery can be changed. Soufyane in [24] proved that when there is a dissipation of the type $\alpha\psi_t$ $(\alpha > 0)$ in the equation that models the rotation angle the system is exponentially stable if only if

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{1.3}$$

Taking into account the condition (1.3) several extensions and generalizations were established, including dissipations like viscoelastic, thermal, memory etc. Among the various references we can cite for example [1, 2, 7, 6, 9, 11, 12, 13, 16, 23].

Still in this context there is only one work related to indefinite dissipation in the Timoshenko system given by Rivera and Racke [17]. In this work, they considered an indefinite dissipation acting on the equation that models the rotation angle, i.e., with a damping mechanism $a(x)\psi_t$ where the function $a(x)$ may change its sign, present only in the (1.2) and proved that the system is exponentially stable under the same conditions used in the positive damping case, and provided

$$\bar{a} = \frac{1}{L}\int_0^L a(x)\,dx > 0 \quad \text{and} \quad \|a - \bar{a}\|_{L^2} < \epsilon, \ \text{for}\, \epsilon \ \text{small enough} . \tag{1.4}$$

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In general, the aforementioned studies that considered one dissipation always did it in the rotation angle equation. The first work that considered the dissipation in the transverse displacement was due to Almeida Junior et al [3]. In this paper the authors studied the Timoshenko system with a constant frictional dissipation acting only in the equation displacement (1.1), ie, they consider the following system

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \bar{a}\varphi_t = 0 \quad \text{in} \quad (0,\infty)\times(0,L), \tag{1.5}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) = 0 \quad \text{in} \quad (0,\infty)\times(0,L), \tag{1.6}$$

with boundary conditions given by

$$\varphi_x(\cdot,0) = \varphi_x(\cdot,L) = \psi(\cdot,0) = \psi(\cdot,L) = 0,$$

where $\bar{a}, \rho_1, \rho_2, b, k > 0$. The main result in [3] asserts that (1.3) is a necessary and sufficient condition for exponentially stability to the system (1.5)-(1.6).

Keeping in mind the last results our aim is to complement early works by establishing the exponential decay when we consider a Timoshenko system with a indefinite damping in the transverse displacement, that is, we consider the following system

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + a(x)\varphi_t = 0 \quad \text{in} \quad (0,\infty)\times(0,L), \tag{1.7}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) = 0 \quad \text{in} \quad (0,\infty)\times(0,L), \tag{1.8}$$

with initial conditions

$$\varphi(0,\cdot) = \varphi_0, \quad \varphi_t(0,\cdot) = \varphi_1 \quad \psi(0,\cdot) = \psi_0, \quad \psi_t(0,\cdot) = \psi_1 \quad \text{in } (0,L) \tag{1.9}$$

and boundary conditions

$$\varphi(\cdot,0) = \varphi(\cdot,L) = \psi_x(\cdot,0) = \psi_x(\cdot,L) = 0 \quad \text{in } (0,\infty). \tag{1.10}$$

We assume that $a \in L^\infty(0,L)$ is a real function that may change its sign and satisfies (1.4) for the part on the exponential stability. Our goal is to proof that the conditions (1.3) and (1.4) are sufficient to yield exponential stability to the system (1.7)-(1.10).

The paper is organized as follows. In Section 2, we state the results on existence and global well posedness to the system (1.7)-(1.10). In Section 3, firstly we discuss the exponential stability in the positive constant damping case and then we finish with our main result to the original system. Finally, in Section 4 we show some numerical results.

2. Existence and regularity

We will study the existence and uniqueness of solution for the Timoshenko system (1.7)-(1.10). Putting $U = (\varphi, \Phi, \psi, \Psi)'$ where the prime is used to denote the transpose. Then U satisfies

$$\begin{cases} U_t = \mathcal{A}U \quad t>0, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1)'$ and $\mathcal{A}$ is the differential operator given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & -\frac{a(x)}{\rho_1}I & k\frac{\partial_x}{\rho_1} & 0 \\ 0 & 0 & 0 & I \\ -\frac{k}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{k}{\rho_2}I & 0 \end{pmatrix}.$$

We denote

$$L^2_*(0,L) = \left\{ u \in L^2(0,L); \ \int_0^L u(x)\,dx = 0 \right\} \quad \text{and} \quad H^1_*(0,L) = H^1(0,L) \cap L^2_*(0,L)$$

and let us introduce the following Hilbert space

$$\mathcal{H} = H^1_0(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2_*(0,L),$$

with the norm given by

$$\|U\|^2_{\mathcal{H}} = \|(\varphi,\Phi,\psi,\Psi)\|^2_{\mathcal{H}} = \rho_1\|\Phi\|^2_{L^2} + b\|\psi_x\|^2_{L^2} + k\|\varphi_x+\psi\|^2_{L^2} + \rho_2\|\Psi\|^2_{L^2}. \tag{2.2}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \{U \in \mathcal{H}; \varphi \in H^2(0,L), \Phi \in H_0^1(0,L), \psi \in H_*^1(0,L), \psi_x \in H_0^1(0,L), \Psi \in H_*^1(0,L)\}.$$

Setting

$$A_\infty = \mathcal{A} - \frac{a_\infty}{\rho_1}\mathcal{B},$$

where $a_\infty = \|a\|_{L^\infty}$ and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is the continuous linear operator given by

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that, $\mathcal{D}(A_\infty) = \mathcal{D}(\mathcal{A})$ and for all $U \in \mathcal{D}(A_\infty)$ we have that

$$Re\big(A_\infty U, U\big)_{\mathcal{H}} = -\int_0^L (a(x) + a_\infty)|\Phi|^2\,dx \le 0,$$

which yields that the operator $\mathcal{A}_\infty$ is dissipative in $\mathcal{H}$. It is not difficult to prove that $0 \in \rho(\mathcal{A}_\infty)$ (more detail, see [21]). Thus, by Lummer-Phillips Theorem, we have that $\mathcal{A}_\infty$ is an infinitesimal generator of a C_0-semigroup of contractions.

Now, using result about pertubation by bounded linear operators (see Theorem 1.1, chapter 3 in [19]) we have that $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup.

Therefore the well posedness of (1.7)-(1.10) is summarized by the following result

Theorem 2.1. *Assume that $U_0 \in \mathcal{D}(\mathcal{A})$, then exists an unique solution $U = (\varphi, \Phi, \psi, \Psi)$ to the system (1.7)-(1.10) satisfying*

$$U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}).$$

Remark 2.1. *The semigroup $S(t)$ generated by $\mathcal{A}$ satisfies*

$$\|S(t)\| \le e^{\frac{a_\infty}{\rho_1}t} \quad \forall\, t \ge 0.$$

In fact, $S(t) = T(t)e^{\frac{a_\infty}{\rho_1}\mathcal{B}t}$ where $T(t)$ is the C_0-semigroup of contractions generated by A_∞.

Remark 2.2. *If we consider $\overline{A}$ for the arising constant coefficient operator instead of $\mathcal{A}$ when $a(x) = \bar{a}$ in (1.7) then $\overline{A}$ is an infinitesimal generator of a C_0-semigroup of contractions associated (1.5)-(1.6) with boundary condition given by (1.10). In particular, the system (1.5)-(1.6) with boundary condition given by (1.10) is well posed.*

3. Exponential stability

In this section we will see that the mathematical hypothesis $\frac{\rho_1}{\rho_2} = \frac{k}{b}$, the average $\overline{a} > 0$ and $\|a - \bar{a}\|_{L^2} < \epsilon$ are sufficient to conclude that the semigroup $S(t) = e^{\mathcal{A}t}$ associated to Timoshenko system with indefinite damping is exponentially stable.

The main tool we use to show the exponential stability is given by the following result due to Gearhart, Pruss and Greiner (see, Theorem 1.11, chapter V in [10]).

Theorem 3.1. *The C_0-semigroup of contractions $S(t) = e^{\mathcal{A}t}$ over a Hilbert space $\mathcal{H}$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{\lambda \in \mathbb{C} : Re\lambda > 0\} \quad \text{and} \quad M := \sup_{Re\lambda > 0} \|(\lambda I - \mathcal{A})^{-1}\| < \infty \tag{3.1}$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of a liner operator $\mathcal{A}$ and I is the identity.

In order to get (3.1) we first need to show the exponential stability to the positive constant damping case which will be done in the next subsection.

3.1. **The constant coefficient case.** In this subsection, we will show that the Timoshenko system given by (1.5)-(1.6) is exponentially stable if $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ is holds and $\overline{a}>0$ ($\overline{a}$ is not necessarily the average of a). In fact, we use the same techniques used in [3], with minor adjustments and presented here only for the sake of making self-sufficient text.

The energy associated with the system (1.5)-(1.6) is given by

$$E(t):=\|U\|^2_{\mathcal{H}}=\rho_1\|\Phi\|^2_{L^2}+b\|\psi_x\|^2_{L^2}+k\|\varphi_x+\psi\|^2_{L^2}+\rho_2\|\Psi\|^2_{L^2}$$

and obeys the following dissipation law

$$\frac{d}{dt}E(t)=-2\bar{a}\|\Phi\|^2_{L^2}\le 0,\quad \forall\ t\ge 0. \tag{3.2}$$

Now, we will establish some lemmas. Let us introduce the functional

$$\mathcal{F}_1(t):=\rho_1 Re\bigg\{\int_0^L \Phi\bar{p}\,dx\bigg\}\qquad\text{where}\qquad p=\varphi+\int_0^L\psi\,dx.$$

Lemma 3.1. *For every $\delta>0$ there is a positive constant $C_{1,\delta}$ such that*

$$\frac{d}{dt}\mathcal{F}_1(t)\le-\frac{k}{2}\int_0^L|\varphi_x+\psi|^2\,dx+C_{1,\delta}\int_0^L|\Phi|^2\,dx+\rho_1\frac{\delta L^2}{2}\int_0^L|\Psi|^2\,dx.$$

Proof. Multiplying the equation (1.5) by $\bar{p}$, integrating by parts and using the boundary conditions, we have

$$\rho_1\int_0^L\Phi_t\bar{p}\,dx+k\int_0^L|\varphi_x+\psi|^2\,dx+\bar{a}\int_0^L\Phi\bar{p}\,dx=0. \tag{3.3}$$

Using that $\Phi_t\bar{p}=\frac{d}{dt}(\Phi\bar{p})-\Phi\bar{p}_t$ in (3.3) and taking the real part in both sides, we have

$$Re\bigg\{\rho_1\frac{d}{dt}\int_0^L\Phi\bar{p}\,dx\bigg\}=\rho_1\int_0^L\Phi\bar{p}_t\,dx-k\int_0^L|\varphi_x+\psi|^2\,dx-\bar{a}Re\bigg\{\int_0^L\Phi\bar{p}\,dx\bigg\}.$$

Hence, from Poincare, Holder and Young's inequalities in the last term on the right we obtain

$$\frac{d}{dt}\mathcal{F}_1(t)\le-\frac{k}{2}\int_0^L|\varphi_x+\psi|^2\,dx+\left(\rho_1+\frac{\bar{a}^2c_p^2}{2k}\right)\int_0^L|\Phi|^2\,dx+\rho_1\underbrace{\int_0^L|\Phi|\left|\int_0^x\Psi\,ds\right|dx}_{\mathcal{I}_1}, \tag{3.4}$$

where $c_p>0$ is Poincare's constant.

Again from Holder and Young's inequalities in I_1, we have that for any $\delta>0$ there is $C_\delta>0$ such that

$$\mathcal{I}_1(t)\le C_\delta\int_0^L|\Phi|^2\,dx+\frac{\delta L^2}{2}\int_0^L|\Psi|^2\,ds.$$

Therefore, using the above estimate in (3.4) our conclusion follows with $C_{1,\delta}=\rho_1(1+C_\delta)+\frac{\bar{a}^2c_p^2}{2k}$. □

Consider the functional

$$\mathcal{F}_2(t):=-\rho_2 Re\bigg\{\int_0^L\Psi\overline{(\varphi_x+\psi)}dx\bigg\}-\frac{b\rho_1}{k}Re\bigg\{\int_0^L\psi_x\overline{\Phi}dx\bigg\}$$

Lemma 3.2. *Assuming that $\frac{k}{\rho_1}=\frac{b}{\rho_2}$. Then $\mathcal{F}_2$ satisfies*

$$\frac{d}{dt}\mathcal{F}_2(t)=-\rho_2\int_0^L|\Psi|^2\,dx+k\int_0^L|\varphi_x+\psi|^2\,dx+\frac{b\bar{a}}{k}Re\bigg\{\int_0^L\overline{\Phi}\psi_x\,dx\bigg\}.$$

Proof. Multiplying the equation (1.6) by $\bar{p}_x=\overline{(\varphi_x+\psi)}$, integrating over $(0,L)$, we have

$$\rho_2\int_0^L\Psi_t\overline{\varphi}_x\,dx+\rho_2\int_0^L\Psi_t\overline{\psi}\,dx+b\int_0^L\psi_x\overline{(\varphi_x+\psi)}_x\,dx+k\int_0^L|\varphi_x+\psi|^2\,dx=0. \tag{3.5}$$

Note that

$$\Psi_t\overline{\varphi}_x=\frac{d}{dt}(\Psi\overline{\varphi}_x)-\Psi\overline{\Phi}_x=\frac{d}{dt}\big[\Psi\overline{(\varphi_x+\psi)}\big]-\Psi_t\overline{\psi}-|\Psi|^2-\Psi\overline{\Phi}_x. \tag{3.6}$$

Replacing (3.6) and using (1.5) in the third term in (3.5) it yields

$$\rho_2\frac{d}{dt}\int_0^L \Psi\overline{(\varphi_x+\psi)}\,dx = \rho_2\int_0^L \Psi\overline{\Phi}_x\,dx + \rho_2\int_0^L |\Psi|^2\,dx - k\int_0^L |\varphi_x+\psi|^2\,dx - \frac{b\bar{a}}{k}\int_0^L \overline{\Phi}\psi_x\,dx - \underbrace{\frac{b\rho_1}{k}\int_0^L \overline{\Phi}_t\psi_x\,dx}_{\mathcal{I}_2(t)}.$$

Rewritting $\mathcal{I}_2(t)$ as

$$\mathcal{I}_2(t) = \frac{b\rho_1}{k}\int_0^L \overline{\Phi}_t\psi_x\,dx = \frac{b\rho_1}{k}\frac{d}{dt}\int_0^L \overline{\Phi}\psi_x\,dx + \frac{b\rho_1}{k}\int_0^L \Psi_x\overline{\Phi}\,dx$$

we get

$$\frac{d}{dt}\Big[\rho_2\int_0^L \Psi\overline{(\varphi_x+\psi)}\,dx + \frac{b\rho_1}{k}\int_0^L \overline{\Phi}\psi_x\,dx\Big] = \rho_1\Big(\frac{\rho_2}{\rho_1}-\frac{b}{k}\Big)\int_0^L \Psi\overline{\Phi}_x\,dx + \rho_2\int_0^L |\Psi|^2\,dx - k\int_0^L |\varphi_x+\psi|^2\,dx - \frac{b\bar{a}}{k}\int_0^L \overline{\Phi}\psi_x\,dx.$$

Using $\frac{k}{\rho_1}=\frac{b}{\rho_2}$ then taking the real part of the previous equality our result follows. □

Finally, let us consider the functional

$$\mathcal{F}_3(t) = \rho_2 Re\Big\{\int_0^L \psi\overline{\Psi}\,dx\Big\}.$$

Lemma 3.3. *The functional $\mathcal{F}_3$ satisfies*

$$\frac{d}{dt}\mathcal{F}_3(t) \le \rho_2\int_0^L |\Psi|^2\,dx - \frac{3b}{4}\int_0^L |\psi_x|^2\,dx + \frac{c_p^2k^2}{b}\int_0^L |\varphi_x+\psi|^2\,dx.$$

Proof. Multiplying (1.6) by $\overline{\psi}$, integrating by parts and using the boundary conditions, we have

$$\rho_2\frac{d}{dt}\int_0^L \psi\overline{\Psi}\,dx = \rho_2\int_0^L |\Psi|^2\,dx - b\int_0^L |\psi_x|^2\,dx - k\int_0^L \overline{\psi}(\varphi_x+\psi)\,dx.$$

Taking the real part of the previous equality and using Cauchy-Schwarz and Young inequalities on the last term on the right our conclusion follows. □

Now we are able to show the main result of this section. For this, we define the following functional

$$\mathcal{G}(t) := N_0E(t) + N_1\mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + \mathcal{F}_3(t),$$

where N_0, N_1 and N_2 are positive constants chosen conveniently so that the functional $\mathcal{G}(t)$ is equivalent to the energy $E(t)$.

Theorem 3.2. *Suppose that $\frac{k}{\rho_1}=\frac{b}{\rho_2}$. So there are constants $M>0$ and $\omega>0$, independent of the initial conditions, such that*

$$E(t) \le ME(0)e^{-\omega t}.$$

Proof. Using the Lemmas 3.1, 3.2 and 3.3 we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) \le{}& -2\bar{a}N_0\int_0^L |\Phi|^2dx - N_1\frac{k}{2}\int_0^L |\varphi_x+\psi|^2\,dx + N_1C_{1,\delta}\int_0^L |\Phi|^2\,dx\\ &+ N_1\rho_1\frac{\delta L^2}{2}\int_0^L |\Psi|^2\,dx - N_2\rho_2\int_0^L |\Psi|^2\,dx + N_2k\int_0^L |\varphi_x+\psi|^2\,dx\\ &+ N_2\frac{b\bar{a}}{k}Re\Big\{\int_0^L \overline{\Phi}\psi_x\,dx\Big\} + \rho_2\int_0^L |\Psi|^2\,dx - \frac{3b}{4}\int_0^L |\psi_x|^2\,dx\\ &+ \frac{c_p^2k^2}{b}\int_0^L |\varphi_x+\psi|^2\,dx.\end{aligned}$$

Applying Holder and Young's inequalities we have

$$\frac{d}{dt}\mathcal{G}(t) \leq -\frac{1}{\rho_1}\left(2\bar{a}N_0 - N_1C_{1,\delta} - N_2^2\frac{b\bar{a}^2}{k^2}\right)\rho_1\int_0^L |\Phi|^2 dx - \frac{1}{k}\left(N_1\frac{k}{2} - N_2k - \frac{c_p^2k^2}{b}\right)k\int_0^L |\varphi_x + \psi|^2\, dx$$
$$-\frac{1}{\rho_2}\left(-N_1\rho_1\frac{\delta L^2}{2} + N_2\rho_2 - \rho_2\right)\rho_2\int_0^L |\Psi|^2\, dx - \frac{b}{2}\int_0^L |\psi_x|^2\, dx.$$

Taking $N_2 > 1$ fixed. First, we choose N_1 large enough $\left(N_1 > \frac{2}{k}(N_2k + \frac{c_p^2k^2}{b})\right)$, δ small enough $\left(\delta < \frac{2\rho_2(N_2-1)}{N_1\rho_1L^2}\right)$ and finally taking N_0 large enough $\left(N_0 > \frac{1}{2\bar{a}}(N_1C_{1,\delta} + N_2^2\frac{b\bar{a}^2}{k^2})\right)$ we conclude that there is $k_0 > 0$, such that

$$\frac{d}{dt}\mathcal{G}(t) \leq -k_0E(t).$$

As $\mathcal{G}(t)$ is equivalent to the energy $E(t)$, there are $M > 0$ and $\omega > 0$ such that

$$E(t) \leq ME(0)e^{-\omega t}.$$

Therefore, the proof is complete. □

Remark 3.1. *From* (3.2) *and* (3.1) *we have that, if U is a solution to $(\lambda - \bar{\mathcal{A}})U = \mathcal{F}$, then there is a positive constant such that*

$$\|U\|_{\mathcal{H}} \leq c\|\mathcal{F}\|_{\mathcal{H}}.$$

3.2. **The Indefinite Case.** We will show that the Timoshenko system given by (1.7)-(1.10) is exponentially stable since $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $\|a - \bar{a}\|_{L^2}$ is small enough which will be guaranteed by verifying (3.1).

Firstly, we will show that for any $\lambda \in \mathbb{C}$ the operator $(\lambda I - \mathcal{A})$ is invertible, that is, for any $F \in \mathcal{H}$ exists $W \in D(\mathcal{A})$ such that $(\lambda I - \mathcal{A})W = \mathcal{F}$, which can be written as

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi)_x + \bar{a}\lambda\varphi = \rho_1\lambda\big(\bar{a} - a(x)\big)\varphi + F_1 \tag{3.7}$$
$$\rho_2\lambda^2\psi - b\psi_{xx} + k\big(\varphi_x + \psi\big) = F_2, \tag{3.8}$$

where $F_1 = \lambda\rho_1 f_1 + \rho_1 f_2 - (\bar{a} - a(x))f_1 + \bar{a}f_1$ and $F_2 = \rho_2 f_4 + \rho_2\lambda f_3$.

Our goal is to determine φ and using (3.8) deduce ψ.

So, from (3.7), we have that

$$\varphi_{xx} - \alpha^2\varphi = \frac{\rho_1\lambda}{k}(a(x) - \bar{a})\varphi - \frac{F_1}{k} - \psi_x,$$

with

$$\alpha^2 = \frac{(\rho_1\lambda^2 + \bar{a}\lambda)}{k}.$$

Now, for each $(v, w) \in H_0^1(0, L) \times H_*^1(0, L)$, we define

$$g = \frac{\rho_1\lambda}{k}(a(x) - \bar{a})v - \frac{F_1}{k} - w_x.$$

The Dirichlet Problem given by

$$\begin{cases} u_{xx}(x) - \alpha^2 u(x) = g(x), \\ u(0) = u(L) = 0, \end{cases}$$

has the following solution

$$u(x) = \mathcal{D}_\alpha(g) = \frac{\rho_1\lambda}{k}\mathcal{D}_\alpha\big((a(x) - \bar{a})v\big) - \frac{1}{k}\mathcal{D}_\alpha(F_1) - \mathcal{D}_\alpha(w_x),$$

where

$$D_\alpha(g) = \frac{1}{\alpha}\int_0^x \sinh\big(\alpha(x - s)\big)g(s)\, ds - \frac{1}{\alpha}\frac{\sinh(\alpha x)}{\sinh(\alpha L)}\int_0^L \sinh\big(\alpha(L - s)\big)g(s)\, ds.$$

Therefore, for each $(v, w) \in H_0^1(0, L) \times H_*^1(0, L)$ we consider the following system

$$\begin{cases} \rho_1\lambda^2\varphi - k(\varphi_x+\psi)_x + \bar{a}\lambda\varphi = \rho_1\lambda\big(\bar{a}-a(x)\big)\mathcal{G}(v,w) + F_1 \\ \rho_2\lambda^2\psi - b\psi_{xx} + k\big(\varphi_x+\psi\big) = F_2 \\ \varphi(0)=\varphi(L)=\psi_x(0)=\psi_x(L)=0, \end{cases} \tag{3.9}$$

where

$$\mathcal{G}(v,w) := \frac{\rho_1\lambda}{k}\mathcal{D}_\alpha\big((a(x)-\bar{a})v\big) - \frac{1}{k}\mathcal{D}_\alpha(F_1) - \mathcal{D}_\alpha(w_x).$$

Observe that (3.9) is related with spectral equation

$$(\lambda I - \overline{A})W = \tilde{F}$$

where $\overline{A}$ is the operator $\mathcal{A}$ with $a(x)=\bar{a}$ and $\tilde{F} = (\rho_1\lambda\big(\bar{a}-a(x)\big)\mathcal{G}(v,w)+F_1, F_2)'$. From Remark 2.2 we conclude that (3.9) has a solution $(\varphi,\psi)\in H_0^1(0,L)\times H_*^1(0,L)$ for any $\lambda\in\mathbb{C}$ such that $Re\,\lambda>0$.

Thus, for each $\lambda\in\mathbb{C}$ with $Re\,\lambda>0$ we can say is well defined the following operator

$$\begin{aligned} P : H_0^1(0,L)\times H_*^1(0,L) &\longrightarrow H_0^1(0,L)\times H_*^1(0,L) \\ \big(v,w\big) &\longmapsto P\big(v,w\big) = \big(\varphi,\psi\big), \end{aligned}$$

where $\big(\varphi,\psi\big)$ is a solution of (3.9), where $H_0^1(0,L)\times H_*^1(0,L)$ is a Hilbert space with norm given by

$$\|(v,w)\|_\lambda^2 := \int_0^L \left(\rho_1|\lambda v|^2 + \rho_2|\lambda w|^2 + b|w_x|^2 + k|v_x+w|^2\right)dx. \tag{3.10}$$

Now, we will show that P is a contraction. Before, we observe that, if we consider $d_1\in\mathbb{R}_+$ such that

$$d_1 > \frac{a_\infty}{\rho_1} + 1$$

then from Remark 2.1 and Theorem 5.3, which can be found in Pazy's book [19], we have that for all $\lambda\in\mathbb{C}$ such that

$$Re\lambda > d_1 \quad \text{we obtain} \quad \lambda\in\rho(\mathcal{A}) \quad \text{and} \quad \|(\lambda I-\mathcal{A})^{-1}\| < 1.$$

It is easy to show that $0\in\rho(\mathcal{A})$. As $\rho(\mathcal{A})$ is an open set, we have that there is $r_1>0$ such that for all $\lambda\in\mathbb{C}$, with $|\lambda|\le r_1$, we have $\lambda\in\rho(\mathcal{A})$, in other words, there is a c_0 such that $\|(\lambda I - A)^{-1}\|\le c_0$.

Henceforth, we will consider $\lambda\in\mathbb{C}$ such that

$$0 < Re\lambda \le d_1 \quad \text{and} \quad |\lambda|\ge r_1. \tag{3.11}$$

Since $0<\mathrm{Re}\lambda\le d_1$ there is $C_{\mathcal{D}}>0$ such that the functional D_α defined in (3.2) satisfies

$$|D_\alpha(g)| < \frac{C_{\mathcal{D}}}{|\lambda|}\|g\|_{L^1} \tag{3.12}$$

(more details see Lemma 2.61 in [21]).

Lemma 3.4. *If $\|a-\bar{a}\|_{L^2}<\epsilon$ with $\epsilon>0$ small enough, then P is a contraction.*

Proof. Let us consider $(\varphi^i,\psi^i) = P(v^i,w^i)$ with $i=1,2$. Putting $(\varphi,\psi) = (\varphi^1-\varphi^2,\psi^1-\psi^2)$ and $(v,w) = (v^1-v^2, w^1-w^2)$ then (φ,ψ) and (v,w) satisfy

$$\rho_1\lambda^2\varphi - k(\varphi_x+\psi)_x + \bar{a}\lambda\varphi = \rho_1\lambda\big(\bar{a}-a(x)\big)\mathcal{G}_1(v,w), \tag{3.13}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + k\big(\varphi_x+\psi\big) = 0, \tag{3.14}$$

$$\varphi(0)=\varphi(L)=\psi_x(0)=\psi_x(L)=0, \tag{3.15}$$

where $\mathcal{G}_1(v,w) = \dfrac{\lambda}{k}\mathcal{D}_\alpha\big((a(x)-\bar{a})v\big) - \mathcal{D}_\alpha(w_x)$.

Multiplying (3.13) by $\overline{\lambda\varphi}$ and (3.14) by $\overline{\lambda\psi}$, respectively, and integrating

$$\rho_1\overline{\lambda}\int_0^L|\lambda\varphi|^2\,dx+\rho_2\overline{\lambda}\int_0^L|\lambda\psi|^2\,dx+b\overline{\lambda}\int_0^L|\psi_x|^2\,dx+k\overline{\lambda}\int_0^L|\varphi_x+\psi|^2\,dx$$
$$+\,\bar a\int_0^L|\lambda\varphi|^2\,dx=\rho_1\lambda\int_0^L\big(\bar a-a(x)\big)\mathcal{G}_1(v,w)\overline{\lambda\varphi}\,dx.$$

Taking the real part in the last identity we have

$$\begin{aligned}\mathrm{Re}\lambda\|(\varphi,\psi)\|_\lambda^2 &= -\bar a\int_0^L|\lambda\varphi|^2\,dx+\mathrm{Re}\Big\{\rho_1\lambda\int_0^L\big(a(x)-\bar a\big)\mathcal{G}_1(v,w)\overline{\lambda\varphi}\,dx\Big\}\\ &\le -\bar a\int_0^L|\lambda\varphi|^2\,dx+\rho_1\int_0^L|a(x)-\bar a||\lambda\mathcal{G}_1(v,w)||\lambda\varphi|\,dx.\end{aligned}\tag{3.16}$$

Multiplying (3.13) and (3.14) by $\overline{\varphi}$ and $\overline{\psi}$ in L^2- norm, respectively, and taking into account (3.10), we have

$$\begin{aligned}\|(\varphi,\psi)\|_\lambda^2 &= -\lambda\bar a\int_0^L|\varphi|^2\,dx+\rho_1\lambda\int_0^L\big(\bar a-a(x)\big)\mathcal{G}_1(v,w)\overline{\varphi}\,dx\\ &\le \frac{\bar a}{|\lambda|}\int_0^L|\lambda\varphi|^2\,dx+\rho_1\int_0^L|\bar a-a(x)||\lambda\mathcal{G}_1(v,w)||\varphi|\,dx.\end{aligned}$$

From (3.11) we know that $|\lambda|>|r_1|$, so

$$\begin{aligned}\|(\varphi,\psi)\|_\lambda^2 &\le \frac{\bar a}{r_1}\int_0^L|\lambda\varphi|^2\,dx+\rho_1\int_0^L|\bar a-a(x)||\lambda\mathcal{G}_1(v,w)||\varphi|\,dx\\ &\le c\Big(\bar a\int_0^L|\lambda\varphi|^2\,dx+\rho_1\int_0^L|\bar a-a(x)||\lambda\mathcal{G}_1(v,w)||\varphi|\,dx\Big)\end{aligned}$$

where $c=\max\{\frac{1}{r_1},1\}$, that is, for $\gamma_0=\frac{1}{c}>0$ we have

$$\gamma_0\|(\varphi,\psi)\|_\lambda^2\le\bar a\int_0^L|\lambda\varphi|^2\,dx+\rho_1\int_0^L|a(x)-\bar a||\lambda\mathcal{G}_1(v,w)||\varphi|\,dx.\tag{3.17}$$

Adding (3.16) and (3.17)

$$\Big(\mathrm{Re}\lambda+\gamma_0\Big)\|(\varphi,\psi)\|_\lambda^2\le\rho_1\int_0^L|a(x)-\bar a||\lambda\mathcal{G}_1(v,w)||\varphi|\,dx+\rho_1\int_0^L|a(x)-\bar a||\lambda\mathcal{G}_1(v,w)||\lambda\varphi|\,dx$$

that is,

$$\gamma_0\|(\varphi,\psi)\|_\lambda^2\le\rho_1\int_0^L|a(x)-\bar a||\lambda\mathcal{G}_1(v,w)|\Big(|\varphi|+|\lambda\varphi|\Big)\,dx\tag{3.18}$$

once $\mathrm{Re}\lambda>0$.

From (3.12) we can estimate $|\lambda\mathcal{G}_1(v,w)|$ as follows

$$\begin{aligned}\big|\lambda\mathcal{G}_1(v,w)\big| &\le \frac{|\lambda|^2}{k}\big|\mathcal{D}_\alpha\big((a(x)-\bar a)v\big)\big|+|\lambda|\big|\mathcal{D}_\alpha(w_x)\big|\\ &\le\sqrt{\rho_1}\frac{C_{\mathcal{D}}}{\sqrt{\rho_1}k}\|a-\bar a\|_{L^2}\|\lambda v\|_{L^2}+\sqrt{b}\frac{C_{\mathcal{D}}}{\sqrt{b}k}\|w_x\|_{L^2}\\ &\le C(\sqrt{\rho_1}\|a-\bar a\|_{L^2}\|\lambda v\|_{L^2}+\sqrt{b}\|w_x\|_{L^2})\end{aligned}$$

where $C=\frac{C_{\mathcal{D}}}{k}\max\Big\{\frac{1}{\sqrt{\rho_1}},\frac{1}{\sqrt{b}}\Big\}$.

Now, if $\|a-\bar a\|_{L^2}\le 1$, then

$$\big|\lambda\mathcal{G}_1(v,w)\big|\le C\|(v,w)\|_\lambda.$$

Using (3.18), we obtain

$$\begin{aligned}\gamma_0\|(\varphi,\psi)\|_\lambda^2 &\leq C\|(v,w)\|_\lambda\left(\int_0^L |a(x)-\bar{a}|(|\varphi|+|\lambda\varphi|)\,dx\right)\\ &\leq C\|(v,w)\|_\lambda\|a-\bar{a}\|_{L^2}(\|\varphi\|_{L^2}+\|\lambda\varphi\|_{L^2}).\end{aligned} \tag{3.19}$$

Note that, from $|\lambda|\geq r_1$ we have

$$\|\varphi\|_{L^2}^2=\frac{1}{|\lambda|^2}\int_0^L|\lambda\varphi|^2\,dx\leq\frac{1}{r_1^2}\|\lambda\varphi\|_{L^2}^2,$$

thus, using the last estimate in (3.19) we conclude that

$$\|(\varphi,\psi)\|_\lambda^2\leq\left[\frac{1}{\gamma_0\sqrt{\rho_1}}\left(\frac{1}{r_1}+1\right)C\right]\|a-\bar{a}\|_{L^2}\|(v,w)\|_\lambda\|(\varphi,\psi)\|_\lambda.$$

Therefore, if $\|a-\bar{a}\|_{L^2}<\min\left\{1,\left[\frac{1}{\gamma_0\sqrt{\rho_1}}\left(\frac{1}{r_1}+1\right)C\right]^{-1}\right\}$ then there is a constant $d<1$ such that

$$\|(\varphi,\psi)\|_\lambda\leq d\|(v,w)\|_\lambda, \tag{3.20}$$

that is, P is a contraction. □

Lemma 3.5. *Under the same hypothesis as lemma 3.4, if (φ,ψ) is a fixed point of P, then (φ,ψ) is solution of (3.7)-(3.8).*

Proof. Consider

$$\hat{\varphi}:=\mathcal{G}(\varphi,\psi)=\frac{\rho_1\lambda}{k}\mathcal{D}_\alpha\big((a(x)-\bar{a})\varphi\big)-\frac{1}{k}\mathcal{D}_\alpha(F_1)-\mathcal{D}_\alpha(\psi_x).$$

As (φ,ψ) is a fixed point of P, then it is solution of (3.9), that is

$$\varphi_{xx}-\alpha^2\varphi=\frac{\lambda}{k}(a(x)-\bar{a})\hat{\varphi}-\frac{1}{k}F_1-\psi_x. \tag{3.21}$$

Futhermore, $\hat{\varphi}$ satisfies

$$\hat{\varphi}_{xx}-\alpha^2\hat{\varphi}=\frac{\lambda}{k}(a(x)-\bar{a})\varphi-\frac{1}{k}F_1-\psi_x. \tag{3.22}$$

Taking $\widetilde{\Phi}=\varphi-\hat{\varphi}$ and subtracting (3.21) from (3.22), we have

$$\begin{cases}\widetilde{\Phi}_{xx}-\alpha^2\widetilde{\Phi}=\dfrac{\lambda}{k}\big(\bar{a}-a(x)\big)\widetilde{\Phi}\\ \widetilde{\Phi}(0)=\widetilde{\Phi}(L)=0.\end{cases} \tag{3.23}$$

Observe that, $\widetilde{\Phi}$ is a solution of Dirichlet Problem, so its is given by

$$\widetilde{\Phi}=\mathcal{D}_\alpha\left(\frac{\lambda}{k}\big(\bar{a}-a(x)\big)\widetilde{\Phi}\right).$$

Now, using (3.12) we can estimate

$$\big|\widetilde{\Phi}\big|=\left|\frac{\lambda}{k}\mathcal{D}_\alpha\left((a(x)-\bar{a})\widetilde{\Phi}\right)\right|\leq\frac{C_{\mathcal{D}}}{k}\|(a-\bar{a})\widetilde{\Phi}\|_{L^1},$$

which give us

$$\|\Phi\|_{L^2}\left(1-\frac{C_{\mathcal{D}}\sqrt{L}}{k}\|a-\bar{a}\|_{L^2}\right)\leq 0.$$

If $\|a-\bar{a}\|_{L^2}<\frac{k}{C_{\mathcal{D}}\sqrt{L}}$, then $\|\widetilde{\Phi}\|_{L^2}=0$. Therefore, our conclusion follows. □

Now we are able to prove the main result of this work.

Theorem 3.3. *Assume (1.3) and (1.4). If $\|a-\bar{a}\|_{L^2}<\epsilon$ with $\epsilon>0$ small enough then the semigroup associated to the system (1.7)-(1.10) is exponentially stable.*

Proof. Using the previous lemmas and keeping in mind (3.11) we have that for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ and any $\mathcal{F} \in \mathcal{H}$ there is a unique solution W to $(\lambda - \mathcal{A})W = \mathcal{F}$, that is, $(\lambda I - \mathcal{A})^{-1}$ exist. Now, our goal is to show that this operator is bounded, more specifically, to show that the second condition of (3.1) is satisfied. So, consider $W = (\varphi, \Phi, \psi, \Psi) = (\lambda I - \mathcal{A})^{-1}\mathcal{F}$, where $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, that is

$$W = (\varphi, \lambda\varphi - f_1, \psi, \lambda\psi - f_3) = (\varphi, \lambda\varphi, \psi, \lambda\psi) + (0, -f_1, 0, -f_3).$$

Thus,

$$\|(\varphi, \psi)\|_\lambda = \|(\varphi, \lambda\varphi, \psi, \lambda\psi)\|_{\mathcal{H}} = \|W - (0, -f_1, 0, -f_3)\|_{\mathcal{H}} \leq \|W\|_{\mathcal{H}} + \|\mathcal{F}\|_{\mathcal{H}}. \tag{3.24}$$

On the other hand, let $\widetilde{W}$ be the solution to $(\lambda I - \bar{\mathcal{A}})\widetilde{W} = \mathcal{F}$, that is,

$$\widetilde{W} = (\widetilde{\varphi}, \widetilde{\Phi}, \widetilde{\psi}, \widetilde{\Psi}) = (\widetilde{\varphi}, \lambda\widetilde{\varphi}, \widetilde{\psi}, \lambda\widetilde{\psi}) - (0, -f_1, 0 - f_3).$$

Then,

$$\begin{aligned} \|W\|_{\mathcal{H}} - \|\widetilde{W}\|_{\mathcal{H}} &\leq \|W - \widetilde{W}\|_{\mathcal{H}} \\ &= \|(\varphi, \lambda\varphi, \psi, \lambda\psi) - (\widetilde{\varphi}, \lambda\widetilde{\varphi}, \widetilde{\psi}, \lambda\widetilde{\psi})\|_{\mathcal{H}} \\ &= \|(\varphi, \psi) - (\widetilde{\varphi}, \widetilde{\psi})\|_\lambda. \end{aligned}$$

From $P(0,0) = (\widetilde{\varphi}, \widetilde{\psi})$, $P(\varphi, \psi) = (\varphi, \psi)$ and the fact that P is a contraction, we have that

$$\|W\|_{\mathcal{H}} - \|\widetilde{W}\|_{\mathcal{H}} \leq \|P(\varphi, \psi) - P(0,0)\|_\lambda \leq d\|(\varphi, \psi)\|_\lambda. \tag{3.25}$$

Hence, from (3.24) and (3.25) follow

$$\|W\|_{\mathcal{H}} - \|\widetilde{W}\|_{\mathcal{H}} \leq d\|W\|_{\mathcal{H}} + d\|\mathcal{F}\|_{\mathcal{H}},$$

that is,

$$(1 - d)\|W\|_{\mathcal{H}} \leq \|\widetilde{W}\|_{\mathcal{H}} + d\|\mathcal{F}\|_{\mathcal{H}}.$$

Finally, from Remark 3.1 follows that there is a positive constant $\bar{c}$ such that

$$\|W\|_{\mathcal{H}} \leq \bar{c}\|\mathcal{F}\|_{\mathcal{H}},$$

that is, $(\lambda I - \mathcal{A})^{-1}$ is bounded. This completes our proof. □

4. Numerical Results

In this section, we present some numerical results illustrating the asymptotic behavior of the energy and as well as the relevance of the condition (1.4) for the exponential decay.

We study numerically here, the decay of the energy. For this, we use Finite Difference (of second order in space and time). Furthermore, the method of $\beta-$Newmark is a second order method preserving the discrete energy always when the discrete system of equations of motion is symmetric (i.e. matrices associated to the system should be symmetric).

4.1. **Finite difference method.** We consider J an integer non-negative and $h = L/(J+1)$ an spatial subdivision of the interval $(0, L)$ given by $0 = x_0 < x_1 < \ldots < x_J < x_{J+1} = L$, with $x_j = jh$ each node of the mesh. We use $\varphi_j(t)$, $\psi_j(t)$, for all $j = 1, 2, \ldots, J$ and $t > 0$ to denote the approximate values of $\varphi(jh, t)$ and $\psi(jh, t)$, respectively. In addition, we denote the discrete operator $\Delta_h\vartheta_j = \frac{\vartheta_{j+1} - 2\vartheta_j + \vartheta_{j-1}}{h^2}$ and $\delta_h\vartheta_j = \frac{\vartheta_{j+1} - \vartheta_{j-1}}{2h}$. We assume the following finite difference scheme applied to system (1.7)-(1.10)

$$\rho_1\varphi_j'' - \kappa\Delta_h\varphi_j - \kappa\frac{\psi_{j+1} - \psi_{j-1}}{2h} + a_j\frac{\psi_{j+1}' + 2\psi_j' + \psi_{j-1}'}{4} = 0, \tag{4.1}$$

$$\rho_2\psi_j'' - b\Delta_h\psi_j + \kappa\frac{\varphi_{j+1} - \varphi_{j-1}}{2h} + \kappa\frac{\psi_{j+1} + 2\psi_j + \psi_{j-1}}{4} = 0, \qquad j = 1, \ldots, J, \tag{4.2}$$

$$\varphi_0 = \varphi_J = \psi_1 - \psi_0 = \psi_{J+1} - \psi_J = 0, \tag{4.3}$$

$$\varphi_j(0) = \varphi_j^0, \ \varphi_j'(0) = \varphi_j^1, \ \psi_j(0) = \psi_j^0, \ \psi_j'(0) = \psi_j^1, \qquad j = 1, \ldots, J, \tag{4.4}$$

where $a_j = a(x_j)$, for $j = 1, \ldots, J$. We note that $\psi(x_j)$ and $\psi'(x_j)$ are approximated by

$$\psi(x_j) \approx \Theta^h_{1/4}\psi_j := \frac{\psi_{j+1} + 2\psi_j + \psi_{j-1}}{4}, \qquad \psi'(x_j) \approx \Theta^h_{1/4}\psi'_j := \frac{\psi'_{j+1} + 2\psi'_j + \psi'_{j-1}}{4},$$

respectively. These approaches are particular cases of the well known θ-scheme with $\theta = 1/4$, in order to obtain uniform observability for the discrete Timoshenko System (see for more details [2, 3]). The discrete Energy of (4.1)-(4.4) is given by

$$E_\Delta(t) = \rho_1 \frac{h}{2}\sum_{j=0}^{J} |\varphi'_j|^2 + \rho_2 \frac{h}{2}\sum_{j=0}^{J} |\psi'_j|^2 + \frac{h}{2}\left[b\left|\frac{\psi_{j+1} - \psi_j}{h}\right|^2 + \kappa\left|\frac{\varphi_{j+1} - \varphi_j}{h} + \frac{\psi_{j+1} + \psi_j}{2}\right|^2\right]. \tag{4.5}$$

4.2. **Equation of motion and time discretization.** The system (4.1)-(4.3) can be rewritten as

$$\mathbf{M}\begin{bmatrix} \ddot{\varphi}_h \\ \ddot{\psi}_h \end{bmatrix} + \mathbf{C}\begin{bmatrix} \dot{\varphi}_h \\ \dot{\psi}_h \end{bmatrix} + \mathbf{K}\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} = \mathbf{0}, \tag{4.6}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system in $\mathcal{M}_{2J}(\mathbb{R})$, and $\varphi_h = (\varphi_1, \ldots, \varphi_J)^\top$, $\psi_h = (\psi_1, \ldots, \psi_J)^\top \in \mathbb{R}^J$.

The Newmark algorithm [18] is based on a set of two relations expressing the forward displacement $[\varphi_h^{n+1}, \psi_h^{n+1}]^\top$ and velocity $[\Phi_h^{n+1}, \Psi_h^{n+1}]^\top = [\dot{\varphi}_h^{n+1}, \dot{\psi}_h^{n+1}]^\top$. The method consists in updating the displacement, velocity and acceleration vectors from current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$,

$$\Phi_h^{n+1} = \Phi_h^n + (1-\gamma)\delta t\, \dot{\Phi}_h^n + \gamma\delta t\, \dot{\Phi}_h^{n+1} \tag{4.7}$$

$$\varphi_h^{n+1} = \varphi_h^n + \left(\frac{1}{2} - \beta\right)\delta t^2\, \dot{\Phi}_h^n + \beta\delta t^2\, \dot{\Phi}_h^{n+1} \tag{4.8}$$

$$\Psi_h^{n+1} = \Psi_h^n + (1-\gamma)\delta t\, \dot{\Psi}_h^n + \gamma\delta t\, \dot{\Psi}_h^{n+1} \tag{4.9}$$

$$\psi_h^{n+1} = \psi_h^n + \left(\frac{1}{2} - \beta\right)\delta t^2\, \dot{\Psi}_h^n + \beta\delta t^2\, \dot{\Psi}_h^{n+1}, \tag{4.10}$$

where β and γ are parameters of the methods that will be fixed later. Replacing (4.7)-(4.10) in the equation of motion (4.6), we obtain

$$\begin{aligned}\left(\mathbf{M} + \gamma\delta t\,\mathbf{C} + \beta\delta t^2\mathbf{K}\right)\begin{bmatrix} \dot{\Phi}_h^{n+1} \\ \dot{\Psi}_h^{n+1} \end{bmatrix} &= -\mathbf{C}\left(\begin{bmatrix} \Phi_h^n \\ \Psi_h^n \end{bmatrix} + (1-\gamma)\delta t\begin{bmatrix} \dot{\Phi}_h^n \\ \dot{\Psi}_h^n \end{bmatrix}\right) \\ &\quad -\mathbf{K}\left(\begin{bmatrix} \varphi_h^n \\ \psi_h^n \end{bmatrix} + \delta t\begin{bmatrix} \Phi_h^n \\ \Psi_h^n \end{bmatrix} + \left(\frac{1}{2} - \beta\right)\delta t^2\begin{bmatrix} \dot{\Phi}_h^n \\ \dot{\Psi}_h^n \end{bmatrix}\right).\end{aligned} \tag{4.11}$$

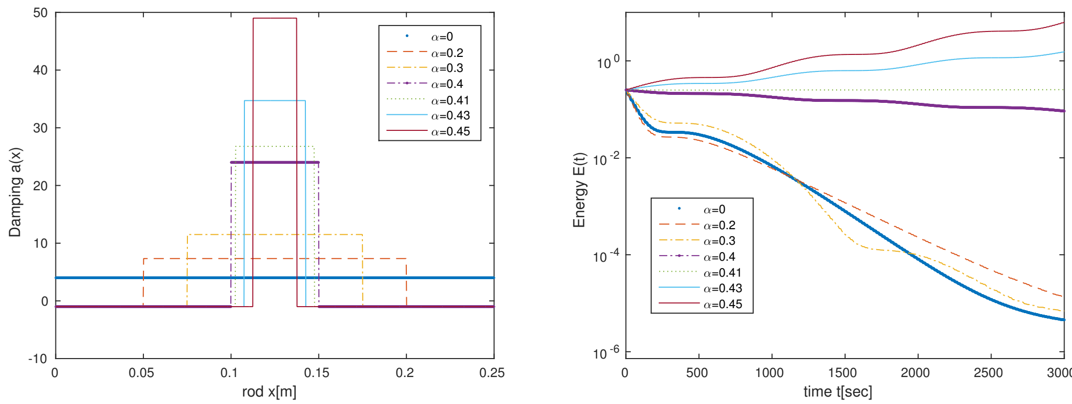
The acceleration $[\dot{\Phi}_h^{n+1}, \dot{\Psi}_h^{n+1}]^\top$ is computed from (4.11), and the velocities $[\Phi_h^{n+1}, \Psi_h^{n+1}]^\top$ are obtained from (4.7) and (4.9), respectively. Finally, the displacement $[\varphi_h^{n+1}, \psi_h^{n+1}]^\top$ follows from (4.8) and (4.10), by simple matrix operations. Thus, the fully discrete energy of the system (4.7)-(4.11) is given by

$$\mathcal{E}_h^n := \frac{1}{2}\left[\Phi_h^\top, \Psi_h^\top\right]\mathbf{M}\begin{bmatrix} \Phi_h \\ \Psi_h \end{bmatrix} + \frac{1}{2}\left[\varphi_h^\top, \psi_h^\top\right]\mathbf{K}\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} \tag{4.12}$$

which is an approximation of energy for the continuous case. The increment of this energy can be expressed in terms of mean values and increments of the displacement and velocity. Then, we choose $\gamma = \frac{1}{2}$ and $\beta = \frac{\gamma}{2}$, reducing the above expression to

$$\mathcal{E}_\delta^{n+1} - \mathcal{E}_\delta^n = -\frac{1}{2}\left\{\left[\Delta\varphi_h^\top, \Delta\psi_h^\top\right]\mathbf{C}\begin{bmatrix} \Delta\varphi_h \\ \Delta\psi_h \end{bmatrix} + \delta t\left[\Phi_h^{n+\frac{1}{2},\top}, \Psi_h^{n+\frac{1}{2},\top}\right]\mathbf{C}\begin{bmatrix} \Phi_h^{n+\frac{1}{2}} \\ \Psi_h^{n+\frac{1}{2}} \end{bmatrix}\right\} \leqslant 0.$$

With this, the fully discrete energy obtained by the $\beta-$Newmark method is decreasing and we expect that its asymptotic behavior be a reflection of the continuous case (see [15] and also [4, 5]).



(A) $a(x) = \begin{cases} h = \frac{1/L+2\alpha}{1-2\alpha} & \text{if } \alpha \leqslant \frac{x}{L} \leqslant 1-\alpha \\ -1 & \text{otherwise.} \end{cases}$

(B) Asymptotic behaviour of the energies in semilog-scale.

FIGURE 1. Asymptotic behavior of the energies for different cases of indefinite dissipation functions; Exponential decay is observed for $\alpha < 0.4$.

4.3. **Numerical examples.** We make simulations with parameters $\rho_1 = \rho_2 = \beta = \kappa = 1$, taking into account the condition (1.3). Our purpose here is to test the asymptotic behavior of the energy for different kinds of indefinite damping of type $a(x)\psi_t$. More precisely, we focus on the following family of damping functions (see Figure 1(A)):

$$a(x) = \begin{cases} h = \dfrac{1/L+2\alpha}{1-2\alpha} & \text{if } \alpha \leqslant \dfrac{x}{L} \leqslant 1-\alpha \\ -1 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0, \frac{1}{2})$. This family of damping functions changes its sign and

$$L\bar{a} = \int_0^L a(x)\, dx = 1 > 0,$$

which is compatible with the condition (1.4). On the other hand, The condition $\|a - \bar{a}\|_{L^2} < \epsilon$, for ϵ small enough, is satisfied or not, depending on the value of α, where the distance between $a(x)$ and $\bar{a}$ in L^2-norm is determined according to Table 1.

α	0.0	0.2	0.3	0.4	0.41	0.43	0.45
$\|a-\bar{a}\|_{L^2}$	0.0	2.0412	3.0619	5.0	5.3359	6.1962	7.5

TABLE 1. Table of distance from the indefinite dissipation function in relation with its average.

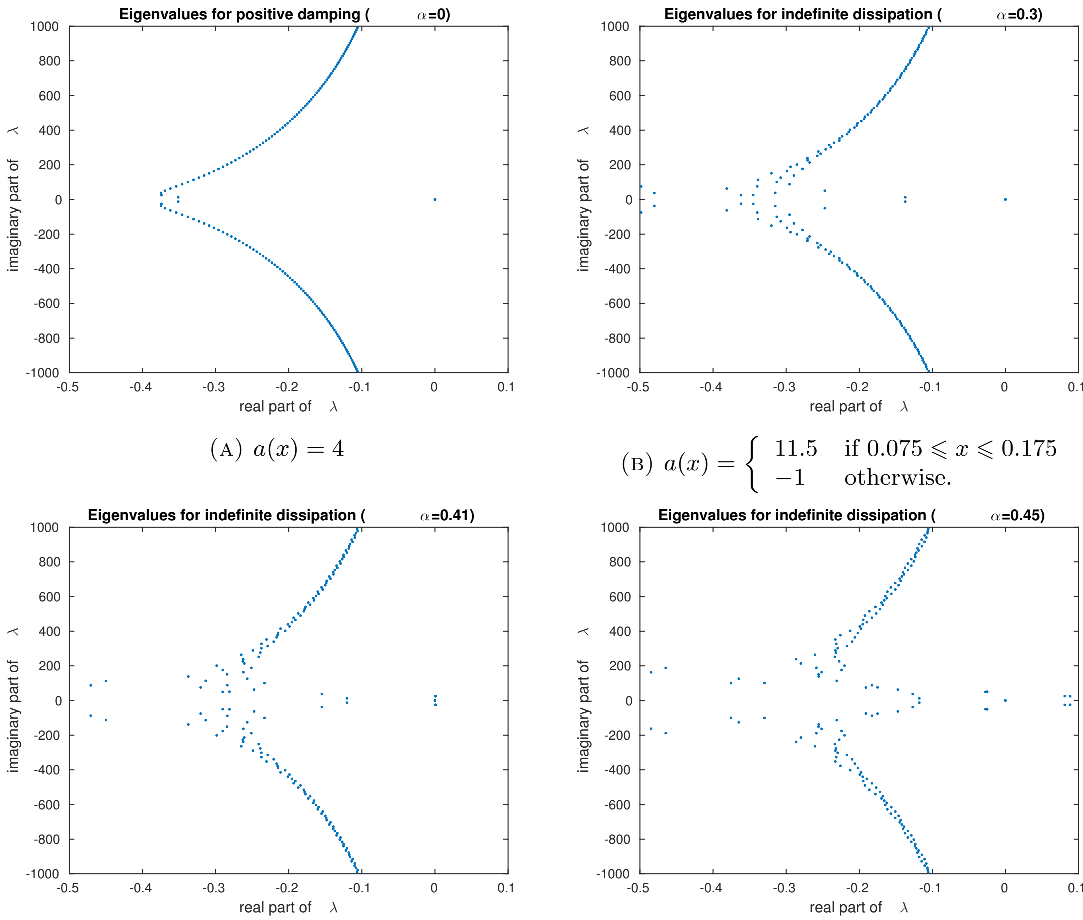
4.4. **Numerical simulation of critical cases and initial conditions.** In order to evidence the importance of the second condition (1.4), which guarantee the exponential decay of energy, we perform here several numerical experiments for different indefinite dissipation functions and a simple initial condition given by

$$\varphi(0,x) = \sin\frac{2\pi x}{L}, \qquad \psi(0,x) = x - \frac{L}{2}, \qquad \varphi_t(0,x) = \psi_t(0,x) = 0. \tag{4.13}$$

Furthermore, we consider $L = 0.25$, $J = 1000$, $h = L/(J+1)$, $T = 10.000$, $\delta t = 1$.

Considering the family of indefinite dissipation functions for different values of α, the results of these simulations are observed in Figure 1(B), where we see exponential decay of the energy for $\alpha \leqslant 0.4$, and lack of exponential decay for $\alpha > 0.4$, reaching even an increase of energy due to a phenomenon of anti-dissipation for cases $\alpha = 0.43$ and $\alpha = 0.45$.

From this graph and based in Table 1, we can interpret the second condition (1.4) for the exponential decay of the energy of a numerical point of view.



(A) $a(x) = 4$

(B) $a(x) = \begin{cases} 11.5 & \text{if } 0.075 \leqslant x \leqslant 0.175 \\ -1 & \text{otherwise.} \end{cases}$

(C) $a(x) = \begin{cases} 26.8 & \text{if } 0.1025 \leqslant x \leqslant 0.1475 \\ -1 & \text{otherwise.} \end{cases}$

(D) $a(x) = \begin{cases} 49 & \text{if } 0.1125 \leqslant x \leqslant 0.1375 \\ -1 & \text{otherwise.} \end{cases}$

FIGURE 2. Eigenvalues for different indefinite dissipations ($L = 0.25$, and $\int_0^L a(x)dx = 1$).

In the proof of Lemma 3.4, it was necessary at least $\|a - \bar{a}\|_{L^2} < 1$ to obtain the exponential decay of energy. On the other hand, in this section we check numerically that if $\|a - \bar{a}\|_{L^2} > 5$, then the energy grows (see curve for $\alpha = 0.41$ in Figure 1(B), and Table 1). As an open problem, it remains the critical value of $\epsilon > 0$ for which there is no longer exponential decay of the energy, independent of the initial condition and the shape of the indefinite damping function. Indeed, it is likely that for a not so smooth initial condition, and perhaps for other family of indefinite damping functions, the energy grows with a smaller value of $\|a - \bar{a}\|_{L^2}$. These numerical examples are not intended to find the critical value of $\epsilon > 0$, but simply to highlight that indeed there exists $\epsilon > 0$ small enough, such that the condition (1.4) is necessary for the exponential decay of the energy.

Now, we made another approach through a numerical analysis of the eigenvalues associated to the Timoshenko System (1.7)-(1.10).

4.5. **Plotting Eigenvalues.** We present numerical results on the linear stability of our system. We use normal mode analysis and set

$$\varphi(x, t) = e^{\lambda t} P(x), \qquad \psi(x, t) = e^{\lambda t} Q(x).$$

Thus, (1.7)-(1.10) become the following eigenvalues problem:

$$\kappa(P_x+Q)_x + a(x)\lambda P = \lambda^2\rho_1 P \tag{4.14}$$

$$bQ_{xx} - \kappa(P_x+Q) = \lambda^2\rho_2 Q \tag{4.15}$$

$$P(0) = P(L) = Q_x(0) = Q_x(L) = 0 \tag{4.16}$$

We approximate $P(x_j)$, $Q(x_j)$ with $j = 1,\dots,J$, by $\mathbf{p}$, $\mathbf{q} \in \mathbb{R}^J$, and we discretize (4.14)-(4.16) by finite difference as in (4.1)-(4.4), obtaining a $4J \times 4J$ matrix equation given by

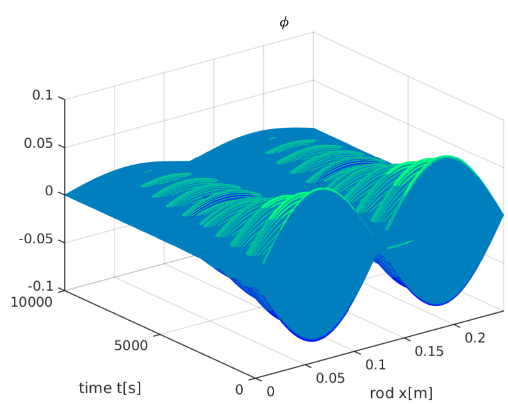
$$\begin{pmatrix} \mathbf{O} & \frac{1}{\rho_1}\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{K}_1 & \mathbf{C} & \mathbf{K}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{1}{\rho_2}\mathbf{I} \\ \mathbf{K}_4 & \mathbf{O} & \mathbf{K}_3 & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{r} \\ \mathbf{q} \\ \mathbf{z} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{p} \\ \mathbf{r} \\ \mathbf{q} \\ \mathbf{z} \end{pmatrix} \tag{4.17}$$

where $\mathbf{K}_i$, with $i = 1,\dots,4$ are the $J \times J$ hyperbolic terms matrices given by $\mathbf{K}_1 = \kappa\Delta_h$, $\mathbf{K}_2 = \mathbf{K}_4^t = \kappa\delta_h$, $\mathbf{K}_3 = b\Delta_h - \kappa\Theta^h_{1/4}$. On the other hand, $\mathbf{C} = diag(a_j)\Theta^h_{1/4}$ correspond to the indefinite dissipation matrix, $\mathbf{O}$ is the null matrix, and $\mathbf{I}$ is the identity.

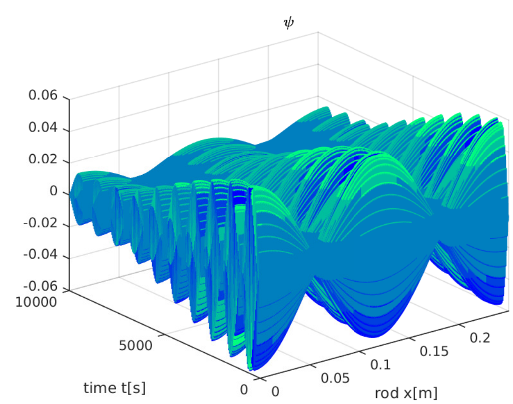
We obtain results for different values of α, in Figure 2. The case $\alpha = 0$ (see Figure 2(A)), corresponds to the possitive and constant damping $a(x) = 4$. In this case, all the eigenvalues are away from the imaginary axis, except the case $\lambda = \pm i\sqrt{\kappa/\rho_2}$, where the eigenfunction corresponds to a constant different to zero. This eigenfunction is easy to avoid in the study of the decay of energy, imposing $\int \psi(x,t)dx = 0$. It is inevitably seen in the graph of the figure 2(A) but since it was not the goal of our study, it was discarded.

The case $\alpha = 0.3$ (see Figure 2(B)), correspond to the damping $a(x) = 11.5$, if $0.075 \leqslant x \leqslant 0.175$, and $a(x) = -1$, otherwise (see Figure 1(A)). This case is similar to the preceding: all the eigenvalues are away from the imaginary axis, except the case $\lambda = \pm i\sqrt{\kappa/\rho_2}$, which was discarded, due to the aim of our study. On the other hand, the case $\alpha = 0.41$ (see Figure 2(C)), correspond to the damping $a(x) = 26.8$, if $0.1025 \leqslant x \leqslant 0.1475$, and $a(x) = -1$, otherwise. In this case we observe four eigenvalues close to the imaginary axis. In fact two of them are on the imaginary axis ($\lambda_{1,2} = \pm i\sqrt{\kappa/\rho_2}$) and the other two have slightly positive real part: $\lambda_{3,4} = 0.0007 \pm i25.63$. Finally, in the case $\alpha = 0.45$ (see Figure 2(D)), there are many eigenvalues with real part postive which is not surprising by the fact that energy is strictly increasing in time, as seen in Figure 1(B).

4.6. **Graph of the solution.** Finally, in this subsection, we show the graph of the complete solution $\varphi(x,t)$ and $\psi(x,t)$ for the case $\alpha = 0.4$, and with the initial condition (4.13). This example corresponds to a critical case when the energy slowly decays according to the graph in Figure 1(B). We observe the decay of both φ and ψ in time in Figure 3. The asymptotic behavior of φ_t and ψ_t is completely analogous.



(A) $\varphi(x,t)$



(B) $\psi(x,t)$

FIGURE 3. Asymptotic behavior of the solution $\varphi(x,t)$ and $\psi(x,t)$ for the case $\alpha = 0.4$.

References

[1] Alabau-Boussouira F. (2007), *Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control*, NoDEA Nonlinear Diferential Equations Appl. 14 , no. 5-6, 643-669.
[2] Almeida Júnior D. S., Muñoz Rivera J. E. , Santos M. L. (2012) , *The stability number of the Timoshenko system with second sound*, J. Diferential Equations 253 , no. 9, 2715-2733.
[3] Almeida Júnior D. S., Santos M. L. and Muñoz Rivera J. E. (2013), *Stability to weakly dissipative Timoshenko systems*, Mathematical Methods in the Applied Sciences , 1965-1976.
[4] Alves M. , Muñoz-Rivera J.E., Sepúlveda M., Vera O. and Zegarra M. (2014), *The asymptotic behaviour of the linear transmission problem in viscoelasticity,* Math. Nachr., 287 , no. 5-6, 483-497.
[5] Alves M., Muñoz-Rivera J., Sepúlveda M. and Vera O. (2014), *Exponential and the lack of exponential stability in transmission problems with localized Kelvin-Voigt dissipation,* SIAM J. Appl. Math., 74 no. 2, 354-365 .
[6] Ammar-Khodja F. , Benabdallah A., Muñoz Rivera J. E. and Racke R. (2003), *Energy decay for Timoshenko systems of memory type*, J. Diferential Equations 194 , no. 1, 82-115.
[7] Ammar-Khodja F. , S. Kerbal and Soufyane A. (2007), *Stabilization of the nonuniform Timoshenko beam*, J. Math. Anal. Appl. 327 , no. 1, 525-538.
[8] Cavalcanti M.M., Domingos Cavalcanti V. N. , Falcão Nascimento F. A., Lasiecka I. , Rodrigues J. H. (2014) , *Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping*, Z. Angew. Math. Phys. 65 1189-1206.
[9] Dell'Oro F. and Pata V. (2014), *On the stability of Timoshenko systems with Gurtin-Pipkin thermal law*, J. Differential Equations 257 , no. 2, 523-548.
[10] Engel K. and Nagel R., *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, NY, 2006.
[11] Fatori L. H. , Monteiro R. N. and Muñoz Rivera J. E. , *Energy decay to Timoshenko's system with thermoelasticity of type III*, Asymptot. Anal. 86 (2014), no. 3-4, 227-247.
[12] Fernandez Sare H. D. and Racke R. (2009), *On the stability of damped Timoshenko systems: Cattaneo versus Fourier law*, Arch. Ration. Mech. Anal. 194 , no. 1, 221-251.
[13] Guesmia A.and S. A. Messaoudi, *A general stability result in a Timoshenko systemwith infnite memory: a new approach*, Math. Methods Appl. Sci. 37 (2014), no. 3, 384-392.
[14] Kim JU, Renardy Y. (1987), *Boundary control of the Timoshenko beam*, SIAM Journal on Control and Optimization , 25(6):1417-1429.
[15] Krenk S. (2006), *Energy conservation in Newmark based time integration algorithms,* Comput. Methods Appl. Mech. Engrg., 195 , no. 44-47, 6110-6124.
[16] Muñoz Rivera J. E. , Racke R.(2002), *Mildly dissipative nonlinear Timoshenko systems global existence and exponential stability*, J. Math. Anal. Appl. 276 , no. 1, 248-278.
[17] Muñoz Rivera J. E. and Racke R. (2008), *Timoshenko Systems with Indefinite Damping*, Journal of Mathematical Analysis and Applications , 1068-1083.
[18] Newmark N. M. (1959) *A method of computation for structural dynamics,* J. Engrg. Mech. Div. ASCE, 85 , 67-94.
[19] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[20] Raposo C. A., Ferreira J. , Santos M. L. and Castro N. N. O. (2005), *Exponential stability for the Timoshenko system with two weak dampings*, Appl. Math. Lett. 18 , no. 5, 535-541.
[21] Sato, T. O.(2016) *Sistema de Timoshenko com amortecimento indefinido na oscilação transversal*, Universidade Estadual de Londrina, Londrina, Dissertação de Mestrado.
[22] Shi D-H, Feng D-X, *Exponential decay of Timoshenko beam with locally distributed feedback*, Proceeding of the 99'IFAC-World Congress, Beijing. Vol F.
[23] Soufyane A. (2009) , *Exponential stability of the linearized nonuniform Timoshenko beam*, Nonlinear Anal. Real World Appl. 10, no. 2, 1016-1020.
[24] Soufyane A.(1999), *Stabilisation de la poutre de Timoshenko*, C. R. Acad. Sci. Paris Sér. I Math. 328 , no. 8, 731-734.

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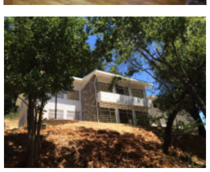

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