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# A posteriori error estimates for a Virtual Elements Method for the Steklov eigenvalue problem. 

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#### Abstract

The paper deals with the a posteriori error analysis of a virtual element method for the Steklov eigenvalue problem. The virtual element method has the advantage of using general polygonal meshes, which allows implementing very efficiently mesh refinement strategies. We introduce a residual type a posteriori error estimator and prove its reliability and efficiency. We use the corresponding error estimator to drive an adaptive scheme. Finally, we report the results of a couple of numerical tests, that allow us to assess the performance of this approach.


Key words: virtual element method, a posteriori error estimates, Steklov eigenvalue problem, polygonal meshes
2000 MSC: $65 \mathrm{~N} 25,65 \mathrm{~N} 30,74 \mathrm{~S} 99$.

## 1. Introduction

The Virtual Element Method (VEM), introduced in [7, 8], is a recent generalization of the Finite Element Method, which is characterized by the capability of dealing with very general polygonal/polyhedral meshes. The interest in numerical methods that can make use of general polytopal meshes has recently undergone a significant growth in the mathematical and engineering literature; among the large number of papers on this subject, we cite as a minimal sample [7, 9, 20, $26,35,36,37]$. Indeed, polytopal meshes can be very useful for a wide range of reasons including meshing of the domain, automatic use of hanging nodes, moving meshes and adaptivity. VEM has been applied successfully in a large range of problems; see for instance $[1,3,6,7,8,10,12,13,16$, $18,29,31,32,33]$.

The object of this paper is to introduce and analyze an a posteriori error estimator of residual type for the virtual element approximation of the Steklov eigenvalue problem. In fact, due to the large flexibility of the meshes to which the virtual element method is applied, mesh adaptivity becomes an appealing feature as mesh refinement strategies can be implemented very efficiently. For instance, hanging nodes can be introduced in the mesh to guarantee the mesh conformity without spreading the refined zones. In fact hanging nodes introduced by the refinement of a neighboring

[^0]element are simply treated as new nodes since adjacent non matching element interfaces are perfectly acceptable. On the other hand, polygonal cells with very general shapes are admissible thus allowing us to adopt simple mesh coarsening algorithms.

The approximation of eigenvalue problems has been the object of great interest from both the practical and theoretical points of view, since they appear in many applications. We refer to [17] and the references therein for the state of art in this subject area. In particular, the Steklov eigenvalue problem, which involves the Laplace operator but is characterized by the presence of the eigenvalue in the boundary condition, appears in many applications; for example, the study of the vibration modes of a structure in contact with an incompressible fluid (see [14]) and the analysis of the stability of mechanical oscillators immersed in a viscous media (see [34]). One of its main applications arises from the dynamics of liquids in moving containers, i.e., sloshing problems (see [15, 19, 23, 24, 28, 39]).

On the other hand, adaptive mesh refinement strategies based on a posteriori error indicators play a relevant role in the numerical solution of partial differential equations in a general sense. For instance, they guarantee achieving errors below a tolerance with a reasonable computer cost in presence of singular solutions. Several approaches have been considered to construct error estimators based on the residual equations (see $[2,27,38]$ and the references therein). In particular, for the Steklov eigenvalue problem we mention $[4,5,25,30,40]$. On the other hand, the design and analysis of a posteriori error bounds for the VEM is a challenging task. References [11, 21] are the only a posteriori error analyses for VEM currently available in the literature. In [11], a posteriori error bounds for the $C^{1}$-conforming VEM for the two-dimensional Poisson problem are proposed. In [21], a posteriori error bounds are introduced for the $C^{0}$-conforming VEM proposed in [22] for the discretization of second order linear elliptic reaction-convection-diffusion problems with non constant coefficients in two and three dimensions.

We have recently developed in [31] a virtual element method for the Steklov eigenvalue problem. Under standard assumptions on the computational domain, we have established that the resulting scheme provides a correct approximation of the spectrum and proved optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. In order to exploit the capability of VEM in the use of general polygonal meshes and its flexibility for the application of mesh adaptive strategies, we introduce and analyze an a posteriori error estimator for the virtual element approximation introduced in [31]. Since normal fluxes of the VEM solution are not computable, they will be replaced in the estimators by a proper projection. As a consequence of this replacement, new additional terms appear in the a posteriori error estimator, which represent the virtual inconsistency of VEM. Similar terms also appear in the other papers for a posteriori error estimates of VEM (see [11, 21]). We prove that the error estimator is equivalent to the error and use the corresponding indicator to drive an adaptive scheme.

The outline of this article is as follows: in Section 2 we present the continuous and discrete formulations of the Steklov eigenvalue problem together with the spectral characterization. Then, we recall the a priori error estimates for the virtual element approximation analyzed in [31]. In Section 3 , we define the a posteriori error estimator and proved its reliability and efficiency. Finally, in Section 4, we report a set of numerical tests that allow us to assess the performance of an adaptive strategy driven by the estimator. We have also made a comparison between the proposed estimator and the standard edge-residual error estimator for a finite element method.

Throughout the article we will denote by $C$ a generic constant independent of the mesh parameter $h$, which may take different values in different occurrences.

## 2. The Steklov eigenvalue problem and its virtual element approximation

Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain with polygonal boundary $\partial \Omega$. Let $\Gamma_{0}$ and $\Gamma_{1}$ be disjoint open subsets of $\partial \Omega$ such that $\partial \Omega=\bar{\Gamma}_{0} \cup \bar{\Gamma}_{1}$ with $\Gamma_{0} \neq \emptyset$. We denote by $n$ the outward unit normal vector to $\partial \Omega$.

We consider the following eigenvalue problem:
Find $(\lambda, w) \in \mathbb{R} \times H^{1}(\Omega), w \neq 0$, such that

$$
\left\{\begin{array}{l}
\Delta w=0 \quad \text { in } \Omega \\
\frac{\partial w}{\partial n}= \begin{cases}\lambda w & \text { on } \Gamma_{0} \\
0 & \text { on } \Gamma_{1}\end{cases}
\end{array}\right.
$$

By testing the first equation above with $v \in H^{1}(\Omega)$ and integrating by parts, we arrive at the following equivalent weak formulation:
Problem 1. Find $(\lambda, w) \in \mathbb{R} \times H^{1}(\Omega), w \neq 0$, such that

$$
\int_{\Omega} \nabla w \cdot \nabla v=\lambda \int_{\Gamma_{0}} w v \quad \forall v \in H^{1}(\Omega) .
$$

According to [31, Theorem 2.1], we know that the solutions $(\lambda, w)$ of the problem above are:

- $\lambda_{0}=0$, whose associated eigenspace is the space of constant functions in $\Omega$;
- a sequence of positive finite-multiplicity eigenvalues $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}}$ such that $\lambda_{k} \rightarrow \infty$.

The eigenfunctions corresponding to different eigenvalues are orthogonal in $L^{2}\left(\Gamma_{0}\right)$. Therefore the eigenfunctions $w^{k}$ corresponding to $\lambda_{k}>0$ satisfy

$$
\begin{equation*}
\int_{\Gamma_{0}} w^{k}=0 \tag{2.1}
\end{equation*}
$$

We denote the bounded bilinear symmetric forms appearing in Problem 1 as follows:

$$
\begin{aligned}
& a(w, v):=\int_{\Omega} \nabla w \cdot \nabla v, \quad w, v \in H^{1}(\Omega) \\
& b(w, v):=\int_{\Gamma_{0}} w v, \quad w, v \in H^{1}(\Omega)
\end{aligned}
$$

Let $\left\{\mathcal{T}_{h}\right\}_{h}$ be a sequence of decompositions of $\Omega$ into polygons $K$. We assume that for every mesh $\mathcal{T}_{h}, \bar{\Gamma}_{0}$ and $\bar{\Gamma}_{1}$ are union of edges of elements $K \in \mathcal{T}_{h}$. Let $h_{K}$ denote the diameter of the element $K$ and $h$ the maximum of the diameters of all the elements of the mesh, i.e., $h:=\max _{K \in \mathcal{T}_{h}} h_{K}$.

For the analysis, we will make as in $[7,31]$ the following assumptions.

- A1. Every mesh $\mathcal{T}_{h}$ consists of a finite number of simple polygons (i.e., open simply connected sets with non self intersecting polygonal boundaries).
- A2. There exists $\gamma>0$ such that, for all meshes $\mathcal{T}_{h}$, each polygon $K \in \mathcal{T}_{h}$ is star-shaped with respect to a ball of radius greater than or equal to $\gamma h_{K}$.
- A3. There exists $\widehat{\gamma}>0$ such that, for all meshes $\mathcal{T}_{h}$, for each polygon $K \in \mathcal{T}_{h}$, the distance between any two of its vertices is greater than or equal to $\widehat{\gamma} h_{K}$.

We consider now a simple polygon $K$ and, for $k \in \mathbb{N}$, we define

$$
\mathbb{B}_{k}(\partial K):=\left\{v \in C^{0}(\partial K):\left.v\right|_{\ell} \in \mathbb{P}_{k}(\ell) \text { for all edges } \ell \subset \partial K\right\}
$$

We then consider the finite-dimensional space defined as follows:

$$
\begin{equation*}
V_{k}^{K}:=\left\{v \in H^{1}(K):\left.v\right|_{\partial K} \in \mathbb{B}_{k}(\partial K) \text { and }\left.\Delta v\right|_{K} \in \mathbb{P}_{k-2}(K)\right\} \tag{2.2}
\end{equation*}
$$

where, for $k=1$, we have used the convention that $\mathbb{P}_{-1}(K):=\{0\}$. We choose in this space the degrees of freedom introduced in [7, Section 4.1]. Finally, for every decomposition $\mathcal{T}_{h}$ of $\Omega$ into simple polygons $K$ and for a fixed $k \in \mathbb{N}$, we define

$$
V_{h}:=\left\{v \in H^{1}(\Omega):\left.v\right|_{K} \in V_{k}^{K} \quad \forall K \in \mathcal{T}_{h}\right\}
$$

In what follows, we will use standard Sobolev spaces, norms and seminorms and also the broken $H^{1}$-seminorm

$$
|v|_{1, h}^{2}:=\sum_{K \in \mathcal{T}_{h}}\|\nabla v\|_{0, K}^{2}
$$

which is well defined for every $v \in L^{2}(\Omega)$ such that $\left.v\right|_{K} \in H^{1}(K)$ for each polygon $K \in \mathcal{T}_{h}$.
We split the bilinear form $a(\cdot, \cdot)$ as follows:

$$
a(u, v)=\sum_{K \in \mathcal{T}_{h}} a^{K}(u, v), \quad u, v \in H^{1}(\Omega)
$$

where

$$
a^{K}(u, v):=\int_{K} \nabla u \cdot \nabla v, \quad u, v \in H^{1}(K)
$$

Due to the implicit space definition, we must have into account that we would not know how to compute $a^{K}(\cdot, \cdot)$ for $u_{h}, v_{h} \in V_{h}$. Nevertheless, the final output will be a local matrix on each element $K$ whose associated bilinear form can be exactly computed whenever one of the two entries is a polynomial of degree $k$. This will allow us to retain the optimal approximation properties of the space $V_{h}$.

With this end, for any $K \in \mathcal{T}_{h}$ and for any sufficiently regular function $\varphi$, we define first

$$
\bar{\varphi}:=\frac{1}{N_{K}} \sum_{i=1}^{N_{K}} \varphi\left(P_{i}\right)
$$

where $P_{i}, 1 \leq i \leq N_{K}$, are the vertices of $K$. Then, we define the projector $\Pi_{k}^{K}: V_{k}^{K} \longrightarrow \mathbb{P}_{k}(K) \subseteq$ $V_{k}^{K}$ for each $v_{h} \in V_{k}^{K}$ as the solution of

$$
\begin{aligned}
a^{K}\left(\Pi_{k}^{K} v_{h}, q\right) & =a^{K}\left(v_{h}, q\right) \quad \forall q \in \mathbb{P}_{k}(K), \\
\overline{\Pi_{k}^{K} v_{h}} & =\overline{v_{h}}
\end{aligned}
$$

On the other hand, let $S^{K}(\cdot, \cdot)$ be any symmetric positive definite bilinear form to be chosen as to satisfy

$$
\begin{equation*}
c_{0} a^{K}\left(v_{h}, v_{h}\right) \leq S^{K}\left(v_{h}, v_{h}\right) \leq c_{1} a^{K}\left(v_{h}, v_{h}\right) \quad \forall v_{h} \in V_{k}^{K} \text { with } \Pi_{k}^{K} v_{h}=0 \tag{2.4}
\end{equation*}
$$

for some positive constants $c_{0}$ and $c_{1}$ independent of $K$. Then, set

$$
a_{h}\left(u_{h}, v_{h}\right):=\sum_{K \in \mathcal{T}_{h}} a_{h}^{K}\left(u_{h}, v_{h}\right), \quad u_{h}, v_{h} \in V_{h}
$$

where $a_{h}^{K}(\cdot, \cdot)$ is the bilinear form defined on $V_{k}^{K} \times V_{k}^{K}$ by

$$
a_{h}^{K}\left(u_{h}, v_{h}\right):=a^{K}\left(\Pi_{k}^{K} u_{h}, \Pi_{k}^{K} v_{h}\right)+S^{K}\left(u_{h}-\Pi_{k}^{K} u_{h}, v_{h}-\Pi_{k}^{K} v_{h}\right), \quad u_{h}, v_{h} \in V_{k}^{K}
$$

Notice that the bilinear form $S^{K}(\cdot, \cdot)$ has to be actually computable for $u_{h}, v_{h} \in V_{k}^{K}$.
The following properties of $a_{h}^{K}(\cdot, \cdot)$ have been established in [7, Theorem 4.1].

- $k$-Consistency:

$$
\begin{equation*}
a_{h}^{K}\left(p, v_{h}\right)=a^{K}\left(p, v_{h}\right) \quad \forall p \in \mathbb{P}_{k}(K), \quad \forall v_{h} \in V_{k}^{K} \tag{2.5}
\end{equation*}
$$

- Stability: There exist two positive constants $\alpha_{*}$ and $\alpha^{*}$, independent of $K$, such that:

$$
\begin{equation*}
\alpha_{*} a^{K}\left(v_{h}, v_{h}\right) \leq a_{h}^{K}\left(v_{h}, v_{h}\right) \leq \alpha^{*} a^{K}\left(v_{h}, v_{h}\right) \quad \forall v_{h} \in V_{k}^{K} \tag{2.6}
\end{equation*}
$$

Now, we are in a position to write the virtual element discretization of Problem 1.
Problem 2. Find $\left(\lambda_{h}, w_{h}\right) \in \mathbb{R} \times V_{h}, w_{h} \neq 0$, such that

$$
a_{h}\left(w_{h}, v_{h}\right)=\lambda_{h} b\left(w_{h}, v_{h}\right) \quad \forall v_{h} \in V_{h}
$$

According to [31, Theorem 3.1] we know that the solutions $\left(\lambda_{h}, w_{h}\right)$ of the problem above are:

- $\lambda_{h 0}=0$, whose associated eigenfunction are the constant functions in $\Omega$.
- $\left\{\lambda_{h k}\right\}_{k=1}^{N_{h}}$, with $N_{h}:=\operatorname{dim}\left\{\left.v_{h}\right|_{\Gamma_{0}}, v_{h} \in V_{h}\right\}-1$, which are positive eigenvalues repeated according to their respective multiplicities.

Moreover, the eigenfunctions corresponding to different eigenvalues are orthogonal in $L^{2}\left(\Gamma_{0}\right)$. Therefore the eigenfunctions $w_{h}^{k}$ corresponding to $\lambda_{h k}>0$ satisfy

$$
\begin{equation*}
\int_{\Gamma_{0}} w_{h}^{k}=0 . \tag{2.7}
\end{equation*}
$$

Let $(\lambda, w)$ be a solution to Problem 1. We assume $\lambda>0$ is a simple eigenvalue and we normalize $w$ so that $\|w\|_{0, \Gamma_{0}}=1$. Then, for each mesh $\mathcal{T}_{h}$, there exists a solution ( $\lambda_{h}, w_{h}$ ) of Problem 2 such that $\lambda_{h} \rightarrow \lambda,\left\|w_{h}\right\|_{0, \Gamma_{0}}=1$ and $\left\|w-w_{h}\right\|_{1, \Omega} \rightarrow 0$ as $h \rightarrow 0$. Moreover, according to (2.1) and (2.7), we have that $w$ and $w_{h}$ belong to the space

$$
V:=\left\{v \in H^{1}(\Omega): \int_{\Gamma_{0}} v=0\right\} .
$$

Let us remark that the following generalized Poincaré inequality holds true in this space: there exists $C>0$ such that

$$
\begin{equation*}
\|v\|_{1, \Omega} \leq C|v|_{1, \Omega} \quad \forall v \in V \tag{2.8}
\end{equation*}
$$

The following a priori error estimates have been proved in [31, Theorems 4.2-4.4]: there exists $C>0$ such that for all $r \in\left[\frac{1}{2}, r_{\Omega}\right)$

$$
\begin{align*}
\left\|w-w_{h}\right\|_{1, \Omega} & \leq C h^{\min \{r, k\}}  \tag{2.9}\\
\left|\lambda-\lambda_{h}\right| & \leq C h^{2 \min \{r, k\}}  \tag{2.10}\\
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} & \leq C h^{\min \{r, 1\} / 2+\min \{r, k\}} \tag{2.11}
\end{align*}
$$

where the constant $r_{\Omega}>\frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. Let us remark that $r_{\Omega}>1$, if $\Omega$ is convex, and $r_{\Omega}:=\frac{\pi}{\omega}$ with $\omega$ being the largest re-entrant angle of $\Omega$, otherwise.

## 3. A posteriori error analysis

The aim of this section is to introduce a suitable residual-based error estimator for the Steklov eigenvalue problem which be fully computable, in the sense that it depends only on quantities available from the VEM solution. Then, we will show its equivalence with the error. For this purpose, we introduce the following definitions and notations.

For any polygon $K \in \mathcal{T}_{h}$, we denote by $\mathcal{E}_{K}$ the set of edges of $K$ and

$$
\mathcal{E}:=\bigcup_{K \in \mathcal{T}_{h}} \mathcal{E}_{K}
$$

We decompose $\mathcal{E}=\mathcal{E}_{\Omega} \cup \mathcal{E}_{\Gamma_{0}} \cup \mathcal{E}_{\Gamma_{1}}$, where $\mathcal{E}_{\Gamma_{0}}:=\left\{\ell \in \mathcal{E}: \ell \subset \Gamma_{0}\right\}, \mathcal{E}_{\Gamma_{1}}:=\left\{\ell \in \mathcal{E}: \ell \subset \Gamma_{1}\right\}$ and $\mathcal{E}_{\Omega}:=\mathcal{E} \backslash\left(\mathcal{E}_{\Gamma_{0}} \cup \mathcal{E}_{\Gamma_{1}}\right)$. For each inner edge $\ell \in \mathcal{E}_{\Omega}$ and for any sufficiently smooth function $v$, we define the jump of its normal derivative on $\ell$ by

$$
\llbracket \frac{\partial v}{\partial n} \rrbracket_{\ell}:=\nabla\left(\left.v\right|_{K}\right) \cdot n_{K}+\nabla\left(\left.v\right|_{K^{\prime}}\right) \cdot n_{K^{\prime}},
$$

where $K$ and $K^{\prime}$ are the two elements in $\mathcal{T}_{h}$ sharing the edge $\ell$ and $n_{K}$ and $n_{K^{\prime}}$ are the respective outer unit normal vectors.

As a consequence of the mesh regularity assumptions, we have that each polygon $K \in \mathcal{T}_{h}$ admits a sub-triangulation $\mathcal{T}_{h}^{K}$ obtained by joining each vertex of $K$ with the midpoint of the ball with respect to which $K$ is starred. Let $\widehat{\mathcal{T}}_{h}:=\bigcup_{K \in \mathcal{T}_{h}} \mathcal{T}_{h}^{K}$. Since we are also assuming A3, $\left\{\widehat{\mathcal{T}}_{h}\right\}_{h}$ is a shape-regular family of triangulations of $\Omega$.

We introduce bubble functions on polygons as follows (see [21]). An interior bubble function $\psi_{K} \in H_{0}^{1}(K)$ for a polygon $K$ can be constructed piecewise as the sum of the cubic bubble functions for each triangle of the sub-triangulation $\mathcal{T}_{h}^{K}$ that attain the value 1 at the barycenter of each triangle. On the other hand, an edge bubble function $\psi_{\ell}$ for $\ell \in \partial K$ is a piecewise quadratic function attaining the value 1 at the barycenter of $\ell$ and vanishing on the triangles $T \in \widehat{\mathcal{T}}_{h}$ that do not contain $\ell$ on its boundary.

The following results which establish standard estimates for bubble functions will be useful in what follows (see [2, 38]).

Lemma 3.1 (Interior bubble functions). For any $K \in \mathcal{T}_{h}$, let $\psi_{K}$ be the corresponding interior bubble function. Then, there exists a constant $C>0$ independent of $h_{K}$ such that

$$
\begin{aligned}
& C^{-1}\|q\|_{0, K}^{2} \leq \int_{K} \psi_{K} q^{2} \leq\|q\|_{0, K}^{2} \quad \forall q \in \mathbb{P}_{k}(K) \\
& C^{-1}\|q\|_{0, K} \leq\left\|\psi_{K} q\right\|_{0, K}+h_{K}\left\|\nabla\left(\psi_{K} q\right)\right\|_{0, K} \leq C\|q\|_{0, K} \quad \forall q \in \mathbb{P}_{k}(K)
\end{aligned}
$$

Lemma 3.2 (Edge bubble functions). For any $K \in \mathcal{T}_{h}$ and $\ell \in \mathcal{E}_{K}$, let $\psi_{\ell}$ be the corresponding edge bubble function. Then, there exists a constant $C>0$ independent of $h_{K}$ such that

$$
C^{-1}\|q\|_{0, \ell}^{2} \leq \int_{\ell} \psi_{\ell} q^{2} \leq\|q\|_{0, \ell}^{2} \quad \forall q \in \mathbb{P}_{k}(\ell)
$$

Moreover, for all $q \in \mathbb{P}_{k}(\ell)$, there exists an extension of $q \in \mathbb{P}_{k}(K)$ (again denoted by $q$ ) such that

$$
h_{K}^{-1 / 2}\left\|\psi_{\ell} q\right\|_{0, K}+h_{K}^{1 / 2}\left\|\nabla\left(\psi_{\ell} q\right)\right\|_{0, K} \leq C\|q\|_{0, \ell}
$$

Remark 3.1. A possible way of extending $q$ from $\ell \in \mathcal{E}_{K}$ to $K$ so that Lemma 3.2 holds is as follows: first we extend $q$ to the straight line $L \supset \ell$ using the same polynomial function. Then, we extend it to the whole plain through a constant prolongation in the normal direction to L. Finally, we restrict the latter to $K$.

The following lemma provides an error equation which will be the starting point of our error analysis. From now on, we will denote by $e:=\left(w-w_{h}\right) \in V$ the eigenfunction error and by

$$
J_{\ell}:= \begin{cases}\frac{1}{2} \llbracket \frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n} \|_{\ell}, & \ell \in \mathcal{E}_{\Omega}  \tag{3.1}\\ \lambda_{h} w_{h}-\frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_{0}} \\ -\frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_{1}}\end{cases}
$$

the edge residuals. Notice that $J_{\ell}$ are actually computable since they only involve values of $w_{h}$ on $\Gamma_{0}$ (which are computable in terms of the boundary degrees of freedom) and $\Pi_{k}^{K} w_{h} \in \mathbb{P}_{k}(K)$ which is also computable.
Lemma 3.3. For any $v \in H^{1}(\Omega)$, we have the following identity:
$a(e, v)=\lambda b(w, v)-\lambda_{h} b\left(w_{h}, v\right)-\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, v\right)+\sum_{K \in \mathcal{T}_{h}}\left[\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) v+\sum_{\ell \in \mathcal{E}_{K}} \int_{\ell} J_{\ell} v\right]$.
Proof. Using that $(\lambda, w)$ is a solution of Problem 1, adding and subtracting $\Pi_{k}^{K} w_{h}$ and integrating by parts, we obtain

$$
\begin{aligned}
a(e, v)= & \lambda b(w, v)-a\left(w_{h}, v\right) \\
= & \lambda b(w, v)-\sum_{K \in \mathcal{T}_{h}}\left[a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, v\right)+a^{K}\left(\Pi_{k}^{K} w_{h}, v\right)\right] \\
= & \lambda b(w, v)-\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, v\right)-\sum_{K \in \mathcal{T}_{h}}\left[-\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) v+\int_{\partial K} \frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n} v\right] \\
= & \lambda b(w, v)-\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, v\right) \\
& +\sum_{K \in \mathcal{T}_{h}}\left[\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) v-\sum_{\ell \in \mathcal{E}_{K} \cap\left(\mathcal{E}_{\Gamma_{0}} \cup \mathcal{E}_{\left.\Gamma_{1}\right)}\right.} \int_{\ell} \frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n} v+\frac{1}{2} \sum_{\ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Omega}} \int_{\ell}\left[\frac{\partial\left(\Pi_{k}^{K} w_{h}\right)}{\partial n}\right]_{\ell} v\right] .
\end{aligned}
$$

Finally, the proof follows by adding and subtracting the term $\lambda_{h} b\left(w_{h}, v\right)$.
For all $K \in \mathcal{T}_{h}$, we introduce the local terms $\theta_{K}$ and $R_{K}$ and the local error indicator $\eta_{K}$ by

$$
\begin{aligned}
\theta_{K}^{2} & :=a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right) \\
R_{K}^{2} & :=h_{K}^{2}\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}^{2} \\
\eta_{K}^{2} & :=\theta_{K}^{2}+R_{K}^{2}+\sum_{\ell \in \mathcal{E}_{K}} h_{K}\left\|J_{\ell}\right\|_{0, \ell}^{2} .
\end{aligned}
$$

We also introduce the global error estimator by

$$
\eta^{2}:=\sum_{K \in \mathcal{T}_{h}} \eta_{K}^{2}
$$

Remark 3.2. The indicators $\eta_{K}$ include the terms $\theta_{K}$ which do not appear in standard finite element estimators. This term, which represent the virtual inconsistency of the method, has been introduced in [11, 21] for a posteriori error estimates of other VEM. Let us emphasize that it can be directly computed in terms of the bilinear form $S^{K}(\cdot, \cdot)$. In fact,

$$
\theta_{K}^{2}=a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)=S^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)
$$

### 3.1. Reliability of the a posteriori error estimator

First, we provide an upper bound for the error.
Theorem 3.1. There exists a constant $C>0$ independent of $h$ such that

$$
\left|w-w_{h}\right|_{1, \Omega} \leq C\left(\eta+\frac{\lambda+\lambda_{h}}{2}\left\|w-w_{h}\right\|_{0, \Gamma_{0}}\right)
$$

Proof. Since $e=w-w_{h} \in V \subset H^{1}(\Omega)$, there exists $e_{I} \in V_{h}$ satisfying (see [31, Proposition 4.2])

$$
\begin{equation*}
\left\|e-e_{I}\right\|_{0, K}+h_{K}\left|e-e_{I}\right|_{1, K} \leq C h_{K}\|e\|_{1, K} \tag{3.2}
\end{equation*}
$$

Then, we have that

$$
\begin{align*}
\left|w-w_{h}\right|_{1, \Omega}^{2} & =a\left(w-w_{h}, e\right) \\
& =a\left(w-w_{h}, e-e_{I}\right)+a\left(w, e_{I}\right)-a_{h}\left(w_{h}, e_{I}\right)+a_{h}\left(w_{h}, e_{I}\right)-a\left(w_{h}, e_{I}\right) \\
& =\underbrace{\lambda b(w, e)-\lambda_{h} b\left(w_{h}, e\right)}_{T_{1}}+\underbrace{\sum_{K \in \mathcal{T}_{h}}\left[\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)\left(e-e_{I}\right)+\sum_{\ell \in \mathcal{E}_{K}} \int_{\ell} J_{\ell}\left(e-e_{I}\right)\right]}_{T_{2}}  \tag{3.3}\\
& -\underbrace{\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, e-e_{I}\right)}_{T_{3}}+\underbrace{a_{h}\left(w_{h}, e_{I}\right)-a\left(w_{h}, e_{I}\right)}_{T_{4}},
\end{align*}
$$

the last equality thanks to Lemma 3.3. Next, we bound each term $T_{i}$ separately.
For $T_{1}$, we use the definition of $b(\cdot, \cdot)$, the fact that $\|w\|_{0, \Gamma_{0}}=\left\|w_{h}\right\|_{0, \Gamma_{0}}=1$, a trace theorem and (2.8) to write

$$
\begin{equation*}
T_{1}=\lambda+\lambda_{h}-\left(\lambda+\lambda_{h}\right) \int_{\Gamma_{0}} w w_{h}=\frac{\lambda+\lambda_{h}}{2}\|e\|_{0, \Gamma_{0}}^{2} \leq C \frac{\lambda+\lambda_{h}}{2}\|e\|_{0, \Gamma_{0}}|e|_{1, \Omega} \tag{3.4}
\end{equation*}
$$

For $T_{2}$, first, we use a local trace inequality (see [13, Lemma 14]) and (3.2) to write

$$
\left\|e-e_{I}\right\|_{0, \ell} \leq C\left(h_{K}^{-1 / 2}\left\|e-e_{I}\right\|_{0, K}+h_{K}^{1 / 2}\left|e-e_{I}\right|_{1, K}\right) \leq C h_{K}^{1 / 2}\|e\|_{1, K}
$$

Hence, using (3.2) again, we have

$$
\begin{align*}
T_{2} & \leq C \sum_{K \in \mathcal{T}_{h}}\left[\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}\left\|e-e_{I}\right\|_{0, K}+\sum_{\ell \in \mathcal{E}_{K}}\left\|J_{\ell}\right\|_{0, \ell}\left\|e-e_{I}\right\|_{0, \ell}\right] \\
& \leq C \sum_{K \in \mathcal{T}_{h}}\left[h_{K}\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}\|e\|_{1, K}+\sum_{\ell \in \mathcal{E}_{K}} h_{K}^{1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\|e\|_{1, K}\right] \\
& \leq C\left\{\sum_{K \in \mathcal{T}_{h}}\left[h_{K}^{2}\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}^{2}+\sum_{\ell \in \mathcal{E}_{K}} h_{K}\left\|J_{\ell}\right\|_{0, \ell}^{2}\right]\right\}^{1 / 2}|e|_{1, \Omega}, \tag{3.5}
\end{align*}
$$

where for the last estimate we have used (2.8).
To bound $T_{3}$, we use the stability property (2.6) and (3.2) to write

$$
\begin{equation*}
T_{3} \leq C \sum_{K \in \mathcal{T}_{h}} a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)^{1 / 2}\|e\|_{1, K} \leq C\left(\sum_{K \in \mathcal{T}_{h}} \theta_{K}^{2}\right)^{1 / 2}|e|_{1, \Omega} \tag{3.6}
\end{equation*}
$$

where for the last estimate we have used Remark 3.2 and (2.8) again.
Finally, to bound $T_{4}$, we add and subtract $\Pi_{k}^{K} w_{h}$ on each $K \in \mathcal{T}_{h}$ and use the $k$-consistency property (2.5):

$$
\begin{align*}
T_{4}= & \sum_{K \in \mathcal{T}_{h}}\left[a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, e_{I}\right)-a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, e_{I}\right)\right] \\
\leq & \sum_{K \in \mathcal{T}_{h}} a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)^{1 / 2} a_{h}^{K}\left(e_{I}, e_{I}\right)^{1 / 2} \\
& +\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)^{1 / 2} a^{K}\left(e_{I}, e_{I}\right)^{1 / 2} \\
\leq & C \sum_{K \in \mathcal{T}_{h}} a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)^{1 / 2}\left|e_{I}\right|_{1, K} \\
\leq & C\left(\sum_{K \in \mathcal{T}_{h}} \theta_{K}^{2}\right)^{1 / 2}|e|_{1, \Omega} \tag{3.7}
\end{align*}
$$

where we have used the stability property (2.6), (3.2) and (2.8) for the last two inequalities.
Thus, the result follows from (3.3)-(3.7).

Although the virtual approximate eigenfunction is $w_{h}$, this function is not known in practice. Instead of $w_{h}$, what can be used as an approximation of the eigenfunction is $\Pi_{h} w_{h}$, where $\Pi_{h}$ is defined for $v_{h} \in V_{h}$ by

$$
\left.\left(\Pi_{h} v_{h}\right)\right|_{K}:=\Pi_{k}^{K} v_{h} \quad \forall K \in \mathcal{T}_{h}
$$

Notice that $\Pi_{h} w_{h}$ is actually computable. The following result shows that an estimate similar to that of Theorem 3.1 holds true for $\Pi_{h} w_{h}$.

Corollary 3.1. There exists a constant $C>0$ independent of $h$ such that

$$
\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h} \leq C\left(\eta+\frac{\lambda+\lambda_{h}}{2}\left\|w-w_{h}\right\|_{0, \Gamma_{0}}\right)
$$

Proof. For each polygon $K \in \mathcal{T}_{h}$, we have that

$$
\left|w-\Pi_{k}^{K} w_{h}\right|_{1, K} \leq\left|w-w_{h}\right|_{1, K}+\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}
$$

Then, summing over all polygons we obtain

$$
\left|w-\Pi_{h} w_{h}\right|_{1, h} \leq C\left(\sum_{K \in \mathcal{T}_{h}}\left|w-w_{h}\right|_{1, K}^{2}+\sum_{K \in \mathcal{T}_{h}}\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}^{2}\right)^{1 / 2}
$$

Now, using (2.4) together with Remark 3.2, we have that

$$
\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}^{2} \leq \frac{1}{c_{0}} S^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)=\frac{1}{c_{0}} \theta_{K}^{2} \leq \frac{1}{c_{0}} \eta_{K}^{2}
$$

Thus, the result follows from Theorem 3.1.

In what follows, we prove a convenient upper bound for the eigenvalue approximation.

Corollary 3.2. There exists a constant $C>0$ independent of $h$ such that

$$
\left|\lambda-\lambda_{h}\right| \leq C\left(\eta+\frac{\lambda+\lambda_{h}}{2}\left\|w-w_{h}\right\|_{0, \Gamma_{0}}\right)^{2}
$$

Proof. From the symmetry of the bilinear forms together with the facts that $a(w, v)=\lambda b(w, v)$ for all $v \in H^{1}(\Omega), a_{h}\left(w_{h}, v_{h}\right)=\lambda_{h} b\left(w_{h}, v_{h}\right)$ for all $v_{h} \in V_{h}$ and $b\left(w_{h}, w_{h}\right)=1$, we have

$$
\begin{align*}
\left|\lambda-\lambda_{h}\right| & =\frac{\left|a\left(w-w_{h}, w-w_{h}\right)-\lambda b\left(w-w_{h}, w-w_{h}\right)+a_{h}\left(w_{h}, w_{h}\right)-a\left(w_{h}, w_{h}\right)\right|}{b\left(w_{h}, w_{h}\right)} \\
& \leq C\left[\left|w-w_{h}\right|_{1, \Omega}^{2}+\left\|w-w_{h}\right\|_{0, \Gamma_{0}}^{2}+\left|a_{h}\left(w_{h}, w_{h}\right)-a\left(w_{h}, w_{h}\right)\right|\right] \\
& \leq C\left[\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|a_{h}\left(w_{h}, w_{h}\right)-a\left(w_{h}, w_{h}\right)\right|\right], \tag{3.8}
\end{align*}
$$

where we have also used a trace theorem and (2.8). We now bound the last term on the right-hand side above using the definition of $a_{h}(\cdot, \cdot)$ and (2.4):

$$
\begin{aligned}
& \left|a_{h}\left(w_{h}, w_{h}\right)-a\left(w_{h}, w_{h}\right)\right| \\
& =\left|\sum_{K \in \mathcal{T}_{h}}\left[a^{K}\left(\Pi_{k}^{K} w_{h}, \Pi_{k}^{K} w_{h}\right)+S^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)\right]-\sum_{K \in \mathcal{T}_{h}} a^{K}\left(w_{h}, w_{h}\right)\right| \\
& \leq\left|\sum_{K \in \mathcal{T}_{h}}\left[a^{K}\left(\Pi_{k}^{K} w_{h}, \Pi_{k}^{K} w_{h}\right)-a^{K}\left(w_{h}, w_{h}\right)\right]\right|+\sum_{K \in \mathcal{T}_{h}} c_{1} a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right) \\
& =\sum_{K \in \mathcal{T}_{h}}\left(1+c_{1}\right) a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right) \\
& \leq\left(1+c_{1}\right) \sum_{K \in \mathcal{T}_{h}}\left(\left|w_{h}-w\right|_{1, K}^{2}+\left|w-\Pi_{k}^{K} w_{h}\right|_{1, K}^{2}\right) .
\end{aligned}
$$

Finally, from the above estimate and (3.8) we obtain

$$
\begin{equation*}
\left|\lambda-\lambda_{h}\right| \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}\right) . \tag{3.9}
\end{equation*}
$$

Hence, we conclude the proof thanks to Corollary 3.1.
According to (2.9) and (2.11), it seems reasonable to expect the term $\left\|w-w_{h}\right\|_{0, \Gamma_{0}}$ in the estimate of Theorem 3.1 to be of higher order than $\left|w-w_{h}\right|_{1, \Omega}$ and hence asymptotically negligible. However this cannot be rigorously derived from (2.9) and (2.11), which are only upper error bounds. In fact, the actual error $\left|w-w_{h}\right|_{1, \Omega}$ could be in principle of higher order than the estimate (2.9).

Our next goal is to prove that the term $\left\|w-w_{h}\right\|_{0, \Gamma_{0}}$ is actually asymptotically negligible in the estimates of Corollaries 3.1 and 3.2. With this aim, we will modify the estimate (2.11) and prove that

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C h^{\min \{r, 1\} / 2}\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right) \tag{3.10}
\end{equation*}
$$

This proof is based on the arguments used in Section 4 from [31]. To avoid repeating them step by step, in what follows we will only report the changes that have to be made in order to prove (3.10).

We define in $H^{1}(\Omega)$ the bilinear form $\widehat{a}(\cdot, \cdot):=a(\cdot, \cdot)+b(\cdot, \cdot)$, which is elliptic [31, Lemma 2.1]. Let $u \in H^{1}(\Omega)$ be the solution of

$$
\widehat{a}(u, v)=b(w, v) \quad \forall v \in H^{1}(\Omega)
$$

Since $a(w, v)=\lambda b(w, v)$ we have that $u=w /(\lambda+1)$. We also define in $V_{h}$ the bilinear form $\widehat{a}_{h}(\cdot, \cdot):=a_{h}(\cdot, \cdot)+b(\cdot, \cdot)$, which is elliptic uniformly in $h\left[31\right.$, Lemma 3.1]. Let $u_{h} \in V_{h}$ be the solution of

$$
\begin{equation*}
\widehat{a}_{h}\left(u_{h}, v_{h}\right)=b\left(w, v_{h}\right) \quad \forall v_{h} \in V_{h} \tag{3.11}
\end{equation*}
$$

The arguments in the proof of Lemma 4.3 from [31] can be easily modified to prove that

$$
\left\|u-u_{h}\right\|_{0, \Gamma_{0}} \leq C h^{\min \{r, 1\} / 2}\left(\left|u-u_{h}\right|_{1, \Omega}+\left|u-\Pi_{h} u_{h}\right|_{1, h}\right)
$$

Then, using this estimate in the proof of Theorem 4.4 from [31] yields

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C h^{\min \{r, 1\} / 2}\left(\left|u-u_{h}\right|_{1, \Omega}+\left|u-\Pi_{h} u_{h}\right|_{1, h}\right) \tag{3.12}
\end{equation*}
$$

Now, since as stated above $u=w /(\lambda+1)$, we have that

$$
\begin{equation*}
\left|u-u_{h}\right|_{1, \Omega} \leq \frac{\left|w-w_{h}\right|_{1, \Omega}}{|\lambda+1|}+\left|\frac{1}{\lambda+1}-\frac{1}{\lambda_{h}+1}\right|\left|w_{h}\right|_{1, \Omega}+\left|\frac{w_{h}}{\lambda_{h}+1}-u_{h}\right|_{1, \Omega} \tag{3.13}
\end{equation*}
$$

For the second term on the right hand side above, we use (3.9) to write

$$
\begin{equation*}
\left|\frac{1}{\lambda+1}-\frac{1}{\lambda_{h}+1}\right|=\frac{\left|\lambda-\lambda_{h}\right|}{|\lambda+1|\left|\lambda_{h}+1\right|} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}\right) \tag{3.14}
\end{equation*}
$$

To estimate the third term we recall first that

$$
\widehat{a}_{h}\left(w_{h}, v_{h}\right)=\left(\lambda_{h}+1\right) b\left(w_{h}, v_{h}\right) \quad \forall v_{h} \in V_{h}
$$

Then, subtracting this equation divided by $\lambda_{h}+1$ from (3.11) we have that

$$
\widehat{a}_{h}\left(u_{h}-\frac{w_{h}}{\lambda_{h}+1}, v_{h}\right)=b\left(w-w_{h}, v_{h}\right) \quad \forall v_{h} \in V_{h}
$$

Hence, from the uniform ellipticity of $\widehat{a}_{h}(\cdot, \cdot)$ in $V_{h}$, we obtain

$$
\left\|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right\|_{1, \Omega}^{2} \leq C\left\|w-w_{h}\right\|_{0, \Gamma_{0}}\left\|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right\|_{0, \Gamma_{0}} \leq C\left\|w-w_{h}\right\|_{0, \Gamma_{0}}\left\|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right\|_{1, \Omega}
$$

Therefore

$$
\begin{equation*}
\left\|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right\|_{1, \Omega} \leq C\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C\left\|w-w_{h}\right\|_{1, \Omega} \leq C\left|w-w_{h}\right|_{1, \Omega} \tag{3.15}
\end{equation*}
$$

the last inequality because of Poincaré inequality (2.8). Then, substituting (3.14) and (3.15) into (3.13) we obtain

$$
\begin{equation*}
\left|u-u_{h}\right|_{1, \Omega} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right) \tag{3.16}
\end{equation*}
$$

For the other term on the right hand side of (3.12) we have

$$
\begin{equation*}
\left|u-\Pi_{h} u_{h}\right|_{1, h} \leq\left|u-u_{h}\right|_{1, \Omega}+\left|u_{h}-\Pi_{h} u_{h}\right|_{1, h} \tag{3.17}
\end{equation*}
$$

whereas

$$
\left|u_{h}-\Pi_{h} u_{h}\right|_{1, h} \leq\left|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right|_{1, \Omega}+\frac{\left|w_{h}-\Pi_{h} w_{h}\right|_{1, h}}{\lambda_{h}+1}+\left|\Pi_{h}\left(\frac{w_{h}}{\lambda_{h}+1}-u_{h}\right)\right|_{1, h}
$$

$$
\begin{aligned}
& \leq 2\left|u_{h}-\frac{w_{h}}{\lambda_{h}+1}\right|_{1, \Omega}+\frac{\left|w-w_{h}\right|_{1, \Omega}}{\lambda_{h}+1}+\frac{\left|w-\Pi_{h} w_{h}\right|_{1, h}}{\lambda_{h}+1} \\
& \leq C\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right)
\end{aligned}
$$

where we have used (3.15) for the last inequality. Substituting this and estimate (3.16) into (3.17) we obtain

$$
\left|u-\Pi_{h} u_{h}\right|_{1, h} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right)
$$

Finally, substituting the above estimate and (3.16) into (3.12), we conclude the proof of the following result.

Lemma 3.4. There exists $C>0$ independent of $h$ such that

$$
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C h^{\min \{r, 1\} / 2}\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right)
$$

Using this result, now it easy to prove that the term $\left\|w-w_{h}\right\|_{0, \Gamma_{0}}$ in Corollaries 3.1 and 3.2 is asymptotically negligible. In fact, we have the following result.

Theorem 3.2. There exist positive constants $C$ and $h_{0}$ such that, for all $h<h_{0}$, there holds

$$
\begin{align*}
& \left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h} \leq C \eta  \tag{3.18}\\
& \left|\lambda-\lambda_{h}\right| \leq C \eta^{2} \tag{3.19}
\end{align*}
$$

Proof. From Lemma 3.4 and Corollary 3.1 we have

$$
\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h} \leq C\left(\eta+h^{\min \{r, 1\} / 2}\left(\left|w-w_{h}\right|_{1, \Omega}+\left|w-\Pi_{h} w_{h}\right|_{1, h}\right)\right)
$$

Hence, it is straightforward to check that there exists $h_{0}>0$ such that for all $h<h_{0}$ (3.18) holds true.

On the other hand, from Lemma 3.4 and (3.18) we have that for all $h<h_{0}$

$$
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C h^{\min \{r, 1\} / 2} \eta
$$

Then, for $h$ small enough, (3.19) follows from Corollary 3.2 and the above estimate.

### 3.2. Efficiency of the a posteriori error estimator

We will show in this section that the local error indicators $\eta_{K}$ are efficient in the sense of pointing out which polygons should be effectively refined.

First, we prove an upper estimate of the volumetric residual term $R_{K}$.
Lemma 3.5. There exists a constant $C>0$ independent of $h_{K}$ such that

$$
R_{K} \leq C\left(\left|w-w_{h}\right|_{1, K}+\theta_{K}\right)
$$

Proof. For any $K \in \mathcal{T}_{h}$, let $\psi_{K}$ be the corresponding interior bubble function. We define $v:=$ $\psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)$. Since $v$ vanishes on the boundary of $K$, it may be extended by zero to the whole domain $\Omega$. This extension, again denoted by $v$, belongs to $H^{1}(\Omega)$ and from Lemma 3.3 we have

$$
a^{K}(e, v)=-a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, \psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)\right)+\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) \psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)
$$

Since $\Delta\left(\Pi_{k}^{K} w_{h}\right) \in \mathbb{P}_{k-2}(K)$, using Lemma 3.1 and the above equality we obtain

$$
\begin{align*}
C^{-1}\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}^{2} & \leq \int_{K} \psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)^{2} \\
& =a^{K}\left(e, \psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)\right)+a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, \psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)\right) \\
& \leq C\left(|e|_{1, K}+\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}\right)\left|\psi_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right)\right|_{1, K} \\
& \leq C h_{K}^{-1}\left(|e|_{1, K}+\theta_{K}\right)\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K} \tag{3.20}
\end{align*}
$$

where, for the last inequality, we have used again Lemma 3.1 and (2.4) together with Remark 3.2. Multiplying the above inequality by $h_{K}$ allows us to conclude the proof.

Next goal is to obtain an upper estimate for the local term $\theta_{K}$.
Lemma 3.6. There exists $C>0$ independent of $h_{K}$ such that

$$
\theta_{K} \leq C\left(\left|w-w_{h}\right|_{1, K}+\left|w-\Pi_{k}^{K} w_{h}\right|_{1, K}\right)
$$

Proof. From the definition of $\theta_{K}$ together with Remark 3.2 and estimate (2.4) we have

$$
\theta_{K} \leq C\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K} \leq C\left(\left|w_{h}-w\right|_{1, K}+\left|w-\Pi_{k}^{K} w_{h}\right|_{1, K}\right)
$$

The proof is complete.
The following lemma provides an upper estimate for the jump terms of the local error indicator.
Lemma 3.7. There exists a constant $C>0$ independent of $h_{K}$ such that

$$
\begin{array}{ll}
h_{K}^{1 / 2}\left\|J_{\ell}\right\|_{0, \ell} \leq C\left(\left|w-w_{h}\right|_{1, K}+\theta_{K}\right) & \forall \ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Gamma_{1}}, \\
h_{K}^{1 / 2}\left\|J_{\ell}\right\|_{0, \ell} \leq C\left(\left|w-w_{h}\right|_{1, K}+\theta_{K}+h_{K}^{1 / 2}\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \ell}\right) & \forall \ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Gamma_{0}}, \\
h_{K}^{1 / 2}\left\|J_{\ell}\right\|_{0, \ell} \leq C \sum_{K^{\prime} \in \omega_{\ell}}\left(\left|w-w_{h}\right|_{1, K^{\prime}}+\theta_{K^{\prime}}\right) & \tag{3.23}
\end{array} \quad \forall \ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Omega},
$$

where $\omega_{\ell}:=\left\{K^{\prime} \in \mathcal{T}_{h}: \ell \in \mathcal{E}_{K^{\prime}}\right\}$.
Proof. First, for $\ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Gamma_{1}}$, we extend $J_{\ell} \in \mathbb{P}_{k-1}(\ell)$ to the element $K$ as in Remark 3.1. Let $\psi_{\ell}$ be the corresponding edge bubble function. We define $v:=J_{\ell} \psi_{\ell}$. Then, $v$ may be extended by zero to the whole domain $\Omega$. This extension, again denoted by $v$, belongs to $H^{1}(\Omega)$ and from Lemma 3.3 we have that

$$
a^{K}(e, v)=-a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, J_{\ell} \psi_{\ell}\right)+\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) J_{\ell} \psi_{\ell}+\int_{\ell} J_{\ell}^{2} \psi_{\ell}
$$

For $J_{\ell} \in \mathbb{P}_{k-1}(\ell)$, from Lemma 3.2 and the above equality we obtain

$$
\begin{aligned}
C^{-1}\left\|J_{\ell}\right\|_{0, \ell}^{2} & \leq \int_{\ell} J_{\ell}^{2} \psi_{\ell} \leq C\left[\left(|e|_{1, K}+\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}\right)\left|\psi_{\ell} J_{\ell}\right|_{1, K}+\left\|\Delta\left(\Pi_{k}^{K} w_{h}\right)\right\|_{0, K}\left\|J_{\ell} \psi_{\ell}\right\|_{0, K}\right] \\
& \leq C\left[\left(| |_{1, K}+\left|w_{h}-\Pi_{k}^{K} w_{h}\right|_{1, K}\right) h_{K}^{-1 / 2}\left\|J_{\ell}\right\|_{0, \ell}+h_{K}^{-1}\left(\theta_{K}+|e|_{1, K}\right) h_{K}^{1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\right] \\
& \leq C h_{K}^{-1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\left(|e|_{1, K}+\theta_{K}\right),
\end{aligned}
$$

where we have used again Lemma 3.2 together with estimate (3.20). Multiplying by $h_{K}^{1 / 2}$ the above inequality allows us to conclude (3.21).

Secondly, for $\ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Gamma_{0}}$, we extend $v:=J_{\ell} \psi_{\ell}$ to $H^{1}(\Omega)$ as in the previous case. Taking into account that in this case $J_{\ell} \in \mathbb{P}_{k}(\ell)$ and $\psi_{\ell}$ is a quadratic bubble function in $K$, from Lemma 3.3 we obtain

$$
a^{K}(e, v)=\lambda \int_{\ell} w J_{\ell} \psi_{\ell}-\lambda_{h} \int_{\ell} w_{h} J_{\ell} \psi_{\ell}-a^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, J_{\ell} \psi_{\ell}\right)+\int_{K} \Delta\left(\Pi_{k}^{K} w_{h}\right) J_{\ell} \psi_{\ell}+\int_{\ell} J_{\ell}^{2} \psi_{\ell}
$$

Then, repeating the previous arguments we obtain

$$
\left|\int_{\ell} J_{\ell}^{2} \psi_{\ell}\right| \leq C\left[\left|\lambda_{h} \int_{\ell} w_{h} J_{\ell} \psi_{\ell}-\lambda \int_{\ell} w J_{\ell} \psi_{\ell}\right|+h_{K}^{-1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\left(\theta_{K}+|e|_{1, K}\right)\right] .
$$

Hence, using Lemma 3.2 and a local trace inequality we arrive at

$$
\begin{aligned}
\left\|J_{\ell}\right\|_{0, \ell}^{2} & \leq C\left[\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \ell}\left\|\psi_{\ell} J_{\ell}\right\|_{0, \ell}+h_{K}^{-1 / 2}\left(\theta_{K}+|e|_{1, K}\right)\left\|J_{\ell}\right\|_{0, \ell}\right] \\
& \leq C h_{K}^{-1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\left(\theta_{K}+|e|_{1, K}+h_{K}^{1 / 2}\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \ell}\right)
\end{aligned}
$$

where we have used Lemma 3.2 again. Multiplying by $h_{K}^{1 / 2}$ the above inequality yields (3.22).
Finally, for $\ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Omega}$, we extend $v:=J_{\ell} \psi_{\ell}$ to $H^{1}(\Omega)$ as above again. Taking into account that $J_{\ell} \in \mathbb{P}_{k-1}(\ell)$ and $\psi_{\ell}$ is a quadratic bubble function in $K$, from Lemma 3.3 we obtain

$$
a(e, v)=-\sum_{K^{\prime} \in \omega_{\ell}} a^{K^{\prime}}\left(w_{h}-\Pi_{k}^{K^{\prime}} w_{h}, J_{\ell} \psi_{\ell}\right)+\sum_{K^{\prime} \in \omega_{\ell}} \int_{K^{\prime}} \Delta\left(\Pi_{k}^{K^{\prime}} w_{h}\right) J_{\ell} \psi_{\ell}+\sum_{K^{\prime} \in \omega_{\ell}} \int_{\ell} J_{\ell}^{2} \psi_{\ell}
$$

Then, proceeding analogously to the previous case we obtain

$$
\left\|J_{\ell}\right\|_{0, \ell}^{2} \leq C h_{K}^{-1 / 2}\left\|J_{\ell}\right\|_{0, \ell}\left[\sum_{K^{\prime} \in \omega_{\ell}}\left(|e|_{1, K^{\prime}}+\theta_{K^{\prime}}\right)\right]
$$

Thus, the proof is complete.

Now, we are in a position to prove an upper bound for the local error indicators $\eta_{K}$.
Theorem 3.3. There exists $C>0$ such that

$$
\eta_{K}^{2} \leq C\left[\sum_{K^{\prime} \in \omega_{K}}\left(\left|w-\Pi_{k}^{K^{\prime}} w_{h}\right|_{1, K^{\prime}}^{2}+\left|w-w_{h}\right|_{1, K^{\prime}}^{2}+\sum_{\ell \in \mathcal{E}_{K} \cap \mathcal{E}_{\Gamma_{0}}} h_{K}\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \ell}^{2}\right)\right]
$$

where $\omega_{K}:=\left\{K^{\prime} \in \mathcal{T}_{h}: K^{\prime}\right.$ and $K$ share an edge $\}$.
Proof. It follows immediately from Lemmas 3.5-3.7.
According to the above theorem, the error indicators $\eta_{K}^{2}$ provide lower bounds of the error terms $\sum_{K^{\prime} \in \omega_{K}}\left(\left|w-\Pi_{k}^{K^{\prime}} w_{h}\right|_{1, K^{\prime}}^{2}+\left|w-w_{h}\right|_{1, K^{\prime}}^{2}\right)$ in the neighborhood $\omega_{K}$ of $K$. For those elements $K$ with an edge on $\Gamma_{0}$, the term $h_{K}\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \ell}^{2}$ also appears in the estimate. Let us remark that it is reasonable to expect this terms to be asymptotically negligible. In fact, this is the case at least for the global estimator $\eta^{2}=\sum_{K \in \mathcal{T}_{h}} \eta_{K}^{2}$ as is shown in the following result.

Corollary 3.3. There exists a constant $C>0$ such that

$$
\eta^{2} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}\right)
$$

Proof. From Theorem 3.3 we have that

$$
\eta^{2} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}+h\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \Gamma_{0}}^{2}\right) .
$$

The last term on the right hand side above is bounded as follows:

$$
\left\|\lambda w-\lambda_{h} w_{h}\right\|_{0, \Gamma_{0}}^{2} \leq 2 \lambda^{2}\left\|w-w_{h}\right\|_{0, \Gamma_{0}}^{2}+2\left|\lambda-\lambda_{h}\right|^{2}
$$

where we have used that $\left\|w_{h}\right\|_{0, \Gamma_{0}}=1$. Now, by using a trace inequality and Poincaré inequality (2.8) we have

$$
\left\|w-w_{h}\right\|_{0, \Gamma_{0}} \leq C\left|w-w_{h}\right|_{1, \Omega}
$$

On the other hand, using the estimate (3.9), we have

$$
\left|\lambda-\lambda_{h}\right|^{2} \leq\left(|\lambda|+\left|\lambda_{h}\right|\right)\left|\lambda-\lambda_{h}\right| \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}\right)
$$

Therefore,

$$
\eta^{2} \leq C\left(\left|w-w_{h}\right|_{1, \Omega}^{2}+\left|w-\Pi_{h} w_{h}\right|_{1, h}^{2}\right)
$$

and we conclude the proof.

## 4. Numerical results

In this section, we will investigate the behavior of an adaptive scheme driven by the error indicator in two numerical tests that differ in the shape of the computational domain $\Omega$ and, hence, in the regularity of the exact solution. With this aim, we have implemented in a MATLAB code a lowest-order VEM $(k=1)$ on arbitrary polygonal meshes following the ideas proposed in [8].

To complete the choice of the VEM, we had to choose the bilinear forms $S^{K}(\cdot, \cdot)$ satisfying (2.4). In this respect, we proceeded as in [7, Section 4.6]: for each polygon $K$ with vertices $P_{1}, \ldots, P_{N_{K}}$, we used

$$
S^{K}(u, v):=\sum_{r=1}^{N_{K}} u\left(P_{r}\right) v\left(P_{r}\right), \quad u, v \in V_{1}^{K}
$$

In all our tests we have initiated the adaptive process with a coarse triangular mesh. In order to compare the performance of VEM with that of a finite element method (FEM), we have used two different algorithms to refine the meshes. The first one is based on a classical FEM strategy for which all the subsequent meshes consist of triangles. In such a case, for $k=1$, VEM reduces to FEM. The other procedure to refine the meshes is described in [11]. It consists of splitting each element into $n$ quadrilaterals ( $n$ being the number of edges of the polygon) by connecting the barycenter of the element with the midpoint of each edge as shown in Figure 1 (see [11] for more details). Notice that although this process is initiated with a mesh of triangles, the successively created meshes will contain other kind of convex polygons, as can be seen in Figures 3 and 7.

(a) Triangle $K$ refined into 3 quadrilaterals.

(b) Pentagon $K$ refined into 5 quadrilaterals.

Figure 1: Example of refined elements for VEM strategy.

Since we have chosen $k=1$, according to the definition of the local virtual element space $V_{1}^{K}$ (cf. (2.2)), the term $R_{K}^{2}:=h_{K}^{2}\left\|\Delta w_{h}\right\|_{0, K}^{2}$ vanishes. Thus, the error indicators reduce in this case to

$$
\eta_{K}^{2}=\theta_{K}^{2}+\sum_{\ell \in \mathcal{E}_{K}} h_{K}\left\|J_{\ell}\right\|_{0, \ell}^{2} \quad \forall K \in \mathcal{T}_{h} .
$$

Let us remark that in the case of triangular meshes, the term $\theta_{K}^{2}:=a_{h}^{K}\left(w_{h}-\Pi_{k}^{K} w_{h}, w_{h}-\Pi_{k}^{K} w_{h}\right)$ vanishes too, since $V_{1}^{K}=\mathbb{P}_{1}(K)$ and hence $\Pi_{k}^{K}$ is the identity. By the same reason, the projection $\Pi_{k}^{K}$ also disappears in the definition (3.1) of $J_{\ell}$. Therefore, for triangular meshes, not only VEM reduces to FEM, but also the error indicator becomes the classical well-known edge-residual error estimator (see [5]):

$$
\eta_{K}^{2}:=\sum_{\ell \in \mathcal{E}_{K}} h_{K}\left\|J_{\ell}\right\|_{0, \ell}^{2} \quad \text { with } \quad J_{\ell}:= \begin{cases}\frac{1}{2} \llbracket \frac{\partial w_{h}}{\partial n} \|_{\ell}, & \ell \in \mathcal{E}_{\Omega}, \\ \lambda_{h} w_{h}-\frac{\partial w_{h}}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_{0}}, \\ -\frac{\partial w_{h}}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_{1}} .\end{cases}
$$

In what follows, we report the results of a couple of tests. In both cases, we will restrict our attention to the approximation of the eigenvalues. Let us recall that according to Corollary 3.2, the global error estimator $\eta^{2}$ provides an upper bound of the error of the computed eigenvalue.

### 4.1. Test 1: Sloshing in a square domain.

We have chosen for this test a problem with known analytical solution. It corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in the domain $\Omega:=(0,1)^{2}$ with a horizontal free surface $\Gamma_{0}$ as shown in Figure 2. The solutions of this problem are

$$
\lambda_{n}=n \pi \tanh (n \pi), \quad w_{n}(x, y)=\cos (n \pi x) \sinh (n \pi y), \quad n \in \mathbb{N} .
$$



Figure 2: Test 1. Sloshing in a square domain.

We have used the two refinement procedures (VEM and FEM ) described above. Both schemes are based on the strategy of refining those elements $K$ which satisfy

$$
\eta_{K} \geq 0.5 \max _{K^{\prime} \in \mathcal{T}_{h}}\left\{\eta_{K^{\prime}}\right\}
$$

Figures 3 and 4 show the adaptively refined meshes obtained with VEM and FEM procedures, respectively.


Figure 3: Test 1. Adaptively refined meshes obtained with VEM scheme at refinement steps $0,1,3$ and 6 .


Figure 4: Test 1. Adaptively refined meshes obtained with FEM scheme at refinement steps $0,1,3$ and 6.

Since the eigenfunctions of this problem are smooth, according to (2.9) we have that $\left|\lambda-\lambda_{h}\right|=$ $\mathcal{O}\left(h^{2}\right)$. Therefore, in case of uniformly refined meshes, $\left|\lambda-\lambda_{h}\right|=\mathcal{O}\left(N^{-1}\right)$, where $N$ denotes the number of degrees of freedom which is the optimal convergence rate that can be attained.

Figure 5 shows the error curves for the computed lowest eigenvalue on uniformly refined meshes and adaptively refined meshes with FEM and VEM schemes. The plot also includes a line of slope -1 , which correspond to the optimal convergence rate of the method $\mathcal{O}\left(N^{-1}\right)$.

Figure 5: Test 1. Error curves of $\left|\lambda_{1}-\lambda_{h 1}\right|$ for uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").


It can be seen from Figure 5 that the three refinement schemes lead to the correct convergence rate. Moreover, the performance of adaptive VEM is slightly better than that of adaptive FEM, while this is also better than uniform FEM.

We report in Table 1, the errors $\left|\lambda_{1}-\lambda_{h 1}\right|$ and the estimators $\eta^{2}$ at each step of the adaptive VEM scheme. We include in the table the terms $\theta^{2}:=\sum_{K \in \mathcal{T}_{h}} \theta_{K}^{2}$ which arise from the inconsistency of VEM and $J^{2}:=\sum_{K \in \mathcal{T}_{h}}\left(\sum_{\ell \in \mathcal{E}_{K}} h_{K}\left\|J_{\ell}\right\|_{0, \ell}^{2}\right)$ which arise from the edge residuals. We also report in the table the effectivity indexes $\left|\lambda_{1}-\lambda_{h 1}\right| / \eta^{2}$.

Table 1: Test 1. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

| $N$ | $\lambda_{h 1}$ | $\left\|\lambda_{1}-\lambda_{h 1}\right\|$ | $\theta^{2}$ | $J^{2}$ | $\eta^{2}$ | $\frac{\left\|\lambda_{1}-\lambda_{h 1}\right\|}{\eta^{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 38 | 3.2499 | 0.1200 | 0 | 0.8245 | 0.8245 | 0.1456 |
| 167 | 3.1644 | 0.0345 | 0.0111 | 0.2469 | 0.2580 | 0.1339 |
| 313 | 3.1450 | 0.0151 | 0.0117 | 0.1108 | 0.1225 | 0.1234 |
| 745 | 3.1355 | 0.0056 | 0.0054 | 0.0427 | 0.0481 | 0.1171 |
| 1540 | 3.1327 | 0.0028 | 0.0033 | 0.0216 | 0.0249 | 0.1113 |
| 3392 | 3.1311 | 0.0013 | 0.0015 | 0.0102 | 0.0117 | 0.1069 |
| 5806 | 3.1307 | 0.0008 | 0.0009 | 0.0064 | 0.0073 | 0.1069 |
| 11973 | 3.1303 | 0.0004 | 0.0005 | 0.0032 | 0.0037 | 0.1075 |

It can be seen from Table 1 that the effectivity indexes are bounded above and below far from zero and that the inconsistency and edge residual terms are roughly speaking of the same order, none of them being asymptotically negligible.

### 4.2. Test 2:

The aim of this test is to assess the performance of the adaptive scheme when solving a problem with a singular solution. In this test $\Omega$ consists of a unit square from which it is subtracted an equilateral triangle as shown in Figure 6. In this case $\Omega$ has a reentrant angle $\omega=\frac{5 \pi}{3}$. Therefore, the Sobolev exponent is $r_{\Omega}:=\frac{\pi}{\omega}=3 / 5$, so that the eigenfunctions will belong to $H^{1+r}(\Omega)$ for all $r<3 / 5$, but in general not to $H^{1+3 / 5}(\Omega)$. Therefore, according to (2.9), using quasi-uniform meshes, the convergence rate for the eigenvalues should be $\left|\lambda-\lambda_{h}\right| \approx \mathcal{O}\left(h^{6 / 5}\right) \approx \mathcal{O}\left(N^{-3 / 5}\right)$. An efficient adaptive scheme should lead to refine the meshes in such a way that the optimal order $\left|\lambda-\lambda_{h}\right|=\mathcal{O}\left(N^{-1}\right)$ could be recovered.


Figure 6: Test 2. Domain $\Omega$.

Figures 7 and 8 show the adaptively refined meshes obtained with the VEM and FEM adaptive schemes, respectively.


Figure 7: Test 2. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1, 4 and 6.


Figure 8: Test 2. Adaptively refined meshes obtained with FEM scheme at refinement steps $0,1,4$ and 6.

In order to compute the errors $\left|\lambda_{1}-\lambda_{h 1}\right|$, due to the lack of an exact eigenvalue, we have used an approximation based on a least squares fitting of the computed values obtained with extremely refined meshes. Thus, we have obtained the value $\lambda_{1}=1.9288$, which has at least four correct significant digits.

We report in Table 2 the lowest eigenvalue $\lambda_{h 1}$ computed with each of the three schemes. Each table includes the estimated convergence rate.

Table 2: Test 2. Eigenvalue $\lambda_{h 1}$ computed with different schemes: uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").

| Uniform FEM |  | Adaptive VEM |  | Adaptive FEM |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | $\lambda_{h 1}$ | $N$ | $\lambda_{h 1}$ | $N$ | $\lambda_{h 1}$ |
| 38 | 2.3083 | 38 | 2.3083 | 38 | 2.3083 |
| 123 | 2.0686 | 58 | 2.0721 | 60 | 2.1067 |
| 437 | 1.9828 | 106 | 1.9960 | 85 | 2.0362 |
| 1641 | 1.9505 | 229 | 1.9592 | 148 | 1.9810 |
| 6353 | 1.9377 | 350 | 1.9467 | 185 | 1.9678 |
| 14137 | 1.9341 | 666 | 1.9384 | 280 | 1.9530 |
| 24993 | 1.9325 | 909 | 1.9354 | 458 | 1.9427 |
| 38291 | 1.9316 | 1340 | 1.9329 | 646 | 1.9382 |
| 55921 | 1.9310 | 2141 | 1.9315 | 895 | 1.9356 |
| 75993 | 1.9306 | 3438 | 1.9306 | 1593 | 1.9325 |
| 99137 | 1.9303 | 5172 | 1.9300 | 2122 | 1.9315 |
| 125353 | 1.9301 | 8014 | 1.9296 | 3178 | 1.9306 |
| 154641 | 1.9299 | 12365 | 1.9293 | 5341 | 1.9298 |
| 187001 | 1.9298 | 19153 | 1.9291 | 7522 | 1.9295 |
| 222433 | 1.9297 | 29403 | 1.9290 | 11124 | 1.9292 |
| Order | $\mathcal{O}\left(N^{-0.68}\right)$ | Order | $\mathcal{O}\left(N^{-1.10}\right)$ | Order | $\mathcal{O}\left(N^{-1.16}\right)$ |
| $\lambda_{1}$ | 1.9288 | $\lambda_{1}$ | 1.9288 | $\lambda_{1}$ | 1.9288 |

It can be seen from Table 2, that the uniform refinement leads to a convergence rate close to that predicted by the theory $\mathcal{O}\left(N^{-3 / 5}\right)$. Instead, Tables 2 show that the adaptive VEM and FEM schemes allow us to recover the optimal order of convergence $\mathcal{O}\left(N^{-1}\right)$. This can be clearly seen from Figure 9, where the three error curves are reported. The plot also includes lines of slopes -1 and $-3 / 5$, which correspond to the convergence rates of each scheme.

Figure 9: Test 2. Error curves of $\left|\lambda_{1}-\lambda_{h 1}\right|$ for uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").


Finally, we report in Table 3 the same information as in Table 1 for this test. Similar conclusions as in the previous test follow from this table.

Table 3: Test 2. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

| $N$ | $\lambda_{h 1}$ | $\left\|\lambda_{1}-\lambda_{h 1}\right\|$ | $\theta^{2}$ | $J^{2}$ | $\eta^{2}$ | $\frac{\left\|\lambda_{1}-\lambda_{h 1}\right\|}{\eta^{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 38 | 2.3083 | 0.3795 | 0 | 2.3181 | 2.3181 | 0.1637 |
| 58 | 2.0721 | 0.1433 | 0.0379 | 0.8231 | 0.8609 | 0.1664 |
| 106 | 1.9960 | 0.0672 | 0.0368 | 0.4188 | 0.4556 | 0.1475 |
| 229 | 1.9592 | 0.0304 | 0.0216 | 0.1942 | 0.2158 | 0.1408 |
| 350 | 1.9467 | 0.0179 | 0.0164 | 0.1359 | 0.1522 | 0.1173 |
| 666 | 1.9384 | 0.0096 | 0.0094 | 0.0749 | 0.0844 | 0.1143 |
| 909 | 1.9354 | 0.0066 | 0.0068 | 0.0556 | 0.0624 | 0.1052 |
| 1340 | 1.9329 | 0.0041 | 0.0047 | 0.0408 | 0.0454 | 0.0907 |
| 2141 | 1.9315 | 0.0027 | 0.0032 | 0.0275 | 0.0308 | 0.0891 |
| 3438 | 1.9306 | 0.0018 | 0.0022 | 0.0178 | 0.0199 | 0.0904 |

## Conclusions

We have derived an a posteriori error indicator for the VEM solution of the Steklov eigenvalue problem. We have proved that it is efficient and reliable. For lowest order elements on triangular meshes, VEM coincides with FEM and the a posteriori error indicators also coincide with the classical ones. However VEM allows using general polygonal meshes including hanging nodes, which is particularly interesting when designing an adaptive scheme. We have implemented such a scheme driven by the proposed error indicators. We have assessed its performance by means of a couple of tests which allow us to confirm that the adaptive scheme yields optimal order of convergence for regular as well as singular solutions.

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## References

[1] B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini and A.Russo, Equivalent projectors for virtual element methods, Comput. Math. Appl., 66, (2013), pp. 376-391.
[2] M. Ainsworth and J.T. Oden, A posteriori error estimation in finite element analysis, In: Pure and Applied Mathematics. Wiley, New York (2000).
[3] P.F. Antonietti, L. Beirão da Veiga, D. Mora and M. Verani, A stream virtual element formulation of the Stokes problem on polygonal meshes, SIAM J. Numer. Anal., 52, (2014), pp. 386-404.
[4] M.G. Armentano, The effect of reduced integration in the Steklov eigenvalue problem, Math. Model. Numer. Anal., 38, (2004), pp. 27-36.
[5] M.G. Armentano and C. Padra, A posteriori error estimates for the Steklov eigenvalue problem, Appl. Numer. Math., 58, (2008), pp. 593-601.
[6] B. Ayuso de Dios, K. Lipnikov and G. Manzini, The nonconforming virtual element method, ESAIM Math. Model. Numer. Anal., 50, (2016), pp. 879-904.
[7] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini and A. Russo, Basic principles of virtual element methods, Math. Models Methods Appl. Sci., 23, (2013), pp. 199-214.
[8] L. Beirão da Veiga, F. Brezzi, L.D. Marini and A. Russo, The hitchhiker's guide to the virtual element method, Math. Models Methods Appl. Sci., 24, (2014), pp. 1541-1573.
[9] L. Beirão da Veiga, K. Lipnikov and G. Manzini, The Mimetic Finite Difference Method for Elliptic Problems, Springer, MS\&A, vol. 11, (2014).
[10] L. Beirão da Veiga, C. Lovadina and D. Mora, A virtual element method for elastic and inelastic problems on polytope meshes, Comput. Methods Appl. Mech. Engrg., 295, (2015) pp. 327-346.
[11] L. Beirão da Veiga and G. Manzini, Residual a posteriori error estimation for the virtual element method for elliptic problems, ESAIM Math. Model. Numer. Anal., 49, (2015), pp. 577-599.
[12] L. Beirão da Veiga, D. Mora and G. Rivera, A virtual element method for ReissnerMindlin plates, $\mathrm{CI}^{2}$ MA preprint 2016-14, available from http://www.ci2ma.udec.cl.
[13] L. Beirão da Veiga, D. Mora, G. Rivera and R. Rodríguez, A virtual element method for the acoustic vibration problem, arXiv:1601.04316v.1, (2016).
[14] A. Bermúdez, R. Rodríguez and D. Santamarina, A finite element solution of an added mass formulation for coupled fluid-solid vibrations, Numer. Math., 87, (2000), pp. 201-227.
[15] A. Bermúdez, R. Rodríguez and D. Santamarina, Finite element computation of sloshing modes in containers with elastic baffle plates, Internat. J. Numer. Methods Engrg., 56, (2003), pp. 447-467.
[16] M.F. Benedetto, S. Berrone, S. Pieraccini and S. Scialò, The virtual element method for discrete fracture network simulations, Comput. Methods Appl. Mech. Engrg., 280, (2014), pp. 135-156.
[17] D. Boffi, Finite element approximation of eigenvalue problems, Acta Numerica, 19, (2010), pp. 1-120.
[18] E. Caceres and G.N. Gatica, A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem, IMA J. Numer. Anal., DOI:10.1093/imanum/drw002 (2016).
[19] J. Canavati and A. Minsoni, A discontinuous Steklov problem with an application to water waves, J. Math. Anal. Appl., 69, (1979), pp. 540-558.
[20] A. Cangiani, E.H. Georgoulis and P. Houston, hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes, Math. Models Methods Appl. Sci., 24, (2014), pp. 2009-2041.
[21] A. Cangiani, E.H. Georgoulis, T. Pryer and O.J. Sutton, A posteriori error estimates for the virtual element method, arXiv:1603.05855v.1, (2016).
[22] A. Cangiani, G. Manzini and O.J. Sutton, Conforming and nonconforming virtual element methods for elliptic problems, IMA J. Numer. Anal., DOI:10.1093/imanum/drw036 (2016).
[23] M. Chiba, Non-linear hydroelastic vibrations of a cylindrical tank with an elastic bottom containing liquid; Part II: linear axisymmetric vibration analysis, J. Fluids Struct., 7, (1993), pp. 57-73.
[24] Y.S. Choun and C.B. Yun, Sloshing characteristics in rectangular tanks with a submerged block, Comput. Struct., 61, (1996), pp. 401-413.
[25] A. Dello Russo and A. Alonso, A posteriori error estimates for nonconforming approximations of Steklov eigenvalue problems, Comput. Math. Appl., 62, (2011), pp. 4100-4117.
[26] D. Di Pietro and A. Ern, A hybrid high-order locking-free method for linear elasticity on general meshes, Comput. Methods Appl. Mech. Eng., 283, (2015), pp. 1-21.
[27] R. Durán, C. Padra and R. Rodríguez, A posteriori error estimates for the finite element approximation of eigenvalue problems, Math. Models Methods Appl. Sci., 13, (2003), pp. 12191229.
[28] D.V. Evans and P. McIver, Resonance frequencies in a container with a vertical baffle, J. Fluid Mech., 175, (1987), pp. 295-307.
[29] A.L. Gain, C. Talischi and G.H. Paulino, On the virtual element method for threedimensional linear elasticity problems on arbitrary polyhedral meshes, Comput. Methods Appl. Mech. Engrg., 282, (2014), pp. 132-160.
[30] E.M. Garau and P. Morin, Convergence and quasi-optimality of adaptive FEM for Steklov eigenvalue problems, IMA J. Numer. Anal., 31, (2011), pp. 914-946.
[31] D. Mora, G. Rivera and R. Rodríguez, A virtual element method for the Steklov eigenvalue problem, Math. Models Methods Appl. Sci., 25, (2015), pp. 1421-1445.
[32] G.H. Paulino and A.L. Gain, Bridging art and engineering using Escher-based virtual elements, Struct. Multidiscip. Optim., 51, (2015), pp. 867-883.
[33] I. Perugia, P. Pietra and A. Russo, A plane wave virtual element method for the Helmholtz problem, ESAIM Math. Model. Numer. Anal., 50, (2016), pp. 783-808.
[34] J. Planchard and J.M. Thomas, On the dynamic stability of cylinders placed in cross-flow, J. Fluids Struct., 7, (1993), pp. 321-339.
[35] S. RJasanow and S. Weisser, Higher order BEM-based FEM on polygonal meshes, SIAM J. Numer. Anal., 50, (2012), pp. 2357-2378.
[36] N. Sukumar and A. Tabarraei, Conforming polygonal finite elements, Internat. J. Numer. Methods Engrg., 61, (2004), pp. 2045-2066.
[37] C. Talischi, G.H. Paulino, A. Pereira and I.F.M. Menezes, Polygonal finite elements for topology optimization: A unifying paradigm, Internat. J. Numer. Methods Engrg., 82, (2010), pp. 671-698.
[38] R. Verfurth, A review of a posteriori error estimate and adaptative mesh-refinement techniques, Wiley-Teubner, Chichester (1996).
[39] E.B.B. Watson and D.V. Evans, Resonance frequencies of a fluid in containers with internal bodies, J. Eng. Math., 25, (1991), pp. 115-135.
[40] H. Xie, A type of multilevel method for the Steklov eigenvalue problem, IMA J. Numer. Anal., 34, (2014), pp. 592-608.

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