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A Banach spaces-based fully mixed virtual element method for the stationary two-dimensional Boussinesq equations*

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Abstract

In this paper we extend recent results obtained for the Navier-Stokes equations to propose and analyze a new fully mixed virtual element method (mixed-VEM) for the stationary two-dimensional Boussinesq equations appearing in non-isothermal flow phenomena. The model consists of a Navier-Stokes type system, modeling the velocity and the pressure of the fluid, coupled to an advection-diffusion equation for the temperature. The variational formulation is based on the introduction of the additional unknowns given by a modified pseudostress tensor, which depends on the pressure, and the diffusive and convective terms of the fluid, and the pseudoheat vector, which involves the temperature, its gradient, and the velocity. As a consequence of the former, the pressure is eliminated from the system, but computed afterwards via a post-processing formula. In turn, for the Galerkin approximation we follow the approach employed in a previous work introducing for the first time an L^p spaces-based mixed-VEM for the Navier-Stokes equations, and couple it with a similar mixed-VEM for the convection-diffusion equation modelling the temperature. The solvability analysis of the resulting coupled discrete scheme is carried out by using appropriate fixed-point arguments, along with the discrete versions of the Babuška-Brezzi theory and the Banach-Nečas-Babuška theorem, both in subspaces of Banach spaces. The first Strang lemma is applied to derive the a priori error estimates for the virtual element solution as well as for the fully computable approximation of the pseudostress tensor, the pseudoheat vector, and the post-processed pressure. Finally, several numerical results, illustrating the performance of the mixed-VEM scheme and confirming the rates of convergence predicted by the theory, are reported.

Key words: Boussinesq problem, pseudostress-based formulation, Banach spaces, mixed virtual element method.

Mathematics Subject Classifications (2020): 65N30, 65N12, 65N15, 65N99, 76M25, 76S05

1 Introduction

The phenomenon of heat transport in a fluid is a significant area of application in modeling mantle convection, stratified oceanic flows and the cooling of electronic devices, which involves three key fields, namely flow velocity, pressure, and temperature. This phenomenon can be mathematically described by a coupled system of Navier–Stokes and advection-diffusion-dominated transport equations called the Boussinesq equation. As we well know, obtaining accurate solutions to coupled systems is

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never an easy task, and this is especially so for nonlinear systems. Hence, experts in computational mathematics are challenged to design new efficient numerical methods which provide highly accurate numerical solutions allowing for preserving the physical properties. Up to now, a variety of numerical approximations has been introduced and investigated for the steady and evolutive Boussinesq equations and its generalizations to the case of temperature-dependent parameters, including finite element method (FEM) [10, 21, 25, 33], mixed and augmented-mixed FEM approaches [2, 1, 23, 24, 19], and virtual element methods [31].

Up to the authors' knowledge, Bernardi et al. [9] constitutes one of the first works in analyzing FEMs for Boussinesq problem. In turn, [27] proposes a dual-mixed FEM for the respective two-dimensional model, where gradients of velocity and temperature are also introduced as additional unknowns. Lately, an augmented mixed-primal FEM, which defines a modified nonlinear pseudostress tensor involving velocity gradient, convective term and the pressure, similarly as done in [18] for a dual-mixed formulation of the Navier–Stokes equation, is presented in [23] for the stationary Boussinesq model. In other words, the augmented scheme for the fluid flow is coupled with a primal scheme for the convection-diffusion equation, thus yielding the aforementioned nonlinear pseudostress, the velocity, the temperature, and the normal derivative of the latter on the boundary, as the main unknowns. Regarding fully mixed approaches, augmented mixed formulations have been introduced for the Boussinesq problem with temperature-dependent and constant viscosity in [1, 24]. In both cases, the analysis is based on the introduction of a pseudostress tensor relating the diffusive and convective terms with the pressure and optimal convergence is proven. However, this approach can be computationally expensive and difficult to implement in practice. To overcome this issue, a new Banach spaces-based mixed FEM for the Boussinesq problem is developed in [19], which allows, on the one hand, to conserve momentum and thermal energy if the external forces belong to the velocity and temperature discrete spaces, respectively, and on the other hand, to compute further variables of interest, such as the fluid vorticity, the fluid velocity gradient, and the heat-flux, through a simple postprocess of the finite element solutions. In this way, no numerical differentiation is applied, and hence no further sources of error arise. A posteriori error analysis of the corresponding formulation is also addressed in [20].

In recent years, a significant number of researchers has concentrated on the extension of polygonal-based numerical methods, such as mimetic finite difference methods [13, 12] and virtual element methods (VEM) [5, 11, 14, 16, 29], for solving diverse models in continuum mechanics. Actually, since the formulation of FEM requires explicit knowledge of the basis functions, these very powerful technique might often be limited (at least in their classical setting) to meshes with simple-geometrical shaped elements, e.g. triangles or quadrilaterals. This constraint is overcome by polytopal element methods such as VEM, which are designed for providing arbitrary order of accuracy on polygonal/polytopal elements, and for which the explicit knowledge of the basis functions is not required. Moreover, its practical implementation relies on suitable projection operators that are computable by their degrees of freedom. Furthermore, the advances in the theory and applications of VEM has also been aware of the fact that additional physically relevant variables, such as stress, velocity gradient, vorticity, heat flux, and others, reveal specific mechanisms of the phenomena, so that they become of primary interest. Hence, the development of mixed VEMs for solving diverse linear and nonlinear problems in, for instance, fluid mechanics, using velocity-pressure [3, 7, 8] and pseudostress-velocity-based formulations [32, 31, 14, 16, 29], has become an interesting research field as well. In particular, a mixed-VEM based on the pseudostress-velocity formulation for the Stokes equation was developed in [14] as a natural extension of previously proposed dual-mixed-FEMs for elliptic equations in divergence form (cf. []). The discussion in [14] includes the virtual finite element subspaces to be employed, the associated interpolation operators, and the respective approximation properties. Moreover, the uniform boundedness of the resulting family of projectors and its corresponding approximation properties are established there. In this way, the classical Babuška–Brezzi theory is applied to prove the well-posedness of the discrete scheme and derive the associated a priori error estimates for the virtual solution as well as for the fully computable projection of it. Later on, this approach was extended to other models in fluid mechanics, such as quasi-Newtonian Stokes [16], linear and nonlinear Brinkman [15, 29], Navier–Stokes [30, 32], and Boussinesq [31], where different formulations were considered. In particular, the Banach spaces-based approach, usually utilized for solving diverse nonlinear problems in continuum mechanics via primal and mixed finite element methods, is extended for the first time in [32] to the virtual element method (VEM) framework and its respective applications. More precisely, an L^p spaces-based mixed VEM for a pseudostress-velocity formulation of the two-dimensional Navier-Stokes equations with Dirichlet boundary conditions, is proposed and analyzed there. The simplicity of the resulting mixed-VEM scheme is reflected by the absence of augmented terms, on the contrary to previous work on this model [30], and by the fact that only a virtual element space for the pseudostress tensor is required since the non-virtual but explicit subspace given by the piecewise polynomial vectors of degree $\leq k$ is employed to approximate the velocity.

According to the above discussion and in order to continue extending the applicability of mixed-VEM to nonlinear models in fluids mechanics, we now generalize the approach from [32] to the case of the Boussinesq problem. More precisely, we basically consider the equations and the resulting variational formulation of the non-augmented version from [24] (see also [19]), and then adapt the approach from [32] to propose, up to our knowledge for the first time, an L^p spaces-based fully mixed-VEM for Boussinesq. In this way, the pseudostress and the velocity of the fluid are approximated by virtual element subspaces of $\mathbb{H}(\mathbf{div}_{4/3})$ and $\mathbf{L}^4$, respectively, whereas virtual element subspaces of $\mathbf{H}(\mathrm{div}_{4/3})$ and L^4 are employed to approximate the pseudoheat vector and the temperature, respectively. Thus, similarly as in the aforementioned references, fixed-point strategies along with discrete versions of the Babuška-Brezzi theory and the Banach-Nečas-Babuška theorem, and a Strang-type lemma, are utilized to develop the solvability analysis and to derive the associated a priori error estimates for the components of the virtual element solution and the postprocessed pressure. The rest of this work is organized as follows. In Section 2 we introduce the model of interest, derive the corresponding variational formulation, and recall its solvability analysis from [19]. In Section 3 we provide most details of the virtual element discretization, including the mesh entities, the degrees of freedom, the construction of the mixed virtual element subspaces, and the main properties of the discrete linear and bilinear forms. Next, in Section 4 we establish the existence and uniqueness of solution of the discrete problem under smallness assumptions on the data. In turn, a priori error estimates and associated rates of convergence for the full solution of the virtual element scheme, as well as for the computable postprocessed approximations of the pseudostress, pseudoheat, and pressure, are derived in Section 5. Finally, several numerical examples illustrating the performance of the mixed-VEM scheme and confirming the rates of convergence predicted by the theory, are reported in Section 6.

1.1 Notations

For any vector fields $\mathbf{v} = (v_1, v_2)^{\mathtt{t}}$ and $\mathbf{w} = (w_1, w_2)^{\mathtt{t}}$, we set the gradient, divergence, and tensor product operators as

$$\boldsymbol{\nabla}\mathbf{v} := (\nabla v_1, \nabla v_2), \quad \mathrm{div}(\mathbf{v}) := \partial_x v_1 + \partial_y v_2, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i\, w_j)_{i,j=1,2}\,,$$

respectively. In addition, denoting by $\mathbb{I}$ the identity matrix of R^2, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})$, $\boldsymbol{\zeta} = (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}\,,$$

which corresponds, respectively, to the transpose, the trace, and the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. Next, given a bounded domain $\mathcal{D} \subset \mathrm{R}^2$ with boundary $\partial\mathcal{D}$, we let $\mathbf{n}$ be the outward unit normal vector on $\partial\mathcal{D}$. Also, given $r \geq 0$ and $1 < p \leq \infty$, we let $\mathrm{W}^{r,p}(\mathcal{D})$ be the standard Sobolev space with norm $\|\cdot\|_{r,p;\mathcal{D}}$ and seminorm $|\cdot|_{r,p;\mathcal{D}}$. In particular, for $r = 0$ we let $\mathrm{L}^p(\mathcal{D}) := \mathrm{W}^{0,p}(\mathcal{D})$ be the usual Lebesgue space, and for $p = 2$ we let $\mathrm{H}^s(\mathcal{D}) := \mathrm{W}^{r,2}(\mathcal{D})$ be the classical Hilbertian Sobolev space with norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. Furthermore, given a generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts, respectively, whose norms and seminorms are denoted exactly as those of M. On the other hand, given $t \in (1, +\infty)$, and letting $\mathbf{div}$ (resp. $\mathbf{rot}$) be the usual divergence operator div (resp. rotational operator rot) acting along the rows of a given tensor, we introduce the non-standard Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) \;:=\; \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\mathbf{v}) \in \mathrm{L}^t(\Omega) \Big\},$$

and

$$\mathbb{H}(\mathbf{div}_t;\Omega) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\},$$

equipped with the usual norms

$$\|\mathbf{v}\|_{\mathrm{div}_t;\Omega} \;:=\; \|\mathbf{v}\|_{0,\Omega} + \|\,\mathrm{div}(\mathbf{v})\|_{0,t;\Omega}, \qquad \forall\, \mathbf{v} \in \mathbf{H}(\mathrm{div}_t;\Omega),$$

and

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,\Omega} + \|\,\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}, \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega),$$

respectively.

2 The model problem and its continuous formulation

Let Ω be a bounded polygonal domain in R^2 with boundary Γ and outward unit normal vector $\mathbf{n}$. We consider the stationary Boussinesq problem, that is, given an external force per unit mass $\mathbf{g} \in \mathbf{L}^\infty(\Omega)$, and the boundary data $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ and $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$, we are interested in finding the velocity $\mathbf{u}$, the pressure p, and the temperature θ of a fluid occupying the region Ω, such that

$$-\nu\,\boldsymbol{\Delta}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\mathbf{u} + \nabla p - \theta\mathbf{g} \;=\; \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$\mathrm{div}\,\mathbf{u} \;=\; 0 \quad \text{in} \quad \Omega, \tag{2.1b}$$

$$-\kappa\,\Delta\theta + \mathbf{u}\cdot\nabla\theta \;=\; 0 \quad \text{in} \quad \Omega\,, \tag{2.1c}$$

where ν and κ denote the viscosity and thermal conductivity of the fluid, respectively. In addition, the model is supplied with the boundary conditions

$$\mathbf{u} \;=\; \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \quad \theta \;=\; \theta_D \quad \text{on} \quad \Gamma\,, \tag{2.2}$$

and the uniqueness condition for the pressure

$$\int_\Omega p \;=\; 0\,. \tag{2.3}$$

Note that, due to the incompressibility of the fluid (cf. (2.1b)), $\mathbf{u}_D$ must satisfy

$$\int_\Gamma \mathbf{u}_D\cdot\mathbf{n} \;=\; 0\,. \tag{2.4}$$

Next, in order to derive our fully-mixed formulation, we need to rewrite equations (2.1a) and (2.1c) as a first-order system. For this purpose, we begin by introducing the pseudostress tensor and pseudoheat vector variables

$$\boldsymbol{\sigma} \;:=\; \nu\boldsymbol{\nabla}\mathbf{u} - \mathbf{u}\otimes\mathbf{u} - (p + c_{\mathbf{u}})\mathbb{I} \quad \text{and} \quad \boldsymbol{\rho} \;:=\; \kappa\nabla\theta - \theta\mathbf{u} \quad \text{in} \quad \Omega\,, \tag{2.5}$$

where $c_{\mathbf{u}}$ is defined by

$$c_{\mathbf{u}} \,:=\, -\,\frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) \,=\, -\,\frac{1}{2|\Omega|}\|\mathbf{u}\|^2_{0,\Omega}. \tag{2.6}$$

By applying the trace operator to both sides of the first equation of (2.5), and utilizing again the incompressibility condition $\mathrm{div}\,\mathbf{u} = \mathrm{tr}(\boldsymbol{\nabla}\mathbf{u}) = 0$ (cf. (2.1b)), one arrives at

$$p \,=\, -\,\frac{1}{2}\big(\,\mathrm{tr}\,\boldsymbol{\sigma} + \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\big) - c_{\mathbf{u}} \quad \text{in} \quad \Omega\,, \tag{2.7}$$

which allows us to eliminate the pressure variable from the rest of the formulation. In turn, according to (2.6) and (2.7), the assumption (2.3) becomes

$$\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0\,. \tag{2.8}$$

Hence, using the deviatoric operator and (2.1b), we can rewrite (2.1a) and (2.1c) together with the associated boundary conditions (2.2), respectively, as

$$\begin{aligned} \nu^{-1}\boldsymbol{\sigma}^{\mathrm{d}} + \nu^{-1}(\mathbf{u}\otimes\mathbf{u}) \quad &= \quad \boldsymbol{\nabla}\mathbf{u} \quad \text{in} \quad \Omega, \\ \mathbf{div}\,\boldsymbol{\sigma} + \theta\mathbf{g} \quad &= \quad \mathbf{0} \quad \text{in} \quad \Omega, \\ \mathbf{u}|_\Gamma = \mathbf{u}_D \qquad &\text{and} \qquad \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0\,, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \kappa^{-1}\boldsymbol{\rho} + \kappa^{-1}\theta\mathbf{u} \quad &= \quad \nabla\theta \quad \text{in} \quad \Omega, \\ \mathrm{div}\,\boldsymbol{\rho} \quad &= \quad 0 \quad \text{in} \quad \Omega, \\ \theta|_\Gamma \quad &= \quad \theta_D\,. \end{aligned} \tag{2.10}$$

Thus, in order to derive a velocity–pseudostress based-mixed formulation for (2.9), we let $\mathbb{X}$ and $\mathbf{Y}$ be suitable test spaces to be defined below, and formally multiply its first and second equations by $\boldsymbol{\tau}\in\mathbb{X}$ and $\mathbf{v}\in\mathbf{Y}$, respectively, so that, using that $\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau} \,=\, \boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}$, we arrive at

$$\nu^{-1}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} + \nu^{-1}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}:\boldsymbol{\tau} - \int_\Omega \boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\tau} \,=\, 0 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{X}, \tag{2.11}$$

and

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \int_\Omega \theta\mathbf{g}\cdot\mathbf{v} \,=\, 0 \qquad \forall\,\mathbf{v}\in\mathbf{Y}, \tag{2.12}$$

Similarly, letting now $\mathbf{X}$ and Y be corresponding test spaces, and multiplying the first and second equations of (2.10) by $\boldsymbol{\eta}\in\mathbf{X}$ and $\varphi\in\mathrm{Y}$ respectively, we obtain

$$\int_\Omega \kappa^{-1}\boldsymbol{\rho}\cdot\boldsymbol{\eta} + \int_\Omega \kappa^{-1}\theta\mathbf{u}\cdot\boldsymbol{\eta} - \int_\Omega \nabla\theta\cdot\boldsymbol{\eta} \,=0 \qquad \forall\,\boldsymbol{\eta}\in\mathbf{X}, \tag{2.13}$$

and

$$\int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{\rho} \,=\, 0 \qquad \forall\,\varphi\in\mathrm{Y}\,. \tag{2.14}$$

Next, regarding the specific choice of the aforementioned spaces, we begin by observing that the first terms of (2.11) and (2.13) are well defined if $\boldsymbol{\sigma}$, $\boldsymbol{\tau}\in\mathbb{L}^2(\Omega)$ and $\boldsymbol{\rho}$, $\boldsymbol{\eta}\in\mathbf{L}^2(\Omega)$, respectively. On the other hand, assuming originally that $\mathbf{u}\in\mathbf{H}^1(\Omega)$ and $\theta\in\mathrm{H}^1(\Omega)$, and given $t, t'\in(1,\infty)$, conjugate to each other, we can integrate by parts the third terms in (2.11) and (2.13) with $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega)$ and $\boldsymbol{\eta}\in\mathbf{H}(\mathrm{div}_t;\Omega)$, respectively, so that using the Dirichlet boundary conditions provided in (2.2), we get

$$\int_\Omega \boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\tau} \,=\, -\int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \langle\boldsymbol{\tau}\,\mathbf{n},\mathbf{u}_D\rangle \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega)\,, \tag{2.15}$$

and

$$\int_\Omega \nabla\theta\cdot\boldsymbol{\eta} \;=\; -\int_\Omega \theta\,\mathrm{div}\,\boldsymbol{\eta} + \langle \boldsymbol{\eta}\cdot\mathbf{n},\theta_D\rangle_\Gamma \quad \forall\,\boldsymbol{\eta}\in\mathbf{H}(\mathrm{div}_t;\Omega)\,, \tag{2.16}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for both the duality pairings $\big(\mathbf{H}^{-1/2}(\Gamma),\mathbf{H}^{1/2}(\Gamma)\big)$ and $\big(\mathrm{H}^{-1/2}(\Gamma),\mathrm{H}^{1/2}(\Gamma)\big)$. Now, from the first terms on the right hand side of the foregoing equations, along with the Sobolev embeddings $\mathbf{H}^1(\Omega)\subset\mathbf{L}^{t'}(\Omega)$ and $\mathrm{H}^1(\Omega)\subset\mathrm{L}^{t'}(\Omega)$, we realize that it actually suffices to look for $\mathbf{u}\in\mathbf{L}^{t'}(\Omega)$ and $\theta\in\mathrm{L}^{t'}(\Omega)$. However, it is clear from (2.11) that its second term is well defined if $\mathbf{u}\in\mathbf{L}^4(\Omega)$, which yields $t'=4$ and thus $t=4/3$. In this way, θ is also sought in $\mathrm{L}^4(\Omega)$, and hence the second term of (2.13) makes sense as well.

According to the foregoing discussion, we now define the spaces

$$\mathbb{X} \;:=\; \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,\quad \mathbf{Y} \;:=\; \mathbf{L}^4(\Omega)\,,\quad \mathbf{X} \;:=\; \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,\quad\text{and}\quad \mathrm{Y} \;:=\; \mathrm{L}^4(\Omega)\,,$$

which are endowed with the norms

$$\|\cdot\|_{\mathbb{X}} \;:=\; \|\cdot\|_{\mathbf{div}_{4/3};\Omega}\,,\quad \|\cdot\|_{\mathbf{Y}} \;:=\; \|\cdot\|_{0,4;\Omega}\,,\quad \|\cdot\|_{\mathbf{X}} \;:=\; \|\cdot\|_{\mathrm{div}_{4/3};\Omega}\,,\quad\text{and}\quad \|\cdot\|_{\mathrm{Y}} \;:=\; \|\cdot\|_{0,4;\Omega}.$$

In addition, in order to deal with the null mean value of $\mathrm{tr}(\boldsymbol{\sigma})$ (cf. third row of (2.9)), we introduce the subspace of $\mathbb{X}$ given by

$$\mathbb{X}_0 \;=\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \;:=\; \Big\{\boldsymbol{\tau}\in\mathbb{X}:\quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;=\; 0\Big\}\,. \tag{2.17}$$

Consequently, replacing (2.15) and (2.16) back into (2.11) and (2.13), respectively, gathering the resulting equations with (2.12) and (2.14), and then realizing, thanks to (2.4), that testing the new (2.11) against $\boldsymbol{\tau}\in\mathbb{X}$ is equivalent to doing it against $\boldsymbol{\tau}\in\mathbb{X}_0$, we deduce that the variational formulation of (2.9) and (2.10) becomes: Find $\boldsymbol{\sigma}\in\mathbb{X}_0$, $\mathbf{u}\in\mathbf{Y}$, $\boldsymbol{\rho}\in\mathbf{X}$, and $\theta\in\mathrm{Y}$, such that

$$\begin{aligned}
a(\boldsymbol{\sigma},\boldsymbol{\tau}) + c(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\mathbf{u}) \;&=\; f(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{X}_0\,,\\
b(\boldsymbol{\sigma},\mathbf{v}) \;&=\; -\int_\Omega \theta\,\mathbf{g}\cdot\mathbf{v} && \forall\,\mathbf{v}\in\mathbf{Y}\,,\\
\widetilde{a}(\boldsymbol{\rho},\boldsymbol{\eta}) + \widetilde{c}(\mathbf{u};\theta,\boldsymbol{\eta}) + \widetilde{b}(\boldsymbol{\eta},\theta) \;&=\; \widetilde{f}(\boldsymbol{\eta}) && \forall\,\boldsymbol{\eta}\in\mathbf{X}\,,\\
\widetilde{b}(\boldsymbol{\rho},\varphi) \;&=\; 0 && \forall\,\varphi\in\mathrm{Y}\,,
\end{aligned} \tag{2.18}$$

where the bilinear forms $a:\mathbb{X}\times\mathbb{X}\to\mathrm{R}$, $b:\mathbb{X}\times\mathbf{Y}\to\mathrm{R}$, $\widetilde{a}:\mathbf{X}\times\mathbf{X}\to\mathrm{R}$, and $\widetilde{b}:\mathbf{X}\times\mathrm{Y}\to\mathrm{R}$ are defined as

$$\begin{aligned}
a(\boldsymbol{\zeta},\boldsymbol{\tau}) \;&:=\; \int_\Omega \nu^{-1}\boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} && \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{X}\,,\\
b(\boldsymbol{\tau},\mathbf{v}) \;&:=\; \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} && \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}\times\mathbf{Y}\,,\\
\widetilde{a}(\boldsymbol{\xi},\boldsymbol{\eta}) \;&:=\; \int_\Omega \kappa^{-1}\boldsymbol{\xi}\cdot\boldsymbol{\eta} && \forall\,\boldsymbol{\xi},\,\boldsymbol{\eta}\in\mathbf{X}\,,\quad\text{and}\\
\widetilde{b}(\boldsymbol{\eta},\psi) \;&:=\; \int_\Omega \psi\,\mathrm{div}\,\boldsymbol{\eta} && \forall\,(\boldsymbol{\eta},\psi)\in\mathbf{X}\times\mathrm{Y}\,,
\end{aligned} \tag{2.19}$$

whereas the linear functionals $f:\mathbb{X}_0\to\mathrm{R}$ and $\widetilde{f}:\mathbf{X}\to\mathrm{X}$ are given by

$$\begin{aligned}
f(\boldsymbol{\tau}) \;&:=\; \langle\boldsymbol{\tau}\,\mathbf{n},\mathbf{u}_D\rangle_\Gamma && \forall\,\boldsymbol{\tau}\in\mathbb{X}_0\,,\quad\text{and}\\
\widetilde{f}(\boldsymbol{\eta}) \;&:=\; \langle\boldsymbol{\eta}\cdot\mathbf{n},\theta_D\rangle_\Gamma && \forall\,\boldsymbol{\eta}\in\mathbf{X}\,.
\end{aligned} \tag{2.20}$$

In turn, for each $\mathbf{z}\in\mathbf{Y}$ we set the bilinear forms $c(\mathbf{z};\cdot,\cdot):\mathbf{Y}\times\mathbb{X}\to\mathrm{R}$ and $\widetilde{c}(\mathbf{z};\cdot,\cdot):\mathrm{Y}\times\mathbf{X}\to\mathrm{R}$ as

$$
\begin{aligned}
c(\mathbf{z};\mathbf{v},\boldsymbol{\tau}) &:= \int_\Omega \nu^{-1}(\mathbf{z}\otimes\mathbf{v})^{\mathtt{d}} : \boldsymbol{\tau} \qquad \forall\,(\mathbf{v},\boldsymbol{\tau}) \in \mathbf{Y}\times\mathbb{X}\,, \quad \text{and} \\
\widetilde{c}(\mathbf{z};\psi,\boldsymbol{\eta}) &:= \int_\Omega \kappa^{-1}\psi\mathbf{z}\cdot\boldsymbol{\eta} \qquad \forall\,(\psi,\boldsymbol{\eta}) \in \mathrm{Y}\times\mathbf{X}\,.
\end{aligned} \tag{2.21}
$$

Next, we consider the product spaces $\mathcal{V} := \mathbb{X}_0 \times \mathbf{Y}$ and $\mathcal{W} := \mathbf{X}\times \mathrm{Y}$ equipped with the norms

$$
\|(\boldsymbol{\tau},\mathbf{v})\|_{\mathcal{V}} := \|\boldsymbol{\tau}\|_{\mathbb{X}} + \|\mathbf{v}\|_{\mathbf{Y}} \qquad \text{and} \qquad \|(\boldsymbol{\eta},\varphi)\|_{\mathcal{W}} := \|\boldsymbol{\eta}\|_{\mathbf{X}} + \|\varphi\|_{\mathrm{Y}},
$$

for any $(\boldsymbol{\tau},\mathbf{v}) \in \mathcal{V}$, $(\boldsymbol{\eta},\varphi)\in\mathcal{W}$, and introduce the bilinear forms $\mathrm{B} : \mathcal{V}\times\mathcal{V}\to \mathrm{R}$ and $\widetilde{\mathrm{B}} : \mathcal{W}\times\mathcal{W}\to\mathrm{R}$ arising from (2.18) after adding the terms involving a and b, and $\widetilde{a}$ and $\widetilde{b}$, respectively, that is

$$
\begin{aligned}
\mathrm{B}\left((\boldsymbol{\zeta},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right) &:= a(\boldsymbol{\zeta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\mathbf{w}) + b(\boldsymbol{\zeta},\mathbf{v}) \qquad \forall\,(\boldsymbol{\zeta},\mathbf{w}),\,(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}\,, \quad \text{and} \\
\widetilde{\mathrm{B}}\left((\boldsymbol{\xi},\psi),(\boldsymbol{\eta},\varphi)\right) &:= \widetilde{a}(\boldsymbol{\xi},\boldsymbol{\eta}) + \widetilde{b}(\boldsymbol{\eta},\psi) + \widetilde{b}(\boldsymbol{\xi},\varphi) \qquad \forall\,(\boldsymbol{\xi},\psi),\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}\,.
\end{aligned} \tag{2.22}
$$

Then, adding separately the first two and the last two rows, respectively, of (2.18), we deduce that our fully mixed variational formulation can be re-stated as: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathcal{V}$ and $(\boldsymbol{\rho},\theta)\in\mathcal{W}$ such that

$$
\begin{aligned}
\mathrm{A}_{\mathbf{u}}\left((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})\right) &= \mathrm{F}_\theta(\boldsymbol{\tau},\mathbf{v}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}\,, \\
\widetilde{\mathrm{A}}_{\mathbf{u}}\left((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\varphi)\right) &= \widetilde{\mathrm{F}}(\boldsymbol{\eta},\varphi) \qquad \forall\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}\,,
\end{aligned} \tag{2.23}
$$

where, for each $\mathbf{z}\in\mathbf{Y}$, the bilinear forms $\mathrm{A}_{\mathbf{z}} : \mathcal{V}\times\mathcal{V}\to\mathrm{R}$ and $\widetilde{\mathrm{A}}_{\mathbf{z}} : \mathcal{W}\times\mathcal{W}\to\mathrm{R}$ are defined by

$$
\begin{aligned}
\mathrm{A}_{\mathbf{z}}\left((\boldsymbol{\zeta},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right) &:= \mathrm{B}\left((\boldsymbol{\zeta},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right) + c(\mathbf{z};\mathbf{w},\boldsymbol{\tau}) \quad \forall\,(\boldsymbol{\zeta},\,\mathbf{w}),\,(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}\,, \quad \text{and} \\
\widetilde{\mathrm{A}}_{\mathbf{z}}\left((\boldsymbol{\xi},\psi),(\boldsymbol{\eta},\varphi)\right) &:= \widetilde{\mathrm{B}}\left((\boldsymbol{\xi},\psi),(\boldsymbol{\eta},\varphi)\right) + \widetilde{c}(\mathbf{z};\psi,\boldsymbol{\eta}) \quad \forall\,(\boldsymbol{\xi},\psi),\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}\,,
\end{aligned} \tag{2.24}
$$

whereas, given $\psi\in\mathrm{Y}$, the linear functionals $\mathrm{F}_\psi\in\mathcal{V}'$ and $\widetilde{\mathrm{F}}\in\mathcal{W}'$ are given by

$$
\begin{aligned}
\mathrm{F}_\psi(\boldsymbol{\tau},\mathbf{v}) &:= f(\boldsymbol{\tau}) - \int_\Omega \psi\,\mathbf{g}\cdot\mathbf{v} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}\,, \quad \text{and} \\
\widetilde{\mathrm{F}}(\boldsymbol{\eta},\varphi) &:= \widetilde{f}(\boldsymbol{\eta}) \qquad \forall\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}\,.
\end{aligned} \tag{2.25}
$$

Here we notice from (2.19) and (2.21), thanks to the Cauchy-Schwarz and Hölder inequalities, that there exist constants $\|a\| = \nu^{-1}$, $\|b\| = 1$, $\|\widetilde{a}\| = \kappa^{-1}$, $\|\widetilde{b}\| = 1$, $\|c\| = \nu^{-1}$, and $\|\widetilde{c}\| = \kappa^{-1}$, such that

$$
\begin{aligned}
\left|a(\boldsymbol{\zeta},\boldsymbol{\tau}\right| &\le \|a\|\,\|\boldsymbol{\zeta}\|_{\mathbb{X}}\,\|\boldsymbol{\tau}\|_{\mathbb{X}} && \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{X}\,, \\
\left|b(\boldsymbol{\tau},\mathbf{v})\right| &\le \|b\|\,\|\boldsymbol{\tau}\|_{\mathbb{X}}\,\|\mathbf{v}\|_{\mathbf{Y}} && \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}\times\mathbf{Y}\,, \\
\left|\widetilde{a}(\boldsymbol{\xi},\boldsymbol{\eta}\right| &\le \|\widetilde{a}\|\,\|\boldsymbol{\xi}\|_{\mathbf{X}}\,\|\boldsymbol{\eta}\|_{\mathbf{X}} && \forall\,\boldsymbol{\xi},\,\boldsymbol{\eta}\in\mathbf{X}\,, \\
\left|\widetilde{b}(\boldsymbol{\eta},\psi)\right| &\le \|\widetilde{b}\|\,\|\boldsymbol{\eta}\|_{\mathbf{X}}\,\|\psi\|_{\mathrm{Y}} && \forall\,(\boldsymbol{\eta},\psi)\in\mathbf{X}\times\mathrm{Y}\,, \\
\left|c(\mathbf{z};\mathbf{w},\boldsymbol{\tau})\right| &\le \|c\|\,\|\mathbf{z}\|_{\mathbf{Y}}\,\|\mathbf{w}\|_{\mathbf{Y}}\,\|\boldsymbol{\tau}\|_{\mathbb{X}} && \forall\,\mathbf{z},\,\mathbf{w}\in\mathbf{Y}, \quad \forall\,\boldsymbol{\tau}\in\mathbb{X}\,, \\
\left|\widetilde{c}(\mathbf{z};\psi,\boldsymbol{\eta})\right| &\le \|\widetilde{c}\|\,\|\mathbf{z}\|_{\mathbf{Y}}\,\|\psi\|_{\mathrm{Y}}\,\|\boldsymbol{\eta}\|_{\mathbf{X}} && \forall\,(\mathbf{z},\psi,\boldsymbol{\eta})\in\mathbf{Y}\times\mathrm{Y}\times\mathbf{X}\,.
\end{aligned} \tag{2.26}
$$

In addition, letting $i_4 : \mathrm{H}^1(\Omega)\to\mathrm{L}^4(\Omega)$ and $\mathbf{i}_4 : \mathbf{H}^1(\Omega)\to\mathbf{L}^4(\Omega)$ be the respective continuous injection, there exist positive constants c_f and $c_{\widetilde{f}}$, depending on $\|\mathbf{i}_4\|$ and $\|i_4\|$, respectively, such that (cf. (2.20))

$$
\|f\| \le c_f\,\|\mathbf{u}_D\|_{1/2,\Gamma} \quad \text{and} \quad \|\widetilde{f}\| \le c_{\widetilde{f}}\,\|\theta_D\|_{1/2,\Gamma}\,. \tag{2.27}
$$

Now, regarding the well-posedness of (2.23) (equivalently, (2.18)), we remark that, up to minor changes caused by the non-homogeneous Dirichlet boundary condition for the velocity, its unique solvability was basically derived in [19]. In particular, it was proved in [19, Section 3.1] that there exist positive constants α_{B} and $\alpha_{\widetilde{\mathrm{B}}}$, depending on ν, κ, and the constants for the continuous inf-sup conditions of b and $\widetilde{b}$, such that

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}} \frac{\mathrm{B}\left((\boldsymbol{\zeta},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right)}{\|(\boldsymbol{\tau},\mathbf{v})\|_{\mathcal{V}}} \;\geq\; \alpha_{\mathrm{B}}\,\|(\boldsymbol{\zeta},\mathbf{w})\|_{\mathcal{V}} \quad \forall\,(\boldsymbol{\zeta},\mathbf{w})\in\mathcal{V},$$

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta},\varphi)\in\mathcal{W}} \frac{\widetilde{\mathrm{B}}\left((\boldsymbol{\xi},\psi),(\boldsymbol{\eta},\varphi)\right)}{\|(\boldsymbol{\eta},\varphi)\|_{\mathcal{W}}} \;\geq\; \alpha_{\widetilde{\mathrm{B}}}\,\|(\boldsymbol{\xi},\psi)\|_{\mathcal{W}} \quad \forall\,(\boldsymbol{\eta},\varphi)\in\mathcal{W},$$

which, along with the boundedness properties of c and $\widetilde{c}$ (cf. (2.26)), yield the inf-sup conditions for the bilinear forms $\mathrm{A}_{\mathbf{z}}$ and $\widetilde{\mathrm{A}}_{\mathbf{z}}$ for sufficiently small $\mathbf{z}$. More precisely, for each $\mathbf{z}\in\mathbf{Y}$ such that

$$\|\mathbf{z}\|_{\mathbf{Y}} \;\leq\; \delta \;:=\; \frac{1}{2}\,\min\left\{\nu\,\alpha_{\mathrm{B}},\kappa\,\alpha_{\widetilde{\mathrm{B}}}\right\}, \tag{2.28}$$

there holds (cf. [19, Lemmas 3.2 and 3.3])

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathcal{V}} \frac{\mathrm{A}_{\mathbf{z}}\left((\boldsymbol{\zeta},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right)}{\|(\boldsymbol{\tau},\mathbf{v})\|_{\mathcal{V}}} \;\geq\; \frac{\alpha_{\mathrm{B}}}{2}\,\|(\boldsymbol{\zeta},\mathbf{w})\|_{\mathcal{V}} \quad \forall\,(\boldsymbol{\zeta},\mathbf{w})\in\mathcal{V},$$

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta},\varphi)\in\mathcal{W}} \frac{\widetilde{\mathrm{A}}_{\mathbf{z}}\left((\boldsymbol{\xi},\psi),(\boldsymbol{\eta},\varphi)\right)}{\|(\boldsymbol{\eta},\varphi)\|_{\mathcal{W}}} \;\geq\; \frac{\alpha_{\widetilde{\mathrm{B}}}}{2}\,\|(\boldsymbol{\xi},\psi)\|_{\mathcal{W}} \quad \forall\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}.$$

Moreover, it is easily seen from (2.22), (2.24), and (2.26), that for each $\mathbf{z}\in\mathbf{Y}$ satisfying (2.28), the bilinear forms $\mathrm{A}_{\mathbf{z}}$ and $\widetilde{\mathrm{A}}_{\mathbf{z}}$ are bounded with corresponding constants $\|\mathrm{A}\|$ (depending only on ν^{-1} and δ) and $\|\widetilde{\mathrm{A}}\|$ (depending only on κ^{-1} and δ), respectively.

In this way, reformulating (2.23) as a fixed-point operator equation, and assuming that $\mathbf{u}_D$ and θ_D are sufficiently small, the well-posedness of (2.23) is obtained as a consequence of the generalized Lax-Milgram lemma (also known as the Banach-Nečas-Babuška theorem) and the Banach fixed-point theorem. More precisely, letting (cf. (2.28))

$$\mathbf{W} \;:=\; \Big\{\mathbf{z}\in\mathbf{Y}:\quad \|\mathbf{z}\|_{\mathbf{Y}} \;\leq\; \delta\Big\}, \tag{2.29}$$

one deduces the existence of positive constants C_0 (depending only on c_f, $c_{\widetilde{f}}$, α_{B}, and $\alpha_{\widetilde{\mathrm{B}}}$) and L_0 (depending only on c_f, $c_{\widetilde{f}}$, α_{B}, $\alpha_{\widetilde{\mathrm{B}}}$, ν, κ, and $\|\mathbf{g}\|_{0,\Omega}$), with which the following result is established.

Theorem 2.1. *Assume that the data satisfy*

$$C_0\left\{\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{g}\|_{0,\Omega}\,\|\theta_D\|_{1/2,\Gamma}\right\} \;\leq\; \delta\,, \quad \textit{and}$$
$$L_0\left\{\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\theta_D\|_{1/2,\Gamma}\right\} \;<\; 1\,.$$

Then, the coupled problem (2.23) *(equivalently,* (2.18)*) has a unique solution* $\big((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\rho},\theta)\big)\in\mathcal{V}\times\mathcal{W}$, *with* $\mathbf{u}\in\mathbf{W}$. *Moreover, there exist positive constants* $\mathcal{C}_0$ *and* $\widetilde{\mathcal{C}}_0$, *depending both only on* c_f, $c_{\widetilde{f}}$, α_{B}, *and* $\alpha_{\widetilde{\mathrm{B}}}$, *such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u})\|_{\mathcal{V}} \;&\leq\; \mathcal{C}_0\left\{\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{g}\|_{0,\Omega}\,\|\theta_D\|_{1/2,\Gamma}\right\}, \quad \textit{and}\\ \|(\boldsymbol{\rho},\theta)\|_{\mathcal{W}} \;&\leq\; \widetilde{\mathcal{C}}_0\,\|\theta_D\|_{1/2,\Gamma}\,.\end{aligned} \tag{2.30}$$

Proof. It reduces to a minor variation of the proof of [19, Theorem 3.2]. □

3 Mixed virtual element approximation

In this section, we describe the virtual element discretization for the 2D version of (2.23) (equivalently, (2.18)) on general polygonal meshes.

3.1 Preliminaries

The aim of this part is to present some preliminary concepts and results to be employed in what follows. For more details, we refer to [11, 5, 14]. We begin by letting $\{\mathcal{K}_h\}_{h>0}$ be a sequence of partitions of Ω into general polygons K with diameter and number of edges denoted by h_K and d_K, respectively, and set, as usual, $h := \max\big\{h_K : \quad K \in \mathcal{K}_h\big\}$. Then, we let $\mathcal{E}_h$ be the set of edges e of $\{\mathcal{K}_h\}_{h>0}$, and let $\mathcal{E}_h^I = \mathcal{E}_h \backslash \partial\Omega$ (resp. $\mathcal{E}_h^B = \mathcal{E}_h \cap \Gamma$) be the set of all interior (resp. boundary) edges. Also, $\mathbf{n}_e^K$ stands for the unit outward normal on any edge $e \in \mathcal{E}_h$ such that $e \subset \partial K$. In addition, following, for example [5], we assume that there exists a constant $C_{\mathcal{K}} > 0$, such that:

i) every $K \in \mathcal{K}_h$ is star-shaped with respect to a ball with radius $\geq C_{\mathcal{K}}\, h_K$, and

ii) for each $K \in \mathcal{K}_h$, the distance between every two vertices of it is $\geq C_{\mathcal{K}}\, h_K$.

It is not difficult to see that the above hypotheses guarantee that each $K \in \mathcal{K}_h$ is simply connected, and that there exists an integer $N_{\mathcal{K}}$ (depending only on $C_{\mathcal{K}}$), such that $d_K \leq N_{\mathcal{K}}$ for all $K \in \mathcal{K}$.

Next, we refer to the L^1-orthogonal projections and its approximation properties. To this end, for any mesh object $\Delta \in \mathcal{K}_h \cup \mathcal{E}_h$, and for any integer $\ell \geq 0$, we let $\mathrm{P}_\ell(\Delta)$ be the space of polynomials defined on Δ of degree $\leq \ell$, with the extended notation $\mathrm{P}_{-1}(\Delta) = \{0\}$. In addition, for each $K \in \mathcal{K}_h$ we let $\mathcal{P}_\ell^K : \mathrm{L}^1(K) \to \mathrm{P}_\ell(K)$ be the L^1-projection operator, which is characterized by the identity

$$\int_K \mathcal{P}_\ell^K(v)\; q \;=\; \int_K v\; q \qquad \forall\, v \in \mathrm{L}^1(K),\ \forall\, q \in \mathrm{P}_\ell(K). \tag{3.1}$$

Similarly, we let $\boldsymbol{\mathcal{P}}_\ell^K : \mathbf{L}^1(K) \to \mathbf{P}_\ell(K)$ and $\mathbb{P}_\ell^K : \mathbb{L}^1(K) \to \mathbb{P}_\ell(K)$ be the vectorial and tensorial versions of $\mathcal{P}_\ell^K$, which are characterized analogously to (3.1).

The approximation properties of $\mathcal{P}_\ell^K$, $\boldsymbol{\mathcal{P}}_\ell^K$, and $\mathbb{P}_\ell^K$ are stated as follows (cf. [32, Lemma 3.1]).

Lemma 3.1. *Let $K \in \mathcal{K}_h$, $p > 1$, and ℓ, s, m be integers such that $\ell \geq 0$ and $0 \leq m \leq s \leq \ell + 1$. Then, there exists a constant C_ℓ, depending only on ℓ and $C_{\mathcal{K}}$, and hence independent of K, such that*

$$\begin{aligned} |v - \mathcal{P}_\ell^K(v)|_{m,p;K} &\leq C_\ell\, h_K^{s-m}\, |v|_{s,p;K} && \forall\, v \in \mathrm{W}^{s,p}(K)\,, \\ |\mathbf{v} - \boldsymbol{\mathcal{P}}_\ell^K(\mathbf{v})|_{m,p;K} &\leq C_\ell\, h_K^{s-m}\, |\mathbf{v}|_{s,p;K} && \forall\, \mathbf{v} \in \mathbf{W}^{s,p}(K)\,, \\ |\boldsymbol{\tau} - \mathbb{P}_\ell^K(\boldsymbol{\tau})|_{m,p;K} &\leq C_\ell\, h_K^{s-m}\, |\boldsymbol{\tau}|_{s,p:K} && \forall\, \boldsymbol{\tau} \in \mathbb{W}^{s,p}(K)\,. \end{aligned} \tag{3.2}$$

We remark now that Lemma 3.1 implies boundedness properties of the projectors $\mathcal{P}_\ell^K$, $\boldsymbol{\mathcal{P}}_\ell^K$, and $\mathbb{P}_\ell^K$. More precisely, taking in particular $m = s$ in (3.2), we deduce the existence of constants M_ℓ, depending only on ℓ and $C_{\mathcal{K}}$ as well, such that for each $K \in \mathcal{K}_h$ there holds

$$\begin{aligned} |\mathcal{P}_\ell^K(v)|_{s,p;K} &\leq M_\ell\, |v|_{s,p;K} && \forall\, v \in \mathrm{W}^{s,p}(K)\,, \\ |\boldsymbol{\mathcal{P}}_\ell^K(\mathbf{v})|_{s,p;K} &\leq M_\ell\, |\mathbf{v}|_{s,p;K} && \forall\, \mathbf{v} \in \mathbf{W}^{s,p}(K)\,, \\ |\mathbb{P}_\ell^K(\boldsymbol{\tau})|_{s,p;K} &\leq M_\ell\, |\boldsymbol{\tau}|_{s,p:K} && \forall\, \boldsymbol{\tau} \in \mathbb{W}^{s,p}(K)\,. \end{aligned} \tag{3.3}$$

Finally, introducing the piecewise polynomial space

$$\mathrm{P}_\ell(\mathcal{K}_h) \;:=\; \Big\{ v \in L^2(\Omega) : \quad v|_K \in \mathrm{P}_\ell(K), \quad \forall\, K \in \mathcal{K}_h \Big\}\,, \tag{3.4}$$

we realize that the properties of $\mathcal{P}_\ell^K$, $\boldsymbol{\mathcal{P}}_\ell^K$, and $\mathbb{P}_\ell^K$ given by Lemma 3.1 (with $m = 0$) and (3.3) (with $s = 0$) easily extend to their global counterparts denoted

$$\mathcal{P}_\ell^h : \mathrm{L}^1(\Omega) \to \mathrm{P}_\ell(\mathcal{K}_h)\,, \quad \boldsymbol{\mathcal{P}}_\ell^h : \mathbf{L}^1(\Omega) \to \mathbf{P}_\ell(\mathcal{K}_h)\,, \quad \text{and} \quad \mathbb{P}_\ell^h : \mathbb{L}^1(\Omega) \to \mathbb{P}_\ell(\mathcal{K}_h)\,,$$

where $\mathbf{P}_\ell(\mathcal{K}_h)$ and $\mathbb{P}_\ell(\mathcal{K}_h)$ are the vector and tensorial versions of (3.4), respectively. Indeed, it is readily seen that for each $K \in \mathcal{K}_h$ there hold

$$\begin{gathered} \mathcal{P}_\ell^h(v)|_K \;=\; \mathcal{P}_\ell^K(v|_K) \quad \forall\, v \in \mathrm{L}^1(\Omega)\,, \quad \boldsymbol{\mathcal{P}}_\ell^h(\mathbf{v})|_K \;=\; \boldsymbol{\mathcal{P}}_\ell^K(\mathbf{v}|_K) \quad \forall\, \mathbf{v} \in \mathbf{L}^1(\Omega)\,, \\ \text{and} \quad \mathbb{P}_\ell^h(\boldsymbol{\tau})|_K \;=\; \mathbb{P}_\ell^K(\boldsymbol{\tau}|_K) \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^1(\Omega)\,. \end{gathered} \tag{3.5}$$

3.2 Discrete spaces

In this section we describe appropriate choices for virtual (both, vector and tensor versions) and non-virtual (both, scalar and vector versions) approximation spaces of the pairs $\big(\mathbb{X}, \mathbf{X}\big)$, and $\big(\mathbf{Y}, \mathrm{Y}\big)$, respectively, for which we first let $\mathrm{rot}(\boldsymbol{\eta}) \;:=\; \partial_x \eta_2 - \partial_y \eta_1$ and $\mathbf{rot}(\boldsymbol{\tau}) \;:=\; (\partial_x \tau_{12} - \partial_y \tau_{11}, \partial_x \tau_{22} - \partial_y \tau_{21})^{\mathtt{t}}$ for all sufficiently smooth vector $\boldsymbol{\eta}$ and tensor $\boldsymbol{\tau}$. Then, given $r \in \mathrm{N}$, we follow [14] and introduce for each $K \in \mathcal{K}_h$ the local virtual spaces

$$\begin{aligned} \mathbf{X}_r(K) \;:=\; \Big\{ \boldsymbol{\eta} \in \mathbf{H}(\mathrm{div}_{4/3}; K) \cap \mathbf{H}(\mathrm{rot}; K) : \quad (\boldsymbol{\eta} \cdot \mathbf{n}_e^K)|_e \in \mathrm{P}_r(e), \quad \forall\, e \subset \partial K\,, \\ \mathrm{div}(\boldsymbol{\eta}) \in \mathrm{P}_r(K)\,, \quad \text{and} \quad \mathrm{rot}(\boldsymbol{\eta}) \in \mathrm{P}_{r-1}(K) \Big\}\,, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \mathbb{X}_r(K) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; K) \cap \mathbb{H}(\mathbf{rot}; K) : \quad (\boldsymbol{\tau} \mathbf{n}_e^K)|_e \in \mathbf{P}_r(e), \quad \forall\, e \subset \partial K\,, \\ \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{P}_r(K)\,, \quad \text{and} \quad \mathbf{rot}(\boldsymbol{\tau}) \in \mathbf{P}_{r-1}(K) \Big\}\,. \end{aligned} \tag{3.7}$$

It is well-known that the vectors $\boldsymbol{\eta} \in \mathbf{X}_r(K)$ and tensors $\boldsymbol{\tau} \in \mathbb{X}_r(K)$ are uniquely determined by the local degrees of freedom given by (cf. [14, 16, 32])

$$\begin{aligned} \widetilde{\mathbf{m}}_{\mathbf{n}_e}(\boldsymbol{\eta}) \;&:=\; \int_e (\boldsymbol{\eta} \cdot \mathbf{n}_e^K)\, q \qquad \forall\, q \in \mathrm{P}_r(e),\ \forall\, e \subset \partial K\,, \\ \widetilde{\mathbf{m}}_{\mathrm{div}}(\boldsymbol{\eta}) \;&:=\; \int_K \boldsymbol{\eta} \cdot \nabla q \qquad \forall\, q \in \mathrm{P}_r(K) \backslash \{1\}\,, \\ \widetilde{\mathbf{m}}_{\mathrm{rot}}(\boldsymbol{\eta}) \;&:=\; \int_K \boldsymbol{\eta} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_{r,\nabla}(K)\,, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \mathbf{m}_{\mathbf{n}_e}(\boldsymbol{\tau}) \;&:=\; \int_e \boldsymbol{\tau} \mathbf{n}_e^K \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_r(e),\ \forall\, e \subset \partial K\,, \\ \mathbf{m}_{\mathbf{div}}(\boldsymbol{\tau}) \;&:=\; \int_K \boldsymbol{\tau} : \boldsymbol{\nabla} \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_r(K) \backslash \{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\}\,, \\ \mathbf{m}_{\mathbf{rot}}(\boldsymbol{\tau}) \;&:=\; \int_K \boldsymbol{\tau} : \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbb{P}_{r,\nabla}(K)\,, \end{aligned} \tag{3.9}$$

respectively, where $\mathbf{P}_{r,\nabla}(K)$ and $\mathbb{P}_{r,\nabla}(K)$ are subspaces of $\mathbf{P}_r(K)$ and $\mathbb{P}_r(K)$, respectively, such that $\mathbf{P}_r(K) = \nabla \mathrm{P}_{r+1}(K) \oplus \mathbf{P}_{r,\nabla}(K)$ and $\mathbb{P}_r(K) = \boldsymbol{\nabla} \mathbf{P}_{r+1}(K) \oplus \mathbb{P}_{r,\nabla}(K)$.

Next, we observe (cf. [11, 5, 14, 32]) that for each $\boldsymbol{\eta} \in \mathbf{X}_r(K)$, and for each $\boldsymbol{\tau} \in \mathbb{X}_r(K)$, the projections $\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})$ and $\mathbb{P}_r^K(\boldsymbol{\tau})$, as well as $\mathrm{div}(\boldsymbol{\eta})$ and $\mathbf{div}(\boldsymbol{\tau})$, are computable using the degrees of freedom defined by (3.8) and (3.9). Moreover, the latter are actually linear functionals that can be evaluated at any $\boldsymbol{\eta} \in \mathbf{W}^{1,1}(K)$ and any $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(K)$, respectively. Then, denoting by $\widetilde{n}_r^K$ and n_r^K the number of degrees of freedom of (3.8) and (3.9), respectively, and gathering them in the sets $\{\widetilde{m}_j^K\}_{j=1}^{\widetilde{n}_r^K}$ and $\{m_j^K\}_{j=1}^{n_r^K}$, we can introduce the interpolation operators $\boldsymbol{\Pi}_r^K : \mathbf{W}^{1,1}(K) \to \mathbf{X}_r(K)$ and $\boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K : \mathbb{W}^{1,1}(K) \to \mathbb{X}_r(K)$, which are defined for each $\boldsymbol{\eta} \in \mathbf{W}^{1,1}(K)$ and for each $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(K)$, as the unique $\boldsymbol{\Pi}_r^K(\boldsymbol{\eta}) \in \mathbf{X}_r(K)$ and $\boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K(\boldsymbol{\tau}) \in \mathbb{X}_r(K)$, such that

$$\begin{aligned} \widetilde{m}_j^K\big(\boldsymbol{\Pi}_r^K(\boldsymbol{\eta})\big) \,&=\, \widetilde{m}_j^K(\boldsymbol{\eta}) \quad \forall\, j \in \big\{1,...,\widetilde{n}_r^K\big\}\,, \quad \text{and} \\ m_j^K\big(\boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K(\boldsymbol{\tau})\big) \,&=\, m_j^K(\boldsymbol{\tau}) \quad \forall\, j \in \big\{1,...,n_r^K\big\}\,, \end{aligned} \tag{3.10}$$

respectively. Regarding the approximation properties of $\boldsymbol{\Pi}_r^K$ and $\boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K$, for each integer $s \in [1, r+1]$ there exist positive constants C and $\widetilde{C}$, independent of $K \in \mathcal{K}_h$, such that (see, e.g., [32, eq. (3.12)])

$$\|\boldsymbol{\eta} - \boldsymbol{\Pi}_r^K(\boldsymbol{\eta})\|_{0,K} \,\le\, C\, h_K^s\, |\boldsymbol{\eta}|_{s,K} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{H}^s(K)\,, \tag{3.11}$$

and

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K(\boldsymbol{\tau})\|_{0,K} \,\le\, \widetilde{C}\, h_K^s\, |\boldsymbol{\tau}|_{s,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K)\,, \tag{3.12}$$

whereas for each integer $s \in [0, r+1]$ there exist positive constants $C_{\mathtt{d}}$ and $\widetilde{C}_{\mathtt{d}}$, independent of K, such that (see, e.g., [32, eq. (3.14)])

$$\|\mathrm{div}\big(\boldsymbol{\eta} - \boldsymbol{\Pi}_r^K(\boldsymbol{\eta})\big)\|_{0,4/3;K} \,\le\, C_{\mathtt{d}}\, h_K^s\, |\mathrm{div}(\boldsymbol{\eta})|_{s,4/3;K} \quad \forall\, \big(\boldsymbol{\eta}, \mathrm{div}(\boldsymbol{\eta})\big) \in \mathbf{W}^{1,1}(K) \times \mathbf{W}^{s,4/3}(K)\,, \tag{3.13}$$

and

$$\|\mathbf{div}\big(\boldsymbol{\tau} - \boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K(\boldsymbol{\tau})\big)\|_{0,4/3;K} \,\le\, \widetilde{C}_{\mathtt{d}}\, h_K^s\, |\mathbf{div}(\boldsymbol{\tau})|_{s,4/3;K} \quad \forall\, \big(\boldsymbol{\tau}, \mathbf{div}(\boldsymbol{\tau})\big) \in \mathbb{W}^{1,1}(K) \times \mathbb{W}^{s,4/3}(K)\,. \tag{3.14}$$

We now introduce the virtual element subspaces of $\mathbf{X}$, $\mathbb{X}$, and $\mathbb{X}_0$ with respect to $\mathcal{K}_h$, namely

$$\mathbf{X}_h \,:=\, \Big\{ \boldsymbol{\eta} \in \mathbf{X} : \quad \boldsymbol{\eta}|_K \,\in\, \mathbf{X}_r(K) \quad \forall\, K \in \mathcal{K}_h \Big\}\,, \tag{3.15}$$

$$\mathbb{X}_h \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{X} : \quad \boldsymbol{\tau}|_K \,\in\, \mathbb{X}_r(K) \quad \forall\, K \in \mathcal{K}_h \Big\}\,, \tag{3.16}$$

and

$$\mathbb{X}_{0,h} \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, \int_\Omega \boldsymbol{\tau} : \mathbb{I} \,=\, 0 \Big\}\,. \tag{3.17}$$

Then, letting $\boldsymbol{\Pi}_r^h : \mathbf{W}^{1,1}(\Omega) \to \mathbf{X}_h$ and $\boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^h : \mathbb{W}^{1,1}(\Omega) \to \mathbb{X}_h$ be the associated global interpolation operators, it is easily seen that for each $K \in \mathcal{K}_h$ there hold

$$\boldsymbol{\Pi}_r^h(\boldsymbol{\eta})|_K \,=\, \boldsymbol{\Pi}_r^K(\boldsymbol{\eta}|_K) \quad \forall\, \boldsymbol{\eta} \in \mathbf{W}^{1,1}(\Omega) \quad \text{and} \quad \boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^h(\boldsymbol{\tau})|_K \,=\, \boldsymbol{\Pi}\!\!\boldsymbol{\Pi}_r^K(\boldsymbol{\tau}|_K) \quad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,1}(\Omega)\,. \tag{3.18}$$

On the other hand, for approximating the temperature and velocity unknowns we simply consider the piecewise polynomial spaces $\mathrm{P}_r(\mathcal{K}_h)$ and $\mathbf{P}_r(\mathcal{K}_h)$ (cf. (3.4)), that is

$$\mathrm{Y}_h \,=\, \mathrm{P}_r(\mathcal{K}_h) \,:=\, \Big\{ \varphi \in \mathrm{Y} : \quad \varphi|_K \in \mathrm{P}_r(K) \quad \forall\, K \in \mathcal{K}_h \Big\}\,, \tag{3.19}$$

and

$$\mathbf{Y}_h \,=\, \mathbf{P}_r(\mathcal{K}_h) \,:=\, \Big\{ \mathbf{v} \in \mathbf{Y} : \quad \mathbf{v}|_K \in \mathbf{P}_r(K) \quad \forall\, K \in \mathcal{K}_h \Big\}\,, \tag{3.20}$$

respectively. We remark from the foregoing definitions that there hold (cf. [15])

$$\mathrm{div}(\mathbf{X}_h) \subseteq \mathrm{Y}_h \quad \text{and} \quad \mathbf{div}(\mathbb{X}_h) \subseteq \mathbf{Y}_h .$$

Furthermore, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set from now on

$$\mathrm{dist}(z, Z_h) := \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z .$$

Hence, according to the local approximation properties provided by the estimates (3.11) up to (3.14), and Lemma 3.1, along with the identities (3.5) and (3.18), we easily derive the following global ones:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ for each integer $s \in [1, r+1]$ there exists a positive constant C, independent of h, such that

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbb{X}_h) \le \|\boldsymbol{\tau} - \boldsymbol{\Pi}_r^h(\boldsymbol{\tau})\|_{\mathbf{div}_{4/3};\Omega} \le C\, h^s \Big\{ |\boldsymbol{\tau}|_{s,\Omega} + |\mathbf{div}(\boldsymbol{\tau})|_{s,4/3;\Omega} \Big\} ,$$

for all $\boldsymbol{\tau} \in \mathbb{X}$ such that $\boldsymbol{\tau} \in \mathbb{H}^s(\Omega)$ and $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(\Omega)$,

$(\mathbf{AP}_h^{\mathbf{u}})$ for each integer $s \in [0, r+1]$ there exists a positive constant C, independent of h, such that

$$\mathrm{dist}(\mathbf{v}, \mathbf{Y}_h) \le \|\mathbf{v} - \boldsymbol{\mathcal{P}}_r^h(\mathbf{v})\|_{0,4;\Omega} \le C\, h^s\, |\mathbf{v}|_{s,4;\Omega} ,$$

for all $\mathbf{v} \in \mathbf{Y}$ such that $\mathbf{v} \in \mathbf{W}^{s,4}(\Omega)$,

$(\mathbf{AP}_h^{\boldsymbol{\rho}})$ for each integer $s \in [1, r+1]$ there exists a positive constant C, independent of h, such that

$$\mathrm{dist}(\boldsymbol{\eta}, \mathbf{X}_h) \le \|\boldsymbol{\eta} - \mathbf{\Pi}_r^h(\boldsymbol{\eta})\|_{\mathrm{div}_{4/3};\Omega} \le C\, h^s \Big\{ |\boldsymbol{\eta}|_{s,\Omega} + |\mathrm{div}(\boldsymbol{\eta})|_{s,4/3;\Omega} \Big\} ,$$

for all $\boldsymbol{\eta} \in \mathbf{X}$ such that $\boldsymbol{\eta} \in \mathbf{H}^s(\Omega)$ and $\mathrm{div}(\boldsymbol{\eta}) \in \mathrm{W}^{s,4/3}(\Omega)$, and

$(\mathbf{AP}_h^{\theta})$ for each integer $s \in [0, r+1]$ there exists a positive constant C, independent of h, such that

$$\mathrm{dist}(\boldsymbol{\varphi}, \mathrm{Y}_h) \le \|\varphi - \mathcal{P}_r^h(\varphi)\|_{0,4;\Omega} \le C\, h^s\, |\varphi|_{s,4;\Omega} ,$$

for all $\varphi \in \mathrm{Y}$ such that $\varphi \in \mathrm{W}^{s,4}(\Omega)$.

3.3 Virtual element scheme

In order to define our virtual element scheme for (2.23), we now introduce, when necessary, computable discrete versions of the bilinear forms and functionals involving the virtual spaces. Following the usual procedure in the virtual element setting, the construction of them is based on the explicit knowledge of the degrees of freedom given by (3.8) and (3.9). In this regard, we first notice from the definitions of the discrete spaces (cf. (3.15) - (3.17), (3.19) - (3.20)), that $b|_{\mathbb{X}_h \times \mathbf{Y}_h}$, $\widetilde{b}|_{\mathbf{X}_h \times \mathrm{Y}_h}$, $f|_{\mathbb{X}_h}$, and $\widetilde{f}|_{\mathbf{X}_h}$ are fully computable as such, without further modifications. The same is valid for the $\mathbf{Y}_h$ evaluation of the ψ-dependent term in the definition of F_ψ (cf. (2.25)) since, being $\mathbf{Y}_h$ non-virtual, each $\mathbf{v} \in \mathbf{Y}_h$ is explicitly known. On the contrary, $a|_{\mathbb{X}_h \times \mathbb{X}_h}$ and $\widetilde{a}|_{\mathbf{X}_h \times \mathbf{X}_h}$, and for each $\mathbf{z} \in \mathbf{Y}_h$, $c(\mathbf{z};\cdot,\cdot)|_{\mathbf{Y}_h \times \mathbf{X}_h}$ and $\widetilde{c}(\mathbf{z};\cdot,\cdot)|_{\mathrm{Y}_h \times \mathbf{X}_h}$ as well, are not computable since the corresponding virtual functions are not known in closed form. In order to overcome this difficulty, and thinking first of a and $\widetilde{a}$, we now resort to the set of degrees of freedom $\big\{m_j^K\big\}_{j=1}^{n_r^K}$ and $\big\{\widetilde{m}_j^K\big\}_{j=1}^{\widetilde{n}_r^K}$, and let $s^K : \mathbb{X}_r(K) \times \mathbb{X}_r(K) \to \mathrm{R}$ and $\widetilde{s}^K : \mathbf{X}_r(K) \times \mathbf{X}_r(K) \to \mathrm{R}$ be the bilinear forms defined by

$$s^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{j=1}^{n_r^K} m_j^K(\boldsymbol{\zeta})\, m_j^K(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{X}_r(K) ,$$

$$\widetilde{s}^K(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{j=1}^{\widetilde{n}_r^K} \widetilde{m}_j^K(\boldsymbol{\xi})\, \widetilde{m}_j^K(\boldsymbol{\eta}) \qquad \forall\, \boldsymbol{\xi},\, \boldsymbol{\eta} \in \mathbf{X}_r(K) .$$

Then, letting a^K, $\widetilde{a}^K$, c^K, and $\widetilde{c}^K$, respectively, be the local versions of a, $\widetilde{a}$, c, and $\widetilde{c}$ (cf. (2.19), (2.21)) for each $K \in \mathcal{K}_h$, we define their computable versions as

$$a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) := a^K\left(\mathcal{P}_r^K(\boldsymbol{\zeta}),\mathcal{P}_r^K(\boldsymbol{\tau})\right) + \nu^{-1}\, s^K\left(\boldsymbol{\zeta}-\mathcal{P}_r^K(\boldsymbol{\zeta}),\boldsymbol{\tau}-\mathcal{P}_r^K(\boldsymbol{\tau})\right) \quad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{X}_r(K)\,, \tag{3.21}$$

$$\widetilde{a}_h^K(\boldsymbol{\xi},\boldsymbol{\eta}) := \widetilde{a}^K\left(\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\xi}),\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\right) + \kappa^{-1}\, \widetilde{s}^K\left(\boldsymbol{\xi}-\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\xi}),\boldsymbol{\eta}-\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\right) \quad \forall\,\boldsymbol{\xi},\,\boldsymbol{\eta}\in\mathbf{X}_r(K)\,, \tag{3.22}$$

$$c_h^K(\mathbf{z};\mathbf{w},\boldsymbol{\tau}) := c^K\left(\mathbf{z};\mathbf{w},\mathcal{P}_r^K(\boldsymbol{\tau})\right) \quad \forall\,(\mathbf{z},\mathbf{w},\boldsymbol{\tau})\in\mathbf{P}_r(K)\times\mathbf{P}_r(K)\times\mathbb{X}_r(K)\,, \tag{3.23}$$

and

$$\widetilde{c}_h^K(\mathbf{z};\psi,\boldsymbol{\eta}) := \widetilde{c}^K\left(\mathbf{z};\psi,\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\right) \quad \forall\,(\mathbf{z},\psi,\boldsymbol{\eta})\in\mathbf{P}_r(K)\times\mathrm{P}_r(K)\times\mathbf{X}_r(K)\,. \tag{3.24}$$

In turn, the associated global forms $a_h : \mathbb{X}_h \times \mathbb{X}_h \to \mathrm{R}$, $\widetilde{a}_h : \mathbf{X}_h \times \mathbf{X}_h \to \mathrm{R}$, $c_h : \mathbf{Y}_h \times \mathbf{Y}_h \times \mathbb{X}_h \to \mathrm{R}$, and $\widetilde{c}_h : \mathbf{Y}_h \times \mathrm{Y}_h \times \mathbf{X}_h \to \mathrm{R}$, are defined in the usual way by summing over all $K \in \mathcal{K}_h$, that is

$$a_h(\boldsymbol{\zeta},\boldsymbol{\tau}) := \sum_{K\in\mathcal{K}_h} a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{X}_h\times\mathbb{X}_h\,, \tag{3.25}$$

$$\widetilde{a}_h(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{K\in\mathcal{K}_h} \widetilde{a}_h^K(\boldsymbol{\xi},\boldsymbol{\eta}) \qquad \forall\,\boldsymbol{\xi},\,\boldsymbol{\eta}\in\mathbf{X}_h\times\mathbf{X}_h\,, \tag{3.26}$$

$$c_h(\mathbf{z};\mathbf{w},\boldsymbol{\tau}) := \sum_{K\in\mathcal{K}_h} c_h^K(\mathbf{z};\mathbf{w},\boldsymbol{\tau}) \qquad \forall\,(\mathbf{z},\mathbf{w},\boldsymbol{\tau})\in\mathbf{Y}_h\times\mathbf{Y}_h\times\mathbb{X}_h\,, \tag{3.27}$$

and

$$\widetilde{c}_h(\mathbf{z};\psi,\boldsymbol{\eta}) := \sum_{K\in\mathcal{K}_h} \widetilde{c}_h^K(\mathbf{z};\psi,\boldsymbol{\eta}) \qquad \forall\,(\mathbf{z},\psi,\boldsymbol{\eta})\in\mathbf{Y}_h\times\mathrm{Y}_h\times\mathbf{X}_h\,. \tag{3.28}$$

According to the above definitions, our virtual element scheme for (2.18) (equivalently, (2.23)) reads as follows: Find $\boldsymbol{\sigma}_h \in \mathbb{X}_{0,h}$, $\mathbf{u}_h \in \mathbf{Y}_h$, $\boldsymbol{\rho}_h \in \mathbf{X}_h$, and $\theta_h \in \mathrm{Y}_h$, such that

$$\begin{aligned}
a_h(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + c_h(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h,\mathbf{u}_h) &= f(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}\,,\\
b(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= -\int_\Omega \theta_h\,\mathbf{g}\cdot\mathbf{v}_h && \forall\,\mathbf{v}_h\in\mathbf{Y}_h\,,\\
\widetilde{a}_h(\boldsymbol{\rho}_h,\boldsymbol{\eta}_h) + \widetilde{c}_h(\mathbf{u}_h;\theta_h,\boldsymbol{\eta}_h) + \widetilde{b}(\boldsymbol{\eta}_h,\theta_h) &= \widetilde{f}(\boldsymbol{\eta}_h) && \forall\,\boldsymbol{\eta}_h\in\mathbf{X}_h\,,\\
\widetilde{b}(\boldsymbol{\rho}_h,\varphi_h) &= 0 && \forall\,\varphi_h\in\mathrm{Y}_h\,.
\end{aligned} \tag{3.29}$$

Next, we set $\mathcal{V}_h := \mathbb{X}_{0,h} \times \mathbf{Y}_h$ and $\mathcal{W}_h := \mathbf{X}_h \times \mathrm{Y}_h$, let B_h and $\widetilde{\mathrm{B}}_h$ be the discrete versions of B and $\widetilde{\mathrm{B}}$ that arise from (2.22) after replacing a and $\widetilde{a}$ by a_h and $\widetilde{a}_h$, respectively, and introduce for each $\mathbf{z}_h \in \mathbf{Y}_h$ the discrete forms

$$\mathrm{A}_{h,\mathbf{z}_h}\left((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) := \mathrm{B}_h\left((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) + c_h(\mathbf{z}_h;\mathbf{w}_h,\boldsymbol{\tau}_h) \tag{3.30}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)$, $(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathcal{V}_h$, and

$$\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}\left((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h)\right) := \widetilde{\mathrm{B}}_h\left((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h)\right) + \widetilde{c}_h(\mathbf{z}_h;\psi_h,\boldsymbol{\eta}_h) \tag{3.31}$$

for all $(\boldsymbol{\xi}_h,\psi_h)$, $(\boldsymbol{\eta}_h,\varphi_h) \in \mathcal{W}_h$. Then, we readily see that (3.29) can be rewritten, equivalently, as: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathcal{V}_h$ and $(\boldsymbol{\rho}_h,\theta_h) \in \mathcal{W}_h$ such that

$$\begin{aligned}
\mathrm{A}_{h,\mathbf{u}_h}\left((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\right) &= \mathrm{F}_{\theta_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) && \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h\,,\\
\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\left((\boldsymbol{\rho}_h,\theta_h),(\boldsymbol{\eta}_h,\varphi_h)\right) &= \widetilde{\mathrm{F}}(\boldsymbol{\eta}_h,\varphi_h) && \forall\,(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h\,,
\end{aligned} \tag{3.32}$$

where F_{θ_h} and $\widetilde{\mathrm{F}}$ are defined as in (2.25) with θ_h instead of ψ there.

4 Solvability analysis

In this section we proceed similarly as in [32, Section 4] to address the well-posedness of (3.29) (equivalently, (3.32)). We begin the analysis with the stability properties of the forms a_h, $\widetilde{a}_h$, b, $\widetilde{b}$, c_h, and $\widetilde{c}_h$. Then, we introduce resolvent operators associated with each one of the decoupled problems forming (3.29), and rewrite the latter as an equivalent fixed-point equation. Finally, we show that the aforementioned operators are well-defined, and apply the classical Banach theorem to conclude the solvability result for (3.29).

4.1 Stability properties

We first recall from [6, eqs. (3.36) and (6.2)] and [11, eq. (5.8)] (see also [14, Lemma 4.5] and [30, Lemma 4.1]) that there exist positive constants c_*, c^*, $\widetilde{c}_*$, and $\widetilde{c}^*$, depending only on $C_{\mathcal{K}}$, such that for each $K \in \mathcal{K}_h$ there hold

$$\begin{aligned} c_* \|\boldsymbol{\tau}\|_{0,K}^2 \,\le\, s^K(\boldsymbol{\tau},\boldsymbol{\tau}) \,\le\, c^* \|\boldsymbol{\tau}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_r(K)\,, \\ \widetilde{c}_* \|\boldsymbol{\eta}\|_{0,K}^2 \,\le\, \widetilde{s}^K(\boldsymbol{\eta},\boldsymbol{\eta}) \,\le\, \widetilde{c}^* \|\boldsymbol{\eta}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\eta} \in \mathbf{X}_r(K)\,. \end{aligned} \tag{4.1}$$

Then, the boundedness properties of a_h (cf. (3.25)) and $\widetilde{a}_h$ (cf. (3.26)) are established as follows.

Lemma 4.1. *There exist positive constants, denoted $|\!|\!|a|\!|\!|$ and $|\!|\!|\widetilde{a}|\!|\!|$, independent of h, such that*

$$\big|a_h(\boldsymbol{\zeta},\boldsymbol{\tau})\big| \,\le\, |\!|\!|a|\!|\!|\, \|\boldsymbol{\zeta}\|_{0,\Omega}\, \|\boldsymbol{\tau}\|_{0,\Omega} \qquad \forall\ \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{X}_h, \tag{4.2a}$$

$$\big|\widetilde{a}_h(\boldsymbol{\xi},\boldsymbol{\eta})\big| \,\le\, |\!|\!|\widetilde{a}|\!|\!|\, \|\boldsymbol{\xi}\|_{0,\Omega}\, \|\boldsymbol{\eta}\|_{0,\Omega} \qquad \forall\ \boldsymbol{\xi},\, \boldsymbol{\eta} \in \mathbf{X}_h. \tag{4.2b}$$

Proof. We employ the Cauchy-Schwarz inequality and the upper estimates from (4.1) to bound s^K and $\widetilde{s}^K$ for each $K \in \mathcal{K}_h$, which yields

$$s^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\le\, \Big\{s^K(\boldsymbol{\zeta},\boldsymbol{\zeta})\Big\}^{1/2} \Big\{s^K(\boldsymbol{\tau},\boldsymbol{\tau})\Big\}^{1/2} \,\le\, c^*\, \|\boldsymbol{\zeta}\|_{0,K}\, \|\boldsymbol{\tau}\|_{0,K} \quad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{X}_r(K)\,, \tag{4.3}$$

and

$$\widetilde{s}^K(\boldsymbol{\xi},\boldsymbol{\eta}) \,\le\, \Big\{\widetilde{s}^K(\boldsymbol{\xi},\boldsymbol{\xi})\Big\}^{1/2} \Big\{\widetilde{s}^K(\boldsymbol{\eta},\boldsymbol{\eta})\Big\}^{1/2} \,\le\, \widetilde{c}^*\, \|\boldsymbol{\xi}\|_{0,K}\, \|\boldsymbol{\eta}\|_{0,K} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\eta} \in \mathbf{X}_r(K)\,. \tag{4.4}$$

Next, we observe that when $m = s = 0$ and $p = 2$, (3.2) and (3.3) hold with boundedness constants $C_\ell = M_\ell = 1$, so that for each $K \in \mathcal{K}_h$ we obtain

$$\|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\|_{0,K} \,\le\, \|\boldsymbol{\tau}\|_{0,K}\,, \quad \text{and} \quad \|\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\|_{0,K} \,\le\, \|\boldsymbol{\tau}\|_{0,K} \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(K)\,, \tag{4.5}$$

$$\|\boldsymbol{\eta} - \boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\|_{0,K} \,\le\, \|\boldsymbol{\eta}\|_{0,K}\,, \quad \text{and} \quad \|\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\|_{0,K} \,\le\, \|\boldsymbol{\eta}\|_{0,K} \quad \forall\, \boldsymbol{\eta} \in \mathbf{L}^2(K)\,. \tag{4.6}$$

In this way, applying Cauchy-Schwarz's inequality and (4.3), respectively, to the first and second terms defining a_h^K (cf. (3.21)), along with the fact that $\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,K} \,\le\, \|\boldsymbol{\tau}\|_{0,K}$ for all $\boldsymbol{\tau} \in \mathbb{L}^2(K)$, and the inequalities from (4.5), we readily find that

$$|a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau})| \,\le\, \nu^{-1}\,(1 + c^*)\, \|\boldsymbol{\zeta}\|_{0,K}\, \|\boldsymbol{\tau}\|_{0,K} \quad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{X}_r(K)\,. \tag{4.7}$$

In turn, proceeding similarly for $\widetilde{a}_h^K$ (cf. (3.22)), but now using (4.4) and (4.6), we deduce that

$$|\widetilde{a}_h^K(\boldsymbol{\xi},\boldsymbol{\eta})| \,\le\, \kappa^{-1}\,(1 + \widetilde{c}^*)\, \|\boldsymbol{\xi}\|_{0,K}\, \|\boldsymbol{\eta}\|_{0,K} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\eta} \in \mathbf{X}_r(K)\,. \tag{4.8}$$

Finally, given $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in \mathbb{X}_h$ and $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbf{X}_h$, we apply (4.7) and (4.8) to their local restrictions, so that summing over all $K \in \mathcal{K}_h$, we arrive at (4.2a) and (4.2b) with the constants $|\!|\!|a|\!|\!| = \nu^{-1}\,(1 + c^*)$ and $|\!|\!|\widetilde{a}|\!|\!| = \kappa^{-1}\,(1 + \widetilde{c}^*)$, respectively. □

Note that (4.2a) and (4.2b), along with the fact that $\|\boldsymbol{\zeta}\|_{0,\Omega} \leq \|\boldsymbol{\zeta}\|_{\mathbb{X}}$ and $\|\boldsymbol{\xi}\|_{0,\Omega} \leq \|\boldsymbol{\xi}\|_{\mathbf{X}}$ for all $(\boldsymbol{\zeta}, \boldsymbol{\xi}) \in \mathbb{X}_h \times \mathbf{X}_h$, guarantee the boundedness of $a_h|_{\mathbb{X}_h \times \mathbb{X}_h}$ and $\widetilde{a}_h|_{\mathbf{X}_h \times \mathbf{X}_h}$ with respect to $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbf{X}}$, respectively.

We now aim to present the coerciveness properties of a_h and $\widetilde{a}_h$, for which we first introduce the kernels of $b|_{\mathbb{X}_{0,h} \times \mathbf{Y}_h}$ and $\widetilde{b}|_{\mathbf{X}_h \times \mathrm{Y}_h}$, namely

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau} \in \mathbb{X}_{0,h} : \quad b(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{Y}_h \Big\},$$

and

$$\mathbf{V}_h := \Big\{ \boldsymbol{\eta} \in \mathbf{X}_h : \quad \widetilde{b}(\boldsymbol{\eta}, \varphi) = 0 \quad \forall\, \varphi \in \mathrm{Y}_h \Big\}.$$

Thus, using that $\mathbf{div}(\mathbb{X}_{0,h}) \subseteq \mathbf{Y}_h$ (cf. (3.7), (3.16), (3.17), (3.20)) and $\mathrm{div}(\mathbf{X}_h) \subseteq \mathrm{Y}_h$ (cf. (3.6), (3.15), (3.19)), it follows that

$$\begin{aligned} \mathbb{V}_h &:= \Big\{ \boldsymbol{\tau} \in \mathbb{X}_{0,h} : \quad \mathbf{div}(\boldsymbol{\tau}) = \quad \text{in} \quad \Omega \Big\}, \quad \text{and} \\ \mathbf{V}_h &:= \Big\{ \boldsymbol{\eta} \in \mathbf{X}_h : \quad \mathrm{div}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \Omega \Big\}. \end{aligned} \tag{4.9}$$

Hence, we are in position to state the following result.

Lemma 4.2. *There exist positive constants α and $\widetilde{\alpha}$, independent of h, such that*

$$\begin{aligned} a_h(\boldsymbol{\tau}, \boldsymbol{\tau}) &\geq \alpha\, \|\boldsymbol{\tau}\|^2_{\mathbb{X}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V}_h\,, \\ \widetilde{a}_h(\boldsymbol{\eta}, \boldsymbol{\eta}) &\geq \widetilde{\alpha}\, \|\boldsymbol{\eta}\|^2_{\mathbf{X}} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{V}_h\,. \end{aligned} \tag{4.10}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{V}_h$ and $K \in \mathcal{K}_h$, we observe, thanks to the definitions of a_h^K (cf. (3.21)) and a^K (cf. (2.19)), along with the lower bound in the first row of (4.1), the fact that $\|\boldsymbol{\tau}\|_{0,\Omega} \geq \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega}$, and the inequality $\|b\|^2 + \|a-b\|^2 \geq \frac{1}{2}\, \|a\|^2$, that there holds

$$\begin{aligned} a_h^K(\boldsymbol{\tau}, \boldsymbol{\tau}) &\geq \nu^{-1} \Big\{ \|\big(\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\big)^{\mathtt{d}}\|^2_{0,K} + c_*\, \|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\|^2_{0,K} \Big\} \\ &\geq \nu^{-1} \Big\{ \|\big(\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\big)^{\mathtt{d}}\|^2_{0,K} + c_*\, \|\boldsymbol{\tau}^{\mathtt{d}} - \big(\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\big)^{\mathtt{d}}\|^2_{0,K} \Big\} \\ &\geq \frac{1}{2\nu}\, \min\big\{1, c_*\big\}\, \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,K}\,. \end{aligned} \tag{4.11}$$

In turn, we recall from [17, Lemma 3.2] (see also [22, eq. (3.43)]) that there exists a positive constant c_1, depending only on Ω, such that (cf. (2.17))

$$\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,4/3;\Omega} \geq c_1\, \|\boldsymbol{\tau}\|^2_{0,\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)\,. \tag{4.12}$$

In this way, bearing in mind the definition of a_h (cf. (3.25)), summing over all $K \in \mathcal{K}_h$ in (4.11), noting that the tensors of $\mathbb{V}_h$ are divergence free (cf. (4.9)), and then employing (4.12), we get the first inequality in (4.10) with

$$\alpha := \frac{c_1}{2\nu}\, \min\big\{1, c_*\big\}\,.$$

On the other hand, for the second one we proceed similarly by invoking now the definitions of $\widetilde{a}_h^K$ (cf. (3.22)), $\widetilde{a}^K$ (cf. (2.19)), and $\widetilde{a}_h$ (cf. (3.26)). Hence, given $\boldsymbol{\eta} \in \mathbf{V}_h$, we use in this case the lower bound in the second row of (4.1), and easily deduce that for each $K \in \mathcal{K}_h$ there holds

$$\widetilde{a}_h^K(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq \kappa^{-1} \Big\{ \|\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\|^2_{0,K} + \widetilde{c}_*\, \|\boldsymbol{\eta} - \boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\eta})\|^2_{0,K} \Big\} \geq \frac{1}{2\kappa}\, \min\big\{1, \widetilde{c}_*\big\}\, \|\boldsymbol{\eta}\|^2_{0,K}\,,$$

from which, summing over all $K \in \mathcal{K}_h$, and recalling from (4.9) that the vector functions of $\mathbf{V}_h$ are divergence free as well, we conclude the remaining inequality with

$$\widetilde{\alpha} := \frac{1}{2\kappa} \min\left\{1, \widetilde{c}_*\right\}.$$

□

Next, we provide the discrete inf-sup conditions for the bilinear forms b and $\widetilde{b}$.

Lemma 4.3. *There exist positive constants β and $\widetilde{\beta}$, independent of h, such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_h} \frac{b(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \beta\,\|\mathbf{v}\|_{\mathbf{Y}} \quad \forall\,\mathbf{v}\in\mathbf{Y}_h\,, \quad \textit{and} \quad \sup_{\mathbf{0}\neq\boldsymbol{\eta}\in\mathbf{X}_h} \frac{\widetilde{b}(\boldsymbol{\eta},\varphi)}{\|\boldsymbol{\eta}\|_{\mathbf{X}}} \geq \widetilde{\beta}\,\|\varphi\|_{\mathrm{Y}} \quad \forall\,\varphi\in\mathrm{Y}_h\,.$$

Proof. The proof of the first inequality can be found in [32, Section 4.2, Lemma 4.9], whereas the second one, being actually a scalar version of the former, follows by similar arguments. □

We end this section with the boundedness properties of c_h and $\widetilde{c}_h$, which coincide with those of c and $\widetilde{c}$ (cf. (2.26)), as stated next. Indeed, we stress that, while c_h and $\widetilde{c}_h$ were originally defined in the discrete spaces (cf. (3.27), (3.28)), they are actually well-defined in $\mathbf{Y}\times\mathbf{Y}\times\mathbb{X}$ and $\mathbf{Y}\times\mathrm{Y}\times\mathbf{X}$, respectively.

Lemma 4.4. *Denoting $|||c||| = \nu^{-1}$ and $|||\widetilde{c}||| = \kappa^{-1}$, there hold*

$$\big|c_h(\mathbf{z};\mathbf{w},\boldsymbol{\tau})\big| \;\leq\; |||c|||\,\|\mathbf{z}\|_{\mathbf{Y}}\,\|\mathbf{w}\|_{\mathbf{Y}}\,\|\boldsymbol{\tau}\|_{\mathbb{X}} \qquad \forall\,(\mathbf{z},\mathbf{w},\boldsymbol{\tau})\in\mathbf{Y}\times\mathbf{Y}\times\mathbb{X}\,, \quad \textit{and} \tag{4.13a}$$

$$\big|\widetilde{c}_h(\mathbf{z};\psi,\boldsymbol{\eta})\big| \;\leq\; |||\widetilde{c}|||\,\|\mathbf{z}\|_{\mathbf{Y}}\,\|\psi\|_{\mathrm{Y}}\,\|\boldsymbol{\eta}\|_{\mathbf{X}} \qquad \forall\,(\mathbf{z},\psi,\boldsymbol{\eta})\in\mathbf{Y}\times\mathrm{Y}\times\mathbf{X}\,. \tag{4.13b}$$

Proof. While the proof of (4.13a) was sketched in [32, Lemma 4.5], for sake of completeness we provide the details here. Indeed, bearing in mind the definitions of c_h (cf. (3.27)), c_h^K (cf. (3.23)), and c^K (cf. (2.21)), and employing Cauchy-Schwarz's inequality and the second estimate in (4.5), we obtain for each $(\mathbf{z},\mathbf{w},\boldsymbol{\tau})\in\mathbf{Y}\times\mathbf{Y}\times\mathbb{X}$

$$\begin{aligned}\big|c_h(\mathbf{z};\mathbf{w},\boldsymbol{\tau})\big| &\leq \sum_{K\in\mathcal{K}_h}\big|c_h^K(\mathbf{z};\mathbf{w},\boldsymbol{\tau})\big| \;=\; \sum_{K\in\mathcal{K}_h}\big|c^K(\mathbf{z};\mathbf{w},\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\big|\\ &\leq \nu^{-1}\sum_{K\in\mathcal{K}_h}\|\mathbf{z}\|_{0,4;K}\,\|\mathbf{w}\|_{0,4;K}\,\|\boldsymbol{\mathcal{P}}_r^K(\boldsymbol{\tau})\|_{0,K}\\ &\leq \nu^{-1}\sum_{K\in\mathcal{K}_h}\|\mathbf{z}\|_{0,4;K}\,\|\mathbf{w}\|_{0,4;K}\,\|\boldsymbol{\tau}\|_{0,K}\,,\end{aligned}$$

from which, applying the discrete version of Cauchy-Schwarz's inequality, we conclude (4.13a). The proof of (4.13b) proceeds similarly by invoking now the definitions of $\widetilde{c}_h$ (cf. (3.28)), $\widetilde{c}_h^K$ (cf. (3.24)), and $\widetilde{c}^K$ (cf. (2.21)), and making use of the second estimate in (4.6). Further details are omitted. □

4.2 The fixed-point strategy

In this section we study the existence and uniqueness of solution of (3.29) (equivalently, (3.32)) by employing a fixed-point strategy. To this end, we first introduce suitable operators associated with each one of the two problems forming the whole nonlinear coupled system. More precisely, we let $\mathbf{S}_h:\mathbf{Y}_h\times\mathrm{Y}_h\to\mathbf{Y}_h$ be the operator defined for each $(\mathbf{z}_h,\psi_h)\in\mathbf{Y}_h\times\mathrm{Y}_h$ by

$$\mathbf{S}_h(\mathbf{z}_h,\psi_h) := \widehat{\mathbf{u}}_h\,,$$

where $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h) \in \mathcal{V}_h$ is the unique solution of the problem arising from the first equation of (3.32) after replacing $(\mathbf{u}_h, \theta_h)$ by $(\mathbf{z}_h, \psi_h)$, that is

$$\mathrm{A}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\big) \,=\, \mathrm{F}_{\psi_h}(\boldsymbol{\tau}_h, \mathbf{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathcal{V}_h\,. \tag{4.14}$$

In turn, we let $\widetilde{\mathbf{S}}_h : \mathbf{Y}_h \to \mathrm{Y}_h$ be the operator defined for each $\mathbf{z}_h \in \mathbf{Y}_h$ by

$$\widetilde{\mathbf{S}}_h(\mathbf{z}_h) \,:=\, \widehat{\theta}_h\,,$$

where $(\widehat{\boldsymbol{\rho}}_h, \widehat{\theta}_h) \in \mathcal{W}_h$ is the unique solution of the problem arising from the second equation of (3.32) after replacing $\mathbf{u}_h$ by $\mathbf{z}_h$, that is

$$\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\rho}}_h, \widehat{\theta}_h), (\boldsymbol{\eta}_h, \varphi_h)\big) \,=\, \widetilde{\mathrm{F}}(\boldsymbol{\eta}_h, \varphi_h) \qquad \forall\, (\boldsymbol{\eta}_h, \varphi_h) \in \mathcal{W}_h\,. \tag{4.15}$$

Hence, defining the operator $\mathbf{T}_h : \mathbf{Y}_h \to \mathbf{Y}_h$ as

$$\mathbf{T}_h(\mathbf{z}_h) \,:=\, \mathbf{S}_h(\mathbf{z}_h, \widetilde{\mathbf{S}}_h(\mathbf{z}_h)) \qquad \forall\, \mathbf{z}_h \in \mathbf{Y}_h\,, \tag{4.16}$$

we realize that solving the discrete scheme (3.29) (equivalently, (3.32)) is equivalent to seeking a fixed point of $\mathbf{T}_h$, that is: Find $\mathbf{u}_h \in \mathbf{Y}_h$ such that

$$\mathbf{T}_h(\mathbf{u}_h) \,=\, \mathbf{u}_h\,. \tag{4.17}$$

In order to conclude that $\mathbf{T}_h$ is in fact well-defined, in the next section we prove that the operators $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$ are, which reduces to establishing the well-posedness of (4.14) and (4.15).

4.2.1 Well-definedness of the operators $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$

We first recall from (2.22) and the description provided in the last paragraph of Section 3.3 that

$$\mathrm{B}_h\left((\boldsymbol{\zeta}_h, \mathbf{w}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\right) \,:=\, a_h(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{w}_h) + b(\boldsymbol{\zeta}_h, \mathbf{v}_h) \tag{4.18}$$

for all $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathcal{V}_h$, and

$$\widetilde{\mathrm{B}}_h\left((\boldsymbol{\xi}_h, \psi_h), (\boldsymbol{\eta}_h, \varphi_h)\right) \,:=\, \widetilde{a}_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) + \widetilde{b}(\boldsymbol{\eta}_h, \psi_h) + \widetilde{b}(\boldsymbol{\xi}_h, \varphi_h) \tag{4.19}$$

for all $(\boldsymbol{\xi}_h, \psi_h)$, $(\boldsymbol{\eta}_h, \varphi_h) \in \mathcal{W}_h$. Then, knowing from Lemmas 4.1, 4.2, and 4.3, that the pairs of bilinear forms (a_h, b) and $(\widetilde{a}_h, \widetilde{b})$ satisfy the hypothesis of the discrete Babuška-Brezzi theory (see, e.g. [26, Proposition 2.42]), straightforward applications of this result yield the discrete inf-sup conditions for B_h and $\widetilde{\mathrm{B}}_h$. This means that there exist positive constants $\alpha_{\mathrm{B},\mathtt{d}}$ (depending only on $|||a|||$, $\|b\| = 1$, α, and β) and $\alpha_{\widetilde{\mathrm{B}},\mathtt{d}}$ (depending only on $|||\widetilde{a}|||$, $\|\widetilde{b}\| = 1$, $\widetilde{\alpha}$, and $\widetilde{\beta}$), such that

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h} \frac{\mathrm{B}_h\left((\boldsymbol{\zeta}_h, \mathbf{w}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\right)}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathcal{V}}} \,\geq\, \alpha_{\mathrm{B},\mathtt{d}}\, \|(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{\mathcal{V}} \qquad \forall\, (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathcal{V}_h\,, \tag{4.20}$$

and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h} \frac{\widetilde{\mathrm{B}}_h\left((\boldsymbol{\xi}_h, \psi_h), (\boldsymbol{\eta}_h, \varphi_h)\right)}{\|(\boldsymbol{\eta}_h, \varphi_h)\|_{\mathcal{W}}} \,\geq\, \alpha_{\widetilde{\mathrm{B}},\mathtt{d}}\, \|(\boldsymbol{\xi}_h, \psi_h)\|_{\mathcal{W}} \qquad \forall\, (\boldsymbol{\xi}_h, \psi_h) \in \mathcal{W}_h\,. \tag{4.21}$$

Next, bearing in mind the definitions of $\mathrm{A}_{h,\mathbf{z}_h}$ (cf. (3.30)) and $\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}$ (cf. (3.31)) for each $\mathbf{z}_h \in \mathbf{Y}_h$, and combining (4.20) and (4.21) with the effect of the extra terms given by $c_h(\mathbf{z}_h; \cdot, \cdot)$ and $\widetilde{c}_h(\mathbf{z}_h; \cdot, \cdot)$, respectively, which means invoking the upper bounds provided by (4.13a) and (4.13b), we arrive at

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h} \frac{\mathrm{A}_{h,\mathbf{z}_h}\left((\boldsymbol{\zeta}_h, \mathbf{w}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\right)}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathcal{V}}} \,\geq\, \Big\{\alpha_{\mathrm{B},\mathtt{d}} - \nu^{-1}\, \|\mathbf{z}_h\|_{\mathbf{Y}}\Big\} \|(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{\mathcal{V}} \quad \forall\, (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathcal{V}_h,$$

and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h} \frac{\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h))}{\|(\boldsymbol{\eta}_h,\varphi_h)\|_{\mathcal{W}}} \,\geq\, \Big\{\alpha_{\widetilde{\mathrm{B}},\mathtt{d}} - \kappa^{-1}\,\|\mathbf{z}_h\|_{\mathbf{Y}}\Big\}\|(\boldsymbol{\xi}_h,\psi_h)\|_{\mathcal{W}} \quad \forall\,(\boldsymbol{\xi}_h,\psi_h)\in\mathcal{W}_h,$$

from which, under the assumption

$$\|\mathbf{z}_h\|_{\mathbf{Y}} \,\leq\, \delta_{\mathtt{d}} \,:=\, \frac{1}{2}\,\min\big\{\nu\,\alpha_{\mathrm{B},\mathtt{d}},\kappa\,\alpha_{\widetilde{\mathrm{B}},\mathtt{d}}\big\}\,, \tag{4.22}$$

we conclude that

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h} \frac{\mathrm{A}_{h,\mathbf{z}_h}((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h))}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathcal{V}}} \,\geq\, \frac{\alpha_{\mathrm{B},\mathtt{d}}}{2}\,\|(\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathcal{V}} \qquad \forall\,(\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathcal{V}_h\,, \tag{4.23}$$

and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h} \frac{\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h))}{\|(\boldsymbol{\eta}_h,\varphi_h)\|_{\mathcal{W}}} \,\geq\, \frac{\alpha_{\widetilde{\mathrm{B}},\mathtt{d}}}{2}\,\|(\boldsymbol{\xi}_h,\psi_h)\|_{\mathcal{W}} \qquad \forall\,(\boldsymbol{\xi}_h,\psi_h)\in\mathcal{W}_h\,. \tag{4.24}$$

Moreover, according to the definitions of $\mathrm{A}_{h,\mathbf{z}_h}$ (cf. (3.30)) and $\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}$ (cf. (3.31)) for each $\mathbf{z}_h\in\mathbf{Y}_h$, which include those of B_h (cf. (4.18)) and $\widetilde{\mathrm{B}}_h$ (cf. (4.19)), and invoking the boundedness properties of a_h, c_h, $\widetilde{a}_h$, and $\widetilde{c}_h$ (cf. Lemmas 4.1 and 4.4), we conclude that $\mathrm{A}_{h,\mathbf{z}_h}$ and $\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}$ are bounded for each $\mathbf{z}_h\in\mathbf{Y}_h$ satisfying (4.22), with corresponding constants $|||\mathrm{A}|||$ (depending only on $|||a|||$, ν^{-1}, and $\delta_{\mathtt{d}}$) and $|||\widetilde{\mathrm{A}}|||$ (depending only on $|||\widetilde{a}|||$, κ^{-1}, and $\delta_{\mathtt{d}}$), respectively.

We are now in position to establish that $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$ are well-defined, equivalently that (4.14) and (4.15) are well-posed.

Lemma 4.5. *For each $\big(\mathbf{z}_h,\psi_h\big)\in\mathbf{Y}_h\times\mathrm{Y}_h$ such that $\mathbf{z}_h$ satisfies (4.22), there exist unique $(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h)\in\mathcal{V}_h$ and $(\widehat{\boldsymbol{\rho}}_h,\widehat{\theta}_h)\in\mathcal{W}_h$ solutions to (4.14) and (4.15), respectively, so that one can define $\mathbf{S}_h\big(\mathbf{z}_h,\psi_h\big):=\widehat{\mathbf{u}}_h\in\mathbf{Y}_h$ and $\widetilde{\mathbf{S}}_h(\mathbf{z}_h):=\widehat{\theta}_h\in\mathrm{Y}_h$. In addition, there exist positive constants $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$, depending only on $\alpha_{\mathrm{B},\mathtt{d}}$ and c_f (cf. (2.27)), and on $\alpha_{\widetilde{\mathrm{B}},\mathtt{d}}$ and $c_{\widetilde{f}}$ (cf. (2.27)), respectively, and hence independent of h, such that there hold the following a priori estimates*

$$\|\mathbf{S}_h(\mathbf{z}_h,\psi_h)\|_{\mathbf{Y}} \,=\, \|\widehat{\mathbf{u}}_h\|_{\mathbf{Y}} \,\leq\, \|(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h)\|_{\mathcal{V}} \,\leq\, C_{\mathbf{S}}\,\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{g}\|_{0,\Omega}\,\|\psi_h\|_{\mathrm{Y}}\Big\}\,, \tag{4.25a}$$

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h)\|_{\mathrm{Y}} \,=\, \|\widehat{\theta}_h\|_{\mathrm{Y}} \,\leq\, \|(\widehat{\boldsymbol{\rho}}_h,\widehat{\theta}_h)\|_{\mathcal{W}} \,\leq\, C_{\widetilde{\mathbf{S}}}\,\|\theta_D\|_{1/2,\Gamma}\,. \tag{4.25b}$$

Proof. Given $\big(\mathbf{z}_h,\psi_h\big)\in\mathbf{Y}_h\times\mathrm{Y}_h$ with $\mathbf{z}_h$ satisfying (4.22), we begin by noticing from (2.25) and (2.27) that $\mathrm{F}_{\psi_h}\in\mathcal{V}'$ and $\widetilde{\mathrm{F}}\in\mathcal{W}'$, with

$$\|\mathrm{F}_{\psi_h}\|_{\mathcal{V}'} \,\leq\, c_f\,\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega}\,\|\psi_h\|_{\mathrm{Y}}\,, \quad \text{and} \quad \|\widetilde{\mathrm{F}}\|_{\mathcal{W}'} \,\leq\, c_{\widetilde{f}}\,\|\theta_D\|_{1/2,\Gamma}\,. \tag{4.26}$$

Therefore, thanks to (4.23), (4.24), the boundedness of $\mathrm{A}_{h,\mathbf{z}_h}$ and $\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}$, and (4.26), straightforward applications of the discrete version of the Banach-Nečas-Babuška theorem (cf. [26, Theorem 2.22]) imply the unique solvabilities of (4.14) and (4.15). In turn, the corresponding a priori estimates read

$$\|(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h)\|_{\mathcal{V}} \,\leq\, \frac{2}{\alpha_{\mathrm{B},\mathtt{d}}}\,\|\mathrm{F}_{\psi_h}\|_{\mathcal{V}'} \quad \text{and} \quad \|(\widehat{\boldsymbol{\rho}}_h,\widehat{\theta}_h)\|_{\mathcal{W}} \,\leq\, \|\widetilde{\mathrm{F}}\|_{\mathcal{W}'}\,,$$

which, along with (4.26), yield (4.25a) and (4.25b), thus completing the proof. □

It follows automatically from the above lemma that $\mathbf{T}_h$ is well-defined, and thus, according to (4.16) and the estimates (4.25a) and (4.25b), we readily obtain

$$\begin{aligned} \|\mathbf{T}_h(\mathbf{z}_h)\|_{\mathbf{Y}} \,=\, \|\mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathbf{S}}_h(\mathbf{z}_h)\big)\|_{\mathbf{Y}} \,\le\, C_{\mathbf{S}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{g}\|_{0,\Omega}\,\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h)\|_{\mathrm{Y}}\right\} \\ \le\, C_{\mathbf{T}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega}\,\|\theta_D\|_{1/2,\Gamma}\right\}, \end{aligned} \tag{4.27}$$

for each $\mathbf{z}_h \in \mathbf{Y}_h$ satisfying (4.22), where $C_{\mathbf{T}}$ is a positive constant depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$.

4.3 Solvability analysis of the fixed-point scheme

Knowing that the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, and hence $\mathbf{T}_h$ as well, are well defined, we now address the solvability of the fixed-point equation (4.17). To this end, and in order to finally apply the Banach theorem, we first establish sufficient conditions under which $\mathbf{T}_h$ maps a closed ball of $\mathbf{Y}_h$ into itself. Indeed, setting (cf. (4.22))

$$\mathbf{W}_h \,:=\, \Big\{\mathbf{z}_h \in \mathbf{Y}_h : \quad \|\mathbf{z}_h\|_{\mathbf{Y}} \,\le\, \delta_{\mathtt{d}}\Big\}, \tag{4.28}$$

we have the following result.

Lemma 4.6. *Assume that the data satisfy*

$$C_{\mathbf{T}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega}\,\|\theta_D\|_{1/2,\Gamma}\right\} \,\le\, \delta_{\mathtt{d}}\,. \tag{4.29}$$

Then, there holds $\mathbf{T}_h(\mathbf{W}_h) \,\subseteq\, \mathbf{W}_h$.

Proof. It is a clear consequence of the definition of $\mathbf{W}_h$ (cf. (4.28)) and the a priori estimate (4.27). □

We now address the continuity properties of $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, and hence of $\mathbf{T}_h$. We begin with that of $\mathbf{S}_h$.

Lemma 4.7. *There exists a positive constant* $\mathrm{L}_{\mathbf{S}}$*, depending only on* $\alpha_{\mathrm{B},\mathtt{d}}$*,* $\|\mathbf{g}\|_{0,\Omega}$*,* ν*, and* $C_{\mathbf{S}}$*, and hence independent of* h*, such that*

$$\|\mathbf{S}(\mathbf{z}_h,\psi_h) - \mathbf{S}(\mathbf{y}_h,\phi_h)\|_{\mathbf{Y}} \,\le\, \mathrm{L}_{\mathbf{S}}\left\{\big(\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_h\|_{\mathrm{Y}}\big)\,\|\mathbf{z}_h - \mathbf{y}_h\|_{\mathbf{Y}} + \|\psi_h - \phi_h\|_{\mathrm{Y}}\right\} \tag{4.30}$$

for all $(\mathbf{z}_h,\psi_h)$, $(\mathbf{y}_h,\phi_h) \in \mathbf{W}_h \times \mathrm{Y}_h$.

Proof. Given $(\mathbf{z}_h,\psi_h)$, $(\mathbf{y}_h,\phi_h) \in \mathbf{W}_h \times \mathrm{Y}_h$, we let $\mathbf{S}_h(\mathbf{z}_h,\psi_h) := \widehat{\mathbf{u}}_h$ and $\mathbf{S}_h(\mathbf{y}_h,\phi_h) := \widehat{\mathbf{w}}_h$, where $(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h) \in \mathcal{V}_h$ and $(\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h) \in \mathcal{V}_h$ are the corresponding unique solutions of (4.14), that is

$$\mathrm{A}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,=\, \mathrm{F}_{\psi_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathcal{V}_h\,, \tag{4.31}$$

$$\mathrm{A}_{h,\mathbf{y}_h}\big((\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,=\, \mathrm{F}_{\phi_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathcal{V}_h\,. \tag{4.32}$$

Then, applying (4.23) to $(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h) - (\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h) \in \mathcal{V}_h$, and using (4.31), we obtain

$$\begin{aligned} \|\mathbf{S}(\mathbf{z}_h,\psi_h) - \mathbf{S}(\mathbf{y}_h,\phi_h)\|_{\mathbf{Y}} \,=\, \|\widehat{\mathbf{u}}_h - \widehat{\mathbf{w}}_h\|_{\mathbf{Y}} \,\le\, \|(\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h) - (\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h)\|_{\mathcal{V}} \\ \le\, \frac{2}{\alpha_{\mathrm{B},\mathtt{d}}} \sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h} \frac{\mathrm{F}_{\psi_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) - \mathrm{A}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathcal{V}}}\,. \end{aligned} \tag{4.33}$$

In turn, adding and subtracting $c_h(\mathbf{y}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)$, and then using (4.32) and the fact that $c_h(\cdot;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)$ is linear, we readily find that

$$\begin{aligned}\mathrm{A}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,&=\, \mathrm{A}_{h,\mathbf{y}_h}\big((\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,-\, c_h(\mathbf{y}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h) \,+\, c_h(\mathbf{z}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)\\ &=\, \mathrm{F}_{\phi_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) \,+\, c_h(\mathbf{z}_h-\mathbf{y}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)\,,\end{aligned}$$

whence

$$\mathrm{F}_{\psi_h}(\boldsymbol{\tau}_h,\mathbf{v}_h) - \mathrm{A}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\zeta}}_h,\widehat{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,=\, \big(\mathrm{F}_{\psi_h}-\mathrm{F}_{\phi_h}\big)(\boldsymbol{\tau}_h,\mathbf{v}_h) - c_h(\mathbf{z}_h-\mathbf{y}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)\,. \tag{4.34}$$

Now, according to the definitions of F_{ψ_h} and F_{ϕ_h} (cf. (2.25)), we easily deduce that

$$\big|\big(\mathrm{F}_{\psi_h}-\mathrm{F}_{\phi_h}\big)(\boldsymbol{\tau}_h,\mathbf{v}_h)\big| \,\le\, \|\mathbf{g}\|_{0,\Omega}\,\|\psi_h-\phi_h\|_{\mathrm{Y}}\,\|\mathbf{v}_h\|_{\mathbf{Y}}\,, \tag{4.35}$$

whereas the boundedness property of c_h (cf. (4.13a)) and the a priori estimate for $\widehat{\mathbf{w}}_h = \mathbf{S}_h(\mathbf{y}_h,\phi_h)$ (cf. (4.25a), Lemma 4.5) yield

$$\big|c_h(\mathbf{z}_h-\mathbf{y}_h;\widehat{\mathbf{w}}_h,\boldsymbol{\tau}_h)\big| \,\le\, \nu^{-1}\,C_{\mathbf{S}}\,\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega}\,\|\phi_h\|_{\mathrm{Y}}\Big\}\,\|\mathbf{z}_h-\mathbf{y}_h\|_{\mathbf{Y}}\,\|\boldsymbol{\tau}_h\|_{\mathbb{X}}\,. \tag{4.36}$$

Finally, employing (4.35) and (4.36) to bound (4.34), and then replacing the resulting estimate back into (4.33), we arrive at (4.30) with $\mathrm{L}_{\mathbf{S}}$ as indicated. □

The continuity property of $\widetilde{\mathbf{S}}$ is established next.

Lemma 4.8. *There exists a positive constant $\mathrm{L}_{\widetilde{\mathbf{S}}}$, depending only on $\alpha_{\widetilde{\mathrm{B}},\mathbf{d}}$, κ, and $C_{\widetilde{\mathbf{S}}}$, and hence independent of h, such that*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h)-\widetilde{\mathbf{S}}_h(\mathbf{y}_h)\|_{\mathrm{Y}} \,\le\, \mathrm{L}_{\widetilde{\mathbf{S}}}\,\|\theta_D\|_{1/2,\Gamma}\,\|\mathbf{z}_h-\mathbf{y}_h\|_{\mathbf{Y}} \qquad \forall\,\mathbf{z}_h,\,\mathbf{y}_h\in\mathbf{W}_h\,. \tag{4.37}$$

Proof. Given $\mathbf{z}_h,\,\mathbf{y}_h\in\mathbf{W}_h$, we let $\widetilde{\mathbf{S}}_h(\mathbf{z}_h) := \widehat{\theta}_h$ and $\widetilde{\mathbf{S}}_h(\mathbf{y}_h) := \widehat{\psi}_h$, where $(\widehat{\rho}_h,\widehat{\theta}_h)\in\mathcal{W}_h$ and $(\widehat{\xi}_h,\widehat{\psi}_h)\in\mathcal{W}_h$ are the corresponding unique solutions of (4.15), that is

$$\widetilde{\mathrm{A}}_{h,\mathbf{z}_h}\big((\widehat{\boldsymbol{\rho}}_h,\widehat{\theta}_h),(\boldsymbol{\eta}_h,\varphi_h)\big) \,=\, \widetilde{\mathrm{F}}(\boldsymbol{\eta}_h,\varphi_h) \qquad \forall\,(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h\,, \tag{4.38}$$

$$\widetilde{\mathrm{A}}_{h,\mathbf{y}_h}\big((\widehat{\boldsymbol{\xi}}_h,\widehat{\psi}_h),(\boldsymbol{\eta}_h,\varphi_h)\big) \,=\, \widetilde{\mathrm{F}}(\boldsymbol{\eta}_h,\varphi_h) \qquad \forall\,(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h\,. \tag{4.39}$$

Then, proceeding analogously to the proof of Lemma 4.7, but now applying the global inf-sup condition (4.24), adding and subtracting $\widetilde{c}_h(\mathbf{y}_h;\widehat{\psi}_h,\boldsymbol{\eta}_h)$, and then employing (4.38), (4.39), and the linearity of $\widetilde{c}_h(\cdot;\widehat{\psi}_h,\boldsymbol{\eta}_h)$, we arrive at

$$\begin{aligned}\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h)-\widetilde{\mathbf{S}}_h(\mathbf{y}_h)\|_{\mathrm{Y}} \,&=\, \|\widehat{\theta}_h-\widehat{\psi}_h\|_{\mathrm{Y}} \,\le\, \|(\widehat{\rho}_h,\widehat{\theta}_h)-(\widehat{\xi}_h,\widehat{\psi}_h)\|_{\mathcal{W}}\\ &\le\, \frac{2}{\alpha_{\widetilde{\mathrm{B}},\mathbf{d}}}\,\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h}\frac{\widetilde{c}_h(\mathbf{y}_h-\mathbf{z}_h;\widehat{\psi}_h,\boldsymbol{\eta}_h)}{\|(\boldsymbol{\eta}_h,\varphi_h)\|_{\mathcal{W}}}\,.\end{aligned} \tag{4.40}$$

Next, thanks to the boundedness property of $\widetilde{c}_h$ (cf. (4.13b)) and the a priori estimate for $\widehat{\psi}_h = \widetilde{\mathbf{S}}_h(\mathbf{y}_h)$ (cf. (4.25b), Lemma 4.5), we get

$$\big|\widetilde{c}_h(\mathbf{y}_h-\mathbf{z}_h;\widehat{\psi}_h,\boldsymbol{\eta}_h)\big| \,\le\, \kappa^{-1}\,C_{\widetilde{\mathbf{S}}}\,\|\theta_D\|_{1/2,\Gamma}\,\|\mathbf{z}_h-\mathbf{y}_h\|_{\mathbf{Y}}\,\|\boldsymbol{\eta}_h\|_{\mathbf{X}}\,,$$

which, replaced back into (4.40), yields (4.37) with $\mathrm{L}_{\widetilde{\mathbf{S}}}$ as announced. □

Having proved Lemmas 4.7 and 4.8, we now utilize them to derive the continuity property of $\mathbf{T}_h$. Indeed, bearing in mind the definition of $\mathbf{T}_h$ (cf. (4.16)), we observe, thanks to (4.30), (4.37), and the a priori estimate (4.25b) (cf. Lemma 4.5), that for each $\mathbf{z}_h,\, \mathbf{y}_h \in \mathbf{W}_h$ there holds

$$
\begin{aligned}
\|\mathbf{T}_h(\mathbf{z}_h) - \mathbf{T}_h(\mathbf{y}_h)\|_{\mathbf{Y}} \,&=\, \|\mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathbf{S}}_h(\mathbf{z}_h)\big) - \mathbf{S}_h\big(\mathbf{y}_h, \widetilde{\mathbf{S}}_h(\mathbf{y}_h)\big)\|_{\mathbf{Y}} \\
&\le\, \mathrm{L}_{\mathbf{S}} \left\{ \big(\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\widetilde{\mathbf{S}}_h(\mathbf{y}_h)\|_{\mathrm{Y}}\big) \|\mathbf{z}_h - \mathbf{y}_h\|_{\mathbf{Y}} + \|\widetilde{\mathbf{S}}_h(\mathbf{z}_h) - \widetilde{\mathbf{S}}_h(\mathbf{y}_h)\|_{\mathrm{Y}} \right\} \\
&\le\, \mathrm{L}_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \big(C_{\widetilde{\mathbf{S}}} + \mathrm{L}_{\widetilde{\mathbf{S}}}\big) \|\theta_D\|_{1/2,\Gamma} \right\} \|\mathbf{z}_h - \mathbf{y}_h\|_{\mathbf{Y}}\,,
\end{aligned}
$$

from which it follows that

$$
\|\mathbf{T}_h(\mathbf{z}_h) - \mathbf{T}_h(\mathbf{y}_h)\|_{\mathbf{Y}} \,\le\, \mathrm{L}_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\theta_D\|_{1/2,\Gamma} \right\} \|\mathbf{z}_h - \mathbf{y}_h\|_{\mathbf{Y}}\,, \tag{4.41}
$$

with a positive constant $\mathrm{L}_{\mathbf{T}}$ depending only on $\mathrm{L}_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$, and $\mathrm{L}_{\widetilde{\mathbf{S}}}$, and hence independent of h.

We are now in position to establish the main result of this section.

Lemma 4.9. *Given $\delta_{\mathbf{d}}$ and $\mathbf{W}_h$ as in (4.22) and (4.28), respectively, assume that, in addition to the hypothesis of Lemma 4.6 (cf. (4.29)), the data satisfy*

$$
\mathrm{L}_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\theta_D\|_{1/2,\Gamma} \right\} \,<\, 1\,. \tag{4.42}
$$

Then, there exists a unique $\mathbf{u}_h \in \mathbf{W}_h$ fixed point of $\mathbf{T}_h$ (cf. (4.17)). Equivalently, (3.29) has a unique solution $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\rho}_h, \theta_h)\big) \in \mathcal{V}_h \times \mathcal{W}_h$ with $\mathbf{u}_h \in \mathbf{W}_h$. Moreover, there exist positive constants $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$, such that there hold

$$
\begin{aligned}
&\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{V}} \,\le\, \mathcal{C} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega}\, \|\theta_D\|_{1/2,\Gamma} \right\}, \quad \textit{and} \\
&\|(\boldsymbol{\rho}_h, \theta_h)\|_{\mathcal{W}} \,\le\, \widetilde{\mathcal{C}}\, \|\theta_D\|_{1/2,\Gamma}\,.
\end{aligned} \tag{4.43}
$$

Proof. It is clear from Lemma 4.6, (4.41), and hypothesis (4.42) that $\mathbf{T}_h$ is a contraction that maps the ball $\mathbf{W}_h$ into itself, and thus a direct application of the Banach fixed-point theorem yields the existence of a unique fixed point $\mathbf{u}_h \in \mathbf{W}_h$ of $\mathbf{T}_h$, and hence, equivalently, the indicated unique solvability of (3.29). In addition, since $\mathbf{u}_h = \mathbf{T}_h(\mathbf{u}_h) = \mathbf{S}\big(\mathbf{u}_h, \widetilde{\mathbf{S}}(\mathbf{u}_h)\big)$, we readily see that $\mathbf{u}_h = \mathbf{S}\big(\mathbf{u}_h, \theta_h\big)$ with $\theta_h = \widetilde{\mathbf{S}}(\mathbf{u}_h)$, whence the a priori estimates in (4.43) follow from (4.25a) and (4.25b). □

5 A priori error analysis

In this section we derive Céa-type estimates and establish the corresponding rates of convergence for the global error $\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{V}} + \|(\boldsymbol{\rho}, \theta) - (\boldsymbol{\rho}_h, \theta_h)\|_{\mathcal{W}}$, and for computable approximations of $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$, and the pressure p, where $\big((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\rho}, \theta)\big) \in \mathcal{V} \times \mathcal{W}$, with $\mathbf{u} \in \mathbf{W}$ (cf. (2.29)), is the unique solution of (2.18) (equivalently, (2.23)), and $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\rho}_h, \theta_h)\big) \in \mathcal{V}_h \times \mathcal{W}_h$, with $\mathbf{u}_h \in \mathbf{W}_h$ (cf. (4.28)), is the unique solution of (3.29) (equivalently, (3.32)).

5.1 The main Céa estimate

We first set the pairs of continuous and discrete schemes arising from (2.23) and (3.32), namely

$$
\begin{aligned}
\mathrm{A}_{\mathbf{u}}\left((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})\right) \;&=\; \mathrm{F}_{\theta}(\boldsymbol{\tau}, \mathbf{v}) \qquad && \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{V}\,, \\
\mathrm{A}_{h,\mathbf{u}_h}\left((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\right) \;&=\; \mathrm{F}_{\theta_h}(\boldsymbol{\tau}_h, \mathbf{v}_h) \qquad && \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathcal{V}_h\,,
\end{aligned}
$$

and

$$\widetilde{\mathrm{A}}_{\mathbf{u}}\,((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\varphi)) \;=\; \widetilde{\mathrm{F}}(\boldsymbol{\eta},\varphi) \qquad \forall\,(\boldsymbol{\eta},\varphi)\in\mathcal{W}\,,$$
$$\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\,((\boldsymbol{\rho}_h,\theta_h),(\boldsymbol{\eta}_h,\varphi_h)) \;=\; \widetilde{\mathrm{F}}(\boldsymbol{\eta}_h,\varphi_h) \qquad \forall\,(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h\,.$$

In this way, knowing that $\mathrm{A}_{\mathbf{u}}$, $\mathrm{A}_{h,\mathbf{u}_h}$, $\widetilde{\mathrm{A}}_{\mathbf{u}}$, and $\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}$ are all bounded, and that $\mathrm{A}_{h,\mathbf{u}_h}$ and $\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}$ satisfy the discrete global inf-sup conditions with constants $\frac{\alpha_{\mathrm{B},\mathbf{d}}}{2}$ and $\frac{\alpha_{\widetilde{\mathrm{B}},\mathbf{d}}}{2}$, respectively, we can apply the first Strang lemma (cf. [26, Lemma 2.27]) to conclude the existence of positive constants $C_{\mathbf{st}}$ (depending only on $\alpha_{\mathrm{B},\mathbf{d}}$ and $\|\mathrm{A}\|$) and $\widetilde{C}_{\mathbf{st}}$ (depending only on $\alpha_{\widetilde{\mathrm{B}},\mathbf{d}}$ and $\|\widetilde{\mathrm{A}}\|$), such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{V}} \,\le\, C_{\mathbf{st}}\Big\{\|\mathrm{F}_\theta-\mathrm{F}_{\theta_h}\|_{\mathcal{V}_h'} \,+\, \inf_{(\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathcal{V}_h}\Big\{\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathcal{V}}\\ +\,\big\|\big(\mathrm{A}_{\mathbf{u}}-\mathrm{A}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),\cdot\big)\big\|_{\mathcal{V}_h'}\Big\}\Big\}\,,\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\|(\boldsymbol{\rho},\theta)-(\boldsymbol{\rho}_h,\theta_h)\|_{\mathcal{W}} \,\le\, \widetilde{C}_{\mathbf{st}} \inf_{(\boldsymbol{\xi}_h,\psi_h)\in\mathcal{W}_h}\Big\{\|(\boldsymbol{\rho},\theta)-(\boldsymbol{\xi}_h,\psi_h)\|_{\mathcal{W}}\\ +\,\big\|\big(\widetilde{\mathrm{A}}_{\mathbf{u}}-\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\xi}_h,\psi_h),\cdot\big)\big\|_{\mathcal{W}_h'}\Big\}\,,\end{aligned} \tag{5.2}$$

where

$$\|\mathrm{F}_\theta-\mathrm{F}_{\theta_h}\|_{\mathcal{V}_h'} \;:=\; \sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h}\frac{\big|\big(\mathrm{F}_\theta-\mathrm{F}_{\theta_h}\big)(\boldsymbol{\tau}_h,\mathbf{v}_h)\big|}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathcal{V}}}\,, \tag{5.3}$$

$$\big\|\big(\mathrm{A}_{\mathbf{u}}-\mathrm{A}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),\cdot\big)\big\|_{\mathcal{V}_h'} \;:=\; \sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h}\frac{\big|\big(\mathrm{A}_{\mathbf{u}}-\mathrm{A}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)\big|}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathcal{V}}}\,, \tag{5.4}$$

and

$$\big\|\big(\widetilde{\mathrm{A}}_{\mathbf{u}}-\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\xi}_h,\psi_h),\cdot\big)\big\|_{\mathcal{W}_h'} \;:=\; \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h}\frac{\big|\big(\widetilde{\mathrm{A}}_{\mathbf{u}}-\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h)\big)\big|}{\|(\boldsymbol{\eta}_h,\varphi_h)\|_{\mathcal{W}}}\,. \tag{5.5}$$

We now aim to bound each one of the consistency terms given by (5.3), (5.4), and (5.5). First, according to the definitions of F_θ and F_{θ_h} (cf. (2.25)), and similarly as for the derivation of (4.35), we easily obtain

$$\|\mathrm{F}_\theta-\mathrm{F}_{\theta_h}\|_{\mathcal{V}_h'} \;\le\; \|\mathbf{g}\|_{0,\Omega}\,\|\theta-\theta_h\|_{\mathrm{Y}}\,. \tag{5.6}$$

On the other hand, bearing in mind the definitions of $\mathrm{A}_{\mathbf{u}}$ (cf. (2.24)) $\mathrm{A}_{h,\mathbf{u}_h}$ (cf. (3.30), (4.18)), $\widetilde{\mathrm{A}}_{\mathbf{u}}$ (cf. (2.24)) and $\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}$ (cf. (3.31), (4.19)), we deduce that for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)$, $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathcal{V}_h$, and for all $(\boldsymbol{\xi}_h,\psi_h)$, $(\boldsymbol{\eta}_h,\varphi_h)\in\mathcal{W}_h$, there hold, respectively,

$$\big(\mathrm{A}_{\mathbf{u}}-\mathrm{A}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \;=\; (a-a_h)(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)+c(\mathbf{u};\mathbf{w}_h,\boldsymbol{\tau}_h)-c_h(\mathbf{u}_h;\mathbf{w}_h,\boldsymbol{\tau}_h)\,, \tag{5.7}$$

and

$$\big(\widetilde{\mathrm{A}}_{\mathbf{u}}-\widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\xi}_h,\psi_h),(\boldsymbol{\eta}_h,\varphi_h)\big) \;=\; (\widetilde{a}-\widetilde{a}_h)(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h)+\widetilde{c}(\mathbf{u};\psi_h,\boldsymbol{\eta}_h)-\widetilde{c}_h(\mathbf{u}_h;\psi_h,\boldsymbol{\eta}_h)\,. \tag{5.8}$$

Next, adding and subtracting $\mathbf{u}$ in the second component of both $c(\mathbf{u};\mathbf{w}_h,\boldsymbol{\tau}_h)$ and $c_h(\mathbf{u}_h;\mathbf{w}_h,\boldsymbol{\tau}_h)$, as well as adding and subtracting $c_h(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)$ to the resulting terms, and then using the boundedness of c and c_h with respective constants $\|c\|=|||c|||=\nu^{-1}$, we get

$$\begin{aligned}\big|c(\mathbf{u};\mathbf{w}_h,\boldsymbol{\tau}_h)-c_h(\mathbf{u}_h;\mathbf{w}_h,\boldsymbol{\tau}_h)\big| \,\le\, \nu^{-1}(\|\mathbf{u}\|_{\mathbf{Y}}+\|\mathbf{u}_h\|_{\mathbf{Y}})\,\|\mathbf{u}-\mathbf{w}_h\|_{\mathbf{Y}}\,\|\boldsymbol{\tau}_h\|_{\mathbb{X}}\\ +\,\nu^{-1}\,\|\mathbf{u}\|_{\mathbf{Y}}\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Y}}\,\|\boldsymbol{\tau}_h\|_{\mathbb{X}} \,+\, \big|(c-c_h)(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)\big|\,.\end{aligned} \tag{5.9}$$

In turn, proceeding analogously with the terms involving $\widetilde{c}$ and $\widetilde{c}_h$ in (5.8), which means adding and subtracting θ in the second component of them, adding and subtracting $\widetilde{c}_h(\mathbf{u};\theta,\boldsymbol{\eta}_h)$, and then employing the boundedness of $\widetilde{c}$ and $\widetilde{c}_h$ with respective constants $\|\widetilde{c}\| = |||\widetilde{c}||| = \kappa^{-1}$, we arrive at

$$\begin{aligned} \left|\widetilde{c}(\mathbf{u};\psi_h,\boldsymbol{\eta}_h) - \widetilde{c}_h(\mathbf{u}_h;\psi_h,\boldsymbol{\eta}_h)\right| \,&\leq\, \kappa^{-1}(\|\mathbf{u}\|_{\mathbf{Y}} + \|\mathbf{u}_h\|_{\mathbf{Y}})\,\|\theta-\psi_h\|_{\mathrm{Y}}\,\|\boldsymbol{\eta}_h\|_{\mathbf{X}} \\ &+\, \kappa^{-1}\,\|\theta\|_{\mathrm{Y}}\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Y}}\,\|\boldsymbol{\eta}_h\|_{\mathbf{X}} \,+\, \left|(\widetilde{c}-\widetilde{c}_h)(\mathbf{u};\theta,\boldsymbol{\eta}_h)\right|. \end{aligned} \tag{5.10}$$

In order to estimate the terms involving $(a-a_h)$ and $(\widetilde{a}-\widetilde{a}_h)$ in (5.7) and (5.8), respectively, as well as to complete the bounds in (5.9) and (5.10), we now state the following lemma, whose proof is basically provided in [32, Section 4.1], as indicated below.

Lemma 5.1. *There exist positive constants C_a and $C_{\widetilde{a}}$, independent of h, such that*

$$\begin{aligned} \left|(a-a_h)(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)\right| \,&\leq\, C_a\,\|\boldsymbol{\zeta}_h - \mathcal{P}_r^h(\boldsymbol{\zeta}_h)\|_{0,\Omega}\,\|\boldsymbol{\tau}_h\|_{0,\Omega} \qquad \forall\,\boldsymbol{\zeta}_h,\,\boldsymbol{\tau}_h\in\mathbb{X}_h\,, \\ \left|(\widetilde{a}-\widetilde{a}_h)(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h)\right| \,&\leq\, C_{\widetilde{a}}\,\|\boldsymbol{\xi}_h - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\xi}_h)\|_{0,\Omega}\,\|\boldsymbol{\eta}_h\|_{0,\Omega} \qquad \forall\,\boldsymbol{\xi}_h,\,\boldsymbol{\eta}_h\in\mathbf{X}_h\,. \end{aligned} \tag{5.11}$$

In addition, there hold

$$\begin{aligned} \left|(c-c_h)(\mathbf{z};\mathbf{v},\boldsymbol{\tau})\right| \,&\leq\, \nu^{-1}\,\|(\mathbf{z}\otimes\mathbf{v}) - \mathcal{P}_r^h(\mathbf{z}\otimes\mathbf{v})\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega} \quad \forall\,(\mathbf{z},\mathbf{v},\boldsymbol{\tau})\in\mathbf{Y}\times\mathbf{Y}\times\mathbb{X}, \\ \left|(\widetilde{c}-\widetilde{c}_h)(\mathbf{z};\psi,\boldsymbol{\eta})\right| \,&\leq\, \kappa^{-1}\,\|(\psi\,\mathbf{z}) - \boldsymbol{\mathcal{P}}_r^h(\psi\,\mathbf{z})\|_{0,\Omega}\,\|\boldsymbol{\eta}\|_{0,\Omega} \qquad \forall\,(\mathbf{z},\psi,\boldsymbol{\eta})\in\mathbf{Y}\times\mathrm{Y}\times\mathbf{X}. \end{aligned} \tag{5.12}$$

Proof. The first inequalities in (5.11) and (5.12) are established in [32, Lemmas 4.4 and 4.6], whereas the proofs of the second ones, being analogous, are omitted. □

The inequalities given in (5.11) are more useful if their upper bounds are expressed in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, respectively, which is done next. In fact, adding and subtracting $\boldsymbol{\sigma}$ and $\mathcal{P}_r^h(\boldsymbol{\sigma})$, and using from the second identity in (4.5) that $\|\mathcal{P}_r^h(\boldsymbol{\tau})\|_{0,\Omega} \,\leq\, \|\boldsymbol{\tau}\|_{0,\Omega}$ for all $\boldsymbol{\tau}\in\mathbb{L}^2(\Omega)$, we easily find that

$$\|\boldsymbol{\zeta}_h - \mathcal{P}_r^h(\boldsymbol{\zeta}_h)\|_{0,\Omega} \,\leq\, \|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, 2\,\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega}\,,$$

which yields

$$\left|(a-a_h)(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)\right| \,\leq\, C_a\left\{\|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, 2\,\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega}\right\}\|\boldsymbol{\tau}_h\|_{0,\Omega}\,. \tag{5.13}$$

Analogously, adding and subtracting $\boldsymbol{\rho}$ and $\boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})$, and using now the second identity in (4.6), we are able to show that

$$\left|(\widetilde{a}-\widetilde{a}_h)(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h)\right| \,\leq\, C_{\widetilde{a}}\left\{\|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \,+\, 2\,\|\boldsymbol{\rho}-\boldsymbol{\xi}_h\|_{0,\Omega}\right\}\|\boldsymbol{\eta}_h\|_{0,\Omega}\,. \tag{5.14}$$

Furthermore, applying (5.12) to the last terms in (5.9) and (5.10), we obtain

$$\left|(c-c_h)(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)\right| \,\leq\, \nu^{-1}\,\|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega}\,\|\boldsymbol{\tau}_h\|_{0,\Omega}\,, \quad \text{and} \tag{5.15}$$

$$\left|(\widetilde{c}-\widetilde{c}_h)(\mathbf{u};\theta,\boldsymbol{\eta}_h)\right| \,\leq\, \kappa^{-1}\,\|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega}\,\|\boldsymbol{\eta}_h\|_{0,\Omega}\,. \tag{5.16}$$

Consequently, using (5.15) to complete the upper bound in (5.9), bounding $\|\mathbf{u}\|_{\mathbf{Y}}+\|\mathbf{u}_h\|_{\mathbf{Y}}$ by $\delta+\delta_{\mathtt{d}}$, and then employing the resulting estimate along with (5.13) to bound the right hand side of (5.7), we deduce, according to (5.4), that there exists a positive constant C_{A}, depending only on C_a, ν, δ, and $\delta_{\mathtt{d}}$, such that

$$\begin{aligned} \|\big(\mathrm{A}_{\mathbf{u}} - \mathrm{A}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),\cdot\big)\|_{\mathcal{V}_h'} \,&\leq\, C_{\mathrm{A}}\Big\{\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathcal{V}} \,+\, \|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \\ &+\, \|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} \,+\, \|\mathbf{u}\|_{\mathbf{Y}}\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Y}}\Big\}\,. \end{aligned} \tag{5.17}$$

Also, following the same logical sequence as for the derivation of (5.17), but now using (5.16) to complete the upper bound in (5.10), bounding $\|\mathbf{u}\|_{\mathbf{Y}} + \|\mathbf{u}_h\|_{\mathbf{Y}}$ as before, and then using the resulting estimate along with (5.14) to bound the right hand side of (5.8), we deduce, according to (5.5), that there exists a positive constant $C_{\widetilde{\mathrm{A}}}$, depending only on $C_{\widetilde{\mathrm{a}}}$, κ, δ, and $\delta_{\mathtt{d}}$, such that

$$\begin{aligned}\|\big(\widetilde{\mathrm{A}}_{\mathbf{u}} - \widetilde{\mathrm{A}}_{h,\mathbf{u}_h}\big)\big((\boldsymbol{\xi}_h,\psi_h),\cdot\big)\|_{\mathcal{W}_h'} \,\le\, C_{\widetilde{\mathrm{A}}}\Big\{&\|(\boldsymbol{\rho},\theta) - (\boldsymbol{\xi}_h,\psi_h)\|_{\mathcal{W}} \,+\, \|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \\ &+\, \|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega} \,+\, \|\theta\|_{\mathrm{Y}}\,\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Y}}\Big\}\,.\end{aligned} \tag{5.18}$$

In this way, employing (5.6) and (5.17) back into (5.1), we deduce the existence of a positive constant $C_{\mathtt{st},\mathrm{A}}$, depending only on $C_{\mathtt{st}}$, $\|\mathbf{g}\|_{0,\Omega}$, and C_{A}, such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{V}} \,\le\, C_{\mathtt{st},\mathrm{A}}\Big\{&\|\theta - \theta_h\|_{\mathrm{Y}} \,+\, \mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathcal{V}_h\big) \,+\, \|\boldsymbol{\sigma} - \mathbb{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \\ &+\, \|(\mathbf{u}\otimes\mathbf{u}) - \mathbb{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} \,+\, \|\mathbf{u}\|_{\mathbf{Y}}\,\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Y}}\Big\}\,.\end{aligned} \tag{5.19}$$

Analogously, utilizing (5.18) back into (5.2), we deduce the existence of a positive constant $C_{\mathtt{st},\widetilde{\mathrm{A}}}$, depending only on $C_{\mathtt{st}}$ and $C_{\widetilde{\mathrm{A}}}$, such that

$$\begin{aligned}\|(\boldsymbol{\rho},\theta) - (\boldsymbol{\rho}_h,\theta_h)\|_{\mathcal{W}} \,\le\, C_{\mathtt{st},\widetilde{\mathrm{A}}}\Big\{&\mathrm{dist}\big((\rho,\theta),\mathcal{W}_h\big) \,+\, \|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \\ &+\, \|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega} \,+\, \|\theta\|_{\mathrm{Y}}\,\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Y}}\Big\}\,.\end{aligned} \tag{5.20}$$

In order to establish the final Céa estimate, we now simplify the underlying notations by setting

$$\vec{\boldsymbol{\sigma}} \,:=\, (\boldsymbol{\sigma},\mathbf{u})\,,\quad \vec{\boldsymbol{\sigma}}_h \,:=\, (\boldsymbol{\sigma}_h,\mathbf{u}_h)\,,\quad \vec{\boldsymbol{\rho}} \,:=\, (\boldsymbol{\rho},\theta)\,,\quad \vec{\boldsymbol{\rho}}_h \,:=\, (\boldsymbol{\rho}_h,\theta_h)\,,$$

and proceed next to suitably combine (5.19) with (5.20). In fact, multiplying (5.19) by $\frac{1}{2\,C_{\mathtt{st},\mathrm{A}}}$, adding the resulting inequality to (5.20), and bounding $\|\mathbf{u}\|_{\mathbf{Y}}$ and $\|\theta\|_{\mathrm{Y}}$ according to the a priori estimates provided by (2.30), we deduce the existence of positive constants $\mathcal{C}_{\mathtt{st}}$ and $\mathcal{D}_{\mathtt{st}}$, depending only on $C_{\mathtt{st},\mathrm{A}}$ and $C_{\mathtt{st},\widetilde{\mathrm{A}}}$ the former, and additionally on $\mathcal{C}_0$ and $\widetilde{\mathcal{C}}_0$ (cf. (2.30)) the later, such that

$$\begin{aligned}\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathcal{V}} + \|\vec{\boldsymbol{\rho}} - \vec{\boldsymbol{\rho}}_h\|_{\mathcal{W}} \,\le\, \mathcal{C}_{\mathtt{st}}\Big\{&\mathrm{dist}\big(\vec{\boldsymbol{\sigma}},\mathcal{V}_h\big) \,+\, \mathrm{dist}\big(\vec{\boldsymbol{\rho}},\mathcal{W}_h\big) \,+\, \|\boldsymbol{\sigma} - \mathbb{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \\ +\, \|(\mathbf{u}\otimes\mathbf{u}) - \mathbb{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} \,+\, &\|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \,+\, \|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega}\Big\} \\ +\, \mathcal{D}_{\mathtt{st}}\Big\{&\|\mathbf{u}_D\|_{1/2,\Gamma} + (1 + \|\mathbf{g}\|_{0,\Omega})\,\|\theta_D\|_{1/2,\Gamma}\Big\}\,\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Y}}\,.\end{aligned} \tag{5.21}$$

Consequently, we are now in a position to state the Céa estimate.

Theorem 5.2. *Assume that the data satisfy*

$$\mathcal{D}_{\mathtt{st}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + (1 + \|\mathbf{g}\|_{0,\Omega})\,\|\theta_D\|_{1/2,\Gamma}\Big\} \,\le\, \frac{1}{2}\,. \tag{5.22}$$

Then, letting $\mathcal{C} \,:=\, 2\,\mathcal{C}_{\mathtt{st}}$*, there holds*

$$\begin{aligned}\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathcal{V}} + \|\vec{\boldsymbol{\rho}} - \vec{\boldsymbol{\rho}}_h\|_{\mathcal{W}} \,\le\, \mathcal{C}\Big\{&\mathrm{dist}\big(\vec{\boldsymbol{\sigma}},\mathcal{V}_h\big) \,+\, \mathrm{dist}\big(\vec{\boldsymbol{\rho}},\mathcal{W}_h\big) \,+\, \|\boldsymbol{\sigma} - \mathbb{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \\ +\, \|(\mathbf{u}\otimes\mathbf{u}) - \mathbb{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} \,+\, &\|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \,+\, \|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega}\Big\}\,.\end{aligned}$$

Proof. It suffices to employ (5.22) in (5.21), and then bound and simplify. □

The associated rates of convergence are established next.

Theorem 5.3. *Assume that for integers $k \in [0, r+1]$ and $s \in [1, r+1]$ there hold $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{s,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{k,4}(\Omega)$, $\boldsymbol{\rho} \in \mathbf{H}^s(\Omega)$, $\mathrm{div}(\boldsymbol{\rho}) \in \mathrm{W}^{s,4/3}(\Omega)$, $\theta \in \mathrm{W}^{k,4}(\Omega)$, $\mathbf{u} \otimes \mathbf{u} \in \mathbb{H}^k(\Omega)$, and $\theta\,\mathbf{u} \in \mathbf{H}^k(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathcal{V}} + \|\vec{\boldsymbol{\rho}} - \vec{\boldsymbol{\rho}}_h\|_{\mathcal{W}} \,\leq\, C\, h^{\min\{k,s\}} \Big\{ |\boldsymbol{\sigma}|_{s,\Omega} \,+\, |\mathbf{div}(\boldsymbol{\sigma})|_{s,4/3;\Omega} \,+\, |\mathbf{u}|_{k,4;\Omega} \\ +\, |\boldsymbol{\rho}|_{s,\Omega} \,+\, |\mathrm{div}(\boldsymbol{\rho})|_{s,4/3;\Omega} \,+\, |\theta|_{s,4;\Omega} \,+\, |\mathbf{u}\otimes\mathbf{u}|_{k,\Omega} \,+\, |\theta\,\mathbf{u}|_{k,\Omega} \Big\}\,.$$

Proof. Since the local approximation properties provided by (3.2) (cf. Lemma 3.1) extend to their global counterparts when $m = 0$, we deduce from them that

$$\begin{aligned} \|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} &\leq C_r\, h^s\, |\boldsymbol{\sigma}|_{s,\Omega}\,, \\ \|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_r^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} &\leq C_r\, h^k\, |\mathbf{u}\otimes\mathbf{u}|_{k,\Omega}\,, \\ \|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} &\leq C_r\, h^s\, |\boldsymbol{\rho}|_{s,\Omega}\,, \quad \text{and} \\ \|(\theta\,\mathbf{u}) - \boldsymbol{\mathcal{P}}_r^h(\theta\,\mathbf{u})\|_{0,\Omega} &\leq C_r\, h^k\, |\theta\,\mathbf{u}|_{k,\Omega}\,. \end{aligned} \tag{5.23}$$

In this way, the proof follows straightforwardly from Theorem 5.2 along with the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, $(\mathbf{AP}_h^{\boldsymbol{\rho}})$, and $(\mathbf{AP}_h^{\theta})$ (cf. Section 3.2), and the estimates given by (5.23). □

5.2 Computable approximations for σ, ρ and p

The computable approximations of $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ are defined as usual by (see, e.g. [32, Section 5.2])

$$\widehat{\boldsymbol{\sigma}}_h \,:=\, \mathcal{P}_r^h(\boldsymbol{\sigma}_h) \quad \text{and} \quad \widehat{\boldsymbol{\rho}}_h \,:=\, \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho}_h)\,.$$

In turn, as suggested by (2.6) and (2.7), the computable pressure is given by

$$\widehat{p}_h \,:=\, -\frac{1}{2}\left(\mathrm{tr}\,\widehat{\boldsymbol{\sigma}}_h + \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\right) + \frac{1}{2\,|\Omega|}\,\|\mathbf{u}_h\|_{0,\Omega}^2\,.$$

Then, adding and subtracting $\mathcal{P}_r^h(\boldsymbol{\sigma})$ and $\boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})$, and using the boundedness properties of $\mathcal{P}_r^h$ and $\boldsymbol{\mathcal{P}}_r^h$ with respect to the $\mathbb{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ norms, respectively, we readily find that

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,\leq\, \|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,, \tag{5.24}$$

and

$$\|\boldsymbol{\rho} - \widehat{\boldsymbol{\rho}}_h\|_{0,\Omega} \,\leq\, \|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \,+\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega}\,. \tag{5.25}$$

In turn, proceeding as in [30, Theorem 5.5, eqs. (5.38) and (5.39)] (see also [32, eq. 5.14]), we deduce the existence of a positive constant C, independent of h, such that

$$\|p - \widehat{p}_h\|_{0,\Omega} \,\leq\, C\Big\{ \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \Big\}\,. \tag{5.26}$$

In this way, it follows from (5.24), (5.25), and (5.26) that there exists a positive constant $\widehat{C}$, such that

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\rho} - \widehat{\boldsymbol{\rho}}_h\|_{0,\Omega} \,+\, \|p - \widehat{p}_h\|_{0,\Omega} \,\leq\, \widehat{C}\Big\{ \|\boldsymbol{\sigma} - \mathcal{P}_r^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\rho} - \boldsymbol{\mathcal{P}}_r^h(\boldsymbol{\rho})\|_{0,\Omega} \\ +\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \Big\}\,. \tag{5.27}$$

We are able to establish now the rates of convergence of $\widehat{\boldsymbol{\sigma}}_h$, $\widehat{\boldsymbol{\rho}}_h$, and $\widehat{p}_h$.

Theorem 5.4. *Under the same regularity assumptions of Theorem 5.3, there exists a positive constant C, independent of h, such that*

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\boldsymbol{\rho} - \widehat{\boldsymbol{\rho}}_h\|_{0,\Omega} + \|p - \widehat{p}_h\|_{0,\Omega} \le C\, h^{\min\{k,s\}} \Big\{ |\boldsymbol{\sigma}|_{s,\Omega} + |\mathbf{div}(\boldsymbol{\sigma})|_{s,4/3;\Omega}$$
$$+ |\mathbf{u}|_{k,4;\Omega} + |\boldsymbol{\rho}|_{s,\Omega} + |\mathrm{div}(\boldsymbol{\rho})|_{s,4/3;\Omega} + |\theta|_{s,4;\Omega} + |\mathbf{u}\otimes\mathbf{u}|_{k,\Omega} + |\theta\,\mathbf{u}|_{k,\Omega} \Big\}.$$

Proof. It follows from (5.27), the estimates in the first and third rows of (5.23), and Theorem 5.3. □

6 Numerical Results

In this section we present three numerical examples illustrating the performance of the fully mixed virtual element method (3.29), which was introduced and analyzed in Sections 3, 4, and 5. In all the computations we consider the pairs of subspaces $(\mathbb{X}_{0,h}, \mathbf{Y}_h)$ and $(\mathbf{X}_h, \mathrm{Y}_h)$ described in Section 3.2 (cf. (3.15) - (3.17), (3.19), and (3.20)) with polynomial degrees $r \in \{0,1,2\}$, and impose the null mean value of the traces of the tensors in $\mathbb{X}_{0,h}$ via a real Lagrange multiplier. Regarding the results to be reported below, we stress that Example 1 is utilized to confirm the theoretical rates of convergence provided by Theorems 5.3 and 5.4, and Examples 2 and 3 are employed to evaluate the effectiveness of the method in the case of practical problems with no analytical solutions available. We also remark that, in the first example, and because of the use of a manufactured solution, additional known terms are added to the right-hand sides of the equations, whereas in the third one, mixed boundary conditions are assumed for the temperature, so that it is worth mentioning that the analysis of the present paper can be extended with minor modifications to this slightly different case.

We begin by introducing additional notations, Firstly, letting $N_h^{\mathtt{ed}}$ and $N_h^{\mathtt{el}}$ be the number of edges and elements, respectively, of $\mathcal{K}_h$, we find that the total number of degrees of freedom (unknowns) of (3.29), denoted by N_h, is given by

$$N_h := 3(r+1)\, N_h^{\mathtt{ed}} + \frac{3}{2}(r+2)(3r+1)\, N_h^{\mathtt{el}} + 1\,.$$

In addition, the individual errors associated to the main unknowns and the postprocessed pressure are denoted and defined, as usual, by

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \quad \mathsf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \widehat{\boldsymbol{\rho}}_h\|_{\mathrm{div}_{4/3};\Omega},$$
$$\mathsf{e}(\theta) := \|\theta - \theta_h\|_{0,4;\Omega}\,, \quad \text{and} \quad \mathsf{e}(p) := \|p - \widehat{p}_h\|_{0,\Omega}\,.$$

In turn, for all $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta\}$, we let $\mathtt{r}(\star) := \dfrac{\log(\mathsf{e}(\star)/\mathsf{e}'(\star))}{\log(h/h')}$ be the experimental rates of convergence, where h and h' denote two consecutive mesh sizes with errors $\mathsf{e}(\star)$ and $\mathsf{e}'(\star)$, respectively.

The nonlinear algebraic system arising from (3.29) is solved by the Newton method with a given tolerance of 1e-6, which means that the respective iterations are stopped when the ℓ^2-norm of the global incremental discrete solutions drop below that value. In this regard, we notice in advance that three iterations are needed for Examples 1 and 2, while at most seven are required for Example 3.

6.1 Example 1: accuracy assessment

Here we consider the parameters $\nu = 1$, $\kappa = 1$, and $\mathbf{g} = (0,1)$, and the domain $\Omega = (0,1)^2$. In addition, we choose the data so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := \begin{pmatrix} x_1^2 \exp(-x_1)(1+x_2)\,(2\sin(1+x_2) + (1+x_2)\cos(1+x_2)) \\ x_1(x_1-2)\exp(-x_1)(1+x_2)^2\sin(1+x_2) \end{pmatrix},$$

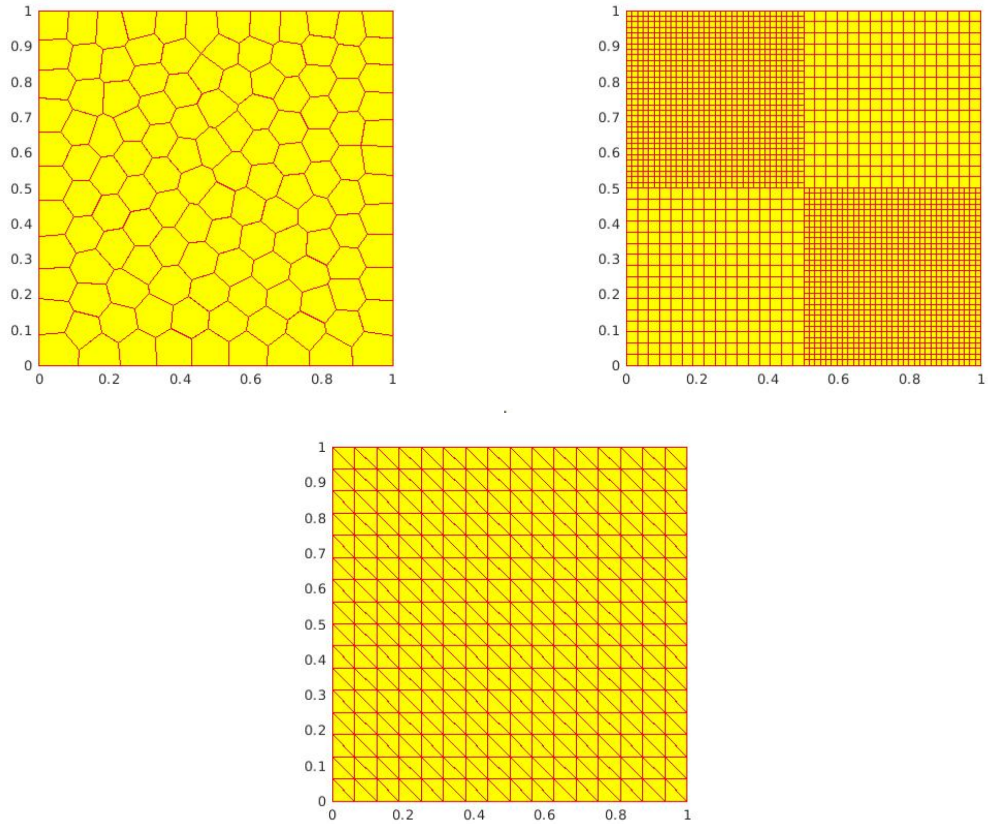

Figure 6.1: Example 1, samples of the kind of meshes employed.

$$p(\mathbf{x}) := \sin(2\pi x_1)\sin(2\pi x_2)\,, \quad \text{and}$$
$$\theta(\mathbf{x}) := \sin(x_1) - \sin(x_2)\,,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$. The computations are performed for each polynomial degree $r \in \{0, 1, 2\}$, by using sequences of successively refined meshes made of hexagons, nonconforming quadrilaterals, and triangles (see Figure 6.1 for a sample of them). In Tables 6.1 up to 6.3 we summarize the convergence history of the fully mixed virtual element method (3.29), from which we realize that, as predicted by Theorems 5.3 and 5.4, the rate of convergence of order $O(h^{r+1})$ is attained by all the unknowns, and for each one of the decompositions of Ω utilized. Furthermore, in order to illustrate the accurateness of the discrete scheme, in Figure 6.2 we display some components of the approximate solution obtained with the polynomial degree $r = 1$ in a mesh made of hexagons.

6.2 Example 2: transient flow passing circular objects

We now use our Boussinesq model for the simulation of transient flow passing one or more circular objects. We take the same parameters from Example 1, and consider the cases described in what follows.

Case i): the domain is given by $\Omega := \Omega_1 \setminus \Omega_2$, where $\Omega_1 := (0, 8) \times (0, 4.5)$ and Ω_2 is the circle with center $(2.25, 2.25)$ and radius 0.25 (see the left-hand picture in Figure 6.3). No-slip velocity condition and temperature $\theta_D = 1$ are enforced on the inner circle boundary $\Gamma_2 := \partial\Omega_2$, whereas the outer

Table 6.1: Example 1, history of convergence using hexagons.

r	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\theta)$	$\mathtt{r}(\theta)$
	1.768e-01	2.620e-01	-	1.362e-01	-	7.335e-01	-	7.531e-02	-	1.795e-01	-
	1.250e-01	1.758e-01	1.150	9.347e-02	1.085	4.538e-01	1.385	5.382e-02	0.969	1.279e-01	0.977
0	8.839e-02	1.224e-01	1.045	5.991e-02	1.283	3.479e-01	0.767	3.565e-02	1.188	8.852e-02	1.061
	6.250e-02	8.021e-02	1.218	4.204e-02	1.022	2.101e-01	1.454	2.460e-02	1.070	6.180e-02	1.036
	4.419e-02	5.382e-02	1.151	2.910e-02	1.061	1.360e-01	1.254	1.718e-02	1.036	4.372e-02	0.998
	3.162e-02	3.666e-02	1.147	2.033e-02	1.072	9.098e-02	1.202	1.218e-02	1.027	3.117e-02	1.010
	2.236e-02	2.539e-02	1.059	1.435e-02	1.004	6.111e-02	1.148	8.538e-03	1.025	2.189e-02	1.020
	1.768e-01	3.005e-02	-	1.172e-02	-	1.125e-01	-	8.251e-03	-	2.975e-03	-
	1.250e-01	1.486e-02	2.032	6.218e-03	1.829	5.460e-02	2.085	4.865e-03	1.524	1.551e-03	1.879
1	8.839e-02	7.258e-03	2.066	2.576e-03	2.542	2.729e-02	2.001	1.940e-03	2.651	7.225e-04	2.204
	6.250e-02	3.605e-03	2.019	1.326e-03	1.916	1.354e-02	2.021	1.079e-03	1.692	3.572e-04	2.032
	4.419e-02	1.779e-03	2.037	6.470e-04	2.070	6.682e-03	2.038	5.300e-04	2.052	1.813e-04	1.955
	3.162e-02	8.768e-04	2.114	3.094e-04	2.203	3.324e-03	2.086	2.547e-04	2.189	9.314e-05	1.990
	2.236e-02	4.361e-04	2.014	1.556e-04	1.982	1.651e-03	2.019	1.303e-04	1.933	4.724e-05	1.958
	1.768e-01	4.864e-03	-	7.582e-04	-	1.938e-02	-	6.167e-04	-	8.048e-05	-
	1.250e-01	1.782e-03	2.897	2.710e-04	2.968	6.928e-03	2.968	2.315e-04	2.826	2.657e-05	3.197
2	8.839e-02	5.921e-04	3.179	7.810e-05	3.589	2.323e-03	3.153	2.698e-05	3.100	8.140e-06	3.413
	6.250e-02	2.110e-04	2.976	2.700e-05	3.064	8.181e-04	3.011	9.158e-06	3.117	2.634e-06	3.255
	4.419e-02	7.338e-05	3.048	9.089e-06	3.141	2.844e-04	3.048	3.182e-06	3.049	9.071e-07	3.075

Table 6.2: Example 1, history of convergence using nonconforming quadrilaterals.

r	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\theta)$	$\mathtt{r}(\theta)$
	1.581e-01	2.992e-01	-	1.457e-01	-	9.611e-01	-	7.586e-02	-	2.030e-01	-
	7.906e-02	1.455e-01	1.039	6.582e-02	1.146	4.535e-01	1.083	3.671e-02	1.047	9.932e-02	1.031
0	3.953e-02	6.395e-02	1.186	3.066e-02	1.102	1.876e-01	1.273	1.768e-02	1.054	4.927e-02	1.011
	1.976e-02	2.827e-02	1.177	1.484e-02	1.046	7.681e-02	1.288	8.662e-03	1.029	2.458e-02	1.003
	1.581e-01	4.756e-02	-	1.508e-02	-	1.760e-01	-	8.852e-03	-	4.020e-03	-
	7.906e-02	1.233e-02	1.947	3.600e-03	2.066	4.575e-02	1.944	2.218e-03	1.996	9.181e-04	2.130
1	3.953e-02	2.990e-03	2.043	8.339e-04	2.110	1.133e-02	2.013	5.390e-04	2.041	2.120e-04	2.114
	1.976e-02	7.289e-04	2.036	2.004e-04	2.057	2.807e-03	2.013	1.319e-04	2.031	5.095e-05	2.056
	1.581e-01	1.063e-02	-	1.418e-03	-	4.361e-02	-	8.250e-04	-	1.720e-04	-
2	7.906e-02	1.393e-03	2.931	1.675e-04	3.081	5.808e-03	2.908	1.005e-04	3.036	1.997e-05	3.106
	3.953e-02	1.695e-04	3.038	1.912e-05	3.131	7.146e-04	3.022	1.168e-05	3.106	2.102e-06	3.247

boundary conditions are $\mathbf{u}_D := (1,0)^{\mathbf{t}}$ and $\theta_D = -1$. Some components of the approximate solution, computed with $r = 0$ in a mesh made of hexagons, are portrayed in Figure 6.4.

Case ii): this transient flow problem involves three circular objects in passing. The domain is considered as in **Case i)**, except that the inner domain Ω_2 is composed by three circles, denoted Ω_3, Ω_4, and Ω_5, with centers $(2.25, 2.25)$, $(4, 1.25)$, and $(4, 3.25)$, respectively, and the same radius 0.25 (see the right-hand picture in Figure 6.3). The boundary conditions applied are as in the previous case. Some components of the numerical solution, computed with $r = 0$ in a mesh made of hexagons, are shown in Figure 6.5. The present simulation suggests that the more the cylinders the more turbulence appears.

Table 6.3: Example 1, history of convergence using triangles.

r	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\theta)$	$\mathtt{r}(\theta)$
	1.768e-01	3.055e-01	-	1.607e-01	-	8.056e-01	-	1.065e-01	-	2.524e-01	-
	8.839e-02	1.597e-01	0.935	6.940e-02	1.211	3.894e-01	1.048	5.421e-02	0.973	1.244e-01	1.021
0	4.419e-02	7.953e-02	1.006	3.283e-02	1.080	1.747e-01	1.156	2.729e-02	0.990	6.193e-02	1.005
	2.210e-02	3.920e-02	1.020	1.615e-02	1.023	7.893e-02	1.146	1.368e-02	0.996	3.093e-02	1.001
	1.768e-01	3.742e-02	-	1.687e-02	-	1.337e-01	-	1.228e-02	-	3.635e-03	-
	8.839e-02	9.651e-03	1.954	3.214e-03	2.392	3.013e-02	2.149	3.388e-03	1.858	6.950e-04	2.386
1	4.419e-02	2.541e-03	1.925	7.169e-04	2.164	7.189e-03	2.067	9.016e-04	1.909	1.578e-04	2.138
	2.210e-02	6.529e-04	1.960	1.731e-04	2.049	1.768e-03	2.023	2.334e-04	1.949	3.838e-05	2.039
	1.768e-01	9.187e-03	-	2.475e-03	-	3.569e-02	-	1.868e-03	-	1.980e-04	-
2	8.839e-02	1.209e-03	2.925	2.038e-04	3.602	4.535e-03	2.976	2.364e-04	2.982	1.998e-05	3.309
	4.419e-02	1.515e-04	2.995	1.972e-05	3.370	5.673e-04	2.998	2.958e-05	2.998	2.347e-06	3.089

6.3 Example 3: natural convection of nanofluid flow in non-convex domains

Here we study the behavior of hybrid nanofluid flow in a non-convex domain with heated walls. This phenomenon has been widely studied with different types of boundary conditions (see, e.g. [4, 28, 34]). In particular, here we are interested in the problem with dimensionless numbers: Find $(\mathbf{u}, p, \theta)$ such that

$$\begin{aligned} -\mathrm{Ra}\ \boldsymbol{\Delta}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\mathbf{u} + \nabla p - \mathrm{Pr}\ \mathrm{Ra}\ \theta\mathbf{g} = \mathbf{0} &\qquad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 &\qquad \text{in } \Omega, \\ -\Delta\theta + \mathbf{u}\cdot\nabla\theta = 0 &\qquad \text{in } \Omega, \end{aligned}$$

where Pr and Ra are the Prandtl and Rayleigh numbers and $\mathbf{g} = (0,1)^{\mathtt{t}}$. The numerical experiments are performed with $r = 0$ for the H-shaped domain Ω displayed in Figure 6.6, considering the mixed boundary conditions for the energy equation that are described next. The left and right vertical walls are kept hot and cold, respectively, with $\theta_D = 1$ and $\theta_D = 0$. Rest of the walls of the domain are adiabatic, which means that we impose there $\boldsymbol{\rho}\cdot\mathbf{n} = 0$. For the flow equations, all boundaries are equipped with no-slip velocity conditions. In addition, we fix the Prandtl number as $\mathrm{Pr} = 1$, and consider three choices, namely small, medium, and large, for the Rayleigh number, which are given by $\mathrm{Ra} = 1$, $\mathrm{Ra} = 1\mathrm{e}4$, and $\mathrm{Ra} = 1\mathrm{e}5$, respectively. In Figure 6.7 we display the approximated stress, velocity, and heat-flux vector magnitudes, along with the approximated pressure and temperature (from top to bottom, respectively), computed in a mesh made of hexagons with $h = 1.58\mathrm{e}\text{-}2$, for the aforementioned values of the Rayleigh number (first to third columns). As can be seen from Figure 6.7 (first column), the flow pattern for the small value of Ra shows a butterfly-like shape inside the domain with a dumbbell-like shape situated in the center. In turn, when Ra increases from 1 to 1e4, and then to 1e5, the flow begins to wander toward the direction of vertical walls spaced apart from one another, and getting closer to the hot wall, as expected.

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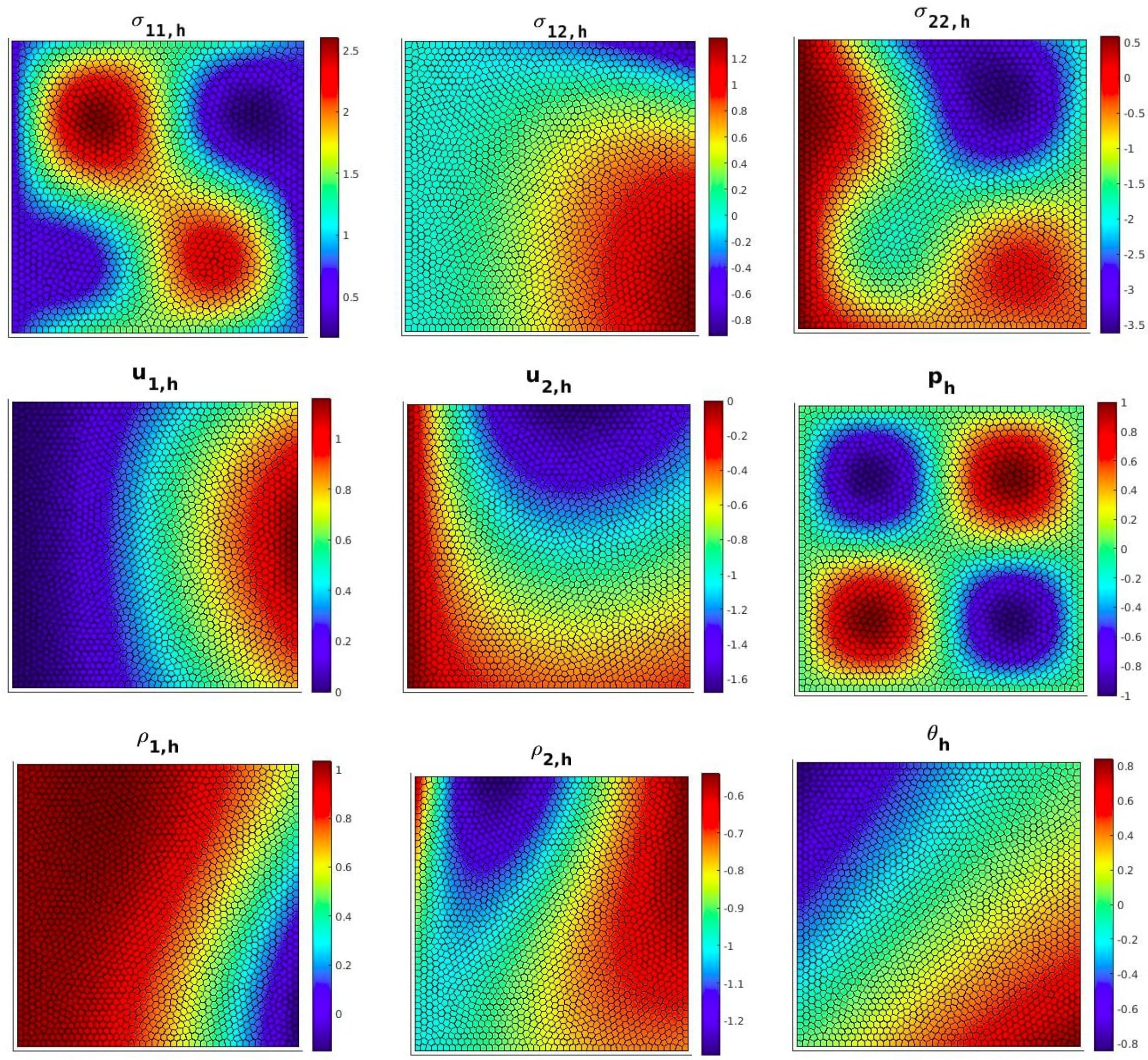

Figure 6.2: Example 1, snapshots of the first, second and fourth components of numerical stress (first row, left to right), two components of velocity and pressure (second row, left to right), and two components of heat flux and temperature (third row, left to right), computed with $r = 1$ in the mesh made of hexagons with $h = 2.236\text{e-}2$.

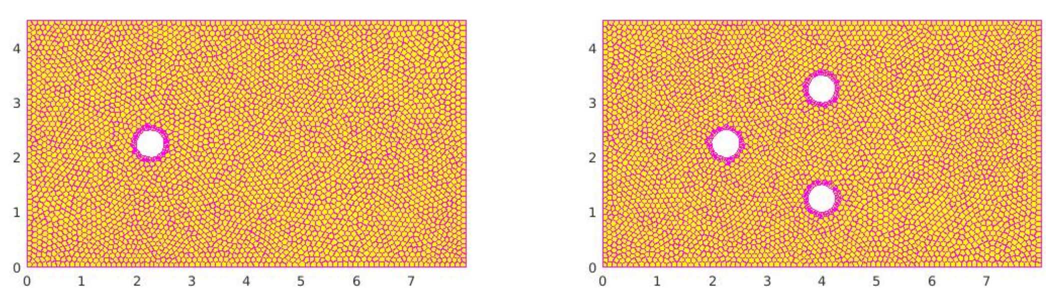

Figure 6.3: Example 2, the domains of cases i) and ii) with meshes made of hexagons.

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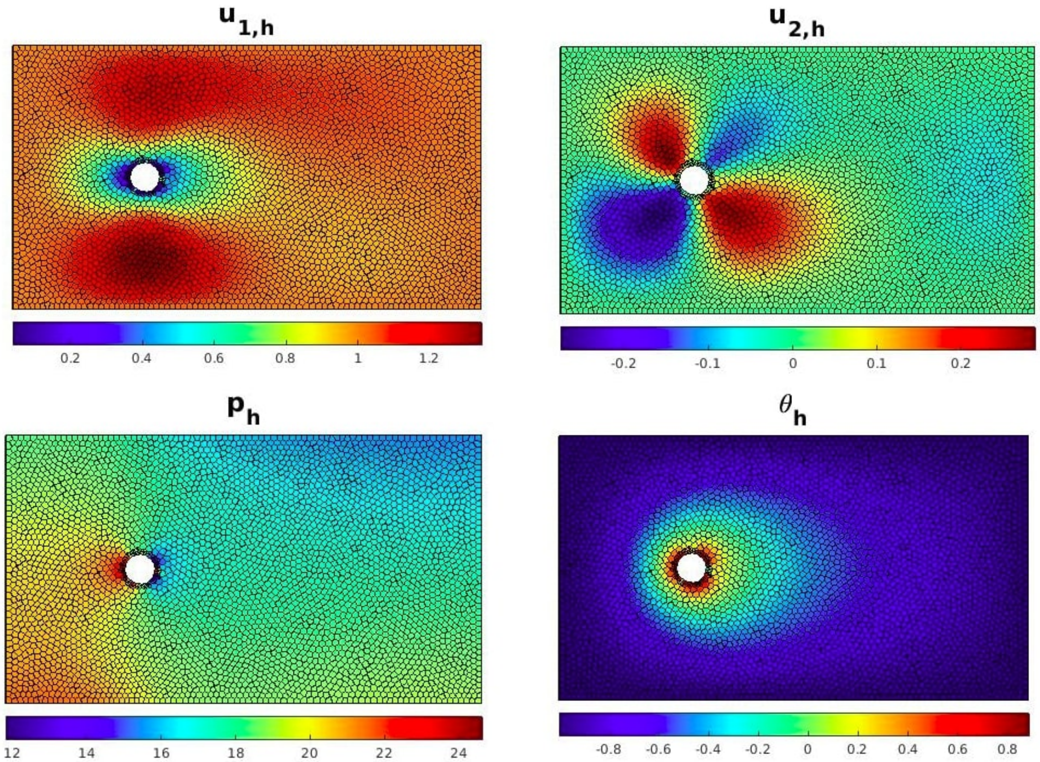

Figure 6.4: Example 2, Case i), snapshots of the first and second components of velocity (first row), and pressure and temperature (second row), computed with $r = 0$ in a mesh made of hexagons with $h = 1.56\text{e-}2$.

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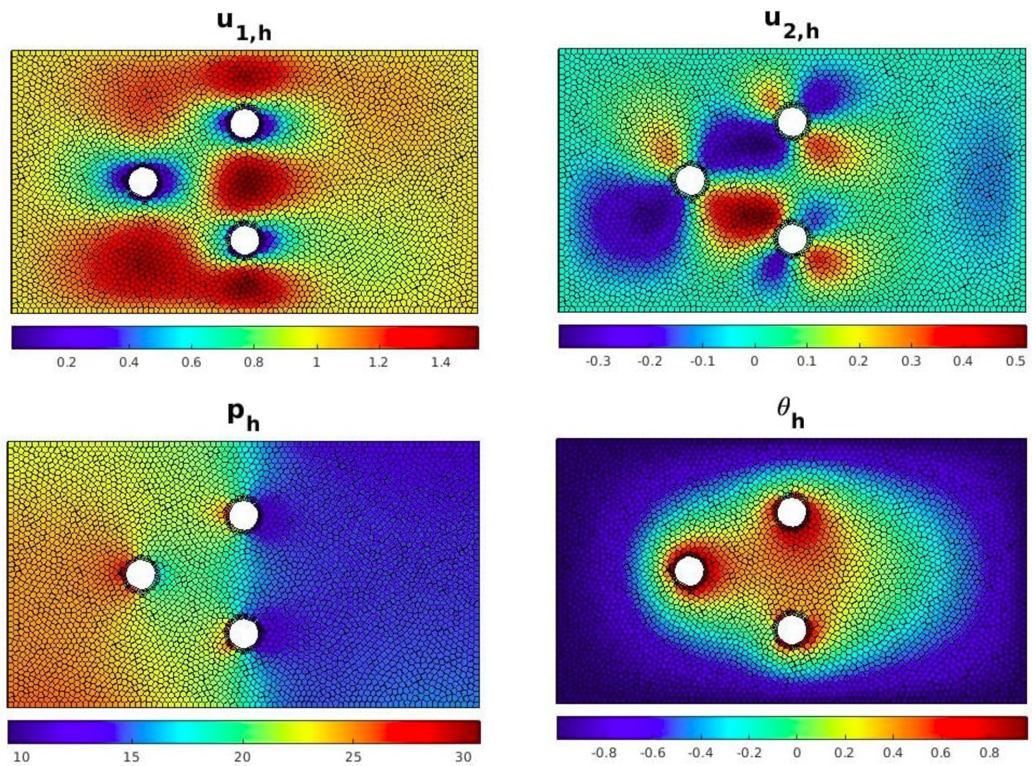

Figure 6.5: Example 2, Case ii), snapshots of the first and second components of velocity (first row), and pressure and temperature (second row), computed with $r = 0$ in a mesh made of hexagons with $h = 1.54\text{e-}2$.

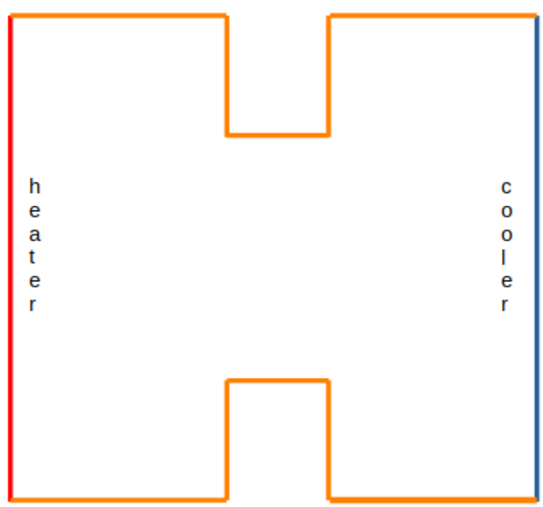

Figure 6.6: Example 3, domain of the model.

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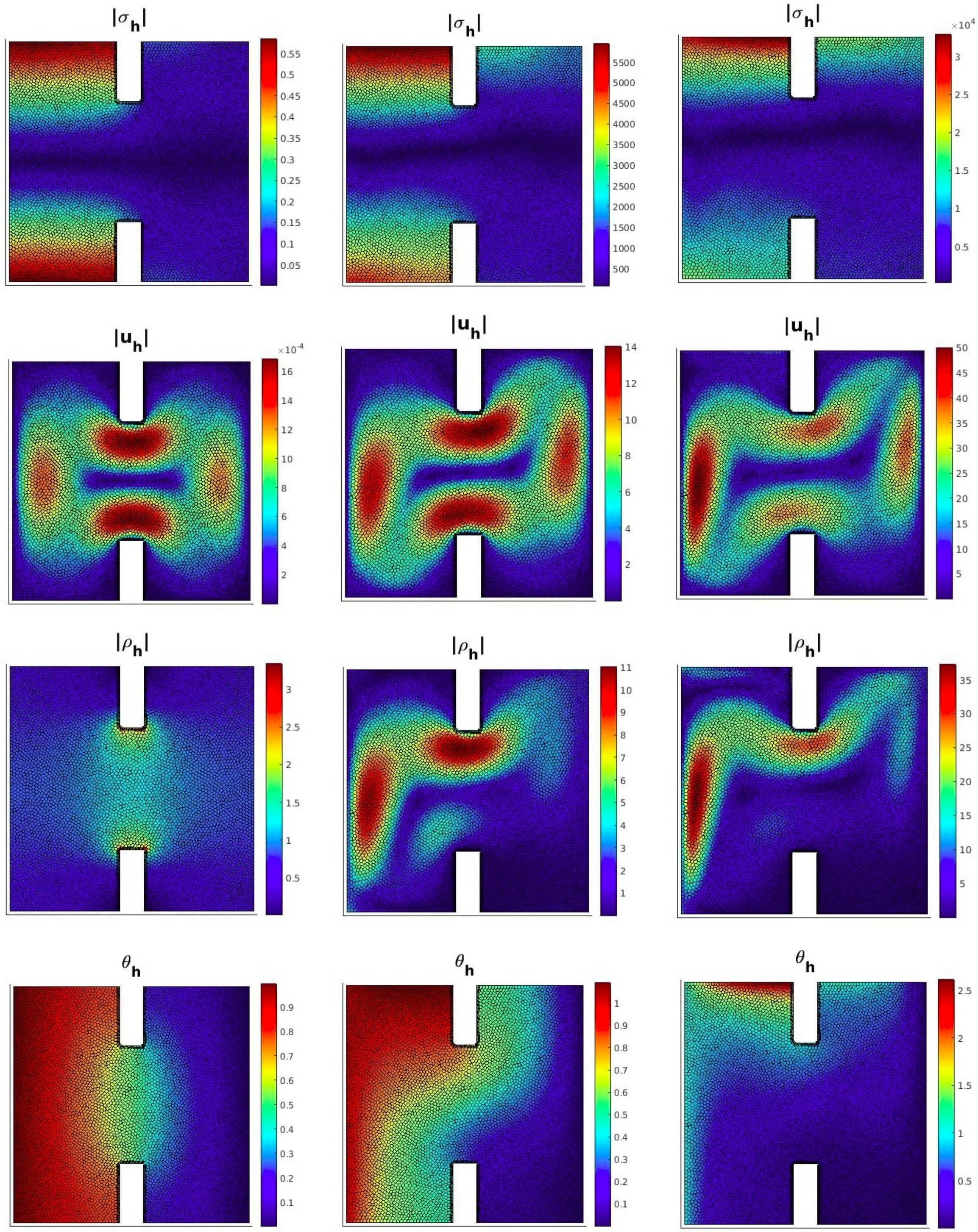

Figure 6.7: Example 3, snapshots of stress, velocity and heat-flux vector magnitudes, along with the temperature (from top to bottom, respectively) for values Ra = 1, 1e4, 1e5 (first to third columns, respectively), computed with $r = 0$ in a mesh made of hexagons with $h = 1.58\text{e-}2$.

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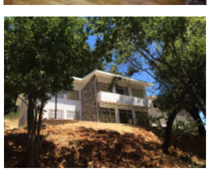
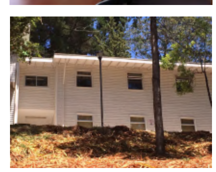
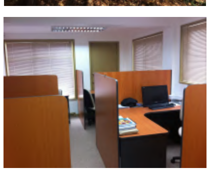
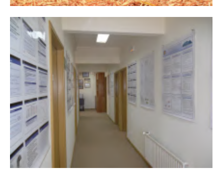

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