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A short journey into the realm of numerical methods for contact in elastodynamics

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Introduction

The main objective of these lecture notes is to provide basic notions about the finite element discretization of frictionless contact in elastodynamics, to point out where the main problems may occur, and what are the possible remedies. They suppose a basic knowledge of elastodynamics and of the finite element method. It is also recommended to have some notions about unilateral contact in elastostatics (the so-called Signorini problem). They cover only frictionless contact, for the sake of simplicity.

Contents

1 Elastic membrane with dynamic contact

Let us introduce a simple model of (frictionless) dynamic contact. For this purpose we need first a domain Ω, that is an open subset of $\mathbb{R}^d$, $d \geq 1$ (in practice, $d = 1, 2$ or 3). We suppose that Ω is non-empty and bounded. We denote by $\Gamma := \partial\Omega$ the boundary of Ω. For $d = 1$ we take Ω as a finite union of open intervals. For $d = 2, 3$, the boundary Γ is supposed polytopal. The boundary is partitionned into three parts: Γ_D (Dirichlet boundary), Γ_N (Neumann boundary) and Γ_C (contact boundary).

1.1 Problem in strong form

We want to solve the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \, \Delta u = f \quad \text{in } \mathbb{R}_*^+ \times \Omega, \tag{1}$$

complemented with the following boundary conditions:

$$u = 0 \quad \text{on } \mathbb{R}_*^+ \times \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}_*^+ \times \Gamma_N, \tag{2}$$

and

$$u \leq 0, \quad \frac{\partial u}{\partial n} \leq 0, \quad u\frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}_*^+ \times \Gamma_C, \tag{3}$$

as well as initial conditions:

$$u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = v_0 \quad \text{in } \Omega. \tag{4}$$

Let us now spend some time to present the protagonists and explain the notations. First the solution is denoted by u and is a scalar field:

$$u : \mathbb{R}^+ \times \overline{\Omega} \to \mathbb{R}.$$

We will denote by $u(t, x)$ its (scalar) value at time $t \geq 0$ and at point $x \in \overline{\Omega}$. The Laplace operator is denoted by Δ. The notation $\frac{\partial}{\partial t}$ is for the partial derivative in time and $\frac{\partial^2}{\partial t^2}$ is for the second-order partial derivative in time. The normal derivative is denoted by $\frac{\partial}{\partial n}$ and is defined as:

$$\frac{\partial}{\partial n} := n \cdot \nabla$$

where ∇ is the gradient operator. The source term in (1) is denoted f ($f : \mathbb{R}_*^+ \times \Omega \to \mathbb{R}$). The coefficient $c > 0$ in front of the Laplace operator in (1) is the wave speed.

We need two initial conditions, and we suppose the displacement is known at the initial time $t = 0$, and equal to u_0, a given function. The same happens for the velocity, prescribed using the function v_0.

Conditions (2) are standard (homogeneous) Dirichlet and Neumann boundary conditions. Conditions (3) on the contact boundary Γ_C are Signorini conditions (see, *e.g.*, [14, 16, 17, 25, 29]). In dimension $d = 2$, Problem (1)–(2)–(3)–(4) can represent the motion in time of a thin elastic membrane, with contact that can occur, or not, on Γ_C (the membrane can not penetrate a prescribed obstacle). In dimension $d = 1$, Problem (1)–(2)–(3)–(4) may represent an elastic bar, that cannot penetrate the rigid ground. To represent more complex situations, for instance three-dimensional elasticity, the Laplace operator needs to be replaced by, for instance, the small strain elasticity operator. We will see later on that, despite the above setting is simplified, it has retained all the worse numerical difficulties that occur in more complex or realistic situations.

This simplified model of contact in elastodynamics and needs to be differentiated from rigid body contact dynamics, where collisions may occur between rigid particles. In this latter situation, the model and the numerical issues are not the same.

For the purpose, it will be convenient to adopt the following convention, that will allow us to see the space-time field u as a function of time, only:

$$u : t \mapsto u(t) \in \overline{\Omega}$$

so we can say that

$$u(t)(x) = (u(t))(x) = u(t, x).$$

1.2 Well-posedness and energy conservation

We recall that for the Signorini problem that represents frictionless static contact, well-posedness is a standard result, coming from the Lions-Stampacchia theory [33] that emerged in the 60s [7, 29]. For contact conditions in the dynamic setting, well-posedness is much more difficult to establish, and, in fact, is still an open problem in a general setting.

However, in dimension one ($d = 1$) existence and uniqueness of the solution to Problem (1)–(2)–(3)–(4) has been proven in details in [10] (see also [9, 15, 30, 32] for other results, in more general settings).

For our purpose, let us just establish, formally, a result of energy conservation for Problem (1)–(2)–(3)–(4). We consider a fixed moment in time $t > 0$ and set $f = 0$ to simplify. We start from equation (1) and multiply each side by $\frac{\partial u}{\partial t}$:

$$\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} - c^2\ (\Delta u)\frac{\partial u}{\partial t} = 0.$$

We use the identity

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = 2\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}$$

to get

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 - c^2\ (\Delta u)\frac{\partial u}{\partial t} = 0.$$

We integrate over the whole domain Ω:

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 - c^2 \int_\Omega (\Delta u)\frac{\partial u}{\partial t} = 0.$$

We then apply the Green formula to the second term:

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 + c^2 \int_\Omega \nabla u \cdot \nabla\left(\frac{\partial u}{\partial t}\right) - c^2 \int_\Gamma (n\cdot\nabla u)\left(\frac{\partial u}{\partial t}\right) = 0. \qquad (5)$$

The first term in (5) can be transformed as

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2.$$

For the second term in (5) let us rewrite it as follows:

$$c^2\int_\Omega \nabla u\cdot\nabla\left(\frac{\partial u}{\partial t}\right) = c^2\int_\Omega \nabla u\cdot\left(\frac{\partial \nabla u}{\partial t}\right) = \frac{c^2}{2}\int_\Omega \left(\frac{\partial \|\nabla u\|^2}{\partial t}\right) = \frac{c^2}{2}\frac{d}{dt}\int_\Omega \|\nabla u\|^2,$$

where $\|\nabla u\|$ denotes the euclidean norm in $\mathbb{R}^d$. The last term in (5), which is the boundary term is split according to the partition into Γ_D, Γ_N and Γ_C and we apply conditions (2) to get:

$$-c^2\int_\Gamma (n\cdot\nabla u)\left(\frac{\partial u}{\partial t}\right) = -c^2\int_{\Gamma_C}(n\cdot\nabla u)\left(\frac{\partial u}{\partial t}\right).$$

Now it is time to introduce the energy associated to Problem (1)–(2)–(3)–(4) as:

$$E(t) := \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial t}(t)\right)^2 + \frac{c^2}{2}\int_\Omega \|\nabla u(t)\|^2 \qquad (6)$$

for $t \geq 0$. People from physics are used to say that this is the sum of a kinetic energy and an elastic energy, but we will not enter into such subtelties here. We will just observe that (5) can be rewritten as

$$\frac{d}{dt}E(t) - c^2\int_{\Gamma_C}(n\cdot\nabla u)\left(\frac{\partial u}{\partial t}\right) = 0. \qquad (7)$$

We finally use the persistency condition

$$(n\cdot\nabla u)\left(\frac{\partial u}{\partial t}\right) = 0 \qquad (8)$$

to conclude from (7) that

$$\frac{d}{dt}E(t) = 0.$$

And finally, by integration in time:

$$E(t) = E(0), \qquad (9)$$

for $t \geq 0$, with $E(0)$ that can be expressed as a function of the initial conditions (4).

If you have been attentive up to now, you have surely noticed that the persistency condition (8) is something nontrivial, that does not derive straightforwardly from Signorini conditions (3). In fact, some extra effort needs to be done to establish them rigorously, even in simple situations, like in one dimension, see [10]. For the moment, we can simply take it as an assumption. Equation (9) traduces then that the contact forces do not produce work, or do not dissipate energy, so that the energy of the elastic bar or elastic membrane is exactly preserve. This is a remarkable property of this system, that we should try to preserve while discretizing. Particularly, this will help in establishing stable discretizations.

2 A few notions about finite elements

There are many ways to discretize Problem (1)–(2)–(3)–(4). Here we follow the most usual one, sometimes called the method of lines. First we discretize the spatial domain, here with finite elements, to obtain an evolution equation in t, for each nonnegative real value of t, and where the approximated unknown (here the displacement) lives in a vector space of finite dimension. This yields a semi-discrete problem in space, for which we will apply later on a time-marching scheme, in order to be able to find an approximate solution on a computer.

Various approximation techniques have been invented from the end of the nineteen century, so as to calculate effective approximate solutions. Among the most popular methods are the *finite difference method*, the *finite element method*, the *finite volume method* and the *spectral method* (but they are many many others in fact, and many more each passing decade). An introductive presentation to such methods is given for instance in the book of Martin Gander and Felix Kwok [18]. For references related to contact and friction in elastodynamics and variational, let us mention for instance [7, 19, 21, 22, 23, 29, 31, 36, 37].

The first step,is to decompose the domain Ω into small, simple cells. This process, in the FEM terminology, is called, *meshing*. So we introduce a *simplicial mesh* as a collection of simplices in $\mathbb{R}^d$ that partition the domain Ω. We denote by K a given simplex in the mesh and denote by $\mathcal{K}_h$ the collection of all the simplices in Ω (we will precise in a few minutes the meaning of the index h). The mesh needs to satisfy two conditions:

1. It should cover exactly the domain Ω:

$$\overline{\Omega} := \bigcup_{K \in \mathcal{K}_h} K,$$

   where the simplices K are supposed to be closed.

2. The intersection of two distinct simplices K and K' is either empty, or a simplex of dimension lower than d, that is common to both K and K'.

For each simplex K in the mesh, h_K denotes the maximal distance between two points inside K (it is the 'size' of K), and

$$h := \max_{K \in \mathcal{K}_h} h_K$$

is the global *size* of the mesh.

Before going further, let us be more explicit about these notions for problems in one or two dimensions. First in one dimension ($d = 1$) and for Ω that is an open interval, the different simplices $K_1, \ldots, K_{N+1}$ are simply closed intervals, that cover Ω, and intersect each other only at both ends. We denote

$$x_i = K_i \cap K_{i+1}$$

the *nodes* of the mesh. Now, for problems in two dimensions, $d = 2$, simplices K are triangles. The intersection between two triangles is either empty, or a common vertex, or a common edge. The vertices of the triangles are also called the *nodes* of the mesh.

Now, from the mesh $\mathcal{K}_h$, we will build a finite dimensional space that we will call V_h, as follows. We define, now for every value of the dimension d, the nodes of the mesh as the vertices of the simplices. Let us denote them by $x_1, \ldots, x_N$. Let us define a family of *basis functions*

$$\varphi_1, \ldots, \varphi_N$$

such that

1. Each φ_i is globally continuous on $\overline{\Omega}$ and its restriction to each simplex K is a piecewise affine function (a polynomial function of degree at most 1):

   $$\varphi_i|_K \in \mathbb{P}_1(K),$$

   where $\mathbb{P}_1(K)$ is the space of (multivariate) polynomials of degree 1.

2. Each φ_i is equal to 1 at node x_i, and equal to 0 for the other nodes:

   $$\varphi_i(x_j) = \delta_{ij},$$

   where δ_{ij} is the Kronecker symbol.

From the two above properties, one can check directly that each φ_i has a *compact support* located on the *patch* ω_i of simplices that share the same vertex x_i. So this function vanishes everywhere except in simplices that are in the neighbourhood of the node x_i.

One can also check that the functions (φ_i) are linearly independent. Then we set

$$V_h := \operatorname{span}(\varphi_1, \ldots, \varphi_N).$$

In dimension 1, (φ_i) is the so-called family of hat functions, and in dimension 2, each φ_i is a pyramid of basis ω_i and height 1. One can also check that V_h

is the space of continuous functions that are affine on each simplex K. The terminology 'Lagrange' finite elements comes from the fact that it relies on piecewise polynomial Lagrange interpolation between the nodes of the mesh. In the sequel, we denote by V_h^D the subspace of V_h made of functions that vanish on the Dirichlet boundary Γ_D:

$$V_h^D := \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_D\}.$$

We will denote by $N_D (\leq N)$ the dimension of this subspace. Moreover, let $u_h(t) \in V_h^D$ be the approximated solution to the dynamic contact Problem (1)–(2)–(3)–(4) at time $t > 0$. We will drop the notation (t) and note $u_h \in V_h^D$ in the following. We write

$$u_h = \sum_{j=1}^{N_D} U_j \varphi_j$$

and set $\mathbf{U}$ the column vector of size N_D that contains all the scalar unknowns (U_j). Remark, that, due to the definition of the basis (φ_i), we get the relationship

$$U_j = u_h(x_j),$$

for any index j. We say that the unknowns are *nodal* unknowns, since they simply represent the value of u_h at interior nodes.

3 Mixed and modified mass methods

Mixed methods are common to discretize contact problems [7, 22, 36] and more generally partial differential equations under constraints. For the Signorini problem (contact of an elastic body in the static case), they lead to stable and optimally convergent discretizations provided a discrete infsup condition is satisfied. So one can hope that such a behavior still holds in the dynamic case but it is not the case. Indeed, when Problem (1)–(2)–(3)–(4) is discretized with a mixed method, an ill-posed differential inclusion is obtained (see for instance [28]) with possibly an infinite number of solutions after each impact. As a result, when a time-marching scheme is applied, it tends to select with its own mood a solution among those possible, which leads ta an awkward numerical approximation. Particularly spurious oscillations contaminate the solution, artificial energy can be created, and what is worse, the numerical solution gets poorer if the time-step is reduced [5, 6, 10, 11, 13, 28].

More precisely, we can follow [28] and [27, Chapter 7]. We take the simple case where $d = 1$ and an elastic bar of length 1 is clamped on one extremity and is subjected to contact (Signorini) conditions on the other. Suppose that the corresponding elastic bar is discretized with only one element ($h = 1$!). In this case a simple mixed formulation yields:

$$m\ddot{U} + kU = \Lambda, \quad \Lambda = [rU - \Lambda]_+, \ \forall r > 0, \tag{10}$$

complemented with initial conditions, and where U (displacement) and Λ (contact stress) are the scalar unknowns. The physical parameter m is the mass associated to the node, and k is the rigidity of the bar element (that acts like a spring in this case). The notation $[\cdot]_+$ stands for the positive part operator:

$$[x]_+ = \max(0, x),$$

for $x \in \mathbb{R}$. Then in [27] follows an explicit calculation, from which is deduced that the discrete problem (10) admits an infinity of solutions.

A first remedy may consists in getting inspired from rigid body dynamics and introduce a restitution coefficient, as done in the Paoli-Schatzman scheme for instance [34, 35]. The difficulty that persists here is to find an ideal value of the restitution coefficient, that has no clear physical meaning here (it is not necessary to get well-posedness of the elastodynamic problem).

The modified mass technique, originated in [28] consists in removing the mass associated with the nodes on the contact boundary Γ_C in the mass matrix $\mathbf{M}$. Indeed we observe in (10) that setting $m = 0$ allows to recover a well-posed problem (static contact). The modified mass method leads to a well-posed problem and allows to conserve a modified energy [28]. Moreover it improves drastically the behavior of the computed solution in practice, even with conservative time-marching schemes, and allows numerical stability and reduced spurious oscillations, that vanish when the discretization gets finer [6, 13]. A theoretical convergence result and some numerical experiments can be found in [10, 11], Many variants of this technique can be found now. Let us turn now to the case of penalized contact.

4 Penalized contact

Basically, and in its simpler form, penalized contact consists in approximating the Signorini conditions (3) by the following (nonlinear) equation:

$$\frac{\partial u}{\partial n} = -\frac{1}{\epsilon}[u]_+, \tag{11}$$

where $[\cdot]_+$ is (still) the positive part and $\epsilon > 0$ the penalty parameter [29]. Roughly speaking, it penalizes the penetration into the rigid support. It is popular because simple and cheap. Moreover, it can be easily combined with an explicit time-marching scheme.

4.1 Semi-discrete problem in time

The semi-discrete in space penalized formulation for the dynamic contact Problem (1)–(2)–(3)–(4) consists in finding a semi-discrete displacement field

$$u_h : t \mapsto u_h(t) \in V_h^D$$

for any $t > 0$ that is solution to

$$\frac{d^2}{dt^2}\int_\Omega u_h(t)v_h + c^2\int_\Omega \nabla u_h(t)\cdot\nabla v_h + \frac{1}{\epsilon}\int_{\Gamma_C}[u_h(t)]_+ v_h = \int_\Omega f(t)v_h,\ t>0, \quad (12)$$

for any $v_h \in V_h^D$. This needs to be complemented by initial conditions that approximate (4) and can be written as

$$u_h(0) = u_{h0}, \quad \frac{du_h}{dt}(0) = v_{h0}. \quad (13)$$

The initial condition $u_{h0} \in V_h$, respectively $v_{h0} \in V_h$, is an approximation of the exact initial displacement u_0, respectively velocity v_0, that can be obtained, for instance, using Lagrange interpolation.

To build the formulation (12), we started from the heat equation (1) and the Green formula. Then we took into account the boundary conditions (2) and the penalized formulation (11) on the contact boundary. Finally, we made use of the Lagrange finite element space V_h, defined in 2, and built from a mesh $\mathcal{K}_h$ of the domain Ω. In practice, the parameter ϵ is defined as

$$\epsilon = \epsilon_0 h,$$

with ϵ_0 user-defined, and set large enough to mimic the contact condition. The scaling as $\mathcal{O}(h)$ is needed for the convergence of the method when h vanishes [3, 29].

Using the Riesz representation theorem, the penalized formulation (12) can be recasted as

$$\mathbf{M}\frac{d^2}{dt^2}\mathbf{U}(t) + \mathbf{B}_\epsilon\mathbf{U}(t) = \mathbf{F}(t), \quad (14)$$

where $\mathbf{B}_\epsilon$ is a nonlinear, Lipschitz operator (since the positive part operator is Lipschitz). Thus, insteady of the ugly, ill-posed, differential inclusion that popped out of the mixed formulation, we obtain here a friendly finite dimensional Lipschitz system of ordinary differential equations, well-posed thanks to the Cauchy-Lipschitz (Picard-Lindelof) theorem.

A second advantage is that a modified energy can be preserved. Indeed, let us define

$$E_h(t) := \frac{1}{2}\int_\Omega \left(\frac{du_h}{dt}(t)\right)^2 + \frac{c^2}{2}\int_\Omega \|\nabla u_h(t)\|^2$$

as a semi-discrete counterpart of E and

$$E_{h,\epsilon}(t) := E_h(t) + \frac{1}{2\epsilon}\int_{\Gamma_C}[u_h(t)]_+^2$$

a modified energy that, roughly speaking, store the work associated to (artificial) penetration. We prove now the following result of (perturbed) energy conservation.

Proposition 1 *Suppose that f is identically equal to zero, then the following energy estimate holds, for all $t \geq 0$:*

$$E_{h,\epsilon}(t) = E_{h,\epsilon}(0) \tag{15}$$

where the initial energy is given as a function of the initial displacement and velocity by

$$E_{h,\epsilon}(0) = \frac{1}{2}\int_\Omega (v_{h0})^2 + \frac{c^2}{2}\int_\Omega \|\nabla u_{h0}\|^2 + \frac{1}{2\epsilon}\int_{\Gamma_C} [u_{h0}]_+^2.$$

Proof. Let u_h be the solution to the penalized problem (12). For an arbitrary $t > 0$, we proceed in the same fashion as in Section 1 and obtain:

$$\frac{d}{dt}E_h(t) + \frac{1}{\epsilon}\int_{\Gamma_C} [u_h(t)]_+ \frac{d}{dt}u_h(t) = 0.$$

Then we use the identity

$$[u_h(t)]_+ \frac{d}{dt}u_h(t) = \frac{1}{2}\frac{d}{dt}[u_h(t)]_+^2$$

and this ends the proof. □

Remark that in the above result, there is no strictly speaking a persistency condition as in the continuous problem, which made the things a little bit more difficult. Of course, the term that appears on the contact boundary mimics a persistency condition when ϵ is small but is much easier to handle. We are now ready to discretize in time the penalized formulation.

4.2 Midpoint for penalized contact

Let us have a look now at discretization in time with the midpoint scheme, well-suited for elastodynamics: for standard boundary conditions, we know it conserves the energy, is unconditionally stable and second-order accurate. We take a time-step $\tau > 0$, and introduce a sequence of (fully discrete) displacements (u_h^n), such that, for each n :

$$u_h^n \simeq u_h(t_n),$$

where $t_n = n\tau$. We introduce identically the notation $\dot{u}_h^n$ for the approximated velocity and $\ddot{u}_h^n$ for the approximated acceleration. It will be convenient as well to use the following notation:

$$u_h^{n+\frac{1}{2}} = \frac{1}{2}(u_h^n + u_h^{n+1})$$

The midpoint scheme applied to the penalized formulation (12) reads, for $n \geq 0$:

$$\int_\Omega \ddot{u}_h^{n+\frac{1}{2}} v_h + c^2\int_\Omega \nabla u_h^{n+\frac{1}{2}} \cdot \nabla v_h + \frac{1}{\epsilon}\int_{\Gamma_C} [u_h^{n+\frac{1}{2}}]_+ v_h = \int_\Omega f(t_{n+\frac{1}{2}}) v_h \tag{16}$$

for $v_h \in V_h$, and along with the updates

$$u_h^{n+1} = u_h^n + \tau \dot{u}_h^{n+\frac{1}{2}} \quad (17)$$

$$\dot{u}_h^{n+1} = \dot{u}_h^n + \tau \ddot{u}_h^{n+\frac{1}{2}} \quad (18)$$

and with initial conditions $u_h^0 = u_{h0}$, $\dot{u}_h^0 = v_{h0}$ and $\ddot{u}_h^0 = a_{h0}$ where a_{h0} is a given acceleration.

An argument for hemicontinuous strongly monotone operator [1] allows to ensure that the fully discrete problem (16) is well-posed whatever the values of $\epsilon > 0$ and $\tau > 0$ are, see [4, 5]. It can be solved then, at each time-step, with, for instance, a semi-smooth Newton method.

Let us now examine if energy preservation and numerical stability can be achieved for such a problem. For this purpose, let us introduce energies associated to the fully discrete solution:

$$E_h^n := \frac{1}{2}\int_\Omega (u_h^n)^2 + \frac{c^2}{2}\int_\Omega \|\nabla u_h^n\|^2$$

as a fully discrete counterpart of E_h and

$$E_{h,\epsilon}^n := E_h^n + \frac{1}{2\epsilon}\int_{\Gamma_C} [u_h^n]_+^2.$$

Let us now check if there can be energy preservation. We choose

$$v_h = \tau \dot{u}_h^{n+\frac{1}{2}} \in V_h$$

in (16) to compute the difference $E_{h,\epsilon}^{n+1} - E_{h,\epsilon}^n$ and get:

$$\begin{aligned} & E_{h,\epsilon}^{n+1} - E_{h,\epsilon}^n \\ = \; & -\frac{1}{\epsilon}\int_{\Gamma_C} [u_h^{n+\frac{1}{2}}]_+ \tau \dot{u}_h^{n+\frac{1}{2}} + \frac{1}{2\epsilon}\int_{\Gamma_C} [u_h^{n+1}]_+^2 - \frac{1}{2\epsilon}\int_{\Gamma_C} [u_h^n]_+^2. \end{aligned}$$

Let us have a closer look at the first term. It can be rewritten using (17):

$$-\frac{1}{\epsilon}\int_{\Gamma_C} [u_h^{n+\frac{1}{2}}]_+ \tau \dot{u}_h^{n+\frac{1}{2}} = -\frac{1}{\epsilon}\int_{\Gamma_C} [u_h^{n+\frac{1}{2}}]_+ (u_h^{n+1} - u_h^n).$$

Now, if we discuss the two cases $u_h^{n+\frac{1}{2}} > 0$ and $u_h^{n+\frac{1}{2}} \leq 0$, this allows to get

$$\begin{aligned} & E_{h,\epsilon}^{n+1} - E_{h,\epsilon}^n \\ = \; & \frac{1}{2\epsilon}\int_{\Gamma_C} H(u_h^{n+\frac{1}{2}}) \left([u_h^{n+1}]_-^2 - [u_h^n]_-^2\right) \\ & + \frac{1}{2\epsilon}\int_{\Gamma_C} H(-u_h^{n+\frac{1}{2}}) \left([u_h^{n+1}]_+^2 - [u_h^n]_+^2\right). \end{aligned}$$

Above $H(\cdot)$ denotes the Heaviside function and $[\cdot]_-$ the negative part (for $x \in \mathbb{R}$, $H(x) = x/|x|$ and $[x]_- = [x]_+ - x$). From the above calculation, we observe

that no energy preservation can be expected because of the contact term. And indeed, in practice, it is observed that the midpoint scheme is not capable of preserving the energy [6].

The morality is that, unfortunately, well-posedness and energy preservation at the semi-discrete level are not enough to ensure stability after application of a time-marching scheme. For penalty, this is not really surprizing, since it regularizes the contact condition and one expects that, when ϵ vanishes the behavior of an (ill-posed) differential inclusion is recovered [13].

A first remedy is to apply once again the modified mass method, and everything gets once again in order. Another possibility, inspired by [20, 24] is to apply a Hybrid-Penalty scheme, where the contact term in (16) is replaced by

$$\frac{1}{\epsilon}\int_{\Gamma_C}\left(H(u_h^n)[u_h^{n+\frac{1}{2}}]_+ + H(-u_h^n)[u_h]_+^{n+\frac{1}{2}}\right)v_h.$$

Roughly speaking, it activates either midpoint or Crank-Nicolson, depending of the state of penetration at the previous time-step. It can be shown with an argument very similar to the above one that this scheme is unconditionnaly stable [6].

5 Nitsche and augmented Lagrangian

Nitsche method is a primal consistent method, that allows to take into account contact condition without any regularization procedure and without any extra unknown (see [7] and references therein for instance). For the semi-discretization in space of Problem (1)–(2)–(3)–(4), we introduce the Nitsche parameter $\gamma_0 > 0$ and write $\gamma_N := \gamma_0/h$. We define a modified bilinear form

$$A_N(u,v) := c^2\int_\Omega \nabla u\cdot\nabla v - c^2\int_{\Gamma_C}\frac{1}{\gamma_N}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}$$

and an operator

$$P_N(v) := \gamma_N v - \frac{\partial v}{\partial n}.$$

The symmetric Nitsche formulation consists in finding a semi-discrete displacement field

$$u_h : t \mapsto u_h(t) \in V_h^D$$

for any $t > 0$ that is solution to

$$\frac{d^2}{dt^2}\int_\Omega u_h(t)v_h + A_N(u_h(t),v_h) + \int_{\Gamma_C}\frac{1}{\gamma_N}[P_N(u_h(t))]_+P_N(v_h) = \int_\Omega f(t)v_h,\ t>0, \tag{19}$$

for any $v_h \in V_h^D$. This needs again to be complemented by initial conditions that approximate (4) and can be written as

$$u_h(0) = u_{h0}, \quad \frac{du_h}{dt}(0) = v_{h0}. \tag{20}$$

When applied to contact in elastodynamics, the Nitsche method provides, as penalty, a well-posed semi-discrete problem is space and the symmetric variant of Nitsche presented here allows the preservation of a modified energy. Conversely to penalty, these properties need a Nitsche parameter γ_0 chosen large enough. When a time-marching scheme is applied, the same difficulties as for penalty appear and it needs to be combined with either a modified mass technique, or a Nitsche-Hybrid scheme. In comparison to penalty, it performs better in the sense it enforces more accurately the contact condition and it is much easier to fix the Nitsche parameter γ_0 than the penalty parameter ϵ_0 (because Nitsche is consistent). See [5, 6, 8] for a detailed study of this method.

Also, more recently, augmented Lagrangian formulations have been deeply revisited as a discretization technique for contact, see for instance [2]. A first study of the augmented Lagrangian in the context of contact in elastodynamics has been carried out in [26].

6 To conclude

We provided an insight into the numerical approximation of elastodynamic contact problems and tried to underline some of the difficulties that one is faced with for this problem. We also introduced some important concepts related to well-posedness and energy preservation. Many details have been skipped and in fact there are many other delicate issues, on both theoretical and applied sides. Among the recent trends in this topic are for instance new time-stepping techniques such as the Bathe scheme (TRBDF2) [26] or explicit schemes [8, 12].

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