UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI²MA)

A hybridizable discontinuous Galerkin method for Stokes/Darcy coupling in dissimilar meshes

Isaac Bermudez, Jaime Manríquez, Manuel Solano

PREPRINT 2024-05

SERIE DE PRE-PUBLICACIONES

A hybridizable discontinuous Galerkin method for Stokes/Darcy coupling in dissimilar meshes *

Isaac Bermúdez † Jaime Manríquez ‡ Manuel Solano §

Abstract

We present and analyze a hybridizable discontinuous Galerkin method for coupling Stokes and Darcy equations, whose domains are discretized by two independent triangulations. This causes non-conformity at the intersection of the subdomains or leaves a gap (unmeshed region) between them. In order to properly couple the two different discretizations and obtain a high order scheme, we propose suitable transmission conditions based on mass conservation and equilibrium of normal forces for matching meshes. Since the meshes do not necessarily coincide, we use the Transfer Path Method to tie them. We establish the well-posedness of the method and provide error estimates where the influences of the non-conformity and the gap are explicit in the constants. Finally, numerical experiments that illustrate the performance of the method are shown.

Keywords: Stokes/Darcy; non–matching meshes; dissimilar meshes; Transfer Path Method; hybrid method; discontinuous Galerkin.

Mathematics Subject Classification (2020): 65N08, 65N15, 65N30, 65N85.

1 Introduction

During the last decade, the development of new non-body-fitted numerical methods for partial differential equations (PDEs) has become of interest in the community, especially with focus on high order schemes. One of the most popular is the cut finite element method (CutFEM). Roughly speaking, the CutFEM method considers a background grid where the domain is immersed and a Nitsche's approach is employed to impose the transmission conditions in the elements cut by the interface. A review can be found in [2] and recent works have also proposed conservative CutFEM schemes [16, 19]. CutFEM requires special quadrature rules to compute the integrals over the interface, in contrast with the recently developed ϕ-FEM method [13, 14, 15]. The main idea there is to introduce an auxiliary variable that depends on the level-set function in such a way that the homogeneous Dirichlet boundary condition is automatically satisfied.

*Supported by ANID-Chile through FONDECYT projects No. 1200569 and 1240183; Centro de Modelamiento Matemático (FB210005); by Becas/Doctorado Nacional 21210582; and by Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción.

†Centro de Investigación en Ingeniería Matemática (CI^2MA) and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ibermudezm.cr@gmail.com, ibermudez@udec.cl

‡Centre for Mathematical Sciences, Lund University, Lund, Sweden, email: jaime.manriquez@math.lth.se

§Centro de Investigación en Ingeniería Matemática (CI^2MA) and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: msolano@ing-mat.udec.cl

A different approach to handle transmission/boundary conditions with unfitted methods is based on a Taylor expansion of the function near the interface/boundary. In this direction, in the literature we can find two methods: the Shifted Boundary Method (SBM) [1, 21] and the Transfer Path Method (TPM) [11, 23, 24]. The former considers a primal formulation and the residual of the Taylor expansion vanishes at discrete level. The TPM, on the other hand, is based on a mixed formulation where the residual of the Taylor expansion does not vanish but it involves the mixed variable that is then approximated by the numerical scheme. Our work focuses on the latter with the aim to demonstrate that the TPM can be a useful technique to handle situations where two meshes of different sizes are apart from each other. In addition, as a byproduct, our analysis also covers the case where the interface is fitted by the two meshes, allowing the presence of hanging nodes.

In several applications, the domain of interest $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, is divided into subdomains where different governing equations are posed. It is not uncommon to mesh each subdomains separately using different meshsizes. For instance, in the case of solid-fluid interactions, the fluid equations are coupled to the elasticity equations via appropriate transmission conditions across an interface, and it is often desirable to have a finer mesh in the region occupied by the fluid compared to the meshsize used for discretizing the solid. When the domain of a PDE is discretized by the union of different computational subdomains, it is possible to identify two configurations. In the first one, the interface is not fitted by the triangulations, generating *dissimilar meshes* with gaps and overlaps appearing between the grids associated to each subdomains, as the one depicted in Figure 1 (left). Therefore, the discrete interfaces of neighboring grids need to be properly connected. In the second configuration the interface is fitted by the grids, but it presents hanging nodes as portrayed in Figure 1 (right). This causes a non-conformity at the intersection of the subdomains in which adjacent elements do not necessarily share a complete face or edge. This is why we prefer to consider a discontinuous Galerkin method (DG) to discretize the PDE. In particular, we focus on the hybridizable DG (HDG) method.

The HDG method, introduced in [5], has the advantage of significantly reducing the globally coupled degrees of freedom that were a major criticism of DG methods for elliptic problems. The only degrees of freedom in HDG are those of the numerical traces on the boundaries between elements, while the remaining unknowns are obtained by solving local problems in each element. Specifically, at the continuous level, intra-element variables can be expressed in terms of inter-element unknowns by solving local problems on each element. These problems, referred to as local solvers, can be discretized using a DG method, leading to the family of methods known as HDG methods.

Furthermore, to the best of our knowledge, there are only two works that analyze the HDG method for non-conforming triangulations [3, 4]. In the first approach the authors perform an analysis for the convection-diffusion equations in non-conforming meshes. In particular, using polynomial approximations of degree k in all elements, they obtained suboptimal order of convergence $h^{k+1/2}$ for the diffusive flux and optimal convergence h^{k+1} for the projection of the error in the scalar variable. The second approach is similar to the first one, but uses the so-called *semimatching* nonconforming meshes. Then, both optimal convergence for the diffusive flux and superconvergence of the projection of the error in the scalar variable is obtained.

In this work, with the aim of developing a high-order method to handle geometries with complex interfaces, we present an HDG method for coupled problems in dissimilar and non-conforming meshes. More precisely, we focus on the coupling of fluid and porous media flows across a discrete interface that does not necessarily match the true interface. To this end, we rely on the TPM [8, 9, 10, 24] originally designed for handling boundary value problems in curved domains, but recently employed for coupling dissimilar meshes in the context of a single PDE in the entire domain [22, 25]. Thus, following a similar scheme developed in [18], we propose and analyze a new method for Stokes/Darcy coupling. More precisely, let Ω_{s} and Ω_{d} be bounded and simply connected polyhedral domains in $\mathbb{R}^n$, $n \in \{2,3\}$, with outward unit normal vectors $\mathbf{n}_{\mathrm{s}}$ and $\mathbf{n}_{\mathrm{d}}$, respectively, such that $\mathcal{I} := \overline{\Omega}_{\mathrm{s}} \cap \overline{\Omega}_{\mathrm{d}}$ is the interface that separates them, and let $\Gamma_\mathrm{s} := \partial\Omega_\mathrm{s}\backslash\mathcal{I}$, $\Gamma_\mathrm{d} := \partial\Omega_\mathrm{d}\backslash\mathcal{I}$. The model consists of two separate groups of equations and a set of coupling terms. In the fluid region Ω_s, the governing equations are those of the Stokes problem, which can be written as follows:

$$\begin{aligned} \mathrm{L}_\mathrm{s} - \nabla\mathbf{u}_\mathrm{s} \,=\, \mathbf{0} \quad \text{in} \quad \Omega_\mathrm{s}, \quad -\nabla\cdot(\nu\mathrm{L}_\mathrm{s} - P_\mathrm{s}\mathrm{I}) \,=\, \mathbf{f}_\mathrm{s} \quad \text{in} \quad \Omega_\mathrm{s}, \quad \nabla\cdot\mathbf{u}_\mathrm{s} \,=\, 0 \quad \text{in} \quad \Omega_\mathrm{s}, \\ \mathbf{u}_\mathrm{s} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{s}, \quad \text{and} \quad \int_{\Omega_\mathrm{s}} P_\mathrm{s} \,=\, 0, \end{aligned} \tag{1.1}$$

where $\nu > 0$ is the fluid dynamic viscosity, $\mathbf{f}_\mathrm{s} \in \mathbf{L}^2(\Omega_\mathrm{s})$ is the volumetric force acting on the fluid, $\mathbf{u}_\mathrm{s}$ is the fluid velocity, L_s is the velocity gradient tensor, P_s is the pressure, and I is the $n \times n$ identity matrix. In turn, in the porous medium region Ω_d we consider the following Darcy model:

$$\mathbf{u}_\mathrm{d} + \kappa\nabla p_\mathrm{d} \,=\, \mathbf{0} \quad \text{in} \quad \Omega_\mathrm{d}, \quad \nabla\cdot\mathbf{u}_\mathrm{d} \,=\, \mathrm{f}_\mathrm{d} \quad \text{in} \quad \Omega_\mathrm{d}, \quad \text{and} \quad p_\mathrm{d} \,=\, 0 \quad \text{on} \quad \Gamma_\mathrm{d}, \tag{1.2}$$

where κ is a tensor valued function, which describes the permeability of Ω_d, satisfies $\kappa^\mathtt{t} = \kappa$, and has $L^\infty(\Omega_\mathrm{d})$ components, $\mathrm{f}_\mathrm{d} \in L^2(\Omega_\mathrm{d})$ is a given source term, and $\mathbf{u}_\mathrm{d}$ and p_d denote the velocity and pressure, respectively. Also, we assume that there exist positive constants $\underline{\kappa}$ and $\overline{\kappa}$ such that $\underline{\kappa} \leqslant \|\kappa\|_{\infty,\Omega_\mathrm{d}} \leqslant \overline{\kappa}$. Finally, the transmission conditions on $\mathcal{I}$ are given by

$$\mathbf{u}_\mathrm{s}\cdot\mathbf{n}_\mathrm{s} + \mathbf{u}_\mathrm{d}\cdot\mathbf{n}_\mathrm{d} = 0 \quad \text{and} \quad (\nu\mathrm{L}_\mathrm{s} - P_\mathrm{s}\mathrm{I})\mathbf{n}_\mathrm{s} \;= p_\mathrm{d}\mathbf{n}_\mathrm{d} \quad \text{on} \quad \mathcal{I}. \tag{1.3}$$

The first equation in (1.3) is based on mass conservation, whereas the second one establishes the balance of normal forces for matching meshes. The analysis studied in this work can be extended with minor modifications to the case when the Beavers–Joseph–Saffman law (see, e.g. [17, 20]) is used. However, for sake of simplicity we choose to avoid this in order to focus solely on the technique applied on the interface.

The manuscript is organized as: in Section 2, we introduce the notation related to the discretization and transferring segments, as well as some preliminaries and definitions related to the computational domain and the approximation spaces. Next, in Section 3, the HDG method is introduced along with the proposed transmission conditions. In Section 4, we show the stability of the method and present the error estimates in Section 5. Finally, several numerical experiments validating the good performance of the method and confirming the rates of convergence are reported in Section 6.

2 Preliminaries

We begin by introducing some preliminary notations related to the geometric discretization, the approximation spaces and the HDG scheme. In turn, we introduce the main tools to address the discretization of the interface.

The computational domain. Let $\Omega^\mathrm{s}_{h_\mathrm{s}}$ and $\Omega^\mathrm{d}_{h_\mathrm{d}}$ be triangulations of the domains Ω_s and Ω_d, with meshsizes h_s, $h_\mathrm{d} > 0$ and boundaries $\mathcal{S}_{\mathrm{s},h_\mathrm{s}}, \mathcal{S}_{\mathrm{d},h_\mathrm{d}}$, respectively. Without loss of generality, we suppose $h_\mathrm{d} \geqslant h_\mathrm{s}$ and drop the sub-index $\star \in \{\mathrm{s},\mathrm{d}\}$ when there is no confusion; for example, we just write Ω^s_h, and Ω^d_h henceforth. We also denote the set of all faces of the triangulation $\Omega^\star_h$ by $\mathcal{E}^\star_h$. Furthermore, since $\overline{\Omega}^\mathrm{s}_h \cap \overline{\Omega}^\mathrm{d}_h$ is not necessarily equal to $\mathcal{I}$, then, for $\star \in \{\mathrm{s},\mathrm{d}\}$, we define the discrete interfaces $\mathcal{I}^\star_h := \mathcal{S}_{\star,h}\backslash\Gamma_{\star,h}$, where $\Gamma_{\star,h}$ denotes the discretization of $\Gamma_\star$. Bearing in mind the above, we consider outward normal vectors for the new interfaces $\mathcal{I}^\mathrm{s}_h$ and $\mathcal{I}^\mathrm{d}_h$, which will be denoted by $\mathbf{n}_{\mathrm{s},h}$ and $\mathbf{n}_{\mathrm{d},h}$, respectively. The family of triangulations $\{\Omega^\star_h\}_{h>0}$ is assumed to be shape-regular, i.e., there exists a constant $\kappa_\star > 0$ such that for all elements $K \in \Omega^\star_h$ and all $h > 0$, $h_K/\rho_K \leqslant \kappa_\star$, where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K. For every element K, we denote by $\mathbf{n}_K$ the outward unit normal vector to K, writting $\mathbf{n}$ instead of $\mathbf{n}_K$ when there is no confusion. In this work, we consider the two configurations depicted in Figure 1. In the first one, a uniform gap of size δ separates the two triangulations. In the second setting, the interface is piecewise flat and both meshes are fitted to it, but with different meshsizes.

Spaces and norms. We use the standard notation for Sobolev spaces and their associated norms and seminorms, where vector-valued functions and their corresponding spaces are denoted in bold face font, and roman font in the tensor-valued case. In addition, let D be an open bounded region of $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. We denote by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ the $L^2(D)$ and $L^2(\partial D)$ inner products, respectively, with induced norms $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$. Given an integer $k \geqslant 0$, we use the usual notation to denote the space of polynomials of degree at most k as $P_k(D)$, and set $\mathbf{P}_k(D) := [P_k(D)]^n$ and $\mathrm{P}_k(D) := [P_k(D)]^{n\times n}$.

We introduce now the finite-dimensional spaces

$$\begin{aligned}
\mathrm{G}_h^{\mathrm{s}} &:= \Big\{\mathrm{G} \in \mathrm{L}^2(\Omega_h^{\mathrm{s}}) : \mathrm{G}|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \Omega_h^{\mathrm{s}}\Big\},\\
\mathbf{V}_h^{\star} &:= \Big\{\mathbf{v} \in \mathbf{L}^2(\Omega_h^{\star}) : \mathbf{v}|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \Omega_h^{\star}\Big\},\\
Q_h^{\star} &:= \Big\{q \in L^2(\Omega_h^{\star}) : q|_K \in P_k(K) \quad \forall\, K \in \Omega_h^{\star}\Big\},
\end{aligned}$$

for intra-element variables, and

$$\begin{aligned}
M_h^{\mathrm{d}} &:= \Big\{\mu \in L^2(\mathcal{E}_h^{\mathrm{d}}) : \mu|_e \in P_k(e) \quad \forall\, e \in \mathcal{E}_{\mathrm{d}}\Big\},\\
\mathbf{M}_h^{\mathrm{s}} &:= \Big\{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h^{\mathrm{s}}) : \boldsymbol{\mu}|_e \in \mathbf{P}_k(e) \quad \forall\, e \in \mathcal{E}_{\mathrm{s}}\Big\}
\end{aligned}$$

for trace variables. We denote by N_h^{d} and $\mathbf{N}_h^{\mathrm{s}}$ the restrictions of M_h^{d} and $\mathbf{M}_h^{\mathrm{s}}$ to the discrete interfaces $\mathcal{I}_h^{\mathrm{d}}$ and $\mathcal{I}_h^{\mathrm{s}}$, respectively. The mesh-dependent inner products are defined as

$$(\cdot,\cdot)_{\Omega_h^\star} := \sum_{K\in\Omega_h^\star} (\cdot,\cdot)_K, \quad \langle\cdot,\cdot\rangle_{\partial\Omega_h^\star} = \sum_{K\in\Omega_h^\star} \langle\cdot,\cdot\rangle_{\partial K} \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\mathcal{I}_h^\star} = \sum_{e\in\mathcal{I}_h^\star} \langle\cdot,\cdot\rangle_e,$$

and their corresponding norms denoted by

$$\|\cdot\|_{\Omega_h^\star} := \left(\sum_{K\in\Omega_h^\star} \|\cdot\|_K^2\right)^{1/2}, \quad \|\cdot\|_{\partial\Omega_h^\star} = \left(\sum_{K\in\Omega_h^\star} \|\cdot\|_{\partial K}^2\right)^{1/2} \quad \text{and} \quad \|\cdot\|_{\mathcal{I}_h^\star} = \left(\sum_{e\in\mathcal{I}_h^\star} \|\cdot\|_e^2\right)^{1/2}.$$

To avoid proliferation of unimportant constants, we use the terminology $a \lesssim b$ whenever $a \leqslant Cb$ and C is a positive constant independent of h and the gap between both discrete interfaces.

Transfer paths. For $\star \in \{\mathrm{s},\mathrm{d}\}$, we introduce a mapping $\boldsymbol{\psi}_\star : \mathcal{I} \to \mathcal{I}_h^\star$, such that for each point $\boldsymbol{x} \in \mathcal{I}$, we associate a point $\boldsymbol{x}_\star = \boldsymbol{\psi}_\star(\boldsymbol{x}) \in \mathcal{I}_h^\star$. We also define a mapping $\boldsymbol{\psi} : \mathcal{I}_h^{\mathrm{d}} \to \mathcal{I}_h^{\mathrm{s}}$ as $\boldsymbol{\psi} = \boldsymbol{\psi}_{\mathrm{s}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}$, which means that for each $\boldsymbol{x}_{\mathrm{d}} \in \mathcal{I}_h^{\mathrm{d}}$, we associate a point $\boldsymbol{x}_{\mathrm{s}} = \boldsymbol{\psi}(\boldsymbol{x}_{\mathrm{d}}) \in \mathcal{I}_h^{\mathrm{s}}$. We denote by $\sigma_\star(\boldsymbol{x}_\star)$ the segment starting at $\boldsymbol{x}$ and ending at $\boldsymbol{x}_\star$, with unit tangent vector $\boldsymbol{t}_\star$ and length $|\sigma_\star(\boldsymbol{x}_\star)|$. Then, keeping in mind the configuration of the interfaces, i.e., piece-wise polynomial if the meshes coincide or flat interfaces for the case with gap, it follows immediately that for each $e \in \mathcal{I}_h^\star$, $\boldsymbol{t}_\star = \mathbf{n}_{\star,h} = \mathbf{n}_\star$ with $\star \in \{\mathrm{s},\mathrm{d}\}$, and $\boldsymbol{t}_{\mathrm{d}} = -\boldsymbol{t}_{\mathrm{s}}$. This means that the direction of the segment $\sigma(\boldsymbol{x}_{\mathrm{d}})$ must be parallel to the normals computed at its ends. Therefore, from now on, we will write $\mathbf{n}_\star$, to refer to the vector associated to $\sigma_\star(\boldsymbol{x}_\star)$, with $\star \in \{\mathrm{s},\mathrm{d}\}$. Then, $\sigma(\boldsymbol{x}_{\mathrm{d}})$ is the segment that starts at $\boldsymbol{x}_{\mathrm{d}}$ and ends at $\boldsymbol{x}_{\mathrm{s}}$, with unit tangent vector $\mathbf{n}_{\mathrm{d}}$ and length $|\sigma(\boldsymbol{x}_{\mathrm{d}})|$. The segment $\sigma(\boldsymbol{x}_{\mathrm{d}})$ is referred as the *transfer path* associated with $\boldsymbol{x}_{\mathrm{d}}$

and is assumed to satisfy two conditions: it does not intersect the interior of another transfer path and its length $|\sigma(\boldsymbol{x}_\mathrm{d})|$ is of order at most $\max\{h_\mathrm{s}, h_\mathrm{d}\} = h_\mathrm{d}$.

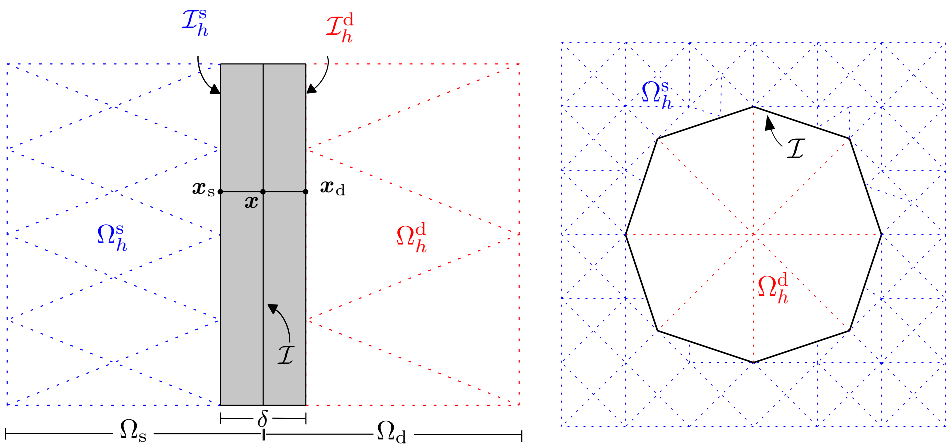


Figure 1: Left: Example of dissimilar meshes separated by a uniform gap of size δ. Right: A piecewise polygonal interface separating two regions discretized by diferent meshes.

Extrapolation operator. The region enclosed by Ω_h^s and Ω_h^d (shaded area in Figure 1) is denoted by $\Omega_h^{\mathtt{ext}}$. We notice that $\Omega_h^{\mathtt{ext}}$ is not meshed and, as a consequence, we do not have an HDG approximation in there. That is why the HDG approximation of the velocity gradient L_s, the pressure field p_s (to be defined below), and the flux $\mathbf{u}_\mathrm{d}$, will be locally extrapolated from the computational domain $\Omega_h^\mathrm{d} \cup \Omega_h^\mathrm{s}$ to $\Omega_h^{\mathtt{ext}}$. More precisely, let q be a tensor, vector, or scalar-valued polynomial function defined on an element K in $\Omega_h^\mathrm{d} \cup \Omega_h^\mathrm{s}$ such that $\overline{K} \cap \overline{\Omega}_h^{\mathtt{ext}} \neq \emptyset$. We define its extrapolation to $\Omega_h^{\mathtt{ext}}$ as

$$\boldsymbol{E}_{q|_K}(\boldsymbol{y}) := q|_K(\boldsymbol{y}) \quad \forall\, \boldsymbol{y} \in \Omega_h^{\mathtt{ext}}. \tag{2.1}$$

Note that the extrapolation function $\boldsymbol{E}_{q|_K}(\boldsymbol{y})$ is a function whose support includes $\Omega_h^{\mathtt{ext}}$, and each element K has its own extrapolation function.

The HDG projections. Let $(\mathrm{L}_\mathrm{s}, \mathbf{u}_\mathrm{s}, p_\mathrm{s}) \in \mathrm{H}^1(\Omega_h^\mathrm{s}) \times \mathbf{H}^1(\Omega_h^\mathrm{s}) \times H^1(\Omega_h^\mathrm{s})$, we recall its HDG projection $\Pi_\mathrm{s}(\mathrm{L}_\mathrm{s}, \mathbf{u}_\mathrm{s}, p_\mathrm{s}) = (\Pi_\mathrm{G}^\mathrm{s}\mathrm{L}_\mathrm{s}, \Pi_\mathbf{V}^\mathrm{s}\mathbf{u}_\mathrm{s}, \Pi_Q^\mathrm{s} p_\mathrm{s})$ as the element of $\mathrm{G}_h^\mathrm{s} \times \mathbf{V}_h^\mathrm{s} \times Q_h^\mathrm{s}$ defined as follows: on an arbitrary element K of Ω_h^s, the values of the projected function on K are determined by requiring that

$$(\Pi_\mathrm{G}^\mathrm{s}\mathrm{L}_\mathrm{s}, \mathrm{G})_K = (\mathrm{L}_\mathrm{s}, \mathrm{G})_K \quad \forall\, \mathrm{G} \in \mathrm{P}_{k-1}(K), \qquad (\Pi_\mathbf{V}^\mathrm{s}\mathbf{u}_\mathrm{s}, \mathbf{v})_K = (\mathbf{u}_\mathrm{s}, \mathbf{v})_K \quad \forall\, \mathbf{v} \in \mathbf{P}_{k-1}(K), \tag{2.2a}$$

$$(\Pi_Q^\mathrm{s} p_\mathrm{s}, q)_K = (p_\mathrm{s}, q)_K \quad \forall\, q \in P_{k-1}(K), \qquad (\mathrm{tr}\Pi_\mathrm{G}^\mathrm{s}\mathrm{L}_\mathrm{s}, q)_K = (\mathrm{tr}\mathrm{L}_\mathrm{s}, q)_K \quad \forall\, q \in P_k(K), \tag{2.2b}$$

$$\langle \nu\Pi_\mathrm{G}^\mathrm{s}\mathrm{L}_\mathrm{s}\mathbf{n}_\mathrm{s} - \Pi_Q^\mathrm{s} p_\mathrm{s}\mathbf{n}_\mathrm{s} - \tau\nu\Pi_\mathbf{V}^\mathrm{s}\mathbf{u}_\mathrm{s}, \boldsymbol{\mu}\rangle_e = \langle \nu\mathrm{L}_\mathrm{s}\mathbf{n}_\mathrm{s} - p_\mathrm{s}\mathbf{n}_\mathrm{s} - \tau\nu\mathbf{u}_\mathrm{s}, \boldsymbol{\mu}\rangle_e \quad \forall\, \boldsymbol{\mu} \in \mathbf{P}_k(e)\, \forall\, e \subset \partial K, \tag{2.2c}$$

where $\tau > 0$ is the stabilization parameter of the HDG method. Furthermore, if $(\mathrm{L}_\mathrm{s}, \mathbf{u}_\mathrm{s}, p_\mathrm{s}) \in \mathrm{H}^{l_\sigma+1}(\Omega_h^\mathrm{s}) \times \mathbf{H}^{l_{\mathbf{u}_\mathrm{s}}+1}(\Omega_h^\mathrm{s}) \times H^{l_\sigma+1}(\Omega_h^\mathrm{s})$, for $l_{\mathbf{u}_\mathrm{s}}, l_\sigma \in [0, k]$, the above projection satisfies (cf. [6, Theorem 2.1]) the following properties:

$$\|\mathbf{u}_\mathrm{s} - \Pi_\mathbf{V}^\mathrm{s}\mathbf{u}_\mathrm{s}\|_K \lesssim h_K^{l_{\mathbf{u}_\mathrm{s}}+1}|\mathbf{u}_\mathrm{s}|_{\mathbf{H}^{l_{\mathbf{u}_\mathrm{s}}+1}(K)} + h_K^{l_\sigma+1}(\tau\nu)^{-1}|\nabla\cdot(\nu\mathrm{L}_\mathrm{s} - p_\mathrm{s}\mathrm{I})|_{\mathbf{H}^{l_\sigma}(K)}, \tag{2.3a}$$

$$\begin{aligned}\|\nu(\mathrm{L}_\mathrm{s} - \Pi_\mathrm{G}^\mathrm{s}\mathrm{L}_\mathrm{s})\|_K + \|p_\mathrm{s} - \Pi_Q^\mathrm{s} p_\mathrm{s}\|_K &\lesssim h_K^{l_\sigma+1}|\nu\mathrm{L}_\mathrm{s} - p_\mathrm{s}\mathrm{I}|_{\mathrm{H}^{l_\sigma+1}(K)} + h_K^{l_u+1}\tau\nu|\mathbf{u}_\mathrm{s}|_{\mathbf{H}^{l_u+1}(K)} \\ &\quad + \tau\nu\|\mathbf{u}_\mathrm{s} - \Pi_\mathbf{V}^\mathrm{s}\mathbf{u}_\mathrm{s}\|_K,\end{aligned} \tag{2.3b}$$

for all $K \in \Omega_h^{\mathrm{s}}$, where I is the identity tensor. Similarly, given $(\mathbf{u}_{\mathrm{d}}, p_{\mathrm{d}}) \in \mathbf{H}^1(\Omega_h^{\mathrm{d}}) \times H^1(\Omega_h^{\mathrm{d}})$, we recall its HDG projection $\Pi_{\mathrm{d}}(\mathbf{u}_{\mathrm{d}}, p_{\mathrm{d}}) = (\Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}}, \Pi_Q^{\mathrm{d}} p_{\mathrm{d}})$ as the element of $\mathbf{V}_h^{\mathrm{d}} \times Q_h^{\mathrm{d}}$ defined as the unique element-wise solution of

$$(\Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}}, \mathbf{v})_K = (\mathbf{u}_{\mathrm{d}}, \mathbf{v})_K \quad \forall\, \mathbf{v} \in \mathbf{P}_{k-1}(K), \quad (\Pi_Q^{\mathrm{d}} p_{\mathrm{d}}, q)_K = (p_{\mathrm{d}}, q)_K \quad \forall\, q \in P_{k-1}(K), \tag{2.4a}$$

$$\langle \Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}} \cdot \mathbf{n}_{\mathrm{d}} + \tau \Pi_Q^{\mathrm{d}} p_{\mathrm{d}}, \mu \rangle_e = \langle \mathbf{u}_{\mathrm{d}} \cdot \mathbf{n}_{\mathrm{d}} + \tau \mathcal{P}^{\mathrm{d}} p_{\mathrm{d}}, \mu \rangle_e \quad \forall\, \mu \in P_k(e), \forall\, e \subset \partial K, \tag{2.4b}$$

for every element $K \in \Omega_h^{\mathrm{d}}$, and, given constants $l_{\mathbf{u}_{\mathrm{d}}}, l_{p_{\mathrm{d}}} \in [0, k]$, if $(\mathbf{u}_{\mathrm{d}}, p_{\mathrm{d}}) \in \mathbf{H}^{l_{\mathbf{u}_{\mathrm{d}}}+1}(\Omega_h^{\mathrm{d}}) \times H^{l_{p_{\mathrm{d}}}+1}(\Omega_h^{\mathrm{d}})$, there hold (cf. [7])

$$\|\mathbf{u}_{\mathrm{d}} - \Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}}\|_K \lesssim h_K^{l_{\mathbf{u}_{\mathrm{d}}}+1} |\mathbf{u}_{\mathrm{d}}|_{\mathbf{H}^{l_{\mathbf{u}_{\mathrm{d}}}+1}(K)} + h_K^{l_{p_{\mathrm{d}}}+1} |p_{\mathrm{d}}|_{H^{l_{p_{\mathrm{d}}}+1}(K)}, \tag{2.5a}$$

$$\|p_{\mathrm{d}} - \Pi_Q^{\mathrm{d}} p_{\mathrm{d}}\|_K \lesssim h_K^{l_{p_{\mathrm{d}}}+1} |p_{\mathrm{d}}|_{H^{l_{p_{\mathrm{d}}}+1}(K)} + h_K^{l_{\mathbf{u}_{\mathrm{d}}}+1} |\nabla \cdot \mathbf{u}_{\mathrm{d}}|_{H^{l_{\mathbf{u}_{\mathrm{d}}}}(K)}, \tag{2.5b}$$

for all $K \in \Omega_h^{\mathrm{d}}$.

3 The HDG method

3.1 The treatment of the pressure

In this section we follow a similar approach to the one developed in [26]. More precisely, since the computational domain Ω_h^{s} does not necessarily coincide with the physical domain Ω_{s}, we introduce a decomposition $P_{\mathrm{s}} = \alpha_{\mathrm{s}} + p_{\mathrm{s}}$ that imposes the zero-mean of the pressure, with $\alpha_{\mathrm{s}} := \frac{1}{|\Omega_h^{\mathrm{s}}|} \int_{\Omega_h^{\mathrm{s}}} P_{\mathrm{s}}$ and $p_{\mathrm{s}} \in L_0^2(\Omega_h^{\mathrm{s}})$ ($L^2(\Omega_h^{\mathrm{s}})$-function with zero mean in Ω_h^{s}). In turn, since P_{s} will be eliminated from the system, we need to rewrite α_{s} in terms of p_{s}. By using the fifth equation of (1.1), we deduce that

$$\alpha_{\mathrm{s}} = \frac{-1}{|\Omega_{\mathrm{s}}|} \int_{\Omega_{\mathrm{s}} \setminus \Omega_h^{\mathrm{s}}} p_{\mathrm{s}}. \tag{3.1}$$

Then, P_{s} can be recovered after the approximation of p_{s} is computed.

3.2 The HDG scheme

The HDG formulation of the coupled system (1.1)-(1.2) reduces to:
Find $(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\mathrm{s},h}, p_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, \mathbf{u}_{\mathrm{d},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}) \in \mathrm{G}_h^{\mathrm{s}} \times \mathbf{V}_h^{\mathrm{s}} \times Q_h^{\mathrm{s}} \times \mathbf{M}_h^{\mathrm{s}} \times \mathbf{V}_h^{\mathrm{d}} \times Q_h^{\mathrm{d}} \times M_h^{\mathrm{d}}$ such that

$$(\mathrm{L}_{\mathrm{s},h}, \mathrm{G}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + (\mathbf{u}_{\mathrm{s},h}, \nabla \cdot \mathrm{G}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} - \langle \widehat{\mathbf{u}}_{\mathrm{s},h}, \mathrm{G}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}} \rangle_{\partial\Omega_h^{\mathrm{s}}} = 0\,, \tag{3.2a}$$

$$(\boldsymbol{\sigma}_{\mathrm{s},h}, \nabla \mathbf{v}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} - \langle \widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \mathbf{v}_{\mathrm{s},h} \rangle_{\partial\Omega_h^{\mathrm{s}}} = (\mathbf{f}_{\mathrm{s}}, \mathbf{v}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}}\,, \tag{3.2b}$$

$$-(\mathbf{u}_{\mathrm{s},h}, \nabla q_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + \langle \widehat{\mathbf{u}}_{\mathrm{s},h} \cdot \mathbf{n}_{\mathrm{s}}, q_{\mathrm{s},h} \rangle_{\partial\Omega_h^{\mathrm{s}}} = 0\,, \tag{3.2c}$$

$$\langle \widehat{\mathbf{u}}_{\mathrm{s},h}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\Gamma_{\mathrm{s},h}} = 0\,, \tag{3.2d}$$

$$(p_{\mathrm{s},h}, 1)_{\Omega_h^{\mathrm{s}}} = 0\,, \tag{3.2e}$$

$$\langle \widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\partial\Omega_h^{\mathrm{s}} \setminus (\Gamma_{\mathrm{s},h} \cup \mathcal{I}_h^{\mathrm{s}})} = 0\,, \tag{3.2f}$$

$$(\kappa^{-1}\mathbf{u}_{\mathrm{d},h}, \mathbf{v}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} - (p_{\mathrm{d},h}, \nabla \cdot \mathbf{v}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \langle \mathbf{v}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h} \rangle_{\partial\Omega_h^{\mathrm{d}}} = 0\,, \tag{3.2g}$$

$$-(\mathbf{u}_{\mathrm{d},h}, \nabla q_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \langle \widehat{\mathbf{u}}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, q_{\mathrm{d},h} \rangle_{\partial\Omega_h^{\mathrm{d}}} = (\mathrm{f}_{\mathrm{d}}, q_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}}\,, \tag{3.2h}$$

$$\langle \widehat{p}_{\mathrm{d},h}, \mu_{\mathrm{d},h} \rangle_{\Gamma_{\mathrm{d},h}} = 0\,, \tag{3.2i}$$

$$\langle \widehat{\mathbf{u}}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h} \rangle_{\partial\Omega_h^{\mathrm{d}} \setminus (\Gamma_{\mathrm{d},h} \cup \mathcal{I}_h^{\mathrm{d}})} = 0\,, \tag{3.2j}$$

for all $(\mathrm{G}_{\mathrm{s},h}, \mathbf{v}_{\mathrm{s},h}, q_{\mathrm{s},h}, \boldsymbol{\mu}_{\mathrm{s},h}, \mathbf{v}_{\mathrm{d},h}, q_{\mathrm{d},h}, \mu_{\mathrm{d},h}) \in \mathrm{G}_h^{\mathrm{s}} \times \mathbf{V}_h^{\mathrm{s}} \times Q_h^{\mathrm{s}} \times \mathbf{M}_h^{\mathrm{s}} \times \mathbf{V}_h^{\mathrm{d}} \times Q_h^{\mathrm{d}} \times M_h^{\mathrm{d}}$, where

$$\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{s},h} &= \nu \mathrm{L}_{\mathrm{s},h} - p_{\mathrm{s},h}\mathrm{I}\,, \\
\widehat{\mathbf{u}}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}} &= \mathbf{u}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}} + \tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h}) \quad \text{on} \quad \partial\Omega_h^{\mathrm{d}}\,, && (3.3a)\\
\widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}} &= \boldsymbol{\sigma}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}} - \tau\nu(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}) \quad \text{on} \quad \partial\Omega_h^{\mathrm{s}}\,, && (3.3b)
\end{aligned}$$

and we recall that τ is a positive stabilization function defined in $\partial\Omega_h^{\mathrm{s}} \cup \partial\Omega_h^{\mathrm{d}}$, assumed to be uniformly bounded. For simplicity of the exposition, we assume τ is constant everywhere. The above equations must be complemented with suitable transmission conditions across the interfaces $\mathcal{I}_h^{\mathrm{s}}$ and $\mathcal{I}_h^{\mathrm{d}}$, which we proceed to derive now and this constitutes the novelty of our work. Indeed, we propose the following conditions:

$$\begin{aligned}
-\langle \widetilde{\mathbf{u}}_{\mathrm{s},h} \cdot \mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h} \rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle \widetilde{\mathbf{u}}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h} \rangle_{\mathcal{I}_h^{\mathrm{d}}} &= 0 \quad \forall\, \mu_{\mathrm{d},h} \in N_h^{\mathrm{d}}, && (3.4a)\\
\langle \widetilde{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\mathcal{I}_h^{\mathrm{s}}} - \langle \widetilde{p}_{\mathrm{d},h}\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\mathcal{I}_h^{\mathrm{s}}} &= 0 \quad \forall\, \boldsymbol{\mu}_{\mathrm{s},h} \in \mathbf{N}_h^{\mathrm{s}}, && (3.4b)
\end{aligned}$$

where $\widetilde{\mathbf{u}}_{\mathrm{s},h}$, $\widetilde{p}_{\mathrm{d},h}$, $\widetilde{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}$, and $\widetilde{\mathbf{u}}_{\mathrm{d},h}$ stand for the approximations of $\mathbf{u}_{\mathrm{s}}|_{\mathcal{I}_h^{\mathrm{s}}}$, $p_{\mathrm{d}}|_{\mathcal{I}_h^{\mathrm{d}}}$, $(\nu\mathrm{L}_{\mathrm{s}} - p_{\mathrm{s}}\mathrm{I})\mathbf{n}_{\mathrm{s}}|_{\mathcal{I}_h^{\mathrm{s}}}$ and $\mathbf{u}_{\mathrm{d}}|_{\mathcal{I}_h^{\mathrm{d}}}$, respectively, based on suitable extensions (constructed below) of $\widehat{\mathbf{u}}_{\mathrm{s},h}$, $\widehat{p}_{\mathrm{d},h}$, $\widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}$ and $\widehat{\mathbf{u}}_{\mathrm{d},h}$ outside their corresponding computational domains. More precisely, employing the transferring technique of [11] (see also [9, 22, 25]), the tilde variables are constructed as follows: let $\boldsymbol{x}_\star \in \mathcal{I}_h^\star$ and its corresponding point $\boldsymbol{x} \in \mathcal{I}$. Integrating the first equation of (1.1) along the transfering path $\sigma_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}})$ and using the first equation of (1.3), we obtain

$$\mathbf{u}_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}}) \cdot \mathbf{n}_{\mathrm{s}} + |\sigma_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}})| \left(\int_0^1 \mathrm{L}_{\mathrm{s}}(\boldsymbol{y}_{\mathrm{s}}(t))\mathbf{n}_{\mathrm{s}} dt \right) \cdot \mathbf{n}_{\mathrm{s}} + (\mathbf{u}_{\mathrm{d}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1})(\boldsymbol{x}_{\mathrm{d}}) \cdot \mathbf{n}_{\mathrm{d}} = 0. \tag{3.5}$$

Similarly, integrating the second equation of (1.2) along the connecting segment $\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})$ and using the second equation of (1.3), it follows that

$$(\nu\mathrm{L}_{\mathrm{s}} - P_{\mathrm{s}}\mathrm{I}) \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1})(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} - p_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})\mathbf{n}_{\mathrm{d}} + |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \left(\int_0^1 \kappa^{-1}\mathbf{u}_{\mathrm{d}}(\boldsymbol{y}_{\mathrm{d}}(t)) \cdot \mathbf{n}_{\mathrm{d}} dt \right) \mathbf{n}_{\mathrm{d}} = \mathbf{0}\,, \tag{3.6}$$

where $\boldsymbol{y}_\star(t) = \boldsymbol{x}_\star + (\boldsymbol{x} - \boldsymbol{x}_\star)t$ with $t \in [0,1]$ being the parametrization of $\sigma_\star(\boldsymbol{x}_\star)$.

Hence, motivated by these expressions and based on the form of the HDG numerical fluxes (3.3a) and (3.3b), we define

$$\begin{aligned}
\widetilde{p}_{\mathrm{d},h}(\boldsymbol{x}_{\mathrm{s}}) &:= \widehat{p}_{\mathrm{d},h}(\boldsymbol{x}_{\mathrm{d}}) - |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_0^1 \kappa^{-1}\boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}}(\boldsymbol{y}_{\mathbf{d}}(t)) \cdot \mathbf{n}_{\mathrm{d}} dt, && (3.7a)\\
\widetilde{\mathbf{u}}_{\mathrm{s},h}(\boldsymbol{x}_{\mathrm{d}}) &:= \widehat{\mathbf{u}}_{\mathrm{s},h}(\boldsymbol{x}_{\mathrm{s}}) + |\sigma_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}})| \int_0^1 \boldsymbol{E}_{\mathrm{L}_{\mathrm{s},h}}(\boldsymbol{y}_{\mathrm{s}}(t))\mathbf{n}_{\mathrm{s}} dt, && (3.7b)\\
\widetilde{\mathbf{u}}_{\mathrm{d},h}(\boldsymbol{x}_{\mathrm{d}}) \cdot \mathbf{n}_{\mathrm{d}} &:= \boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}(\boldsymbol{x}_{\mathrm{d}}) \cdot \mathbf{n}_{\mathrm{d}} + \tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h})(\boldsymbol{x}_{\mathrm{d}}), && (3.7c)\\
\widetilde{\boldsymbol{\sigma}}_{\mathrm{s},h}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} &:= \boldsymbol{E}_{\boldsymbol{\sigma}_{\mathrm{s},h}} \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} - \alpha_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}} - \nu\tau(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h})(\boldsymbol{x}_{\mathrm{s}}), && (3.7d)
\end{aligned}$$

where $\alpha_{\mathrm{s},h} := \dfrac{1}{|\Omega_{\mathrm{s}}|} \displaystyle\int_{\Omega_{\mathrm{s}} \setminus \Omega_h^{\mathrm{s}}} \boldsymbol{E}_{p_{\mathrm{s},h}}$, and $\boldsymbol{E}$ denotes the local extrapolation defined in (2.1).

In the particular case of matching interfaces, namely $\mathcal{I} = \overline{\Omega}_h^{\mathrm{s}} \cap \overline{\Omega}_h^{\mathrm{d}} = \mathcal{I}_h^{\mathrm{s}} = \mathcal{I}_h^{\mathrm{d}}$, the transmission conditions (3.4) become

$$\begin{aligned}
\langle \widehat{\mathbf{u}}_{\mathrm{s},h} \cdot \mathbf{n}_{\mathrm{s}}, \mu_{\mathrm{d},h} \rangle_{\mathcal{I}} + \langle \widehat{\mathbf{u}}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h} \rangle_{\mathcal{I}} &= 0 \quad \forall\, \mu_{\mathrm{d},h} \in N_h^{\mathrm{d}}, && (3.8a)\\
\langle \widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\mathcal{I}} - \langle \widehat{p}_{\mathrm{d},h}\mathbf{n}_{\mathrm{d}}, \boldsymbol{\mu}_{\mathrm{s},h} \rangle_{\mathcal{I}} &= 0 \quad \forall\, \boldsymbol{\mu}_{\mathrm{s},h} \in \mathbf{N}_h^{\mathrm{s}} && (3.8b)
\end{aligned}$$

and the resulting HDG formulation is very similar to the one presented in [18].

4 Stability analysis

In this section we show a stability estimate associated with (3.2) and (3.3). For that, we recall some important estimates and assumptions required to carry out our analysis.

Further notation and auxiliary estimates. Let $\star \in \{\mathrm{s}, \mathrm{d}\}$. Given a face $e \in \mathcal{I}_h^\star$ belonging to the element $K_e \in \Omega_h^\star$, we define the *extrapolation patch* as $K_e^{\mathtt{ext}} := \Big\{ \boldsymbol{x} + \mathbf{n}_\star t \,:\, 0 \leqslant t \leqslant |\sigma_\star(\boldsymbol{x})|,\, \boldsymbol{x} \in e \Big\}$, and denote by $h_e^\perp$ (resp. δ_e) the largest distance of a point inside K_e (resp. $K_e^{\mathtt{ext}}$) to the plane determined by the face e. In other words, $h_e^\perp = \max_{\boldsymbol{x}\in K_e} |\mathrm{dist}(\boldsymbol{x}, e)|$, $\delta_e = \max_{\boldsymbol{x}\in e} |\sigma_\star(\boldsymbol{x})|$, where $\mathrm{dist}(\boldsymbol{x}, e)$ denotes the distance from $\boldsymbol{x}$ to the face e. We note that δ_e is a measure of the local size of the gap and $\delta := \max_e \delta_e$ is an upper bound of the size of the gap. We define the ratio $r_e := \delta_e / h_e^\perp$and, for $e \in \mathcal{I}_h^{\mathrm{s}} \cup \mathcal{I}_h^{\mathrm{d}}$, $\boldsymbol{\mathcal{N}}^k := \Big\{ \mathbf{q} \in \mathbf{P}_k(K_e^{\mathtt{ext}}), \quad \mathbf{q}\cdot\mathbf{n}_e \neq 0 \text{ on each } e \subset \partial K_e^{\mathtt{ext}} \Big\}$, where we denoted by $\mathbf{n}_e$ the interior normal vector to $K_e^{\mathtt{ext}}$ along the face e, that is, the exterior normal vector to K_e pointing in the direction of $K_e^{\mathtt{ext}}$. We can then introduce the constants

$$C_e^{\mathtt{ext}} := \frac{1}{\sqrt{r_e}} \sup_{\mathbf{q}\in\boldsymbol{\mathcal{N}}^k} \frac{\|\mathbf{q}\cdot\mathbf{n}_e\|_{K_e^{\mathtt{ext}}}}{\|\mathbf{q}\cdot\mathbf{n}_e\|_{K_e}}, \quad C_e^{\mathtt{inv}} := h_e^\perp \sup_{\mathbf{q}\in\boldsymbol{\mathcal{N}}^k} \frac{\|\partial_{\mathbf{n}_e}\mathbf{q}\|_{K_e^{\mathtt{ext}}}}{\|\mathbf{q}\cdot\mathbf{n}_e\|_{K_e}}. \tag{4.1}$$

As proved in [9, Lemma A.2], these constants are independent of the meshsize, but depend on the polynomial degree k. The superscripts in $C_e^{\mathtt{ext}}$ and $C_e^{\mathtt{inv}}$ refer to an extrapolation constant and an inverse inequality constant.

On the other hand, proceeding as in [9], we introduce the following auxiliary functions: let $e \in \mathcal{I}_h^\star$ that belongs to K_e and $K_e^{\mathtt{ext}}$. For a function $\mathbf{q}$, we define

$$\Lambda_{\mathbf{q}|_{K_e}}(\boldsymbol{x}_\star) := \frac{1}{|\sigma_\star(\boldsymbol{x}_\star)|} \int_0^{|\sigma_\star(\boldsymbol{x}_\star)|} \big(\mathbf{q}_{K_e}(\boldsymbol{x}_\star + \mathbf{n}_\star y) - \mathbf{q}_{K_e}(\boldsymbol{x}_\star) \big) \cdot \mathbf{n}_\star dy, \tag{4.2}$$

for $\star \in \{\mathrm{s}, \mathrm{d}\}$, where $\boldsymbol{x}_{\mathrm{d}} \in e$ and $\boldsymbol{x}_{\mathrm{s}} \in \mathcal{I}_h^{\mathrm{s}}$ are connected by the segment $\sigma(\boldsymbol{x}_{\mathrm{d}})$. They satisfy (cf. [9, Lemma 5.2]),

$$\big\| |\sigma_\star|^{1/2} \Lambda_{\mathbf{q}|_{K_e}} \big\|_e \leqslant \frac{1}{\sqrt{3}} r_e^{3/2} C_e^{\mathtt{ext}} C_e^{\mathtt{inv}} \|\mathbf{q}\|_{K_e} \qquad \forall\, \mathbf{q} \in \mathbf{P}_k(K_e), \tag{4.3}$$

$$\big\| |\sigma_\star|^{1/2} \Lambda_{\mathbf{q}|_{K_e}} \big\|_e \leqslant \frac{1}{\sqrt{3}} r_e \big\| h_e^\perp \partial_n \mathbf{q}\cdot\mathbf{n} \big\|_{K_e^{\mathtt{ext}}} \qquad \forall\, \mathbf{q} \in \mathbf{H}^1(K_e^{\mathtt{ext}}). \tag{4.4}$$

Another important tool in the analysis of this method, which is based on the Taylor series expansion of a function defined on $\mathcal{I}$ around a point $\mathcal{I}_h^\star$, is the following lemma ([25, Lemma 2.1] or [22, Lemma 2]).

Lemma 1. *Suppose that $\boldsymbol{\psi}_\star : \mathcal{I} \to \mathcal{I}_h^\star$ is a bijection for each $\star \in \{\mathrm{s}, \mathrm{d}\}$. The following assertions hold true: If $\boldsymbol{\phi}_\star \in \mathbf{H}^2(\Omega_\star)$ and $\Phi_\star := \nabla \boldsymbol{\phi}_\star$, then*

$$\big\| |\sigma_\star|^{-1/2} (\boldsymbol{\phi}_\star - \boldsymbol{\phi}_\star \circ \boldsymbol{\psi}_\star^{-1}) + |\sigma_\star|^{1/2} (\Phi_\star \circ \boldsymbol{\psi}_\star^{-1}) \mathbf{n}_\star \big\|_{\mathcal{I}_h^\star} \lesssim \delta \|\boldsymbol{\phi}_\star\|_{\mathbf{H}^2(\Omega_\star)}, \tag{4.5}$$

$$\big\| |\sigma_\star|^{-1/2} (\boldsymbol{\phi}_\star - \boldsymbol{\phi}_\star \circ \boldsymbol{\psi}_\star^{-1}) \big\|_{\mathcal{I}_h^\star} \lesssim \delta^{1/2} \|\boldsymbol{\phi}_\star\|_{\mathbf{H}^2(\Omega_\star)}, \tag{4.6}$$

If $\Phi_\star \in \mathrm{H}^1(\Omega_\star)$ then

$$\big\| |\sigma_\star|^{-1/2} (\Phi_\star - \Phi_\star \circ \boldsymbol{\psi}_\star^{-1}) \mathbf{n}_\star \big\|_{\mathcal{I}_h^\star} \lesssim \|\Phi_\star\|_{\mathrm{H}^1(\Omega_\star)}. \tag{4.7}$$

Let $e \in \mathcal{I}$, $e_\star = \boldsymbol{\psi}_\star(e) \in \mathcal{I}_h^\star$ and $K_{e_\star}$ the element to which $e_\star$ belongs. If $p \in P_k(K_{e_\star})$, then

$$\|p - p \circ \boldsymbol{\psi}_\star^{-1}\|_{e_\star} \lesssim C_e^{\mathtt{ext}} \delta_e h_e^{-3/2} \|p\|_{K_{e_\star}}. \tag{4.8}$$

Finally, we recall the discrete trace inequality (cf. [12, Lemma 1.21]): if ϕ is a scalar, vector or tensor-valued polynomial in K_e, then

$$\|\phi\|_e \leqslant C_e^{\mathrm{tr}} h_e^{-1/2} \|\phi\|_{K_e}, \tag{4.9}$$

where C_e^{tr} is independent of the meshsize but depends on the polynomial degree. We stress that the identities and inequalities established throughout this section hold true for the tensor, vector or scalar-valued cases as required.

Assumptions. In this section, we state the assumptions under which the the stability and error analysis hold. Some of them are technical assumptions that allow us to simplify the analysis, whereas the others establish the relation between the gap size δ and the mesh size h required to guarantee convergence and optimality of the method. More precisely, we assume that

(A.1) $\Omega_h^{\mathrm{s}} \cap \Omega_h^{\mathrm{d}} = \emptyset$, that is, there is no overlap between the subdomains;

(A.2) the mappings $\boldsymbol{\psi}_\star : \mathcal{I} \to \mathcal{I}_\star$ for $\star \in \{\mathrm{s}, \mathrm{d}\}$, and $\boldsymbol{\psi} : \mathcal{I}_h^{\mathrm{d}} \to \mathcal{I}_h^{\mathrm{s}}$ are bijections;

(A.3) $4\nu^{-1}\tau^{-1/2} C_{\delta_\star,h} \max_{e\in\mathcal{I}_h^\star} \left(\delta_e^{-10/14}\right) + 8 \max_{e\in\mathcal{I}_h^\star} \left(\widehat{C}_e^\star \delta_e^{12/7} h_e^{-3} (C_e^{\texttt{ext}})^2 \tau^{-1}\right) \leqslant \dfrac{1}{4}$, where $C_{\delta_\star,h}$ depends on h and δ (cf. Lemma 2), and $\widehat{C}_e^\star$ is a positive constant appearing in the proof of Lemma 4;

(A.4) $C_{\delta,h} := \widetilde{C}_1 C_{\delta,h}^{\mathcal{S}} + \left((C_{\delta,h}^{p})^2 + \widetilde{C}_2\right) \widetilde{C}_{is}$ is small enough, where $C_{\delta,h}^{\mathcal{S}} := h^2 + (C_{\delta,h}^{u_{\mathrm{s}},L_{\mathrm{s}}})^2 + (C_{\delta,h}^{\widehat{u}_{\mathrm{s}},\widehat{p}_{\mathrm{d}}})^2$,

$$\widetilde{C}_1 := \max\{\nu, 1\} \sum_{\star\in\{\mathrm{s},\mathrm{d}\}} \left(4 \max_{e\in\mathcal{I}_h^\star} \left(\left(C_e^{\mathrm{tr}}\right)^2 \delta_e^{2/7} h_e^{-1} \tau\right) + \max_{e\in\mathcal{I}_h^\star} \left(\frac{(C_e^{\mathrm{tr}})^{-2} h_e^{-\beta}}{2}\right)\right),$$

$$\widetilde{C}_2 := \frac{8}{\nu} \max_{e\in\mathcal{I}_h^{\mathrm{s}}} \left(\widehat{C}_e^{\mathrm{s}} \delta_e^{12/7} h_e^{-3} (C_e^{\texttt{ext}})^2 \tau^{-1}\right) + 4\frac{C_\alpha^2}{\nu} \max_{e\in\mathcal{I}_h^{\mathrm{s}}} \left(\delta_e^{-2/7}\right);$$

$$C_{\delta,h}^{u_{\mathrm{s}},L_{\mathrm{s}}} := \max\{\nu^{-1}, 1\} \sum_{\star\in\{\mathrm{s},\mathrm{d}\}} \left(\max_{e\in\mathcal{I}_h^\star} \left(\delta_e h_e^{-3/2} C_e^{\texttt{ext}}\right) + \max_{e\in\mathcal{I}_h^\star} \left(\delta_e^{1/2} C_{\delta_\star,h}^{1/2}\right)\right),$$

$$C_{\delta,h}^{\widehat{u}_{\mathrm{s}},\widehat{p}_{\mathrm{d}}} := \max\{\nu^{-1}, 1\} \sum_{\star\in\{\mathrm{s},\mathrm{d}\}} \left(\max_{e\in\mathcal{I}_h^\star} \left(\delta_e^{6/7} \tau^{-1/2}\right) + \max_{e\in\mathcal{I}_h^\star} \left(\delta_e^{5/14} \tau^{-1/2}\right)\right),$$

$$C_{\delta,h}^{p} := \nu^{-1} \max_{e\in\mathcal{I}_h^{\mathrm{s}}} \left(\delta_e h_e^{-3/2} C_e^{\texttt{ext}}\right) + \nu^{-1} C_\alpha,$$

$\widetilde{C}_{is}$ is related to an inf-sup condition (see Lemma 6), β is a nonnegative parameter whose range will be determined later, and C_α is positive constant, which will appear in the proof of Lemma 4.

Assumptions **(A.1)** and **(A.2)** hold true, for instance, in the illustrations of Figure 1. Note that the purpose of **(A.1)** is to simplify the analysis. The Assumption **(A.2)** is the key to "tie" the interfaces $\mathcal{I}_h^{\mathrm{s}}$ and $\mathcal{I}_h^{\mathrm{d}}$ that cause the gap. The remaining assumptions are smallness assumptions that relate the meshsize and the size of the gap. For example, **(A.3)** is always satisfied for h small enough if $\delta \lesssim h^{7/4}$. To analyze the feasibility of other assumptions, let us write $\delta = C_{\mathrm{g}} h^{1+\gamma}$ with $C_{\mathrm{g}} \geqslant 0$ and $\gamma > 0$ constants independent of the meshsize. Assumption **(A.4)** is satisfied for all $\gamma > 3/4$ and $\beta \in [0, 2] \cap [0, 2\gamma - 1)$, if h is small enough, as we will explain in Corollaries 2 and 3. These are the strongest assumptions, since they indicate that our analysis holds if the gap size is at most of order $h^{7/4}$, however we will present numerical evidence suggesting that the method is still optimal when the gap is of order h. Finally, we highlight that the remaining constants are defined in the subsequent results presented below. In turn, in order to begin with the analysis, we establish the following result.

Lemma 2. *Suppose that Assumptions (A.1)-(A.2) hold, then it follows that*

$$\big\||\sigma_{\rm s}|^{-1/2}(\widetilde{\mathbf{u}}_{{\rm s},h}\circ\boldsymbol{\psi}^{-1}-\widehat{\mathbf{u}}_{{\rm s},h})\big\|_{\mathcal{I}_h^{\rm s}} \;\leqslant\; C_{\delta_{\rm s},h}^{1/2}\,\big\|{\rm L}_{{\rm s},h}\big\|_{\Omega_h^{\rm s}},\quad \textit{and} \tag{4.10a}$$

$$\big\||\sigma_{\rm d}|^{-1/2}(\widetilde{p}_{{\rm d},h}\circ\boldsymbol{\psi}-\widehat{p}_{{\rm d},h})\big\|_{\mathcal{I}_h^{\rm d}} \;\leqslant\; C_{\delta_{\rm d},h}^{1/2}\,\big\|\kappa^{-1}\mathbf{u}_{{\rm d},h}\big\|_{\Omega_h^{\rm d}}, \tag{4.10b}$$

where $C_{\delta_\star,h}=2\max_{e\in\mathcal{I}_h^\star}\left(\delta_e h_e^{-1}(C_e^{\rm tr})^2+\dfrac{1}{3}\delta_e^3h_e^{-3}\kappa_\star^3(C_e^{\tt ext}C_e^{\tt inv})^2\right)$ *for* $\star\in\{{\rm s},{\rm d}\}$.

Proof. It suffices to prove for $\star={\rm d}$, since the proof for $\star={\rm s}$ follows almost verbatim. We begin by stressing that $\boldsymbol{x}(y)=\boldsymbol{x}_{\rm d}+\mathbf{n}_{\rm d}y$, for all $y\in[0,|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|]$. From (3.7a) we deduce that

$$\begin{aligned}\widetilde{p}_{{\rm d},h}(\boldsymbol{x}_{\rm s}) &= \widehat{p}_{{\rm d},h}(\boldsymbol{x}_{\rm d})-\int_0^{|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|}\kappa^{-1}\Big(\boldsymbol{E}_{\mathbf{u}_{{\rm d},h}}(\boldsymbol{x}_{\rm d}+\mathbf{n}_{\rm d}y)-\mathbf{u}_{{\rm d},h}(\boldsymbol{x}_{\rm d})\Big)\cdot\mathbf{n}_{\rm d}dy-\kappa^{-1}|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|\mathbf{u}_{{\rm d},h}(\boldsymbol{x}_{\rm d})\cdot\mathbf{n}_{\rm d}\\ &= \widehat{p}_{{\rm d},h}(\boldsymbol{x}_{\rm d})-\kappa^{-1}|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|\Lambda_{\mathbf{u}_{{\rm d},h}}(\boldsymbol{x}_{\rm d})-\kappa^{-1}|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|\mathbf{u}_{{\rm d},h}(\boldsymbol{x}_{\rm d})\cdot\mathbf{n}_{\rm d},\end{aligned}$$

where we have used (4.2). This implies that $\mathbf{u}_{{\rm d},h}(\boldsymbol{x}_{\rm d})\cdot\mathbf{n}_{\rm d}=-\kappa|\sigma_{\rm d}(\boldsymbol{x}_{\rm d})|^{-1}(\widetilde{p}_{{\rm d},h}\circ\boldsymbol{\psi}-\widehat{p}_{{\rm d},h})-\Lambda_{\mathbf{u}_{{\rm d},h}}(\boldsymbol{x}_{\rm d})$.

By the Cauchy–Schwarz and Young inequalities, and the estimate (4.3), we obtain

$$\begin{aligned}\||\sigma_{\rm d}|^{-1/2}(\widetilde{p}_{{\rm d},h}\circ\boldsymbol{\psi}-\widehat{p}_{{\rm d},h})\|^2_{\mathcal{I}_h^{\rm d}} &\leqslant 2\left(\|\kappa^{-1}|\sigma_{\rm d}|^{1/2}\mathbf{u}_{{\rm d},h}\|^2_{\mathcal{I}_h^{\rm d}}+\|\kappa^{-1}|\sigma_{\rm d}|^{1/2}\Lambda_{\mathbf{u}_{{\rm d},h}}\|^2_{\mathcal{I}_h^{\rm d}}\right)\\ &\leqslant 2\Big(\|\kappa^{-1}|\sigma_{\rm d}|^{1/2}\mathbf{u}_{{\rm d},h}\|^2_{\mathcal{I}_h^{\rm d}}+\frac{1}{3}\max_{e\in\mathcal{I}_h^{\rm d}}\left(r_e^3(C_e^{\tt ext}C_e^{\tt inv})^2\right)\|\kappa^{-1}\mathbf{u}_{{\rm d},h}\|^2_{\mathcal{I}_h^{\rm d}}\Big).\end{aligned}$$

Finally, by the discrete trace inequality and the fact that $r_e\leqslant\delta_e h_e^{-1}\kappa_{\rm d}$, where we recall that $\kappa_{\rm d}$ is the mesh regularity constant of $\Omega_h^{\rm d}$, we obtain (4.10b). We omit further details. $\square$

In order to use this analysis to establish both, well-posedness and error bounds, we consider the problem (3.2), but (3.2a) and (3.2g) are replaced by

$$({\rm L}_{{\rm s},h},{\rm G}_{{\rm s},h})_{\Omega_h^{\rm s}}+(\mathbf{u}_{{\rm s},h},\nabla\cdot{\rm G}_{{\rm s},h})_{\Omega_h^{\rm s}}-\langle\widehat{\mathbf{u}}_{{\rm s},h},{\rm G}_{{\rm s},h}\mathbf{n}_{\rm s}\rangle_{\partial\Omega_h^{\rm s}} \;=\; ({\rm J}_{\rm s},{\rm G}_{{\rm s},h})_{\Omega_h^{\rm s}}, \tag{4.11a}$$

$$(\kappa^{-1}\mathbf{u}_{{\rm d},h},\mathbf{v}_{{\rm d},h})_{\Omega_h^{\rm d}}-(p_{{\rm d},h},\nabla\cdot\mathbf{v}_{{\rm d},h})_{\Omega_h^{\rm d}}+\langle\mathbf{v}_{{\rm d},h}\cdot\mathbf{n}_{\rm d},\widehat{p}_{{\rm d},h}\rangle_{\partial\Omega_h^{\rm d}} \;=\; (\mathbf{J}_{\rm d},\mathbf{v}_{{\rm d},h})_{\Omega_h^{\rm d}}, \tag{4.11b}$$

where ${\rm J}_{\rm s}\in{\rm L}^2(\Omega_h^{\rm s})$, $\mathbf{J}_{\rm d}\in\mathbf{L}^2(\Omega_h^{\rm d})$ are given functions orthogonal to polynomials of degree $k-1$ and (3.4) is replaced by

$$\langle\widetilde{\mathbf{u}}_{{\rm s},h}\cdot\mathbf{n}_{\rm s},\mu_{{\rm d},h}\rangle_{\mathcal{I}_h^{\rm d}}+\langle\widetilde{\mathbf{u}}_{{\rm d},h}\cdot\mathbf{n}_{\rm d},\mu_{{\rm d},h}\rangle_{\mathcal{I}_h^{\rm d}} \;=\; \langle j_{\rm d}^{\rm nc}+j_{\rm d}^{\delta},\mu_{{\rm d},h}\rangle_{\mathcal{I}_h^{\rm d}}\quad\forall\,\mu_{{\rm d},h}\in N_h^{\rm d}, \tag{4.12a}$$

$$\langle\widetilde{\boldsymbol{\sigma}}_{{\rm s},h}\mathbf{n}_{\rm s},\boldsymbol{\mu}_{{\rm s},h}\rangle_{\mathcal{I}_h^{\rm s}}-\langle\widetilde{p}_{{\rm d},h}\mathbf{n}_{\rm d},\boldsymbol{\mu}_{{\rm s},h}\rangle_{\mathcal{I}_h^{\rm s}} \;=\; \langle\mathbf{j}_{\rm s}^{\rm nc}+\mathbf{j}_{\rm s}^{\delta},\boldsymbol{\mu}_{{\rm s},h}\rangle_{\mathcal{I}_h^{\rm s}}\quad\forall\,\boldsymbol{\mu}_{{\rm s},h}\in\mathbf{N}_h^{\rm s}, \tag{4.12b}$$

where $j_{\rm d}^{\rm nc}$ and $\mathbf{j}_{\rm s}^{\rm nc}$ are given functions associated with the non-conformity that occurs at the interface, belonging to $L^2(\mathcal{I}_h^{\rm d})$ and $\mathbf{L}^2(\mathcal{I}_h^{\rm s})$, respectively. Similarly, $j_{\rm d}^{\delta}$ and $\mathbf{j}_{\rm s}^{\delta}$ are associated with the gap between the discrete interfaces $\mathcal{I}_h^{\rm s}$ and $\mathcal{I}_h^{\rm d}$, also belonging to $L^2(\mathcal{I}_h^{\rm d})$ and $\mathbf{L}^2(\mathcal{I}_h^{\rm s})$, respectively. In particular, to show well-posedness, ${\rm J}_{\rm s}$, $\mathbf{J}_{\rm d}$, $j_{\rm d}^{\rm nc}$, $\mathbf{j}_{\rm s}^{\rm nc}$, $j_{\rm d}^{\delta}$ and $\mathbf{j}_{\rm s}^{\delta}$ vanish, whereas they are related to projection errors when proving the error bounds.

4.1 An energy argument

Before presenting the energy estimate, we proceed to deduce how the transmission conditions (3.4) connect $\langle\widehat{\boldsymbol{\sigma}}_{{\rm s},h}\mathbf{n}_{\rm s},\widehat{\mathbf{u}}_{{\rm s},h}\rangle_{\mathcal{I}_h^{\rm s}}$ and $\langle\widehat{\mathbf{u}}_{{\rm d},h}\cdot\mathbf{n}_{\rm d},\widehat{p}_{{\rm d},h}\rangle_{\mathcal{I}_h^{\rm d}}$. To this end, we define $\mathbb{T}=-\langle\widehat{\boldsymbol{\sigma}}_{{\rm s},h}\mathbf{n}_{\rm s},\widehat{\mathbf{u}}_{{\rm s},h}\rangle_{\mathcal{I}_h^{\rm s}}+\langle\widehat{\mathbf{u}}_{{\rm d},h}\cdot\mathbf{n}_{\rm d},\widehat{p}_{{\rm d},h}\rangle_{\mathcal{I}_h^{\rm d}}$ and we write it in terms associated with the mismatch between $\mathcal{I}_h^{\rm s}$ and $\mathcal{I}_h^{\rm d}$. More precisely, we prove the following lemma.

Lemma 3. *It holds that*

$$\begin{aligned}\mathbb{T} =& \langle \big(\boldsymbol{E}_{\boldsymbol{\sigma}_{\mathrm{s},h}} \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1} - \boldsymbol{\sigma}_{\mathrm{s},h}\big)\, \mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \langle \big(\mathbf{u}_{\mathrm{d},h} - \boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}\big) \cdot \mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \alpha_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & \langle (\widetilde{p}_{\mathrm{d},h} \circ \boldsymbol{\psi} - \widehat{p}_{\mathrm{d},h})\mathbf{n}_{\mathrm{d}}, \widehat{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \langle (\widetilde{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}^{-1} - \widehat{\mathbf{u}}_{\mathrm{s},h}) \cdot \mathbf{n}_{\mathrm{s}}, \widehat{p}_{\mathrm{d},h} \circ \boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & + \langle j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \nu\langle \delta_e^{2/7}\tau\widehat{\mathbf{u}}_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \nu\langle \delta_e^{2/7}\tau(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}), \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} - \nu\langle \delta_e^{2/7}\tau\mathbf{u}_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & + \langle \delta_e^{2/7}\tau\widehat{p}_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle \delta_e^{2/7}\tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h}), \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \delta_e^{2/7}\tau p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}}.\end{aligned}$$

Before proving this result, we point out that in the particular case when $\mathcal{I}_h^{\mathrm{s}} = \mathcal{I}_h^{\mathrm{d}}$, we have coincident meshes free of hanging nodes, from which it is easily seen that $\mathbb{T} = 0$.

Proof. It follows straightforwardly from simple algebraic manipulations. Indeed, we first use the definition of the numerical fluxes (3.3) to rewrite the two last terms in (3.7c) and (3.7d). Next, using the conditions transmission (4.12), we obtain

$$\begin{aligned}\mathbb{T} =& -\langle \boldsymbol{\sigma}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \langle \mathbf{u}_{\mathrm{d},h} \cdot \mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle \tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h}), \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle \tau\nu(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}), \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ =& \langle \big(\boldsymbol{E}_{\boldsymbol{\sigma}_{\mathrm{s},h}} \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1} - \boldsymbol{\sigma}_{\mathrm{s},h}\big)\, \mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \langle \big(\mathbf{u}_{\mathrm{d},h} - \boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}\big) \cdot \mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \alpha_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & + \langle j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle (\widetilde{p}_{\mathrm{d},h} \circ \boldsymbol{\psi} - \widehat{p}_{\mathrm{d},h})\mathbf{n}_{\mathrm{d}}, \widehat{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & + \langle (\widetilde{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}^{-1} - \widehat{\mathbf{u}}_{\mathrm{s},h}) \cdot \mathbf{n}_{\mathrm{s}}, \widehat{p}_{\mathrm{d},h} \circ \boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}.\end{aligned}$$

Finally, in order to obtain the terms $\nu\|\delta_e^{1/7}\tau^{1/2}\widehat{\mathbf{u}}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}}$ and $\|\delta_e^{1/7}\tau^{1/2}\widehat{p}_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}}$ on the left-hand side of the stability estimate, we add

$$\begin{aligned}0 =& \nu\langle \delta_e^{2/7}\tau\widehat{\mathbf{u}}_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} + \nu\langle \delta_e^{2/7}\tau(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}), \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} - \nu\langle \delta_e^{2/7}\tau\mathbf{u}_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ & + \langle \delta_e^{2/7}\tau\widehat{p}_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \langle \delta_e^{2/7}\tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h}), \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle \delta_e^{2/7}\tau p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}},\end{aligned}$$

which finishes the proof. □

In what follows, we define

$$\begin{aligned}\mathcal{Q}\big(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\big) :=& \Big\{ \nu\|\mathrm{L}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \|\kappa^{-1}\mathbf{u}_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}} + \nu\|\tau^{1/2}(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h})\|^2_{\partial\Omega_h^{\mathrm{s}}} \\ & + \|\delta_e^{1/7}\tau^{1/2}\widehat{p}_{\mathrm{d},h}\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \|\tau^{1/2}(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h})\|^2_{\partial\Omega_h^{\mathrm{d}}} + \nu\|\delta_e^{1/7}\tau^{1/2}\widehat{\mathbf{u}}_{\mathrm{s},h}\|^2_{\mathcal{I}_h^{\mathrm{s}}}\Big\}^{1/2},\end{aligned} \tag{4.13}$$

and provide an upper bound for this energy term $\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})$. This bound depends, in addition to the sources, on the norms of the approximations of the velocity $\mathbf{u}_{\mathrm{s},h}$, pressures $p_{\mathrm{s},h}$, and $p_{\mathrm{d},h}$. Also, for a facet e of $\mathcal{E}_h^{\star}$, we consider $\boldsymbol{\mathcal{P}}^{\star} : \mathbf{L}^2(e) \to \mathbf{P}_k(e)$ and $\mathcal{P}^{\star} : L^2(e) \to P_k(e)$ the respective $\mathbf{L}^2$ and L^2 orthogonal projections. In abuse of notation, the global projections will be also denoted by $\boldsymbol{\mathcal{P}}^{\star}$ and $\mathcal{P}^{\star}$.

Lemma 4. *Assume that* $\mathrm{f}_{\mathrm{d}} = 0$ *and* $\mathbf{f}_{\mathrm{s}} = \mathbf{0}$*. If assumptions* ***(A.1)*** – ***(A.3)*** *hold and* $\star \in \{\mathrm{s}, \mathrm{d}\}$*, then*

$$\begin{aligned}\frac{5}{16}\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2 \leqslant& \widetilde{C}_1\big(\|\mathbf{u}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}}\big) + \widetilde{C}_2\|p_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + 4\|\kappa\mathbf{J}_{\mathrm{d}}\|^2_{\Omega_h^{\mathrm{d}}} + 4\nu\|\mathrm{J}_{\mathrm{s}}\|^2_{\Omega_h^{\mathrm{s}}} \\ & + \|h_e^{(\beta-1)/2}(j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \|h_e^{(\beta-1)/2}(\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{s}}},\end{aligned} \tag{4.14}$$

where β *is a non-negative parameter whose range will be chosen later.*

Proof. Taking $\mathrm{G}_{\mathrm{s},h} = \nu \mathrm{L}_{\mathrm{s},h}$, $\mathbf{v}_{\mathrm{s},h} = \mathbf{u}_{\mathrm{s},h}$, $q_{\mathrm{s},h} = p_{\mathrm{s},h}$, and $\boldsymbol{\mu}_{\mathrm{s},h} = \widehat{\mathbf{u}}_{\mathrm{s},h}$ in (4.11a), (3.2b), (3.2c), and (3.2d), along with (3.3b), we easily obtain

$$\nu\|\mathrm{L}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \nu\|\tau^{1/2}(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h})\|^2_{\partial\Omega_h^{\mathrm{s}}} - \langle \widehat{\boldsymbol{\sigma}}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} = (\nu \mathrm{J}_{\mathrm{s}}, \mathrm{L}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}}. \tag{4.15}$$

Similarly, taking $\mathbf{v}_{\mathrm{d},h} = \mathbf{u}_{\mathrm{d},h}$, $q_{\mathrm{d},h} = p_{\mathrm{d},h}$, $\mu_{\mathrm{d},h} = \widehat{p}_{\mathrm{d},h}$ in (4.11b), (3.2h), (3.2i), and (3.2j), along with (3.3a), we find that

$$(\kappa^{-1}\mathbf{u}_{\mathrm{d},h}, \mathbf{u}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \|\tau^{1/2}(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h})\|^2_{\partial\Omega_h^{\mathrm{d}}} + \langle \widehat{\mathbf{u}}_{\mathrm{d},h}\cdot\mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} = (\mathbf{J}_{\mathrm{d}}, \mathbf{u}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}}. \tag{4.16}$$

Next, adding (4.16) and (4.15), it follows that

$$\begin{aligned}&\nu\|\mathrm{L}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \nu\|\tau^{1/2}(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h})\|^2_{\partial\Omega_h^{\mathrm{s}}} + (\kappa^{-1}\mathbf{u}_{\mathrm{d},h}, \mathbf{u}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \|\tau^{1/2}(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h})\|^2_{\partial\Omega_h^{\mathrm{d}}} + \mathbb{T}\\ &\quad= (\nu \mathrm{J}_{\mathrm{s}}, \mathrm{L}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + (\mathbf{J}_{\mathrm{d}}, \mathbf{u}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}}.\end{aligned}$$

In this way, by combining this identity with the expression for $\mathbb{T}$ given in Lemma 3, we obtain

$$\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2 = \sum_{i=1}^{9} I_i + (\nu \mathrm{J}_{\mathrm{s}}, \mathrm{L}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + (\mathbf{J}_{\mathrm{d}}, \mathbf{u}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}}, \tag{4.17}$$

where $\mathcal{Q}$ is the energy term defined in (4.13), and

$$\begin{aligned}
I_1 &:= -\langle(\widetilde{p}_{\mathrm{d},h}\circ\boldsymbol{\psi} - \widehat{p}_{\mathrm{d},h})\mathbf{n}_{\mathrm{d}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\circ\boldsymbol{\psi}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, & I_2 &:= -\langle(\widetilde{\mathbf{u}}_{\mathrm{s},h}\circ\boldsymbol{\psi}^{-1} - \widehat{\mathbf{u}}_{\mathrm{s},h})\cdot\mathbf{n}_{\mathrm{s}}, \widehat{p}_{\mathrm{d},h}\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}},\\
I_3 &:= -\langle\big(\boldsymbol{E}_{\boldsymbol{\sigma}_{\mathrm{s},h}}\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1} - \boldsymbol{\sigma}_{\mathrm{s},h}\big)\,\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & I_4 &:= -\langle\big(\mathbf{u}_{\mathrm{d},h} - \boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}}\circ\boldsymbol{\psi}_{\mathrm{d}}^{-1}\big)\cdot\mathbf{n}_{\mathrm{d}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}},\\
I_5 &:= \langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & I_6 &:= -\langle j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}},\\
I_7 &:= \langle\alpha_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & I_8 &:= -\nu\langle\delta_e^{2/7}\tau(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}), \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}},\\
I_9 &:= -\langle\delta_e^{2/7}\tau(p_{\mathrm{d},h} - \widehat{p}_{\mathrm{d},h}), \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, & I_{10} &:= \nu\langle\delta_e^{2/7}\tau\mathbf{u}_{\mathrm{s},h}, \widehat{\mathbf{u}}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}},\\
I_{11} &:= \langle\delta_e^{2/7}\tau p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}}.
\end{aligned}$$

Now, recalling the properties of κ^{-1}, τ and ν, we apply the Cauchy–Schwarz and Young inequalities to bound each of these terms as follows. First,

$$\begin{aligned}
I_1 &\leqslant 4\tau^{-1/2}\nu^{-1}C_{\delta_{\mathrm{d}},h}\|\delta_e^{-5/14}\kappa^{-1}\mathbf{u}_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}} + \frac{\nu}{16}\|\delta_e^{1/7}\tau\widehat{\mathbf{u}}_{\mathrm{s},h}\|^2_{\mathcal{I}_h^{\mathrm{s}}}, \quad \text{and}\\
I_2 &\leqslant 4\tau^{-1/2}C_{\delta_{\mathrm{s}},h}\|\delta_e^{-5/14}\mathrm{L}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \frac{1}{16}\|\delta_e^{1/7}\tau\widehat{p}_{\mathrm{d},h}\|^2_{\mathcal{I}_h^{\mathrm{d}}},
\end{aligned}$$

where we have used (4.10b), and (4.10a), respectively. For I_3 and I_4, it follows from the estimate (4.8), that there exist positive constants $\widehat{C}_{\mathrm{s}}^{e}$ and $\widehat{C}_{e}^{\mathrm{d}}$, such that

$$\begin{aligned}
I_3 &\leqslant \frac{8}{\nu}\max_{e\in\mathcal{I}_h^{\mathrm{s}}}\left(\widehat{C}_e^{\mathrm{s}}\delta_e^{12/7}h_e^{-3}(C_e^{\mathtt{ext}})^2\tau^{-1}\right)\left(\nu^2\|\mathrm{L}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}}\right) + \frac{\nu}{16}\|\delta_e^{1/7}\tau^{1/2}\widehat{\mathbf{u}}_{\mathrm{s},h}\|^2_{\mathcal{I}_h^{\mathrm{s}}}, \quad \text{and}\\
I_4 &\leqslant 4\max_{e\in\mathcal{I}_h^{\mathrm{d}}}\left(\widehat{C}_e^{\mathrm{d}}\delta_e^{12/7}h_e^{-3}(C_e^{\mathtt{ext}})^2\tau^{-1}\right)\|\kappa^{-1}\mathbf{u}_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}} + \frac{1}{16}\|\delta_e^{1/7}\tau^{1/2}\widehat{p}_{\mathrm{d},h}\|^2_{\mathcal{I}_h^{\mathrm{d}}}.
\end{aligned}$$

For I_5 and I_6, we have

$$\begin{aligned}
I_5 &\leqslant \|h_e^{(\beta-1)/2}(\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{s}}} + \frac{1}{2}\|h_e^{(1-\beta)/2}(\widehat{\mathbf{u}}_{\mathrm{s},h} - \mathbf{u}_{\mathrm{s},h})\|^2_{\mathcal{I}_h^{\mathrm{s}}} + \max_{e\in\mathcal{I}_h^{\mathrm{s}}}\left(\frac{(C_e^{\mathrm{tr}})^{-2}h_e^{-\beta}}{2}\right)\|\mathbf{u}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}}, \quad \text{and}\\
I_6 &\leqslant \|h_e^{(\beta-1)/2}(j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \frac{1}{2}\|h_e^{(1-\beta)/2}(\widehat{p}_{\mathrm{d},h} - p_{\mathrm{d},h})\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \max_{e\in\mathcal{I}_h^{\mathrm{d}}}\left(\frac{(C_e^{\mathrm{tr}})^{-2}h_e^{-\beta}}{2}\right)\|p_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}}.
\end{aligned}$$

Next, noticing that $I_7 \leqslant \|\alpha_{s,h}\|_{\mathcal{I}_h^s}\|\hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h^s}$, and $\|\alpha_{s,h}\|_{\mathcal{I}_h^s} \leqslant C_\alpha \|p_{s,h}\|_{\Omega_h^s}$, where

$$C_\alpha := |\mathcal{I}_h^s|^{1/2} \frac{|\Omega_s \backslash \Omega_h^s|^{1/2}}{|\Omega_s|} \max_{e \in \mathcal{I}_h^s} \left(\delta_e^{1/2} h_e^{-1/2} C_e^{\texttt{ext}} \right),$$

it follows that

$$I_7 \leqslant 4\frac{C_\alpha^2}{\nu} \|\delta_e^{-1/7} p_{s,h}\|_{\Omega_h^s}^2 + \frac{\nu}{16} \|\delta_e^{1/7} \tau^{1/2} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h^s}^2.$$

For I_8 and I_9, we easily obtain

$$\begin{aligned} I_8 &\leqslant 4\nu \|\delta_e^{1/7} \tau^{1/2} (\mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h})\|_{\mathcal{I}_h^s}^2 + \frac{\nu}{16} \|\delta_e^{1/7} \tau^{1/2} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h^s}^2, \quad \text{and} \\ I_9 &\leqslant 4\|\delta_e^{1/7} \tau^{1/2} (p_{d,h} - \widehat{p}_{d,h})\|_{\mathcal{I}_h^d}^2 + \frac{1}{16} \|\delta_e^{1/7} \tau^{1/2} \widehat{p}_{d,h}\|_{\mathcal{I}_h^d}^2. \end{aligned}$$

In turn, using the discrete trace inequality (4.9), we find that

$$\begin{aligned} I_{10} &\leqslant 4\nu \max_{e \in \mathcal{I}_h^s} \left(\left(C_e^{\text{tr}}\right)^2 \delta_e^{2/7} h_e^{-1} \tau \right) \|\mathbf{u}_{s,h}\|_{\Omega_h^s}^2 + \frac{\nu}{16} \|\delta_e^{1/7} \tau^{1/2} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h^s}^2, \\ I_{11} &\leqslant 4 \max_{e \in \mathcal{I}_h^d} \left(\left(C_e^{\text{tr}}\right)^2 \delta_e^{2/7} h_e^{-1} \tau \right) \|p_{d,h}\|_{\Omega_h^d}^2 + \frac{1}{16} \|\delta_e^{1/7} \tau^{1/2} \widehat{p}_{d,h}\|_{\mathcal{I}_h^d}^2. \end{aligned}$$

Finally, rearranging terms and bearing in mind the Assumption **(A.3)**, we obtain (4.14). We omit further details. □

Our next goal is to provide an estimate for the L^2-norm of $\mathbf{u}_{s,h}$, $p_{s,h}$, and $p_{d,h}$. To bound $\|\mathbf{u}_{s,h}\|_{\Omega_h^s}$ and $\|p_{d,h}\|_{\Omega_h^d}$ we employ a duality argument, whereas for $\|p_{s,h}\|_{\Omega_h^s}$ we use an inf-sup condition.

4.2 A duality argument

In order to estimate $\|\mathbf{u}_{s,h}\|_{\Omega_h^s} + \|p_{d,h}\|_{\Omega_h^d}$, we now proceed as in [25, 22, 18] and incorporate a suitable auxiliary problem. More precisely, in what follows we consider the continuous problem (1.1)-(1.2)-(1.3) with sources given by $\mathbf{f}_s := \Theta_s \in \mathbf{L}^2(\Omega_s)$ and $\mathrm{f}_d := \Theta_d \in L^2(\Omega_d)$, that is:

$$\Phi_s = \nabla \boldsymbol{\phi}_s,\ \nabla \cdot (\nu \Phi_s - \widetilde{\varphi}_s \mathrm{I}) = \boldsymbol{\Theta}_s,\ \nabla \cdot \boldsymbol{\phi}_s = 0 \quad \text{in} \quad \Omega_s,\ \boldsymbol{\phi}_s = 0 \quad \text{on} \quad \Gamma_s \quad \text{and} \quad \int_{\Omega_s} \widetilde{\varphi}_s = 0, \tag{4.18}$$

$$\nabla \cdot \boldsymbol{\phi}_d = \Theta_d, \quad \boldsymbol{\phi}_d + \kappa \nabla \varphi_d = \mathbf{0} \quad \text{in} \quad \Omega_d \quad \text{and} \quad \varphi_d = 0 \quad \text{on} \quad \Gamma_d, \tag{4.19}$$

$$\boldsymbol{\phi}_s \cdot \mathbf{n}_s + \boldsymbol{\phi}_d \cdot \mathbf{n}_d = 0, \quad (\nu \Phi_s - \widetilde{\varphi}_s \mathrm{I})\mathbf{n}_s = \varphi_d \mathbf{n}_d \quad \text{on} \quad \mathcal{I}. \tag{4.20}$$

In addition, we proceed to carry the same decomposition performed for the pressure of the continuous problem, that is $\widetilde{\varphi}_s = \widetilde{\alpha}_s + \varphi_s$, where $\varphi_s \in L_0^2(\Omega_h^s)$, and $\widetilde{\alpha}_s = \dfrac{-1}{|\Omega_s|} \displaystyle\int_{\Omega_s \backslash \Omega_h^s} \varphi_s$, is the constant similar to (3.1). Furthermore, suppose that elliptic regularity holds, that is,

$$\nu \|\Phi_s\|_{\mathrm{H}^1(\Omega_s)} + \nu \|\boldsymbol{\phi}_s\|_{\mathbf{H}^2(\Omega_s)} + \|\varphi_s\|_{H^1(\Omega_s)} + \|\boldsymbol{\phi}_d\|_{\mathbf{H}^1(\Omega_d)} + \|\varphi_d\|_{H^2(\Omega_d)} \leqslant C_{\mathbf{r}} \Big\{ \|\Theta_s\|_{\Omega_s} + \|\Theta_d\|_{\Omega_d} \Big\}. \tag{4.21}$$

Lemma 5. *Suppose Assumptions* **(A)** *and* (4.21) *hold true. There exist $h_0 \in (0,1)$ and a positive constant C such that, for all $h < h_0$,*

$$\begin{aligned} \|\mathbf{u}_{s,h}\|_{\Omega_h^s}^2 + \|p_{d,h}\|_{\Omega_h^d}^2 \leqslant C \Big\{ & C_{\delta,h}^{\mathcal{S}} \mathcal{Q}(\mathrm{L}_{s,h}, \mathbf{u}_{\star,h}, \hat{\mathbf{u}}_{s,h}, p_{d,h}, \widehat{p}_{d,h})^2 + \|\mathbf{j}_s^{\text{nc}} + \mathbf{j}_s^{\delta}\|_{\mathcal{I}_h^s}^2 + \|j_d^{\text{nc}} + j_d^{\delta}\|_{\mathcal{I}_h^d}^2 \\ & + (C_{\delta,h}^p)^2 \|p_{s,h}\|_{\Omega_h^s}^2 + h^{2\min\{1,k\}} \|\mathbf{J}_d\|_{\Omega_h^d}^2 + h^{2\min\{1,k\}} \|\mathrm{J}_s\|_{\Omega_h^s}^2 \Big\}, \end{aligned} \tag{4.22}$$

where $C_{\delta,h}^{\mathcal{S}}$ and $C_{\delta,h}^p$ are constants defined in ***(A.4)****.*

Proof. We proceeded as in [18, Lemma 4.6]. First, from (4.18) and (4.19), we have

$$(\mathbf{u}_{s,h}, \Theta_s)_{\Omega_h^s} + (p_{d,h}, \Theta_d)_{\Omega_h^d} = (\mathbf{u}_{s,h}, \nabla\cdot(\nu\Phi_s - \widetilde{\varphi}_s \mathrm{I}))_{\Omega_h^s} + (p_{d,h}, \nabla\cdot\boldsymbol{\phi}_d)_{\Omega_h^d} + (\nu \mathrm{L}_{s,h}, \Phi_s - \nabla\boldsymbol{\phi}_s)_{\Omega_h^s} - (\mathbf{u}_{d,h}, \kappa^{-1}\boldsymbol{\phi}_d + \nabla\varphi_d)_{\Omega_h^d}.$$

Now, using the properties of the HDG projectors, L^2-projectors, and performing some algebraic manipulations along with the transmission conditions given by (1.3) and (4.12), we deduce that

$$\begin{aligned}
&(\mathbf{u}_{s,h}, \Theta_s)_{\Omega_h^s} + (p_{d,h}, \Theta_d)_{\Omega_h^d}\\
&= -(\nu \mathrm{L}_{s,h}, \Pi_G^s\Phi_s - \Phi_s)_{\Omega_h^s} - (\mathrm{J}_s, \nu\Pi_G^s\Phi_s)_{\Omega_h^s} + (\mathbf{J}_d, \Pi_{\mathbf{V}}^d\boldsymbol{\phi}_d - \boldsymbol{\phi}_d)_{\Omega_h^d} - (\mathbf{J}_d, \kappa\nabla\varphi_d)_{\Omega_h^d} + \langle \boldsymbol{\phi}_d\cdot\mathbf{n}_d, \widehat{p}_{d,h}\rangle_{\mathcal{I}_h^d}\\
&\quad + (\kappa^{-1}\mathbf{u}_{d,h}, \Pi_{\mathbf{V}}^d\boldsymbol{\phi}_d - \boldsymbol{\phi}_d)_{\Omega_h^d} + \langle \widehat{\mathbf{u}}_{s,h}, (\nu\Phi_s - \mathrm{I}\widetilde{\varphi}_s)\mathbf{n}_s\rangle_{\mathcal{I}_h^s} - \langle(\nu\widehat{\mathrm{L}}_{s,h} - \mathrm{I}\widehat{p}_{s,h})\mathbf{n}_s, \boldsymbol{\phi}_s\rangle_{\partial\Omega_h^s} - \langle \widehat{\mathbf{u}}_{d,h}\cdot\mathbf{n}_d, \varphi_d\rangle_{\partial\Omega_h^d}\\
&= \mathbb{T}_1 + \mathbb{T}_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{T}_1 =& -(\nu \mathrm{L}_{s,h}, \Pi_G^s\Phi_s - \Phi_s)_{\Omega_h^s} - (\mathrm{J}_s, \nu(\Pi_G^s\Phi_s - \Phi_s))_{\Omega_h^s} - (\mathrm{J}_s, \nu(\Phi_s - \boldsymbol{P}_{k-1}(\Phi_s)))_{\Omega_h^s} + (\mathbf{J}_d, \Pi_{\mathbf{V}}^d\boldsymbol{\phi}_d - \boldsymbol{\phi}_d)_{\Omega_h^d}\\
&- (\mathbf{J}_d, \kappa(\nabla\varphi_d - P_{k-1}(\nabla\varphi_d)))_{\Omega_h^d} + (\kappa^{-1}\mathbf{u}_{d,h}, \Pi_{\mathbf{V}}^d\boldsymbol{\phi}_d - \boldsymbol{\phi}_d)_{\Omega_h^d},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{T}_2 =& \langle(\mathbf{u}_{d,h} - \boldsymbol{E}_{\mathbf{u}_{d,h}}\circ\boldsymbol{\psi}_d^{-1})\cdot\mathbf{n}_d, \mathcal{P}^d\varphi_d\rangle_{\mathcal{I}_h^d} + \langle\boldsymbol{\mathcal{P}}^s\boldsymbol{\phi}_s, (\boldsymbol{E}_{\boldsymbol{\sigma}_{s,h}}\circ\boldsymbol{\psi}_s^{-1} - \boldsymbol{\sigma}_{s,h})\,\mathbf{n}_s\rangle_{\mathcal{I}_h^s} - \langle\mathbf{j}_s^{nc} + \mathbf{j}_s^{\delta}, \boldsymbol{\mathcal{P}}^s\boldsymbol{\phi}_s\rangle_{\mathcal{I}_h^s}\\
&- \langle j_d^{nc} + j_d^{\delta}, \mathcal{P}^d\varphi_d\rangle_{\mathcal{I}_h^d} + \langle(\widehat{p}_{d,h}\circ\boldsymbol{\psi}^{-1} - \widetilde{p}_{d,h})\,\mathbf{n}_d, \boldsymbol{\mathcal{P}}^s\boldsymbol{\phi}_s\rangle_{\mathcal{I}_h^s} + \langle(\widetilde{\mathbf{u}}_{s,h} - \widehat{\mathbf{u}}_{s,h}\circ\boldsymbol{\psi})\cdot\mathbf{n}_s, \mathcal{P}^d\varphi_d\rangle_{\mathcal{I}_h^d}\\
&+ \langle\widehat{\mathbf{u}}_{s,h}, (\mathcal{P}^d\varphi_d - \varphi_d)\mathbf{n}_d\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^s} + \langle(\boldsymbol{\mathcal{P}}^s\boldsymbol{\phi}_s - \boldsymbol{\phi}_s)\circ\boldsymbol{\psi}\cdot\mathbf{n}_s, \widehat{p}_{d,h}\rangle_{\mathcal{I}_h^d} + \langle\widehat{\mathbf{u}}_{s,h}\circ\boldsymbol{\psi}, (\varphi_d\circ\boldsymbol{\psi}_d^{-1} - \varphi_d)\,\mathbf{n}_d\rangle_{\mathcal{I}_h^d}\\
&- \langle\widehat{p}_{d,h}\circ\boldsymbol{\psi}^{-1}, (\boldsymbol{\phi}_s\circ\boldsymbol{\psi}_s^{-1} - \boldsymbol{\phi}_s)\cdot\mathbf{n}_s\rangle_{\mathcal{I}_h^s} + \langle\widehat{\mathbf{u}}_{s,h}, (\nu\Phi_s - \varphi_s\mathrm{I} - (\nu\Phi_s - \varphi_s\mathrm{I})\circ\boldsymbol{\psi}_s^{-1})\mathbf{n}_s\rangle_{\mathcal{I}_h^s}\\
&+ \langle\widehat{\mathbf{u}}_{s,h}, \widetilde{\alpha}_s\mathbf{n}_s\rangle_{\mathcal{I}_h^s} + \langle\widehat{p}_{d,h}, (\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\circ\boldsymbol{\psi}_d^{-1})\cdot\mathbf{n}_d\rangle_{\mathcal{I}_h^d} - \langle\alpha_{s,h}\mathbf{n}_s, \boldsymbol{\mathcal{P}}^s\boldsymbol{\phi}_s\rangle_{\mathcal{I}_h^s}.
\end{aligned}$$

Here $\boldsymbol{P}_{k-1}|_{\widetilde{K}}$ and $P_{k-1}|_K$ are the projections $\mathbf{L}^2(\widetilde{K})$ and $L^2(K)$ onto $\mathbf{P}_{k-1}(\widetilde{K})$ and $P_{k-1}(K)$ for each $\widetilde{K}\in\Omega_h^s$ and $K\in\Omega_h^d$. Hence, applying the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\mathbb{T}_1 \leqslant& \Big\{\|\mathrm{L}_{s,h}\|_{\Omega_h^s} + \|\mathrm{J}_s\|_{\Omega_h^s} + (1+\overline{\kappa})\|\mathbf{J}_d\|_{\Omega_h^d} + \|\kappa^{-1}\mathbf{u}_{d,h}\|_{\Omega_h^d}\Big\}\Big\{\|\nu(\Pi_G^s\mathrm{L}_s - \Phi_s)\|_{\Omega_h^s}\\
&+\|\nu(\Phi_s - \boldsymbol{P}_{k-1}(\Phi_s))\|_{\Omega_h^s} + \|\Pi_{\mathbf{V}}^d\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{\Omega_h^d} + \|\nabla\varphi_d - P_{k-1}(\nabla\varphi_d)\|_{\Omega_h^d}\Big\}.
\end{aligned}$$

Thus, invoking [12, Lemma 1.58], the approximation properties given by (2.3) and (2.5), and the regularity assumption (cf. (4.21)) we obtain

$$\begin{aligned}
\mathbb{T}_1 \leqslant& C_1 h\Big\{\|\mathrm{L}_{s,h}\|_{\Omega_h^s} + \|\mathrm{J}_s\|_{\Omega_h^s} + (1+\overline{\kappa})\|\mathbf{J}_d\|_{\Omega_h^d} + \|\kappa^{-1}\mathbf{u}_{d,h}\|_{\Omega_h^d}\Big\}\Big\{\|\nu\Phi_s\|_{1,\Omega_h^s} + \|\varphi_s\|_{1,\Omega_h^s}\\
&+ \|\nu\boldsymbol{\phi}_s\|_{2,\Omega_h^s} + \|\Theta_s\|_{\Omega_h^s} + \|\boldsymbol{\phi}_d\|_{1,\Omega_h^d} + \|\varphi_d\|_{2,\Omega_h^d}\Big\}\\
\leqslant& 2C_1C_{\mathrm{r}}h\Big\{\|\mathrm{L}_{s,h}\|_{\Omega_h^s} + \|\mathrm{J}_s\|_{\Omega_h^s} + (1+\overline{\kappa})\|\mathbf{J}_d\|_{\Omega_h^d} + \|\kappa^{-1}\mathbf{u}_{d,h}\|_{\Omega_h^d}\Big\}\Big\{\|\Theta_s\|_{\Omega_h^s} + \|\Theta_d\|_{\Omega_h^d}\Big\}. \qquad (4.23)
\end{aligned}$$

In turn, for $\mathbb{T}_2$ we write

$$\begin{aligned}
\mathbb{B}_1 &:= \langle(\mathbf{u}_{\mathrm{d},h} - \boldsymbol{E}_{\mathbf{u}_{\mathrm{d},h}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}) \cdot \mathbf{n}_{\mathrm{d}}, \mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, & \mathbb{B}_2 &:= \langle \boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}, (\boldsymbol{E}_{\boldsymbol{\sigma}_{\mathrm{s},h}} \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1} - \boldsymbol{\sigma}_{\mathrm{s},h})\,\mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, \\
\mathbb{B}_3 &:= -\langle \mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}, \boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & \mathbb{B}_4 &:= -\langle j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}, \mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, \\
\mathbb{B}_5 &:= \langle(\widehat{p}_{\mathrm{d},h} \circ \boldsymbol{\psi}^{-1} - \widetilde{p}_{\mathrm{d},h})\,\mathbf{n}_{\mathrm{d}}, \boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & \mathbb{B}_6 &:= \langle(\widetilde{\mathbf{u}}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}) \cdot \mathbf{n}_{\mathrm{s}}, \mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, \\
\mathbb{B}_7 &:= \langle \widehat{\mathbf{u}}_{\mathrm{s},h}, (\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}} - \varphi_{\mathrm{d}})\mathbf{n}_{\mathrm{d}} \circ \boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & \mathbb{B}_8 &:= \langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}} - \boldsymbol{\phi}_{\mathrm{s}}) \circ \boldsymbol{\psi} \cdot \mathbf{n}_{\mathrm{s}}, \widehat{p}_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, \\
\mathbb{B}_9 &:= \langle \widehat{\mathbf{u}}_{\mathrm{s},h} \circ \boldsymbol{\psi}, (\varphi_{\mathrm{d}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1} - \varphi_{\mathrm{d}})\,\mathbf{n}_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, & \mathbb{B}_{10} &:= -\langle \widehat{p}_{\mathrm{d},h} \circ \boldsymbol{\psi}^{-1}, (\boldsymbol{\phi}_{\mathrm{s}} \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1} - \boldsymbol{\phi}_{\mathrm{s}}) \cdot \mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, \\
\mathbb{B}_{11} &:= \langle \widehat{\mathbf{u}}_{\mathrm{s},h}, (\nu\Phi_{\mathrm{s}} - \varphi_{\mathrm{s}}\mathrm{I} - (\nu\Phi_{\mathrm{s}} - \varphi_{\mathrm{s}}\mathrm{I}) \circ \boldsymbol{\psi}_{\mathrm{s}}^{-1})\mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & \mathbb{B}_{12} &:= \langle \widehat{p}_{\mathrm{d},h}, (\boldsymbol{\phi}_{\mathrm{d}} - \boldsymbol{\phi}_{\mathrm{d}} \circ \boldsymbol{\psi}_{\mathrm{d}}^{-1}) \cdot \mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}, \\
\mathbb{B}_{13} &:= -\langle \alpha_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}, & \mathbb{B}_{14} &:= \langle \widehat{\mathbf{u}}_{\mathrm{s},h}, \widetilde{\alpha}_{\mathrm{s}}\mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}.
\end{aligned}$$

In this way, we bound each of these terms applying the Cauchy–Schwarz and trace inequalities, and the regularity Assumption (cf. (4.21)). Indeed, note that

$$\begin{aligned}
\mathbb{B}_3 &\lesssim \nu^{-1}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}, & \mathbb{B}_4 &\lesssim \|j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{d}}}\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}, \\
\mathbb{B}_{13} &\lesssim \nu^{-1}C_\alpha\|p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}, & \mathbb{B}_{14} &\lesssim \nu^{-1}\delta_e^{5/14}\|\tau\delta_e^{1/7}\widehat{\mathbf{u}}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_{\mathrm{s}}}.
\end{aligned}$$

In turn, thanks to Lemma 1, it follows that

$$\begin{aligned}
\mathbb{B}_1 &\lesssim \max_{e\in\mathcal{I}_h^{\mathrm{d}}}\left(\delta_e h_e^{-3/2}C_e^{\mathsf{ext}}\right)\|\mathbf{u}_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}}\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}, & \mathbb{B}_9 &\lesssim \delta\|\widehat{\mathbf{u}}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}}\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}, \\
\mathbb{B}_2 &\lesssim \nu^{-1}\max_{e\in\mathcal{I}_h^{\mathrm{s}}}\left(\delta_e h_e^{-3/2}C_e^{\mathsf{ext}}\right)\|\boldsymbol{\sigma}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}, & \mathbb{B}_{10} &\lesssim \nu^{-1}\delta\|\widehat{p}_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}, \\
\mathbb{B}_{11} &\lesssim \delta^{1/2}\|\widehat{\mathbf{u}}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}, & \mathbb{B}_{12} &\lesssim \delta^{1/2}\|\widehat{p}_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}}\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}.
\end{aligned}$$

Next, from (4.10a) and (4.10b), we obtain

$$\mathbb{B}_6 \lesssim \nu^{-1}\delta_e^{1/2}C_{\delta_{\mathrm{s},h}}^{1/2}\|\nu\mathrm{L}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}, \qquad \mathbb{B}_5 \lesssim \nu^{-1}\delta_e^{1/2}C_{\delta_{\mathrm{d},h}}^{1/2}\|\kappa^{-1}\mathbf{u}_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{s}}}\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}.$$

On the other hand, from [12, Lemma 1.59], we find that

$$\begin{aligned}
\mathbb{B}_7 &\lesssim \left(h_e^{3/2}\|\widehat{\mathbf{u}}_{\mathrm{s},h} - \mathbf{u}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}} + h_e(C_e^{\mathrm{tr}})^{-1}\|\mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\right)\|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}, \\
\mathbb{B}_8 &\lesssim \left(h_e^{3/2}\|\widehat{p}_{\mathrm{d},h} - p_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}} + h_e(C_e^{\mathrm{tr}})^{-1}\|p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}}\right)\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}.
\end{aligned}$$

In summary, adding up all the estimates for $\mathbb{B}_i$ $(i = 1, \ldots, 14)$, we deduce that

$$\begin{aligned}
\mathbb{T}_2 \lesssim \Big\{ & C_{\delta,h}^{u_{\mathrm{s}},L_{\mathrm{s}}}\left(\|\kappa^{-1}\mathbf{u}_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}} + \|\mathrm{L}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\right) + C_{\delta,h}^{\widehat{u}_{\mathrm{s}},\widehat{p}_{\mathrm{d}}}\left(\nu\|\delta_e^{1/7}\tau^{1/2}\widehat{\mathbf{u}}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}} + \|\delta_e^{1/7}\tau^{1/2}\widehat{p}_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}}\right) \\
& + C_{\delta,h}^{p}\|p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + \|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{s}}} + \|j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{d}}} + h_e^{3/2}\|\widehat{\mathbf{u}}_{\mathrm{s},h} - \mathbf{u}_{\mathrm{s},h}\|_{\mathcal{I}_h^{\mathrm{s}}} \\
& + h_e(C_e^{\mathrm{tr}})^{-1}\|\mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + h_e^{3/2}\|\widehat{p}_{\mathrm{d},h} - p_{\mathrm{d},h}\|_{\mathcal{I}_h^{\mathrm{d}}} + h_e(C_e^{\mathrm{tr}})^{-1}\|p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}}\Big\}\Big\{\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}} + \|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}\Big\},
\end{aligned} \tag{4.24}$$

where $C_{\delta,h}^{u_{\mathrm{s}},L_{\mathrm{s}}}$, $C_{\delta,h}^{\widehat{u}_{\mathrm{s}},\widehat{p}_{\mathrm{d}}}$, and $C_{\delta,h}^{p}$ are the constants that have been defined in **(A.4)**. In this way, from (4.23) and (4.24), we find that

$$\begin{aligned}
& (\mathbf{u}_{\mathrm{s},h}, \Theta_{\mathrm{s}})_{\Omega_h^{\mathrm{s}}} + (p_{\mathrm{d},h}, \Theta_{\mathrm{d}})_{\Omega_h^{\mathrm{d}}} \\
& \lesssim \Big\{\left(h + C_{\delta,h}^{u_{\mathrm{s}},L_{\mathrm{s}}} + C_{\delta,h}^{\widehat{u}_{\mathrm{s}},\widehat{p}_{\mathrm{d}}}\right)\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h}) + C_{\delta,h}^{p}\|p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + \|j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{d}}} \\
& \quad + \|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{s}}} + h^{\min\{1,k\}}\left(\|\mathbf{J}_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}} + \|\mathrm{J}_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}\right) + h\left(\|p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}} + \|\mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}\right)\Big\}\Big\{\|\Theta_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}} + \|\Theta_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}\Big\}.
\end{aligned}$$

Hence, taking $\Theta_{\mathrm{s}} = \mathbf{u}_{\mathrm{s},h}$ and $\Theta_{\mathrm{d}} = p_{\mathrm{d},h}$ leads to the required inequality (4.22). □

Lemma 6. *Assume that* $\mathrm{f}_{\mathrm{d}} = 0$ *and* $\mathbf{f}_{\mathrm{s}} = \mathbf{0}$*. It holds that*

$$\|p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} \leqslant \widetilde{C}_{is}\Big\{\nu\|\mathrm{L}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + \nu\|\tau^{1/2}\left(\mathbf{u}_{\mathrm{s},h} - \widehat{\mathbf{u}}_{\mathrm{s},h}\right)\|_{\partial\Omega_h^{\mathrm{s}}}\Big\}, \tag{4.25}$$

where $\widetilde{C}_{is} = \widetilde{\beta}\max\left\{1, \max_{K\in\Omega_h^{\mathrm{s}}}(\tau h_K)^{1/2}\right\}$*, and* $\widetilde{\beta}$ *is a positive constant independent of* h*.*

Proof. See [26, Lemma 2] or [22, Lemma 10]. □

We are now in position to establish the main result of this section.

Theorem 1. *Suppose Assumptions* **(A)** *and elliptic regularity (cf.* (4.21)*) hold true. If* τ *is of order one,* $k \geqslant 1$ *and* $h < 1$*, there exists* $h_0 \in (0,1)$*, such that for all* $h < h_0$*,*

$$\begin{aligned}
&\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2 \\
&\quad\lesssim \|h_e^{(\beta-1)/2}(j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \|h_e^{(\beta-1)/2}(\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{s}}} + \|\mathbf{J}_{\mathrm{d}}\|^2_{\Omega_h^{\mathrm{d}}} + \|\nu\mathrm{J}_{\mathrm{s}}\|^2_{\Omega_h^{\mathrm{s}}},
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\|\mathbf{u}_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{d},h}\|^2_{\Omega_h^{\mathrm{d}}} \lesssim\ &\Big\{C^{\mathcal{S}}_{\delta,h}\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2 + \|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta}\|^2_{\mathcal{I}_h^{\mathrm{s}}} + \|j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta}\|^2_{\mathcal{I}_h^{\mathrm{d}}} \\
&+ (C^{p}_{\delta,h})^2\|p_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} + h^{2\min\{1,k\}}\|\mathbf{J}_{\mathrm{d}}\|^2_{\Omega_h^{\mathrm{d}}} + h^{2\min\{1,k\}}\|\mathrm{J}_{\mathrm{s}}\|^2_{\Omega_h^{\mathrm{s}}}\Big\}, \quad \text{and}
\end{aligned} \tag{4.27}$$

$$\|p_{\mathrm{s},h}\|^2_{\Omega_h^{\mathrm{s}}} \lesssim \nu\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2. \tag{4.28}$$

Proof. We first employ the estimate obtained in Lemma 5 to bound the first and second terms of the right-hand side of Lemma 4. Next, using the estimate for $\|p_{\mathrm{s},h}\|$ (cf. (4.25)), we obtain

$$\begin{aligned}
(1 - C_{\delta,h})\,\mathcal{Q}(\mathrm{L}_{\mathrm{s},h}, \mathbf{u}_{\star,h}, \widehat{\mathbf{u}}_{\mathrm{s},h}, p_{\mathrm{d},h}, \widehat{p}_{\mathrm{d},h})^2 \lesssim\ &\|h_e^{(\beta-1)/2}(j_{\mathrm{d}}^{\mathrm{nc}} + j_{\mathrm{d}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{d}}} + \|h_e^{(\beta-1)/2}(\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} + \mathbf{j}_{\mathrm{s}}^{\delta})\|^2_{\mathcal{I}_h^{\mathrm{s}}} \\
&+ \|\mathbf{J}_{\mathrm{d}}\|^2_{\Omega_h^{\mathrm{d}}} + \|\nu\mathrm{J}_{\mathrm{s}}\|^2_{\Omega_h^{\mathrm{s}}},
\end{aligned}$$

where $C_{\delta,h}$ is the constant defined in **(A.4)**, since $\delta < 1$, τ is of order one, $k \geqslant 1$ and $h < 1$. Hence, (4.26) follows by Assumption **(A.4)**. Finally, making use of (4.22) and (4.25), we get (4.27) and (4.28), which finishes the proof. □

Corollary 1. The HDG scheme (3.2) has a unique solution.

Proof. We first note that the existence of the solution follows from its uniqueness. Thus, it suffices to show that when the right-hand sides of (3.2) vanish, then $\mathrm{L}_{\mathrm{s},h}$, $\mathbf{u}_{\mathrm{d},h}$, $p_{\mathrm{d},h}$, $\mathbf{u}_{\mathrm{s},h}$, $\widehat{p}_{\mathrm{d},h}$, $\widehat{\mathbf{u}}_{\mathrm{s},h}$ also vanish. Indeed, assuming that $\mathrm{f}_{\mathrm{d}} = 0$, $\mathbf{f}_{\mathrm{s}} = \mathbf{0}$, $\mathbf{J}_{\mathrm{d}} = \mathbf{0}$, $\mathrm{J}_{\mathrm{s}} = \mathbf{0}$, $j_{\mathrm{d}} = 0$, and $\mathbf{j}_{\mathrm{s}} = \mathbf{0}$, we deduce from Theorem 1 that $\mathrm{L}_{\mathrm{s},h} = \mathbf{0}$, $\mathbf{u}_{\mathrm{d},h} = \mathbf{0}$, $p_{\mathrm{d},h} = \widehat{p}_{\mathrm{d},h} = 0$, $\mathbf{u}_{\mathrm{s},h} = \widehat{\mathbf{u}}_{\mathrm{s},h} = \mathbf{0}$, and $p_{\mathrm{s},h} = 0$. In turn, we notice from Lemma 2 that $\widetilde{p}_{\mathrm{d},h} = 0$, and $\widetilde{\mathbf{u}}_{\mathrm{s},h} = 0$, which completes the proof. □

4.3 Semi-aligned discrete interfaces

With the aim of improving the estimate given in Theorem 1, in this section, we consider a specific structure of nonconforming meshes, where the discrete interfaces satisfy the following: if $\boldsymbol{v}_{\mathrm{d}}$ is a vertex in $\mathcal{I}_h^{\mathrm{d}}$, then $\boldsymbol{\psi}(\boldsymbol{v}_{\mathrm{d}})$ is a vertex $\boldsymbol{v}_{\mathrm{s}}$ in $\mathcal{I}_h^{\mathrm{s}}$ (see Fig 1). In this way, we refer to $\mathcal{I}_h^{\mathrm{d}}$ and $\mathcal{I}_h^{\mathrm{s}}$ as *semi-aligned discrete interfaces.*

Under this condition and the fact that in the error analysis $\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}} := (\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}} - p_{\mathrm{d}}) \circ \boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{d}}$ and $j_{\mathrm{d}}^{\mathrm{nc}} := (\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s}}) \circ \boldsymbol{\psi} \cdot \mathbf{n}_{\mathrm{s}}$ (see eqs. (5.4) in Lemma 8), it is easy to see that $\langle j_{\mathrm{d}}^{\mathrm{nc}}, \mu_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}}$ in (4.12a) vanishes. In fact, since $h_{\mathrm{s}} \leqslant h_{\mathrm{d}}$, the term $\mu_{\mathrm{d},h}\mathbf{n}_{\mathrm{s}} \circ \boldsymbol{\psi}^{-1}$ belongs to $\mathbf{N}_h^{\mathrm{s}}$ and thus $\langle j_{\mathrm{d}}^{\mathrm{nc}}, \mu_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} =$

$\langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}}-\mathbf{u}_{\mathrm{s}}),\mu_{\mathrm{d},h}\mathbf{n}_{\mathrm{s}}\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}=0$. Furthermore, in what follows we show that under this configuration $\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}$ and $\langle j_{\mathrm{d}}^{\mathrm{nc}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}$ also vanish. Indeed,

$$\begin{aligned}\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}&=\langle(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\cdot\mathbf{n}_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}-\langle p_{\mathrm{d}}\mathbf{n}_{\mathrm{d}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\rangle_{\mathcal{I}_h^{\mathrm{d}}}\\&=\langle(p_{\mathrm{d}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\cdot\mathbf{n}_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}-\langle p_{\mathrm{d}}\mathbf{n}_{\mathrm{d}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\rangle_{\mathcal{I}_h^{\mathrm{d}}}=0.\end{aligned}$$

On the other hand, taking into account that $h_{\mathrm{s}}\leqslant h_{\mathrm{d}}$, we have

$$\begin{aligned}\langle j_{\mathrm{d}}^{\mathrm{nc}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}&=\langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{s}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}-\langle\mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{s}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}\\&=\langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}-\langle\mathbf{u}_{\mathrm{s}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}=0,\end{aligned}$$

since $\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{s}}\in\mathbf{M}_h^{\mathrm{s}}$. All these identities imply an improvement of the estimate of Lemma 5. In fact, we recall that $\mathbb{B}_3=-\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}-\langle\mathbf{j}_{\mathrm{s}}^{\delta},\boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}$. Then, we have

$$\begin{aligned}\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}&=\langle\mathbf{j}_{\mathrm{s}}^{nc},\boldsymbol{\mathcal{P}}^{\mathrm{s}}\boldsymbol{\phi}_{\mathrm{s}}-\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}-\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\circ\boldsymbol{\psi}^{-1}-\boldsymbol{\phi}_{\mathrm{s}}\rangle_{\mathcal{I}_h^{\mathrm{s}}}+\langle\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}},\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})\circ\boldsymbol{\psi}^{-1}\rangle_{\mathcal{I}_h^{\mathrm{s}}}\\&\lesssim h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}\,|\boldsymbol{\phi}_{\mathrm{s}}|_{\mathbf{H}^2(\Omega_{\mathrm{s}})}+\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}\|\boldsymbol{\mathcal{P}}^{\mathrm{d}}(\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi})-\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi}\|_{\mathcal{I}_h^{\mathrm{d}}}\\&\lesssim h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}\,|\boldsymbol{\phi}_{\mathrm{s}}|_{\mathbf{H}^2(\Omega_{\mathrm{s}})}+h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}|\boldsymbol{\phi}_{\mathrm{s}}\circ\boldsymbol{\psi}|_{\mathbf{H}^{3/2}(\mathcal{I}_h^{\mathrm{d}})}\\&\lesssim\nu^{-1}h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}\,\|\Theta_{\mathrm{s}}\|_{\Omega_{\mathrm{s}}},\end{aligned}$$

which implies that

$$\mathbb{B}_3\lesssim\nu^{-1}h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}\,\|\Theta_{\mathrm{s}}\|_{\Omega_{\mathrm{s}}}+\nu^{-1}\|\mathbf{j}_{\mathrm{s}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{s}}}\,\|\Theta_{\mathrm{s}}\|_{\Omega_{\mathrm{s}}}.\tag{4.29}$$

On the other hand, we have that

$$\mathbb{B}_4=-\langle j_{\mathrm{d}}^{\mathrm{nc}},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}-\langle j_{\mathrm{d}}^{\delta},\mathcal{P}^{\mathrm{d}}\varphi_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}}\leqslant\|j_{\mathrm{d}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{d}}}\,\|\Theta_{\mathrm{d}}\|_{\Omega_{\mathrm{d}}}.\tag{4.30}$$

We now have the following result.

Lemma 7. *Under the same assumptions of Lemma 5, if we have semi-aligned discrete interfaces, we get*

$$\begin{aligned}\|\mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}^2+\|p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}}^2\leqslant C\Big\{&C_{\delta,h}^{\mathcal{S}}\mathcal{Q}(\mathrm{L}_{\mathrm{s},h},\mathbf{u}_{\star,h},\widehat{\mathbf{u}}_{\mathrm{s},h},p_{\mathrm{d},h},\widehat{p}_{\mathrm{d},h})^2+h^{3/2}\|\mathbf{j}_{\mathrm{s}}^{\mathrm{nc}}\|_{\mathcal{I}_h^{\mathrm{s}}}^2+\|\mathbf{j}_{\mathrm{s}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{s}}}^2+\|j_{\mathrm{d}}^{\delta}\|_{\mathcal{I}_h^{\mathrm{d}}}^2\\&+(C_{\delta,h}^{p})^2\|p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}}^2+h^{2\min\{1,k\}}\|\mathbf{J}_{\mathrm{d}}\|_{\Omega_h^{\mathrm{d}}}^2+h^{2\min\{1,k\}}\|\mathrm{J}_{\mathrm{s}}\|_{\Omega_h^{\mathrm{s}}}^2\Big\}.\end{aligned}\tag{4.31}$$

Proof. The proof follows almost verbatim as in Lemma 5, but now considering the estimates (4.29) and (4.30). □

5 Error Analysis

Our first goal in this section is to derive the error estimates of the proposed method. We employ the stability estimate deduced in previous sections. In what follows, we introduce the projection of the errors, namely $\varepsilon^{\mathrm{L}_{\mathrm{s}}}:=\Pi_{\mathrm{G}}^{\mathrm{s}}\mathrm{L}_{\mathrm{s}}-\mathrm{L}_{\mathrm{s},h}$, $\varepsilon^{\mathbf{u}_\star}:=\Pi_{\mathbf{V}}^{\star}\mathbf{u}_\star-\mathbf{u}_{\star,h}$, $\varepsilon^{\widehat{\mathbf{u}}_\star}:=\boldsymbol{\mathcal{P}}^{\star}\mathbf{u}_\star-\widehat{\mathbf{u}}_{\star,h}$, $\varepsilon^{p_\star}:=\Pi_{P}^{\star}p_\star-p_{\star,h}$, $\varepsilon^{\widehat{p}_{\mathrm{d}}}:=\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}-\widehat{p}_{\mathrm{d},h}$, and $\varepsilon^{\alpha_{\mathrm{s}}}:=\Pi_{Q}^{\mathrm{s}}\alpha_{\mathrm{s}}-\alpha_{\mathrm{s},h}$. In turn, the error of the projections are given by $\mathrm{I}_{\mathrm{L}}^{\mathrm{s}}:=\mathrm{L}_{\mathrm{s}}-\Pi_{\mathrm{G}}^{\mathrm{s}}\mathrm{L}_{\mathrm{s}}$, $\mathrm{I}_{\mathbf{u}}^{\star}:=\mathbf{u}_\star-\Pi_{\mathbf{V}}^{\star}\mathbf{u}_\star$, $\mathrm{I}_{p}^{\star}:=p_\star-\Pi_{P}^{\star}p_\star$, and $\mathrm{I}_{\alpha}^{\mathrm{s}}:=\alpha_{\mathrm{s}}-\Pi_{Q}^{\mathrm{s}}\alpha_{\mathrm{s}}$, and note that $\mathrm{I}_{\alpha}^{\mathrm{s}}=0$.

Lemma 8. *The projection of the errors satisfies*

$$(\varepsilon^{\mathrm{L_s}}, \mathrm{G}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + (\varepsilon^{\mathbf{u}_{\mathrm{s}}}, \nabla\cdot \mathrm{G}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} - \langle \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}}, \mathrm{G}_{\mathrm{s},h}\mathbf{n}_{\mathrm{s}}\rangle_{\partial\Omega_h^{\mathrm{s}}} = -(\mathrm{I}_{\mathrm{L}}^{\mathrm{s}}, \mathrm{G}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}}, \tag{5.1a}$$

$$(\varepsilon^{\mathrm{L_s}}, \nabla \mathbf{v}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} - (\varepsilon^{p_{\mathrm{s}}}, \nabla\cdot\mathbf{v}_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} - \langle(\nu\varepsilon^{\widehat{\mathrm{L}}_{\mathrm{s}}} - \varepsilon^{\widehat{p}_{\mathrm{s}}})\mathbf{n}_{\mathrm{s}}, \mathbf{v}_{\mathrm{s},h}\rangle_{\partial\Omega_h^{\mathrm{s}}} = 0, \tag{5.1b}$$

$$-(\varepsilon^{\mathbf{u}_{\mathrm{s}}}, \nabla q_{\mathrm{s},h})_{\Omega_h^{\mathrm{s}}} + \langle \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}}\cdot\mathbf{n}_{\mathrm{s}}, q_{\mathrm{s},h}\rangle_{\partial\Omega_h^{\mathrm{s}}} = 0, \tag{5.1c}$$

$$\langle \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}}, \boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\Gamma_{\mathrm{s},h}} = 0, \tag{5.1d}$$

$$\langle(\nu\varepsilon^{\widehat{\mathrm{L}}_{\mathrm{s}}} - \varepsilon^{\widehat{p}_{\mathrm{s}}})\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\partial\Omega_h^{\mathrm{s}}\setminus(\Gamma_{\mathrm{s},h}\cup\mathcal{I}_h^{\mathrm{s}})} = 0, \tag{5.1e}$$

$$(\kappa^{-1}\varepsilon^{\mathbf{u}_{\mathrm{d}}}, \mathbf{v}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} - (\varepsilon^{p_{\mathrm{d}}}, \nabla\cdot\mathbf{v}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \langle \mathbf{v}_{\mathrm{d},h}\cdot\mathbf{n}_{\mathrm{d}}, \varepsilon^{\widehat{p}_{\mathrm{d}}}\rangle_{\partial\Omega_h^{\mathrm{d}}} = -(\kappa^{-1}\mathrm{I}_{\mathbf{u}}^{\mathrm{d}}, \mathbf{v}_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}}, \tag{5.1f}$$

$$-(\varepsilon^{\mathbf{u}_{\mathrm{d}}}, \nabla q_{\mathrm{d},h})_{\Omega_h^{\mathrm{d}}} + \langle \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{d}}}\cdot\mathbf{n}_{\mathrm{d}}, q_{\mathrm{d},h}\rangle_{\partial\Omega_h^{\mathrm{d}}} = 0, \tag{5.1g}$$

$$\langle \varepsilon^{\widehat{p}_{\mathrm{d}}}, \mu_{\mathrm{d},h}\rangle_{\Gamma_{\mathrm{d},h}} = 0, \tag{5.1h}$$

$$\langle \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{d}}}\cdot\mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h}\rangle_{\partial\Omega_h^{\mathrm{d}}\setminus(\Gamma_{\mathrm{d},h}\cup\mathcal{I}_h^{\mathrm{d}})} = 0, \tag{5.1i}$$

for all $(\mathrm{G}_{\mathrm{s},h}, \mathbf{v}_{\mathrm{s},h}, q_{\mathrm{s},h}, \boldsymbol{\mu}_{\mathrm{s},h}, \mathbf{v}_{\mathrm{d},h}, q_{\mathrm{d},h}, \mu_{\mathrm{d},h}) \in \mathrm{G}_h^{\mathrm{s}}\times\mathbf{V}_h^{\mathrm{s}}\times Q_h^{\mathrm{s}}\times\mathbf{M}_h^{\mathrm{s}}\times\mathbf{V}_h^{\mathrm{d}}\times Q_h^{\mathrm{d}}\times M_h^{\mathrm{d}}$, *and*

$$\varepsilon^{\widehat{\mathbf{u}}_{\mathrm{d}}}\cdot\mathbf{n}_{\mathrm{d}} = \varepsilon^{\mathbf{u}_{\mathrm{d}}}\cdot\mathbf{n}_{\mathrm{d}} + \tau(\varepsilon^{p_{\mathrm{d}}} - \varepsilon^{\widehat{p}_{\mathrm{d}}}) \quad \text{on} \quad \partial\Omega_h^{\mathrm{d}}, \tag{5.2a}$$

$$\varepsilon^{\widehat{\boldsymbol{\sigma}}_{\mathrm{s}}}\mathbf{n}_{\mathrm{s}} = \nu\varepsilon^{\mathrm{L_s}}\mathbf{n}_{\mathrm{s}} - \varepsilon^{p_{\mathrm{s}}}\mathbf{I}\mathbf{n}_{\mathrm{s}} - \tau\nu(\varepsilon^{\mathbf{u}_{\mathrm{s}}} - \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}}) \quad \text{on} \quad \partial\Omega_h^{\mathrm{s}}. \tag{5.2b}$$

Moreover, for $\boldsymbol{x}_\star \in \mathcal{I}_h^\star$, *let*

$$\varepsilon^{\widetilde{p}_{\mathrm{d}}}(\boldsymbol{x}_{\mathrm{s}}) := \varepsilon^{\widehat{p}_{\mathrm{d}}}(\boldsymbol{x}_{\mathrm{d}}) - |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_0^1 \boldsymbol{E}_{\varepsilon^{\mathbf{u}_{\mathrm{d}}}}(\boldsymbol{y}_{\mathrm{d}}(t))\cdot\mathbf{n}_{\mathrm{d}}dt, \tag{5.3a}$$

$$\varepsilon^{\widetilde{\mathbf{u}}_{\mathrm{s}}}(\boldsymbol{x}_{\mathrm{d}}) := \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}}(\boldsymbol{x}_{\mathrm{s}}) + |\sigma_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}})| \int_0^1 \boldsymbol{E}_{\varepsilon^{\mathrm{L_s}}}(\boldsymbol{y}_{\mathrm{s}}(t))\mathbf{n}_{\mathrm{s}}dt, \tag{5.3b}$$

$$\varepsilon^{\widetilde{\mathbf{u}}_{\mathrm{d}}}(\boldsymbol{x}_{\mathrm{d}}) := \boldsymbol{E}_{\varepsilon^{\mathbf{u}_{\mathrm{d}}}}\circ\boldsymbol{\psi}_{\mathrm{d}}^{-1}(\boldsymbol{x}_{\mathrm{d}})\cdot\mathbf{n}_{\mathrm{d}} + \tau(\varepsilon^{p_{\mathrm{d}}} - \varepsilon^{\widehat{p}_{\mathrm{d}}})(\boldsymbol{x}_{\mathrm{d}}), \tag{5.3c}$$

$$\varepsilon^{\widetilde{\boldsymbol{\sigma}}_{\mathrm{s}}}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} := (\nu\boldsymbol{E}_{\varepsilon^{\mathrm{L_s}}} - \boldsymbol{E}_{\varepsilon^{p_{\mathrm{s}}}}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} - \nu\tau(\varepsilon^{\mathbf{u}_{\mathrm{s}}} - \varepsilon^{\widehat{\mathbf{u}}_{\mathrm{s}}})(\boldsymbol{x}_{\mathrm{s}}) - \varepsilon^{\alpha_{\mathrm{s}}}\mathbf{n}_{\mathrm{s}}. \tag{5.3d}$$

They satisfy

$$\begin{aligned}\langle \varepsilon^{\widetilde{\mathbf{u}}_{\mathrm{s}}}\cdot\mathbf{n}_{\mathrm{s}} + \varepsilon^{\widetilde{\mathbf{u}}_{\mathrm{d}}}\cdot\mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} &= \langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s}})\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{s}}, \mu_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} - \langle|\sigma_{\mathrm{s}}|\Lambda_{\mathrm{I}_{\mathrm{L}}^{\mathrm{s}}}\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h}\rangle_{\mathcal{I}_h^{\mathrm{d}}} \\ &+\langle|\sigma_{\mathrm{s}}|\mathrm{I}_{\mathrm{L}}^{\mathrm{s}}\circ\boldsymbol{\psi}\mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h}\mathbf{n}_{\mathrm{d}}\rangle_{\mathcal{I}_h^{\mathrm{d}}} + \left\langle\left(\mathrm{I}_{\mathbf{u}}^{\mathrm{d}} - \mathrm{I}_{\mathbf{u}}^{\mathrm{d}}\circ\boldsymbol{\psi}_{\mathrm{d}}^{-1}\right)\cdot\mathbf{n}_{\mathrm{d}}, \mu_{\mathrm{d},h}\right\rangle_{\mathcal{I}_h^{\mathrm{d}}} \quad \forall\, \mu_{\mathrm{d},h}\in N_h^{\mathrm{d}},\end{aligned} \tag{5.4a}$$

$$\begin{aligned}\langle \varepsilon^{\widetilde{\boldsymbol{\sigma}}_{\mathrm{s}}}\mathbf{n}_{\mathrm{s}} - \varepsilon^{\widetilde{p}_{\mathrm{d}}}\mathbf{n}_{\mathrm{d}}, \boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} &= -\langle(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}} - p_{\mathrm{d}})\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{d}} + \kappa^{-1}|\sigma_{\mathrm{d}}|\Lambda_{\mathrm{I}_{\mathbf{u}}^{\mathrm{d}}}\mathbf{n}_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1}, \boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \\ -\langle\kappa^{-1}|\sigma_{\mathrm{d}}|\mathrm{I}_{\mathbf{u}}^{\mathrm{d}}\circ\boldsymbol{\psi}^{-1} &- \left((\nu\mathrm{I}_{\mathrm{L}}^{\mathrm{s}} - \mathrm{I}_p^{\mathrm{s}}\mathrm{I}) - (\nu\mathrm{I}_{\mathrm{L}}^{\mathrm{s}} - \mathrm{I}_p^{\mathrm{s}}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}\right)\mathbf{n}_{\mathrm{s}}, \boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}_h^{\mathrm{s}}} \quad \forall\,\boldsymbol{\mu}_{\mathrm{s},h}\in\mathbf{N}_h^{\mathrm{s}}.\end{aligned} \tag{5.4b}$$

Proof. The identities (5.1)-(5.2) are straightforwardly obtained from the definition of the projection of the errors and (3.2) (see also [6, Lemma 3.1] and [7, Lemma 3.1]). Now, let $\boldsymbol{x}_{\mathrm{s}}\in\mathcal{I}_h^{\mathrm{s}}$. By (3.7a) and (2.1), we rewrite (5.3a) as

$$\varepsilon^{\widetilde{p}_{\mathrm{d}}}(\boldsymbol{x}_{\mathrm{s}}) = \mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}}) + |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_0^1 \kappa^{-1}\boldsymbol{E}_{\Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}}}(\boldsymbol{y}_{\mathrm{d}}(t))\cdot\mathbf{n}_{\mathrm{d}}dt - \widetilde{p}_{\mathrm{d},h}(\boldsymbol{x}_{\mathrm{s}}).$$

Moreover,

$$|\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_0^1 \kappa^{-1}\boldsymbol{E}_{\Pi_{\mathbf{V}}^{\mathrm{d}}\mathbf{u}_{\mathrm{d}}}(\boldsymbol{y}(t))\cdot\mathbf{n}_{\mathrm{d}}dt = -|\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_{\mathbf{0}}^1 \kappa^{-1}\mathrm{I}_{\mathbf{u}}^{\mathrm{d}}(\boldsymbol{y}(t))\cdot\mathbf{n}_{\mathrm{d}} + |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})| \int_{\mathbf{0}}^1 \kappa^{-1}\mathbf{u}_{\mathrm{d}}(\boldsymbol{y}(t))\cdot\mathbf{n}_{\mathrm{d}},$$

from which, proceeding in a similar way as in the proof of Lemma 2, we obtain

$$|\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})|\int_0^1 \kappa^{-1}\boldsymbol{E}_{\Pi^{\mathrm{d}}_{\mathbf{V}}\mathbf{u}_{\mathrm{d}}}(\boldsymbol{y}(t))\cdot\mathbf{n}_{\mathrm{d}}dt = \kappa^{-1}|\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})|\Lambda_{\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}}(\boldsymbol{x}_{\mathrm{d}}) - |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})|\kappa\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}(\boldsymbol{x}_{\mathrm{d}})\cdot\mathbf{n}_{\mathrm{d}} + |\sigma_{\mathrm{d}}(\boldsymbol{x}_{\mathrm{d}})|\int_{\mathbf{0}}^1 \kappa^{-1}\mathbf{u}_{\mathrm{d}}(\boldsymbol{y}(t))\cdot\mathbf{n}_{\mathrm{d}}.$$

On the other hand, from (5.3d), (2.1), and (5.3d) we find that

$$\begin{aligned}\varepsilon^{\tilde{\boldsymbol{\sigma}}_{\mathrm{s}}}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} := &-(\nu\widetilde{\mathrm{L}}_{\mathrm{s},h}-\widetilde{p}_{\mathrm{s},h}\mathrm{I})(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} + (\nu\Pi^{\mathrm{s}}_{\mathrm{G}}\mathrm{L}_{\mathrm{s}} - \Pi^{\mathrm{s}}_{Q}p_{\mathrm{s}}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}(\boldsymbol{x}_{\mathrm{s}})\mathbf{n}_{\mathrm{s}} - \nu\tau(\Pi^{\mathrm{s}}_{\mathbf{V}}\mathbf{u}_{\mathrm{s}} - \boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}})(\boldsymbol{x}_{\mathrm{s}})\\ &-\boldsymbol{E}_{\Pi^{\mathrm{s}}_{Q}\alpha_{\mathrm{s}}}\mathbf{n}_{\mathrm{s}}.\end{aligned}$$

Hence, for $\boldsymbol{\mu}_{\mathrm{s},h}\in\mathbf{N}^{\mathrm{s}}_h$, the above expressions together with (2.2c) and the definition of the L^2-projection $\mathcal{P}^{\mathrm{d}}$ imply

$$\begin{aligned}\langle\varepsilon^{\tilde{\boldsymbol{\sigma}}_{\mathrm{s}}}\mathbf{n}_{\mathrm{s}} - \varepsilon^{\tilde{p}_{\mathrm{d}}}\mathbf{n}_{\mathrm{d}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h} =& \langle(\nu\Pi^{\mathrm{s}}_{\mathrm{G}}\mathrm{L}_{\mathrm{s}} - \Pi^{\mathrm{s}}_{Q}p_{\mathrm{s}}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}\mathbf{n}_{\mathrm{s}} + ((\nu\mathrm{L}_{\mathrm{s}}-p_{\mathrm{s}}\mathrm{I})\mathbf{n}_{\mathrm{s}} - (\nu\Pi^{\mathrm{s}}_{\mathrm{G}}\mathrm{L}_{\mathrm{s}} - \Pi^{\mathrm{s}}_{Q}p_{\mathrm{s}}\mathrm{I})\mathbf{n}_{\mathrm{s}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h}\\ &-\langle(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}-p_{\mathrm{d}})\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{d}} - \kappa^{-1}(|\sigma_{\mathrm{d}}|\Lambda_{\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}}\mathbf{n}_{\mathrm{d}} + |\sigma_{\mathrm{d}}|\mathrm{I}^{\mathrm{d}}_{\mathbf{u}})\circ\boldsymbol{\psi}^{-1},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h}\\ &-\langle\alpha_{\mathrm{s}}\mathbf{n}_{\mathrm{s}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h} - \langle p_{\mathrm{d}}\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}\mathbf{n}_{\mathrm{d}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h},\end{aligned}$$

from which, employing (3.6) in the third term of the right hand side of the above expression, we obtain (5.4b). In turn, for $\boldsymbol{x}_{\mathrm{d}}\in\mathcal{I}^{\mathrm{d}}_h$, the identity (5.4a) is proved in a completely analogous way. We omit further details. □

We observe that the above equiations are similar to those of the HDG scheme (3.2), where $\mathrm{I}^{\mathrm{s}}_{\mathrm{L}}$, $\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}$, 0, and $\mathbf{0}$ play the role of J_{s}, $\mathbf{J}_{\mathrm{d}}$, f_{d} and $\mathbf{f}_{\mathrm{s}}$, respectively. Moreover,

$$\begin{aligned}\langle j^{\mathrm{nc}}_{\mathrm{d}} + j^{\delta}_{\mathrm{d}},\mu_{\mathrm{d},h}\rangle_{\mathcal{I}^{\mathrm{d}}_h} =& \langle(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s}})\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{s}} - |\sigma_{\mathrm{s}}|\Lambda_{\mathrm{I}^{\mathrm{s}}_{\mathrm{L}}}\circ\boldsymbol{\psi}\cdot\mathbf{n}_{\mathrm{d}},\mu_{\mathrm{d},h}\rangle_{\mathcal{I}^{\mathrm{d}}_h} + \langle|\sigma_{\mathrm{s}}|\mathrm{I}^{\mathrm{s}}_{\mathrm{L}}\circ\boldsymbol{\psi}\mathbf{n}_{\mathrm{d}},\mu_{\mathrm{d},h}\mathbf{n}_{\mathrm{d}}\rangle_{\mathcal{I}^{\mathrm{d}}_h}\\ &+\langle\left(\mathrm{I}^{\mathrm{d}}_{\mathbf{u}} - \mathrm{I}^{\mathrm{d}}_{\mathbf{u}}\circ\boldsymbol{\psi}_{\mathrm{d}}^{-1}\right)\cdot\mathbf{n}_{\mathrm{d}},\mu_{\mathrm{d},h}\rangle_{\mathcal{I}^{\mathrm{d}}_h},\end{aligned}\tag{5.5}$$

$$\begin{aligned}\langle\mathbf{j}^{\mathrm{nc}}_{\mathrm{s}} + \mathbf{j}^{\delta}_{\mathrm{s}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h} =& -\langle(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}-p_{\mathrm{d}})\circ\boldsymbol{\psi}^{-1}\mathbf{n}_{\mathrm{d}} + \kappa^{-1}|\sigma_{\mathrm{d}}|\Lambda_{\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}}\mathbf{n}_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h}\\ &-\langle\kappa^{-1}|\sigma_{\mathrm{d}}|\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}\circ\boldsymbol{\psi}^{-1} - \left((\nu\mathrm{I}^{\mathrm{s}}_{\mathrm{L}} - \mathrm{I}^{\mathrm{s}}_{p}\mathrm{I}) - (\nu\mathrm{I}^{\mathrm{s}}_{\mathrm{L}} - \mathrm{I}^{\mathrm{s}}_{p}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1}\right)\mathbf{n}_{\mathrm{s}},\boldsymbol{\mu}_{\mathrm{s},h}\rangle_{\mathcal{I}^{\mathrm{s}}_h}.\end{aligned}\tag{5.6}$$

Hence, we consider the result of Theorem 1 applied to this context. More precisely, we notice that

$$\begin{aligned}\|h_e^{(\beta-1)/2}(j^{\mathrm{nc}}_{\mathrm{d}} + j^{\delta}_{\mathrm{d}})\|^2_{\mathcal{I}^{\mathrm{d}}_h} \lesssim& \|h_e^{(\beta-1)/2}(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi} - \mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi})\|^2_{\mathcal{I}^{\mathrm{d}}_h} + \|h_e^{(\beta-1)/2}|\sigma_{\mathrm{s}}|\Lambda_{\mathrm{I}^{\mathrm{s}}_{\mathrm{L}}}\circ\boldsymbol{\psi}\|^2_{\mathcal{I}^{\mathrm{d}}_h}\\ &+\|h_e^{(\beta-1)/2}|\sigma_{\mathrm{s}}|\mathrm{I}^{\mathrm{s}}_{\mathrm{L}}\circ\boldsymbol{\psi}\|^2_{\mathcal{I}^{\mathrm{d}}_h} + \|h_e^{(\beta-1)/2}\left(\mathrm{I}^{\mathrm{d}}_{\mathbf{u}} - \mathrm{I}^{\mathrm{d}}_{\mathbf{u}}\circ\boldsymbol{\psi}_{\mathrm{d}}^{-1}\right)\|^2_{\mathcal{I}^{\mathrm{d}}_h},\end{aligned}\tag{5.7}$$

and

$$\begin{aligned}\|h_e^{(\beta-1)/2}(\mathbf{j}^{\mathrm{nc}}_{\mathrm{s}} + \mathbf{j}^{\delta}_{\mathrm{s}})\|^2_{\mathcal{I}^{\mathrm{s}}_h} \lesssim& \|h_e^{(\beta-1)/2}(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1} - p_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1})\|^2_{\mathcal{I}^{\mathrm{s}}_h} + \|h_e^{(\beta-1)/2}\kappa^{-1}|\sigma_{\mathrm{d}}|\Lambda_{\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}}\circ\boldsymbol{\psi}^{-1}\|^2_{\mathcal{I}^{\mathrm{s}}_h}\\ &+\|h_e^{(\beta-1)/2}\kappa^{-1}|\sigma|\mathrm{I}^{\mathrm{d}}_{\mathbf{u}}\circ\boldsymbol{\psi}^{-1}\|^2_{\mathcal{I}^{\mathrm{s}}_h} + \|h_e^{(\beta-1)/2}(\nu\mathrm{I}^{\mathrm{s}}_{\mathrm{L}} - \mathrm{I}^{\mathrm{s}}_{p}\mathrm{I} - (\nu\mathrm{I}^{\mathrm{s}}_{\mathrm{L}} - \mathrm{I}^{\mathrm{s}}_{p}\mathrm{I})\circ\boldsymbol{\psi}_{\mathrm{s}}^{-1})\|^2_{\mathcal{I}^{\mathrm{s}}_h}.\end{aligned}\tag{5.8}$$

and observe that the first terms in (5.7) and (5.8) can be bounded using the approximation properties of the L^2-projection over N^{d}_h and $\mathbf{N}^{\mathrm{s}}_h$, respectively, that is there exist constants $C_{\mathrm{n}\star}\geqslant 0$, for $\star\in\{\mathrm{s},\mathrm{d}\}$, independent of h, such that

$$\|h_e^{(\beta-1)/2}(\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi} - \mathbf{u}_{\mathrm{s}}\circ\boldsymbol{\psi})\|^2_{\mathcal{I}^{\mathrm{d}}_h} \leqslant C_{\mathrm{ns}}h^{2l_{\mathbf{u}_{\mathrm{s}}}+\beta}|\mathbf{u}_{\mathrm{s}}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\mathrm{s}}}+1}(\Omega_{\mathrm{s}})},\tag{5.9}$$

$$\|h_e^{(\beta-1)/2}(\mathcal{P}^{\mathrm{d}}p_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1} - p_{\mathrm{d}}\circ\boldsymbol{\psi}^{-1})\|^2_{\mathcal{I}^{\mathrm{s}}_h} \leqslant C_{\mathrm{nd}}h^{2l_{p_{\mathrm{d}}}+\beta}|p_{\mathrm{d}}|^2_{H^{l_{p_{\mathrm{d}}}+1}(\Omega_{\mathrm{d}})},\tag{5.10}$$

where the constants $C_{\rm ns}$ and $C_{\rm nd}$ take into account the nonconformity between the computational interfaces.
On the other hand, following the proof of [25, Theorem 4.2] we deduce from (4.3), a scaling argument to bound the $L^2(\mathcal{I}_h^\star)$−norm in terms of its $L^2(\Omega_h^\star)$−norm, and Lemma 1, that

$$\|h_e^{(\beta-1)/2}(j_{\rm d}^{\rm nc}+j_{\rm d}^{\delta})\|^2_{\mathcal{I}_h^{\rm d}} \lesssim C_{\rm ns}h^{2l_{\mathbf{u}_{\rm s}}+\beta}|\mathbf{u}_{\rm s}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})} + h^\beta\Big(\max_{e\in\mathcal{I}_h^{\rm s}}(\delta_e^4h_e^{-4}) + \max_{e\in\mathcal{I}_h^{\rm s}}(\delta_e h_e^{-1})^2\Big)\|\mathrm{I}_{\rm L}^{\rm s}\|^2_{\Omega_h^{\rm s}} + h^\beta\max_{e\in\mathcal{I}_h^{\rm d}}(\delta_e h_e^{-1})|\kappa^{-1}\mathrm{I}_{\mathbf{u}}^{\rm d}|^2_{\mathbf{H}^1(\Omega_h^{\rm d})},$$

and

$$\|h_e^{(\beta-1)/2}(\mathbf{j}_{\rm s}^{\rm nc}+\mathbf{j}_{\rm s}^{\delta})\|^2_{\mathcal{I}_h^{\rm s}} \lesssim C_{\rm nd}h^{2l_{p_{\rm d}}+\beta}|p_{\rm d}|^2_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + h^\beta\max_{e\in\mathcal{I}_h^{\rm s}}(\delta_e h_e^{-1})|\nu\mathrm{I}_{\rm L}^{\rm s}-\mathrm{I}_p^{\rm s}\mathrm{I}|^2_{\mathbf{H}^1(\Omega_h^{\rm s})} + h^\beta\Big(\max_{e\in\mathcal{I}_h^{\rm d}}(\delta_e^4h_e^{-4}) + \max_{e\in\mathcal{I}_h^{\rm d}}(\delta_e h_e^{-1})\Big)\|\mathrm{I}_{\mathbf{u}}^{\rm d}\|^2_{\Omega_h^{\rm d}}.$$

Next, by Assumptions **(A)**, the fact that $(\delta_e^4h_e^{-4}h^\beta)$ and $(\delta_e h_e^{-1}h^\beta)$ are bounded, with $\star\in\{\rm s,d\}$, it follows that

$$\|h_e^{(\beta-1)/2}(j_{\rm d}^{\rm nc}+j_{\rm d}^{\delta})\|^2_{\mathcal{I}_h^{\rm d}} \lesssim C_{\rm ns}h^{2l_{\mathbf{u}_{\rm s}}+\beta}|\mathbf{u}_{\rm s}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})} + \delta^4h^{-4+\beta}\|\mathrm{I}_{\rm L}^{\rm s}\|^2_{\Omega_h^{\rm s}} + \delta h^{-1+\beta}|\mathrm{I}_{\mathbf{u}}^{\rm d}|^2_{\mathbf{H}^1(\Omega_h^{\rm d})}, \quad \text{and}$$
$$\|h_e^{(\beta-1)/2}(\mathbf{j}_{\rm s}^{\rm nc}+\mathbf{j}_{\rm s}^{\delta})\|^2_{\mathcal{I}_h^{\rm s}} \lesssim C_{\rm nd}h^{2l_{p_{\rm d}}+\beta}|p_{\rm d}|^2_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + \delta h^{-1+\beta}|\nu\mathrm{I}_{\rm L}^{\rm s}-\mathrm{I}_p^{\rm s}\mathrm{I}|^2_{\mathrm{H}^1(\Omega_h^{\rm s})} + \delta^4h^{-4+\beta}\|\mathrm{I}_{\mathbf{u}}^{\rm d}\|^2_{\Omega_h^{\rm d}}.$$

Then, bearing in mind the above, the estimates from Theorem 1 applied to (5.1) become

$$\mathcal{Q}(\varepsilon^{\rm L_s},\boldsymbol{\varepsilon}^{\mathbf{u}_\star},\varepsilon^{\hat{\mathbf{u}}_{\rm s}},\varepsilon^{p_{\rm d}},\varepsilon^{\hat{p}_{\rm d}})^2 \lesssim C_{\rm ns}h^{2l_{\mathbf{u}_{\rm s}}+\beta}|\mathbf{u}_{\rm s}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})} + \delta h^{-1+\beta}|\mathrm{I}_{\mathbf{u}}^{\rm d}|^2_{\mathbf{H}^1(\Omega_h^{\rm d})} + \big(\delta^4h^{-4+\beta}+1\big)\|\mathrm{I}_{\mathbf{u}}^{\rm d}\|^2_{\Omega_h^{\rm d}} + C_{\rm nd}h^{2l_{p_{\rm d}}+\beta}|p_{\rm d}|^2_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + \delta h^{-1+\beta}|\nu\mathrm{I}_{\rm L}^{\rm s}-\mathrm{I}_p^{\rm s}\mathrm{I}|^2_{\mathrm{H}^1(\Omega_h^{\rm s})} + \big(\nu^{-2}\delta^4h^{-4+\beta}+1\big)\|\nu\mathrm{I}_{\rm L}^{\rm s}\|^2_{\Omega_h^{\rm s}},$$

and

$$\|\varepsilon^{\mathbf{u}_{\rm s}}\|^2_{\Omega_h^{\rm s}} + \|\varepsilon^{p_{\rm d}}\|^2_{\Omega_h^{\rm d}} \lesssim \big(C_{\delta,h}^{\mathcal{S}} + (C_{\delta,h}^{p})^2\big)\mathcal{Q}(\varepsilon^{\rm L_s},\boldsymbol{\varepsilon}^{\mathbf{u}_\star},\varepsilon^{\hat{\mathbf{u}}_{\rm s}},\varepsilon^{p_{\rm d}},\varepsilon^{\hat{p}_{\rm d}})^2 + \delta|\nu\mathrm{I}_{\rm L}^{\rm s}-\mathrm{I}_p^{\rm s}\mathrm{I}|^2_{\mathrm{H}^1(\Omega_h^{\rm s})} + \delta|\mathrm{I}_{\mathbf{u}}^{\rm d}|^2_{\mathbf{H}^1(\Omega_h^{\rm d})} + \big(\delta^4h^{-3}+h^2\big)\|\mathrm{I}_{\mathbf{u}}^{\rm d}\|^2_{\Omega_h^{\rm d}} + \big(\delta^4h^{-3}\nu^{-2}+h^2\big)\|\nu\mathrm{I}_{\rm L}^{\rm s}\|^2_{\Omega_h^{\rm s}} + C_{\rm ns}h^{2l_{\mathbf{u}_{\rm s}}+1}|\mathbf{u}_{\rm s}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})} + C_{\rm nd}h^{2l_{p_{\rm d}}+1}|p_{\rm d}|^2_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})}.$$

which, along with the properties of the HDG projectors (cf. (2.3)-(2.5)), we obtain the following result.

Theorem 2. *Suppose that Assumption* **(A)** *and elliptic regularity hold true. If τ is of order one, $k\geqslant 1$ and $(\mathrm{L}_{\rm s},\mathbf{u}_\star,p_\star)\in\mathrm{H}^{l_\sigma+1}(\Omega_h^{\rm s})\times\mathbf{H}^{l_{\mathbf{u}_\star}+1}(\Omega_h^\star)\times H^{l_{p_\star}+1}(\Omega_h^\star)$, for $l_{\mathbf{u}_\star}$, l_σ and $l_{p_\star}\in[0,k]$, $\star\in\{\rm s,d\}$. There exists $h_0\in(0,1)$ such that, for all $h<h_0$, it holds*

$$\|\mathbf{u}_{\rm d}-\mathbf{u}_{{\rm d},h}\|_{\Omega_h^{\rm d}} + \|\nu(\mathrm{L}_{\rm s}-\mathrm{L}_{{\rm s},h})\|_{\Omega_h^{\rm s}} + \|p_{\rm s}-p_{{\rm s},h}\|_{\Omega_h^{\rm s}}$$
$$\lesssim\Big(C_{\rm nd}^{\frac12}h^{l_{p_{\rm d}}+\frac{\beta}{2}} + h^{l_{p_{\rm d}}+1} + \delta^{1/2}h^{l_{p_{\rm d}}-\frac{1-\beta}{2}}\Big)|p_{\rm d}|_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + \Big(h^{l_\sigma+1}+\delta^{1/2}h^{l_\sigma-\frac{1-\beta}{2}}\Big)|\nu\mathrm{L}_{\rm s}-p_{\rm s}\mathrm{I}|_{\mathrm{H}^{l_\sigma+1}(\Omega_{\rm s})} + \Big(C_{\rm ns}^{\frac12}h^{l_{\mathbf{u}_{\rm s}}+\frac{\beta}{2}} + h^{l_{\mathbf{u}_{\rm s}}+1} + \delta^{1/2}h^{l_{\mathbf{u}_{\rm s}}-\frac{1-\beta}{2}}\Big)|\mathbf{u}_{\rm s}|_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})} + \Big(h^{l_{\mathbf{u}_{\rm d}}+1}+\delta^{1/2}h^{l_{\mathbf{u}_{\rm d}}-\frac{1-\beta}{2}}\Big)|\mathbf{u}_{\rm d}|_{\mathbf{H}^{l_{\mathbf{u}_{\rm d}}+1}(\Omega_{\rm d})},$$

$$\big(\|\varepsilon^{\mathbf{u}_{\rm s}}\|^2_{\Omega_h^{\rm s}}+\|\varepsilon^{p_{\rm d}}\|^2_{\Omega_h^{\rm d}}\big)^{\frac12} \lesssim \Big(C_{\rm nd}^{1/2}\delta h^{\frac{-3+\beta}{2}} + C_{\rm nd}^{1/2}h^{1/2} + \delta h^{-1/2} + h^2\Big)h^{l_{p_{\rm d}}}|p_{\rm d}|_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + \Big(\delta^{3/2}h^{\frac{\beta-4}{2}} + \delta^{1/2} + h^2\Big)h^{l_\sigma}|\nu\mathrm{L}_{\rm s}-p_{\rm s}\mathrm{I}|_{\mathrm{H}^{l_\sigma+1}(\Omega_{\rm s})} + \Big(\delta^{3/2}h^{\frac{\beta-4}{2}} + \delta^{1/2} + h^2\Big)h^{l_{\mathbf{u}_{\rm d}}}|\mathbf{u}_{\rm d}|_{\mathbf{H}^{l_{\mathbf{u}_{\rm d}}+1}(\Omega_{\rm d})} + \Big(C_{\rm ns}^{1/2}\delta h^{\frac{-3+\beta}{2}} + C_{\rm ns}^{1/2}h^{1/2} + \delta h^{-1/2} + h^2\Big)h^{l_{\mathbf{u}_{\rm s}}}|\mathbf{u}_{\rm s}|_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})},$$

$$\|\mathbf{u}_{\rm s}-\mathbf{u}_{{\rm s},h}\|_{\Omega_h^{\rm s}} + \|p_{\rm d}-p_{{\rm d},h}\|_{\Omega_h^{\rm d}} \lesssim \Big(\|\varepsilon^{\mathbf{u}_{\rm s}}\|^2_{\Omega_h^{\rm s}}+\|\varepsilon^{p_{\rm d}}\|^2_{\Omega_h^{\rm d}}\Big)^{\frac12} + \Big(+h_{\rm d}^{2(l_{p_{\rm d}}+1)}|p_{\rm d}|^2_{H^{l_{p_{\rm d}}+1}(\Omega_{\rm d})} + h_{\rm s}^{2(l_{\mathbf{u}_{\rm s}}+1)}|\mathbf{u}_{\rm s}|^2_{\mathbf{H}^{l_{\mathbf{u}_{\rm s}}+1}(\Omega_{\rm s})}\Big)^{1/2}.$$

Corollary 2. *Suppose that the assumptions of Theorem 2 hold and* $(\mathrm{L_s}, \mathbf{u}_\star, p_\star) \in \mathrm{H}^{k+1}(\Omega_h^{\mathrm{s}}) \times \mathbf{H}^{k+1}(\Omega_h^\star) \times H^{k+1}(\Omega_h^\star)$. *Let* $\delta = C_{\mathrm{g}} h^{1+\gamma}$ *with* $C_{\mathrm{g}} \geqslant 0$ *and* $\gamma \in (3/4, 3]$, *and* $\beta \in [0,2] \cap [0, 2\gamma - 1)$, *with* $\star \in \{\mathrm{s}, \mathrm{d}\}$. *There hold*

$$\|\mathbf{u}_{\mathrm{d}} - \mathbf{u}_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}} + \|\nu(\mathrm{L_s} - \mathrm{L}_{\mathrm{s},h})\|_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{s}} - p_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} \lesssim (C_{\mathrm{ns}}^{1/2} + C_{\mathrm{nd}}^{1/2}) h^{k+\frac{\beta}{2}} + h^{k+1}\big(1 + C_{\mathrm{g}}^{1/2} h^{\frac{\gamma+\beta-2}{2}}\big),$$

$$\big(\|\varepsilon^{\mathbf{u}_{\mathrm{s}}}\|^2_{\Omega_h^{\mathrm{s}}} + \|\varepsilon^{p_{\mathrm{d}}}\|^2_{\Omega_h^{\mathrm{d}}}\big)^{1/2} \lesssim h^{k+1}\big(h + (C_{\mathrm{nd}}^{1/2} + C_{\mathrm{ns}}^{1/2}) h^{-1/2} + C_{\mathrm{g}}^{1/2} h^{\frac{\gamma-1}{2}} + \big(C_{\mathrm{ns}}^{1/2} + C_{\mathrm{nd}}^{1/2}\big) C_{\mathrm{g}} h^{\gamma + \frac{\beta-3}{2}}\big),$$

$$\|\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{d}} - p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}} \lesssim h^{k+1}\big(1 + (C_{\mathrm{nd}}^{1/2} + C_{\mathrm{ns}}^{1/2}) h^{-1/2} + C_{\mathrm{g}}^{1/2} h^{\frac{\gamma-1}{2}} + \big(C_{\mathrm{ns}}^{1/2} + C_{\mathrm{nd}}^{1/2}\big) C_{\mathrm{g}} h^{\gamma + \frac{\beta-3}{2}}\big).$$

We now explain the consequences of this corollary for some particular cases.

- **(C.1)** *No-gap and no hanging nodes.* In this case, $C_{\mathrm{ns}} = C_{\mathrm{nd}} = C_{\mathrm{g}} = 0$ and our result shows optimal order of convergence of h^{k+1} for all the variables and order h^{k+2} for the projection of the errors $\varepsilon^{\mathbf{u}_{\mathrm{s}}}$ and $\varepsilon^{p_{\mathrm{d}}}$, as expected.
- **(C.2)** *No-gap and hanging nodes.* In this case, $C_{\mathrm{g}} = 0$, but $C_{\mathrm{ns}} \neq 0$ and $C_{\mathrm{nd}} \neq 0$. In this situation, since $C_{\mathrm{g}} = 0$, we can take $\beta = 2$. Therefore, we obtain optimal order of convergence of h^{k+1} for the variables $\mathbf{u}_{\mathrm{d}}$, $\mathrm{L_s}$ and p_{s}, but suboptimal order of $h^{k+1/2}$ for the projection of the errors $\varepsilon^{\mathbf{u}_{\mathrm{s}}}$ and $\varepsilon^{p_{\mathrm{d}}}$ and also for the errors in $\mathbf{u}_{\mathrm{s}}$ and p_{d}.
- **(C.3)** *Gap δ of order h^2 and no hanging nodes.* Here $\gamma = 1$, which implies that $\beta = 1 - \epsilon$ for all $\epsilon > 0$, whereas the nonconformity constants C_{ns} and C_{nd} vanish. This yields, for all the variables, order of convergence $h^{k+1-\epsilon}$ for all $\epsilon > 0$.
- **(C.4)** *Gap δ of order h^2 and hanging nodes.* Again, $\gamma = 1$ and $\beta = 1 - \epsilon$ for all $\epsilon > 0$, but now $C_{\mathrm{ns}} \neq 0$ and $C_{\mathrm{nd}} \neq 0$. Therefore, an order of convergence of $h^{k+1/2-\epsilon}$ for all $\epsilon > 0$ is attained for the variables $\mathbf{u}_{\mathrm{d}}$, $\mathrm{L_s}$ and p_{s} and $h^{k+1/2}$ for all the other variables.
- **(C.5)** *Gap δ of order $h^{7/4}$ and no hanging nodes.* In this case $\gamma = 3/4$, $\beta = 1/2 - \epsilon$ for all $\epsilon > 0$, and $C_{\mathrm{ns}} = C_{\mathrm{nd}} = 0$. Then, the order of convergence is $h^{k+5/8-\epsilon}$ for all $\epsilon > 0$ for the variables $\mathbf{u}_{\mathrm{d}}$, $\mathrm{L_s}$ and p_{s} and $h^{k+7/8}$ for the rest of the variables.
- **(C.6)** *Gap δ of order $h^{7/4}$ and hanging nodes.* Here $\gamma = 3/4$ and $\beta = 1/2 - \epsilon$ for all $\epsilon > 0$, whereas $C_{\mathrm{ns}} \neq 0$ and $C_{\mathrm{nd}} \neq 0$. Hence, an order of convergence of $h^{k+1/4-\epsilon}$ for all $\epsilon > 0$ is attained for the variables $\mathbf{u}_{\mathrm{d}}$, $\mathrm{L_s}$ and p_{s} and $h^{k+1/2}$ for the rest of the variables.

We observe that the introduction of the constant β as exponent in the first term of the right-hand side of the first equation of Corollary 2 slightly improves the theoretical convergence rate of the variables involved there, despite the presence of the non-conformity constants.

We end this section by considering the particular case where the discrete interfaces $\mathcal{I}_h^{\mathrm{d}}$ and $\mathcal{I}_h^{\mathrm{s}}$ satisfy the requirement set out in Section 4.3. In this case, Lemma 7 suggests an improvement of $h^{3/2}$ in the term associated to $\mathbf{j}_{\mathrm{s}}^{nc}$, whereas the term associated to $j_{\mathrm{d}}^{\mathrm{nc}}$ vanishes. Thus, the *semi-aligned* variant of Corollary 2 is established as follows.

Corollary 3. *Let us consider the same assumptions as in Corollary 2. If the discrete interfaces are semi-aligned, then*

$$\big(\|\varepsilon^{\mathbf{u}_{\mathrm{s}}}\|^2_{\Omega_h^{\mathrm{s}}} + \|\varepsilon^{p_{\mathrm{d}}}\|^2_{\Omega_h^{\mathrm{d}}}\big)^{1/2} \lesssim h^{k+1}\big(h + C_{\mathrm{nd}}^{1/2} h + C_{\mathrm{g}}^{1/2} h^{\frac{\gamma-1}{2}} + C_{\mathrm{nd}}^{1/2} C_{\mathrm{g}} h^{\gamma + \frac{\beta-3}{2}}\big),$$

$$\|\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s},h}\|_{\Omega_h^{\mathrm{s}}} + \|p_{\mathrm{d}} - p_{\mathrm{d},h}\|_{\Omega_h^{\mathrm{d}}} \lesssim h^{k+1}\big(1 + C_{\mathrm{nd}}^{1/2} h + C_{\mathrm{g}}^{1/2} h^{\frac{\gamma-1}{2}} + C_{\mathrm{nd}}^{1/2} C_{\mathrm{g}} h^{\gamma + \frac{\beta-3}{2}}\big).$$

Let us comment on the consequences of this result.

(D.1) *No-gap and hanging nodes.* Here, $C_{\mathrm{g}} = 0$, but $C_{\mathrm{ns}} \neq 0$ and $C_{\mathrm{nd}} \neq 0$. Therefore, optimal order of convergence of h^{k+1} for all the variables and order h^{k+2} for the projection of the errors $\varepsilon^{\mathbf{u}_{\mathrm{s}}}$ and $\varepsilon^{p_{\mathrm{d}}}$, which improves the power result in **(C.2)**.

(D.2) *Gap $\delta = h^2$ and hanging nodes.* Here, $\gamma = 1$ and $\beta = 1 - \epsilon$ for all $\epsilon > 0$, but now $C_{\mathrm{ns}} \neq 0$ and $C_{\mathrm{nd}} \neq 0$. Therefore, we improve the order of convergence stated in **(C.4)** since now $h^{k+1-\epsilon}$ for all $\epsilon > 0$ is attained for the projection of the errors $\varepsilon^{\mathbf{u}_{\mathrm{s}}}$ and $\varepsilon^{p_{\mathrm{d}}}$ and also for the errors in $\mathbf{u}_{\mathrm{s}}$ and p_{d}.

(D.3) *Gap $\delta = h^{7/4}$ and hanging nodes.* In this case, there is no improvement in the order of convergence compared to case **(C.6)**.

6 Numerical results

We consider four numerical results we the aim of illustrating the convergence of our HDG method presented in Section 3.2 for the two dimensional case. In all of them, we consider the computational domain $\Omega = \Omega_{\mathrm{s}} \cup \Omega_{\mathrm{d}} \cup \Sigma$, where $\Omega_{\mathrm{s}} = (0,1) \times (1/2, 1)$, $\Omega_{\mathrm{d}} = (0,1) \times (0, 1/2)$, and $\Sigma = (0,1) \times \{1/2\}$. In turn, we approach our numerical examples by two computational subdomains $\Omega_h^{\mathrm{s}} = (0,1) \times (1/2 + \delta, 1)$, $\Omega_h^{\mathrm{d}} = (0,1) \times (0, 1/2 - \delta)$, i.e., two rectangular meshed subdomains separated by a flat interface centered at $y = 1/2$. In addition, we define the manufactured exact solution:

$$P_{\mathrm{s}} = \exp(-x-y) - 2\exp(-2)(\exp(1/2)-1)^2(\exp(1/2)+1), \quad \mathbf{u}_{\mathrm{s}} = \begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ \cos(\pi x)\cos(\pi y) \end{pmatrix},$$

$$p_{\mathrm{d}} = \cos(\pi x)\sin(\pi y), \quad \text{and} \quad \mathbf{u}_{\mathrm{d}} = \begin{pmatrix} \pi\sin(\pi x)\sin(\pi y) \\ -\pi\cos(\pi x)\cos(\pi y) \end{pmatrix}.$$

Also, hereafter we take $\kappa = \mathbb{I}$, $\nu = 1$ and the stabilization parameter $\tau \equiv 1$. Subsequently, we define the errors:

$$e(\mathrm{L}) = \|\mathrm{L}_{\mathrm{s}} - \mathrm{L}_{\mathrm{s},h}\|_{\Omega_{\mathrm{s}}}, \qquad e(\mathbf{u}) = \left(\|\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{s},h}\|^2_{0,\Omega_{\mathrm{s}}} + \|\mathbf{u}_{\mathrm{d}} - \mathbf{u}_{\mathrm{d},h}\|^2_{\Omega_{\mathrm{d}}}\right)^{1/2},$$

$$e(p) = \left(\|P_{\mathrm{s}} - P_{\mathrm{s},h}\|^2_{\Omega_{\mathrm{s}}} + \|p_{\mathrm{d}} - p_{\mathrm{d},h}\|^2_{\Omega_{\mathrm{d}}}\right)^{1/2}, \quad \widehat{e} = \left(\|\boldsymbol{\mathcal{P}}^{\mathrm{s}}\mathbf{u}_{\mathrm{s}} - \widehat{\mathbf{u}}_{\mathrm{s},h}\|^2_{\partial\Omega_{\mathrm{s}}} + \left\|\mathcal{P}^{\mathrm{d}} p_{\mathrm{d}} - \widehat{p}_{\mathrm{d},h}\right\|^2_{\partial\Omega_{\mathrm{d}}}\right)^{1/2},$$

$$P_{\mathrm{s},h} = p_{\mathrm{s},h} - \frac{1}{|\Omega_{\mathrm{s}}|}\int_{\Omega_{\mathrm{s}} \setminus \Omega_h^{\mathrm{s}}} \boldsymbol{E}_{p_{\mathrm{s},h}},$$

where $\boldsymbol{E}$ denotes the local extrapolation defined in (2.1) and $p_{\mathrm{s},h}$ is the discrete pressure of our HDG scheme that satisfies (3.2e). Next, the experimental convergence rates are set as

$$r = -2\frac{\log(\widetilde{e}/e)}{\log(\widetilde{N}/N)},$$

where e and $\widetilde{e}$ denote errors computed on two consecutive meshes with N and $\widetilde{N}$ elements, respectively.

6.1. No gap. In our first numerical experiment, we take $\delta = 0$ and consider two different scenarios: one free of hanging nodes and one containing hanging nodes on the discrete interfaces. For the first scenario (case **(C.1)** above), the results in Table 1, confirm the theoretical rate of convergence for all variables provided by Corollary 2 (for $k = 4$, the errors calculated for $\widehat{e}$ are affected by round-off errors). For the second scenario, we take coarser meshes for Ω_{d} such that each interface edge corresponds to two interface edges on the meshes for Ω_{s}, effectively introducing one hanging node per side. This corresponds to case **(D.1)**. Table 2 shows optimal order of convergence for all variables in this case, observing superconvergence h^{k+2} in the numerical trace, which is the theoretical order of convergence provided by Corollary 3.

k	N	$e(\mathrm{L})$	r	$e(\mathbf{u})$	r	$e(p)$	r	$\widehat{e}$	$\widehat{r}$
1	56	8.09e-02	*	8.05e-02	*	4.86e-02	*	3.34e-03	*
	212	2.00e-02	2.10	2.13e-02	2.00	1.28e-02	2.00	3.57e-04	3.36
	800	5.49e-03	1.94	5.62e-03	2.01	3.33e-03	2.03	5.16e-05	2.91
	3216	1.35e-03	2.01	1.40e-03	2.00	8.07e-04	2.04	6.30e-06	3.02
	12716	3.49e-04	1.97	3.54e-04	2.00	2.04e-04	2.00	8.39e-07	2.93
2	56	1.21e-02	*	1.05e-02	*	4.97e-03	*	2.29e-04	*
	212	1.29e-03	3.37	1.18e-03	3.29	6.19e-04	3.13	1.00e-05	4.70
	800	1.63e-04	3.12	1.52e-04	3.09	7.93e-05	3.10	6.54e-07	4.11
	3216	2.07e-05	2.97	1.89e-05	2.99	9.82e-06	3.00	4.27e-08	3.92
	12716	2.73e-06	2.95	2.47e-06	2.96	1.28e-06	2.97	2.97e-09	3.88
3	56	9.50e-04	*	8.42e-04	*	4.11e-04	*	9.09e-06	*
	212	6.14e-05	4.11	5.47e-05	4.11	2.64e-05	4.12	2.35e-07	5.49
	800	4.89e-06	3.81	4.15e-06	3.88	1.91e-06	3.96	9.93e-09	4.77
	3216	2.88e-07	4.07	2.52e-07	4.03	1.13e-07	4.06	2.95e-10	5.05
	12716	1.97e-08	3.90	1.65e-08	3.97	7.49e-09	3.95	1.04e-11	4.87
4	56	1.63e-04	*	1.29e-04	*	4.87e-05	*	6.81e-07	*
	212	3.47e-06	5.78	2.90e-06	5.70	1.28e-06	5.47	6.61e-09	6.96
	800	1.04e-07	5.29	8.92e-08	5.24	4.09e-08	5.19	1.01e-10	6.30
	3216	3.37e-09	4.93	2.90e-09	4.92	1.29e-09	4.97	1.73e-12	5.84
	12716	1.21e-10	4.84	9.94e-11	4.91	4.51e-11	4.88	2.01e-13	3.13

Table 1: History of convergence of the HDG method for $\delta = 0$ and without hanging nodes

k	N	$e(\mathrm{L})$	r	$e(\mathbf{u})$	r	$e(p)$	r	$\widehat{e}$	$\widehat{r}$
1	34	9.06e-02	*	2.95e-01	*	1.57e-01	*	1.68e-02	*
	134	2.09e-02	2.14	6.39e-02	2.23	3.64e-02	2.13	1.99e-03	3.11
	506	5.58e-03	1.99	1.62e-02	2.06	1.03e-02	1.91	2.42e-04	3.17
	2000	1.36e-03	2.05	4.32e-03	1.93	2.69e-03	1.95	3.33e-05	2.89
	7974	3.50e-04	1.96	1.07e-03	2.01	6.61e-04	2.03	4.15e-06	3.01
2	34	1.28e-02	*	6.99e-02	*	3.31e-02	*	2.25e-03	*
	134	1.34e-03	3.29	9.08e-03	2.98	3.71e-03	3.19	1.48e-04	3.97
	506	1.66e-04	3.14	9.68e-04	3.37	4.86e-04	3.06	7.25e-06	4.54
	2000	2.09e-05	3.02	1.22e-04	3.01	6.39e-05	2.95	4.73e-07	3.97
	7974	2.74e-06	2.94	1.53e-05	3.01	7.89e-06	3.02	3.05e-08	3.97
3	34	1.06e-03	*	1.51e-02	*	6.58e-03	*	2.33e-04	*
	134	6.49e-05	4.07	6.90e-04	4.50	3.39e-04	4.32	5.24e-06	5.53
	506	4.97e-06	3.87	4.52e-05	4.10	2.18e-05	4.13	1.75e-07	5.12
	2000	2.90e-07	4.13	3.57e-06	3.69	1.51e-06	3.89	7.19e-09	4.65
	7974	1.97e-08	3.89	2.16e-07	4.06	9.23e-08	4.04	2.19e-10	5.05
4	34	1.71e-04	*	2.62e-03	*	1.10e-03	*	2.20e-05	*
	134	3.66e-06	5.60	1.17e-04	4.53	3.81e-05	4.91	4.56e-07	5.65
	506	1.07e-07	5.32	2.54e-06	5.77	9.89e-07	5.50	5.25e-09	6.72
	2000	3.42e-09	5.01	7.53e-08	5.12	3.37e-08	4.92	7.69e-11	6.15
	7974	1.22e-10	4.82	2.50e-09	4.93	1.07e-09	4.99	1.31e-12	5.89

Table 2: History of convergence of the HDG method for $\delta = 0$ and with hanging nodes

6.2. Gap of order $h^{7/4}$. According to Corollary 2, γ must be larger than 3/4. In this experiment, we want to observe the behavior of the errors when γ is equal to 3/4, which means δ of order $h^{7/4}$.

Similarly to the previous example, we divide this experiment in two different scenarios. First, we suppose again that the meshes are free of "hanging nodes" (case **(C.5)**), i.e., there is a one-to-one face bijection between the two interfaces, which means that $C_{\text{ns}} = C_{\text{nd}} = 0$. The behaviour of the errors reported in Table 3 is better than the prediction of Corollary 2. In fact, we observe optimal rates for all the variables and superconvergence in $\widehat{e}$. For the second scenario, we add hanging nodes

k	N	$e(\mathrm{L})$	r	$e(\mathbf{u})$	r	$e(p)$	r	$\widehat{e}$	$\widehat{r}$
	60	1.21e-01	∗	9.86e-02	∗	4.59e-02	∗	1.64e-02	∗
	212	2.62e-02	2.43	2.40e-02	2.24	1.31e-02	1.98	2.52e-03	2.96
1	816	6.18e-03	2.15	5.57e-03	2.17	3.27e-03	2.06	3.66e-04	2.87
	3228	1.45e-03	2.11	1.41e-03	2.00	8.17e-04	2.02	4.22e-05	3.14
	12772	3.55e-04	2.04	3.55e-04	2.01	2.05e-04	2.01	5.86e-06	2.87
	60	1.61e-02	∗	1.57e-02	∗	2.93e-02	∗	1.65e-03	∗
	212	2.24e-03	3.13	2.32e-03	3.03	2.18e-03	4.11	2.34e-04	3.10
2	816	2.15e-04	3.47	2.17e-04	3.52	1.61e-04	3.87	1.85e-05	3.76
	3228	2.28e-05	3.27	2.10e-05	3.39	1.44e-05	3.51	1.04e-06	4.18
	12772	2.82e-06	3.04	2.52e-06	3.09	1.48e-06	3.31	7.73e-08	3.79
	60	3.14e-03	∗	3.56e-03	∗	1.19e-03	∗	3.29e-04	∗
	212	1.59e-04	4.72	2.27e-04	4.36	6.29e-05	4.65	1.87e-05	4.54
3	816	8.93e-06	4.27	5.88e-06	5.42	3.35e-06	4.35	5.41e-07	5.26
	3228	3.51e-07	4.71	2.64e-07	4.52	1.46e-07	4.56	1.08e-08	5.69
	12772	2.05e-08	4.13	1.68e-08	4.00	7.86e-09	4.24	3.60e-10	4.95
	60	2.53e-04	∗	3.64e-04	∗	6.16e-04	∗	1.75e-05	∗
	212	9.90e-06	5.13	1.54e-05	5.01	1.08e-05	6.40	7.87e-07	4.92
4	816	2.25e-07	5.61	3.01e-07	5.84	1.77e-07	6.10	1.35e-08	6.03
	3228	4.27e-09	5.76	3.86e-09	6.34	3.23e-09	5.82	1.27e-10	6.79
	12772	1.31e-10	5.07	1.05e-10	5.24	6.69e-11	5.64	2.30e-12	5.83

Table 3: History of convergence of the HDG method for $\delta = h^{7/4}$ and without hanging nodes

on the bottom mesh as before, introducing one per interface edge. This corresponds to case **(D.3)**. In Table 4 we show the results for this case. We again observe optimal and superconvergent rates, which is better than what the theory predicted.

6.2. Gap of order h. We now consider the last numerical example taking a gap $\delta = h$ and that the meshes are free of "hanging nodes". Since $\gamma = 0$, this case is not covered by the analysis but we observe that the approximations of all the variables converge with optimal order according to Table 5. However, the superconvergence of the numerical trace variables is lost.

References

[1] N. M. Atallah, C. Canuto, and G. Scovazzi, *The high-order shifted boundary method and its analysis*, Computer Methods in Applied Mechanics and Engineering, 394 (2022), p. 114885.

[2] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing, *Cutfem: Discretizing geometry and partial differential equations*, International Journal for Numerical Methods in Engineering, 104 (2015), pp. 472–501.

k	N	$e(\mathrm{L})$	r	$e(\mathbf{u})$	r	$e(p)$	r	$\widehat{e}$	$\widehat{r}$
	36	1.25e-01	*	2.90e-01	*	1.11e-01	*	3.69e-02	*
	134	2.81e-02	2.28	6.47e-02	2.28	3.55e-02	1.73	3.89e-03	3.42
1	514	6.33e-03	2.22	1.71e-02	1.98	9.84e-03	1.91	5.29e-04	2.97
	2010	1.46e-03	2.15	4.32e-03	2.01	2.66e-03	1.92	5.86e-05	3.23
	8018	3.57e-04	2.04	1.06e-03	2.03	6.59e-04	2.02	7.54e-06	2.97
	36	1.98e-02	*	1.30e-01	*	3.61e-02	*	1.37e-02	*
	134	2.40e-03	3.21	8.98e-03	4.07	3.95e-03	3.36	3.82e-04	5.45
2	514	2.37e-04	3.44	1.10e-03	3.13	5.12e-04	3.04	4.08e-05	3.33
	2010	2.41e-05	3.35	1.34e-04	3.08	6.67e-05	2.99	3.17e-06	3.75
	8018	2.87e-06	3.08	1.62e-05	3.05	8.13e-06	3.04	1.97e-07	4.02
	36	3.83e-03	*	1.29e-02	*	4.41e-03	*	6.33e-04	*
	134	2.11e-04	4.41	1.05e-03	3.81	3.42e-04	3.89	5.32e-05	3.77
3	514	9.93e-06	4.54	5.84e-05	4.30	2.29e-05	4.02	1.62e-06	5.19
	2010	3.74e-07	4.81	3.45e-06	4.15	1.52e-06	3.98	3.40e-08	5.67
	8018	2.07e-08	4.18	2.15e-07	4.01	9.33e-08	4.03	6.59e-10	5.70
	36	1.36e-03	*	1.41e-02	*	2.08e-03	*	7.79e-04	*
	134	1.27e-05	7.11	1.02e-04	7.50	3.28e-05	6.32	1.69e-06	9.34
4	514	3.70e-07	5.26	3.27e-06	5.11	1.15e-06	4.99	7.05e-08	4.72
	2010	7.17e-09	5.79	1.00e-07	5.12	3.66e-08	5.05	1.55e-09	5.59
	8018	1.59e-10	5.51	2.82e-09	5.16	1.12e-09	5.04	2.07e-11	6.25

Table 4: History of convergence of the HDG method for $\delta = h^{7/4}$ and with hanging nodes

k	N	$e(\mathrm{L})$	r	$e(\mathbf{u})$	r	$e(p)$	r	$\widehat{e}$	$\widehat{r}$
	60	9.13e-02	*	8.36e-02	*	4.54e-02	*	9.34e-03	*
	216	2.48e-02	2.04	2.21e-02	2.08	1.26e-02	2.01	2.26e-03	2.22
1	804	6.92e-03	1.94	5.59e-03	2.09	3.54e-03	1.93	5.48e-04	2.16
	3242	1.69e-03	2.02	1.39e-03	1.99	9.16e-04	1.94	1.26e-04	2.10
	12874	4.22e-04	2.02	3.52e-04	2.00	2.26e-04	2.03	3.03e-05	2.07
	60	1.81e-02	*	1.07e-02	*	2.36e-02	*	1.76e-03	*
	216	2.10e-03	3.37	2.13e-03	2.52	2.03e-03	3.83	2.14e-04	3.29
2	804	3.08e-04	2.92	3.40e-04	2.79	2.82e-04	3.01	3.27e-05	2.86
	3242	4.37e-05	2.80	4.02e-05	3.06	3.95e-05	2.82	4.12e-06	2.97
	12874	5.90e-06	2.90	5.28e-06	2.94	4.95e-06	3.01	5.65e-07	2.88
	60	1.70e-03	*	2.03e-03	*	8.05e-04	*	1.71e-04	*
	216	1.38e-04	3.92	1.58e-04	3.98	5.28e-05	4.25	1.43e-05	3.88
3	804	1.36e-05	3.53	6.17e-06	4.94	5.68e-06	3.39	9.05e-07	4.20
	3242	7.36e-07	4.18	3.03e-07	4.32	3.52e-07	3.99	4.77e-08	4.22
	12874	4.59e-08	4.02	1.93e-08	3.99	2.02e-08	4.15	2.99e-09	4.01
	60	6.05e-04	*	1.87e-04	*	7.72e-04	*	3.27e-05	*
	216	8.92e-06	6.59	1.27e-05	4.19	9.91e-06	6.80	6.80e-07	6.05
4	804	4.02e-07	4.72	5.97e-07	4.66	4.21e-07	4.81	2.63e-08	4.95
	3242	1.49e-08	4.73	1.63e-08	5.16	1.50e-08	4.78	7.80e-10	5.05
	12874	5.35e-10	4.83	5.40e-10	4.94	4.74e-10	5.01	2.82e-11	4.81

Table 5: History of convergence of the HDG method for $\delta = h$ and without hanging nodes

[3] Y. Chen and B. Cockburn, *Analysis of variable-degree hdg methods for convection–diffusion*

equations. part i: general nonconforming meshes, IMA Journal of Numerical Analysis, 32 (2012), pp. 1267–1293.

[4] ——, *Analysis of variable-degree hdg methods for convection-diffusion equations. part ii: Semi-matching nonconforming meshes*, Mathematics of Computation, 83 (2014), pp. 87–111.

[5] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous galerkin, mixed, and continuous galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, 47 (2009), pp. 1319–1365.

[6] B. Cockburn, J. Gopalakrishnan, N. C. Nguyen, J. Peraire, and F.-J. Sayas, *Analysis of HDG methods for Stokes flow*, Math. Comp., 80 (2011), pp. 723–760.

[7] B. Cockburn, J. Gopalakrishnan, and F.-J. Sayas, *A projection-based error analysis of HDG methods*, Mathematics of Computation, 79 (2010), pp. 1351–1367.

[8] B. Cockburn, D. Gupta, and F. Reitich, *Boundary-conforming discontinuous Galerkin methods via extensions from subdomains*, Journal of Scientific Computing, 42 (2010), pp. 144–184.

[9] B. Cockburn, W. Qiu, and M. Solano, *A priori error analysis for HDG methods using extensions from subdomains to achieve boundary conformity*, Mathematics of Computation, 83 (2014), pp. 665–699.

[10] B. Cockburn, F.-J. Sayas, and M. Solano, *Coupling at a distance HDG and BEM*, SIAM Journal on Scientific Computing, 34 (2012), pp. A28–A47.

[11] B. Cockburn and M. Solano, *Solving Dirichlet boundary-value problems on curved domains by extensions from subdomains*, SIAM Journal on Scientific Computing, 34 (2012), pp. A497–A519.

[12] D. A. Di Pietro and A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Mathématiques et Applications, 69., Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.

[13] M. Duprez, V. Lleras, and A. Lozinski, *A new ϕ-FEM approach for problems with natural boundary conditions*, Numerical Methods for Partial Differential Equations, 39 (2023), pp. 281–303.

[14] ——, *ϕ-FEM: an optimally convergent and easily implementable immersed boundary method for particulate flows and Stokes equations*, ESAIM: Mathematical Modelling and Numerical Analysis, 57 (2023), pp. 1111–1142.

[15] M. Duprez and A. Lozinski, *ϕ-FEM: A finite element method on domains defined by level-sets*, SIAM Journal on Numerical Analysis, 58 (2020), pp. 1008–1028.

[16] T. Frachon, E. Nilsson, and S. Zahedi, *Divergence-free cut finite element methods for Stokes flow*, 2023.

[17] G. N. Gatica, R. Oyarzúa, and F.-J. Sayas, *Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem*, Mathematics of Computation, 80 (2011), pp. 1911–1948.

[18] G. N. Gatica and F. A. Sequeira, *Analysis of the HDG method for the Stokes–Darcy coupling*, Numerical Methods for Partial Differential Equations, 33 (2017), pp. 885–917.

[19] M. G. Larson and S. Zahedi, *Conservative cut finite element methods using macroelements*, Computer Methods in Applied Mechanics and Engineering, 414 (2023), p. 116141.

[20] R. Li, J. Li, X. Liu, and Z. Chen, *A weak Galerkin finite element method for a coupled Stokes-Darcy problem*, Numerical Methods for Partial Differential Equations, 33 (2017), pp. 1352–1373.

[21] A. Main and G. Scovazzi, *The shifted boundary method for embedded domain computations. Part I: Poisson and Stokes problems*, Journal of Computational Physics, 372 (2018), pp. 972–995.

[22] J. Manríquez, N.-C. Nguyen, and M. Solano, *A dissimilar non-matching HDG discretization for Stokes flows*, Computer Methods in Applied Mechanics and Engineering, 399 (2022), p. 115292.

[23] R. Oyarzúa, M. Solano, and P. Zúñiga, *A high order mixed-FEM for diffusion problems on curved domains*, Journal of Scientific Computing, 79 (2019), pp. 49–78.

[24] W. Qiu, M. Solano, and P. Vega, *A high order HDG method for curved-interface problems via approximations from straight triangulations*, Journal of Scientific Computing, 69 (2016), pp. 1384–1407.

[25] M. Solano, S. Terrana, N.-C. Nguyen, and J. Peraire, *An HDG method for dissimilar meshes*, IMA Journal of Numerical Analysis, 42 (2021), pp. 1665–1699.

[26] M. Solano and F. Vargas, *A high order HDG method for Stokes flow in curved domains*, Journal of Scientific Computing, 79 (2019), pp. 1505–1533.

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2023 - 2024

2023-25 Franz Chouly, Patrick Hild, Yves Renard: *Lagrangian and Nitsche methods for frictional contact*

2023-26 Sergio Caucao, Gabriel N. Gatica, Juan P. Ortega: *A three-field mixed finite element method for the convective Brinkman–Forchheimer problem with varying porosity*

2023-27 Raimund Bürger, Yessennia Martínez, Luis M. Villada: *Front tracking and parameter identification for a conservation law with a space-dependent coefficient modeling granular segregation*

2023-28 Marie Haghebaert, Beatrice Laroche, Mauricio Sepúlveda: *Study of the numerical method for an inverse problem of a simplified intestinal crypt*

2023-29 Rodolfo Araya, Fabrice Jaillet, Diego Paredes, Frederic Valentin: *Generalizing the Multiscale Hybrid-Mixed Method for Reactive-Advective-Diffusive Equations*

2023-30 Jessika Camaño, Ricardo Oyarzúa, Miguel Serón, Manuel Solano: *A mass conservative finite element method for a nonisothermal Navier-Stokes/Darcy coupled system*

2023-31 Franz Chouly, Hao Huang, Nicolás Pignet: *HHT-α and TR-BDF2 schemes for Nitsche-based discrete dynamic contact*

2024-01 Sergio Caucao, Gabriel N. Gatica, Saulo Medrado, Yuri D. Sobral: *Nonlinear twofold saddle point-based mixed finite element methods for a regularized $\mu(I)$-rheology model of granular materials*

2024-02 Julio Careaga, Gabriel N. Gatica, Cristian Inzunza, Ricardo Ruiz-Baier: *New Banach spaces-based mixed finite element methods for the coupled poroelasticity and heat equations*

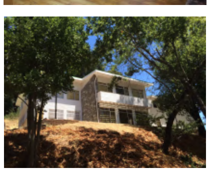
2024-03 Harold D. Contreras, Paola Goatin, Luis M. Villada: *A two-lane bidirectional nonlocal traffic model*

2024-04 Rommel Bustinza, Matteo Cicuttin, Ariel Lombardi: *A Hybrid High-Order method for the mixed Steklov eigenvalue problem*

2024-05 Isaac Bermudez, Jaime Manríquez, Manuel Solano: *A hybridizable discontinuous Galerkin method for Stokes/Darcy coupling in dissimilar meshes*

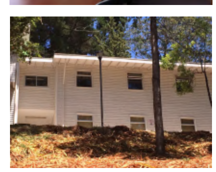
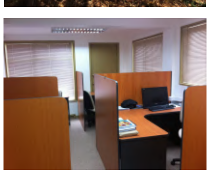
Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



Centro de Investigación en Ingeniería Matemática (CI²MA)

Universidad de Concepción

Casilla 160-C, Concepción, Chile

Tel.: 56-41-2661324/2661554/2661316

http://www.ci2ma.udec.cl