Centro de Investigación en Ingeniería Matemática

Universidad de Concepción

10th CI²MA Focus Seminar
"Numerical Methods for Hyperbolic and Related Problems"
Supported by Conicyt projects Anillo ACT1118 (ANANUM) and Fondecyt 11140708 and 1130154

October 21, 2015
Auditorio Alamiro Robledo
Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción

Organizers: Raimund Bürger and Luis Miguel Villada

Programme

14.00 **Aníbal Coronel** (Universidad del Bío-Bío, Chillán): A systematic construction of the solution for the Riemann problem for the Burgers equation with discontinuous source

14.30 **Mauricio Sepúlveda** (Universidad de Concepción): Convergence of a level-set algorithm in scalar conservation laws

15.00 **Christophe Chalons** (Université de Versailles Saint-Quentin-en-Yvelines, France): On all-regime Lagrange-Riemann numerical schemes for compressible fluids systems

15.30 **Luis Miguel Villada** (Universidad del Bío-Bío, Concepción): On second-order antidiffusive Lagrangian-remap schemes for multispecies kinematic flow models

16.00 **Lihki Rubio** (Universidad de Concepción): Polynomial viscosity methods for multispecies kinematic flow models

16.30 **COFFEE BREAK**

17.00 **Raimund Bürger** (Universidad de Concepción): A hybrid stochastic Galerkin method for uncertainty quantification applied to a conservation law modelling a clarifier-thickener unit with several random sources

17.30 **María del Carmen Martí** (Universidad de Concepción): Finite difference WENO schemes for multiphase flow in porous media

18.00 **Sudarshan Kennetinkara** (Universidad de Concepción): Discontinuous approximation of viscous two-phase flow in porous media

18.30 **Patricio Cumsille** (Universidad del Bío-Bío, Chillán): Numerical solution of a novel biofilm growth model

20.30 **Seminar Dinner**

Seminar participants who would like to join dinner should register with CI²MA secretary:

Ms Angelina Fritz, CI²MA
E-mail: afritz@ci2ma.udec.cl, Phone: (041) 266 1324

Abstracts

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A SYSTEMATIC CONSTRUCTION OF THE SOLUTION FOR THE RIEMANN PROBLEM FOR THE BURGERS EQUATION WITH DISCONTINUOUS SOURCE

ANIBAL CORONEL

Abstract. This paper is concerned with the explicit construction of the Riemann problem arising in the theory of Radiation Hydrodynamics:

$$u_t + \left(\frac{u^2}{2}\right)_x = g(x), \quad u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the source term and the initial condition are defined as follows

$$g(x) = g_R H(x) + g_L H(-x), \quad u_0(x) = u_R H(x) + u_L H(-x), \quad (g_L, g_R, u_L, u_R) \in \mathbb{R}^4,$$

with H is the Heviside function, i.e. $H(x) = 0$ for $x \in \mathbb{R}^-$ and $H(x) = 1$ for $x \in \mathbb{R}_0^+$. The main result of this work is this work is the following theorem: "*Denote by u_i for $i \in \{1, \dots, 43\}$ each of the posible entropic solutions of the Riemann problem. Then, there exists a partition $\{\mathbb{U}_1, \dots, \mathbb{U}_{43}\}$ of $\mathbb{R}^4$ such u_i is the entropic solution if and only if $(u_L, u_R, g_L, g_R) \in \mathbb{U}_i$.*" The proof is developed in sixty Lemmas. First, we apply the characteristics method and introduce a classification of the different type of waves. A systematic discussion of the all possible type of waves at $t = 0$, implies the existence of sixty type of solutions. Then, we analyze in detail the analytic construction of these solution types. Basically, and in a broad sense, a shock or a rarefaction wave are formed at $t = 0$. The evolution of shock curve is completely characterized by analyzing the initial value problem obtained by the Rankine-Hugoniot condition. The rarefaction wave solution is explicitly obtained by the characteristics method. Here, a subcase of rarefaction wave, called "vacuum wave", requires a regularization of the source term and the initial condition before of the application of the characteristics method. Finally, by a unification of the sixty Lemmas we obtain the partition $\{\mathbb{U}_1, \dots, \mathbb{U}_{43}\}$.

This contribution is a joint work with A. Tello (Chillán, Chile) and M. Sepúlveda (Concepción, Chile).

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CONVERGENCE OF A LEVEL-SET ALGORITHM IN SCALAR CONSERVATION LAWS

MAURICIO SEPÚLVEDA

Abstract. This contribution is concerned with the convergence of the level-set algorithm introduced by Aslam [1] for tracking the discontinuities in scalar conservation laws in the case of linear or strictly convex flux function. The numerical method is deduced by the level-set representation of the entropy solution: the zero of a level-set function is used as an indicator of the discontinuity curves and two auxiliary states, which are assumed continuous through the discontinuities, are introduced. We rewrite the numerical level-set algorithm as a procedure consisting of three big steps: (a) initialization, (b) evolution and (c) reconstruction. In (a) we choose an entropy admissible level-set representation of the initial condition. In (b), for each iteration step, we solve an uncoupled system of three equations and select the entropy admissible level-set representation of the solution profile at the end of the time iteration. In (c) we reconstruct the entropy solution by using the level-set representation. We prove the convergence of the numerical solution to the entropy solution in L^p_{loc} for every $p \geq 1$, using L^∞-weak BV estimates and a cell entropy inequality. In addition, some numerical examples focused on the elementary wave interaction are presented.

These results are based on recent joint work with Anibal Coronel and Patricio Cumsille (Universidad del Bío-Bío - Chillán).

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Partially supported by Fondecyt 1140676.

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ON ALL-REGIME LAGRANGE-REMAP NUMERICAL SCHEMES FOR COMPRESSIBLE FLUIDS SYSTEMS

CHRISTOPHE CHALONS

Abstract. It is the purpose of this contribution to provide an overview on recent advances in the development of all-regime Lagrange-Remap numerical schemes for compressible fluids systems with source terms. We will consider in particular the case of large friction coefficients and the case of low-Mach numbers. More precisely, we will present a discretization strategy for gas dynamics equations for unstrutctured grids based a Lagrange-Remap approach that does not involve any moving mesh. A natural semi-implicit extension of the method that allows to remain stable under a CFL condition involving only the material velocity will be given, together with an extremely simple modification that allows to provide an accurate and stable solver for simulations involving low-Mach regions in the flow.

This contribution is based on a series of joint works [1], [2], [3], [4] with Mathieu Girardin and Samuel Kokh. These works were performed during M. Girardin's PhD thesis.

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[4] Chalons C., Girardin M., Kokh S., An all-regime Lagrange-Projection like scheme for 2D homogeneous models for two-phase flows on unstructured meshes (submitted, 2015)

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ON SECOND-ORDER ANTIDIFFUSIVE LAGRANGIAN-REMAP SCHEMES FOR MULTISPECIES KINEMATIC FLOW MODELS

LUIS M. VILLADA

Abstract. This talk deals the numerical approximation of the solutions of multi-species kinematic flow models. These models are strongly coupled nonlinear first-order conservation laws with various applications like sedimentation of a polydisperse suspension in a viscous fluid, or traffic flow modeling. Since the eigenvalues and eigenvectors of the corresponding flux Jacobian matrix have no closed algebraic form, this is a challenging issue. A new class of simple schemes based on a Lagrangian-Eulerian decomposition (the so-called Lagrangian-remap (LR) schemes) was recently advanced in [2] for traffic flow models with nonnegative velocities, and extended to models of polydisperse sedimentation in [3]. Since they are only first-order accurate, we propose in [4] an extension to second-order accuracy using quite standard MUSCL and Runge-Kutta techniques. Numerical illustrations are presented for both applications and involving eleven species (sedimentation) and nine species (traffic) respectively.

This contribution is based on a serie of joint works [2, 3, 4] with Raimund Bürger (Concepción) and Christophe Chalons (Versailles, France).

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POLYNOMIAL VISCOSITY METHODS FOR MULTISPECIES KINEMATIC FLOW MODELS

LIHKI RUBIO

ABSTRACT. Multispecies kinematic flow models are defined by systems of strongly coupled, nonlinear first-order conservation laws. They arise in various applications including sedimentation of a polydisperse suspensions and multiclass vehicular traffic. Their numerical approximation is a challenge since the eigenvalues and eigenvectors of the corresponding flux Jacobian matrix have no closed algebraic form. It is demostrated that a recently introduced class of fast first-order finite volume solvers, called PVM (polynomial viscosity matrix) methods [M.J. Castro Díaz and E. Fernández-Nieto, SIAM J Sci Comput 34 (2012), A2173–A2196] can be adapted to multispecies kinematic flows. PVM methods have the advantage that they only need some information about the eigenvalues of the flux Jacobian, and no spectral decomposition of a Roe matrix is needed. In fact, the so-called interlacing property (of eigenvalues with known velocity functions), which holds for several important multispecies kinematic flow models, provides sufficient information for the implementation of PVM methods. Several variants of PVM methods (differing in polynomial degree and the underlying quadrature formula to approximate the Roe matrix) are compared by numerical experiments. It turns out that PVM methods are competitive in accuracy and efficiency with several existing methods, including the HLL method and a spectral WENO scheme that is based on the same interlacing property.

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A HYBRID STOCHASTIC GALERKIN METHOD FOR UNCERTAINTY QUANTIFICATION APPLIED TO A CONSERVATION LAW MODELLING A CLARIFIER-THICKENER UNIT WITH SEVERAL RANDOM SOURCES

RAIMUND BÜRGER

Abstract. The continuous sedimentation process in a clarifier-thickener can be described by a scalar nonlinear conservation law for the local solids volume fraction whose flux density function is discontinuous with respect to spatial position due to feed and discharge mechanisms. In the applications of this model, which include mineral processing and wastwater treatment, the rate and composition of the feed flow cannot be given deterministically. Efficient numerical simulation is required to quantify the effect of uncertainty in these control parameters in terms of the response of the clarifier-thickener system. Thus, the problem at hand is one of uncertainty quantification for nonlinear hyperbolic problems with several random perturbations. To solve it, a hybrid stochastic Galerkin (HSG) method is devised that extends the classical polynomial chaos approximation by multiresolution discretization in the stochastic space. The approach leads to a deterministic hyperbolic system for a finite number of stochastic moments which is however partially decoupled and thus allows efficient parallelisation. The complexity of the problem is further reduced by stochastic adaptivity. For the approximate solution of the resulting high-dimensional system a finite volume scheme is introduced. Several numerical experiments are presented.

This contribution is based on recent joint work [1] with A. Barth, I. Kröker and C. Rohde (Universität Stuttgart, Germany), see also [2, 3, 4].

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FINITE-DIFFERENCE WENO SCHEMES FOR MULTIPHASE FLOW IN POROUS MEDIA

MARÍA DEL CARMEN MARTÍ

Abstract. Mathematical models of multiphase flow are useful in some engineering applications like enhanced oil recovery or filtration of pollutants into subsurface. It is the purpose of this work to derive a numerical technique for the simulation of multi-dimensional multiphase flow motion in a homogeneus porous medium under the condition of vertical equilibrium, neglecting the capillary effects.

It is expected that discontinuities and/or sharp gradients will develop when capillary effects are neglected or are small. In order to obtain high-order finite-difference conservative schemes for the approximate solution of the equations of the system we use Shu and Osher's technique [3], for which the conservation property of the spatial discretization is obtained by flux reconstructions, obtained using the WENO reconstruction scheme [1, 2]. The key idea in the numerical technique proposed is to define a compatible discretization for the fluxes of the convective term, defined using Darcy's law, in order to maintain their divergence-free character not only on the continuous scale but also on the discrete scale, assuring the conservation of the sum of the concentrations through time evolution.

This contribution is based on an ongoing joint work with F. Guerrero and P. Mulet (Universitat de València, Spain) and R. Bürger (Universidad de Concepción, Chile).

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This work is partially supported by Proyecto FONDECYT de Postdoctorado 3150140.

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DISCONTINUOUS APPROXIMATION OF VISCOUS TWO-PHASE FLOW IN HETEROGENEOUS POROUS MEDIA

SUDARSHAN KENETTINKARA

Abstract. Two phase flow in porous media is well known for its high importance in many of the industrial and engineering applications like petroleum reservoir, sedimentation process, water management in polymer electrolyte fuel cells, environmental remediation etc. By the two phase flow we mean the simultaneous flow of two fluids (for instance oil and water) and a porous medium is a substance that contains pores, or spaces between solid material through which liquid or gas can pass. This flow process is modelled by a set of partial differential equations of which the exact solution could be used to describe the flow process or to predict the behaviour of the flow in advance. Unfortunately for several reasons its hard to obtain such solutions, however on the other hand the entire industry increasingly relies on the numerical simulation of the model equations. Apparently the complexities in the flow process make it hard to conduct an accurate numerical simulation of the underlying physical model. In this talk we present the Runge-Kutta Discontinuous Galerkin (RKDG) and Discontinuous Finite Volume Element (DFVE) methods which are applied to a coupled flow-transport problem describing the immiscible displacement of a viscous incompressible fluid in a non-homogeneous porous medium. The model problem consists of a nonlinear pressure-velocity equation assuming Brinkman flow, coupled to a non-linear hyperbolic equation governing the mass balance (saturation equation). The mass conservation properties inherent to finite volume-based methods motivate a DFVE scheme for the approximation of the Brinkman flow in combination with a RKDG method for the spatio-temporal discretization of the saturation equation. Also we present the stability of the scheme for the saturation equation together with a few numerical results.

Joint work with: Raimund Burger, CI^2MA, Universidad de Concepcion, Concepcion, Chile. Sarvesh Kumar, Department of Mathematics, IIST, Trivandrum, India. Ricardo Ruiz-Baier, Mathematical Institute, Oxford University, Andrew Wiles Building, Woodstock Road, Oxford, UK.

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NUMERICAL SOLUTION OF A NOVEL BIOFILM GROWTH MODEL

PATRICIO CUMSILLE, JUAN A. ASENJO, AND CARLOS CONCA

Abstract. In this contribution we simulate biofilm structures ("finger-like", as well as, compact structures) as a result of microbial growth in different environmental conditions. At the same time, the numerical method that we use in order to carry out the computational simulations is new to the biological community, as far as we know. The use of our model sheds light on the biological process of biofilm formation since it simulates some central issues of biofilm growth: the *pattern formation of heterogeneous structures, such as finger-like structures*, in a substrate-transport-limited regime, and the formation of more compact structures, in a growth-limited-regime. The main advantage of our approach is that we consider several of the most relevant aspects of biofilm modeling, particularly, the existence and evolution of a biofilm-liquid interface. At the same time, in order to perform numerical simulations, we have used sophisticated numerical techniques based on mixing the immersed interface method and the level-set method, which are well described in the present work.

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El trabajo es financiado por PIA-CONICYT PFBasal-01.

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