# FINITE ELEMENT POTENTIALS

ANA ALONSO RODRÍGUEZ[1] AND ALBERTO VALLI[2]

ABSTRACT. Determining the necessary and sufficient conditions for assuring that a vector field defined in a bounded domain $\Omega \subset \mathbb{R}^3$ is the gradient of a scalar potential or the curl of a vector potential is one of the most classical problem of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. [2]):

- a vector field is the gradient of a scalar potential if and only if it is curl free and its line integral is vanishing on all the closed curves that give a basis of the first homology group of $\overline{\Omega}$;
- a vector field is the curl of a vector potential if and only if it is divergence free and its flux is vanishing across all the closed surfaces that give a basis of the second homology group of $\overline{\Omega}$, or, equivalently, across (all but one) the connected components of $\partial\Omega$.

Less interesting is the problem of finding a vector field with assigned divergence $f$: this problem is very simply solved by taking the gradient of the solution $\varphi$ of the elliptic problem $\Delta\varphi = f$ in $\Omega$, $\varphi$ vanishing on the boundary $\partial\Omega$; no compatibility conditions on $f$ are needed, no topological properties of $\Omega$ come into play.

However, a less clarified situation takes shape when, given a suitable *finite element* vector field, we want to furnish an explicit and efficient procedure for constructing its *finite element* scalar potential and vector potential. Note also that at this level the construction of a finite element vector field with an assigned divergence comes back on the table: in fact, the gradient of a (standard) finite element approximate solution of $\Delta\varphi = f$ has a distributional divergence which is not a function, and therefore this divergence cannot be equal to an assigned finite element.

The aim of this paper is to furnish a simple and efficient way for constructing finite elements with assigned gradient, or curl, or divergence. Some simple notions of homology theory and graph theory applied to the finite element mesh are basic tools for devising the solution algorithms.

## REFERENCES

[1] A. ALONSO RODRÍGUEZ, AND A. VALLI, *Finite element potentials*, Appl. Numer. Math., (2014), to appear.

[2] J. CANTARELLA, D. DETURCK, AND H. GLUCK, *Vector calculus and the topology of domains in 3-space*, Amer. Math. Monthly, 109(2) (2002) 409–442.

[1] e-mail: alonso@science.unitn.it
Department of Mathematics, University of Trento, Trento, Italy.

[2] e-mail: valli@science.unitn.it
Department of Mathematics, University of Trento, Trento, Italy.