Numerical approximation of Maxwell equations in low-frequency regime

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Contents

1 Introduction **1**

2 Eddy currents and eddy current model **5**

2.1 Maxwell system in harmonic regime . . . 5
2.2 Eddy current model . . . 7

3 Eddy current problem with input current intensities as boundary data **9**

3.1 Model problem . . . 9
3.2 A magnetic field formulation . . . 11
3.3 Introducing a magnetic potential . . . 17
3.4 Discretization . . . 19
3.5 Numerical experiments . . . 23

4 A mixed formulation **27**

4.1 Avoiding cut surfaces . . . 27
4.2 Eddy current problems involving inner source currents . . . 31

5 A potential formulation **37**

5.1 Eddy current problem . . . 38
5.2 The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation . . . 39
5.3 Variational formulation. Existence and uniqueness of solution . 42
5.4 Numerical approximation . . . 46

Bibliography **49**

Chapter 1

Introduction

Numerical simulation plays an important role in electrical engineering to optimize the design and operation conditions of electromagnetic devices such us electrical machines, induction heating systems, transformers, etc. The behavior of these devices is governed by Maxwell equations in low-frequency regime. In such a case, the electric displacement in Maxwell-Ampère's Law can be neglected leading to the so called *eddy current model*. Sometimes, the geometry of the device and the presence of symmetric operational conditions, allow solving a two-dimensional problem (plane or axisymmetric), which leads to important savings in computational effort. However, a full three-dimensional analysis is needed in many real engineering problems.

Among the numerical methods found in the literature to approximate the eddy current problem, the finite element method is the most extended. Its main advantages are its geometric flexibility and the richness in theoretical mathematical tools useful to analyze the approximation of the problem. We notice, however, that Maxwell equations concern the whole space; so it is necessary to define suitable boundary conditions in order to use the finite element method. Because of this, we can also find an important number of papers in the literature which couple the finite element method with the boundary element method (BEM-FEM methods) combining the advantages and disadvantages of each of them.

The finite element method was introduced in electrical engineering calculations in the seventies and, since then, it has been applied to the simulation of a great variety of electromagnetic problems in static and transient state in two and three dimensions. Development and analysis of finite element methods for the eddy current problem began later and this is a subject which was deeply studied during the last decade. The recent book by Alonso and Valli [5] is an excellent reference on this subject, which includes an extensive and updated list of references.

The aim of these notes is to discuss different formulations of the eddy current problem and their finite element approximation, as well as its application to particular electrical engineering problems. The outline is as follows. In Section 2 we recall the time-domain and the time-harmonic Maxwell equations. Then, we derive and discuss very briefly the eddy current model.

In Section 3, we analyze an eddy current problem that arises from the modeling of a metallurgical arc-furnace. The final aim of this section is to propose and analyze a finite element method to solve the low-frequency harmonic Maxwell equations in a bounded domain containing conductors and dielectrics when a source current pass through the conductors, using realistic boundary conditions (the term *realistic* is used in the sense that they can be actually measured in practice). The resulting eddy current problem is formulated in terms of the magnetic field. This formulation is discretized by using Nédélec edge finite elements on a tetrahedral mesh. Error estimates are easily obtained when the curl-free condition is imposed explicitly on the elements in the dielectric domain.

Then, a magnetic scalar potential is introduced to impose this curl-free condition. This amounts to the so called magnetic field/magnetic potential hybrid formulation introduced by Bossavit and Vérité in [15]. The discrete counterpart of this formulation leads to an important saving in computational effort. Problems related with the topology are also considered; more precisely, the possibility of having a non simply connected dielectric domain is taken into account. Most of the material of this section has been taken from References [8, 9].

Although the approach described in Section 3 is one of the least expensive, it needs of cut surfaces and meshes respecting these cuts, which in certain practical situations can be very hard to build. One alternative approach to avoid such cuts has been proposed by Alonso *et al.* in [3]. It is based on imposing the curl free constraint for the magnetic field in the dielectric domain by means of an appropriate Lagrange multiplier (instead of introducing a magnetic potential), which leads to a mixed formulation of the problem. After discretization, the direct implementation of this approach leads to a linear system with a singular matrix. However the problem turns out to be equivalent to a third formulation involving an additional Lagrange multiplier, which leads to a well-posed (although larger) system of linear equations.

We show in Section 4 how this approach can be used to solve the same kind of problems analyzed in Section 3. We prove that, in such a case, both resulting discrete mixed formulations are exactly equivalent to those of Section 3. Then, we show how the same methodology can be used to solve an eddy current problem in which the current source is imposed in the interior of the computational domain by means of a generalized Ohm's law. Let us remark that this is the original setting for which this approach was introduced and analyzed in [3]. Such a source current can be easily handled with this mixed formulation without the need of solving an additional magnetostatic problem or of introducing appropriate vector potentials as in other approaches. Most of the material of this section has been taken from References [3, 7].

Finally, in Section 5, we introduce and analyze a potential formulation of the eddy current problem. This kind of formulations, based on writing the physical vector fields in terms of scalar and vector potentials, have been the first ones that have been solved by means of finite elements methods (see, for instance, [10, 26, 29]). However, the corresponding mathematical analysis is much more scarce.

Different potentials have been used for the eddy current problem: a vector potential $\boldsymbol{A}$ for the magnetic induction field, a scalar potential V for the electric field in the conducting domain, a scalar potential ψ for the magnetic field in the dielectric domain, a vector potential $\boldsymbol{T}$ for the current density, etc. A hierarchy of formulations involving some of these potentials have been discussed by Bíró and Preis [11]. In particular, they conclude that the so-called $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation is the most convenient in terms of computer cost.

We analyze this formulation in Section 5 and prove that it leads to an elliptic problem, which can be discretized by standard (piecewise polynomial and continuous) finite elements. However, we also show that the method converges to the correct solution, only if the connected components of the domain for the vector potential $\boldsymbol{A}$ are convex polyhedra. This domain can be chosen freely as far as it contains all the conductors and the current source support. For the sake of computer cost, it is typically chosen as the union of as small as possible disjoint polyhedra containing each of the connected components of these domains. According to this analysis, these polyhedra has to be chosen necessarily convex for the method to work properly. Most of the material of this section has been taken from References [1, 11].

Chapter 2

Eddy currents and eddy current model

In this section, we will describe briefly the eddy currents and introduce the eddy current model. With this aim, first we recall the full Maxwell equations system in harmonic regime.

2.1 ▪ Maxwell system in harmonic regime

The complete *Maxwell system* of electromagnetism equations reads as follows:

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} - \mathbf{curl}\,\boldsymbol{\mathcal{H}} &= -\boldsymbol{\mathcal{J}} && \text{(Maxwell-Ampère's law)},\\
\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} &= \mathbf{0} && \text{(Faraday's law)},\\
\operatorname{div} \boldsymbol{\mathcal{B}} &= 0 && \text{(Gauss' magnetic law)},\\
\operatorname{div} \boldsymbol{\mathcal{D}} &= \rho && \text{(Gauss' electric law)},
\end{aligned}$$

where we have used the standard notation:

- $\boldsymbol{\mathcal{D}}$ is the electric displacement,
- $\boldsymbol{\mathcal{E}}$ is the electric field,
- $\boldsymbol{\mathcal{B}}$ is the magnetic induction,
- $\boldsymbol{\mathcal{H}}$ is the magnetic field,
- $\boldsymbol{\mathcal{J}}$ is the current density,
- ρ is the electric charge density (which vanishes in any dielectric domain).

We use boldface letters to denote vector fields and variables, as well as vector-valued operators, throughout the paper.

Moreover, we also have to impose constitutive laws for $\boldsymbol{\mathcal{B}}$ in terms of $\boldsymbol{\mathcal{H}}$ and for $\boldsymbol{\mathcal{D}}$ in terms of $\boldsymbol{\mathcal{E}}$, that we will assume linear:

$$\begin{aligned}
\boldsymbol{\mathcal{B}} &= \mu \boldsymbol{\mathcal{H}},\\
\boldsymbol{\mathcal{D}} &= \epsilon \boldsymbol{\mathcal{E}},
\end{aligned}$$

as well as Ohm's law:

$$\boldsymbol{\mathcal{J}} = \sigma \boldsymbol{\mathcal{E}}.$$

The coefficients of these equations are:

- μ the magnetic permeability,
- ϵ the electric permittivity,
- σ the electric conductivity.

In all that follows we will assume these coefficients to be scalars (which correspond to isotropic media), although not necessarily constant. The magnetic permeability μ and the electric permittivity ϵ are always strictly positive, whereas the electric conductivity σ is strictly positive in conductors, but vanishes in dielectrics.

We will restrict the present analysis to bounded domains, so that the equations above must be completed with appropriate boundary conditions that we will discuss below. Furthermore, to close the system, some source term has to be imposed, too. This can be done in different ways: either by means of boundary conditions, which in turn involves either prescribed input currents or voltage drops (cf. Section 3), or by fixing an applied current density on some subdomain (cf. Section 5).

In the last case, Ohm's law has to be substituted by the so-called *generalized Ohm's law*:

$$\boldsymbol{\mathcal{J}} = \sigma \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}}_{\mathrm{S}},$$

where $\boldsymbol{\mathcal{J}}_{\mathrm{S}}$ is the imposed source current density. Note that as a consequence of Maxwell-Ampére's and Gauss' laws and the fact that the charge density and the electric conductivity both vanish on dielectric domains, the imposed source current has to be divergence-free in these domains:

$$\operatorname{div} \boldsymbol{\mathcal{J}}_{\mathrm{S}} = 0 \qquad \text{in dielectrics.}$$

In what follows, we will focus on problems where the physical quantities vary periodically with time. This typically happens when alternating source currents are considered. For instance, if an applied source current is alternating, then it can be written as

$$\boldsymbol{\mathcal{J}}_{\mathrm{S}}(\boldsymbol{x}, t) = \boldsymbol{J}_*(x) \cos(\omega t + \phi).$$

where $\boldsymbol{J}_*(x)$ is the amplitude, $\omega > 0$ the angular frequency and ϕ the phase angle. This can be equivalently written in the following more convenient form:

$$\boldsymbol{\mathcal{J}}_{\mathrm{S}}(\boldsymbol{x}, t) = \mathrm{Re}\left[\boldsymbol{J}_*(x)\, \mathrm{e}^{i(\omega t + \phi)}\right] = \mathrm{Re}\left[\boldsymbol{J}_{\mathrm{S}}(x)\, \mathrm{e}^{i\omega t}\right],$$

where $\boldsymbol{J}_{\mathrm{S}} := \boldsymbol{J}_*(x)\, \mathrm{e}^{i\phi}$ is the complex-valued amplitude, which takes into account both, the real amplitude and the phase angle.

In such a case, all the other electromagnetic fields have a similar steady-state form:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{x}, t) = \mathrm{Re}\left[\boldsymbol{F}(\boldsymbol{x})\, \mathrm{e}^{i\omega t}\right],$$

with $\boldsymbol{F}$ being the complex-valued amplitudes (which are called the *phasors*) of the respective quantities.

Substituting these expressions in the Maxwell system and using the constitutive laws to eliminate $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$, we arrive at the *time-harmonic Maxwell equations* with an applied current source:

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{H} - i\omega\epsilon \boldsymbol{E} &= \boldsymbol{J}, \\ i\omega\mu \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0}, \\ \operatorname{div} \mu \boldsymbol{H} &= 0, \\ \operatorname{div} \epsilon \boldsymbol{E} &= \rho, \\ \boldsymbol{J} &= \sigma \boldsymbol{E} + \boldsymbol{J}_{\mathrm{s}}. \end{aligned}$$

Note that the third equation is not independent but follows from the second one. In turn, the fourth equation is not independent in the dielectric domain, where it follows from the first one and the facts that, in dielectrics, the conductivity vanishes and the imposed source current has to be divergence-free.

2.2 ▪ Eddy current model

As stated by the Faraday's law, a time-variation of the magnetic field generates an electric field. Therefore, a current density $\boldsymbol{\mathcal{J}} = \sigma \boldsymbol{\mathcal{E}}$ is induced in each conductor. This is the so-called *eddy current*. This phenomenon and the related heating of the conductor, was observed by the French physicist L. Foucault in the mid of the nineteenth century. Because of this, eddy currents are also known as *Foucault currents*.

These currents are relevant in applications. On one side, they generate heat in conductors according to the *Joule's law*: $\mathcal{Q} = \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}$. This heat can have a productive use in some applications (like in induction furnaces), while in others should be avoided (like to avoid overheating of electrical devices).

Moreover, eddy currents also generate the so-called *Lorentz forces*, which act on conducting media: $\boldsymbol{f} = \boldsymbol{\mathcal{J}} \times \boldsymbol{\mathcal{B}}$. This forces can be used to drive metal conforming processes (electromagnetic forming). They also drive the motion of melted fluids in magnetohydrodynamics processes.

Another typical application of eddy currents is in non-destructive testing of materials. We do not extend in this respects and refer to [5, Section 9] for a detailed description of all these topics.

In all these applications, it can be checked that the time derivative of the displacement field is negligible with respect to the other terms of the Maxwell-Ampére's law in the conductive domain. To quantify this, we refer again to [5, Section 1.2], where a much more complete discussion on this issue can be found.

The system of equations obtained by disregarding the displacement current term $\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t}$ (or, equivalently, $i\omega\epsilon\boldsymbol{\mathcal{E}}$ in the harmonic regime) is called the eddy current model (or the magnetoquasistatic model) of the Maxwell equations.

In the time-harmonic case, the resulting set of equations is therefore

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J}, \\ i\omega\mu \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0}, \\ \operatorname{div}(\mu \boldsymbol{H}) &= 0, \\ \operatorname{div}(\epsilon \boldsymbol{E}) &= \rho, \\ \boldsymbol{J} &= \sigma \boldsymbol{E}. \end{aligned}$$

Once more, the third equation can be dropped out, since it is a consequence of the second one. Instead, the fourth equation in the dielectric domain (actually $\operatorname{div}(\epsilon \boldsymbol{E}) = 0$, because ρ vanishes in dielectrics) is no longer a consequence of the first one. In fact, the term $i\omega\epsilon\boldsymbol{\mathcal{E}}$ (from which $\operatorname{div}(\epsilon \boldsymbol{E}) = 0$ follows in the harmonic full Maxwell system) has been deleted in the first equation. In its turn, in the conductors domain, the fourth equation is not needed to have a well posed problem. Thus, it can be decoupled and eventually used to compute ρ, once $\boldsymbol{E}$ is obtained from the other equations.

Furthermore, when an applied source current $\boldsymbol{J}_{\mathrm{s}}$ is imposed, the last equation must be substituted as for the full Maxwell system by the generalized Ohm's law:

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_{\mathrm{s}}.$$

In such a case, once more, the condition $\operatorname{div} \boldsymbol{J}_{\mathrm{s}} = 0$ has to be assumed in the dielectric domain.

As we will see in the following sections, the above system has to be completed with appropriate boundary conditions and with some topological constraints (except in case of topologically trivial domains) to lead to a well-posed problem (see [5]).

Chapter 3

Eddy current problem with input current intensities as boundary data

3.1 ▪ Model problem

Let us consider again the low-frequency harmonic Maxwell equations:

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}, \qquad \text{(Ampère's law)}, \tag{3.1}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \boldsymbol{0}, \qquad \text{(Faraday's law)}, \tag{3.2}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0, \tag{3.3}$$

$$\operatorname{div}(\epsilon\boldsymbol{E}) = 0 \qquad \text{(in dielectrics)}, \tag{3.4}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E}, \qquad \text{(Ohm's law)}, \tag{3.5}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{J}$ the current density, ω the angular frequency, μ the magnetic permeability, ϵ the electric permittivity and σ the electric conductivity, which vanishes in dielectrics. Notice that the constraint (3.3) is not an independent equation but a consequence of (3.2).

We are interested in solving these equations in a simply connected bounded three-dimensional domain Ω, which consists of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. For the sake of clarity we refer to the configuration shown in Figure 3.1, which is a sketch of a metallurgical electric furnace. We denote $\Omega_{\mathrm{C}}^1, \dots, \Omega_{\mathrm{C}}^N$ the connected components of Ω_{C}, which correspond to the different electrodes of the furnace in Figure 3.1. We also assume that $\bar{\Omega}_{\mathrm{C}}^1, \dots, \bar{\Omega}_{\mathrm{C}}^N$ are mutually disjoint and that Ω_{D} and $\partial\Omega_{\mathrm{D}}$ are connected.

The domain Ω is assumed to have a Lipschitz-continuous connected boundary $\partial\Omega$, which splits into two parts: $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{D}}$, with $\Gamma_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ and $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ being the outer boundaries of the conducting and dielectric domains, respectively. We denote $\Gamma_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$, the interface between dielectrics and conductors. We also denote by $\boldsymbol{n}$, $\boldsymbol{n}_{\mathrm{C}}$ and $\boldsymbol{n}_{\mathrm{D}}$ the outer unit normal vectors to $\partial\Omega$, $\partial\Omega_{\mathrm{C}}$ and $\partial\Omega_{\mathrm{D}}$, respectively.

We assume that the outer boundary of each electrode, $\partial\Omega_{\mathrm{C}}^n \cap \partial\Omega$ $(n = 1, \dots, N)$, has two connected components, both with non-zero measure: the current entrance, $\Gamma_{\boldsymbol{J}}^n$, where the electrode is connected to a bar supplying alternating electric current, and the electrode tip, $\Gamma_{\boldsymbol{E}}^n$, where an electric arc arises. Finally, we denote $\Gamma_{\boldsymbol{J}} := \Gamma_{\boldsymbol{J}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{J}}^N$ and $\Gamma_{\boldsymbol{E}} := \Gamma_{\boldsymbol{E}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{E}}^N$. We also assume that $\Gamma_{\boldsymbol{J}} \cap \Gamma_{\boldsymbol{E}} = \emptyset$.

Maxwell equations (3.1)–(3.5) concern the whole space but we are only interested in a bounded domain, so it is necessary to define suitable boundary conditions. In

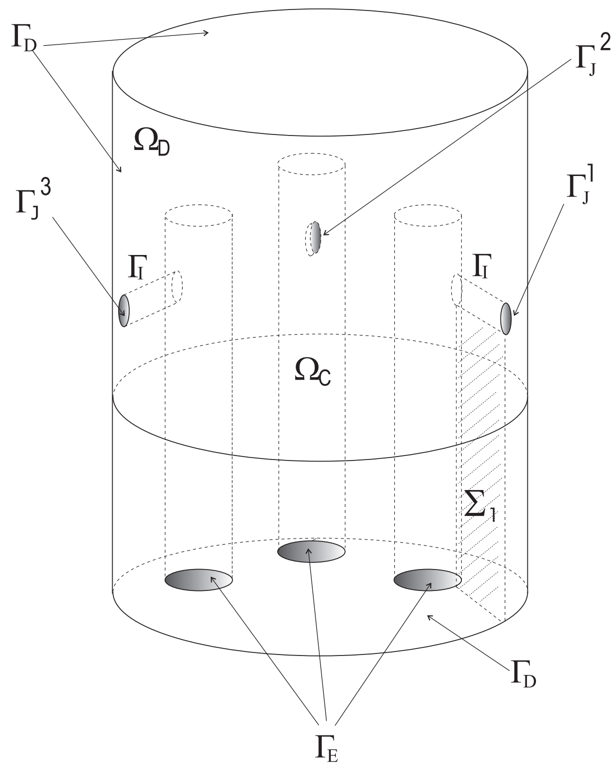

Figure 3.1. *Sketch of the domain.*

fact, this need represents the main difficulty to solve the problem in a bounded domain. From a mathematical point of view, a natural set of boundary conditions for the weak formulation written in terms of the magnetic field consist of giving $\boldsymbol{H} \times \boldsymbol{n}$ on Γ_{D} and $\boldsymbol{E} \times \boldsymbol{n}$ on Γ_{C} (see [8]). While these boundary conditions are easy to handle from mathematical and computational points of view, it is not so easy to obtain the former from the physical data, which usually reduces either to the input current intensities or to the voltage drops on each electrode.

In principle, we focus on input current intensities and, following Bossavit [14], we consider the following boundary conditions:

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{E}}, \tag{3.6}$$

$$\int_{\Gamma_{\boldsymbol{J}}^n} \boldsymbol{J} \cdot \boldsymbol{n} = I_n \quad \text{on } \Gamma_{\boldsymbol{J}}^n, \qquad n = 1, \ldots, N, \tag{3.7}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{J}}, \tag{3.8}$$

$$\mu \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3.9}$$

where the only data are the current intensities I_n on each current entrance.

The boundary condition (3.6) is the natural one to model the current free exit on the electrode tips, whereas (3.7) accounts for the input intensities through each bar. Conditions (3.8) and (3.9) have been proposed by Bossavit in [14] in a more general setting. They will appear as natural boundary conditions of our weak formulation of the problem. The former implies the assumption that the electric current is normal to the surface on the current entrance, whereas the latter means that the magnetic field is tangential to the boundary. Of course, condition (3.9) is not always fulfilled, but it is a good approximation in the case motivating this study.

The case that the boundary data are the voltage drops on each electrode is not too different and will be discussed in what follows by means of several remarks.

3.2 ▪ A magnetic field formulation

We introduce a weak formulation in terms of the magnetic field to solve the eddy current model with the boundary conditions (3.6)–(3.9). First, note that by virtue of (3.1), the boundary condition (3.7) can be equivalently written

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{n} = I_n \quad \text{on } \Gamma_{\boldsymbol{J}}^n, \qquad n = 1, \ldots, N.$$

This will become an essential boundary condition of the formulation. Then, let us consider a smooth test function $\boldsymbol{G}$ such that

$$\mathbf{curl}\, \boldsymbol{G} = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}} \qquad \text{and} \qquad \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = 0, \quad n = 1, \ldots, N. \tag{3.10}$$

From (3.2) we have

$$i\omega \int_\Omega \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \bar{\boldsymbol{G}} = 0. \tag{3.11}$$

Moreover, from (3.2) and (3.9), we have that $\mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{n} = i\omega\mu \boldsymbol{H} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Hence, since $\mathrm{curl}_\Gamma\, \boldsymbol{E} := \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{n}$ is the tangential (scalar) curl operator on the simply-connected surface $\partial\Omega$, we can assert that there exists a sufficiently smooth function V defined in Ω up to a constant, such that $V|_{\partial\Omega}$ is a surface potential of the tangential component of $\boldsymbol{E}$; namely,

$$\boldsymbol{E} \times \boldsymbol{n} = -\,\mathbf{grad}\, V \times \boldsymbol{n} \qquad \text{on } \partial\Omega.$$

On the other hand, (3.6) and (3.8) imply that V must be constant on each connected component of $\Gamma_{\boldsymbol{J}}$ and $\Gamma_{\boldsymbol{E}}$. Furthermore, in our model case, because of the geometry of the furnace, we may assume that the potential is the same on the whole $\Gamma_{\boldsymbol{E}}$ (see Remark 3.2.1 below for a more general case). Hence, V can be chosen to be null on $\Gamma_{\boldsymbol{E}}$. Then, we can transform the second term of (3.11) by using Green's formulas as follows:

$$\int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \bar{\boldsymbol{G}} = \int_\Omega \boldsymbol{E} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} - \int_{\partial\Omega} \boldsymbol{E} \times \boldsymbol{n} \cdot \bar{\boldsymbol{G}} = \int_\Omega \boldsymbol{E} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}, \tag{3.12}$$

because

$$\int_{\partial\Omega} \boldsymbol{E} \times \boldsymbol{n} \cdot \bar{\boldsymbol{G}} = -\int_{\partial\Omega} \mathbf{grad}\, V \times \boldsymbol{n} \cdot \bar{\boldsymbol{G}} = \int_\Omega \mathbf{grad}\, V \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} = \int_{\partial\Omega} V\, \mathbf{curl}\, \bar{\boldsymbol{G}} \cdot \boldsymbol{n} = 0,$$

where, in the last equality, we have used that $V = 0$ on $\Gamma_{\boldsymbol{E}}$, that V is constant on each $\Gamma_{\boldsymbol{J}}^n$ and (3.10).

Now, by substituting (3.12) in (3.11), we obtain

$$i\omega \int_\Omega \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_\Omega \boldsymbol{E} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} = 0.$$

Moreover, because of the first equation in (3.10), the second integral above reduces to the conducting domain Ω_{C}, where (3.1) and (3.5) lead to $\boldsymbol{E} = \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}$. Thus, we finally obtain

$$i\omega \int_\Omega \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\bar{\boldsymbol{G}} = 0 \tag{3.13}$$

for all $\boldsymbol{G}$ sufficiently smooth that satisfy (3.10).

***Remark* 3.2.1.** In general, the (constant) electric potentials on each connected component of $\Gamma_{\boldsymbol{E}}$ cannot be assumed to be equal. In such a case they have to be prescribed as additional boundary conditions on each of these connected components, $\Gamma_{\boldsymbol{E}}^1, \dots, \Gamma_{\boldsymbol{E}}^N$, except on one of them where the potential is taken to be zero. This leads to a new term on the right-hand side of the weak formulation (3.13), namely,

$$i\omega \int_\Omega \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\bar{\boldsymbol{G}} = \sum_{n=1}^{N-1} V_{\boldsymbol{E}}^n \int_{\Gamma_{\boldsymbol{E}}^n} \mathbf{curl}\,\bar{\boldsymbol{G}} \cdot \boldsymbol{n},$$

where $V_{\boldsymbol{E}}^n$ are the corresponding prescribed constant potentials ($V_{\boldsymbol{E}}^N = 0$).

For any surface Γ without boundary, we denote by $\mathrm{H}^{-1/2}(\Gamma)$ the dual space of $\mathrm{H}^{1/2}(\Gamma)$ and by $\langle \cdot, \cdot \rangle_\Gamma$ the corresponding duality pairing. For any open surface $S \subset \Gamma$ we denote by $\mathrm{H}_{00}^{1/2}(S) := \left\{\xi \in \mathrm{L}^2(S) :\ \widehat{\xi} \in \mathrm{H}^{1/2}(\Gamma)\right\}$, where $\widehat{\xi}$ is the extension by zero of ξ to Γ, and by $\mathrm{H}_{00}^{-1/2}(S)$ its dual space. Clearly, $\mathrm{H}^{-1/2}(\Gamma) \hookrightarrow \mathrm{H}_{00}^{-1/2}(S)$.

Let

$$\boldsymbol{\mathcal{X}} := \left\{\boldsymbol{G} \in \mathrm{H}(\mathbf{curl}, \Omega) :\ \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\right\}.$$

For all $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$, we have that $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial\Omega)$ and $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0$ in $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})$. In fact, let $\varphi \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{D}})$, $\widehat{\varphi}$ its extension by zero to $\partial\Omega$ and $\widetilde{\varphi} \in \mathrm{H}^1(\Omega)$ such that $\widetilde{\varphi}|_\Gamma = \widehat{\varphi}$ and $\widetilde{\varphi} = 0$ in Ω_{C}. Then,

$$\langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, \widetilde{\varphi} \rangle_{\partial\Omega} = \int_\Omega \mathbf{curl}\,\boldsymbol{G} \cdot \mathbf{grad}\,\widetilde{\varphi} = 0,$$

because $\mathbf{curl}\,\boldsymbol{G}$ vanishes in Ω_{D} and $\widetilde{\varphi}$ in Ω_{C}.

Therefore, for all $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$, $\langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n}$ is well defined. Indeed, let ζ_n be any smooth function defined on $\partial\Omega$ such that $\zeta_n|_{\Gamma_{\boldsymbol{J}}^m} = \delta_{mn}$ and $\zeta_n = 0$ on $\Gamma_{\boldsymbol{E}}$ (such a function exists because $\Gamma_{\boldsymbol{J}}^1, \dots, \Gamma_{\boldsymbol{J}}^N, \Gamma_{\boldsymbol{E}}$ are all mutually disjoint). Then, $\langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} := \langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, \zeta_n \rangle_{\partial\Omega}$ is well defined and its value does not depend on $\zeta_n|_{\Gamma_{\mathrm{D}}}$.

Given a vector $\boldsymbol{I} := (I_1, \dots, I_N) \in \mathbb{C}^N$ of (complex) input intensities through each bar, let

$$\boldsymbol{\mathcal{V}}(\boldsymbol{I}) := \left\{\boldsymbol{G} \in \boldsymbol{\mathcal{X}} :\ \langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n,\ n = 1, \dots, N\right\}.$$

This is a closed linear manifold of $\boldsymbol{\mathcal{X}}$ with associated subspace

$$\boldsymbol{\mathcal{V}}(\mathbf{0}) = \left\{ \boldsymbol{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} = 0, \ n = 1, \dots, N \right\}.$$

Lemma 3.1. *For all $\boldsymbol{I} \in \mathbb{C}^n$, $\boldsymbol{\mathcal{V}}(\boldsymbol{I})$ is non empty.*

Proof. For $n = 1, \dots, N$, let $u_n \in \mathrm{H}^1(\Omega_{\mathrm{C}}^n)$ be the unique solution of

$$\begin{aligned} -\Delta u_n &= 0 \quad \text{in } \Omega_{\mathrm{C}}^n, \\ \frac{\partial u_n}{\partial \boldsymbol{n}_{\mathrm{C}}} &= \begin{cases} \dfrac{I_n}{\operatorname{meas}(\Gamma_{\boldsymbol{J}}^n)} & \text{on } \Gamma_{\boldsymbol{J}}^n, \\ 0 \quad \text{on } \Gamma_{\mathrm{I}} \cap \partial\Omega_{\mathrm{C}}^n, \end{cases} \\ u_n &= 0 \quad \text{on } \Gamma_{\boldsymbol{E}}^n. \end{aligned}$$

Namely,

$$u_n \in \mathrm{H}^1_{\Gamma_{\boldsymbol{E}}^n}(\Omega_{\mathrm{C}}^n): \qquad \int_{\Omega_{\mathrm{C}}^n} \mathbf{grad}\, u_n \cdot \mathbf{grad}\, v = \int_{\Gamma_{\boldsymbol{J}}^n} \frac{I_n}{\operatorname{meas}(\Gamma_{\boldsymbol{J}}^n)}\, v \qquad \forall v \in \mathrm{H}^1_{\Gamma_{\boldsymbol{E}}^n}(\Omega_{\mathrm{C}}^n).$$

Let $\boldsymbol{F} \in \mathrm{L}^2(\Omega)^3$ be defined by $\boldsymbol{F}|_{\Omega_{\mathrm{C}}^n} := \mathbf{grad}\, u_n$, $n = 1, \dots, N$, and $\boldsymbol{F}|_{\Omega_{\mathrm{D}}} := \mathbf{0}$. Since $\operatorname{div}(\mathbf{grad}\, u_n) = 0$ in Ω_{C}^n and $\mathbf{grad}\, u_n \cdot \boldsymbol{n}_{\mathrm{C}} = 0$ on $\Gamma_{\mathrm{I}} \cap \partial\Omega_{\mathrm{C}}^n$, $\boldsymbol{F} \in \mathrm{H}(\operatorname{div}, \Omega)$ and $\operatorname{div} \boldsymbol{F} = 0$ in Ω. Then, since $\partial\Omega$ is connected, we know from Theorem I.3.4 of [22] that there exists a vector potential $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$ satisfying $\mathbf{curl}\, \boldsymbol{G} = \boldsymbol{F}$ in Ω.

Therefore, $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$ because $\boldsymbol{F} = \mathbf{0}$ in Ω_{D}. Moreover, by using ζ_n as defined above and $\widetilde{\zeta}_n \in \mathrm{H}^1_{\Gamma_{\boldsymbol{E}}}(\Omega)$ an extension of ζ_n vanishing in Ω_{C}^m, $m \neq n$, we have

$$\begin{aligned} \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} &= \langle \boldsymbol{F} \cdot \boldsymbol{n}, \zeta_n \rangle_{\partial\Omega} = \int_{\Omega} \boldsymbol{F} \cdot \mathbf{grad}\, \widetilde{\zeta}_n = \int_{\Omega_{\mathrm{C}}^n} \boldsymbol{F} \cdot \mathbf{grad}\, \widetilde{\zeta}_n \\ &= \int_{\Omega_{\mathrm{C}}^n} \mathbf{grad}\, u_n \cdot \mathbf{grad}\, \widetilde{\zeta}_n = \int_{\Gamma_{\boldsymbol{J}}^n} \frac{I_n}{\operatorname{meas}(\Gamma_{\boldsymbol{J}}^n)} = I_n. \end{aligned}$$

Thus $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$ and we conclude the proof. □

Now, we are in a position to write properly the weak formulation (3.13) of our problem and to prove that it is well posed:

Problem 3.2.1. *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$ such that*

$$i\omega \int_{\Omega} \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} = 0 \qquad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\mathbf{0}).$$

Let $a : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}} \longrightarrow \mathbb{C}$ be the sesquilinear continuous form of this problem:

$$a(\boldsymbol{H}, \boldsymbol{G}) := i\omega \int_{\Omega} \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}.$$

It clearly satisfies the following $\boldsymbol{\mathcal{X}}$-ellipticity property:

$$|a(\boldsymbol{G}, \boldsymbol{G})| \geq \alpha \|\boldsymbol{G}\|^2_{\mathrm{H}(\mathbf{curl}, \Omega)} \qquad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{X}}. \tag{3.14}$$

As a consequence we have the following result.

Theorem 3.2. *For each $\boldsymbol{I} \in \mathbb{C}^N$ Problem 3.2.1 has a unique solution $\boldsymbol{H}$.*

Proof. Since $\boldsymbol{\mathcal{V}}(\boldsymbol{I})$ is not empty, let $\boldsymbol{H}^{\boldsymbol{I}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$ and consider the translation $\widehat{\boldsymbol{H}} = \boldsymbol{H} - \boldsymbol{H}^{\boldsymbol{I}}$. Then problem 3.2.1 is equivalent to finding $\widehat{\boldsymbol{H}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{0})$ such that

$$a(\widehat{\boldsymbol{H}}, \boldsymbol{G}) = -a(\boldsymbol{H}^{\boldsymbol{I}}, \boldsymbol{G}) \qquad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{0}),$$

and this problem has a unique solution because of the ellipticity inequality (3.14) and the Lax-Milgram Lemma. □

Once the magnetic field $\boldsymbol{H}$ is known, the current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$ can be readily computed in the conductors by means of (3.1) and (3.5), respectively. These are the magnitudes actually needed in most applications.

Our next goal is to show that the solution of Problem 3.2.1 satisfies somehow equations (3.1)–(3.9). To this aim, we introduce an equivalent mixed formulation of Problem 3.2.1. Let $b : \boldsymbol{\mathcal{X}} \times \mathbb{C}^N \longrightarrow \mathbb{C}$ be the sesquilinear form defined by

$$b(\boldsymbol{G}, \boldsymbol{W}) := \sum_{n=1}^{N} \bar{W}_n \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma^n_{\boldsymbol{J}}},$$

where $\boldsymbol{W} = (W_1, \ldots, W_N) \in \mathbb{C}^N$. In the following mixed formulation of Problem 3.2.1, the constraints $\langle \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma^n_{\boldsymbol{J}}} = I_n$ are imposed by means of a Lagrange multiplier.

Problem 3.2.2. *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H} \in \boldsymbol{\mathcal{X}}$ and $\boldsymbol{V} \in \mathbb{C}^N$ such that*

$$a(\boldsymbol{H}, \boldsymbol{G}) + b(\bar{\boldsymbol{G}}, \bar{\boldsymbol{V}}) = 0 \qquad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{X}}, \tag{3.15}$$

$$b(\boldsymbol{H}, \boldsymbol{W}) = \boldsymbol{I} \cdot \bar{\boldsymbol{W}} \qquad \forall \boldsymbol{W} \in \mathbb{C}^N. \tag{3.16}$$

Theorem 3.3. *Given $\boldsymbol{I} \in \mathbb{C}^N$, let $\boldsymbol{H} \in \boldsymbol{\mathcal{X}}$ be the solution of Problem 3.2.1. Then there exists a unique $\boldsymbol{V} \in \mathbb{C}^N$ such that $(\boldsymbol{H}, \boldsymbol{V})$ is the only solution of Problem 3.2.2.*

Proof. Since a is $\boldsymbol{\mathcal{X}}$-elliptic, to conclude the well posedness of Problem 3.2.2, we only need to prove the corresponding *inf-sup* condition for b (see, for instance, [22]). With this end, let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ be the canonical basis of $\mathbb{C}^N$. Because of Lemma 3.1, $\exists \boldsymbol{G}_n \in \boldsymbol{\mathcal{V}}(\boldsymbol{e}_n)$, $n = 1, \ldots, N$. Then, given $\boldsymbol{W} \in \mathbb{C}^N$, let $\boldsymbol{G}^{\boldsymbol{W}} := \sum_{n=1}^N W_n \boldsymbol{G}_n$. Hence, $\|\boldsymbol{G}^{\boldsymbol{W}}\|_{\boldsymbol{\mathcal{X}}} \leq |\boldsymbol{W}| \left(\sum_{n=1}^N \|\boldsymbol{G}_n\|^2_{\boldsymbol{\mathcal{X}}} \right)^{1/2}$ and $b(\boldsymbol{G}^{\boldsymbol{W}}, \boldsymbol{W}) = |\boldsymbol{W}|^2$. Consequently,

$$\sup_{\boldsymbol{G} \in \boldsymbol{\mathcal{X}}} \frac{b(\boldsymbol{G}, \boldsymbol{W})}{\|\boldsymbol{G}\|_{\boldsymbol{\mathcal{X}}}} \geq \frac{b(\boldsymbol{G}^{\boldsymbol{W}}, \boldsymbol{W})}{\|\boldsymbol{G}^{\boldsymbol{W}}\|_{\boldsymbol{\mathcal{X}}}} \geq \beta |\boldsymbol{W}|, \qquad \text{with} \quad \beta := \left(\sum_{n=1}^N \|\boldsymbol{G}_n\|^2_{\boldsymbol{\mathcal{X}}} \right)^{-1/2}.$$

Therefore, Problem 3.2.2 has a unique solution. Thus we conclude the proof from the fact that clearly any solution of this problem also solves Problem 3.2.1. □

In the following theorem we show that the solution of Problem 3.2.2 satisfies the Maxwell equations (3.1)–(3.5) (with the exception of those involving $\boldsymbol{E}$ in Ω_{D}, which is not an unknown of the problem) and the boundary conditions (3.6)–(3.9).

Theorem 3.4. *Let* $(\boldsymbol{H}, \boldsymbol{V}) \in \boldsymbol{\mathcal{X}} \times \mathbb{C}^N$ *be the solution of Problem 3.2.2. Let* $\boldsymbol{J} := \mathbf{curl}\,\boldsymbol{H}$ *and* $\boldsymbol{E} := (\frac{1}{\sigma}\boldsymbol{J})|_{\Omega_{\mathrm{C}}}$. *Then the following properties hold true:*

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \quad \text{in } \Omega, \tag{3.17}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega_{\mathrm{C}}, \tag{3.18}$$

$$\boldsymbol{J} = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.19}$$

$$\langle \boldsymbol{J}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n \quad \text{on } \Gamma_{\boldsymbol{J}}^n, \qquad n = 1, \dots, N, \tag{3.20}$$

$$\mu\boldsymbol{H}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{3.21}$$

Moreover, $\exists V_* \in \mathrm{H}^1(\Omega)$ *such that* $V_*|_{\Gamma_{\boldsymbol{J}}^n} = V_n$, $n = 1, \dots, N$, $V_*|_{\Gamma_{\boldsymbol{E}}} = 0$ *and*

$$\boldsymbol{E}\times\boldsymbol{n} = -\,\mathbf{grad}\,V_* \times \boldsymbol{n} \quad \text{in } \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{C}})^3, \tag{3.22}$$

Hence, in particular,

$$\boldsymbol{E}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{E}} \qquad \text{and} \qquad \boldsymbol{E}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{J}}.$$

Proof. Given $v \in \mathcal{D}(\Omega) := \{v \in \mathcal{C}^\infty(\Omega) : \operatorname{supp} v \subset \Omega\}$, $\mathbf{grad}\,v \in \boldsymbol{\mathcal{V}}(\mathbf{0})$. Then, (3.15) yields

$$\int_\Omega \mu\boldsymbol{H}\cdot\mathbf{grad}\,\bar{v} = 0.$$

Consequently, (3.17) holds true.

Now, let $\boldsymbol{G} \in \mathcal{D}(\Omega)^3$ be such that $\operatorname{supp}\boldsymbol{G} \subset \Omega_{\mathrm{C}}$. Then, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\mathbf{0})$ too and (3.15) yields

$$i\omega\int_{\Omega_{\mathrm{C}}} \mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} = 0.$$

Hence, $\boldsymbol{E} := (\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H})|_{\Omega_{\mathrm{C}}}$ satisfies (3.18).

Equation (3.19) follows from the definition of $\boldsymbol{J}$ and the fact that $\boldsymbol{H} \in \boldsymbol{\mathcal{X}}$, whereas equation (3.20) follows from (3.16).

To prove (3.21), notice that $\mu\boldsymbol{H} \in \mathrm{H}(\operatorname{div}, \Omega)$ because of (3.17). Then $\mu\boldsymbol{H}\cdot\boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial\Omega)$ and, given $v \in \mathcal{C}^\infty(\bar{\Omega})$, we have

$$\langle \mu\boldsymbol{H}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega} = \int_\Omega \operatorname{div}(\mu\boldsymbol{H})\bar{v} + \int_\Omega \mu\boldsymbol{H}\cdot\mathbf{grad}\,\bar{v} = 0,$$

because of (3.17), (3.15) and the fact that $\mathbf{grad}\,v \in \boldsymbol{\mathcal{V}}(\mathbf{0})$. Then, (3.21) holds true.

Finally, let $V_* := \sum_{n=1}^N V_n\widetilde{\zeta}_n$ with $\widetilde{\zeta}_n \in \mathrm{H}^1_{\Gamma_{\boldsymbol{E}}}(\Omega)$ as defined in the proof of Lemma 3.1 (i.e., $\widetilde{\zeta}_n|_{\Gamma_{\boldsymbol{J}}^n} = 1$, $\widetilde{\zeta}_n|_{\Gamma_{\boldsymbol{E}}^n} = 0$ and $\widetilde{\zeta}_n|_{\Omega_{\mathrm{C}}^m} = 0$ for all $m \neq n$). Then, $V_* \in \mathrm{H}^1(\Omega)$, $V_*|_{\Gamma_{\boldsymbol{J}}^n} = V_n$, $n = 1, \dots, N$, and $V_*|_{\Gamma_{\boldsymbol{E}}} = 0$. On the other hand, notice that $\boldsymbol{E} \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}})$ because of (3.18) and consequently $\boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}} \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{C}})^3$. Hence, to prove (3.22), it is enough to show that $\langle \boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}} = -\langle \mathbf{grad}\,V_*\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}}$ $\forall \boldsymbol{v} \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{C}})^3$ (namely, $\forall \boldsymbol{v} \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{C}})^3$ that vanishes on Γ_{I}).

Given one such $\boldsymbol{v}$, notice that there exists $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$ vanishing in Ω_{D} and such that $\boldsymbol{G}|_{\partial\Omega_{\mathrm{C}}} = \boldsymbol{v}$. Then $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$ and, from (3.15), (3.18), Green's formula and the fact that $\boldsymbol{E} = \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}$ in Ω_{C}, we obtain

$$\begin{aligned}
0 &= i\omega\int_{\Omega}\mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}}\boldsymbol{E}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} + b(\bar{\boldsymbol{G}},\bar{\boldsymbol{V}}) \\
&= i\omega\int_{\Omega_{\mathrm{C}}}\mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}}\mathbf{curl}\,\boldsymbol{E}\cdot\bar{\boldsymbol{G}} + \left\langle \boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{G}|_{\partial\Omega_{\mathrm{C}}}\right\rangle_{\partial\Omega_{\mathrm{C}}} + b(\bar{\boldsymbol{G}},\bar{\boldsymbol{V}}) \\
&= \langle \boldsymbol{E}\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}} + \langle \mathbf{grad}\,V_*\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}},
\end{aligned}$$

the last equality because

$$b(\bar{\boldsymbol{G}},\bar{\boldsymbol{V}}) = \left\langle \mathbf{curl}\,\bar{\boldsymbol{G}}\cdot\boldsymbol{n}, \bar{V}_*\right\rangle_{\partial\Omega} = \int_{\Omega_{\mathrm{C}}}\mathbf{curl}\,\bar{\boldsymbol{G}}\cdot\mathbf{grad}\,V_* = \langle \mathbf{grad}\,V_*\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}},$$

which in its turn follows from the definitions of b and V_*, the fact that $\boldsymbol{G}$ vanishes in Ω_{D} and Green's formulas. Therefore, we conclude the proof. □

***Remark* 3.2.2.** Equation (3.22) shows that the physical meaning of V_n is the electric potential on $\Gamma_{\boldsymbol{J}}^n$, assuming this potential vanishes on $\Gamma_{\boldsymbol{E}}$ (namely, V_n is the voltage drop on the electrode Ω_{C}^n). If the available data are these voltage drops V_n instead of the corresponding input currents I_n, then the problem to be solved reduces to equation (3.15): Given $\boldsymbol{V}\in\mathbb{C}^N$, find $\boldsymbol{H}\in\boldsymbol{\mathcal{X}}$ such that

(3.23)

$$i\omega\int_{\Omega}\mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} = -\sum_{n=1}^{N}V_n\left\langle \mathbf{curl}\,\bar{\boldsymbol{G}}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_{\boldsymbol{J}}^n} \qquad \forall \boldsymbol{G}\in\boldsymbol{\mathcal{X}}.$$

This is clearly a well posed problem because of the ellipticity of a on the whole $\boldsymbol{\mathcal{X}}$ (cf. (3.14)) and the Lax-Milgram Lemma.

***Remark* 3.2.3.** The theorems above show that Problem 3.2.1 allows us to determine uniquely the electric field $\boldsymbol{E}$ in the conductors. Instead, this field is not determined in the dielectrics. Indeed, from the eddy current equations (3.1)–(3.5) and the boundary conditions (3.6)–(3.9), we obtain the following equations for $\boldsymbol{E}|_{\Omega_{\mathrm{D}}}$:

$$\mathbf{curl}\,\boldsymbol{E} = -i\omega\mu\boldsymbol{H} \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.24}$$

$$\operatorname{div}(\epsilon\boldsymbol{E}) = 0 \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.25}$$

$$\boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{E}|_{\Omega_{\mathrm{C}}}\times\boldsymbol{n} \quad \text{on } \Gamma_{\mathrm{I}}, \tag{3.26}$$

$$\mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{D}}, \tag{3.27}$$

The latter arises from (3.24) and (3.21), whereas the previous to the latter follows from the facts that $\boldsymbol{E}|_{\Omega_{\mathrm{C}}}$ is already known and $\boldsymbol{E}$ is globally in $\mathrm{H}(\mathbf{curl},\Omega)$.

Additional boundary conditions on Γ_{D} seem to be needed to determine a unique solution, even in the simplest case of a topologically trivial Ω_{D} (i.e., when Ω_{D} is simply connected with a connected boundary, which does not correspond to our problem). A natural condition would be to impose

$$\boldsymbol{E}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{3.28}$$

from which (3.27) follows by means of the Stokes Theorem. Existence of solution to (3.24)–(3.28) has been studied in [5, Section 3.5] in a general topological setting, in which case a number of additional constraints related with the topology of $\Omega_{\rm D}$ must be also added.

However, the fact that $\boldsymbol{E}|_{\Omega_{\rm D}}$ is not determined by the present eddy current model is not a drawback in most applications, where the typical goal is to model the behavior of conductors.

3.3 ▪ Introducing a magnetic potential

In this section we show how Problem 3.2.1 can be transformed by introducing a (scalar) magnetic potential, which will allow us to replace the magnetic field in the dielectric domain $\Omega_{\rm D}$.

We assume that for each connected component of the conducting domain, $\Omega_{\rm C}^n$, there exists a connected "cut" surface $\Sigma_n \subset \Omega_{\rm D}$, which is a manifold with boundary such that $\partial\Sigma_n \subset \partial\Omega_{\rm D}$ and $\widetilde{\Omega}_{\rm D} := \Omega_{\rm D} \setminus \bigcup_{n=1}^{n=N} \Sigma_n$ is simply connected (see, for instance, [6]). We also assume that $\bar{\Sigma}_n \cap \bar{\Sigma}_m = \emptyset$ for $n \neq m$ (see Figure 3.1) and that the boundary of each current entrance surface, Γ_J^n, is a simple closed curve, that we denote by γ_n.

We denote the two faces of each Σ_n by Σ_n^- and Σ_n^+, and fix a unit normal $\boldsymbol{n}_n$ on Σ_n as the "outer" normal to $\Omega_{\rm D} \setminus \Sigma_n$ along Σ_n^+. We choose an orientation for each γ_n by taking its initial and end points on Σ_n^- and Σ_n^+, respectively. We denote by $\boldsymbol{t}_n$ the corresponding unit vector tangent to γ_n.

For any function $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\rm D})$, we denote by

$$[\![\widetilde{\Psi}]\!]_{\Sigma_n} := \widetilde{\Psi}|_{\Sigma_n^-} - \widetilde{\Psi}|_{\Sigma_n^+}$$

the jump of $\widetilde{\Psi}$ through Σ_n along $\boldsymbol{n}_n$. The gradient of $\widetilde{\Psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{\rm D})$ can be extended to $\mathrm{L}^2(\Omega_{\rm D})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

Let Θ be the linear subspace of $\mathrm{H}^1(\widetilde{\Omega}_{\rm D})$ defined by

$$\Theta = \left\{ \widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\rm D}) : \; [\![\widetilde{\Psi}]\!]_{\Sigma_n} = \text{constant}, \; n = 1, \dots N \right\}.$$

Then, for $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\rm D})$, we have that $\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \in \mathrm{H}(\mathbf{curl}, \Omega_{\rm D})$ if and only if $\widetilde{\Psi} \in \Theta$, in which case $\mathbf{curl}(\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) = \mathbf{0}$ (see Lemma 3.11 in [6]). Actually, the kernel of the operator $\mathbf{curl}: \; \mathrm{H}(\mathbf{curl}, \Omega_{\rm D}) \longrightarrow \mathrm{L}^2(\Omega_{\rm D})^3$ is given by

$$\mathrm{Ker}\,(\mathbf{curl}) = \widetilde{\mathbf{grad}}\,\Theta = \mathbf{grad}\,\mathrm{H}^1(\Omega_{\rm D}) \oplus \boldsymbol{\mathcal{K}}_{\rm T}, \tag{3.29}$$

where $\boldsymbol{\mathcal{K}}_{\rm T}$ is the space of the so-called *Neumann harmonic fields* in $\Omega_{\rm D}$ defined by

$$\boldsymbol{\mathcal{K}}_{\rm T} := \left\{ \boldsymbol{G} \in \mathrm{L}^2(\Omega_{\rm D})^3 : \; \mathbf{curl}\,\boldsymbol{G} = \mathbf{0}, \; \mathrm{div}(\mu \boldsymbol{G}) = 0 \text{ in } \Omega_{\rm D}, \text{ and } \boldsymbol{G} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega_{\rm D} \right\}.$$

A basis of this space is given by the set of functions $\{\widetilde{\mathbf{grad}}\,\widetilde{\Phi}_n, \; n = 1, \dots, N\}$, where, for each n, $\widetilde{\Phi}_n \in \mathrm{H}^1(\Omega_{\rm D} \backslash \Sigma_n)$ is a solution of

$$\int_{\widetilde{\Omega}_{\rm D}^n} \mu\, \widetilde{\mathbf{grad}}\,\widetilde{\Phi}_n \cdot \mathbf{grad}\,\bar{\Psi} = 0 \quad \forall \Psi \in \mathrm{H}^1(\Omega_{\rm D}), \tag{3.30}$$

$$[\![\widetilde{\Phi}_n]\!]_{\Sigma_n} = 1. \tag{3.31}$$

By using the Lax-Milgram Lemma, it is straightforward to see that $\widetilde{\Phi}_n$ is uniquely defined in $\mathrm{H}^1(\Omega_{\rm D} \backslash \Sigma_n)/\mathbb{C}$. (See, for instance, again [6].)

Therefore, according to (3.29), for all $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$, there exist unique constants c_n, $n = 1, \dots, N$, and a unique scalar field $\Psi \in \mathrm{H}^1(\Omega_{\mathrm{D}})/\mathbb{C}$, such that $\boldsymbol{G}|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}$, with $\widetilde{\Psi} \in \Theta$ given by $\widetilde{\Psi} = \Psi + \sum_{n=1}^{N} c_n \widetilde{\Phi}_n$. In such a case, we say that $\widetilde{\Psi}$ a *multivalued potential* of $\boldsymbol{G}$ in Ω_{D} (although actually the potential $\widetilde{\Psi}$ is multivalued only on the cut surfaces). Furthermore, because of (3.31), the constants c_n are the jumps of $\widetilde{\Psi}$ across the respective cuts Σ_n. Consequently, given $\widetilde{\Psi} \in \Theta$, we have that $\widetilde{\Psi} \in \mathrm{H}^1(\Omega)$ if and only if $[\![\widetilde{\Psi}]\!]_{\Sigma_n} = 0$ for $n = 1, \dots, N$.

We use the following notation: given $\boldsymbol{G}_{\mathrm{C}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$ and $\boldsymbol{G}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$, $(\boldsymbol{G}_{\mathrm{C}}|\boldsymbol{G}_{\mathrm{D}})$ denotes the field $\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3$ defined by $\boldsymbol{G}|_{\Omega_{\mathrm{C}}} := \boldsymbol{G}_{\mathrm{C}}$ and $\boldsymbol{G}|_{\Omega_{\mathrm{D}}} := \boldsymbol{G}_{\mathrm{D}}$. We denote by $\boldsymbol{\mathcal{Y}}$ the linear space given by

$$\boldsymbol{\mathcal{Y}} := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}}) \times (\Theta/\mathbb{C}) : \ (\boldsymbol{G}|\, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}, \Omega) \right\}.$$

Then $(\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}$ if and only if $(\boldsymbol{G}|\, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}) \in \boldsymbol{\mathcal{X}}$.

When a multivalued magnetic potential is used in the dielectric domain, the boundary condition (3.7) can be imposed by fixing its jumps on the cut surfaces. Indeed, if $(\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}$ is smooth enough for the following integrals to make sense, we have

$$(3.32) \quad \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_J^n} = \int_{\Gamma_J^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = \int_{\gamma_n} \boldsymbol{G} \cdot \boldsymbol{t}_n = \int_{\gamma_n} \widetilde{\mathbf{grad}}\, \widetilde{\Psi} \cdot \boldsymbol{t}_n = [\![\widetilde{\Psi}]\!]_{\Sigma_n},$$

where we have used the Stokes Theorem and the fact that $\boldsymbol{G} \times \boldsymbol{n} = \widetilde{\mathbf{grad}}\, \widetilde{\Psi} \times \boldsymbol{n}$ on $\Gamma_{\mathrm{I}} \supset \gamma_n$.

Because of this, given $\boldsymbol{I} \in \mathbb{C}^N$, it is natural to search the solution of our problem in

$$\boldsymbol{\mathcal{W}}(\boldsymbol{I}) := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}} : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = I_n, \ n = 1, \dots, N \right\}.$$

Note that the associated linear subspace is given by

$$\begin{aligned} \boldsymbol{\mathcal{W}}(\mathbf{0}) &:= \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}} : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = 0, \ n = 1, \dots, N \right\} \\ &= \left\{ (\boldsymbol{G}, \Psi) \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}}) \times \left(\mathrm{H}^1(\Omega_{\mathrm{D}})/\mathbb{C} \right) : \ (\boldsymbol{G}|\, \mathbf{grad}\, \Psi) \in \mathrm{H}(\mathbf{curl}, \Omega) \right\}. \end{aligned}$$

Then, Problem 3.2.1 can be written in terms of the magnetic potential as follows:

Problem 3.3.1. *Given* $\boldsymbol{I} \in \mathbb{C}^N$*, find* $(\boldsymbol{H}, \widetilde{\Phi}) \in \boldsymbol{\mathcal{W}}(\boldsymbol{I})$ *such that*

$$i\omega \int_{\Omega_{\mathrm{C}}} \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + i\omega \int_{\Omega_{\mathrm{D}}} \mu\, \widetilde{\mathbf{grad}}\, \widetilde{\Phi} \cdot \widetilde{\mathbf{grad}}\, \bar{\widetilde{\Psi}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} = 0 \qquad \forall (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{W}}(\mathbf{0}).$$

Problem 3.3.1 is the well-known magnetic field/magnetic potential hybrid formulation introduced by Bossavit and Vérité in [15], adapted to our eddy current problem with input current intensities as boundary data. One main advantage with respect to Problem 3.2.1 lies in the fact that a vector field is replaced by a scalar one in the dielectric domain.

***Remark* 3.3.1.** When the available data are the voltage drops V_n instead of the corresponding input currents I_n, the magnetic field/magnetic potential formulation of (3.23) reads as follows: Given $\boldsymbol{V} \in \mathbb{C}^N$, find $(\boldsymbol{H}, \widetilde{\Phi}) \in \boldsymbol{\mathcal{Y}}$ such that

$$(3.33) \quad i\omega \int_{\Omega_{\mathrm{C}}} \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + i\omega \int_{\Omega_{\mathrm{D}}} \mu \, \widetilde{\mathbf{grad}}\, \widetilde{\Phi} \cdot \widetilde{\mathbf{grad}}\, \bar{\widetilde{\Psi}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \bar{\boldsymbol{G}} = -\sum_{n=1}^{N} V_n \left\langle \mathbf{curl}\, \bar{\boldsymbol{G}} \cdot \boldsymbol{n}, 1 \right\rangle_{\Gamma_{\boldsymbol{J}}^n} \qquad \forall (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}.$$

In such a case, the constants jumps $[\![\widetilde{\Phi}_n]\!]_{\Sigma_n}$ (which correspond to the intensities I_n on each current entrance $\Gamma_{\boldsymbol{J}}^n$) are additional unknowns that will be computed while solving the problem above.

3.4 ▪ Discretization

In this section, we introduce a discretization of Problem 3.2.1 and prove its convergence. Then, we show that the obtained discrete problem is equivalent to a more convenient discrete version of Problem 3.3.1.

We employ "edge" finite elements to approximate the magnetic field; more precisely, the lowest-order finite elements of the family introduced by Nédélec in [27].

We assume that Ω, Ω_{C} and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in Ω_{C} or in Ω_{D} (h stands as usual for the corresponding mesh-size).

The magnetic field is approximated in each tetrahedron K by a polynomial vector field in the space

$$\boldsymbol{\mathcal{N}}_K := \left\{ \boldsymbol{G}_h \in \mathcal{P}_1(K)^3 : \; \boldsymbol{G}_h(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}, \; \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3, \; \boldsymbol{x} \in K \right\}.$$

An explicit computation shows that vector fields of this type have constant tangential components along each straight line in the Euclidean space. Moreover, the tangential components along the edges of K can be taken as the degrees of freedom defining an element in $\boldsymbol{\mathcal{N}}_K$.

These elements are H(**curl**)-conforming in the sense that, for all $\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_K$, their tangential traces on each triangular face F of K depend only on the degrees of freedom of $\boldsymbol{G}_h$ on the three edges of F. So, if we set

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{ \boldsymbol{G}_h \in \mathrm{H}(\mathbf{curl}, \Omega) : \; \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{N}}_K \quad \forall K \in \mathcal{T}_h \},$$

the elements in this space are piecewise linear vector fields with tangential traces that are continuous through the faces of the mesh. This is the lowest-order Nédélec finite element space introduced in [27]. See [22] for a detailed mathematical analysis and [13] for useful implementation issues.

We introduce the finite-dimensional space

$$\boldsymbol{\mathcal{X}}_h := \{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \; \mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0} \;\text{ in } \Omega_{\mathrm{D}} \} \subset \boldsymbol{\mathcal{X}},$$

and, for $\boldsymbol{I} \in \mathbb{C}^N$, the linear manifold

$$\boldsymbol{\mathcal{V}}_h(\boldsymbol{I}) := \left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{X}}_h : \; \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n} = I_n, \; n = 1, \ldots, N \right\} \subset \boldsymbol{\mathcal{V}}(\boldsymbol{I}).$$

Then we define the discrete problem as follows.

Problem 3.4.1. *Given* $\boldsymbol{I} \in \mathbb{C}^N$, *find* $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$ *such that*

$$i\omega \int_\Omega \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + \int_{\Omega_\mathrm{C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}). \tag{3.34}$$

In the following theorem we prove existence and uniqueness of solution for this problem under mild smoothness assumptions on the solution of Problem 3.2.1. Moreover, an error estimate is deduced from the standard finite element approximation theory.

Theorem 3.5. *Let us assume that the solution of Problem 3.2.1 satisfies* $\boldsymbol{H}|_{\Omega_\mathrm{C}} \in \mathrm{H}^r(\mathbf{curl}, \Omega_\mathrm{C})$ *and* $\boldsymbol{H}|_{\Omega_\mathrm{D}} \in \mathrm{H}^r(\Omega_\mathrm{D})^3$, *with* $r \in (\frac{1}{2}, 1]$. *Then, Problem 3.4.1 has a unique solution* $\boldsymbol{H}_h$ *and*

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} \le C h^r \left[\|\boldsymbol{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega_\mathrm{C})} + \|\boldsymbol{H}\|_{\mathrm{H}^r(\Omega_\mathrm{D})^3} \right],$$

where C *is a strictly positive constant independent of* h *and* $\boldsymbol{H}$.

Proof. Under the assumptions of this theorem on $\boldsymbol{H}$, its Nédélec interpolant, $\boldsymbol{H}^\mathrm{I}$, is well defined and satisfies

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{H}^\mathrm{I} \cdot \boldsymbol{n} = \int_{\gamma_n} \boldsymbol{H}^\mathrm{I} \cdot \boldsymbol{t}_n = \int_{\gamma_n} \boldsymbol{H} \cdot \boldsymbol{t}_n = \langle \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n,$$

because of a density argument, the Stokes Theorem and the definition of $\boldsymbol{H}^\mathrm{I}$. Moreover, in Ω_D, since $\mathbf{curl}\, \boldsymbol{H} = 0$, we have that $\mathbf{curl}\, \boldsymbol{H}^\mathrm{I} = 0$ too. (See [25] and [6] for the definition and the properties that we have used of the Nédélec interpolant.) Then $\boldsymbol{H}^\mathrm{I} \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$. Consequently, $\boldsymbol{\mathcal{V}}_h(\boldsymbol{I}) \neq \emptyset$ and, given that a is $\boldsymbol{\mathcal{X}}$-elliptic, Problem 3.4.1 has a unique solution $\boldsymbol{H}_h$. Moreover, by using Cea's Lemma, it is easy to check that

$$\begin{aligned}\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} &\le C \inf_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})} \|\boldsymbol{H} - \boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} \le C \|\boldsymbol{H} - \boldsymbol{H}^\mathrm{I}\|_{\mathrm{H}(\mathbf{curl},\Omega)} \\ &\le C h^r \left[\|\boldsymbol{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega_\mathrm{C})} + \|\boldsymbol{H}\|_{\mathrm{H}^r(\Omega_\mathrm{D})^3} \right],\end{aligned}$$

the latter because of the standard approximation results for the Nédélec interpolant (see [25, 6]). Thus, we conclude the proof. □

***Remark* 3.4.1.** The smoothness assumption on the solution $\boldsymbol{H}$ of Problem 3.2.1 is not actually necessary to prove that Problem 3.4.1 has a unique solution. However, such an assumption is needed for the error estimate.

In what follows we show how to impose efficiently the curl-free condition in the definition of $\boldsymbol{\mathcal{X}}_h$. We do it by introducing a discrete multivalued magnetic potential in the dielectric domain.

We assume that the cut surfaces Σ_n are polyhedral and that the meshes are compatible with them, in the sense that each Σ_n is a union of faces of tetrahedra $K \in \mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Omega_\mathrm{D}} := \{K \in \mathcal{T}_h :\ K \subset \Omega_\mathrm{D}\}$ can also be seen as a mesh of $\widetilde{\Omega}_\mathrm{D}$

in which vertices, edges and faces on Σ_n^+ and the corresponding ones on Σ_n^- are different.

First, we introduce an approximation of the space Θ. Let

$$\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) := \left\{ \widetilde{\Psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ \widetilde{\Psi}_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}} \right\}$$

and consider the finite-dimensional subspace of Θ given by

$$\Theta_h := \left\{ \widetilde{\Psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = \text{constant}, \ n = 1, \dots N \right\}.$$

The following lemma shows that the curl-free vector fields in $\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})$ admit a multivalued potential in Θ_h.

Lemma 3.6. *Let $\boldsymbol{G}_h \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$. Then $\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})$ with $\mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0}$ in Ω_{D} if and only if there exists $\widetilde{\Psi}_h \in \Theta_h$ such that $\boldsymbol{G}_h = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h$ in Ω_{D}. Such $\widetilde{\Psi}_h$ is unique up to an additive constant.*

Proof. According to (3.29), $\mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0}$ in Ω_{D} if and only if there exists $\widetilde{\Psi}_h \in \Theta$ such that $\boldsymbol{G}_h = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h$ in $\widetilde{\Omega}_{\mathrm{D}}$. Moreover, since $\widetilde{\Omega}_{\mathrm{D}}$ is connected, then $\widetilde{\Psi}_h$ is unique up to an additive constant. Now, let $K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}}$ be a tetrahedron of the mesh. A direct calculation shows that $\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_K$ with $\mathbf{curl}\, \boldsymbol{G}_h|_K = \mathbf{0}$ if and only if $\boldsymbol{G}_h|_K \in \mathcal{P}_0(K)^3$, or, equivalently, if and only if $\widetilde{\Psi}_h|_K \in \mathcal{P}_1(K)^3$. Thus the lemma follows from the definition of Θ_h. □

Let us introduce the following finite-dimensional subsets of $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{W}}(\boldsymbol{I})$, $\boldsymbol{I} \in \mathbb{C}^N$, respectively:

$$\boldsymbol{\mathcal{Y}}_h := \left\{ (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}}) \times (\Theta_h/\mathbb{C}) : \ (\boldsymbol{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl}, \Omega) \right\},$$

$$\boldsymbol{\mathcal{W}}_h(\boldsymbol{I}) := \left\{ (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h : \ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = I_n, \ n = 1, \dots, N \right\}.$$

Next, we define a new discrete problem as follows.

Problem 3.4.2. *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $(\boldsymbol{H}_h, \widetilde{\Phi}_h) \in \boldsymbol{\mathcal{W}}_h(\boldsymbol{I})$ such that*

$$i\omega \int_{\Omega_{\mathrm{C}}} \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + i\omega \int_{\Omega_{\mathrm{D}}} \mu \, \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \cdot \widetilde{\mathbf{grad}}\, \bar{\widetilde{\Psi}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h = 0 \qquad \forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{W}}_h(\mathbf{0}).$$

The following theorem shows that Problems 3.4.1 and 3.4.2 are equivalent.

Theorem 3.7. *Given $\boldsymbol{I} \in \mathbb{C}^N$, $\boldsymbol{H}_h$ is a solution of Problem 3.4.1 if and only if there exists $\widetilde{\Phi}_h \in \Theta_h$ such that $\boldsymbol{H}_h|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h$ and $(\boldsymbol{H}_h|_{\Omega_{\mathrm{C}}}, \widetilde{\Phi}_h)$ is a solution of Problem 3.4.2*

Proof. Let $\boldsymbol{H}_h$ be a solution of Problem 3.4.1. According to Lemma 3.6, there exists $\widetilde{\Phi}_h \in \Theta_h$ such that $\boldsymbol{H}_h = \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h$ in Ω_{D}. Moreover, since $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$, because of (3.32) and the definition of $\boldsymbol{\mathcal{W}}_h(\boldsymbol{I})$, we have that $(\boldsymbol{H}_h|_{\Omega_{\mathrm{C}}}, \widetilde{\Phi}_h) \in \boldsymbol{\mathcal{W}}_h(\boldsymbol{I})$. Analogously, $\forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{W}}_h(\mathbf{0})$, $(\boldsymbol{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{V}}_h(\mathbf{0})$. Then, because of (3.34), $(\boldsymbol{H}_h|_{\Omega_{\mathrm{C}}}, \widetilde{\Phi}_h)$ is a solution of Problem 3.4.2.

Now, since the bilinear form of Problem 3.4.2 is $\boldsymbol{\mathcal{Y}}_h$-elliptic, $(\boldsymbol{H}_h|_{\Omega_C}, \widetilde{\Phi}_h)$ is the unique solution of Problem 3.4.2. Then both problems are equivalent. □

Problem 3.4.2 leads to an important saving in computational effort, since it involves a scalar instead of a vector field in the dielectric domain. However, its implementation requires to impose the following constraints:

- $(\boldsymbol{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl}, \Omega)$, which arises in the definition of $\boldsymbol{\mathcal{Y}}_h$;
- $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = \text{constant}$, $n = 1, \dots, N$, which arise in the definition of Θ_h.

To impose the first one we use that, for $(\boldsymbol{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl}, \Omega)$, we have that $\boldsymbol{G}_h \times \boldsymbol{n} = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h \times \boldsymbol{n}$ on Γ_{I} and, hence,

$$\int_\ell \boldsymbol{G}_h \cdot \boldsymbol{t}_\ell = \int_\ell \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h \cdot \boldsymbol{t}_\ell = \widetilde{\Psi}_h(P_\ell^+) - \widetilde{\Psi}_h(P_\ell^-) \qquad \forall \text{ edge } \ell \text{ of } \mathcal{T}_h : \ \ell \subset \Gamma_{\mathrm{I}},$$

where P_ℓ^- and P_ℓ^+ are the end points of ℓ and $\boldsymbol{t}_\ell$ is the unit tangent vector pointing from P_ℓ^- to P_ℓ^+. Then the degrees of freedom of $\boldsymbol{G}_h$ associated with the edges $\ell \subset \Gamma_{\mathrm{I}}$ can be easily eliminated by static condensation in terms of those of $\widetilde{\Psi}_h$ corresponding to the vertices of the mesh on Γ_{I}.

Regarding the second constraint, for each cut surface Σ_n we in principle distinguish the degrees of freedom of $\widetilde{\Psi}_h$ on Σ_n^+ from those on Σ_n^-. Then, the latter are eliminated by using

$$\widetilde{\Psi}_h|_{\Sigma_n^-} = \widetilde{\Psi}_h|_{\Sigma_n^+} + [\![\widetilde{\Psi}_h]\!]_{\Sigma_n},$$

with $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = I_n$ for the trial functions and $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0$ for the test functions.

We refer to [8] for further implementation details.

***Remark* 3.4.2.** The electric potentials V_n do not appear in this formulation. An alternative that allows computing them is the following: To discretize Problem 3.2.2, substituting $\boldsymbol{H}_h|_{\Omega_D}$ by $\widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h$, with $\widetilde{\Phi}_h \in \Theta_h$, but without imposing the condition $[\![\widetilde{\Phi}_h]\!]_{\Sigma_n} = I_n$ on the multivalued magnetic potential. In this case $[\![\widetilde{\Phi}_h]\!]_{\Sigma_n}$ are additional unknowns that must be also computed and the test functions $\widetilde{\Psi}_h$ are also taken in Θ_h without imposing $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0$.

***Remark* 3.4.3.** In case the available data are the voltage drops V_n, instead of the corresponding input currents I_n, the discretization of (3.33) reads as follows: Given $\boldsymbol{V} \in \mathbb{C}^N$, find $(\boldsymbol{H}_h, \widetilde{\Phi}_h) \in \boldsymbol{\mathcal{Y}}_h$ such that

$$i\omega \int_{\Omega_C} \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + i\omega \int_{\Omega_D} \mu\, \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \cdot \widetilde{\mathbf{grad}}\, \bar{\widetilde{\Psi}}_h + \int_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h$$
$$= -\sum_{n=1}^N V_n \langle \mathbf{curl}\, \bar{\boldsymbol{G}}_h \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_J^n} \qquad \forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h.$$

In this case, the jumps $[\![\widetilde{\Phi}_h]\!]_{\Sigma_n} = I_n$ are unknowns of the problem that have to be computed and the test functions $\widetilde{\Psi}_h$ have to be taken in Θ_h without imposing $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0$.

3.5 ▪ Numerical experiments

In what follows we report some numerical results obtained with a code which implements in MATLAB the method described above.

As a first numerical experiment, we have solved a particular problem with a known analytical solution to validate the computer code and to test the performance and convergence properties of the method. We have considered a domain Ω containing a conductor Ω_{C} and a dielectric Ω_{D} as shown in Figure 3.2.

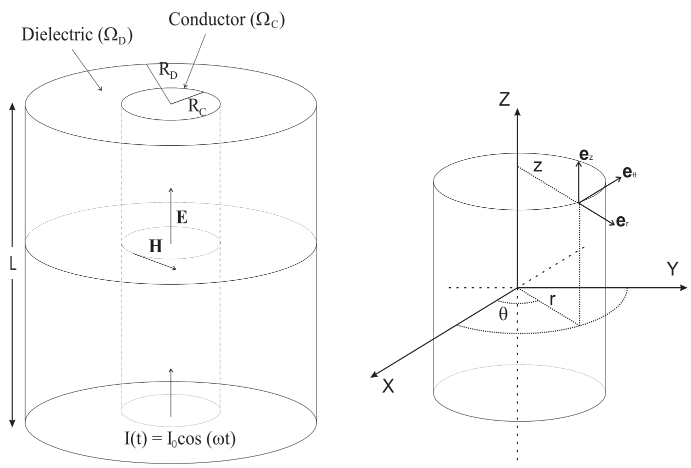

Figure 3.2. *Sketch of the domain. Coordinate system.*

We assume that $\bar{\Omega}_{\mathrm{C}}$ and $\bar{\Omega} = \bar{\Omega}_{\mathrm{C}} \cup \bar{\Omega}_{\mathrm{D}}$ are coaxial cylinders of radius R_{C} and R_{D}, respectively, and height L. To obtain the data for a test problem in this domain with known analytical solution, we consider that Ω_{C} and Ω are bounded sections of respective infinite cylinders. The electric conductivity σ is taken constant in Ω_{C} and the magnetic permeability μ constant in the whole Ω. We consider that an alternating current $\boldsymbol{J}$ goes through the conductor Ω_{C} in the direction of its axis; this current is assumed to be axially symmetric with an intensity $I(t) = I_0 \cos(\omega t)$.

We analyze this problem using a cylindrical coordinate system (r, θ, z) with the z-axis coinciding with the common axis of both cylinders (see Figure 3.2). We denote $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ the unit vectors in the corresponding coordinate directions.

Because of the assumed conditions on $\boldsymbol{J}$, only the z-component of the electric field $\boldsymbol{E} = \frac{1}{\sigma}\boldsymbol{J}$ does not vanish in the conductor. Moreover, it depends on the radial coordinate r, but is independent of the other two coordinates z and θ. Consequently, only the θ-component of the magnetic field $\boldsymbol{H} = \frac{i}{\omega\mu}\,\mathbf{curl}\,\boldsymbol{E}$ does not vanish and it also depends only on the coordinate r. In fact, taking into account that for a vector field $\boldsymbol{F} = F_r(r,\theta,z)\boldsymbol{e}_r + F_\theta(r,\theta,z)\boldsymbol{e}_\theta + F_z(r,\theta,z)\boldsymbol{e}_z$, the **curl** operator in cylindrical coordinates reads

$$\mathbf{curl}\,\boldsymbol{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)\boldsymbol{e}_z,$$

we have that $\boldsymbol{H}(r,\theta,z) = H_\theta(r)\boldsymbol{e}_\theta$, with $H_\theta = -\frac{i}{\omega\mu}\frac{dE_z}{dr}$. Moreover, eliminating $\boldsymbol{E}$ in (3.2) from (3.5) and (3.1), we have that H_θ satisfies the ordinary differential equation

$$i\omega\mu H_\theta(r) - \frac{d}{dr}\left(\frac{1}{\sigma r}\frac{d}{dr}\left(rH_\theta(r)\right)\right) = 0, \qquad 0 < r < R_{\mathrm{C}},$$

and the boundary conditions

$$|H_\theta(0)| < \infty \qquad \text{and} \qquad H_\theta(R_{\mathrm{C}}) = \frac{I_0}{2\pi R_{\mathrm{C}}},$$

where the latter follows from the fact that $I_0 = \int_{\Gamma_J} \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{e}_z = \int_{\partial\Gamma_J} H_\theta$, where Γ_J is the current entrance.

To solve this problem, we perform the change of variable $x = \gamma r$, where $\gamma := \sqrt{i\omega\mu\sigma} \in \mathbb{C}$. Thus, we obtain the equation

$$x^2\frac{d^2}{dx^2}\widetilde{H}_\theta(x) + x\frac{d}{dx}\widetilde{H}_\theta(x) - (x^2+1)\widetilde{H}_\theta(x) = 0, \qquad 0 < x < \gamma R_{\mathrm{C}},$$

where $\widetilde{H}_\theta(x) = H_\theta(x/\gamma)$. This is a Bessel equation, whose solution is given by $\widetilde{H}_\theta(x) = \alpha \mathrm{I}_1(x)$, with I_1 being the modified Bessel function of the first kind and α a constant to be obtained from the boundary condition at $x = \gamma R_{\mathrm{C}}$. Thus, the magnetic field in the conductor is given by

$$\boldsymbol{H}(r,\theta,z) = \frac{I_0}{2\pi R_{\mathrm{C}}}\frac{\mathrm{I}_1(\gamma r)}{\mathrm{I}_1(\gamma R_{\mathrm{C}})}\boldsymbol{e}_\theta, \qquad r \in (0, R_{\mathrm{C}}), \quad \theta \in [0, 2\pi], \quad z \in \mathbb{R}.$$

On the other hand, the magnetic field in the dielectric domain is also of the form $\boldsymbol{H}(r,\theta,z) = H_\theta(r)\boldsymbol{e}_\theta$ (see, for instance, [28]) with H_θ satisfying now

$$\frac{1}{r}\frac{d}{dr}\left(rH_\theta(r)\right) = 0, \qquad r > R_{\mathrm{C}},$$

and the boundary condition $H_\theta(R_{\mathrm{C}}) = \frac{I_0}{2\pi R_{\mathrm{C}}}$, which follows from the continuity of H_θ. Then,

$$H_\theta(r) = \frac{I_0}{2\pi r}, \qquad r \geq R_{\mathrm{C}}.$$

Let us remark that $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy automatically the boundary conditions (3.6)–(3.9).

Moreover, from this expression, it is also possible to know the multivalued magnetic potential $\widetilde{\Phi}$ which corresponds to the magnetic field in the dielectric domain. Indeed, taking into account the expression of the gradient operator in cylindrical coordinates of a function $f(r,\theta,z)$,

$$\mathbf{grad}\, f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z,$$

we obtain

$$\widetilde{\Phi}(r,\theta,z) = \frac{I_0}{2\pi}\theta, \qquad r > R_{\mathrm{C}}, \quad \theta \in [0, 2\pi], \quad z \in \mathbb{R}.$$

Notice that the scalar potential depends only on the variable θ and experiments a jump of magnitude I_0 across the cut surface Σ placed at $\theta = 0$.

For the numerical test, we have used the following geometrical and physical data: $R_{\mathrm{C}} = 1\,\mathrm{m}$; $R_{\mathrm{D}} = 2\,\mathrm{m}$; $L = 1\,\mathrm{m}$; $\sigma = 151565.8\,(\Omega\mathrm{m})^{-1}$; $\mu = \mu_0 = 4\pi 10^{-7}\,\mathrm{Hm}^{-1}$ (magnetic permeability of free space); $I_0 = 62000\,\mathrm{A}$; $\omega = 50\,\mathrm{Hz}$.

To determine the order of convergence, the numerical method has been used on several successively refined meshes and we have compared the obtained numerical solutions with the analytical one. Figure 3.3 shows a log-log plot of the errors measured in $\mathrm{H}(\mathbf{curl}, \Omega)$-norm versus the number of degrees of freedom (d.o.f.). The slope of the line shows a clear linear dependence on the mesh-size. These $\mathcal{O}(h)$ errors agree with the theoretical results, since the solution is smooth, and, hence, the hypotheses of Theorem 3.5 are fulfilled for $r = 1$.

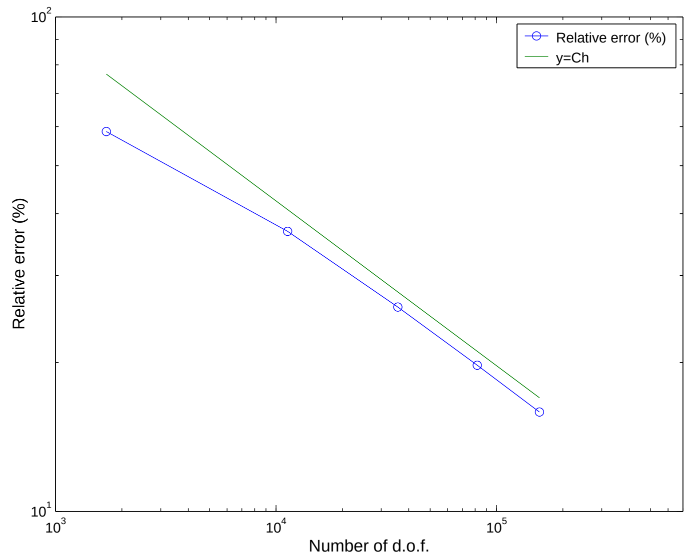

Figure 3.3. *Error versus number of d.o.f. (log-log scale).*

As a second numerical test, we have applied the method to an electric furnace with three electrodes, similar to that sketched in Figure 3.1, with the following dimensions: furnace diameter: 8.88 m; furnace height: 2 m; electrodes diameter: 1 m; electrodes height: 1.25 m; distance from the center of each electrode to the furnace wall: 3 m].

We have considered ELSA compound electrodes (see [17]) which consist of a graphite cylindrical core (diameter: 0.4 m) and an outer part of Söderberg paste. The electric current enter the electrodes through copper bars of rectangular section ($0.07\,\mathrm{m} \times 0.25\,\mathrm{m}$).

The physical parameters we have used are the following: $\sigma = 10^6\,(\Omega\mathrm{m})^{-1}$ for graphite; $\sigma = 10^4\,(\Omega\mathrm{m})^{-1}$ for Söderberg paste; $\sigma = 0.5 \times 10^7\,(\Omega\mathrm{m})^{-1}$ for copper; $\mu = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$; $\omega = 2\pi \times 50\,\mathrm{Hz}$; one-phase intensities $I_n = 7 \times 10^4\,\mathrm{A}$ for each electrode.

Figures 3.4 and 3.5 show the intensity of the computed current density $|\boldsymbol{J}_h| := |\mathbf{curl}\,\boldsymbol{H}_h|$ in the conductor domain Ω_{C} and the computed magnetic potential $\widetilde{\Phi}_h$ in the dielectric domain Ω_{D}, respectively. Figures 3.6 and 3.7 show $|\boldsymbol{J}_h|$ in horizontal and vertical sections of one of the electrodes, respectively.

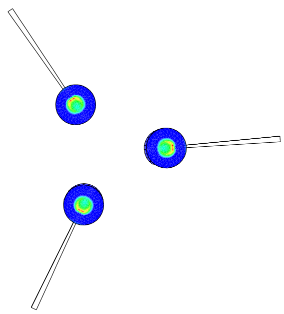

Figure 3.4. *Intensity of the current density,* $|\boldsymbol{J}_h|$*, in the conductors.*

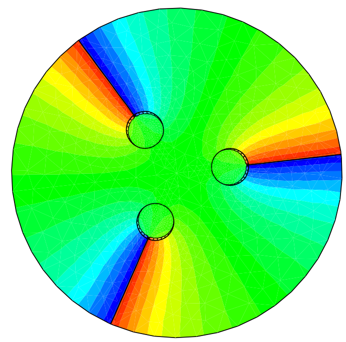

Figure 3.5. *Magnetic potential* $\widetilde{\Phi}_h$ *in the dielectric.*

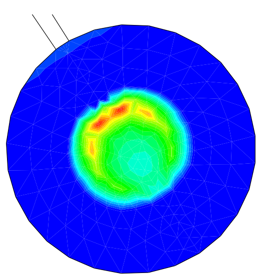

Figure 3.6. $|\boldsymbol{J}_h|$*: Horizontal section of one of the electrodes.*

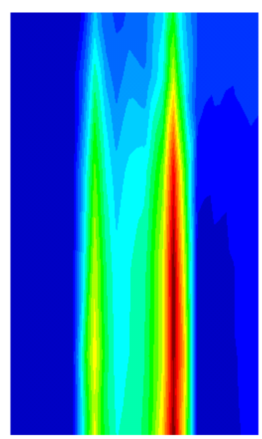

Figure 3.7. $|\boldsymbol{J}_h|$*: Vertical section of one of the electrodes.*

Chapter 4

A mixed formulation

4.1 ▪ Avoiding cut surfaces

Solving Problem 3.4.2 is a good alternative to obtain an approximate solution of the eddy current problem. The only drawback is that it needs finite element meshes involving cuts, which sometimes can be difficult to build. In what follows we will introduce a mixed discrete formulation of the same eddy current problem analyzed in the previous section. The main advantage of this mixed formulation is that it does not need any cut. We will show that it is completely equivalent to Problem 3.4.1 and, hence, also to Problem 3.4.2.

This mixed formulation has been previously analyzed in [3] for other boundary conditions and source terms, without establishing any relation with a magnetic field/magnetic scalar potential discretization as that of Problem 3.4.1. The formulation is based on using a Lagrange multiplier to impose the curl-free constraint in the dielectric instead of introducing the scalar potential in Ω_{D}, so that cuts are not required in the mesh.

For each $\boldsymbol{I} \in \mathbb{C}^N$ we introduce the linear manifold of $\boldsymbol{\mathcal{N}}_h(\Omega)$

$$\boldsymbol{\mathcal{U}}_h(\boldsymbol{I}) := \left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n} = I_n,\ n = 1, \dots, N \right\}$$

with associated subspace $\boldsymbol{\mathcal{U}}_h(\mathbf{0})$. The discrete mixed problem reads as follows.

Problem 4.1.1. *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H}_h \in \boldsymbol{\mathcal{U}}_h(\boldsymbol{I})$ and $\boldsymbol{A}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ such that*

(4.1)

$$i\omega \int_\Omega \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{D}}} \boldsymbol{A}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}),$$

(4.2)

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{H}_h \cdot \bar{\boldsymbol{Z}}_h = 0 \qquad \forall \boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right).$$

For each $\boldsymbol{I} \in \mathbb{C}^N$ it is easy to find $\boldsymbol{H}_h^{\boldsymbol{I}} \in \boldsymbol{\mathcal{U}}_h(\boldsymbol{I})$. In fact, it is enough to take the degrees of freedom corresponding to the constant values of $\boldsymbol{H}_h^{\boldsymbol{I}} \cdot \boldsymbol{t}_n$ on each edge

$\ell \subset \gamma_n$ so that $\int_{\gamma_n} \boldsymbol{H}_h^{\boldsymbol{I}} \cdot \boldsymbol{t}_n = I_n$, $n = 1, \dots, N$, and the rest of them arbitrarily. Therefore, a translation argument similar to that used in the proof of Theorem 3.2 allows us to show that Problem 4.1.1 is equivalent to a standard discrete mixed problem. Thus, we only need to check the ellipticity in the kernel and the inf-sup condition to conclude that it is well posed. The former follows from the facts that the kernel is given by

$$\left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}) : \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \bar{\boldsymbol{Z}}_h = 0 \quad \forall \boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right) \right\} \\ = \left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}) : \mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \right\} = \boldsymbol{\mathcal{V}}_h(\mathbf{0}) \subset \boldsymbol{\mathcal{X}}$$

and the bilinear form a is elliptic on the whole $\boldsymbol{\mathcal{X}}$ (cf. (3.14)). Thus, there only remains to check the following inf-sup condition.

Proposition 4.1. *There exist a constant $\beta_h > 0$ such that*

$$\sup_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}):\ \boldsymbol{G}_h \neq \mathbf{0}} \frac{\left| \int_{\Omega_{\mathrm{D}}} \boldsymbol{Z}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h \right|}{\|\boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)}} \geq \beta_h \|\boldsymbol{Z}_h\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \qquad \forall \boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right).$$

Proof. Since $\dim \boldsymbol{\mathcal{U}}_h(\mathbf{0}) < \infty$, it is enough to prove that for all non-vanishing $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ there exists $\boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0})$ such that $\int_{\Omega_{\mathrm{D}}} \boldsymbol{Z}_h \cdot \mathbf{curl}\, \boldsymbol{G}_h \neq 0$. Let $\boldsymbol{U}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ be such that $\boldsymbol{Z}_h = \mathbf{curl}\, \boldsymbol{U}_h$ in Ω_{D}. In general $\boldsymbol{U}_h \notin \boldsymbol{\mathcal{U}}_h(\mathbf{0})$, but if we are able to find $\boldsymbol{Y}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ satisfying $\mathbf{curl}\, \boldsymbol{Y}_h = \mathbf{0}$ in Ω_{D} and $\int_{\gamma_n} \boldsymbol{Y}_h \cdot \boldsymbol{t}_n = -\int_{\gamma_n} \boldsymbol{U}_h \cdot \boldsymbol{t}_n$, $n = 1, \dots, N$, it is straightforward to check that $\boldsymbol{G}_h := \boldsymbol{U}_h + \boldsymbol{Y}_h$ satisfies the above requirements. Such a $\boldsymbol{Y}_h$ can be defined as follows: $\boldsymbol{Y}_h := -\sum_{m=1}^{N} \left(\int_{\gamma_m} \boldsymbol{U}_h \cdot \boldsymbol{t}_m \right) \boldsymbol{Y}_h^m$, where $\boldsymbol{Y}_h^m \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is such that $\boldsymbol{Y}_h^m|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m$, with $\widetilde{\Phi}_h^m \in \Theta_h$ satisfying $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = \delta_{nm}$, $n, m = 1, \dots, N$. □

Now we are in a position to conclude the well posedness of Problem 4.1.1.

Proposition 4.2. *For each $\boldsymbol{I} \in \mathbb{C}^N$ Problem 4.1.1 has a unique solution $(\boldsymbol{H}_h, \boldsymbol{A}_h)$.*

By repeating the arguments from the proof of Theorem 5.2 from [3], it is not difficult to prove that the inf-sup condition from Proposition 4.1 holds uniformly in h, which would allow us to prove an error estimate for the solution to Problem 4.1.1. However, this is not actually necessary in our case, since such error estimate is a direct consequence of Theorem 3.5 and the following equivalence result.

Proposition 4.3. *Given $\boldsymbol{I} \in \mathbb{C}^N$, a discrete field $\boldsymbol{H}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is solution of Problem 3.4.1 (and, equivalently, of Problem 3.4.2) if and only if there exists $\boldsymbol{A}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ such that $(\boldsymbol{H}_h, \boldsymbol{A}_h)$ solves Problem 4.1.1.*

Proof. Since each problem has a unique solution, it is enough to prove that if $(\boldsymbol{H}_h, \boldsymbol{A}_h)$ solves Problem 4.1.1, then $\boldsymbol{H}_h$ solves Problem 3.4.1. For this purpose, let us take $\boldsymbol{Z}_h = \mathbf{curl}\, \boldsymbol{H}_h$ as test function in (4.2). We deduce that $\mathbf{curl}\, \boldsymbol{H}_h = \mathbf{0}$ in Ω_{D} and, consequently, $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$. Finally, we complete the proof by testing (4.1) with $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}) \subset \boldsymbol{\mathcal{U}}_h(\mathbf{0})$. □

Although Problem 4.1.1 has a unique solution, its direct implementation leads to a singular linear system. Indeed, when the functions $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ are written as $\boldsymbol{Z}_h = \mathbf{curl}\,\boldsymbol{U}_h$, with $\boldsymbol{U}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})$, such $\boldsymbol{U}_h$ is clearly not unique and this leads to a singular matrix. However, as stated in [3, Remark 5.1], since the kernel of this matrix is well separated from the rest of the spectrum, a conjugate gradient type method will work for its numerical solution.

An alternative leading to a system with a non-singular matrix, was also proposed in [3]. Let $\mathcal{Q}_h(\Omega_{\rm D})$ be the space of piecewise constant functions in $\mathcal{T}_h^{\Omega_{\rm D}}$:

$$\mathcal{Q}_h(\Omega_{\rm D}) := \left\{ q_h \in \mathrm{L}^2(\Omega_{\rm D}) : \ q_h|_K \in \mathcal{P}_0(K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\rm D}} \right\}.$$

Let $\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$ be the space of lowest-order 3D Crouzeix-Raviart elements that vanish at the mid-points of the faces lying on $\partial\Omega_{\rm D}$:

$$\begin{aligned}\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D}) := \Big\{ q_h \in \mathrm{L}^2(\Omega_{\rm D}) : \ & q_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\rm D}}, \\ & q_h \text{ is continuous at the centroid of each face } F \in \mathcal{F}_{\rm int} \\ & \text{and } q_h \text{ vanishes at the centroid of each face } F \in \mathcal{F}_{\partial\Omega_{\rm D}} \Big\},\end{aligned}$$

where $\mathcal{F}_{\rm int}$ denote the set of inner faces of the mesh $\mathcal{T}_h^{\Omega_{\rm D}}$ and $\mathcal{F}_{\partial\Omega_{\rm D}}$ the set of faces lying on the boundary $\partial\Omega_{\rm D}$. We recall that we have assumed that $\partial\Omega_{\rm D}$ is connected. The point values at the centroids of the inner faces $F \in \mathcal{F}_{\rm int}$ can be taken as the degrees of freedom defining an element in $\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$.

For $q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$, let $\mathbf{grad}_h q_h$ denote the vector field in $\mathcal{Q}_h^3(\Omega_{\rm D})$ defined by

$$(\mathbf{grad}_h q_h)|_K := \mathbf{grad}(q_h|_K) \qquad \forall K \in \mathcal{T}_h^{\Omega_{\rm D}}.$$

The following result has been proved in [24, Theorem 4.9] (see also [3, Lemma 5.4] for $\partial\Omega_{\rm D}$ non connected).

Lemma 4.4. *The following decomposition holds true and is orthogonal in $L^2(\Omega_{\rm D})^3$:*

$$\mathcal{Q}_h^3(\Omega_{\rm D}) = \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) \oplus \mathbf{grad}_h\big(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\big).$$

Proof. First, we prove the orthogonality. Let $\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})$ and $q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$. Integrating by parts, we have

$$\begin{aligned}\int_{\Omega_{\rm D}} \mathbf{curl}\,\boldsymbol{G}_h \cdot \mathbf{grad}_h q_h &= \sum_{K\in\mathcal{T}_h^{\Omega_{\rm D}}} \int_K \mathbf{curl}\,\boldsymbol{G}_h \cdot \mathbf{grad}\, q_h = \sum_{K\in\mathcal{T}_h^{\Omega_{\rm D}}} \int_{\partial K} \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n}_K\, q_h \\ &= -\sum_{F\in\mathcal{F}_{\rm int}} \int_F \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n}_F \, [\![q_h]\!]_F + \sum_{F\in\mathcal{F}_{\partial\Omega_{\rm D}}} \int_F \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n}_{\partial\Omega_{\rm D}}\, q_h,\end{aligned}$$

where $\boldsymbol{n}_K$, $\boldsymbol{n}_F$ and $\boldsymbol{n}_{\partial\Omega_{\rm D}}$ are unit vectors, outer normal to ∂K, normal to $F \in \mathcal{F}_{\rm int}$ and outer normal to $\partial\Omega_{\rm D}$, respectively. Moreover, for each inner face $F \in \mathcal{F}_{\rm int}$, if K and K' are the tetrahedra sharing F so that $\boldsymbol{n}_F$ points from K to K', then the jump of q_h across F is defined as $[\![q_h]\!]_F := (q_h|_{K'})|_F - (q_h|_K)|_F$. (Notice that $\boldsymbol{n}_F \, [\![q_h]\!]_F$ is independent of the choice of the unit normal to F.)

The two terms on the right-hand side above vanish because, for each $F \in \mathcal{F}_{\rm int}$, $\mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n}_F$ is constant and $[\![q_h]\!]_F$ is a linear function vanishing at the centroid of F, whereas for each $F \in \mathcal{F}_{\partial\Omega_{\rm D}}$ the same happens with $\mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n}_{\partial\Omega_{\rm D}}$ and q_h, respectively. Thus we conclude the claimed orthogonality.

Since $\mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right)$ and $\mathbf{grad}_h\left(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\right)$ both are subspaces of $\mathcal{Q}_h^3(\Omega_{\rm D})$, to end the proof it is enough to show that the dimensions of $\mathcal{Q}_h^3(\Omega_{\rm D})$ and $\mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right)\oplus \mathbf{grad}_h\left(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\right)$ coincide. Let N_K be the number of tetrahedra of the mesh $\mathcal{T}_h^{\Omega_{\rm D}}$, N_F the total number of faces and N_F^∂ the number of faces lying on $\partial\Omega_{\rm D}$. The following identity can be easily proved by induction on the number of elements of the mesh:

$$4N_K = 2N_F - N_F^\partial. \tag{4.3}$$

It is easy to check that if $q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$ and $\mathbf{grad}_h\, q_h = \mathbf{0}$, then $q_h = 0$. Hence,

$$\dim \mathbf{grad}_h(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})) = \dim \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D}) = N_F - N_F^\partial.$$

On the other hand, clearly

$$\dim\, \mathcal{Q}_h^3(\Omega_{\rm D}) = 3N_K.$$

To evaluate $\dim\, \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right)$, we introduce the lowest-order Raviart-Thomas space:

$$\boldsymbol{\mathcal{RT}}_h(\Omega) := \left\{\boldsymbol{G}_h \in \mathrm{H}(\mathrm{div},\Omega) :\ \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{RT}}(K) \quad \forall K \in \mathcal{T}_h\right\},$$

where

$$\boldsymbol{\mathcal{RT}}(K) := \left\{\boldsymbol{G}_h \in \mathcal{P}_1(K)^3 :\ \boldsymbol{G}_h(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x},\ \boldsymbol{a} \in \mathbb{C}^3,\ b \in \mathbb{C},\ \boldsymbol{x} \in K\right\}.$$

An explicit computation shows that vector fields of this type have constant normal components along each plane of the Euclidean space. Moreover, the normal components along the faces of K can be taken as the degrees of freedom defining an element in $\boldsymbol{\mathcal{RT}}(K)$. Thus $\dim\, \boldsymbol{\mathcal{RT}}_h(\Omega) = N_F$.

Since

$$\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D}) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{RT}}_h(\Omega_{\rm D}) \xrightarrow{\mathrm{div}} \mathcal{Q}_h(\Omega_{\rm D}) \longrightarrow 0$$

is an exact sequence, $\dim\, \boldsymbol{\mathcal{RT}}_h(\Omega_{\rm D}) = \dim\, \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) + \dim\, \mathcal{Q}_h(\Omega_{\rm D})$. Therefore,

$$\dim\, \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) = \dim\, \boldsymbol{\mathcal{RT}}_h(\Omega_{\rm D}) - \dim\, \mathcal{Q}_h(\Omega_{\rm D}) = N_F - N_K$$

and, by virtue of (4.3),

$$\begin{aligned}\dim\, \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) + \dim\, \mathbf{grad}_h\left(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\right) &= N_F - N_K + N_F - N_F^\partial \\ &= 3N_K = \dim\, \mathcal{Q}_h^3(\Omega_{\rm D}).\end{aligned}$$

Thus, we conclude that $\mathcal{Q}_h^3(\Omega_{\rm D}) = \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) \oplus \mathbf{grad}_h\left(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\right)$. □

Consider the following discrete problem.

Problem 4.1.2. *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H}_h \in \boldsymbol{\mathcal{U}}_h(\boldsymbol{I})$, $\boldsymbol{A}_h \in \mathcal{Q}_h^3(\Omega_{\rm D})$ and $p_h \in$*

$\mathcal{CR}_h^0(\Omega_{\mathrm{D}})$ *such that*

(4.4)
$$i\omega\int_\Omega \mu \boldsymbol{H}_h\cdot\bar{\boldsymbol{G}}_h+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_h\cdot\mathbf{curl}\,\bar{\boldsymbol{G}}_h+\int_{\Omega_{\mathrm{D}}}\boldsymbol{A}_h\cdot\mathbf{curl}\,\bar{\boldsymbol{G}}_h=0 \qquad \forall\boldsymbol{G}_h\in\boldsymbol{\mathcal{U}}_h(\mathbf{0}),$$

(4.5)
$$\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{H}_h\cdot\bar{\boldsymbol{Z}}_h+\int_{\Omega_{\mathrm{D}}}\mathbf{grad}_h p_h\cdot\bar{\boldsymbol{Z}}_h=0 \qquad \forall\boldsymbol{Z}_h\in\mathcal{Q}_h^3(\Omega_{\mathrm{D}}),$$

(4.6)
$$\int_{\Omega_{\mathrm{D}}}\boldsymbol{A}_h\cdot\mathbf{grad}_h\bar{q}_h=0 \qquad \forall q_h\in\mathcal{CR}_h^0(\Omega_{\mathrm{D}}).$$

Next result shows that Problems 4.1.2 is equivalent to Problems 4.1.1 and, hence, to Problems 3.4.1 and 3.4.2, too.

Proposition 4.5. *Let* $\boldsymbol{I}\in\mathbb{C}^N$. *If* $(\boldsymbol{H}_h,\boldsymbol{A}_h)$ *is the solution of Problem 4.1.1, then* $(\boldsymbol{H}_h,\boldsymbol{A}_h,0)$ *solves Problem 4.1.2. Conversely, if* $(\boldsymbol{H}_h,\boldsymbol{A}_h,p_h)$ *solves Problem 4.1.2, then* $p_h=0$ *and* $(\boldsymbol{H}_h,\boldsymbol{A}_h)$ *is the solution of Problem 4.1.1.*

Proof. Let $(\boldsymbol{H}_h,\boldsymbol{A}_h)$ be the solution of Problem 4.1.1. Then $(\boldsymbol{H}_h,\boldsymbol{A}_h,0)$ satisfies (4.4) and (4.5), the latter by virtue of Lemma 4.4. On the other hand, (4.6) follows from the fact that $\boldsymbol{A}_h\in\mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ and Lemma 4.4 again. Conversely, let $(\boldsymbol{H}_h,\boldsymbol{A}_h,p_h)$ solve Problem 4.1.2. By testing (4.5) with $\boldsymbol{Z}_h=\mathbf{grad}_h p_h$, it follows from Lemma 4.4 that $\mathbf{grad}_h\,p_h=\mathbf{0}$ and, hence, $p_h=0$. The same lemma and (4.6) imply that $\boldsymbol{A}_h\in\mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$. Hence, for $p_h=0$, (4.4) and (4.5) shows that $(\boldsymbol{H}_h,\boldsymbol{A}_h)$ solves Problem 4.1.1. □

As a consequence of the above proposition and the well-posedness of Problem 4.1.1, it follows that Problem 4.1.2 has also a unique solution. Thus, using standard basis for the finite element spaces leads to a linear system with a nonsingular matrix. On the other hand, the approximation properties proved for Problem 3.4.1, automatically lead to optimal order error estimates for the component $\boldsymbol{H}_h$ of the solution to Problem 4.1.2.

4.2 ▪ Eddy current problems involving inner source currents

In other eddy current problems (e.g., in non-destructive testing) the source current is fixed in a bounded subdomain. This typically happens, for instance, when the source current is produced by a coil with a large number of turns, which is practically not affected by the presence of other conductors.

In such a case, in the low-frequency harmonic Maxwell equations (3.1)–(3.5), Ohm's law ($\boldsymbol{J}=\sigma\boldsymbol{E}$) is replaced by the so called *generalized Ohm's law*:

$$\boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_{\mathrm{S}},$$

where $\boldsymbol{J}_{\mathrm{S}}$ is the imposed source current (whereas $\sigma\boldsymbol{E}$ is the eddy current induced in conductors).

A source current $\boldsymbol{J}_{\mathrm{S}}$ imposed on the conductor domain is easy to handle. It leads to a problem similar to the analyzed above, but with an additional right-hand side arising from this imposed current (see [5]). However, in many applications the support of the imposed current $\boldsymbol{J}_{\mathrm{S}}$ is contained in the dielectric domain. This happens for instance with coils in which the eddy current induced in the same coil is usually disregarded.

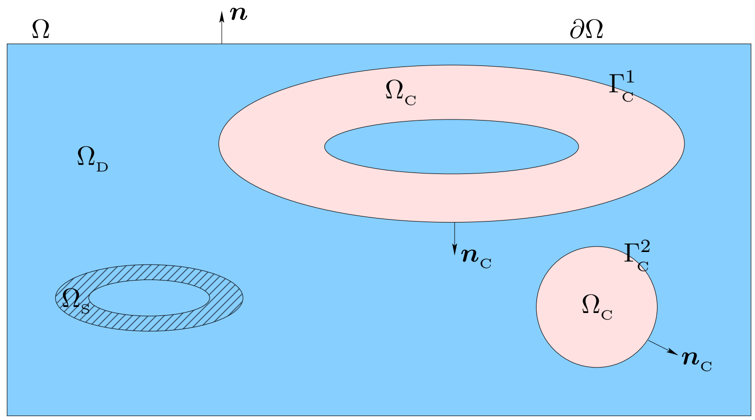

Figure 4.1. *Two-dimensional sketch of the domains.*

In such a case, the whole problem is posed in a bounded domain Ω sufficiently large to contain all the relevant conductors Ω_{C} and the source current support $\bar{\Omega}_{\mathrm{S}}$, with boundary $\partial\Omega = \partial\Omega_{\mathrm{D}}$ sufficiently far from them so that vanishing boundary conditions can be assumed. Moreover, the domain Ω is chosen topologically trivial (i.e., simply connected with a connected boundary; see Fig 4.1 for a two-dimensional sketch). Thus, we are led to the following equations:

$$\mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}} \quad \text{in } \Omega, \tag{4.7}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega, \tag{4.8}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \quad \text{in } \Omega, \tag{4.9}$$

$$\operatorname{div}(\epsilon\boldsymbol{E}) = 0 \quad \text{in } \Omega_{\mathrm{D}}, \tag{4.10}$$

$$\mu\boldsymbol{E}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{4.11}$$

As observed in Remark 3.2.3, additional constraints should also be imposed to determine uniquely the electric field $\boldsymbol{E}$ in the dielectric domain Ω_{D} (see [5]). Moreover, as already mentioned, (4.9) is a consequence of (4.8). On the other hand, notice that the boundary condition

$$\mu\boldsymbol{H}\cdot\boldsymbol{n} = \mathbf{0} \qquad \text{on } \partial\Omega$$

is recovered from (4.11) by using (4.8) and the Stokes theorem.

As a consequence of (4.7), since the conductivity σ vanishes in dielectrics, the source current $\boldsymbol{J}_{\mathrm{S}}$ has to be divergence-free in Ω_{D} and $\int_{\Gamma_{\mathrm{I}}^j}\boldsymbol{J}_{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{C}} = 0$, $j = 1,\dots,J$, with $\boldsymbol{n}_{\mathrm{C}}$ being the outer unit normal to Ω_{C} and $\Gamma_{\mathrm{I}}^1,\dots,\Gamma_{\mathrm{I}}^J$ the connected components of the interface Γ_{I} (which now are closed surfaces contained in Ω). We will make the stringent assumption that the support $\bar{\Omega}_{\mathrm{S}}$ is contained in the interior of the dielectric domain, in which case

$$\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\operatorname{div}^0,\Omega_{\mathrm{S}}) := \left\{\boldsymbol{G}\in \mathrm{L}^2(\Omega_{\mathrm{S}})^3 : \ \operatorname{div}\boldsymbol{G} = 0 \text{ in } \Omega_{\mathrm{S}} \text{ and } \boldsymbol{G}\cdot\boldsymbol{n}_{\mathrm{S}} = 0 \text{ on } \partial\Omega_{\mathrm{S}}\right\},$$

where $\boldsymbol{n}_{\mathrm{S}}$ is the unit vector outer normal to $\partial\Omega_{\mathrm{S}}$.

Proceeding as in Section 3.2, we can derive a magnetic field formulation, which allows determining a fortiori the eddy current $\boldsymbol{J} = \sigma\boldsymbol{E}$ in the conductors. However, the magnetic field no longer belongs to $\boldsymbol{\mathcal{V}}(\boldsymbol{I})$ as above, but to the set

$$\widetilde{\boldsymbol{\mathcal{V}}}(\boldsymbol{J}_{\mathrm{S}}) := \left\{\boldsymbol{G} \in \mathrm{H}(\mathbf{curl}, \Omega) : \ \mathbf{curl}\,\boldsymbol{G} = \boldsymbol{J}_{\mathrm{S}} \text{ in } \Omega_{\mathrm{D}}\right\}.$$

In fact, by repeating the steps used to derive Problem 3.2.1 we obtain the following:

Problem 4.2.1. *Given $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$, find $\boldsymbol{H} \in \widetilde{\boldsymbol{\mathcal{V}}}(\boldsymbol{J}_{\mathrm{S}})$ such that*

$$i\omega \int_{\Omega} \mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} = 0 \qquad \forall \boldsymbol{G} \in \widetilde{\boldsymbol{\mathcal{V}}}(\mathbf{0}).$$

The existence and uniqueness of solution follows immediately from the Lax-Milgram lemma and the fact that for all $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$ there exists $\boldsymbol{H}_{\mathrm{S}} \in \widetilde{\boldsymbol{\mathcal{V}}}(\boldsymbol{J}_{\mathrm{S}})$. To prove the latter, it is enough to extend $\boldsymbol{J}_{\mathrm{S}}$ by zero to the whole Ω and to use, for instance, [22, Theor. I.3.4] to find $\boldsymbol{H}_{\mathrm{S}} \in \mathrm{H}(\mathbf{curl}, \Omega)$ such that $\mathbf{curl}\,\boldsymbol{H}_{\mathrm{S}} = \boldsymbol{J}_{\mathrm{S}}$.

When a particular $\boldsymbol{H}_{\mathrm{S}} \in \widetilde{\boldsymbol{\mathcal{V}}}(\boldsymbol{J}_{\mathrm{S}})$ is available, one can write $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{S}} + \widehat{\boldsymbol{H}}$ with $\widehat{\boldsymbol{H}} \in \widetilde{\boldsymbol{\mathcal{V}}}(\mathbf{0}) \equiv \boldsymbol{\mathcal{X}}$, so that it can be written as $\widehat{\boldsymbol{H}} = \widetilde{\mathbf{grad}}\,\widetilde{\Phi}$ for an appropriate multivalued magnetic potential $\widetilde{\Phi} \in \Theta$. Thus, one can derive a problem for $\widehat{\boldsymbol{H}}$ which could be analyzed and discretized as in Sections 3.3 and 3.4.

However, the computation of such $\boldsymbol{H}_{\mathrm{S}}$ is not straightforward. One possibility is to use Biot-Savart law (see, for instance, [5, Sec. 5.4.1]). In what follows, instead of pursuing this approach further, we will show that the mixed formulation analyzed in Section 4 adapts perfectly well to this problem. In fact, this mixed formulation was originally proposed and analyzed in [3] for a problem of this kind and it reads as follows:

Problem 4.2.2. *Given $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$, find $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{A} \in \mathbf{curl}\left(\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})\right)$ such that*

$$i\omega \int_{\Omega} \mu\boldsymbol{H}\cdot\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{D}}} \boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{G}} = 0 \qquad \forall \boldsymbol{G} \in \mathrm{H}(\mathbf{curl}, \Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{H}\cdot\bar{\boldsymbol{Z}} = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}\cdot\bar{\boldsymbol{Z}} \qquad \forall \boldsymbol{Z} \in \mathbf{curl}\left(\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})\right).$$

To prove that Problem 4.2.2 is well posed, first notice that $\mathbf{curl}\left(\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})\right) = \left\{\boldsymbol{G} \in \mathrm{H}(\mathrm{div}^0, \Omega_{\mathrm{D}}) : \ \int_{\Gamma_{\mathrm{I}}^j} \boldsymbol{G}\cdot\boldsymbol{n} = 0, \ j = 1, \dots, J\right\}$ endowed with the $\mathrm{L}^2(\Omega_{\mathrm{D}})$-norm is a Hilbert space. Thus, we only need to check the ellipticity in the kernel and the inf-sup condition. The former follows from the fact that the kernel is given by $\{\boldsymbol{G} \in \mathrm{H}(\mathbf{curl}, \Omega) : \ \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\} = \boldsymbol{\mathcal{X}}$, space in which the bilinear form a is elliptic (cf. (3.14)). The latter follows from the fact that for all $\boldsymbol{Z} \in \mathbf{curl}\left(\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})\right)$ there exists $\boldsymbol{F} \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})$ such that $\boldsymbol{Z} = \mathbf{curl}\,\boldsymbol{F}$ and $\|\boldsymbol{F}\|_{\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})} \leq C\,\|\boldsymbol{Z}\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})}$ (the proof of this is essentially contained in [4]; see also [21]). Finally $\boldsymbol{F}$ can be continuously extended from $\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})$ into $\mathrm{H}(\mathbf{curl}, \Omega)$ (see again [4]), which allows us to conclude that there exists $\beta > 0$ such that, for all $\boldsymbol{Z} \in \mathbf{curl}\big(\mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})\big)$,

$$\sup_{\boldsymbol{G}\in \mathrm{H}(\mathbf{curl},\Omega):\ \boldsymbol{G}\neq \mathbf{0}} \frac{\left|\int_{\Omega_{\mathrm{D}}} \boldsymbol{Z}\cdot \mathbf{curl}\,\bar{\boldsymbol{G}}\right|}{\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl},\Omega)}} \geq \frac{\left|\int_{\Omega_{\mathrm{D}}} \boldsymbol{Z}\cdot \mathbf{curl}\,\bar{\boldsymbol{F}}\right|}{\|\boldsymbol{F}\|_{\mathrm{H}(\mathbf{curl},\Omega)}} \geq \beta \|\boldsymbol{Z}\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}\,.$$

The finite element discretization of this problem is as follows.

Problem 4.2.3. *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$*, find* $\boldsymbol{H}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ *and* $\boldsymbol{A}_h \in \mathbf{curl}\big(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\big)$ *such that*

$$i\omega \int_{\Omega} \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{D}}} \boldsymbol{A}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{H}_h \cdot \bar{\boldsymbol{Z}}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}} \cdot \bar{\boldsymbol{Z}}_h \qquad \forall \boldsymbol{Z}_h \in \mathbf{curl}\big(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\big).$$

Problem 4.2.3 satisfies the ellipticity in the discrete kernel and the discrete inf-sup condition with constants independent of h. The former follows by repeating the arguments used for the continuous kernel. The latter has been proved in [3, Lemma 5.3] for a problem with different boundary conditions, but the same arguments apply to the present case. Consequently, Problem 4.2.3 has a unique solution, for which error estimates follow from the classical Babuška-Brezzi theory (see for instance [22]).

As in the case of Problem 4.1.1, the direct implementation of Problem 4.2.3 leads to an underdetermined linear system, which anyway could be solved by a conjugate gradient type method. The alternative three-field mixed formulation used for Problem 4.1.1 also works in this case.

Problem 4.2.4. *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$*, find* $\boldsymbol{H}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$*,* $\boldsymbol{A}_h \in \mathcal{Q}_h^3(\Omega_{\mathrm{D}})$ *and* $p_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\mathrm{D}})$ *such that*

$$i\omega \int_{\Omega} \mu \boldsymbol{H}_h \cdot \bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{D}}} \boldsymbol{A}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{H}_h \cdot \bar{\boldsymbol{Z}}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{grad}_h p_h \cdot \bar{\boldsymbol{Z}}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}} \cdot \bar{\boldsymbol{Z}}_h \qquad \forall \boldsymbol{Z}_h \in \mathcal{Q}_h^3(\Omega_{\mathrm{D}}),$$

$$\int_{\Omega_{\mathrm{D}}} \boldsymbol{A}_h \cdot \mathbf{grad}_h \bar{q}_h = 0 \qquad \forall q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\mathrm{D}}).$$

Next result shows that Problems 4.2.4 is equivalent to Problems 4.2.3.

Proposition 4.6. *Let* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$*. If* $(\boldsymbol{H}_h, \boldsymbol{A}_h)$ *is the solution of Problem 4.2.3, then there exists a unique* $p_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\mathrm{D}})$ *such that* $(\boldsymbol{H}_h, \boldsymbol{A}_h, p_h)$ *solves Problem 4.2.4. Conversely, if* $(\boldsymbol{H}_h, \boldsymbol{A}_h, p_h)$ *solves Problem 4.2.4, then* $(\boldsymbol{H}_h, \boldsymbol{A}_h)$ *is the solution of Problem 4.2.3.*

Proof. The proof is based on Lemma 4.4. We omit it because it runs almost identical to that of Proposition 4.5. The only difference is that, now, p_h does not necessarily vanish, but is the solution of the following well posed discrete problem:

$$p_h \in \mathcal{CR}_h^0(\Omega_{\rm D}): \quad \int_{\Omega_{\rm D}} \mathbf{grad}_h p_h \cdot \mathbf{grad}_h \bar{q}_h = \int_{\Omega_{\rm S}} \boldsymbol{J}_{\rm S} \cdot \mathbf{grad}_h \bar{q}_h \quad \forall q_h \in \mathcal{CR}_h^0(\Omega_{\rm D}).$$

□

As a consequence of the above proposition and the well-posedness of Problem 4.2.3, it follows that Problem 4.2.4 has also a unique solution. Thus, using standard basis for the finite element spaces leads to a linear system with a nonsingular matrix. On the other hand, the error estimates valid for Problem 4.2.3 automatically hold for the solution of Problem 4.2.4.

Chapter 5

A potential formulation

The first attempts to numerically solve the eddy current problem were based on so-called *potential formulations*. In spite of the fact that these are the most frequently used in applications, there is only a small number of papers dealing with their mathematical analysis. Among them, we mention a paper by Alonso *et al.* [2], where the well-posedness of some of these formulations is analyzed, and another one by Bíró and Valli [12] with the analysis of one such formulation in a general topological setting.

Different potentials have been used for the eddy current problem: a vector potential $\boldsymbol{A}$ for the magnetic induction field, a scalar potential V for the electric field in the conducting domain, a scalar potential ψ for the magnetic field in dielectric domains, etc. A hierarchy of formulations involving these potentials have been discussed by Bíró and Preis in [11] and they conclude that the so-called $\boldsymbol{A}, V-\boldsymbol{A}-\psi$ formulation, which involves all of them, is the most convenient in terms of computer cost. Numerical experiments illustrating the performance of this approach are also reported in this reference.

In what follows, we provide a rigorous mathematical analysis of this formulation. Under rather general topological conditions, we prove that it leads to a well-posed problem, which can be numerically approximated by standard nodal finite elements. We also prove error estimates for the resulting numerical method. These estimates are valid as long as the three potentials are sufficiently smooth.

The smoothness of the scalar potentials V and ψ only relies on those of the original physical variables of the problem: the electric and the magnetic fields, respectively. Instead, the smoothness of the vector potential $\boldsymbol{A}$ also depends on the geometry of the domain chosen to define this non-physical variable. In principle this domain can be freely taken, as far as it contains the conductors and the source current support. However, it has also to be chosen so that its connected components are convex polyhedra, to ensure that the smoothness of $\boldsymbol{A}$ is only determined by the regularity of the magnetic induction field $\boldsymbol{B} = \mu\boldsymbol{H}$.

Because of this, we make such a choice for the domain of $\boldsymbol{A}$, which is not restrictive in practice. However, it is convenient to choose it as small as possible, because the magnetic field is written in terms of the more economical scalar potential ψ outside this domain. Thus, in the applications, the domain of $\boldsymbol{A}$ typically consists of a union of disjoint boxes, as small as possible, containing the current source and the conductors.

5.1 ▪ Eddy current problem

We consider the eddy current problem of determining the electromagnetic fields induced in a three-dimensional conducting domain Ω_C by a given source current density $\boldsymbol{J}_S$. We assume that the support of $\boldsymbol{J}_S$ is compact and disjoint with Ω_C. As above, we restrict the problem to a bounded domain Ω containing both, Ω_C and the support of $\boldsymbol{J}_S$, such that appropriate vanishing boundary conditions can be imposed on its boundary. To this aim, we choose the geometry of Ω as simple as possible (e.g., simply connected with a connected boundary). See Fig. 5.1 for a two-dimensional sketch.

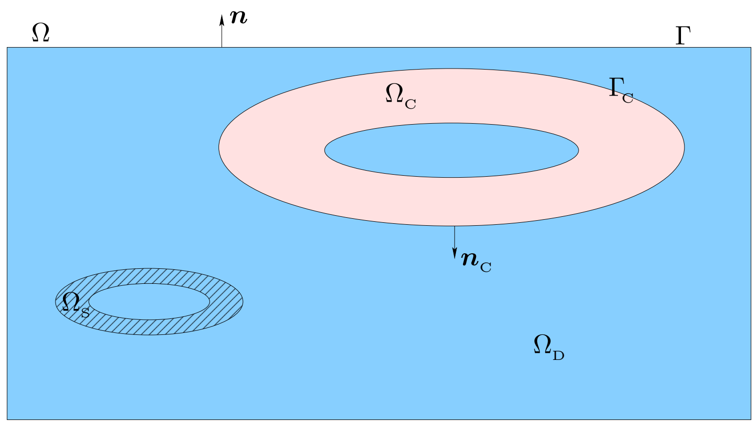

Figure 5.1. *Two-dimensional sketch of the domain.*

Let $\Omega_C \subset \mathbb{R}^3$ be an open and bounded set with boundary Γ_C. Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with a connected boundary Γ, such that $\overline{\Omega}_C \subset \Omega$. We suppose that both, Ω and Ω_C are either Lipschitz polyhedra or domains with $\mathcal{C}^{1,1}$ boundaries. We denote by $\boldsymbol{n}$ and $\boldsymbol{n}_C$ the outward unit normal vectors to Ω and Ω_C, respectively, and by $\Omega_D := \Omega \setminus \overline{\Omega}_C$ the subdomain of Ω occupied by dielectric material, which includes the support of the source current Ω_S (see Fig. 5.1).

The eddy current problem reads as follows:

Problem 5.1.1. *Given* $\boldsymbol{J}_S \in \mathrm{H}_0(\mathrm{div}^0, \Omega_S)$, *find* $\boldsymbol{E}$ *and* $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ *such that:*

$$\mathbf{curl}\,\boldsymbol{H} = \sigma \boldsymbol{E} \quad \text{in } \Omega_C, \tag{5.1}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}_S \quad \text{in } \Omega_D, \tag{5.3}$$

$$\mathrm{div}(\mu\boldsymbol{H}) = 0 \quad \text{in } \Omega, \tag{5.4}$$

$$\mathrm{div}(\epsilon\boldsymbol{E}) = 0 \quad \text{in } \Omega_D, \tag{5.5}$$

$$\boldsymbol{H} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \tag{5.6}$$

The unknowns $\boldsymbol{E}$ and $\boldsymbol{H}$ are the magnetic and electric fields, respectively. The magnetic permeability μ and the conductivity σ are assumed to be bounded functions satisfying:

$$0 < \mu_{\min} \le \mu \le \mu_{\max} \qquad \text{in } \Omega,$$
$$0 < \sigma_{\min} \le \sigma \le \sigma_{\max} \qquad \text{in } \Omega_{\mathrm{C}}.$$

The source of the problem is the current density $\boldsymbol{J}_{\mathrm{S}}$ whose support is assumed to be contained in Ω_{D}, Notice that for (5.3) to make sense, $\boldsymbol{J}_{\mathrm{S}}$ has to belong to $\mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$.

As in the previous sections, our goal is to determine $\boldsymbol{E}$ in the conductor domain Ω_{C} and $\boldsymbol{H}$ in the whole Ω, but not $\boldsymbol{E}$ in the dielectric domain Ω_{D}.

5.2 ▪ The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation

In this section we recall a classical formulation of the eddy current problem in terms of three potentials, $\boldsymbol{A}$, V and ψ, which was introduced by Leonard and Rodger [23]. We refer to Bíró and Preis [11] for a detailed discussion, which also includes numerical tests showing the efficiency of this approach.

First, we introduce a vector potential $\boldsymbol{A}$ for the magnetic induction field $\boldsymbol{B} = \mu\boldsymbol{H}$ in a subdomain Ω_A of Ω containing the conducting domain Ω_{C} and the support Ω_{S} of the source current to be determined. This subdomain does not need to be connected, but each of its connected components has to be convex; the reason for such constraint will be discussed at the end of Section 5.4 below. On the other hand, for the sake of discretization, it is convenient to choose Ω_A polyhedral; moreover, outside Ω_A, we will use a scalar potential which will consequently require much less degrees of freedom for its discretization. Because of this, Ω_A will be chosen as small as possible, but with convex polyhedral connected components containing Ω_{C} and Ω_{S} (see Fig. 5.2).

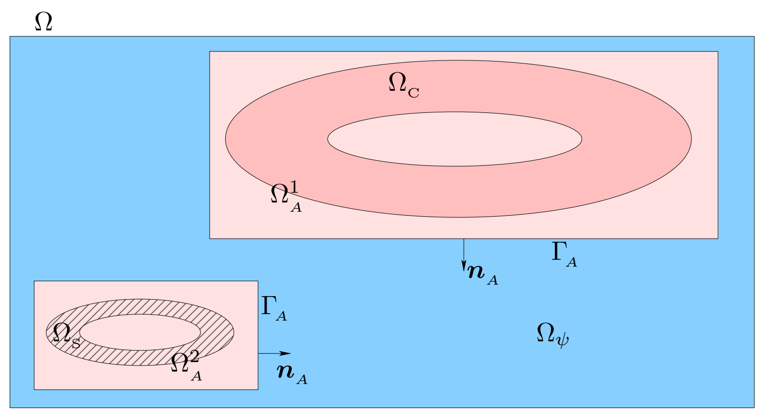

Figure 5.2. *Two-dimensional sketch of the domains for the different potentials.*

Let $\Omega_A \subset \mathbb{R}^3$ be an open set satisfying

$$\overline{\Omega}_{\mathrm{C}} \cup \operatorname{supp} \boldsymbol{J}_{\mathrm{S}} \subset \Omega_A \qquad \text{and} \qquad \overline{\Omega}_A \subset \Omega. \tag{5.7}$$

We denote by Ω^j_A, $j = 1, \ldots, m_A$, the connected components of Ω_A. We assume that each Ω^j_A is a convex polyhedron and that $\overline{\Omega}^j_A$ are mutually disjoint. We denote by Γ_A the boundary of Ω_A and by $\boldsymbol{n}_A$ its outward unit normal vector (see Fig. 5.2).

As a consequence of [22, Theorem I.3.5.], equation (5.4) implies that there exist unique $\boldsymbol{A}_j \in \mathrm{H}(\mathbf{curl}, \Omega_A^j)$ such that

$$\mu \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{A}_j \qquad \text{in } \Omega_A^j, \tag{5.8}$$

$$\operatorname{div} \boldsymbol{A}_j = 0 \qquad \text{in } \Omega_A^j, \tag{5.9}$$

$$\boldsymbol{A}_j \cdot \boldsymbol{n}_A = 0 \qquad \text{on } \partial\Omega_A^j. \tag{5.10}$$

Thus, if we define $\boldsymbol{A} : \Omega_A \longrightarrow \mathbb{C}$ by

$$\boldsymbol{A}|_{\Omega_A^j} := \boldsymbol{A}_j, \qquad j = 1, \ldots, m_A,$$

then $\boldsymbol{A}$ belongs to the space

$$\boldsymbol{\mathcal{Z}} := \mathrm{H}_0(\operatorname{div}, \Omega_A) \cap \mathrm{H}(\mathbf{curl}, \Omega_A),$$

whose natural norm is given by

$$\|\boldsymbol{Z}\|^2_{\boldsymbol{\mathcal{Z}}} := \|\boldsymbol{Z}\|^2_{0,\Omega_A} + \|\operatorname{div} \boldsymbol{Z}\|^2_{0,\Omega_A} + \|\mathbf{curl}\, \boldsymbol{Z}\|^2_{0,\Omega_A}.$$

Next, from (5.2) and (5.8), we have that

$$\mathbf{curl}(\boldsymbol{E} + i\omega \boldsymbol{A}) = \mathbf{0} \quad \text{in } \Omega_C. \tag{5.11}$$

Thus, according to [11] we introduce an electric scalar potential $V \in \mathrm{H}^1(\Omega_C)$, such that

$$\boldsymbol{E} = -i\omega \boldsymbol{A} - i\omega\, \mathbf{grad}\, V \qquad \text{in } \Omega_C. \tag{5.12}$$

Notice that if the connected components of Ω_C are not all simply connected, in principle we do not have the right to introduce such an electric potential. In fact, in such a case, the space

$$\mathrm{H}(\mathbf{curl}^0, \Omega_C) := \{\boldsymbol{Z} \in \mathrm{H}(\mathbf{curl}, \Omega_C) : \ \mathbf{curl}\, \boldsymbol{Z} = 0 \text{ in } \Omega_C\}$$

also contains gradients of potentials multivalued on the respective cut surfaces into Ω_C, which are not gradients of functions in $\mathrm{H}^1(\Omega_C)$ (analogously to what was shown in Section 3.3). In what follows we will show that, anyway, such a $V \in \mathrm{H}^1(\Omega_C)$ always exists.

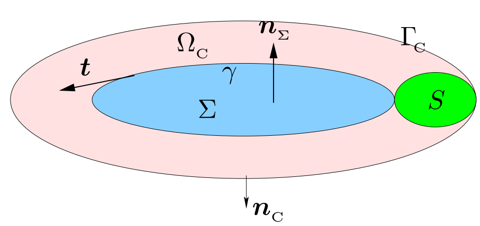

Figure 5.3. *Non-simply connected conductor domain* Ω_C.

To make the argument simpler, we restrict ourselves to the case of a non-simply connected domain as that shown in Fig. 5.3. In such a case, Let S be a cut surface of Ω_C such that $\widetilde{\Omega}_C := \Omega_C \setminus S$ is simply connected. We denote by S^- and S^+ the two faces of this surface. Let γ be the curve shown in Fig. 5.3, i.e., the boundary of a corresponding cut surface Σ of Ω_{D}. This cut surface is chosen so that $\Sigma \subset \Omega_A$. The orientation of the curve is given by the unit vector $\boldsymbol{t}$, which is chosen so that γ goes from S^- to S^+. Let $\boldsymbol{n}_\Sigma$ be the unit vector normal to Σ as shown in Fig. 5.3.

Let $\widetilde{\varphi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{C}})$ be the solution of the following elliptic problem:

$$\llbracket \widetilde{\varphi} \rrbracket_S := \widetilde{\varphi}|_{S^-} - \widetilde{\varphi}|_{S^+} = 1,$$
$$\int_{\widetilde{\Omega}_{\mathrm{C}}} \mu\, \widetilde{\mathbf{grad}}\, \widetilde{\varphi} \cdot \mathbf{grad}\, \bar{\psi} = 0 \qquad \forall \psi \in \mathrm{H}^1(\Omega_{\mathrm{C}}).$$

Proceeding as in Section 3.3, we have that

$$\mathrm{H}(\mathbf{curl}^0, \Omega_{\mathrm{C}}) = \mathbf{grad}\left(\mathrm{H}^1(\Omega_{\mathrm{C}})\right) \oplus \left\langle \widetilde{\mathbf{grad}}\, \widetilde{\varphi} \right\rangle.$$

Therefore, the fact that $\mathbf{curl}(\boldsymbol{E} + i\omega \boldsymbol{A}) = \mathbf{0}$ in Ω_{C} in principle implies that there exist $V \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ and $\alpha \in \mathbb{C}$ such that

$$\boldsymbol{E} + i\omega \boldsymbol{A} = -i\omega \left(\mathbf{grad}\, V + \alpha\, \widetilde{\mathbf{grad}}\, \widetilde{\varphi} \right) \qquad \text{in } \Omega_{\mathrm{C}}.$$

However,

$$\int_\gamma (\boldsymbol{E} + i\omega \boldsymbol{A}) \cdot \boldsymbol{t} = -i\omega \int_\gamma \mathbf{grad}\, V \cdot \boldsymbol{t} - i\omega\alpha \int_\gamma \widetilde{\mathbf{grad}}\, \widetilde{\varphi} \cdot \boldsymbol{t} = i\omega\alpha \llbracket \widetilde{\varphi} \rrbracket_S = i\omega\alpha,$$

whereas, because of the Stokes Theorem and (5.11),

$$\int_\gamma (\boldsymbol{E} + i\omega \boldsymbol{A}) \cdot \boldsymbol{t} = \int_\Sigma \mathbf{curl}(\boldsymbol{E} + i\omega \boldsymbol{A}) \cdot \boldsymbol{n}_\Sigma = 0.$$

Hence, we conclude that $\alpha = 0$ and, consequently, that there exists $V \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ such that (5.12) holds true.

Notice that, from (5.1),

$$\operatorname{div}\left(-i\omega\sigma \boldsymbol{A} - i\omega\sigma\, \mathbf{grad}\, V\right) = 0 \qquad \text{in } \Omega_{\mathrm{C}}.$$

Moreover, since $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$, (5.1) and (5.3) also imply that

$$(i\omega\sigma \boldsymbol{A} + i\omega\sigma\, \mathbf{grad}\, V) \cdot \boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}.$$

These last two equations will be also collected in the potential formulation.

Equation (5.12) determines the electric potential V on each connected component of Ω_{C} up to an additive constant. Thus, if Ω_{C} has m_{C} connected components Ω_{C}^j, then the natural space for V is

$$\mathcal{M} := \prod_{j=1}^{m_{\mathrm{C}}} \mathrm{H}^1(\Omega_{\mathrm{C}}^j)/\mathbb{C},$$

endowed with the norm $\|\mathbf{grad}\, V\|_{0,\Omega_{\mathrm{C}}}$.

Finally, we introduce a magnetic scalar potential ψ in

$$\Omega_\psi := \Omega \setminus \overline{\Omega}_A$$

(see Fig. 5.2). To do this, notice that since Ω_A is a disjoint union of convex sets with $\overline{\Omega}_A \subset \Omega$ and Ω is simply connected, it turns out that Ω_ψ is simply connected too. Therefore, from (5.3) and (5.7) we know that there exists a scalar potential $\psi \in \mathrm{H}^1(\Omega_\psi)$ (unique up to an additive constant) such that

$$\boldsymbol{H} = \omega \,\mathbf{grad}\, \psi \qquad \text{in } \Omega_\psi.$$

Moreover, by virtue of the boundary condition (5.6), the surface gradient of this scalar potential $\mathbf{grad}_\Gamma \psi := \boldsymbol{n} \times \mathbf{grad}\, \psi \times \boldsymbol{n} = \mathbf{0}$ on Γ, so that ψ has to be constant on this boundary. Therefore, we may choose $\psi \in \mathrm{H}^1_\Gamma(\Omega_\psi)$ and thus ψ is uniquely determined.

Thus, we are led to the following formulation of Problem 5.1.1 in terms of the potentials $\boldsymbol{A} \in \boldsymbol{\mathcal{Z}}$, $V \in \mathcal{M}$ and $\psi \in \mathrm{H}^1_\Gamma(\Omega_\psi)$:

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) + i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{C}}, \tag{5.13}$$

$$\operatorname{div}(-i\omega\sigma\boldsymbol{A} - i\omega\sigma\,\mathbf{grad}\,V) = 0 \qquad \text{in } \Omega_{\mathrm{C}}, \tag{5.14}$$

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) = \boldsymbol{J}_{\mathrm{S}} \qquad \text{in } \Omega_A \setminus \overline{\Omega}_{\mathrm{C}}, \tag{5.15}$$

$$\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\Big|_{\Omega_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} - \left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\Big|_{\Omega_A \setminus \overline{\Omega}_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{C}}, \tag{5.16}$$

$$\operatorname{div}(\mu\,\mathbf{grad}\,\psi) = 0 \qquad \text{in } \Omega_\psi, \tag{5.17}$$

$$\operatorname{div}\boldsymbol{A} = 0 \qquad \text{in } \Omega_A, \tag{5.18}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{5.19}$$

$$\mathbf{curl}\,\boldsymbol{A} \cdot \boldsymbol{n}_A - \omega\mu\,\mathbf{grad}\,\psi \cdot \boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{5.20}$$

$$\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_A - \omega\,\mathbf{grad}\,\psi \times \boldsymbol{n}_A = \mathbf{0} \qquad \text{on } \Gamma_A, \tag{5.21}$$

$$(i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V) \cdot \boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}. \tag{5.22}$$

Let us remark that (5.16) and (5.21) are consequences of the fact that $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$, whereas (5.20) follows from the fact that $\mu\boldsymbol{H} \in \mathrm{H}(\operatorname{div}, \Omega)$, which in its turn is a consequence of (5.4)

5.3 ▪ Variational formulation. Existence and uniqueness of solution

The aim of this section is to give a variational formulation of problem (5.13)–(5.22) and to prove its well-posedness.

First, we recall some results settled in [16] for Lipschitz domains. We write these results for Ω_A, as will be used in the sequel. The tangential trace operator $\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u}|_{\Gamma_A} \times \boldsymbol{n}_A$ is a bounded linear operator from $\mathrm{H}(\mathbf{curl}, \Omega_A)$ onto $\mathrm{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma_A)$. The tangential projection $\pi_\tau(\boldsymbol{v}) := \boldsymbol{n}_A \times \boldsymbol{v}|_{\Gamma_A} \times \boldsymbol{n}_A$ is a bounded linear operator from $\mathrm{H}(\mathbf{curl}, \Omega_A)$ onto $\mathrm{H}^{-1/2}(\operatorname{rot}_\Gamma, \Gamma_A)$. Thus, the duality pairing between $\mathrm{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma_A)$ and $\mathrm{H}^{-1/2}(\operatorname{rot}_\Gamma, \Gamma_A)$ is well defined by

$$\langle \gamma_\tau(\boldsymbol{u}), \pi_\tau(\boldsymbol{v}) \rangle_{\Gamma_A} := \int_{\Omega_A} \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{v} - \int_{\Omega_A} \boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathrm{H}(\mathbf{curl}, \Omega_A).$$

For any $\boldsymbol{w} \in \mathrm{H}(\mathbf{curl}, \Omega_\psi)$, its tangential trace on Γ_A also belongs to $\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$ and, consequently, $\langle \boldsymbol{w} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{v})\rangle_{\Gamma_A}$ is also well defined.

To obtain a variational formulation of problem (5.13)–(5.22), notice that by virtue of (5.13), (5.15) and (5.16) we have that $\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A} \in \mathrm{H}(\mathbf{curl}, \Omega_A)$ and, for all $\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}$,

$$\int_{\Omega_A} \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) \cdot \bar{\boldsymbol{Z}} = -i\omega \int_{\Omega_C} \sigma\,(\boldsymbol{A} + \mathbf{grad}\,V) \cdot \bar{\boldsymbol{Z}} + \int_{\Omega_A} \boldsymbol{J}_S \cdot \bar{\boldsymbol{Z}}.$$

Integrating by parts the left-hand side above and using (5.18) and (5.21) lead to

$$\begin{aligned}(5.23)\quad &\int_{\Omega_A} \frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + i\omega \int_{\Omega_C} \sigma \boldsymbol{A} \cdot \bar{\boldsymbol{Z}} \\ &+ i\omega \int_{\Omega_C} \sigma\,\mathbf{grad}\,V \cdot \bar{\boldsymbol{Z}} - \omega\,\langle \mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{Z})\rangle_{\Gamma_A} = \int_{\Omega_A} \boldsymbol{J}_S \cdot \bar{\boldsymbol{Z}} \qquad \forall \boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}.\end{aligned}$$

On the other hand, from (5.14), by integrating by parts and using (5.22) we obtain

$$(5.24)\qquad i\omega \int_{\Omega_C} \sigma \boldsymbol{A} \cdot \mathbf{grad}\,\bar{U} + i\omega \int_{\Omega_C} \sigma\,\mathbf{grad}\,V \cdot \mathbf{grad}\,\bar{U} = 0 \qquad \forall U \in \mathrm{H}^1(\Omega_C).$$

Finally, for any $\varphi \in \mathrm{H}^1_\Gamma(\Omega_\psi)$, from (5.17), by integrating by parts and using (5.20) we obtain

$$\omega \int_{\Omega_\psi} \mu\,\mathbf{grad}\,\psi \cdot \mathbf{grad}\,\bar{\varphi} + \langle \mathbf{curl}\,\boldsymbol{A} \cdot \boldsymbol{n}_A, \bar{\varphi}\rangle_{\Gamma_A} = 0,$$

Now, let $\varphi^* \in \mathrm{H}^1(\Omega)$ be an extension of φ to the whole Ω. Hence,

$$\langle \mathbf{curl}\,\boldsymbol{A} \cdot \boldsymbol{n}_A, \bar{\varphi}\rangle_{\Gamma_A} = \int_{\Omega_A} \mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{grad}\,\bar{\varphi}^* = \left\langle \mathbf{grad}\,\bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}.$$

Therefore, we obtain

$$(5.25)\quad \omega \int_{\Omega_\psi} \mu\,\mathbf{grad}\,\psi \cdot \mathbf{grad}\,\bar{\varphi} + \left\langle \mathbf{grad}\,\bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A} = 0 \qquad \forall \varphi \in \mathrm{H}^1_\Gamma(\Omega_\psi).$$

Equations (5.23)–(5.25) provide a variational formulation of (5.13)–(5.22). To prove that this formulation has a unique solution, we write it in a more compact form. With this end, let $\mathscr{A}$ be the bilinear form defined on $\boldsymbol{\mathcal{Z}} \times \mathcal{M} \times \mathrm{H}^1_\Gamma(\Omega_\psi)$ by

$$\begin{aligned}&\mathscr{A}((\boldsymbol{A}, V, \psi), (\boldsymbol{Z}, U, \varphi)) \\ &:= \int_{\Omega_A} \frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + \omega^2 \int_{\Omega_\psi} \mu\,\mathbf{grad}\,\psi \cdot \mathbf{grad}\,\bar{\varphi} \\ &\quad + i\omega \int_{\Omega_C} \sigma\,(\boldsymbol{A} + \mathbf{grad}\,V) \cdot \left(\bar{\boldsymbol{Z}} + \mathbf{grad}\,\bar{U}\right) \\ &\quad - \omega\,\langle \mathbf{grad}\,\psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{Z})\rangle_{\Gamma_A} + \omega\,\left\langle \mathbf{grad}\,\bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}.\end{aligned}$$

Then, (5.23)–(5.25) can be equivalently written as follows:

Problem 5.3.1. *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$, *find* $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{Z}} \times \mathcal{M} \times \mathrm{H}^1_\Gamma(\Omega_\psi)$ *such that*

$$\mathscr{A}((\boldsymbol{A}, V, \psi), (\boldsymbol{Z}, U, \varphi)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{S}} \cdot \bar{\boldsymbol{Z}} \qquad \forall (Z, U, \varphi) \in \boldsymbol{\mathcal{Z}} \times \mathcal{M} \times \mathrm{H}^1_\Gamma(\Omega_\psi).$$

Theorem 5.1. *Problem 5.3.1 has a unique solution.*

Proof. It is enough to show that $\mathscr{A}$ is elliptic, since, in such a case, the theorem follows from the Lax-Milgram's Lemma.

To prove the ellipticity, for $(\boldsymbol{Z}, U, \varphi) \in \boldsymbol{\mathcal{Z}} \times \mathcal{M} \times \mathrm{H}^1_\Gamma(\Omega_\psi)$ we write

$$\begin{aligned}\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi)) = &\int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{Z}|^2 + |\mathrm{div}\, \boldsymbol{Z}|^2 \right) + \omega^2 \int_{\Omega_\psi} \mu \, |\mathbf{grad}\, \varphi|^2 \\ &+ i\omega \left\{ \int_{\Omega_{\mathrm{C}}} \sigma \left(|\boldsymbol{Z}|^2 + |\mathbf{grad}\, U|^2 \right) + 2 \int_{\Omega_{\mathrm{C}}} \sigma \, \mathrm{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \right. \\ &\left. + 2\, \mathrm{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A} \right\}.\end{aligned}$$

Thus,

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 = (a + \omega^2 b)^2 + \omega^2 (c + 2d)^2,$$

where

$$a := \int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{Z}|^2 + |\mathrm{div}\, \boldsymbol{Z}|^2 \right), \qquad b := \int_{\Omega_\psi} \mu \, |\mathbf{grad}\, \varphi|^2,$$
$$c := \int_{\Omega_{\mathrm{C}}} \sigma \left(|\boldsymbol{Z}|^2 + |\mathbf{grad}\, U|^2 \right), \qquad d := e + f,$$

with

$$e := \int_{\Omega_{\mathrm{C}}} \sigma \, \mathrm{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \qquad \text{and} \qquad f := \mathrm{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A}.$$

Next, we proceed as in [12] and use the elementary inequality

$$(c + 2d)^2 \geq \rho c^2 - 8\rho d^2 \qquad \forall c, d \in \mathbb{R}, \ \forall \rho \in (0, 1/2],$$

to obtain

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))|^2 \geq a^2 + \omega^4 b^2 + \omega^2 (\rho c^2 - 8\rho d^2) \qquad \forall \rho \in (0, 1/2].$$

Now, since[1]

$$a \geq \frac{K}{\mu_{\max}} \|\boldsymbol{Z}\|^2_{\boldsymbol{\mathcal{Z}}} \qquad \text{and} \qquad b \geq \mu_{\min} \|\mathbf{grad}\, \varphi\|^2_{0,\Omega_\psi},$$

[1]For the first inequality see for instance, Lemma I.3.6 from [22].

with $K > 0$ independent of $\boldsymbol{Z}$, we have

$$|\mathscr{A}((\boldsymbol{Z},U,\varphi),(\boldsymbol{Z},U,\varphi))|^2 \geq \frac{K^2}{\mu_{\max}^2}\|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{Z}}}^4 + \omega^4\mu_{\min}^2\|\mathbf{grad}\,\varphi\|_{0,\Omega_\psi}^4 + \omega^2\rho\left(\int_{\Omega_{\mathrm{C}}}\sigma\,|\mathbf{grad}\,U|^2\right)^2 - 16\omega^2\rho(e^2+f^2).$$

To estimate the last term on the right-hand side above, notice first that, for all $\varepsilon > 0$,

$$e^2 \leq \left(\int_{\Omega_{\mathrm{C}}}\left|\sigma\,\mathbf{grad}\,U\cdot\bar{\boldsymbol{Z}}\right|\right)^2 \leq \frac{\varepsilon}{2}\left(\int_{\Omega_{\mathrm{C}}}\sigma\,|\mathbf{grad}\,U|^2\right)^2 + \frac{1}{2\varepsilon}\left(\int_{\Omega_{\mathrm{C}}}\sigma\,|\boldsymbol{Z}|^2\right)^2.$$

On the other hand, $\exists C > 0$ independent of φ and $\boldsymbol{Z}$ such that

$$\begin{aligned} f^2 &\leq \|\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_{A}\|_{\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma_A)}^2\left\|\pi_\tau(\bar{\boldsymbol{Z}})\right\|_{\mathrm{H}^{-1/2}(\mathrm{rot}_\Gamma,\Gamma_A)}^2 \\ &\leq C\left(\|\mathbf{grad}\,\varphi\|_{0,\Omega_\psi}^4 + \|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{Z}}}^4\right). \end{aligned}$$

Therefore, by combining the last three inequalities and taking ε and ρ small enough, we obtain that $\exists\alpha > 0$ such that, $\forall(\boldsymbol{Z},U,\varphi)\in\boldsymbol{\mathcal{Z}}\times\mathcal{M}\times\mathrm{H}_\Gamma^1(\Omega_\psi)$,

$$|\mathscr{A}((\boldsymbol{Z},U,\varphi),(\boldsymbol{Z},U,\varphi))|^2 \geq \alpha\left(\|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{Z}}}^4 + \|\mathbf{grad}\,U\|_{0,\Omega_{\mathrm{C}}}^4 + \|\mathbf{grad}\,\varphi\|_{0,\psi}^4\right),$$

which allows us to conclude the ellipticity of $\mathscr{A}$. □

To end this section, we prove that the unique solution of Problem 5.3.1 is actually a solution of the strong form of the problem given by equations (5.13)–(5.22).

Theorem 5.2. *The solution* $(\boldsymbol{A},V,\psi)$ *of Problem 5.3.1 satisfies* (5.13)–(5.22).

Proof. Clearly the solution $(\boldsymbol{A},V,\psi)$ of Problem 5.3.1 satisfies (5.23)–(5.25).

Now, let $\xi\in\mathrm{H}^1(\Omega_A)$ be a solution of the compatible Neumann problem $\Delta\xi = \mathrm{div}\,\boldsymbol{A}$ in Ω_A, $\partial\xi/\partial\boldsymbol{n}_A = 0$ on Γ_A. By testing (5.23) with $\boldsymbol{Z} = \mathbf{grad}\,\xi\in\boldsymbol{\mathcal{Z}}$, we obtain (5.18) by using (5.24) (since $\xi|_{\Omega_{\mathrm{C}}}\in\mathcal{M}$) and $\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\mathbf{grad}\,\xi)\rangle_{\Gamma_A} = 0$ (which is a consequence of the definition of the duality pairing).

On the other hand, by testing (5.23)–(5.25) with smooth functions supported in adequate domains and proceeding in the standard way, it is easy to verify equations (5.13)–(5.17), (5.20) and (5.22). Since (5.19) is imposed in the definition of the space $\boldsymbol{\mathcal{Z}}$, there only remains to prove (5.21) in $\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma_A)$; namely, that for all $\boldsymbol{\zeta}\in\mathrm{H}(\mathbf{curl},\Omega_A)$,

$$\left\langle\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{\zeta})\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{\zeta})\rangle_{\Gamma_A} = 0. \tag{5.26}$$

To do this, notice first that by substituting (5.18) in (5.23), integrating by parts and having into account (5.13) and (5.15), we obtain

$$\left\langle\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{Z})\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{Z})\rangle_{\Gamma_A} = 0 \qquad \forall\boldsymbol{Z}\in\boldsymbol{\mathcal{Z}}.$$

Next, for $\boldsymbol{\zeta} \in \mathrm{H}(\mathbf{curl}, \Omega_{A})$, let φ be a solution of the following auxiliary problem:

$$\varphi \in \mathrm{H}^1(\Omega_{A})/\mathbb{C}: \qquad \int_{\Omega_{A}} \mathbf{grad}\,\varphi \cdot \mathbf{grad}\,\bar{\chi} = \int_{\Omega_{A}} \boldsymbol{\zeta} \cdot \mathbf{grad}\,\bar{\chi} \qquad \forall \chi \in \mathrm{H}^1(\Omega_{A})/\mathbb{C}.$$

Hence, $\mathrm{div}(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi) = 0$ in Ω_{A} and $(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi) \cdot \boldsymbol{n}_{A} = 0$ on Γ_{A}. Consequently, $\boldsymbol{Z} := \boldsymbol{\zeta} - \mathbf{grad}\,\varphi \in \boldsymbol{\mathcal{Z}}$ and using it as a test function in the equation above we obtain

$$\left\langle \frac{1}{\mu} \mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_{A}, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi) \right\rangle_{\Gamma_{A}} - \langle \mathbf{grad}\,\psi \times \boldsymbol{n}_{A}, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\,\varphi) \rangle_{\Gamma_{A}} = 0.$$

Now, from (5.13) and (5.15), we have

$$\begin{aligned} \left\langle \frac{1}{\mu} \mathbf{curl}\,\boldsymbol{A} \times \boldsymbol{n}_{A}, \pi_\tau(\mathbf{grad}\,\varphi) \right\rangle_{\Gamma_{A}} &= \int_{\Omega_{A}} \mathbf{curl} \left(\frac{1}{\mu} \mathbf{curl}\,\boldsymbol{A} \right) \cdot \mathbf{grad}\,\bar{\varphi} \\ &= -\int_{\Omega_{\mathrm{C}}} (i\omega\sigma \boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V) \cdot \mathbf{grad}\,\bar{\varphi} \\ &\quad + \int_{\Omega_{A}} \boldsymbol{J}_{\mathrm{S}} \cdot \mathbf{grad}\,\bar{\varphi} \\ &= 0, \end{aligned}$$

where, for the last step, we have used integration by parts, (5.14), (5.22), the assumption that $\boldsymbol{J}_{\mathrm{S}}$ is divergence-free and (5.7).

Thus, using again that $\langle \mathbf{grad}\,\psi \times \boldsymbol{n}_{A}, \pi_\tau(\mathbf{grad}\,\varphi) \rangle_{\Gamma_{A}}$ vanishes, (5.26) follows from the last two equations and we conclude the proof. ∎

5.4 ▪ Numerical approximation

In this section we describe and analyze a finite element method to approximate the solution of Problem 5.3.1. We assume that all the domains are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}$ be a family of tetrahedral meshes of Ω such that, for each mesh, all the elements $T \in \mathcal{T}_h$ are completely included in one of the three subdomains $\overline{\Omega}_{\mathrm{C}}$, $\overline{\Omega}_{A} \setminus \Omega_{\mathrm{C}}$ or $\overline{\Omega}_{\psi}$.

Consider the following finite element spaces:

$$\begin{aligned} \boldsymbol{\mathcal{Z}}_h &:= \left\{ \boldsymbol{Z}_h \in \boldsymbol{\mathcal{Z}} : \ \boldsymbol{Z}_h|_T \in \mathcal{P}_m^3 \ \ \forall T \in \mathcal{T}_h : \ T \subset \overline{\Omega}_{A} \right\}, \\ \mathcal{M}_h &:= \left\{ U_h \in \mathcal{M} : \ U_h|_T \in \mathcal{P}_m \ \ \forall T \in \mathcal{T}_h : \ T \subset \overline{\Omega}_{\mathrm{C}} \right\}, \\ \mathcal{Q}_h &:= \left\{ \varphi_h \in \mathrm{H}^1(\Omega_\psi) : \ \varphi_h|_T \in \mathcal{P}_m \ \ \forall T \in \mathcal{T}_h : \ T \subset \overline{\Omega}_\psi \right\}, \\ \mathcal{Q}_{\Gamma,h} &:= \left\{ \varphi_h \in \mathcal{Q}_h : \ \varphi_h|_\Gamma = 0 \right\}, \end{aligned}$$

where $\mathcal{P}_m$, $m \geq 1$, is the set of polynomials of degree not greater than m.

Thus, we are led to the following discrete problem:

Problem 5.4.1. *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{S}})$*, find* $(\boldsymbol{A}_h, V_h, \psi_h) \in \boldsymbol{\mathcal{Z}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}$ *such that*

$$\mathscr{A}((\boldsymbol{A}_h, V_h, \psi_h), (\boldsymbol{Z}_h, U_h, \varphi_h)) = \int_{\Omega_{A}} \boldsymbol{J}_{\mathrm{S}} \cdot \bar{\boldsymbol{Z}}_h \qquad \forall (\boldsymbol{Z}_h, U_h, \varphi_h) \in \boldsymbol{\mathcal{Z}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}.$$

The existence and uniqueness of the solution of this discrete problem is again an immediate consequence of the ellipticity of $\mathscr{A}$, proved in the proof of Theorem 5.2, and the Lax-Milgram Lemma. Moreover, if the solution of the continuous problem is smooth enough, the standard finite element error analysis techniques yield the following result:

Theorem 5.3. *Let $(\boldsymbol{A}, V, \psi)$ and $(\boldsymbol{A}_h, V_h, \psi_h)$ be the solutions of problems 5.3.1 and 5.4.1, respectively. If $\boldsymbol{A} \in \mathrm{H}^{1+s}(\Omega_A)^3$, $V \in \mathrm{H}^{1+s}(\Omega_\mathrm{C})$ and $\psi \in \mathrm{H}^{1+s}(\Omega_\psi)$ with $s > 0$, then there exists a strictly positive constant C, independent of h, $\boldsymbol{A}$, V and ψ, such that*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{Z}}} + \|\mathbf{grad}\,(V - V_h)\|_{0,\Omega_\mathrm{C}} + \|\mathbf{grad}\,(\psi - \psi_h)\|_{0,\Omega_\psi} \\ \leq Ch^r \left(\|\boldsymbol{A}\|_{1+s,\Omega_A} + \|V\|_{1+s,\Omega_\mathrm{C}} + \|\psi\|_{1+s,\Omega_\psi} \right),$$

with $r := \min\{m, s\}$.

Proof. It is a direct consequence of the ellipticity of $\mathscr{A}$, Cea's lemma and the approximation properties of the Lagrange interpolant (see, for instance, [18]). □

To end the paper we discuss the need of choosing the domain Ω_A of the vector potential so that its connected components be convex. For simplicity, in what follows we take Ω_A connected, but all the statements hold true for each of its connected components. So let Ω_A be simply connected with a connected boundary.

According to [22, Theorem I.3.4], since $\mathrm{div}(\mu\boldsymbol{H}) = 0$ in Ω, there exists $\boldsymbol{\Phi} \in \mathrm{H}^1(\Omega)^3$ satisfying:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{\Phi} &= \mu\boldsymbol{H} && \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{\Phi} &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, according to Remark I.3.12 of the same reference, if $\mu\boldsymbol{H} \in \mathrm{H}^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi} \in \mathrm{H}^{1+p}(\Omega)^3$.

Therefore, by virtue of (5.8)–(5.10), there holds:

$$\begin{aligned} \mathbf{curl}(\boldsymbol{A} - \boldsymbol{\Phi}) &= \mathbf{0} && \text{in } \Omega_A, \\ \mathrm{div}(\boldsymbol{A} - \boldsymbol{\Phi}) &= 0 && \text{in } \Omega_A, \\ (\boldsymbol{A} - \boldsymbol{\Phi}) \cdot \boldsymbol{n}_A &= -\boldsymbol{\Phi} \cdot \boldsymbol{n}_A && \text{on } \Gamma_A. \end{aligned}$$

The first equation above and the simple-connectedness of Ω_A implies that there exists a unique $\chi \in \mathrm{H}^1(\Omega_A)/\mathbb{C}$ such that $\boldsymbol{A} - \boldsymbol{\Phi} = \mathbf{grad}\,\chi$ in Ω_A, whereas the remaining equations imply that χ is the solution of the following compatible Neumann problem:

$$\begin{aligned} \Delta\chi &= 0 && \text{in } \Omega_A, \\ \frac{\partial\chi}{\partial\boldsymbol{n}_A} &= -\boldsymbol{\Phi} \cdot \boldsymbol{n}_A && \text{on } \Gamma_A. \end{aligned}$$

The Neumann data of this problem will be in general smooth on each polygonal face F of Γ_A, since Γ_A is an arbitrary polyhedral surface within the dielectric domain. In fact, if $\mu\boldsymbol{H} \in \mathrm{H}^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi}|_F \cdot \boldsymbol{n}_A \in \mathrm{H}^{\frac{1}{2}+p}(F)$ for all faces F.

Therefore, if Ω_{A} is a convex polyhedron, then there exists $q > 0$ such that $\chi \in \mathrm{H}^{2+q}(\Omega_{A})$ (see [20]). Consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\,\chi \in \mathrm{H}^{1+s}(\Omega_{A})^3,$$

with $s := \min\{p, q\} > 0$. Conversely, if Ω_{A} were a non-convex polyhedron, then, in general, $\chi \notin \mathrm{H}^2(\Omega_{A})$ and, consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\,\chi \notin \mathrm{H}^1(\Omega_{A})^3.$$

In such a case, Theorem 5.3 would become meaningless.

Moreover, $\widehat{\boldsymbol{\mathcal{Z}}} := \left\{\boldsymbol{Z} \in \mathrm{H}^1(\Omega_{A})^3 : \ \boldsymbol{Z} \cdot \boldsymbol{n}_{A} = 0 \text{ on } \Gamma_{A}\right\}$ is a closed subspace of $\boldsymbol{\mathcal{Z}}$ (see [19]). When Ω_{A} is a polyhedron, it is well-known that $\widehat{\boldsymbol{\mathcal{Z}}} = \boldsymbol{\mathcal{Z}}$ if and only if Ω_{A} is convex (see [22, Theorem I.3.9] and [19]).

The finite element space $\boldsymbol{\mathcal{Z}}_h$ is clearly a subspace of $\widehat{\boldsymbol{\mathcal{Z}}}$. Therefore, when Ω_{A} is a convex polyhedron, it makes sense to approximate $\boldsymbol{A} \in \boldsymbol{\mathcal{Z}}$ by finite elements from $\boldsymbol{\mathcal{Z}}_h$. Instead, if Ω_{A} were not convex, then there would be no hope of approximating $\boldsymbol{A}$ by finite elements from $\boldsymbol{\mathcal{Z}}_h$. Indeed, as stated above, in general $\boldsymbol{A} \notin \mathrm{H}^1(\Omega_{A})^3$ in such a case. Hence, $\boldsymbol{A}$ would not belong to the closed set $\widehat{\boldsymbol{\mathcal{Z}}}$ containing the finite element spaces $\boldsymbol{\mathcal{Z}}_h$ for all meshes. So, there could not exist $\boldsymbol{A}_h$ such that $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{Z}}} \to 0$ as h goes to zero.

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