Gordan-type alternative theorems and vector optimization revisited *

Fabián Flores-Bazán, Fernando Flores-Bazán and Cristián Vera

Abstract Theorems of the alternative has proved to be one of the most powerful tools in optimization theory. They provide existence of Lagrange multipliers, (strong) duality results, linear scalarizations of various classes of solutions to vector optimzation problems. This chapter is devoted to this last part of applications.
The chapter starts by recalling the (1957) Fan-Glicksberg-Hoffman alternative theorem for convex functions. Then, many equivalent formulations to a general Gordan-type alternative theorem valid for (not neccesarily pointed) convex cones with possibly empty interior, are established. They will be expressed in terms of quasi relative interior. Several classes of generalized convexity for sets and for vector valued mappings, are revisited.
Applications to linear characterizations of weakly efficient, (Benson) proper efficient solutions, and to characterize the Fritz-John type optimality condition in vector optimization, are discussed. Finally, we also present some recent developments about proper efficiency.

---

* This research, for the first author, was supported in part by CONICYT-Chile through FONDECYT 110-0667 and BASAL Project, CMM, Universidad de Chile.

Fabián Flores-Bazán
Departamento de Ingeniería Matemática, CIIMA, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile, e-mail: fflores@ing-mat.udec.cl

Fernando Flores-Bazán
Departamento de Matemática, Facultad de Ciencias, Universidad del Bío Bío, Av. Collao 1202, Casilla 5-C, Concepción, Chile. e-mail: fflores@ubiobio.cl

Cristián Vera
Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, e-mail: cristianv@ucsc.cl

1 Introduction an formulation of the problem

Alternative theorems have proved to be important in deriving key results in optimization theory like the existence of Lagrange multipliers, duality results, scalarization of vector functions,etc. Since the pionering result due to Julius Farkas in 1902 concerning his alternative lema which is well known in linear programming, or even the elder alternative result established by Paul Gordan in 1873, many mathematicians have made a lot of effort to generalize both results in a nonlinear setting. To these author's knowledge the first Gordan type result for convex functions is due to Fan, Glicksberg and Hoffman [13] and was established in 1957. Such a result says the following:

Let $K \subseteq \mathbb{R}^n$ be convex, and $f_i : K \to \mathbb{R}$, $i = 1, \ldots, m$, be convex functions. Then, exactly one of the following two systems has solution:

(a) $f_i(x) < 0$, $i = 1, \ldots, m$, $x \in K$;
(b) $p \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, $\sum_{i=1}^m p_i f_i(x) \geq 0$ for all $x \in K$.

After that, the problem without the convexity became an interesting challenge in mathematics.
To be precise, let us consider a real locally convex topological vector space Y and a closed convex cone $P \subseteq Y$ such that $\text{int}\, P \neq \emptyset$. We denote by Y^* the topological dual space of Y, and by P^* the (positive) polar cone of P. Given a nonempty set $A \subseteq Y$, a Gordan-type alternative theorem asserts the validity of exactly one of the following assertions:

$$\exists\, a \in A \text{ such that } a \in -\text{int}\, P; \tag{1}$$

$$\exists\, p^* \in P^*,\ p^* \neq \mathbf{0}, \text{ such that } \langle p^*, a \rangle \geq 0 \ \ \forall\, a \in A. \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y and Y^* and $\text{int}\, P$ denotes the topological interior of P. We recall that P^* is defined by

$$P^* = \{p^* \in Y^* : \ \langle p^*, p \rangle \geq 0 \ \ \forall\, p \in P\}.$$

The closedness and convexity of the cone P is equivalent to $P = P^{**}$ by the bipolar theorem. In this case,

$$p \in P \iff \langle p^*, p \rangle \geq 0 \ \ \forall\, p^* \in P^*.$$

Moreover,

$$p \in \text{int}\, P \iff \langle p^*, p \rangle > 0 \ \ \forall\, p^* \in P^* \setminus \{\mathbf{0}\}. \tag{3}$$

Via the last equation, we see that the inconsistency of assertions (1) and (2) is straightforward, whereas the validity of (2) by assuming that (1) does not hold, requires a careful analysis due to the lack of convexity of A.
In fact, because of many applications, one of our purposes in this chapter is to avoid convexity and to allow convex cones possibly with empty topological interior. The latter happens for instance if $(1 < p < +\infty)$

$$P = L^p_+ \doteq \{u \in L^p(\Omega) : \ u \geq 0 \quad \text{a.e. } x \in \Omega\},$$

or if P is of the form $P = Q \times \{\mathbf{0}\}$ with $\text{int}\, Q \neq \emptyset$.

A good substitute for the interior is the quasi interior and even the quasi-relative interior. Borwein and Lewis in [5] introduced the quasi-relative interior of a convex set $A \subseteq Y$, although the concept of quasi interior was introduced earlier. We use both notions in order to deal with convex cones with possibly empty interior. In this situation, the convex hull arises naturally.

One of the main goals of the present chapter is to characterize those sets A for which the negation of (1) implies (2). The negation of (1) means

$$A \cap (-\text{int}\, P) = \emptyset, \tag{4}$$

which is equivalent to

$$\overline{\text{cone}}(A + P) \cap (-\text{int}\, P) = \emptyset. \tag{5}$$

Therefore, by assuming the convexity of $\overline{\text{cone}}(A+P)$, a standard separation theorem of convex sets provides the existence of p^* satisfying (2): this fact was proved in [42], see also [32, 22, 43] for additional sufficiency conditions of alternative theorems. In [14, Theorem 4.1] is established that such a convexity assumption is necessary and sufficient to get the implication (4) $\Longrightarrow$ (2) provided the space is two dimensional; whereas it is far to being necessary in dimension greater than or equal to three [14, Example 3.8]. We shall revise that alternative theorem in dimension two for convex cones having possibly empty interior, as well as various equivalences to the above convexity assumption.

This chapter is organized as follows. Section 2 gives the necessary basic definitions together with some elementary results about cones: in particular, when P is a halfspace, a complete answer to the validity of a Gordan-type alternative theorem is given, see Corollary 2. In Section 3, we establish several equivalent formulations to the Gordan-type alternative theorems valid for (not necessarily pointed or closed) convex cones with possibly empty (topological) interior, see Theorem 1 and Corollary 3. This is given in terms of quasi interior and quasi relative interior. We also compare various of the previously introduced notions of generalized convexity for sets and vector functions. As a consequence of these results, we are able to derive and strenghten several of the already known alternative theorems. Section 4 establishes an optimal alternative theorem in 2-dimension for a cone with possibly empty interior under a regularity assumption, which always holds if the interior is nonempty, see Theorem 4.1. Section 5 is devoted to applications. One of them is devoted to characterize those mappings $F : K \to \mathbb{R}^2$ for which a equivalence between

$$\bigcup_{p^* \in P^* \setminus \{0\}} \text{argmin}_K \langle p^*, F(\cdot)\rangle \ \ (\text{resp. } \bigcup_{p^* \in \text{int}\, P^*} \text{argmin}_K \langle p^*, F(\cdot)\rangle)$$

and E_W (resp. E_{pr}, the properly efficient set) holds, where E_W denotes the set of weakly efficient solutions to F on K. Such an equivalence is expected to be useful for developing a well-posedness theory in vector optimization as in [12]. In addition, as another application, we revise the Fritz-John optimality conditions for a class of nonconvex vector minimization problems. Finally, we also present some recent developments about proper efficiency.

2 Basic definitions and preliminares

Throughout the paper, X will be a vector space and Y a real locally convex topological vector space, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Y and its topological dual space, Y^*. Given $x, y \in X$ we set $[x,y] = \{tx + (1-t)y : t \in [0,1]\}$. The segments $]x,y]$, $]x,y[$, etc., are defined analogously.

A set $P \subseteq Y$ is said to be a *cone* if $tP \subseteq P \ \forall\ t \geq 0$; given $A \subseteq Y$, $\text{cone}(A)$ stands for the smallest cone containing A, that is,

$$\text{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A: obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(A)}$, where $\overline{A}$ denotes the closure of A. Furthermore, we set

$$\text{cone}_+(A) \doteq \bigcup_{t > 0} tA.$$

Evidently, $\text{cone}(A) = \text{cone}_+(A) \cup \{0\}$ and therefore $\overline{\text{cone}(A)} = \overline{\text{cone}_+(A)}$. In [32, 42, 43, 33] the notation $\text{cone}(A)$ instead of $\text{cone}_+(A)$ is employed.

Given a convex set $A \subseteq Y$ and $x \in A$, $N_A(x)$ stands for the *normal cone* to A at x, defined by $N_A(x) = \{\xi \in Y^* : \langle \xi, a - x \rangle \leq 0, \ \ \forall\ a \in A\}$.

Definition 1. We say that $x \in A$ is a (see for instance [7]):

- *quasi interior point* of A, denoted by $x \in \text{qi}\, A$, if $\overline{\text{cone}}(A - x) = Y$, or equivalently, $N_A(x) = \{\mathbf{0}\}$;
- *quasi relative interior point* of A, denoted by $x \in \text{qri}\, A$, if $\overline{\text{cone}}(A - x)$ is a linear subspace of Y, or equivalently, $N_A(x)$ is a linear subspace of Y^*.
- ([31, 44]) *core point* of A, denoted by $x \in \text{core}\, A$, if $\text{cone}(A - x) = Y$.
- ([6, 18, 44]) *intrinsic core point* of A, denoted by $x \in \text{icr}\, A$, if $\text{cone}(A - x)$ is a linear subspace of Y.
- ([31]) *strong-quasi relative interior point* of A, denoted by $x \in \text{sqri}\, A$, if $\text{cone}(A - x)$ is a closed linear subspace of Y.

For any convex set A, we have that ([25, 7]) $\text{qi}\, A \subseteq \text{qri}\, A$ and, $\text{int}\, A \neq \emptyset$ implies $\text{int}\, A = \text{qi}\, A$. Similarly, if $\text{qi}\, A \neq \emptyset$, then $\text{qi}\, A = \text{qri}\, A$. Moreover [5], if Y is a finite dimensional space, then $\text{qi}\, A = \text{int}\, A$ and $\text{qri}\, A = \text{ri}\, A$, where $\text{ri}\, A$ means the relative interior of A, which is the interior of A with respect to the affine hull of A. In addition,

$$\text{core}\, A \subseteq \text{sqri}\, A \subseteq \text{qri}\, A \ \text{ and } \ \text{core}\, A \subseteq \text{qi}\, A \subseteq \text{qri}\, A.$$

Let $B \subseteq Y$ another convex set. Then

$$\text{qri } A + \text{qri } B \subseteq \text{qri}(A+B);\ \text{qri } A \times \text{qri } B = \text{qri}(A \times B);\ \text{qri}(A-x) = \text{qri } A - x;$$

$$\text{qri}(tA) = t\text{qri } A\ \ \forall\, t \in \mathbb{R};\ \text{qri } A = A,\ \text{provided } A \text{ is affine};\ \text{qri}(\text{qri } A) = \text{qri } A;$$

$$\overline{\text{qri } A} = \overline{A};\ \overline{\text{cone}}(\text{qri } A) = \overline{\text{cone}}\, A, \quad \text{if qri } A \neq \emptyset.$$

Thus, all results in this paper involving qi A are also true for int A, provided the latter set is nonempty. On the other hand, the cone l^p_+ has nonempty quasi interior, but its interior (and even the relative algebraic interior) is empty for all $p \in [1,+\infty[$. Likewise, the core and even the strong quasi relative interior of L^p_+ is empty. Quasi relative interior points share some properties of the interior points; for instance, if $x \in \text{qri } A$ and $y \in A$ then $[x,y[\subseteq \text{qri } A$. In particular, qri A is convex.

If P is a closed convex cone, then it is easy to check that $x \in \text{qi } P$ if and only if $\langle x^*, x\rangle > 0$ for all $x^* \in P^* \setminus \{\mathbf{0}\}$, or equivalently if the set $B = \{x^* \in P^* : \langle x^*, x\rangle = 1\}$ is a w*-closed base for P^* (we recall that a convex set B is called a base for P^* if $\mathbf{0}$ is not in the w*-closure of B and $P^* = \text{cone}(B)$). If $P \neq Y$, then $0 \notin \text{qi } P$. Note also that $\text{qi } P = \text{cone}_+(\text{qi } P)$ and $P + \text{qi } P = \text{qi } P$.

In the rest of the paper, $\{\mathbf{0}\} \neq P \subsetneq Y$ will be a convex cone.

Some elementary properties of sets and cones are collected in the next proposition.

Proposition 1. *Let $A, K \subseteq Y$ be any nonempty sets.*

(a) $\overline{\text{co}}(A) = \overline{\text{co}}(\overline{A})$, $\overline{\text{cone}}(\overline{A}) = \overline{\text{cone}}(A)$;
(b) *if A is open then* $\text{cone}_+(A)$ *is open;*
(c) $\text{cone}(\text{co}(A)) = \text{co}(\text{cone}(A))$; $\text{cone}_+(\text{co}(A)) = \text{co}(\text{cone}_+(A))$;
(d) $\text{co}(A+K) = \text{co}(A) + K$ *provided K is convex;*
(e) $\text{cone}_+(A+K) = \text{cone}_+(A) + K$ *provided K is such that $tK \subseteq K\ \forall\, t > 0$;*
(f) $\overline{A+K} = \overline{A+\overline{K}}$;
(g) $K \subseteq \overline{\text{cone}}(A+K)$ *provided K is a cone;*
(h) $\text{cone}(A+K) \subseteq \text{cone}(A) + K \subseteq \overline{\text{cone}}(A+K)$ *provided K is a cone; if additionally $0 \in A$, then*

$$\text{cone}(A+K) = \text{cone}(A) + K;$$

In the following, K is a convex cone such that $\text{int } K \neq \emptyset$*.*
(i) $\overline{A + \text{int } K} = \overline{A+K}$, $\text{int}\overline{A+K} = A + \text{int}K = \text{int}(A+K)$;
(j) $\overline{\text{cone}}(A + \text{qri } P) = \overline{\text{cone}}(A+P)$, *provided P is a convex cone with* $\text{qri } P \neq \emptyset$*.*
(k) $\text{cone}_+(A + \text{int } K)$ *is convex* $\Longleftrightarrow$ $\text{cone}(A) + \text{int } K$ *is convex* $\Longleftrightarrow$ $\overline{\text{cone}}(A+K)$ *is convex.*

Proof. (a), (b), (c), (d), and (e) are straightforward.
(f): Since $K \subseteq \overline{K}$, we have $\overline{A+K} \subseteq \overline{A+\overline{K}}$. On the other hand, it is not difficult to obtain $A + \overline{K} \subseteq \overline{A+K}$, which completes both inclusions.
(g): For any fix $a \in A$, every $x \in K$ can be obtained as the limit of $\frac{1}{n}(a+nx)$. Hence $K \subseteq \overline{\text{cone}}(A+K)$.

(h): The first inclusion is obvious. According to (e), $\mathrm{cone}_+(A)+K=\mathrm{cone}_+(A+K)\subseteq\overline{\mathrm{cone}}(A+K)$, which along with (g) prove the second inclusion. The remaining equality is trivial.
(i): The first part follows from (f), and the other is in [36, 8].
(j): $\overline{\mathrm{cone}}(A+\mathrm{qri}\,P)=\overline{\mathrm{cone}}(\overline{A+\mathrm{qri}\,P})=\overline{\mathrm{cone}}(\overline{A+\overline{\mathrm{qri}\,P}})=\overline{\mathrm{cone}}(\overline{A+\overline{P}})=\overline{\mathrm{cone}}(A+P)$.
(k): By (e) and (i),

$$\mathrm{cone}_+(A+\mathrm{int}\,K)=\mathrm{cone}_+(A)+\mathrm{int}\,K=\mathrm{int}(\mathrm{cone}_+(A)+K)=\mathrm{int}(\overline{\mathrm{cone}_+(A)+K})$$
$$=\mathrm{int}(\overline{\mathrm{cone}(A)+K})=\mathrm{int}(\mathrm{cone}(A)+K)=\mathrm{cone}(A)+\mathrm{int}\,K.$$

This proves the first equivalence. We also obtain

$$\mathrm{int}(\overline{\mathrm{cone}}(A+K))=\mathrm{int}(\overline{\mathrm{cone}_+(A)+K})=\mathrm{cone}_+(A)+\mathrm{int}\,K=\mathrm{cone}_+(A+\mathrm{int}\,K),$$

proving the equivalence between the first and third sets. □

Remark 1. Proposition $1(k)$ does not hold with qi P in the place of int P. Indeed, let $Y=l^1$ and $P=l^1_+$. Then $\mathrm{qi}\,l^1_+=\left\{(\alpha_i)_{i\in\mathbb{N}}:\alpha_i>0\right\}$ while $\mathrm{int}\,l^1_+=\emptyset$. Set

$$A=l^1\setminus\left(-\mathrm{qi}\,l^1_+\right)=\left\{(\alpha_i)_{i\in\mathbb{N}}:\exists\, i\in\mathbb{N}\text{ with }\alpha_i\geq 0\right\}.$$

Each $(a_i)_{i\in\mathbb{N}}\in l^1$ can be written as a limit of a sequence of elements each of which has a finite number of nonzero coordinates. Thus $\overline{A}=l^1$ and $\overline{\mathrm{cone}}(A+l^1_+)=l^1$ is convex. However, one can readily check that $\mathrm{cone}_+(A+\mathrm{qi}\;P)=A+\mathrm{qi}\;P=\left\{(\alpha_i)_{i\in\mathbb{N}}:\exists\, i\in\mathbb{N}\text{ with }\alpha_i>0\right\}$ is not convex.

Proposition 2. *Let $\emptyset\neq A\subseteq Y$. The following assertions hold:*

(a) $\alpha A+(1-\alpha)A\subseteq\overline{\mathrm{cone}}(A)\ \forall\ \alpha\in\ ]0,1[\ \Longleftrightarrow\overline{\mathrm{cone}}(A)$ is convex $\Longleftrightarrow\mathrm{co}(A)\subseteq\overline{\mathrm{cone}}(A)$;
(b) $\alpha A+(1-\alpha)A\subseteq\mathrm{cone}(A)\ \forall\ \alpha\in\ ]0,1[\ \Longleftrightarrow\mathrm{cone}(A)$ *is convex* $\Longleftrightarrow\mathrm{co}(A)\subseteq\mathrm{cone}(A)$;
(c) $\alpha A+(1-\alpha)A\subseteq\mathrm{cone}_+(A)\ \forall\ \alpha\in\ ]0,1[\ \Longleftrightarrow\mathrm{cone}_+(A)$ *is convex* $\Longleftrightarrow\mathrm{co}(A)\subseteq\mathrm{cone}_+(A)$.

Proof. (a): Let x_i, $i=1,2$, such that there are nets $\{t_i^\alpha\}_{\alpha\in\Lambda}$, $\{x_i^\alpha\}_{\alpha\in\Lambda}$ such that $t_i^\alpha\geq 0$, $x_i^\alpha\in A$ and $t_i^\alpha x_i^\alpha\to x_i$, $\alpha\in\Lambda$. We may assume $t_i^\alpha>0$ for all α, $i=1,2$. For any fixed $\lambda\in\ ]0,1[$, set $t^\alpha=\lambda t_1^\alpha+(1-\lambda)t_2^\alpha>0$. Then

$$\lambda t_1^\alpha x_1^\alpha+(1-\lambda)t_2^\alpha x_2^\alpha=t^n\Big(\frac{\lambda t_1^\alpha}{t^\alpha}x_1^\alpha+\frac{(1-\lambda)t_2^\alpha}{t^\alpha}x_2^\alpha\Big)\in t^\alpha\mathrm{co}\,A\subseteq t^\alpha\overline{\mathrm{cone}(A)}=\overline{\mathrm{cone}(A)}.$$

Hence, $\lambda x_1+(1-\lambda)x_2\in\overline{\mathrm{cone}(A)}$. This proves the first implication; the next one results from the inclusion $A\subseteq\overline{\mathrm{cone}}(A)$, and the remaining implication to close the circle is a consequence of $\alpha A+(1-\alpha)A\subseteq\mathrm{co}(A)$.
(b): Let x_1,x_2 be in $\mathrm{cone}(A)$ and $\lambda\in\ ]0,1[$. Then $x_i=t_ik_i$ for some $t_i>0$ (if $t_i=0$

for some i, there is nothing to prove), $k_i \in A$. Hence, setting $t = \lambda t_1 + (1-\lambda)t_2 > 0$, we have

$$\lambda x_1 + (1-\lambda)x_2 = \lambda t_1 k_1 + (1-\lambda)t_2 k_2 = t\Big(\frac{\lambda t_1}{t}k_1 + \frac{(1-\lambda)t_2}{t}k_2\Big) \in \mathrm{cone}(A),$$

proving the first implication; the second one follows from the inclusion $A \subseteq \mathrm{cone}(A)$, whereas the remaining implication is straightforward.
(c): The only implication to close the circle we have to check corresponds to the first one, and it is a consequence of (b) since $\mathrm{cone}(A)$ is convex if and only if $\mathrm{cone}_+(A)$ is convex. □

Corollary 1. *Let $\emptyset \neq A \subseteq Y$, P be a convex cone. The following assertions hold:*

(a) $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}_+(A+P) \quad \forall\ \alpha \in\ ]0,1[\iff \mathrm{cone}_+(A+P)$ *is convex* $\iff \mathrm{co}(A) \subseteq \mathrm{cone}_+(A+P)$;
(b) $\alpha A + (1-\alpha)A + P \subseteq \mathrm{cone}(A+P) \quad \forall\ \alpha \in\ ]0,1[\iff \mathrm{cone}(A+P)$ *is convex* $\iff \mathrm{co}(A) + P \subseteq \mathrm{cone}(A+P)$;
(c) $\alpha A + (1-\alpha)A + \mathrm{int}\,P \subseteq \mathrm{cone}_+(A+\mathrm{int}\,P) \ \forall\ \alpha \in\ ]0,1[\iff \mathrm{cone}_+(A+\mathrm{int}\,P)$ *is convex* $\iff \mathrm{co}(A) + \mathrm{int}\,P \subseteq \mathrm{cone}_+(A+\mathrm{int}\,P)$;
(d) $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}_+(A+\mathrm{int}\,P) \ \forall\ \alpha \in\ ]0,1[\iff \mathrm{cone}_+(A+\mathrm{int}\,P)$ *is convex* $\iff \mathrm{co}(A) \subseteq \mathrm{cone}_+(A+\mathrm{int}\,P)$;
(e) $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}(A) + P \quad \forall\ \alpha \in\ ]0,1[\iff \mathrm{cone}(A) + P$ *is convex* $\iff \mathrm{co}(A) \subseteq \mathrm{cone}(A) + P$;
(f) $\alpha A + (1-\alpha)A \subseteq \overline{\mathrm{cone}}(A+P) \ \forall\ \alpha \in\ ]0,1[\iff \overline{\mathrm{cone}}(A+P)$ is convex $\iff \mathrm{co}(A) \subseteq \overline{\mathrm{cone}}(A+P)$.

Proof. (a), (b) follow from (c) and (b), respectively, of the previous proposition applied to $A+P$; (c) is a consequence of (c) by taking $A + \mathrm{int}\,P$.
(e): One implication for the second equivalence results from the inclusions $A \subseteq \mathrm{cone}(A) \subseteq \mathrm{cone}(A) + P$; whereas the other follows from the following (use Proposition 1(c))

$$\mathrm{co}(\mathrm{cone}(A)+P) = \mathrm{co}(\mathrm{cone}(A)) + P = \mathrm{cone}(\mathrm{co}(A)) + P$$

$$\subseteq \mathrm{cone}(\mathrm{cone}(A)+P) + P \subseteq \mathrm{cone}(A) + P.$$

The first equivalence is straighforward.
(f): It comes from (a) of the previous proposition applied to $A+P$ and the fact that $P + \overline{\mathrm{cone}}(A+P) \subseteq \overline{\mathrm{cone}}(A+P)$. □

Part (f) already appeared in [32].

Remark 2. From Proposition 1(h)), we obtain

$$\overline{\mathrm{cone}}(A+\mathrm{int}\,P) = \overline{\mathrm{cone}(A)+P} = \overline{\mathrm{cone}}(A+P). \tag{6}$$

The next proposition gives us a way for finding a sufficient condition to get $\mathrm{co}(A) \cap (-\mathrm{int}\,P) = \emptyset$, say, the convexity of $\overline{\mathrm{cone}}(A+P)$.

Proposition 3. *Let $A \subseteq Y$ be a nonempty set and $P \subsetneq Y$ be a convex cone such that* $\operatorname{int} P \neq \emptyset$. *The following assertions hold:*

(a) $A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow \operatorname{cone}_+(A) \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow \overline{A} \cap (-\operatorname{int} P) = \emptyset$;
(b) $A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow A_0 \cap (-\operatorname{int} P) = \emptyset,\ \forall A_0,\ A + \operatorname{int} P \subseteq A_0 \subseteq \operatorname{cone}_+(A+P)$;

Proof. It is straightforward. □

Remark 3. On combining (a) and (b), we obtain

$$A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow B \cap (-\operatorname{int} P) = \emptyset,$$

for $B = A + \operatorname{int}\ P, A+P, \operatorname{cone}_+(A), \operatorname{cone}(A)+P, \operatorname{cone}(A+P), \operatorname{cone}_+(A+\operatorname{int}\ P)$, $\operatorname{cone}(A)$, $\operatorname{cone}(A) + \operatorname{int} P$, and certainly all of their closures.

The case when P is a halfspace deserves a special formulation.

Lemma 1. *Let $P \subsetneq Y$ be a closed and convex cone satisfying* $\operatorname{int} P \neq \emptyset$. *The following assertions are equivalent:*

(a) $P = Y \setminus -\operatorname{int} P$;
(b) $P \cup (-P) = Y$;
(c) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ P = \Big\{ p \in Y : \langle p^*, p \rangle \geq 0 \Big\}$.

Proof. $(a) \Longrightarrow (b)$: Obviously $P \cup (-P) \subseteq Y$. Take any $y \in Y \setminus P$, then by assumption $y \in -\operatorname{int} P \subseteq -P$, as required.
$(b) \Longrightarrow (c)$: Since $P \neq Y$, we take $p_0 \notin P$. Then, by an usual separation theorem for convex sets, there exist $p^* \in Y^*$, $p^* \neq \mathbf{0}$, $\alpha \in \mathbb{R}$, such that

$$\langle p^*, p_0 \rangle < \alpha < \langle p^*, p \rangle \ \ \forall\, p \in P.$$

Hence $\alpha < 0$ and therefore $\langle p^*, p \rangle \geq 0$ for all $p \in P$, showing that $p^* \in P^* \setminus \{\mathbf{0}\}$ and

$$P \subseteq \Big\{ p \in Y :\ \langle p^*, p \rangle \geq 0 \Big\}. \tag{7}$$

Assume now that there exists $p \in Y \setminus P$ such that $\langle p^*, p \rangle \geq 0$. Since P is closed, there exists $\varepsilon > 0$ such that $p - \varepsilon p_0 \in Y \setminus P \subseteq -P$. Thus,

$$0 \leq \langle p^*, -p \rangle + \varepsilon \langle p^*, p_0 \rangle \leq \varepsilon \langle p^*, p_0 \rangle < 0,$$

reaching a contradiction. This proves the reverse inclusion in (7), which completes the proof of (c).
$(c) \Longrightarrow (a)$: Simply take into account that in this case $\operatorname{int} P = \{ p \in Y : \langle p^*, p \rangle > 0\}$. □

If P is a halfspace we obtain an alternative theorem whatever the set A satisfies $A \cap (-\operatorname{int} P) = \emptyset$, as the following result shows.

Corollary 2. *Let $A \subseteq Y$ be any nonempty set, and $P \subsetneq Y$ be a closed convex cone satisfying $P \cup (-P) = Y$. Then,* $\text{int } P \neq \emptyset$*,* $P = Y \setminus -\text{int } P$*, and $A + P$ is convex and so $A + \text{int } P$ is also convex. Consequently, the sets* $\text{cone}(A+P)$, $\text{cone}(A + \text{int } P)$, $\text{cone}(A) + P$*, are convex. Furthermore, either* $\overline{\text{cone}}(A+P) = P$*, or* $\text{cone}_+(A + \text{int } P) = Y$ *and therefore* $\overline{\text{cone}}(A+P) = Y$.
Moreover, the following assertions are equivalent.

(a) $A \cap (-\text{int } P) = \emptyset$;
(b) $A \subseteq P$;
(c) $\text{co}(A) \subseteq P$;
(d) $\text{co}(A) \cap (-\text{int } P) = \emptyset$;
(e) $\text{co}(A + \text{int } P) \subseteq \text{int } P$.

Proof. Obviously $Y \setminus P \subseteq -P$, and since P is closed, we conclude that $\text{int } P \neq \emptyset$. Let $a_i \in A$, $i = 1,2$. We may assume $a_1 \in a_2 + P$. Since $a_2 + P$ is convex, we obtain that $[a_1, a_2] \subseteq a_2 + P$. Thus, $[a_1,a_2] + P \subseteq a_2 + P + P \subseteq a_2 + P$. This proves the convexity of $A+P$, and so $\text{int}(A+P) = A + \text{int } P$ is also convex. The convexity of $\text{cone}(A)+P$ is a consequence of Corollary 1(e) since $\text{cone}_+(A+P)$ is convex.
Let us prove the last part. By (g) of Proposition 1, $P \subseteq \overline{\text{cone}}(A+P)$. If $\overline{\text{cone}}(A+P) \setminus P \neq \emptyset$, then there exists $x \in Y \setminus P = -\text{int } P$ and nets $\{t_\alpha\}_{\alpha\in\Lambda}$, $\{a_\alpha\}_{\alpha\in\Lambda}$, $\{p_\alpha\}_{\alpha\in\Lambda}$ satisfying $t_\alpha > 0$, $a_\alpha \in A$, $p_\alpha \in P$ such that $t_\alpha(a_\alpha + p_\alpha) \to x$. Thus, we may assume $t_\alpha(a_\alpha + p_\alpha) \in -\text{int } P$ for all $\alpha \in \Lambda$. This implies that $0 \in A + \text{int } P$. It turns out that $\text{cone}_+(A + \text{int } P) = Y$.
The equivalences between (a), (b), (c) and (d), follow from the fact $P = Y \setminus -\text{int } P$ (see the previous lemma). Clearly (b) implies (e); let us prove (a) from (e): if $x \in A \cap (-\text{int } P)$ then

$$\mathbf{0} = x + (-x) \in A + \text{int } P \subseteq \text{co}(A + \text{int } P) \subseteq \text{int } P.$$

Thus, $\mathbf{0} \in \text{int } P$, which implies that $P = Y$, a contradiction. □

3 Equivalent formulations of Gordan-type alternative theorems

The main goal of this section is to establish equivalent formulations of Gordan-type alternative theorems valid for (not necessarily pointed or closed) convex cones with possibly empty interior. This will be carried out via quasi relative and topological interior.

We recall the definition of pointedness for a cone that is not necessarily convex (see for instance [30]).

Definition 2. A cone $K \subseteq Y$ is called *pointed* if $x_1 + \cdots + x_k = \mathbf{0}$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \cdots = x_k = \mathbf{0}$.

It is easy to see that a cone K is pointed if, and only if $\text{co}(K) \cap (-\text{co}(K)) = \{\mathbf{0}\}$ if, and only if $\mathbf{0}$ is a extremal point of $\text{co}(K)$.

3.1 Via quasi-relative interior

We start by noticing that

$$\operatorname{qri}(\operatorname{cone}(\operatorname{co}(A)+P)) \subseteq \operatorname{qri}(\overline{\operatorname{cone}}(\operatorname{co}(A)+P)). \tag{8}$$

Next theorem subsumes most alternative theorems existing in the literature.

Theorem 1. *Let* $\emptyset \neq A \subseteq Y \neq \{\mathbf{0}\}$ *and* P *be a convex cone such that*

$$\operatorname{qri}(\operatorname{cone}(\operatorname{co}(A)+P)) \neq \emptyset \neq \operatorname{qri}[\operatorname{co}((A+P)\cup\{0\})].$$

Let us consider the following statements:

(a) $\mathbf{0} \notin \operatorname{qri}(\overline{\operatorname{cone}}(\operatorname{co}(A)+P))$;
(b) $\mathbf{0} \notin \operatorname{qri}[\operatorname{co}((A+P)\cup\{\mathbf{0}\})]$;
(c) $\mathbf{0} \notin \operatorname{qri}(\operatorname{cone}(\operatorname{co}(A)+P))$;
(d) $\exists\ p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \ \ \forall\ a \in A$, *with strict inequality for some* $\tilde{a} \in \operatorname{co}(A)+P$.

In case $\operatorname{qi}(\operatorname{co}(A)+P) \neq \emptyset$, *consider also*

(e) $\mathbf{0} \notin \operatorname{qri}(\operatorname{co}(A)+P)$;
(f) $\operatorname{cone}(\operatorname{qri}(\operatorname{co}(A)+P))$ *is pointed.*

The following hold:

$$(a) \Longleftrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d) \Longrightarrow (e) \Longrightarrow (f) \Longrightarrow (g).$$

Proof. The first two equivalences are a consequences of the following equalities:

$$\overline{\operatorname{cone}}[\operatorname{co}((A+P)\cup\{\mathbf{0}\})] = \overline{\operatorname{co}[\operatorname{cone}((A+P)\cup\{\mathbf{0}\})]} = \overline{\operatorname{co}[\operatorname{cone}(A+P)]} = \overline{\operatorname{cone}}[\operatorname{co}(A+P)]$$

$$= \overline{\operatorname{cone}}[\operatorname{cone}(\operatorname{co}(A)+P)] = \overline{\operatorname{cone}}[\overline{\operatorname{cone}}(\operatorname{co}(A)+P)].$$

$(c) \Longleftrightarrow (d)$: See the proof of Proposition 2.16 in [5].
$(c) \Longrightarrow (e)$: It is obvious.
$(e) \Longrightarrow (f)$: Let $x,\ -x \in \operatorname{cone}(\operatorname{qi}(\operatorname{co}(A)+P))$, $x \neq \mathbf{0}$. Thus, $x,\ -x \in \operatorname{cone}_+(\operatorname{qri}(\operatorname{co}(A)+P)$. Then

$$\mathbf{0} = \frac{1}{2}x + \frac{1}{2}(-x) \in \operatorname{cone}_+(\operatorname{qri}(\operatorname{co}(A)+P)).$$

Hence, $\mathbf{0} \in \operatorname{qri}(\operatorname{co}(A)+P)$, proving the desired implication. □

Remark 4. Assume that $\operatorname{qi} P \neq \emptyset$. Since $\operatorname{co}(A) + \operatorname{qi} P \subseteq \operatorname{qi}(\operatorname{co}(A)+P)$, then

$$\begin{array}{ccc} \operatorname{cone}(\operatorname{qi}(\operatorname{co}(A)+P)) \text{ is pointed} & \Longrightarrow & \operatorname{cone}(\operatorname{co}(A)+\operatorname{qi} P) \text{ is pointed} \\ & & \Updownarrow \\ \operatorname{co}(A) \cap (-\operatorname{qi} P) = \emptyset & \Longleftrightarrow & \operatorname{cone}(A+\operatorname{qi} P) \text{ is pointed} \end{array}$$

where the last equivalence comes from [14].

By observing that

$$\mathrm{qi}[\mathrm{co}((A+P)\cup\{\mathbf{0}\})]\subseteq \mathrm{qi}(\mathrm{cone}(\mathrm{co}(A)+P)), \tag{9}$$

the preceding theorem implies the following result

Corollary 3. *Let $\emptyset \neq A \subseteq Y \neq \{0\}$ and P be a convex cone such that*

$$\mathrm{qi}[\mathrm{co}((A+P)\cup\{0\})]\neq\emptyset.$$

The following assertions are equivalent:

(a) $\mathbf{0}\notin \mathrm{qi}(\overline{\mathrm{cone}}(\mathrm{co}(A)+P))$;
(b) $\mathbf{0}\notin \mathrm{qi}[\mathrm{co}((A+P)\cup\{\mathbf{0}\})]$;
(c) $\mathbf{0}\notin \mathrm{qi}(\mathrm{cone}(\mathrm{co}(A)+P))$;
(d) $\exists\, p^*\in P^*\setminus\{\mathbf{0}\}$ *such that* $\langle p^*,a\rangle\geq 0,\ \forall\, a\in A$.

3.2 Via topological interior

Before establishing a similar result for topological interior, we state the following properties sharing by convex cones.

Proposition 4. *Let $\emptyset \neq A \subseteq Y$. Let $P \subsetneq Y$ be a convex cone. The following assertions hold.*

(a) $\mathrm{cone}_+(\mathrm{int}(A+P))\subseteq \mathrm{int}(\mathrm{cone}_+(A+P))$; *the equality holds provided* $\mathrm{int}\,P\neq\emptyset$.
(b) $\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P)))=\mathrm{int}(\mathrm{co}(\mathrm{cone}_+(A+P)))=\mathrm{int}(\mathrm{co}(\mathrm{cone}(A+P)))=$

$$=\mathrm{int}(\overline{\mathrm{cone}}(\mathrm{co}(A)+P)).$$

(c) $\overline{\mathrm{cone}}(A+P)=\overline{\mathrm{cone}}((A\cup\{\mathbf{0}\})+P)$;
(d) *If* $\overline{\mathrm{cone}}(A+P)$ *is convex then* $\overline{\mathrm{cone}}(\mathrm{co}(A)+P)=\overline{\mathrm{cone}}(A+P)$.

Proof. (a): The inclusion is immediate. For the other, take any $x\in\mathrm{int}(\mathrm{cone}_+(A+P))$ and $v\in\mathrm{int}\,P$, we can choose $\varepsilon>0$ such that $x-\varepsilon v\in\mathrm{cone}_+(A+P)$. It follows easily that $x\in\mathrm{cone}_+(\mathrm{int}(A+P))=\mathrm{cone}_+(A+\mathrm{int}\,P)$ by Proposition 1(e), proving our claim.
(b): It follows from the following chain of inclusions:

$$\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P)))\subseteq\mathrm{int}(\overline{\mathrm{cone}}(\mathrm{co}(A+P)))=\mathrm{int}(\overline{\mathrm{cone}_+}(\mathrm{co}(A+P)))=$$

$$\mathrm{int}(\mathrm{cone}_+(\mathrm{co}(A+P)))\subseteq\mathrm{int}(\mathrm{cone}(\mathrm{co}(A+P)))=\mathrm{int}(\mathrm{co}(\mathrm{cone}(A+P)))\subseteq\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P))).$$

(c): This follows from the fact $P\subseteq\overline{\mathrm{cone}}(A+P)$ by Proposition 1(d). □

Next example shows an instance where the inclusion in (a) of the previous proposition may be strict if $\mathrm{int}\,P=\emptyset$ but $\mathrm{int}(A+P)\neq\emptyset$; the second instance shows we cannot delete the closures in (c).

Example 1. Let us consider in $\mathbb{R}^2$, the cone $P = \{(t,0) \in \mathbb{R}^2 : t \geq 0\}$.

(a) Let $A = \text{co}(\{(0,1),(0,0)\}) \cup \{(-1,1)\}$. It is easy to see that $\text{int}(A+P) \neq \emptyset$, $\text{cone}_+(\text{int}(A+P)) \subsetneq \text{int}(\text{cone}_+(A+P))$. See Figure 1.
(b) Take $A = \{(0,1),(0,2)\}$. Then, we obtain $\text{cone}(A+P) \subsetneq \text{cone}((A \cup \{0\})+P)$.

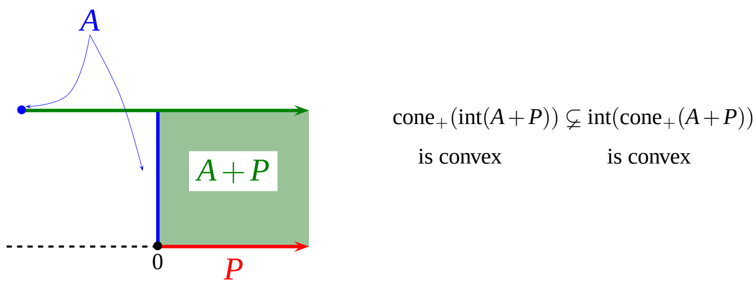


Fig. 1 Example 1(a)

Next result is the analogue to Theorem 1 when topological interior is employed. Likewise, it allows us to deal with cones having possibly empty interior.

Theorem 2. *Let $\emptyset \neq A \subseteq Y \neq \{\mathbf{0}\}$ and P be a convex cone such that*

$$\text{int}[\text{co}((A+P) \cup \{\mathbf{0}\})] \neq \emptyset.$$

The following statements are equivalent:

(a) $\mathbf{0} \notin \text{int}(\overline{\text{cone}}(\text{co}(A)+P))$;
(b) $\mathbf{0} \notin \text{int}(\text{co}(\overline{\text{cone}}(A+P)))$;
(c) $\mathbf{0} \notin \text{int}(\text{co}(\text{cone}(A+P)))$;
(d) $\mathbf{0} \notin \text{int}[\text{co}((A+P) \cup \{\mathbf{0}\})]$;
(e) $\mathbf{0} \notin \text{int}(\text{co}(\text{cone}_+(A+P)))$;
(f) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \;\; \forall\, a \in A$.

In case $\text{int}(\text{co}(A)+P) \neq \emptyset$, *consider also*

(g) $\mathbf{0} \notin \text{int}(\text{co}(A)+P)$;
(h) $\text{cone}(\text{int}(\text{co}(A)+P))$ *is pointed.*

In case $\text{int}\, P \neq \emptyset$, $(h) \iff \text{cone}(A+\text{int}\, P)$ is pointed; $(g) \iff \text{co}(A) \cap (-\text{int}\, P) = \emptyset$.

Proof. The equivalences between (a), (b), (c), (d), (e) and (f) follows from Corollary 3 and Theorem 1.
$(h) \Longrightarrow (g)$: If $0 \in \text{int}(\text{co}(A)+P)$, then it easy to check that $Y = \text{cone}_+(\text{int}(\text{co}(A)+P))$.
$(g) \Longrightarrow (f)$: It is a consequence of a standar convex separation theorem.

We now prove the last equivalence in case $\text{int}\, P \neq \emptyset$. Indeed, from (a) and (e) of Proposition 1, it follows that $\text{cone}(\text{int}(\text{co}(A)+P)) = \text{co}(\text{cone}(A+\text{int}\, P))$. Taking into the account the remark made after Definition 2, the result follows □

The alternative theorems proved in [33, 43, 42], [10, Theorem 1.79] (where A is the image of a vector-valued function) are consequences of the following result.

Theorem 3. *Assume that* $\text{int}(\text{cone}_+(A+P)) \neq \emptyset$ *and*

$$\text{int}(\text{co}(\overline{\text{cone}}(A+P))) = \text{int}(\text{cone}_+(A+P)). \tag{10}$$

Then, exactly one of the following assertions holds:

(a) $\mathbf{0} \in \text{int}(\text{cone}_+(A+P))$;
(b) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \ \ \forall\, a \in A$.

Proof. It is a consequence of the first part of Theorem 2. □

Some results from [16], where $\text{int}\, P = \emptyset$, are also recovered.
When $\text{int}\, P \neq \emptyset$, the convexity of $\overline{\text{cone}}(A+P)$, or equivalently, of $\text{cone}_+(A+\text{int}\, P)$, implies that (10) is fulfilled, by virtue of Propositions $4(b)$ and $1(k)$. This yields the following result, which already appears in [33, 43, 42], [10, Theorem 1.79] (where A is the image of a vector-valued function).

Corollary 4. *Assume that* $\text{int}\, P \neq \emptyset$. *If* $\overline{\text{cone}}(A+P)$ *is convex, then, exactly one of the following assertions holds:*

(a) $A \cap (-\text{int}\, P) \neq \emptyset$;
(b) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \ \ \forall\, a \in A$.

An example showing the convexity of $\overline{\text{cone}}(A+P)$ is not necessary for the validity of the previous alternative theorem, is exhibited in [14].

Let us consider in addition to $F: C \to Y$ and a closed convex cone $P \subsetneq Y$ with $\text{int}\, P \neq \emptyset$, another mapping $G: C \to Z$, with Z being another real locally convex topological vector space and a closed convex cone $Q \subsetneq Z$.

Corollary 5. ([33]) *Assume that* $\overline{\text{cone}}((F\times G)(C)+(P\times Q))$ *is convex and*

$$\text{int}(\overline{\text{cone}}((F\times G)(C)+(P\times Q))) = \text{int}(\text{cone}_+((F\times G)(C)+(P\times Q))) \neq \emptyset.$$

If the following system is inconsistent:

$$x \in C,\ F(x) \in -\text{int}\, P,\ G(x) \in -Q,$$

then there exists $(p^*, q^*) \in (P^* \times Q^*) \setminus \{(\mathbf{0},\mathbf{0})\}$ *such that*

$$\langle p^*, F(x)\rangle + \langle q^*, G(x)\rangle \geq 0 \ \ \forall\, x \in C.$$

The converse assertion is true if $p^* \neq \mathbf{0}$.

Proof. If the above system has no solution, then

$$(0,0) \notin \operatorname{int}(\operatorname{cone}_+((F\times G)(C)+(P\times Q))).$$

Then, Theorem 2 applies. □

A standar constraint qualification implying $p^* \neq \mathbf{0}$ is $\overline{\operatorname{cone}}(G(C)+Q)=Z$. When $\operatorname{int} Q \neq \emptyset$ the latter is implied by the condition: $G(x_0) \in -\operatorname{int} Q$ for some $x_0 \in C$.

In view of previous results, the following notion arise in a natural way. It seems to be the most general among the relaxed notions of convexity that were used in alternative theorems.

Definition 3. Let $P \subseteq Y$ be a closed convex cone with nonempty interior. A set $A \subseteq Y$ is called *nearly subconvexlike* if $\overline{\operatorname{cone}}(A+P)$ is convex.

The previous notion was introduced originally in [42] when A is the image of set-valued mappings, and further developed in [32].

Proposition $2(a)$ provides a characterization of near subconvexlikeness already appeared in [32]. When $\operatorname{int} P \neq \emptyset$, several necessary and sufficient conditions for having near subconvexlikeness appear in [10, Proposition 1.76] and [14, Proposition 3.5]. In particular, the presubconvexlikeness which is a transcription of an analogous definition for vector-valued functions given in [45], is nothing else that nearly subconvexlikeness, see Proposition 6 below. We also know that (Proposition $1(k)$)

$$\overline{\operatorname{cone}}(A+P) \text{ is convex} \Longleftrightarrow \operatorname{cone}_+(\operatorname{int}(A+P)) \text{ is convex.}$$

However, if $\operatorname{int} P = \emptyset$ but $\operatorname{int}(A+P) \neq \emptyset$, one can show that there is not any relationship between the convexity of $\operatorname{cone}_+(\operatorname{int}(A+P))$ and the convexity of $\overline{\operatorname{cone}}(A+P)$, see Figures 2 and 3.

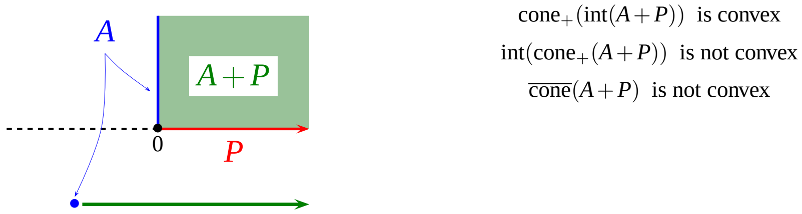


Fig. 2 $\operatorname{cone}_+(\operatorname{int}(A+P))$ convex $\not\Rightarrow$ $\overline{\operatorname{cone}}(A+P)$ convex

Another interesting class of mappings arising in deriving alternative theorems is the following. Given a convex set $C \subseteq X$, with X being a locally convex topological vector space, a mapping $F : C \to Y$ is called $*$-*quasiconvex* [22] if $\langle x^*, F(\cdot)\rangle$ is quasiconvex for all $x^* \in P^*$. It is called *naturally-P-quasiconvex* [38] if for all $x,y \in C$, $F([x,y]) \subseteq [F(x),F(y)]-P$. Both classes coincide as shows in [14, Proposition 3.9], [15, Theorem 2.3]. It is still valid if P has empty interior.

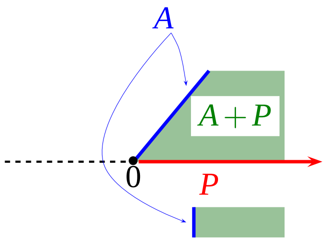


Fig. 3 $\overline{\text{cone}}(A+P)$ convex $\not\Rightarrow$ $\text{cone}_+(\text{int}(A+P))$ convex

In [22] it is proven that a Gordan-type alternative theorem holds for $A = F(C)$ under the $*$-quasiconvexity of F and the assumption

$$\forall p^* \in P^*, \text{ the restriction of } \langle p^*, F(\cdot)\rangle \text{ on any line segment of } C \text{ is lower semicontinuous.} \tag{11}$$

We will see the naturally P-quasiconvexity of F along with (11) imply the convexity of $F(C)+P$; in particular, F is nearly subconvexlike, and so the alternative theorems of [22] and [38] are consequences from Theorem 2.

Proposition 5. ([14]) *Let $\emptyset \neq C \subseteq X$ be any convex set, $\emptyset \neq P \subsetneq Y$ be a closed convex cone and $F : C \to Y$ be naturally-P-quasiconvex and satisfying* (11). *Then*

$$\forall\, x,y \in C,\;\; [F(x),F(y)] \subseteq F([x,y]) + P. \tag{12}$$

Consequently, $F(C)+P$ is convex.

Next result suplements Proposition 1.76 in [10] and Proposition 3.5 in [14].

Proposition 6. *Let $\emptyset \neq A \subseteq Y$, $P \subseteq Y$ be a convex convex with* int $P \neq \emptyset$. *The following assertions are equivalent.*

(a) *A is nearly subconvexlike;*
(b) $\text{cone}_+(\text{int}(A+P))$ *is convex;*
(c) $\text{cone}(A) + \text{int}\, P$ *is convex;*
(d) $\exists\, u \in \text{int}\, P$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\;]0,1[$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that*

$$\varepsilon u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P; \tag{13}$$

(e) $\exists\, u \in Y$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\;]0,1[$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that* (13) *holds;*

(f) $\forall\, u \in \operatorname{int} P$, $\forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[$, $\exists\, \rho > 0$ *such that*

$$u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P;$$

(g) $\forall\, u \in \operatorname{int} P$, $\forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[$, $\exists\, \rho > 0$ *such that*

$$u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + \operatorname{int} P;$$

(h) $\forall\, x_1,\ x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\exists\, u \in \operatorname{int} P$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that* (13) *holds.*

Proof. From Proposition 1 we get the equivalences between (a), (b) and (c). The equivalences $(a) \Longleftrightarrow (d) \Longleftrightarrow (e)$ are proved in Proposition 3.5 in [14], whereas $(c) \Longleftrightarrow (g)$ is proved in Proposition 1.76 in [10]. The remaining implications $(g) \Longrightarrow (f) \Longrightarrow (d) \Longrightarrow (h) \Longrightarrow (g)$ are straightforward. □

Remark 5. Assertion (d) refers to the notion of generalized subconvexlikeness introduced in [41], see also [43]; whereas (e) corresponds to the notion of presubconvexlikeness which is a transcription of an analogous definition for vector-valued functions given in [45].

Proposition 5 shows that any naturally-P-quasiconvex function satisfying (11) is nearly-subconvexilke. One can give some examples showing the converse is not true in general, see [14].

4 A bidimensional optimal alternative theorem and a characterization of two-dimensionality

The bidimensional setting deserves a special treatment since, as we will see, the convexity of $\overline{\text{cone}}(A+P)$ is not only sufficient (see Theorem 3) but also a necessary condition to have a Gordan-type alternative theorem. In such a case, we refer it as an optimal alternative theorem, valid for convex cones with possibly empty interior under a regularity assumption. This is expressed in the next theorem.

Theorem 4. *Let* $P \subseteq \mathbb{R}^2$ *be a convex cone,* $A \subseteq \mathbb{R}^2$ *such that* $\operatorname{int}(\text{cone}_+(A+P)) \neq \emptyset$, *and*

$$\operatorname{int}(\overline{\text{cone}}(A+P)) = \operatorname{int}(\text{cone}_+(A+P)). \tag{14}$$

The following assertions are equivalent:

(a) $\mathbf{0} \notin \operatorname{int}(\text{cone}_+(A+P))$ *and* $\overline{\text{cone}}(A+P)$ *is convex;*
(b) $\mathbf{0} \notin \operatorname{int}(\text{cone}_+(A+P))$ *and* $\text{cone}_+(A+P)$ *is convex, provided* $\mathbf{0} \in A+P$;
(c) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0\ \ \forall\, a \in A$.

Notice that when $\operatorname{int} P \neq \emptyset$ *condition* (14) *is superfluous.*

Proof. $(a) \Longrightarrow (c)$: It follows from Theorem 3; $(b) \Longrightarrow (a)$ is evident.
$(c) \Longrightarrow (a)$, $(c) \Longrightarrow (b)$: (We do not need (14)) The first part of (a) is a consequence of Theorem 2(b). For the second part, we re-write the proof of [14, Theorem 4.1] with obvious changes. Certainly, $\langle p^*, a\rangle \geq 0$ for all $a \in \text{cone}(A+P)$. Choose $u \in P \setminus \{0\}$. Let $y, z \in A$. Then obviously

$$\text{cone}(\{y\}) + \text{cone}(\{u\}) = \{\lambda y + \mu u : \lambda, \mu \geq 0\}$$

is a closed convex cone containing y and u and contained in $\overline{\text{cone}}(A+P)$ (if $\mathbf{0} \in A+P$, it is contained in $\text{cone}_+(A+P)$). The same is true for the cone $\text{cone}(\{z\}) + \text{cone}(\{u\})$. The two cones have the line $\text{cone}(\{u\})$ in common and their union is contained in $\overline{\text{cone}}(A+P)$, thus it is contained in the halfspace $\left\{x \in \mathbb{R}^2 : \langle p^*, x\rangle \geq 0\right\}$. Hence, the set $B \doteq (\text{cone}(\{y\}) + \text{cone}(\{u\})) \cup (\text{cone}(\{z\}) + \text{cone}(\{u\}))$ is a convex cone. Since $y, z \in B$ we deduce that $[y, z] \subseteq B \subseteq \overline{\text{cone}}(A+P)$. Thus $\alpha A + (1-\alpha)A \subseteq \overline{\text{cone}}(A+P)$ for all $\alpha \in \ ]0,1[$, proving the convexity of $\overline{\text{cone}}(A+P)$ by Corollary 1(f). In case $0 \in A+P$, we get $\alpha A + (1-\alpha)A \subseteq \text{cone}_+(A+P)$, proving the convexity of $\text{cone}_+(A+P)$ by Corollary 1(a).

□

When $\text{int}\, P \neq \emptyset$ a more precise formulation of the previous theorem may be obtained.

Theorem 5. ([14, *Theorem* 4.1]) *Let* $P \subseteq \mathbb{R}^2$ *be a convex closed cone such that* $\text{int}\, P \neq \emptyset$, *and* $A \subseteq \mathbb{R}^2$ *be any nonempty set satisfying* $A \cap (-\text{int}\, P) = \emptyset$. *Then the following assertions are equivalent:*

(a) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \ \ \forall\, a \in A$;
(b) $\text{cone}(A+P)$ *is convex;*
(c) $\text{cone}(A + \text{int}\, P)$ *is convex;*
(d) $\text{cone}(A) + P$ *is convex;*
(e) $\overline{\text{cone}}(A+P)$ *is convex.*

Next result has its own importance from a functional analysis point of view. Indeed, such a result characterizes the two-dimensionality of any space where a Gordan-type alternative theorem holds.

Theorem 6. ([14, *Theorem* 4.2]) *Let* Y *be a locally convex topological vector space and* $P \subseteq Y$ *be a closed, convex cone such that* $\text{int}\, P \neq \emptyset$ *and* $\text{int}\, P^* \neq \emptyset$. *The following assertions are equivalent:*

(a) *for all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \overline{\text{cone}}(A+P) \ \text{ is convex;}$$

(b) *for all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \text{cone}(A) + P \ \text{ is convex;}$$

(c) *for all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \operatorname{cone}(A + \operatorname{int} P) \text{ is convex;}$$

(d) Y *is at most two-dimensional.*

Remark 6. The assumption $\operatorname{int} P^* \neq \emptyset$ (which corresponds to pointedness of P when Y is finite-dimensional) cannot be removed. Indeed, let $P = \{y \in Y : \langle p^*, y\rangle \geq 0\}$ where $p^* \in Y^* \setminus \{\mathbf{0}\}$. Then $P^* = \operatorname{cone}(\{p^*\})$, $\operatorname{int} P^* = \emptyset$. For any nonempty $A \subseteq Y$, the set $A + P$ is convex by Corollary 2. Thus, (a) in Theorem 6 holds no matter the dimension of the space Y is.

5 Applications to vector optimization

One of the important issues in optimization concerns the characterization of various notions of solutions to vector optimization problems through linear scalarization. This will be done for Benson proper efficiency and weak efficiency in case of bi-criteria problems. For an theoretical treatment of these notions and others solution concepts, we refer the books [26, 20]. The last subsection will be devoted to characterize the Fritz-John optimality condition.

In what follows, for a real-valued function h, by $\operatorname{argmin}_K h$ we mean the set of minima of h on K. Let X be a real vector space, $\emptyset \neq K \subseteq X$, Y be a real normed vector space. Given a vector function $F : K \to Y$ and a convex cone, possibly with empty interior, $P \subseteq Y$, we immediately obtain the following result.

Theorem 7. *Let* $K \subseteq X$, F *as above, and* P *a convex cone. Assume that*

$$\operatorname{int}(\operatorname{co}(F(K)) - F(\bar{x}) + P) \neq \emptyset.$$

The following assertions are equivalent:

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*,\, p^* \neq 0} \operatorname{argmin}_K \langle p^*, F(\cdot)\rangle;$$

(b) $\operatorname{cone}(\operatorname{int}(\operatorname{co}(F(K)) - F(\bar{x}) + P))$ *is pointed;*

(b') *In case* $\operatorname{int} P \neq \emptyset$, $(b) \Longleftrightarrow \operatorname{cone}(F(K) - F(\bar{x}) + \operatorname{int} P)$ *is pointed, as observed in Theorem* 2.

Proof. It follows from Theorem 2 applied to $A = F(K) - F(\bar{x})$. □

5.1 Characterizing weakly efficient solutions through linear scalarization of bicriteria problems

Here, we assume that int $P \neq \emptyset$. We say that $\bar{x} \in K$ is a *weakly efficient point* of F on K, shortly $\bar{x} \in E_W$, if

$$F(x) - F(\bar{x}) \notin -\mathrm{int}\, P, \ \ \forall\, x \in K. \tag{15}$$

Clearly

$$\bar{x} \in E_W \Longleftrightarrow (F(K) - F(\bar{x})) \cap (-\mathrm{int}\, P) = \emptyset. \tag{16}$$

$$\Longleftrightarrow \overline{\mathrm{cone}}(F(K) - F(\bar{x}) + P) \cap (-\mathrm{int}\, P) = \emptyset.$$

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 5.

Theorem 8. *Let $\emptyset \neq K \subseteq X$ and P be a convex cone having nonempty interior with $Y = \mathbb{R}^2$. Then, the following assertions are equivalent:*

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq \mathbf{0}} \mathrm{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\bar{x} \in E_W$ *and* $\mathrm{cone}_+(F(K) - F(\bar{x}) + \mathrm{int}\, P)$ *is convex;*
(c) $\bar{x} \in E_W$ *and* $\mathrm{cone}_+(F(K) - F(\bar{x}) + P)$ *is convex;*
(d) $\bar{x} \in E_W$ *and* $\mathrm{cone}(F(K) - F(\bar{x}) + P)$ *is convex;*
(e) $\bar{x} \in E_W$ *and* $\mathrm{cone}(F(K) - F(\bar{x})) + P$ *is convex.*

5.1.1 The Pareto case

We consider $P = \mathbb{R}^2_+$ and denote $\mathbb{R}^2_{++} \doteq \mathrm{int}\ \mathbb{R}^2_+$. Given a vector mapping $F = (f_1, f_2) : K \to \mathbb{R}^2$, we consider the problem of finding

$$\bar{x} \in K : F(x) - F(\bar{x}) \notin -\mathbb{R}^2_{++}, \quad \forall x \in K. \tag{17}$$

Let $\bar{x} \in E_W$ and for $i = 1, 2$, set

$$S_i^-(\bar{x}) \doteq \{x \in K : \ f_i(x) < f_i(\bar{x})\}; \ S_i^+(\bar{x}) \doteq \{x \in K : \ f_i(x) > f_i(\bar{x})\};$$

$$S_i^=(\bar{x}) \doteq \{x \in K : \ f_i(x) = f_i(\bar{x})\}.$$

Taking into account Theorem 8, we write $F(K) - F(\bar{x}) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$. It follows that

$$\mathrm{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++}) = \mathrm{cone}_+(\Omega_1) \cup \mathrm{cone}_+(\Omega_2) \cup \mathrm{cone}_+(\Omega_3),$$

where

$$\Omega_1 \doteq \bigcup_{x\in S_1^-(\bar{x})} [(f_1(x)-f_1(\bar{x}), f_2(x)-f_2(\bar{x}))+\mathbb{R}^2_{++}];$$

$$\Omega_2 \doteq \bigcup_{x\in S_1^=(\bar{x})} [(f_1(x)-f_1(\bar{x}), f_2(x)-f_2(\bar{x}))+\mathbb{R}^2_{++}];$$

$$\Omega_3 \doteq \bigcup_{x\in S_1^+(\bar{x})} [(f_1(x)-f_1(\bar{x}), f_2(x)-f_2(\bar{x}))+\mathbb{R}^2_{++}].$$

Whenever $S_1^+(\bar{x})\cap S_2^-(\bar{x})\neq\emptyset$ and $S_1^-(\bar{x})\cap S_2^+(\bar{x})\neq\emptyset$, we set

$$\alpha \doteq \inf_{x\in S_1^+(\bar{x})\cap S_2^-(\bar{x})} \frac{f_2(x)-f_2(\bar{x})}{f_1(x)-f_1(\bar{x})},\quad \beta \doteq \sup_{x\in S_1^-(\bar{x})\cap S_2^+(\bar{x})} \frac{f_2(x)-f_2(\bar{x})}{f_1(x)-f_1(\bar{x})}. \tag{18}$$

Clearly, $-\infty\le\alpha<0$ and $-\infty<\beta\le 0$.
The following figures can be obtained directly

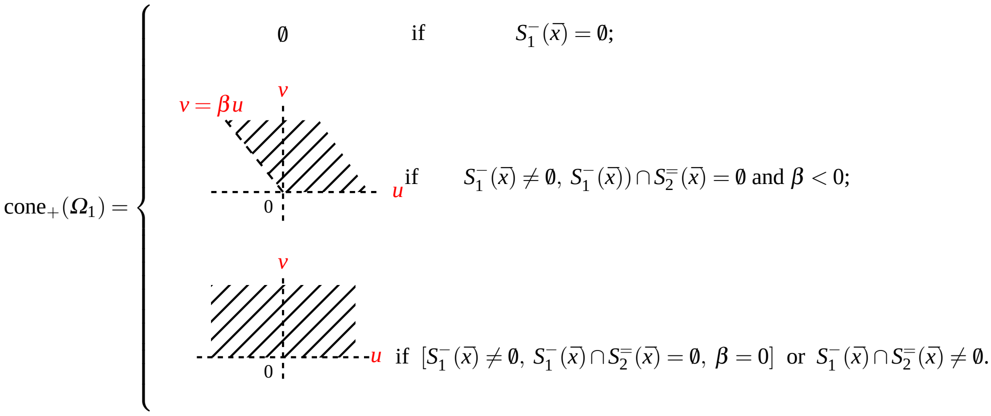


Fig. 4 To visualize Theorem 9

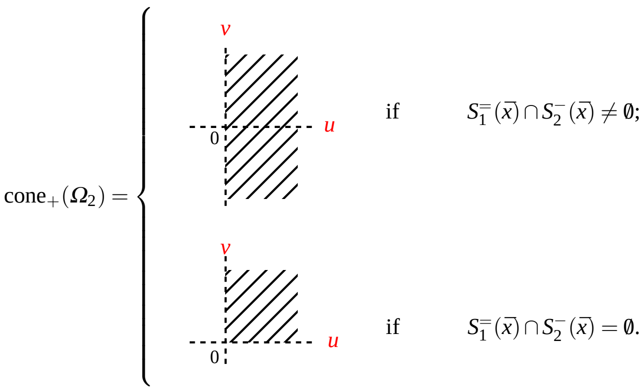
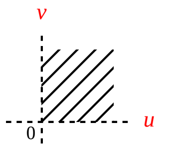


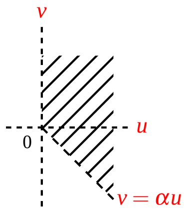
Fig. 5 To visualize Theorem 9

$\mathrm{cone}_+(\Omega_3) = $

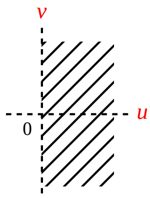
- $\emptyset$ if $S_1^+(\bar{x}) = \emptyset$;

- 

if $S_1^+(\bar{x}) \neq \emptyset$ and $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$;

- 

if $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) \neq \emptyset$ and $\alpha > -\infty$;

- 

if $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) \neq \emptyset$ and $\alpha = -\infty$.

Fig. 6 To visualize Theorem 9

Notice that

$$S_1^-(\bar{x}) \cap S_2^+(\bar{x}) = \emptyset \Longleftrightarrow S_1^-(\bar{x}) \subseteq S_2^=(\bar{x}); \;\; S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset \Longleftrightarrow S_2^-(\bar{x}) \subseteq S_1^=(\bar{x}).$$

The following theorem is immediate from the expressions of $\mathrm{cone}_+(\Omega_i)$, $i = 1,2,3$.

Theorem 9. *Assume that $\bar{x} \in E_W$. Then, $\mathrm{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++})$ is convex if, and only if any of the following assertions hold:*

(a) $S_1^-(\bar{x}) = \emptyset$;
(b) $S_1^-(\bar{x}) \neq \emptyset$, $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) = \emptyset$, $\beta < 0$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

$(b1)$ $S_1^+(\bar{x}) = \emptyset$, *or*
$(b2)$ $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$, *or*
$(b3)$ $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) \neq \emptyset$, $\alpha > -\infty$, $\beta \leq \alpha$;

(c) $S_1^-(\bar{x}) \neq \emptyset$, $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) = \emptyset$, $\beta = 0$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

$(c1)$ $S_1^+(\bar{x}) = \emptyset$, *or*
$(c2)$ $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$;

(d) $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) \neq \emptyset$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

$(d1)$ $S_1^+(\bar{x}) = \emptyset$, *or*
$(d2)$ $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$.

Proof. We omit the long but easy proof once we get Figures 4, 5 and 6. □

We also notice that

$$[S_1^+(\bar{x}) = \emptyset \text{ and } S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset] \Longrightarrow S_2^-(\bar{x}) = \emptyset;$$

and

$$[S_1^+(\bar{x}) \neq \emptyset \text{ and } S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset] \Longrightarrow S_2^-(\bar{x}) = \emptyset.$$

Both implications assert that (b_1) (along with (b)), (b_2) (along with (b)), (c) and (d) of the previous theorem imply $S_2^-(\bar{x}) = \emptyset$. On the other hand,

$$S_i^-(\bar{x}) = \emptyset \Longleftrightarrow \bar{x} \in \operatorname{argmin}_K f_i.$$

Thus, next corollary, which follows from (b_3) (along with (b)) of Theorem 9, excludes situations the other situations of such a theorem.

Corollary 6. *Let us consider problem* (17) *and assume that* $\bar{x} \notin \operatorname{argmin}_K f_i$, $i = 1, 2$. *Then,*

(a)

$$\bar{x} \in \bigcup_{(p_1^*, p_2^*) \in \mathbb{R}_+^2 \setminus \{(0,0)\}} \operatorname{argmin}_K (p_1^* f_1 + p_2^* f_2)$$

if and only if $\bar{x} \in E_W$ *and* (b_3) *(along with* (b)*) of Theorem 9 is satisfied.*

(b) *If* $\bar{x} \in E_W$ *and* (b_3) *(along with* (b)*) holds, then any* $-\alpha \leq p_1^* \leq -\beta$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K (p_1^* f_1 + f_2).$$

5.2 Characterizing properly efficient solutions through linear scalarization of bicriteria problems

We say that $\bar{x} \in K$ is (Benson) *properly efficient point* of F on K ([2]), in short $\bar{x} \in E_{pr}$, if

$$\overline{\text{cone}}(F(K) - F(\bar{x}) + P) \cap (-P) = \{0\}. \tag{19}$$

One can easily check that if E_{pr} is nonempty, then P is pointed.

Setting

$$P^{*i} \doteq \Big\{ p^* \in Y^*,\ \langle p^*, p \rangle > 0,\ \ \forall\, p \in P \setminus \{0\} \Big\},$$

it can be seen that

$$\bigcup_{p^* \in P^{*i}} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle \subseteq E_{pr}. \tag{20}$$

Conversely, if $\bar{x} \in E_{pr}$ and $\overline{\text{cone}}(F(K) - F(\bar{x}) + P)$ is convex then

$$\bar{x} \in \bigcup_{p^* \in P^{*i}} \text{argmin}_K \langle p^*, F(\cdot) \rangle,$$

provided P is locally compact (use the separation result for convex cones [3, Proposition 3]).

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 4 and the remarks above.

Theorem 10. *Let $K \subseteq X$ be a convex set and F as above with $P \subseteq \mathbb{R}^2$ being a pointed, closed, convex cone. Assume that*

$$\text{int}(F(K) - F(\bar{x}) + P) \neq \emptyset.$$

Then, the following assertions are equivalent:

(a)

$$\bar{x} \in \bigcup_{p^* \in \text{int } P^*} \text{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\bar{x} \in E_{pr}$ *and* $\overline{\text{cone}}(F(K) - F(\bar{x}) + P)$ *is convex;*
(c) $\bar{x} \in E_{pr}$ *and* $\text{cone}(F(K) - F(\bar{x}) + P)$ *is convex.*

5.2.1 The Pareto case

We now particularize $P = \mathbb{R}^2_+$. Given a vector mapping $F = (f_1, f_2) : K \to \mathbb{R}^2$, we consider the problem of finding

$$\bar{x} \in K: \ \overline{\text{cone}}(F(K) - F(\bar{x}) + \mathbb{R}^2_+) \cap (-\mathbb{R}^2_+) = \{(0,0)\}, \tag{21}$$

Let $\bar{x} \in E_{pr}$ and for $i = 1, 2$, consider the sets $S_i^-(\bar{x})$, $S_i^+(\bar{x})$ and $S_i^=(\bar{x})$ as defined in the previous subsection.

By (k) of Proposition 1, the convexity of $\overline{\text{cone}}(F(K) - F(\bar{x}) + \mathbb{R}^2_+)$ is equivalent to the convexity of $\text{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++})$. Thus, by writing $F(K) - F(\bar{x}) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, we can use the same expressions for $\text{cone}_+(\Omega_i)$, $i = 1, 2, 3$ computed in the preceding section. The fact that $\bar{x} \in E_{pr}$ allows us to conclude that α, β (as defined in (18)) satisfy $-\infty < \alpha < 0$, $-\infty < \beta < 0$, and

$$S_1^-(\bar{x}) \subseteq S_2^+(\bar{x}); \ \ S_2^-(\bar{x}) \subseteq S_1^+(\bar{x}).$$

Thus, the preceding expressions for $\text{cone}(\Omega_i)$, $i = 1, 2, 3$, reduces to

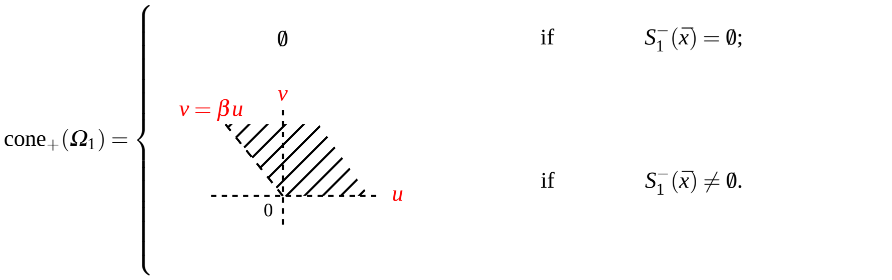


Fig. 7 To visualize Theorem 11

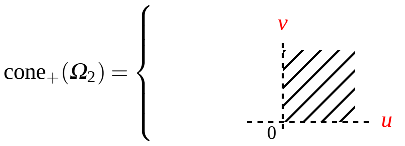


Fig. 8 To visualize Theorem 11

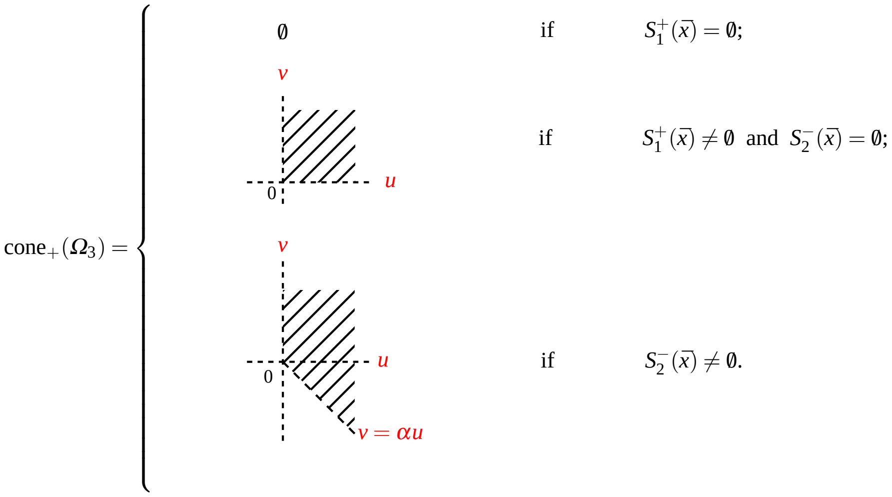


Fig. 9 To visualize Theorem 11

Theorem 11. *Assume that* $\bar{x} \in E_{pr}$. *Then,* $\overline{\text{cone}}(F(C) - F(\bar{x}) + \mathbb{R}^2_+)$ *is convex if, and only if either* (a) *or* (b) *holds. Here,*

(a) $S_1^-(\bar{x}) \neq \emptyset$ *and, either* $S_1^+(\bar{x}) = \emptyset$ *or* $[S_2^-(\bar{x}) = \emptyset,\, S_1^+(\bar{x}) \neq \emptyset]$ *or* $[S_2^-(\bar{x}) \neq \emptyset,\, \beta \leq \alpha]$;
(b) $S_1^-(\bar{x}) = \emptyset$.

Proof. The proof is easy once we get Figures 7, 8 and 9. □

Corollary 7. *Let us consider problem* (21). *Then,*

$$\bar{x} \in \bigcup_{(p_1^*,p_2^*)\in\mathbb{R}^2_{++}} \operatorname{argmin}_K(p_1^* f_1 + p_2^* f_2)$$

if and only if either (a) *or* (b) *holds, where*

(a) $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) \neq \emptyset$ *and, either* $S_1^+(\bar{x}) = \emptyset$ *or* $[S_2^-(\bar{x}) = \emptyset, S_1^+(\bar{x}) \neq \emptyset]$ *or* $[S_2^-(\bar{x}) \neq \emptyset, \beta \leq \alpha]$;
(b) $\bar{x} \in E_{pr}$ *and* $S_1^-(\bar{x}) = \emptyset$.

Corollary 8. *Let us consider problem* (21). *Then,*

(a) *If* $\bar{x} \in E_{pr}, S_1^-(\bar{x}) \neq \emptyset$ *and, either* $S_1^+(\bar{x}) = \emptyset$ *or* $[\, S_2^-(\bar{x}) = \emptyset, S_1^+(\bar{x}) \neq \emptyset \,]$, *then any* p_1^* *such that* $0 < p_1^* \leq -\beta$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K(p_1^* f_1 + f_2).$$

(b) *If* $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) \neq \emptyset$ *and* $[\, S_2^-(\bar{x}) \neq \emptyset, \beta \leq \alpha \,]$, *then any* p_1^* *such that* $-\alpha \leq p_1^* \leq -\beta$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K(p_1^* f_1 + f_2).$$

(c) *If* $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) = \emptyset$ *and* $S_2^-(\bar{x}) \neq \emptyset$, *then any* p_1^* *such that* $-\alpha \leq p_1^*$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K(p_1^* f_1 + f_2).$$

(d) *If* $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) = \emptyset$ *and* $S_2^-(\bar{x}) = \emptyset$, *then any* $(p_1^*, p_2^*) \in \mathbb{R}^2_{++}$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K(p_1^* f_1 + p_2^* f_2).$$

5.3 Characterizing the Fritz-John type optimality conditions

For simplicity we now consider X to be a real normed vector space. It is well known that if $\bar{x}$ is a local minimum point for the real-valued differentiable function F on K, then

$$\nabla F(\bar{x}) \in (T(K;\bar{x}))^*. \tag{22}$$

Here, K is a (not necessarily convex) set, $T(C;\bar{x})$ denotes the *contingent cone* of C at $\bar{x} \in C$, defined as the set of vectors v such that there exist $t_k \downarrow 0$, $v_k \in X$, $v_k \to v$ such that $\bar{x} + t_k v_k \in C$ for all k; recall that C^* denotes the (positive) polar cone of C. It is now our purpose to extend the previous optimality condition to the vector case without smoothness assumptions. More precisely, let $K \subseteq X$ be closed and consider a mapping $F : K \to \mathbb{R}^m$. A vector $\bar{x} \in K$ is a local weakly efficient solution for F on K, if there exists an open neighborhood V of $\bar{x}$ such that

$$(F(K \cap V) - F(\bar{x})) \cap (-\operatorname{int} P) = \emptyset. \tag{23}$$

Following [37], we say that a function $h : X \to \mathbb{R}$ admits a *Hadamard directional derivative* at $\bar{x} \in X$ in the direction v if

$$\lim_{(t,u)\to(0^+,v)} \frac{h(\bar{x}+tu)-h(\bar{x})}{t} \in \mathbb{R}.$$

In this case, we denote such a limit by $dh(\bar{x};v)$.

If $F=(f_1,f_2,\ldots,f_m)$, we set

$$\mathscr{F}(v) \doteq ((df_1(\bar{x};v),\ldots,df_m(\bar{x};v)), \;\; \mathscr{F}(T(K;\bar{x})) \doteq \{\mathscr{F}(v)\in\mathbb{R}^n:\; v\in T(K;\bar{x})\}.$$

It is known that if $df_i(\bar{x};\cdot)$, $i=1,\ldots,m$, do exist in $T(K;\bar{x})$, and $\bar{x}\in K$ is a local weakly efficient solution for F on K, i.e., $\bar{x}$ satisfies (23), then (see for instance Lemma 3.2 of [37])

$$(df_1(\bar{x};v),\ldots,df_n(\bar{x};v)) \in \mathbb{R}^n\setminus -\text{int}\, P, \;\; \forall\, v\in T(K;\bar{x}), \tag{24}$$

or equivalently,

$$\mathscr{F}(T(K;\bar{x}))\cap(-\text{int}\, P)=\emptyset.$$

The following theorems provide complete characterizations for the validity of (a) as a necessary condition for $\bar{x}$ to be a local weakly efficient solution for F on K.

Theorem 12. *Let $K\subseteq X$ be a closed set, $P\subseteq\mathbb{R}^n$ be a closed convex cone such that $\text{int}\, P\neq\emptyset$ and $P\neq\mathbb{R}^n$. Assume that $\bar{x}\in K$ and $df_i(\bar{x};\cdot)$, $i=1,\ldots,m$, do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:*

(a) $\exists\,(\alpha_1^*,\ldots,\alpha_m^*)\in P^*\setminus\{0\}$, $\alpha_1^* df_1(\bar{x},v)+\ldots+\alpha_m^* df_n(\bar{x},v)\geq 0 \;\; \forall\, v\in T(K;\bar{x})$;
(b) $\text{cone}(\mathscr{F}(T(K;\bar{x}))+\text{int}\, P)$ *is pointed.*

Proof. We obtain the desired result from Theorem 2. □

When $Y=\mathbb{R}^2$, more precise formulations can be obtained from Theorem 5.

Theorem 13. *Let $K\subseteq X$ be a closed set, $P\subseteq\mathbb{R}^2$ be a closed convex cone such that $\text{int}\, P\neq\emptyset$. Assume that $\bar{x}\in K$ and $df_i(\bar{x};\cdot)$, $i=1,2$, do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:*

(a) $\exists\,(\alpha_1^*,\alpha_2^*)\in P^*\setminus\{(0,0)\}$, $\alpha_1^* df_1(\bar{x},v)+\alpha_2^* df_2(\bar{x},v)\geq 0 \;\; \forall\, v\in T(K;\bar{x})$;
(b) $\mathscr{F}(T(K;\bar{x}))\cap(-\text{int}\, P)=\emptyset$ *and* $\text{cone}(\mathscr{F}(T(K;\bar{x}))+\text{int}\, P)$ *is convex.*

Proof. We apply Theorem 5 to obtain the desired result. □

We can go further when differentiability conditions are imposed.

Proposition 7. *Assume that $P=\mathbb{R}^m_+$ and $f_i:\mathbb{R}^n\to\mathbb{R}$ be differentiable for $i=1,\ldots,m$, and $\bar{x}\in\mathbb{R}^n$. Then, for any set $A\subseteq\mathbb{R}^n$,*

$$\mathscr{F}(A)\cap(-\text{int}\,\mathbb{R}^m_+)=\emptyset \iff \max_{1\leq i\leq m}\langle\nabla f_i(\bar{x}),v\rangle\geq 0 \;\; \forall\, v\in\overline{A},$$

and the following statements are equivalent:

(a) $\text{cone}(\mathscr{F}(T(K;\bar{x}))+\text{int}\,\mathbb{R}^n_+)$ is pointed;

(b) $\mathscr{F}(\overline{\text{co}}(T(K;\bar{x}))) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset$;
(c) $\max_{1\le i\le m} \langle \nabla f_i(\bar{x}), v\rangle \geq 0 \ \ \forall\, v \in \overline{\text{co}}(T(K;\bar{x}))$;
(d) $\text{co}(\{\nabla f_i(\bar{x}) :\ i = 1,\dots,m\}) \cap (T(K;\bar{x}))^* \neq \emptyset$.

Proof. The first part is a consequence of the linearity of $\mathscr{F}$:

$$\mathscr{F}(v) = (\langle \nabla f_1(\bar{x}), v\rangle, \dots, \langle \nabla f_m(\bar{x}), v\rangle).$$

We already know that

$$\text{cone}(\mathscr{F}(T(K;\bar{x})) + \text{int}\,\mathbb{R}^m_+)) \text{ is pointed } \iff \text{co}(\mathscr{F}(T(K;\bar{x})) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset.$$

It is not difficult to prove that $\text{co}(\mathscr{F}(T(K;\bar{x}))) = \mathscr{F}(\text{co}(T(K;\bar{x})))$ and

$$\begin{aligned}\mathscr{F}(\text{co}(T(K;\bar{x}))) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset &\iff \mathscr{F}(\overline{\text{co}}(T(K;\bar{x}))) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset \\ &\iff \overline{\mathscr{F}(\text{co}(T(K;\bar{x})))} \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset.\end{aligned}$$

This and the fact that (a) of Theorem 12 amounts to writing

$$\text{co}(\{\nabla f_i(\bar{x}) :\ i = 1,\dots,m\}) \cap (T(K;\bar{x}))^* \neq \emptyset,$$

we get all the remaining equivalences. □

We apply the previous proposition to get the following result.

Theorem 14. *Let $K \subseteq X$ be a closed set, Assume that $\bar{x} \in K$ and $f_i : \mathbb{R}^n \to \mathbb{R}^2$ are differentiable functions for $i = 1,2,\dots,m$. Then, the following assertions are equivalent:*

(a) $\mathscr{F}(T(K;\bar{x}) \cap (-\text{int}\mathbb{R}^2_+) = \emptyset$ *and* $\text{cone}(\mathscr{F}(T(K;\bar{x})) + \mathbb{R}^2_+)$ *is convex;*
(b) $\text{co}(\{\nabla f_i(\bar{x}) :\ i = 1,2\}) \cap (T(K;\bar{x}))^* \neq \emptyset$;

Before going on some remarks are in order. Certainly, if $T(K;\bar{x})$ is convex, then (d) is a necessary optimality condition for $\bar{x}$ to be a local weakly efficient solution (this fact was point out earlier in [39], see also [9]). Thus, (d) could be considered a natural extension of (22). However, next example shows that (d) is not a necessary optimality condition if $T(K;\bar{x})$ is not convex. The second example shows an instance where (d) holds without the convexity of $T(K;\bar{x})$.

Example 2. Take the (modified) example from [1], see also [9, 40]:

$$K = \{(x_1,x_2) :\ (x_1 + 2x_2)(2x_1 + x_2) \le 0\},\ f_i(x_1,x_2) = x_i,\ \bar{x} = (0,0) \in E_W.$$

In this case $T(K;\bar{x}) = K$, which is nonconvex, $(T(K;\bar{x}))^* = \{(0,0)\}$, and therefore (d) does not hold since $\text{co}(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}) = \text{co}(\{(1,0),(0,1)\})$. Since $\mathscr{F}(v) = v$, the set

$$\text{cone}(\mathscr{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t\ge 0} t(T(K;\bar{x}) + \mathbb{R}^2_+).$$

is nonconvex.

Example 3. Consider the same mapping F as before and

$$K=\{(x_1,x_2)\in\mathbb{R}^2_+ : x_1x_2=0\},\ \bar{x}=(0,0)\in E_W.$$

Then, (d) holds since in this case, $T(K;\bar{x})=K$, $(T(K;\bar{x}))^*=\mathbb{R}^2_+$. Here, the set

$$\text{cone}(\mathscr{F}(T(K;\bar{x}))+\mathbb{R}^2_+)=\bigcup_{t\geq 0} t(T(K;\bar{x})+\mathbb{R}^2_+)$$

is convex.

6 More about proper efficiency

We now present some recent developments about proper efficiency. As before, throughout this section we consider a nonempty set $A\subsetneq Y$, with Y being a locally convex topological vector space. In addition, we are given a convex cone $P\subsetneq Y$. We say that $\bar{a}\in A$ is a

- *Benson proper efficient point* if $\overline{\text{cone}}(A-\bar{a}+P)\cap(-P)=\{\mathbf{0}\}$. This is the definition given in Benson [2]. and the set is denoted by $E_{\text{pr}}(A,P)$.
- *Borwein proper efficient point* if $\overline{\text{cone}}(A-\bar{a})\cap(-P)=\{\mathbf{0}\}$. This notion is introduced in [4] when P is pointed.

Evidently every Benson proper efficient point is also a Borwein efficient.

Proper efficiency is introduced in order to avoid efficient points satisfying some abnormal properties, in particular, efficient points for which at least one objective function exists for which the marginal trade-off between it and each of the other objective functions is infinitely large, [17], or if one prefers efficient points that allow more satisfactory characterization in terms of linear/sublinear scalarization, for instance. The starting point was the pionering work by Kuhn and Tucker in multiojective programming problems [24].

Benson and Borwein efficiency coincide if P has a compact base, see [11]; whereas in general it is not true, as shows Example 4.3 in [11]. We say that B is a *base* for P if B is convex, $0\notin\overline{B}$ and $P=\text{cone}(B)$. Obviously, the existence of a base for P implies its pointedness; likewise if $E_{\text{pr}}(A,P)\neq\emptyset$.

When the corresponding scalar function which is involved in the characterization of proper efficiency, is a continuous seminorm, we refer to [11]. This result is based in the following theorem

Theorem 15. ([*11, Theorem* 2.3]) *Let P and Q be cones in Y satisfying $P\cap Q=\{\mathbf{0}\}$, and either (a) P be a weak-closed and Q have a weak-compact base or (b) P be closed and Q have a compact base. Then, there is a pointed convex cone C such that $Q\setminus\{\mathbf{0}\}\subseteq \text{int}\, C$ and $C\cap P=\{\mathbf{0}\}$.*

Now, we present some results on interior of a polar cone, and afterwards, dual characterizations and scalarizations for Benson proper efficiency. To that purpose, we recall that Y^* is the topological dual of Y. For any convex cone $P \subseteq Y$, the quasi interior of P^*, is defined as

$$\text{qi}\, P^* = P^{*i} \doteq \{y^* \in Y^* : \langle y^*, y\rangle > 0\ \forall\ p \in P \setminus \{\mathbf{0}\}\}.$$

A convex cone P with int $P \neq \emptyset$ is said to be a solid cone. Moreover, a convex cone P has a base if and only if $P^{*i} \neq \emptyset$. For a base B of P, we define B^{st} to be the set

$$B^{st} \doteq \{y^* \in Y^* : \inf_{b\in B} \langle y^*, b\rangle > 0\ \ \}.$$

For any locally convex topological vector space Y, we have various ways of introducing a locally convex topology on the dual Y^*. If $\mathscr{M}$ is any total saturated class of bounded subsets of Y ([19, 23, 35]), the topology of uniform convergence on the sets M of $\mathscr{M}$ is a locally convex topology on Y^*. We denote it by $\tau_{\mathscr{M}}$. Obviously $\{M^\circ : M \in \mathscr{M}\}$ is a 0-neighborhood base in $(Y^*, \tau_{\mathscr{M}})$. Particularly, we denote the topologies on Y^* of uniform convergence on bounded subsets, weakly compact (absolutely) convex subsets, and finite subsets of Y by $\beta(Y^*,Y)$, $\tau(Y^*,Y)$, and $\sigma(Y^*,Y)$, which are called the strong topology, Mackey topology, and weak topology, respectively.

Lemma 2. $([29, Lemma\ 2.1])$ *Let $P \subseteq Y$ be a convex cone. If there exist a locally convex topology $\mathscr{T}$ on Y^* such that* $\text{int}_{\mathscr{T}} P^* \neq \emptyset$*, where* $\text{int}_{\mathscr{T}} P^*$ *denotes the interior of P^* in $(Y^*, \mathscr{T})$, then* $\text{int}_{\mathscr{T}}\, P^* \subseteq P^{*i}$.

Theorem 16. $([29, Theorem\ 2.1])$. *Let $P \subseteq Y$ be a convex cone. Then,* $\text{int}_{\tau_{\mathscr{M}}}\, P^* \neq \emptyset$ *if and only if P has a base $B \in \mathscr{M}$. In this case,* $\text{int}_{\tau_{\mathscr{M}}}\, P^* = B^{st}$.

Similar expressions hold for $\tau(Y^*,Y)$ and $\beta(Y^*,y)$, for details, see [21, Theorem 3.8.6], [28, Theorem 2.3], [28, Theorem 2.2].

We now give the following general dual characterization and scalarization for Benson proper efficiency.

Theorem 17. $([29, Theorem\ 3.1.])$ *Let $P \subseteq Y$ be a closed convex cone, $\bar{a} \in A \subseteq Y$. Then the following statements are equivalent:*

(a) $\bar{a} \in E_{\text{pr}}(\text{co}\, A, P)$;
(b) $(P^* - P^* \cap (A-\bar{a})^*)$ *is dense in $(Y^*, \mathscr{T})$ where $\mathscr{T}$ is any locally convex topology on Y^* which is compatible with the dual pair (Y^*,Y) (i.e., $(Y^*, \mathscr{T})^* = Y$);*
(c) *for any weakly compact convex set $K \subseteq P$ and $0 \notin K$, there exists $p^* \in P^* \cap K^{st}$ such that $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle\ \forall\ a \in A$;*
(d) *for any $p \in P \setminus \{0\}$ there exists $p^* \in P^*$ such that $\langle p^*, p\rangle > 0$ and $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle\ \forall\ a \in A$.*

Theorem 18. $([29, Theorem\ 3.2])$ *Let $P \subseteq Y$ be a closed convex cone and $\bar{a} \in A \subseteq Y$. If there exists a locally convex topology $\mathscr{T}$ on Y^* such that $(Y^*, \mathscr{T})^* = Y$ and* $\text{int}_{\mathscr{T}} P^* \neq \emptyset$ *then the following statements are equivalent:*

(a) $\bar{a} \in E_{\rm pr}({\rm co}\, A, P)$;
(b) *there exists* $p^* \in {\rm int}_{\mathscr{T}} P^*$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle, \ \forall\, a \in A$;
(c) *there exists* $p^* \in P^{*i}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle, \ \forall\, a \in A$.

Corollary 9. ([29, *Corollary* 3.1]) *Let* $C \subseteq Y$ *be a closed convex cone with a weakly compact base* B *and* $\bar{a} \in A \subseteq Y$. *Then the following statements are equivalent:*

(a) $\bar{a} \in E_{\rm pr}({\rm co}\, A, P)$;
(b) *there exists* $p^* \in B^{st}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle \ \forall\, a \in A$;
(c) *there exists* $p^* \in C^{*i}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \bar{a}\rangle \ \forall\, a \in A$.

A recent notion of proper efficiency was introduced in [27]. It is equivalent to strict efficiency, strong efficiency and to super efficiency as shown in [27, Proposition 2.2], provided P is a convex cone with a (convex) bounded base.

References

1. B. Aghezzaf, M. Hachimi, On a gap between multiobjective optimization and scalar optimization, J. Optim. Theory Appl., **109**, 431–435(2001)
2. H.P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, J. Math. Anal. and Appli., **71**, 232–241(1979)
3. J. Borwein, Proper efficient points for maximizations with respect to cones, SIAM, J. Control and Optimization, **15**, 57–63(1977)
4. J. Borwein, The geometry of Pareto efficiency over cones, Math. Oper. Statist. Ser. Optim., **11**, 235–248(1980)
5. J. M. Borwein, A. S. Lewis, Partially finite convex programming, part I: Quasi relative interiors and duality theory, Math. Program. A, **57**, 15–48(1992)
6. J. Borwein, R. Goebel, Notions of relative interior in Banach spaces, *J. Math. Sci.*, **115** (2003), 2542–2553.
7. R. I. Bot, E. R. Csetnek, G. Wanka, Regularity conditions via quasi-relative interior in convex programming,SIAM, J. Optim., **19**, 217–233(2008)
8. W. W. Brecker, G. Kassay, A systematization of convexity concepts for sets and functions, J. Convex Analysis, **4**, 109–127(1997)
9. M. Castellani, M. Pappalardo, About a gap between multiobjective optimization and scalar optimization, J. Optim. Theory Appl., **109**, 437–439(2001)
10. Guang-ya Chen, Xuexiang Huang, Xiaoqi Yang, Vector Optimization, Set-Valued and Variational Analysis, Springer-Verlag, Berlin Heidelberg, (2005).
11. J. P. Dauer, O. A. Saleh, A characterization of proper points as solutions of sublinear optimization problems, J. Math. Anal. and Appli., **178**, 227–246(1993)
12. S. Deng, Coercivity properties and well posedness in vector optimization, RAIRO Oper. Res., **37**, 195–208(2003)
13. K. Fan, I. Glicksberg, A. J. Hoffman, Systems of inequalities involving convex functions, Proc. Amer. Math. Soc., **8**, 617–622(1957)
14. F. Flores-Bazán, N. Hadjisavvas, C. Vera, An optimal alternative theorem and applications to mathematical programming, J. Global Optim., **37**, 229–243(2007)
15. F. Flores-Bazán, C. Vera, Unifying efficiency and weak efficiency in generalized quasiconvex vector minimization on the real-line, Int. J. Optim.: Theory, Methods Appl., **1** No 3, 247–265(2009)
16. J. B. G. Frenk, G. Kassay, On classes of generalized convex functions. Gordan-Farkas type theorems, and Lagrangian duality, J. Optim. Theory Appl., **102**, 315–343(1999)

17. A. M. Geoffrion, Proper efficiency and the theory of vector maximization, J. Math. Anal. and Appli., **22**, 618–630(1968)
18. R. B. Holmes, Geometric Functional Analysis, Springer, Berlin (1975).
19. J. Horváth, Topological Vector Spaces and Distributions, Addison-Wesley, Reading, MA (1966).
20. J. Jahn, Vector optimization, theory, applications, and extensions, Springer-Verlag, Berlin (2004).
21. G. Jameson, Ordered linear spaces, Springer-Verlag, Berlin (1970).
22. V. Jeyakumar, W. Oettli, M. Natividad, A solvability theorem for a class of quasiconvex mappings with applications to optimization, J. Math. Anal. and Appli., **179**, 537–546(1993)
23. G. Köthe, Topological vector spaces I, Springer-Verlag, Berlin, (1970).
24. H. W. Kuhn, A. W. Tucker, Nonlinear programming, in "Proceedings, Second Berkeley Symposium on Mathematical Statistics and Probability", Univ. of California Press, Berkeley, California, 481–492(1951)
25. M. A. Limber, R. K. Goodrich, Quasi interiors, Lagrange multipliers, and L^p spectral estimation with lattice bounds, J. Optim. Theory Appl., **78**, 143–161(1993)
26. D. T. Luc, Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems, **319**, Springer-Verlag, New york, Berlin (1989).
27. A. Guerraggio, D. T. Luc, Properly maximal points in product spaces, Math. Oper. Res., **31**, 305–315(2006)
28. J.-H. Qiu, On solidness of polar cones, J. Optim. Theory Appl. **109**, 199–214(2001).
29. J.-H. Qiu, Dual characterization and scalarization for Benson proper efficiency, SIAM J. Optim. **19**, 144–162(2008).
30. R. T. Rockafellar, R. J-B., Variational Analysis, Springer-Verlag, Berlin Hedelberg (1998).
31. R. T. Rockafellar, Conjugate Duality and Optimization, CBMS Reg. Conf- Ser. Math. **16** SIAM, Philadelphia (1974).
32. P.H. Sach, Nearly subconvexlikeness set-valued maps and vector optimization problems, J. Optim. Theory Appl., **119**, 335–356(2003)
33. P. H. Sach, New generalized convexity notion for set-valued maps and applications to vector applications, J. Optim. Theory Appl., **125** , 157–179(2005)
34. Y. Sawaragi, H. Nakayama, Y. Tanino, Theory of Multiobjective Optimization, Academic Press, New York (1985).
35. H. H. Schaefer, Topological vector spaces, Springer-Verlag, New York (1971).
36. W. Song, Duality in Set-Valued Optimization, Dissertationes Mathematicae, **375**, 1–67(1998)
37. A. Taa, Optimality conditions for vector mathematical programming via a theorem of the alternative, J. Math. Anal. and Appli., **233**, 233–245(1999)
38. T. Tanaka, General quasiconvexities, cones saddle points and minimax theorem for vector-valued functions, J. of Optimization Theory and Applications, **81**, 355–377(1994)
39. S. Y. Wang, A note on optimality conditions in multiobjective programming, Systems Science and Mathematical Sciences, **1**, 184–190(1988)
40. S. Y. Wang, F. M. Yang, A gap between multiobjective optimization and scalar optimization, J. Optim. Theory Appl., **68**, 389–391(1991)
41. X. M. Yang, Alternative theorems and optimality conditions with weakened convexity, Opsearch, **29**, 115–125(1992)
42. X. M. Yang, D. Li, S. Y. Wang, Nearly-subconvexlikeness in vector optimization with set-valued functions, J. Optim. Theory Appl., **110**, 413–427(2001)
43. X. M. Yang, X. Q. Yang, G. Y. Chen, Theorems of the alternative and optimization with set-valued maps, J. Optim. Theory Appl., **107**, 627–640(2000)
44. C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).
45. R. Zeng, A general Gordan alternative theorem with weakened convexity and its application, Optimization, **51**, 709–717(2002)