A geometric characterization of strong duality in nonconvex quadratic programming with linear and nonconvex quadratic constraints

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Abstract

We first establish a relaxed version of Dines theorem associated to quadratic minimization problems with finitely many linear equality and a single (nonconvex) quadratic inequality constraints. The case of unbounded optimal valued is also discussed. Then, we characterize geometrically the strong duality, and some relationships with the conditions employed in Finsler theorem are established. Furthermore, necessary and sufficient optimality conditions with or without the Slater assumption are derived. Our results can be used to situations where none of the results appearing elsewhere are applicable. In addition, a revisited theorem due to Frank and Wolfe along with that due to Eaves is established for asymptotically linear sets.

Key words. Strong duality, Nonconvex optimization, Quadratic programming, Relaxed Dines's theorem.

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1 Introduction

Given a subset C of a finite dimensional space $\mathbb{R}^n$, and functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, let us consider the following minimization problem:

$$\mu \doteq \inf_{\substack{g(x)\leq 0\\ x\in C}} f(x). \tag{1}$$

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The Lagrangian dual problem associated to (1) is

$$\nu \doteq \sup_{\lambda \geq 0} \inf_{x \in C} [f(x) + \lambda g(x)], \tag{2}$$

We say Problem (1) has a (Lagrangian) *zero duality gap* if the optimal values of (1) and (2) coincide, that is, $\mu = \nu$. Problem (1) is said to have *strong duality* if it has a zero duality gap and Problem (2) admits a solution. In general, the lack of convexity makes the problem of characterizing strong duality very difficult.

Quadratic functions have proved to be very important in applications (telecommunications, robust control [30, 35], trust region problems [19, 36]) and enjoy very nice properties. After the result ($C = \mathbb{R}^n$) due to Gay [19] and Sorensen [36] concerning a characterization of solutions for a special quadratic optimization problem without any convexity assumptions, several authors extended such a result for general quadratic optimization with a single inequality constraint. In particular, we mention the work by Moré [31] who considered the general case of a single equality constraint and then used it to cover the single inequality constraint under the standard Slater condition. Moré actually provided necessary and sufficient optimality conditions for a point to be optimal under no convexity conditions. Certainly, this may be seen as a strong duality-type result.

More recently, when $C = \mathbb{R}^n$ with g being a quadratic function that is not identically zero, the authors in [23] prove that, (1) has strong duality for each quadratic function f if, and only if there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$, that is, the standard Slater condition holds. Unlike this result and many others established in [22, 24, 26, 25, 27], our approach allows us to derive conditions on the pair, f and g jointly, that ensure that (1) has strong duality without satisfying the Slater condition, and under no convexity assumptions on f or g. This is carried out by further developing the geometric approach introduced in [15], where strong duality is characterized under a single inequality constraint for any (not necessarily quadratic) functions f and g. We actually characterize completely the strong duality in the presence of finitely many linear equality and a single quadratic inequality constraints without convexity assumptions or Slater condition (Theorem 3.5), and derive necessary and sufficient optimality conditions.

Among the main results showing some of the nice properties of quadratic functions we mention two of them. The first one is due to Dines [12] (see also [34]) and it ensures the convexity of the set $\{(f(x), g(x)) \in \mathbb{R}^2 : x \in \mathbb{R}^n\}$ for any homogeneous quadratic functions f and g. For general quadratic non homogeneous functions we provide a relaxed version of this result, see Theorem 3.3 when μ is finite, and when $\mu = -\infty$ it is provided conditions under a Dines-type result holds. A second result showing another nice property of these functions is that due to Frank and Wolfe [17], which asserts that any quadratic function bounded from below on a nonempty (possibly unbounded) polyhedral set attains its infimum value. We establish several equivalences (including that due to Frank and Wolfe) for a larger family of sets than polyhedral, whose proof uses elementary analysis and it is related to that by Blum and Oettli [5], being suitable for expository purposes; whereas the original proof of Frank and Wolfe requires a decomposition theorem for convex polyhedra.

More precisely, in the present paper, we deal with the case where f and g are quadratic functions and $C = H^{-1}(d) = \{x \in \mathbb{R}^n : \ Hx = d\}$ where H is a real matrix of order $m \times n$ and $d \in \mathbb{R}^m$, and the regularized Lagrangian dual problem is considered. It means that instead of considering the standard Lagrangian dual problem

$$\sup_{\lambda \geq 0, \gamma \in \mathbb{R}^m} \inf_{x \in \mathbb{R}^n} [f(x) + \lambda g(x) + \gamma(Hx - d)], \tag{3}$$

we choose the regularized Lagrangian dual problem

$$\sup_{\lambda \geq 0} \inf_{x \in H^{-1}(d)} [f(x) + \lambda g(x)], \tag{4}$$

which is more suitable for our purpose since there are instances, specially in trust-region problems, showing a non zero duality gap between (1) and (3) against with the zero duality gap between (1) and (4), even if the Slater condition holds, as stated in [27].

Apart from these characterizations several sufficient conditions of the zero duality gap for convex programs have been established in the literature, see [18, 1, 2, 41, 6, 8, 9, 37, 32].

The paper is structured as follows. Section 2 provides the formulation of the problem we are going to discuss along with a characterization of a separation between a convex set and an open cone in terms of the convexity of the conic hull of some sets. It contains also a Dines-type theorem when the optimal value $\mu = -\infty$. The main Section 3 starts by proving a relaxed version of Dines theorem when μ is finite, along with a geometric characterization of strong duality for the minimization problem with finitely many linear equality and a single quadratic inequality constraints without convexity or Slater assumptions. This will serve to obtain necessary and sufficient optimality condition, both under or without Slater condition. Some relationships with the conditions employed in Finsler theorem are also established. Section 4 presents a refinement and an improvement of the Frank and Wolfe theorem and that due to Eaves for asymptotically linear sets. In Section 5 some necessary conditions for existence are derived.

2 Basic notations and formulation of the problem

This section will provides the necessary notations to be employed throughout this paper along with the formulation of the problem in the nonquadratic situation.

Given a set $A \subseteq \mathbb{R}^n$, its closure is denoted by $\overline{A}$; its convex hull by $\mathrm{co}(A)$ which is the smallest convex set containing A; its topological interior by $\mathrm{int}\, A$. We set $\mathrm{cone}(A) \doteq \bigcup_{t\geq 0} tA$, being the smallest cone containing A; $\mathrm{cone}_+(A) \doteq \bigcup_{t>0} tA$, and $\overline{\mathrm{cone}}(A) \doteq \overline{\bigcup_{t\geq 0} tA}$. Obviously, $\mathrm{cone}(A) = \mathrm{cone}_+(A) \cup \{0\}$.

Furthermore, A^* stands for the (non-negative) polar cone of A which is defined by

$$A^* \doteq \{\xi \in \mathbb{R}^n : \ \langle \xi, a\rangle \geq 0 \ \ \forall \ a \in A\},$$

where $\langle\cdot,\cdot\rangle$ means the scalar or inner product in $\mathbb{R}^n$; P is a cone if $tP \subseteq P$ for all $t \geq 0$.

Another notion to be used in the last section is the asymptotic cone of a set K, denoted by K^∞, and defined by

$$K^\infty \doteq \{v \in \mathbb{R}^n : \ \exists \ t_k \downarrow 0, \ \exists \ x_k \in K, \ t_k x_k \to v\}.$$

When K is closed and convex we get $K^\infty = \{v \in \mathbb{R}^n : \ x_0 + tv \in K, \ \ \forall \ t > 0\}$ for any $x_0 \in K$.

Finally we set $\mathbb{R}^2_{++} \doteq \mathrm{int}\ \mathbb{R}^2_+$.

2.1 The general case with finite optimal value

In this subsection we assume the real-valued functions f and g are defined in a Hausdorff topological space X, and C is a nonempty subset of X.

We associate to Problem (1) the usual linear Lagrangian

$$L(\gamma, \lambda, x) \doteq \gamma f(x) + \lambda g(x),$$

where $\gamma \geq 0$ and $\lambda \geq 0$ are called the Lagrange multiplier. By setting $K = \{x \in C : \ g(x) \leq 0\}$, we obtain the trivial inequality

$$\gamma \inf_{x\in K} f(x) \geq \inf_{x\in K} L(\gamma,\lambda,x) \geq \inf_{x\in C} L(\gamma,\lambda,x), \quad \forall \ \gamma \geq 0, \ \forall \ \lambda \geq 0. \tag{5}$$

In order to get the equality, we need to find conditions under which the reverse inequality holds, that is, we must have:

$$\gamma(f(x) - \mu) + \lambda g(x) \geq 0 \quad \forall \ x \in C. \tag{6}$$

This will imply strong duality once we get $\gamma > 0$, and by recalling that $\mu = \inf_{x\in K} f(x)$. By setting $F(x) \doteq (f(x), g(x))$ and so $F(C) = \{(f(x), g(x)) \in \mathbb{R}^2 : \ x \in C\}$ along with

$\rho \doteq (\gamma, \lambda)$, the previous inequality can be written as

$$\langle \rho, a \rangle \geq 0 \quad \forall\, a \in F(C) - \mu(1,0). \tag{7}$$

The following result, which is important by itself, characterizes completely (7). Part of this result was established in [16, Theorem 4.1].

Theorem 2.1. *Let* $P \subseteq \mathbb{R}^2$ *be a convex closed cone such that* $\mathrm{int}\ P \neq \emptyset$*, and* $A \subseteq \mathbb{R}^2$ *be any nonempty set. Then the following assertions are equivalent:*

(a) $\exists\ \lambda \in P^* \setminus \{0\}$, $\langle \lambda, a \rangle \geq 0 \quad \forall\ a \in A$;

(b) $A \cap (-\mathrm{int}\ P) = \emptyset$ *and* $\overline{\mathrm{cone}}(A + P)$ *is convex;*

(c) $A \cap (-\mathrm{int}\ P) = \emptyset$ *and* $\mathrm{cone}_+(A + \mathrm{int}\ P)$ *is convex;*

(d) $A \cap (-\mathrm{int}\ P) = \emptyset$ *and* $\mathrm{cone}(A + \mathrm{int}\ P)$ *is convex;*

(e) $\mathrm{cone}(A + \mathrm{int}\ P)$ *is pointed;*

(f) $\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset$.

Proof. Obviously (c) $\Longrightarrow$ (d) $\Longrightarrow$ (b).
(f) $\Longrightarrow$ (a) It follows from a simple use of a separation result of convex sets.
(a) $\Longrightarrow$ (b): Clearly $\langle \lambda, x \rangle \geq 0$ for all $x \in \mathrm{cone}(A+P)$. Choose $u \in \mathrm{int}\ P$. Let $y, z \in A$. Then obviously

$$\mathrm{cone}(\{y\}) + \mathrm{cone}(\{u\}) = \{sy + tu : s,\ t \geq 0\}$$

is a closed convex cone containing y and u and contained in $\mathrm{cone}(A + P)$. The same is true for the cone $\mathrm{cone}(\{z\}) + \mathrm{cone}(\{u\})$. The two cones have the line $\mathrm{cone}(\{u\})$ in common and their union is contained in $\mathrm{cone}(A + P)$, thus it is contained in the halfspace $\{x \in \mathbb{R}^2 : \ \langle \lambda, x \rangle \geq 0\}$. Hence, the set $B \doteq (\mathrm{cone}(\{y\}) + \mathrm{cone}(\{u\})) \cup (\mathrm{cone}(\{z\}) + \mathrm{cone}(\{u\}))$ is a convex cone. Since $y, z \in B$ we deduce that $[y, z] \subseteq B \subseteq \mathrm{cone}(A + P)$. Thus $\mathrm{co}(A) \subseteq \overline{\mathrm{cone}}(A + P)$, from which we infer that $\overline{\mathrm{cone}}(A + P)$ is convex since $P \subseteq \overline{\mathrm{cone}}(A + P)$ holds as well.
(b) $\Longleftrightarrow$ (c): Obviously (c) implies (b). If $\overline{\mathrm{cone}}(A + P)$ is convex then ([18])

$$\mathrm{int}(\overline{\mathrm{cone}}(A + P)) = \mathrm{int}(\overline{\mathrm{cone}_+(A) + P}) = \mathrm{cone}_+(A) + \mathrm{int}\ P = \mathrm{cone}_+(A + \mathrm{int}\ P)$$

is convex as well.
(c) $\Longrightarrow$ (e): Let $x, -x \in \mathrm{cone}(A + \mathrm{int}\ P)$. Then $x = t_1(a_1 + p_1,\ -x = t_2(a_2 + p_2)$ for some $t_i \geq 0$, $a_i \in A$, $p_i \in \mathrm{int}\ P$ for $i = 1, 2$. Assuming $t_i > 0$, for $i = 1, 2$, we have $x, -x \in \mathrm{cone}_+(A + \mathrm{int}\ P)$. By convexity, $0 = x + (-x) \in \mathrm{cone}_+(A + \mathrm{int}\ P)$, which

implies that $0 \in A + \text{int } P$, contradicting the first part of (c).
$(e) \implies (f)$: Assume on the contrary that $\text{co}(A) \cap (-\text{int } P) \neq \emptyset$. Then, there exist $a_i \in A$, $p_0 \in \text{int } P$, $\alpha_i \geq 0$, satisfying $\sum_{i=1}^m \alpha_i = 1$ and $0 = \sum_{i=1}^m \alpha_i a_i + p_0$. Thus, $0 = \sum_{i=1}^m \alpha_i (a_i + p_0)$. By pointedness, we get $\alpha_i(a_i + p_0) = 0$ for all $i = 1, \cdots, m$. Hence, $0 = a_j + p_0 \in A + \text{int } P$ for some j, which implies that $\text{cone}_+(A + \text{int } P) = \mathbb{R}^2$, contradicting (e). $\square$

Remark 2.2. *An example showing that the preceding result is not valid in dimension higher than two is given in [15]. This paper also proved that two dimensionality characterizes the validity of the equivalences in Theorem 2.1 for all sets A.*

By virtue of the preceding result and following the reasoning developed in [15], we need to split the set $\text{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++})$. To that purpose, some notations are in order. By setting $K \doteq \{x \in C : \ g(x) \leq 0\}$, we get $K = S_g^-(0) \cup S_g^=(0)$, where

$$S_g^-(0) \doteq \{x \in C : \ g(x) < 0\}, \ S_g^=(0) \doteq \{x \in C : \ g(x) = 0\}, S_g^+(0) \doteq \{x \in C : \ g(x) > 0\}.$$

Similarly, we define

$$S_f^-(\mu) \doteq \{x \in C : \ f(x) < \mu\}, \ S_f^+(\mu) \doteq \{x \in C : \ f(x) > \mu\},$$

$$S_f^=(\mu) \doteq \{x \in C : \ f(x) = \mu\}.$$

Furthermore, whenever $S_g^-(0) \cap S_f^+(\mu) \neq \emptyset$ and $S_f^-(\mu) \neq \emptyset$, we set

$$r \doteq \inf_{x \in S_f^+(\mu) \cap S_g^-(0)} \frac{g(x)}{f(x) - \mu}, \quad s \doteq \sup_{x \in S_f^-(\mu)} \frac{g(x)}{f(x) - \mu}.$$

Evidently, $-\infty \leq r < 0$, $-\infty < s \leq 0$. Notice that

$$x \in S_f^-(\mu) \Longrightarrow x \in S_g^+(0).$$

The latter and other basic facts about the previous sets are collected in the next proposition.

Proposition 2.3. *Let $\mu \in \mathbb{R}$, we have the following:*

(a) $C = K \iff S_g^+(0) = \emptyset$;

(b) $[\operatorname*{argmin}_K f \cap S_g^-(0) = \emptyset$ and $S_f^+(\mu) \cap S_g^-(0) = \emptyset] \iff S_g^-(0) = \emptyset$;

(c) $S_f^+(\mu) \cap S_g^-(0) = \emptyset \iff S_g^-(0) \subseteq \operatorname*{argmin}_K f$;

(d) $S_f^-(\mu) = \emptyset \Longleftrightarrow \mu = \inf_{x \in C} f(x)$;

(*e*) $S_f^-(\mu) \subseteq S_g^+(0)$.

Proof. (a), (c) and (e) are straightforward.
(b): Suppose on the contrary that $S_g^-(0) \neq \emptyset$. Then, by assumption every $x \in C$ such that $g(x) < 0$ satisfies $f(x) \leq \mu$. Thus $f(x) = \mu$ yielding a contradiction. The other implication is obvious.
(d): It follows by noticing that $S_f^-(\mu) = \emptyset$ if and only if $f(x) \geq \mu$ for all $x \in C$. ☐

We now proceed to split the set $\text{cone}[F(C) - \mu(1,0) + \mathbb{R}^2_{++}]$ by writing $F(C) - \mu(1,0) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$. This gives

$$\text{cone}(F(C) - \mu(1,0) + \mathbb{R}^2_{++}) = \text{cone}(\Omega_1) \cup \text{cone}(\Omega_2) \cup \text{cone}(\Omega_3), \tag{8}$$

where

$$\Omega_1 \doteq \bigcup_{x \in \underset{K}{\text{argmin}}\ f \cap S_g^=(0)} [(0,0) + \mathbb{R}^2_{++}] \cup \bigcup_{x \in \underset{K}{\text{argmin}}\ f \cap S_g^-(0)} [(0, g(x)) + \mathbb{R}^2_{++}];$$

$$\Omega_2 \doteq \bigcup_{x \in S_f^+(\mu) \cap S_g^-(0)} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}] \cup \bigcup_{x \in S_f^+(\mu) \cap S_g^=(0)} [(f(x) - \mu, 0) + \mathbb{R}^2_{++}];$$

$$\Omega_3 \doteq \bigcup_{x \in S_f^-(\mu)} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}] \cup \bigcup_{x \in S_f^=(\mu) \cap S_g^+(0)} [(0, g(x)) + \mathbb{R}^2_{++}] \cup$$
$$\cup \bigcup_{x \in S_f^+(\mu) \cap S_g^+(0)} [(f(x) - \mu, g(x)) + \mathbb{R}^2_{++}].$$

This decomposition will be used in Section 3.

2.2 The general case with unbounded optimal value

We continue by considering real-valued functions defined in a Hausdorff topological space X.
The case $\mu = -\infty$ deserves a special attention and it will be discussed in this subsection. First of all, it is not difficult to check that

$$\mu = -\infty \iff (F(C) + \mathbb{R}^2_+) \cap [(\rho, 0) - (\mathbb{R}_{++} \times \{0\})] \neq \emptyset, \quad \forall\, \rho \in \mathbb{R}. \tag{9}$$

By denoting $S_f^-(0) \doteq \{x \in C : \ f(x) < 0\}$ and $S_f^+(0) \doteq \{x \in C : \ f(x) > 0\}$, we set

$$\gamma \doteq \inf_{x \in S_g^-(0)} \frac{g(x)}{f(x)},$$

whenever $S_g^-(0) \neq \emptyset$. Furthermore, set

$$W \doteq \{(u,v) \in \mathbb{R}^2 : \ v > \gamma u, \ v \leq 0\}, \quad \text{if} \quad \gamma \in \mathbb{R}.$$

The following theorem establishes the geometric structure of the set $\text{cone}[F(C)+\mathbb{R}^2_{++}]$ in case $\mu=-\infty$.

Theorem 2.4. *Let $\mu=-\infty$. Then*

$$\mathbb{R}\times\mathbb{R}_+\subseteq F(C)+\mathbb{R}^2_+. \tag{10}$$

Furthermore,

(*a*) *If $S_g^-(0)=\emptyset$ then $F(C)+\mathbb{R}^2_+=\mathbb{R}\times\mathbb{R}_+$.*

(*b*) *If $S_g^-(0)\cap S_f^-(0)\neq\emptyset$ then $\text{cone}_+[F(C)+\mathbb{R}^2_+]=\mathbb{R}^2$.*

(*c*) *If $S_g^-(0)\neq\emptyset$ and $S_g^-(0)\cap S_f^-(0)=\emptyset$ then $-\infty=\inf\limits_{\substack{g(x)=0\\ x\in C}} f(x)$, $-\infty\leq\gamma<0$ and*

  (*c*1) $\text{cone}_+[F(C)+\mathbb{R}^2_{++}]=(\mathbb{R}\times\mathbb{R}_{++})\cup W$ *if* $-\infty<\gamma<0$;

  (*c*2) $\text{cone}_+[F(C)+\mathbb{R}^2_{++}]=(\mathbb{R}\times\mathbb{R}_{++})\cup(\mathbb{R}_{++}\times\mathbb{R})$ *if* $\gamma=-\infty$.

Proof. Let us prove (10). Take any $(\xi_1,\xi_2)\in\mathbb{R}\times\mathbb{R}_+$; by (9), there exist $x\in C$, $p\geq 0$, $q\geq 0$, $r>0$, such that $f(x)+p=\xi_1-r$ and $g(x)+q=0$. It follows that

$$(\xi_1,\xi_2)=(f(x),g(x))+(p+r,q+\xi_2)\in F(C)+\mathbb{R}^2_+.$$

(*a*): Since $g(x)\geq 0$ for all $x\in C$, we obtain

$$F(C)+\mathbb{R}^2_+\subseteq[f(C)\times g(C)]+\mathbb{R}^2_+\subseteq(\mathbb{R}\times\mathbb{R}_+)+\mathbb{R}^2_+=\mathbb{R}\times\mathbb{R}_+.$$

(*b*): By assumption, there exists $x_0\in C$ satisfying $g(x_0)<0$ and $f(x_0)<0$. This implies that $(0,0)\in F(x_0)+\mathbb{R}^2_{++}$, which gives $\text{cone}_+(F(x_0)+\mathbb{R}^2_{++})=\mathbb{R}^2$ and therefore the conclusion follows.
(*c*): By assumption, $f(x)\geq 0$ for all $x\in S_g^-(0)$, which implies that $-\infty=\inf\limits_{\substack{g(x)=0\\ x\in C}} f(x)$ and $-\infty\leq\gamma<0$.
(*c*1): Let $(u_0,v_0)\in W$. Then, $u_0>0$ and there exists $x_0\in C$ satisfying $g(x_0)<0$ and

$$\gamma\leq\frac{g(x_0)}{f(x_0)}<\frac{v_0}{u_0}.$$

We choose $\varepsilon>0$ satisfying $v_0 f(x_0)=u_0 g(x_0)+\varepsilon(u_0-v_0)$ and write

$$u_0=\frac{u_0}{f(x_0)+\varepsilon}(f(x_0)+\varepsilon),\quad v_0=\frac{u_0}{f(x_0)+\varepsilon}(g(x_0)+\varepsilon).$$

This proves that $(u_0,v_0)\in\text{cone}_+[F(C)+\mathbb{R}^2_{++}]$. This result along with (10) prove one inclusion in (*c*1).
For the other inclusion we reason as follows. Take any $(u_0,v_0)\in\text{cone}_+[F(C)+\mathbb{R}^2_{++}]$.

Then, for some $(p, q) \in \mathbb{R}^2_{++}$, $t_0 > 0$, $x_0 \in C$, we have $u_0 = t_0(f(x_0) + p)$ and $v_0 = t_0(g(x_0)+q)$. If $(u_0, v_0) \notin \mathbb{R}\times\mathbb{R}_{++}$ then $v_0 \le 0$. This implies that $g(x_0) < g(x_0)+q \le 0$, and therefore, by assumption, $f(x_0) \ge 0$. Clearly $f(x_0) > 0$ since otherwise $\gamma = -\infty$. Hence $\gamma \le \dfrac{g(x_0)}{f(x_0)}$, and so

$$\gamma u_0 = \gamma t_0(f(x_0) + p) \le t_0 g(x_0) + \gamma t_0 p < t_0 g(x_0) < t_0(g(x_0) + q) = v_0,$$

showing that $(u_0, v_0) \in W$. Hence, the proof of $(c1)$ is completed.
$(c2)$: It is similar to $(c1)$. □

3 The quadratic non-homogeneous case with linear and quadratic constraints

In this section we consider the case of quadratic functions defined in a finite dimensional space $\mathbb{R}^n$. Problems arising in telecommunications, robust control [30, 35], trust region [19, 36], may be modeled via quadratic non-homogeneous functions.
Consider the following quadratic optimization problem:

$$\mu \doteq \inf\Big\{\frac{1}{2}x^\top Ax + a^\top x + \alpha : \ \frac{1}{2}x^\top Bx + b^\top x + \beta \le 0,\ Hx = d\Big\}, \tag{11}$$

where A, B are symmetric matrices of order n; $a,\ b \in \mathbb{R}^n$; $d \in \mathbb{R}^m$; $\alpha, \beta \in \mathbb{R}$, and H is a real matrix of order $m \times n$.
Setting, $C \doteq H^{-1}(d) \doteq \{x \in \mathbb{R}^n : \ Hx = d\}$, it is known that

$$C = x_0 + \ker H, \ \ \forall\, x_0 \in C.$$

Let

$$f(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha,\ g(x) = \frac{1}{2}x^\top Bx + b^\top x + \beta.$$

One of the most important results concerning quadratic functions refers to Dine's theorem [12] (motivated by Finsler theorem [14]), which ensures that

$$\Big\{(x^\top Ax, x^\top Bx) : \ x \in \mathbb{R}^n\Big\} \ \text{ is convex.}$$

This result does not hold in the non-homogeneous case as the next example shows.

Example 3.1. *Take* $f(x, y) = x + y - x^2 - y^2 - 2xy$, $g(x, y) = x^2 + y^2 + 2xy - 1$, *and consider the set* $M \doteq \{(f(x, y), g(x, y)) \in \mathbb{R}^2 : \ (x, y) \in \mathbb{R}^2\}$. *Clearly* $(0, 0) = (f(0, 1), g(0, 1)) \in M$ *and* $(-2, 0) = (f(-1, 0), g(-1, 0)) \in M$. *We claim that* $(-1, 0) = \frac{1}{2}(0, 0) + \frac{1}{2}(-2, 0) \ \notin\ F(\mathbb{R}^2)$. *Indeed, if* $-1 = f(x, y)$ *and* $0 = g(x, y)$, *then* $|x + y| = 1$

and $x+y=0$*, reaching a contradiction. Hence,* $(-1,0)\in \mathrm{co}\, F(\mathbb{R}^2)\setminus F(\mathbb{R}^2)$*, showing that* $F(\mathbb{R}^2)$ *is nonconvex. More precisely, one can check that*

$$F(\mathbb{R}^2)=\{(t-t^2,t^2-1):\ t\in\mathbb{R}\}.$$

Let us consider the minimization problem

$$\mu \doteq \inf_{\substack{g(x,y)\leq 0\\ (x,y)\in\mathbb{R}^2}} f(x,y)$$

We claim that $\mu=-2$*. Indeed,*

$$\begin{aligned} f(x,y)+2+\frac{3}{2}g(x,y) &= x+y-(x+y)^2+2+\frac{3}{2}((x+y)^2-1)\\ &= 2(x+y+1)^2\geq 0,\ \forall\ (x,y)\ \in\ \mathbb{R}^2. \end{aligned}$$

In particular, if (x,y) *is such that* $g(x,y)\leq 0$*, we obtain* $f(x,y)\geq -2=f(-1,0)$*, proving our claim. We actually have* $\operatorname*{argmin}_K f=\{(x,y)\in\mathbb{R}^2:\ x+y=-1\}$ *and* $\lambda=\frac{3}{2}$ *is a Lagrange multiplier. Furthermore,* $S_f^+(\mu)\cap S_g^-(0)=\{(x,y)\in\mathbb{R}^2:\ |x+y|<1\}$ *and* $S_f^-(\mu)=\{(x,y)\in\mathbb{R}^2:\ x+y<-1\}\cup\{(x,y)\in\mathbb{R}^2:\ 2<x+y\}$*. Therefore*

$$r=\inf_{|x+y|<1}\frac{(x+y)^2-1}{x+y-(x+y)^2+2}=\inf_{|t|<1}\frac{t^2-1}{t-t^2+2}=\inf_{|t|<1}-\frac{t-1}{t-2}=-\frac{2}{3};$$

$$s=\sup_{(x,y)\in S_f^-(\mu)}\frac{(x+y)^2-1}{x+y-(x+y)^2+2}=-\frac{2}{3}.$$

However, we can prove a relaxed version of Dines theorem. To that purpose the next result, valid for quadratic functions, will play an important role.

Proposition 3.2. [*25, Theorem* 3.6] *Let* $f,g:\mathbb{R}^n\to\mathbb{R}$ *be any quadratic functions not necessarily homogeneous, let* $x_0\in\mathbb{R}^n$*, and let* S_0 *be a subspace of* $\mathbb{R}^n$*. Then exactly one of the following statements holds:*

(*a*) $\exists\ x\in x_0+S_0,\ f(x)<0,\ g(x)<0;$

(*b*) $\exists\ (\lambda_1,\lambda_2)\in\mathbb{R}^2_+\setminus\{(0,0)\},\ \lambda_1 f(x)+\lambda_2 g(x)\geq 0,\ \ \forall\ x\in x_0+S_0.$

On combining the preceding result and Theorem 2.1, we obtain the following theorem which may be considered as a relaxed version of the Dines theorem and, according to the author's knowledge, it is new in the literature. Obviously our result is weaker than that provided by Dines when f and g are homogeneous quadratic functions (for the case $\mu=-\infty$ we refer Theorem 2.4).

Theorem 3.3. (*Relaxed Dines theorem*) *Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ be any quadratic functions as above, and $C \doteq H^{-1}(d) = x_0 + \ker\, H$. If $\mu \in \mathbb{R}$, then*

$$\mathrm{cone}(F(C) - \rho(1,0) + \mathbb{R}^2_{++}) \ \text{ is convex for all } \rho \le \mu.$$

Proof. Since there is no x satisfying $g(x) < 0$, $Hx = d$ and $f(x) - \rho < 0$, by Proposition 3.2 we obtain the existence of $(\gamma, \lambda) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ such that

$$\gamma(f(x) - \rho) + \lambda g(x) \ge 0 \quad \forall\, x \in x_0 + \ker\, H = C. \tag{12}$$

The desired result is a consequence of Theorem 2.1. □

In case $\mu = -\infty$ Theorem 2.4 provides a complete description of $\mathrm{cone}_+(F(C) + \mathbb{R}^2_+)$; in particular, it establishes conditions under which $\mathrm{cone}_+(F(C) + \mathbb{R}^2_+)$ is convex.

We next present an application of the previous theorem to derive the S-lemma for any (not necessarily homogeneous) quadratic functions already appeared in [33, Theorem 2.2]; [25, Corollary 3.7]). Some variants of the S-lemma may be found in [11].

Theorem 3.4. (*The S-lemma*) *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be any quadratic functions and assume that there is $\bar{x} \in C \doteq H^{-1}(d)$ such that $g(\bar{x}) < 0$. Then, (a) and (b) are equivalent:*

(a) *There is no $x \in C$ such that*

$$f(x) < 0, \ \ g(x) \le 0.$$

(b) *There is $\lambda \ge 0$ such that*

$$f(x) + \lambda g(x) \ge 0, \quad \forall\, x \in C.$$

Proof. Obviously $(b) \Longrightarrow (a)$ always holds. Assume therefore that (a) is satisfied. This means that $x \in C$, $g(x) \le 0$ implies $f(x) \ge 0$, that is, $0 \le \mu \doteq \inf\limits_{x \in K} f(x)$. It follows that

$$\mathrm{cone}[F(C) - \mu(1,0) + \mathbb{R}^2_{++}] \cap \mathcal{H} = \emptyset,$$

$\mathcal{H} \doteq \{(u,v) \in \mathbb{R}^2 \ : u < 0, v \le 0\}$. By the previous theorem $\mathrm{cone}[F(C) - \mu(1,0) + \mathbb{R}^2_{++}]$ is convex, and so by a separation theorem, there exist $(\gamma, \lambda) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ and $\alpha \in \mathbb{R}$ such that

$$\gamma(f(x) - \mu + p) + \lambda(g(x) + q) \ge \alpha \ge \gamma u + \lambda v, \ \forall\, x \in C, \ \forall\, (p,q) \in \mathbb{R}^2_{++}, \ \forall\, u < 0, \ \forall\, v \le 0.$$

This implies $\alpha \geq 0$, $\gamma \geq 0$ and $\lambda \geq 0$. Thus

$$\gamma(f(x) - \mu) + \lambda g(x) \geq 0, \quad \forall\, x \in C,$$

that is, $\gamma f(x) + \lambda g(x) \geq \gamma\mu \geq 0, \quad \forall\, x \in C$. The Slater condition yields $\gamma > 0$, completing the proof of the theorem. $\square$

The important case, when f and g are quadratic, with $C = \mathbb{R}^n$, was studied by Yakubovich [38, 39], see the survey by Pólik and Terlaky in [33, Theorem 2.2]. Its proof uses the Dines theorem which asserts the convexity of the set $\{(f(x), g(x)) \in \mathbb{R}^2 : \; x \in \mathbb{R}^n\}$ when f and g are homogeneous quadratic functions.

We observe that (12) for $\rho = \mu$ amounts to writing that

$$(\gamma, \lambda) \in [\text{cone}(F(C) - \mu(1, 0) + \mathbb{R}^2_{++})]^*, \tag{13}$$

and we also get

$$\text{cone}(F(C) - \mu(1, 0) + \mathbb{R}^2_{++}) = \text{cone}_+(F(C) - \mu(1, 0) + \mathbb{R}^2_{++}) \cup \{(0, 0)\}.$$

The slightly dark region in Figures 1 and 2 represents $\text{cone}_+(F(C) - \mu(1, 0) + \mathbb{R}^2_{++})$.

Taking into account the splitting (8) introduced in Subsection 2.1, we establish the main theorem which is new in the literature and describes all the situations may happen when considering quadratic minimization problems with finitely many linear equality and a single quadratic inequality constraints. It provides also the solution set of the regularized Lagrangian dual (4).

Theorem 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ be the quadratic functions $f(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha$, $g(x) = \frac{1}{2}x^\top Bx + b^\top x + \beta$, and $C = H^{-1}(d)$ as above. Let $(\gamma, \lambda) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$ and μ be finite. Then, exactly one of the following assertions holds:*

(*a*1) *If either $\operatorname*{argmin}_K f \cap S^-_g(0) \neq \emptyset$ or $[S^+_f(\mu) \cap S^-_g(0) \neq \emptyset$ with $r = -\infty]$, then $S^-_f(\mu) = \emptyset$ and*

$$\text{cone}_+(F(C) - \mu(1, 0) + \mathbb{R}^2_{++}) = \{(u, v) \in \mathbb{R}^2 : \; u > 0\}.$$

Hence,

$$\gamma(f(x) - \mu) + \lambda g(x) \geq 0, \; \forall\, x \; \in C \Longleftrightarrow \gamma > 0, \; \lambda = 0.$$

(*a*2) *If $\operatorname*{argmin}_K f \cap S^-_g(0) = \emptyset$, $S^+_f(\mu) \cap S^-_g(0) = \emptyset = S^-_f(\mu)$, then*

$$\text{cone}_+(F(C) - \mu(1, 0) + \mathbb{R}^2_{++}) = \{(u, v) \in \mathbb{R}^2 : \; u > 0, \; v > 0\}.$$

Hence

$$\gamma(f(x) - \mu) + \lambda g(x) \geq 0, \; \forall\, x \in C \Longleftrightarrow (\gamma, \lambda) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}.$$

($a3$) *If* $S_f^+(\mu)\cap S_g^-(0)\neq\emptyset$ *with* $-\infty<r<0$ *and* $S_f^-(\mu)\neq\emptyset$, *then* $\operatorname*{argmin}_K f\cap S_g^-(0)=\emptyset$, $s\leq r$ *and*

$$\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})=\{(u,v)\in\mathbb{R}^2:\ v>ru,\ v>su\}.$$

Hence,

$$\gamma(f(x)-\mu)+\lambda g(x)\geq 0,\ \forall\ x\ \in\ C\Longleftrightarrow\gamma>0,\ \ -\frac{1}{s}\gamma\leq\lambda\leq-\frac{1}{r}\gamma.$$

($a4$) *If* $S_f^+(\mu)\cap S_g^-(0)=\emptyset$, $S_f^-(\mu)\neq\emptyset$ *with* $-\infty<s<0$, *then* $\operatorname*{argmin}_K f\cap S_g^-(0)=\emptyset$ *and*

$$\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})=\{(u,v)\in\mathbb{R}^2:\ v>su,\ v>0\}.$$

Hence

$$\gamma(f(x)-\mu)+\lambda g(x)\geq 0,\ \forall\ x\in C\Longleftrightarrow\gamma\geq 0,\ \lambda\geq-\frac{1}{s}\gamma,\ \lambda\neq 0.$$

($a5$) *If* $\operatorname*{argmin}_K f\cap S_g^-(0)=\emptyset$, $S_f^+(\mu)\cap S_g^-(0)\neq\emptyset$ *with* $-\infty<r<0$ *and* $S_f^-(\mu)=\emptyset$, *then*

$$\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})=\{(u,v)\in\mathbb{R}^2:\ v>ru,\ u>0\}.$$

Hence,

$$\gamma(f(x)-\mu)+\lambda g(x)\geq 0,\ \forall\ x\ \in C\Longleftrightarrow\gamma>0,\ 0\leq\lambda\leq-\frac{1}{r}\gamma.$$

($a6$) *If* $S_f^-(\mu)\neq\emptyset$, $s=0$, *then* $\operatorname*{argmin}_K f\cap S_g^-(0)=\emptyset$ *and*

$$\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})=\{(u,v)\in\mathbb{R}^2:\ v>0\}.$$

Hence

$$\gamma(f(x)-\mu)+\lambda g(x)\geq 0,\ \forall\ x\in C\Longleftrightarrow\gamma=0,\ \lambda>0.$$

Proof. Since the proof uses a frequent application of the convexity of $\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, the splitting (8) and (13) along with Figures 1, 2, we simply prove ($a1$) and ($a2$) just to give an idea of the reasoning to be employed.

($a1$): By assumptions and due to the convexity of $\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, looking at Figure 1($a1$), we immediately get that $S_f^-(\mu)=\emptyset$, and so

$$\text{cone}_+(F(C)-\mu(1,0)+\mathbb{R}^2_{++})=\{(u,v)\in\mathbb{R}^2:\ u>0\}.$$

From this, the equivalence follows in view of (13).

($a2$): It is a consequence of the splitting (8) and (13).

As mentioned above all other assertions follow in similar way by taking into account the convexity of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, (8) and (13), see Figures 1, 2. $\square$

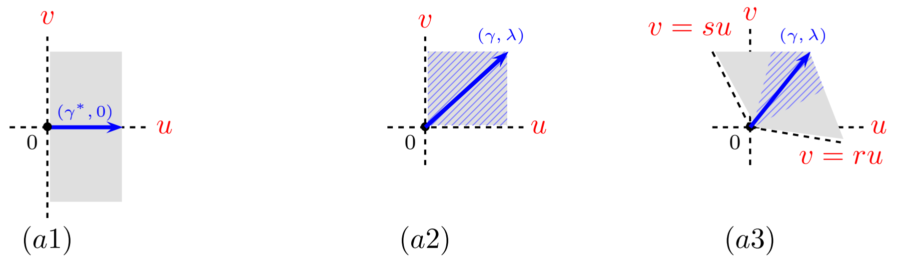

Figure 1: Theorem 3.5:$(a1)$, $(a2)$, $(a3)$

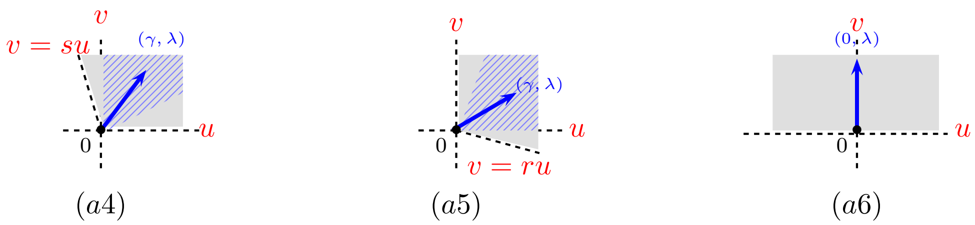

Figure 2: Theorem 3.5:$(a4)$, $(a5)$, $(a6)$

One can check that Example 3.1 satisfies $(a3)$, since

$$(0,-2) \in S_f^-(\mu) \quad \text{and} \quad \operatorname*{argmin}_K f \cap S_g^-(0) = \emptyset.$$

We have also obtained $r = s = -\dfrac{2}{3}$; therefore $\lambda = \dfrac{2}{3}\gamma$.

Before providing a characterization of strong duality, some preliminaries are necessary for linking the behaviour of the Hessians of f and g and the number r and s. We first provide a necessary condition to have $\mu \in \mathbb{R}$.

Proposition 3.6. *Assume that μ is finite. Then,*

$$0 \neq v \in \ker H, \; v^\top Bv \leq 0 \implies v^\top Av \geq 0. \tag{14}$$

Proof. Let $v \in \ker H$, $v \neq 0$, we distinguish the discussion into two cases: $v^\top Bv < 0$ and $v^\top Bv = 0$.

In the first case, given $x \in H^{-1}(d)$, we obtain $g(x+tv) = g(x) + t\nabla g(x)^\top v + \frac{t^2}{2}v^\top Bv \to -\infty$ as $|t| \to +\infty$ since $v^\top Bv < 0$. Thus, there exists $t_1 > 0$ such that $x + tv \in S_g^-(0)$ for all $|t| \geq t_1$, which gives $f(x) + t\nabla f(x)^\top v + \frac{t^2}{2}v^\top Av = f(x+tv) \geq \mu$ for all $|t| \geq t_1$ since $x + tv \in H^{-1}(d)$. On dividing by t^2 and letting $t \to +\infty$, we get $v^\top Av \geq 0$.

Now assume that $v^\top Bv = 0$, and suppose on the contrary that $v^\top Av < 0$. This yields, given any $x \in H^{-1}(d)$, $f(x+tv) \to -\infty$ for all $|t| \to +\infty$. Then $g(x+tv) = g(x) + t\nabla g(x)^\top v > 0$ for all $|t|$ sufficiently large, which implies that $\nabla g(x)^\top v = 0$, and therefore $g(x) = g(x+tv) > 0$ for all $t \in \mathbb{R}$ and all $x \in H^{-1}(d)$. This cannot happen if we choose x satisfying in addition $g(x) \leq 0$. $\square$

The necessary condition (14) given in the previous proposition is stronger than the condition

$$0 \neq v \in \ker H,\ v^\top Bv = 0 \implies v^\top Av \geq 0. \tag{15}$$

This is related to a relaxed version of Finsler's theorem due to Moré [31, Theorem 2.3] and independently to Hamburger [20]: assume that B be indefinite, then (i) and (ii) below are equivalent:

(i) $v \in \ker H,\ v^\top Bv = 0 \implies v^\top Av \geq 0$.

(ii) $\exists\ t \in \mathbb{R}$ such that $A + tB$ is positive semidefinite on $\ker H$.

Proposition 3.7. *Let $v \in \ker H$, $v^\top Av = 0$ and $v^\top Bv < 0$, and assume that μ is finite. Then,*

(a) $\exists\ t_1 > 0$ *such that* $x + tv \ \in\ S_g^-(0)$, $\forall\ x \in H^{-1}(d)$ *and* $\forall\ |t| \geq t_1$;

(b) $\nabla f(x)^\top v = 0,\ \forall\ x \in\ H^{-1}(d)$, *or equivalently,* $f(x+tv) = f(x)\ \forall\ x \in\ H^{-1}(d)$ *and* $\forall\ t \in \mathbb{R}$, *or equivalently,* $\exists\ y \in \mathbb{R}^m$ *such that* $Av = H^\top y$ *and* $d^\top y + a^\top v = 0$;

(c) $S_f^-(\mu) = \emptyset$, *and therefore* $\mu = \inf\limits_{x \in H^{-1}(d)} f(x)$ *with* $\operatorname*{argmin}\limits_{H^{-1}(d)} f \neq \emptyset$;

(d) $S_f^+(\mu) \neq \emptyset \implies r = -\infty$.

Proof. (a): Let $x \in\ H^{-1}(d)$. Then, $g(x+tv) = g(x) + t\nabla g(x)^\top v + \frac{t^2}{2} v^\top Bv \to -\infty$ as $|t| \to +\infty$ since $v^\top Bv < 0$. Thus, there exists $t_1 > 0$ such that $x + tv \ \in\ S_g^-(0)$ for all $|t| \geq t_1$.

(b): For the first equivalence; from (a), $f(x+tv) \geq \mu$ for all $|t| \geq t_1$ because of $x + tv \ \in\ H^{-1}(d)$ and $g(x+tv) < 0$. By writting $\mu \leq f(x+tv) = f(x) + t\nabla f(x)^\top v$, we conclude that $\nabla f(x)^\top v = 0$, and therefore $f(x+tv) = f(x)$ for all $t \in \mathbb{R}$.

One implication for the second equivalence is as follows. By noticing that $H^{-1}(d) = x_0 + \ker H$ for all $x_0 \in H^{-1}(d)$, the equality $(Ax+a)^\top v = 0$ for all $x \in H^{-1}(d)$ implies that $Av \in (\ker H)^\perp = H^\top(\mathbb{R}^m)$. Thus, there exists $y \in \mathbb{R}^m$ such that $Av = H^\top y$ and therefore

$$0 = x^\top Av + a^\top v = x^\top H^\top y + a^\top v = d^\top y + a^\top v.$$

The remaining implication is obvious.

(c): It follows from (a) and (b), along with Proposition 2.3 and Corollary 4.3.

(d): Take any $x_0 \ \in\ S_f^+(\mu)$. Then, from (b) it follows that $f(x_0 + tv) = f(x_0) > \mu$ for all $|t| \geq t_1$. For such t, (a) implies that $x_0 + tv \ \in\ S_f^+(\mu) \cap S_g^-(0)$. Hence, since

$$r \leq \frac{g(x_0) + t\nabla g(x_0)^\top v + \frac{t^2}{2} v^\top Bv}{f(x_0) - \mu}, \quad \forall\ |t| \geq t_1,$$

we infer that $r = -\infty$. $\square$

Proposition 3.8. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be quadratic functions as above: $f(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha$, $g(x) = \frac{1}{2}x^\top Bx + b^\top x + \beta$, $C = H^{-1}(d)$ with μ finite. Then,*

$$(a)\ \ r = -\infty \implies \begin{cases} \operatorname*{argmin}_K f \neq \emptyset, \quad \text{or;} \\ \mu = \inf\limits_{x \in H^{-1}(d)} f(x) \text{ with } \operatorname*{argmin}_{H^{-1}(d)} f \neq \emptyset, \quad \text{or;} \\ \exists\ x_k \in\ S_f^+(\mu) \cap S_g^-(0) : \|x_k\| \to +\infty,\ \frac{x_k}{\|x_k\|} \to v \in \ker\, H,\ \text{and} \\ \ v^\top Av = 0,\ v^\top Bv = 0. \end{cases}$$

$$(b)\ \ s = 0 \implies \begin{cases} \operatorname*{argmin}_K f \cap S_g^=(0) \neq \emptyset, \quad \text{or;} \\ \exists\ x_k \in\ S_f^-(\mu) : \|x_k\| \to +\infty,\ \frac{x_k}{\|x_k\|} \to v \in \ker\, H,\ \text{and} \\ v^\top Av = 0, \quad v^\top Bv = 0. \end{cases}$$

Proof. (a): By assumption, there exists a sequence $x_k \in S_f^+(\mu) \cap S_g^-(0)$ such that

$$\lim_{k \to +\infty} \frac{g(x_k)}{f(x_k) - \mu} = -\infty.$$

We distinguish two cases.

Case 1. $\sup\limits_{k \in \mathbb{N}} \|x_k\| < +\infty$. Up to a subsequence we may assume that $x_k \to x_0$ as $k \to +\infty$. Thus, $g(x_0) \leq 0$ and $f(x_0) \geq \mu$. The case $g(x_0) = 0$, $f(x_0) = \mu$ (resp. $g(x_0) < 0$, $f(x_0) = \mu$) yields $\operatorname*{argmin}_K f \cap S_g^=(0) \neq \emptyset$ (resp. $\operatorname*{argmin}_K f \cap S_g^-(0) \neq \emptyset$). The other situations cannot occur since

$$\lim_{k \to +\infty} \frac{g(x_k)}{f(x_k) - \mu} = \begin{cases} \dfrac{g(x_0)}{f(x_0) - \mu} \neq -\infty & , \text{ if } g(x_0) < 0, \quad f(x_0) > \mu; \\ \\ 0 & , \text{ if } g(x_0) = 0, \quad f(x_0) > \mu. \end{cases}$$

Case 2. $\sup\limits_{k \in \mathbb{N}} \|x_k\| = +\infty$. Then, we can assume that

$$\|x_k\| \to +\infty, \quad \frac{x_k}{\|x_k\|} \to v, \quad \text{as } k \to +\infty, \tag{16}$$

and therefore $v \in \ker\, H$, $v^\top Av \geq 0$ and $v^\top Bv \leq 0$. Moreover, we obtain, as in Case 1,

$$\lim_{k \to +\infty} \frac{g(x_k)}{f(x_k) - \mu} = \begin{cases} \dfrac{v^\top Bv}{v^\top Av} \neq -\infty & , \text{ if } v^\top Av > 0,\ v^\top Bv < 0; \\ \\ 0 & , \text{ if } v^\top Av > 0,\ v^\top Bv = 0. \end{cases}$$

Hence, we must have $v^\top Av = 0$ and $v^\top Bv \leq 0$. In case $v^\top Av = 0$ and $v^\top Bv < 0$, we apply Proposition 3.7(c) to get the second posibility of (a).

(b): We have the existence of a sequence $x_k \in S_f^-(\mu)$ such that

$$\lim_{k \to +\infty} \frac{g(x_k)}{f(x_k) - \mu} = 0.$$

We likewise distinguish two cases.

Case 1. $\sup_{k\in\mathbb{N}} \|x_k\| < +\infty$. Up to a subsequence, we obtain $x_k \to x_0 \in H^{-1}(d)$, $g(x_0) \geq 0$ and $f(x_0) \leq \mu$. Since $g(x_0) = 0$ and $f(x_0) < \mu$ is impossible, and because of

$$\lim_{k\to+\infty} \frac{g(x_k)}{f(x_k)-\mu} = \begin{cases} \dfrac{g(x_0)}{f(x_0)-\mu} \neq 0 & , \text{ if } g(x_0) > 0, \quad f(x_0) < \mu; \\ \\ -\infty & , \text{ if } g(x_0) > 0, \quad f(x_0) = \mu, \end{cases}$$

we must have $g(x_0) = 0$ and $f(x_0) = \mu$.

Case 2. $\sup_{k\in\mathbb{N}} \|x_k\| = +\infty$. Passing to a subsequence, if necessary, we have (16), and therefore $v \in \ker H$, $v^\top Av \leq 0$ and $v^\top Bv \geq 0$. As in (a), we get necessarily $v^\top Av \leq 0$ and $v^\top Bv = 0$. The conclusion follows after noticing that $v^\top Av < 0$ and $v^\top Bv = 0$ cannot occur by Proposition 3.6. □

In view of Propositions 3.7, 3.6 and 3.8, the following conditions arise:

- $$[0 \neq v \in \ker H,\ v^\top Bv \leq 0] \implies v^\top Av > 0; \tag{17}$$
- $$[0 \neq v \in \ker H,\ v^\top Bv = 0] \implies v^\top Av > 0; \tag{18}$$
- $$[v \in \ker H,\ v^\top Av = 0 = v^\top Bv] \implies v = 0; \tag{19}$$
- $$[0 \neq v \in \ker H,\ v^\top Bv = 0] \implies v^\top Av \neq 0; \tag{20}$$
- $$0 \neq v \in \ker H \implies [v^\top Av \neq 0 \text{ or } v^\top Bv \neq 0]. \tag{21}$$

Clearly,

$$(17) \implies (18) \implies (19) \iff (20) \iff (21).$$

By Finsler's theorem [14] (see also [20]), condition (18) is equivalent to:

$$\exists\, t \in \mathbb{R},\ A + tB \text{ is positive definite on } \ker H. \tag{22}$$

When this condition is satisfied it is said that the Simultaneous Diagonalization property holds, since it implies the existence of a nonsingular matrix C such that both $C^\top AC$ and $C^\top BC$ are diagonal [21, Theorem 7.6.4]. Such an assumption allowed the authors in [4] to re-write the original problem in a more tractable one.

In [40] when $H = 0$ and $d = 0$, some relationships between (15), (18), (19) and the Yakuvobich S-lemma (with quadratic homogeneous functions) are estalished. They are related with the non-strict Finsler's, strict Finsler's and Finsler-Calabi's theorem, respectively.

Under assumption (20), (b) of Proposition 3.8, implies the following corollary

Corollary 3.9. *Assume that f, g be as above with $C = H^{-1}(d)$ and $\mu \in \mathbb{R}$. If $s = 0$ and (20) is satisfied then $\operatorname*{argmin}_K f \cap S_g^{=}(0) \neq \emptyset$ and strong duality does not hold.*

Proof. If $s = 0$ then by (b) of Proposition 3.8 we obtain that either $\operatorname*{argmin}_K f \cap S_g^{=}(0) \neq \emptyset$ or there exists $0 \neq v \in \ker H$ satisfying $v^\top Av = 0$ and $v^\top Bv = 0$. By assumption the second situation is not possible, and therefore the first holds proving the desired result. The lack of strong duality is a consequence of $(a6)$ in Theorem 3.5. $\square$

In contrast to a similar result due to Moré [31] where condition (17) (stronger than (20)) is imposed, our corollary applies to situations where Theorem 3.3 in [31] does not.

Corollary 3.10. *Assume that f, g be as above with $C = H^{-1}(d)$ and $\mu \in \mathbb{R}$. If $r = -\infty$ and (20) is satisfied then strong duality holds and either $\operatorname*{argmin}_K f \neq \emptyset$ or $\mu = \inf_{x \in H^{-1}(d)} f(x)$ with $\operatorname*{argmin}_{H^{-1}(d)} f \neq \emptyset$.*

Proof. It is a direct consequence of (a) in Proposition 3.8. $\square$

Next result, which is new, on one hand characterizes the regularized strong duality without requiring the nonemptiness of $\operatorname*{argmin}_K f$, and where the Slater condition may fail, and on the other, gives a sufficient or necessary condition in terms of inequality systems.

Theorem 3.11. *Let μ be finite with $C = H^{-1}(d)$. Let us consider the following assertions:*

(a) *$\operatorname*{argmin}_K f \cap S_g^{=}(0) = \emptyset$ and (21) holds;*

(b) *strong duality holds;*

(c) *either $S_f^{-}(\mu) = \emptyset$ or $[S_f^{-}(\mu) \neq \emptyset$ with $s < 0]$ holds;*

(d) *either $\inf_{x \in C} f(x) = \mu$ or $[v \in \ker H,\ v^\top Av \leq 0 \implies v^\top Bv \geq 0]$* holds.

Then, we have the following relationships:

$$(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longrightarrow (d).$$

Proof. $(a) \implies (c)$: We have to check that $s < 0$. If on the contrary, $s = 0$, by using Proposition 3.8(b) we get a contradiction.
$(b) \implies (c)$: Suppose that $S_f^{-}(\mu) \neq \emptyset$. Strong duality implies the existence of $\lambda_0 \geq 0$ such that $f(x) + \lambda_0 g(x) \geq \mu$ for all $x \in C$, which yields $\lambda_0 > 0$. Indeed, if $\lambda_0 = 0$, the

previous inequality gives $f(x)-\mu \geq 0$ for all $x \in C$, which is impossible if $S_f^-(\mu) \neq \emptyset$. Now, suppose that $s=0$. Then, there exists $\bar{x} \in S_f^-(\mu) \neq \emptyset$ such that

$$\frac{g(\bar{x})}{f(\bar{x})-\mu} > -\frac{1}{\lambda_0}.$$

It follows that $f(\bar{x})+\lambda_0 g(\bar{x}) < \mu$, giving a contradiction; this proves that $s>0$.
$(c) \Longrightarrow (b)$: It is simply a consequence of Theorem 3.5 by looking at those items where $\gamma^* > 0$ is possible.
$(c) \Longrightarrow (d)$: If $S_f^-(\mu)=\emptyset$ then $f(x) \geq \mu$ for all $x \in C$ by Proposition 3.2. Assume now that $S_f^-(\mu) \neq \emptyset$ and $s<0$. Due to the convexity of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$, we obtain $\operatorname*{argmin}_K f \cap S_g^-(0)=\emptyset$. We consider two cases: $S_g^-(0)=\emptyset$ or $S_g^-(0) \neq \emptyset$. Obviously in the first case, the implication in (d) holds vacuously. If $S_g^-(0) \neq \emptyset$, it follows that $S_f^+(\mu) \cap S_g^-(0) \neq \emptyset$ since otherwise $S_g^-(0)=\emptyset$ by Proposition 2.3(b). Thus, we must have $-\infty < s \leq r < 0$ again by the convexity of $\text{cone}(F(C)-\mu(1,0)+\mathbb{R}^2_{++})$. From Proposition 3.7$(d)$ it follows that $v \in \ker\, H,\ v^\top Bv<0 \Longrightarrow v^\top Av \neq 0$, which together with (14) yields the desired implication. $\square$

Example 3.1 shows that the implication $(c) \Longrightarrow (a)$ may be false, and the next instance shows the second part of (d) does not necessarily imply the second part of (c).

Example 3.12. *Let $C=\mathbb{R}^n$, $f(x_1,x_2)=x_1+x_2$ and $g(x_1,x_2)=(x_1+x_2)^2$. Clearly it satisfies the second part of (d), but it holds $S_f^-(\mu) \neq \emptyset$ with $s=0$. Indeed, $K=\{(0,0)\}$ and*

$$S_f^-(\mu)=\{(x_1,x_2) \in \mathbb{R}^2 : \ x_1+x_2<0\}.$$

Hence,

$$s=\sup_{x_1+x_2<0} \frac{(x_1+x_2)^2}{x_1+x_2}=0,$$

and the strong duality does not hold, since for any $\lambda>0$, the inequality

$$x_1+x_2+\lambda(x_1+x_2)^2 \geq 0, \quad \forall\ (x_1,x_2) \in \mathbb{R}^2$$

yields a contradiction. This agrees with $(a6)$ of Theorem 3.5.

Next example illustrates a situation where our main Theorem 3.5 applies, exhibiting that strong duality holds without satisfying the Slater condition: there exists $x_0 \in H^{-1}(d)$ such that $g(x_0)<0$.

Example 3.13. *Take* $H(x_1, x_2) = x_1 - x_2$, $d = 0$, $f(x_1, x_2) = 2x_1^2 - x_2^2$, $g(x_1, x_2) = x_1^2 - x_2^2$. *Here,* $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2,\ x_1^2 - x_2^2 \leq 0\} = \{(t, t) \in \mathbb{R}^2 :\ t \in \mathbb{R}\}$. *Clearly,* $S_g^-(0) = \emptyset = S_f^-(\mu)$ *and* $\mu = 0$ *with* $\operatorname{argmin}_K f = \{(0, 0)\}$. *According to* (*a*2) *of Theorem 3.5, we conclude that strong duality holds by choosing any* $\lambda^* \geq 0$.

Given a vector subspace $P \subseteq \mathbb{R}^n$, we recall that a symmetric matrix A is positive semidefinite on P if $x^\top Ax \geq 0$ for all $x \in P$. By $M^\perp$ we mean the orthogonal subspace of $M \subseteq \mathbb{R}^m$, that is, $M^\perp = \{\xi \in \mathbb{R}^m :\ \langle \xi, x\rangle = 0\ \ \forall\ x \in M\}$. Next theorem, which is new in the literature, considers non-convex situations.

Theorem 3.14. *Let* f *and* g *be quadratic functions as above,* μ *finite and* $\bar{x}$ *feasible for* (11). *Set* $C = \{x \in \mathbb{R}^n :\ Hx = d\}$. *The following assertions are equivalent:*

(*a*) $\bar{x}$ *is a solution to* (*11*) *and either* $S_f^-(\mu) = \emptyset$ *or* $[S_f^-(\mu) \neq \emptyset$ *with* $s < 0]$ *holds*;

(*b*) $\exists\ \lambda \geq 0\ \exists\ y \in \mathbb{R}^m$ *such that* $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) + H^\top y = 0$, $\lambda g(\bar{x}) = 0$, $A + \lambda B$ *is positive semidefinite on* ker H.

Proof. (*a*) $\Longrightarrow$ (*b*): By Theorem 3.11, strong duality holds, thus, there exists $\lambda \geq 0$ such that

$$f(\bar{x}) + \lambda g(\bar{x}) \leq f(\bar{x}) = \inf_{x \in C}(f(x) + \lambda g(x)).$$

This implies that $\lambda g(\bar{x}) = 0$ and $\bar{x}$ is a minimum for $L(x) = f(x) + \lambda g(x)$ on C. The necessary optimality condition yields

$$\langle \nabla f(\bar{x}) + \lambda \nabla g(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall\ x \in C.$$

Since $x - \bar{x} \in \ker H$ for all $x \in C$, we obtain $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) \in (\ker H)^\perp = H^\top(\mathbb{R}^m)$. Thus, there exists $y \in \mathbb{R}^m$ such that $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) + H^\top y = 0$. On the other hand, we also have $f(x) + \lambda g(x) \geq f(\bar{x})$ for all $x \in C$, which gives $\lambda g(\bar{x}) = 0$ and $v^\top(A + \lambda B)v \geq 0$ for all $v \in \ker H$, i.e., $A + \lambda B$ is positive semidefinite on ker H.
(*b*) $\Longrightarrow$ (*a*): Setting $L(x) = f(x) + \lambda g(x)$, $x \in C$, we write

$$L(x) - L(\bar{x}) = \langle \nabla f(\bar{x}) + \lambda^* \nabla g(\bar{x}), x - \bar{x}\rangle + \frac{1}{2}\langle (A + \lambda B)(x - \bar{x}), x - \bar{x}\rangle.$$

By taking into account that $x - \bar{x} \in \ker H$ for all $x \in C$ and the assumptions, the previous equality implies that

$$f(x) \geq L(x) \geq L(\bar{x}) = f(\bar{x}) + \lambda g(\bar{x}) = f(\bar{x}), \quad \forall\ x \in C,\ g(x) \leq 0,$$

which yields $f(x) \geq f(\bar{x})$, proving that $\bar{x}$ is a solution to (11). □

By applying Theorem 3.3, we re-obtain Theorem 3.8 in [25] which generalizes the Moré theorem [31, Theorem 3.4].

Corollary 3.15. [*25, Theorem* 3.8] (*Under Slater condition*) *Let f, g be any quadratic functions with $\mu \in \mathbb{R}$. Assume that $Hx_0 = d$ and $g(x_0) < 0$ for some $x_0 \in \mathbb{R}^n$, and let $\bar{x} \in K$ (feasible for problem* (11)*). Then, the following assertions are equivalent:*

(*a*) $\bar{x} \in \underset{K}{\text{argmin}}\ f$;

(*b*) $\exists\ \lambda \geq 0\ \exists\ y \in \mathbb{R}^m$ *such that* $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) + H^\top y = 0$, $\lambda g(\bar{x}) = 0$ *and* $A + \lambda B$ *is positive semidefinite on* $\ker H$.

Proof. In case $S_f^-(\mu) = \emptyset$ the result is a consequence of Theorem 3.14. If $S_f^-(\mu) \neq \emptyset$ we need to check that $s < 0$ and the result again is a consequence of Theorem 3.14. Suppose on the contrary that $s = 0$. Then, by the convexity of $\text{cone}(F(\mathbb{R}^n) - \mu(1, 0) + \mathbb{R}_{++})$ (see Theorem 3.3 or ($a6$) of Theorem 3.5), we must have

$$S_f^+(\mu) \cap S_g^-(0) = \emptyset \quad \text{and} \quad \underset{K}{\text{argmin}}\ f \cap S_g^-(0) = \emptyset.$$

This implies that $S_g^-(0) = \emptyset$ by Proposition 2.3, contradicting the Slater condition. Therefore, $s < 0$, and the conclusion follows. $\square$

For completeness we establish a characterization of solutions when Slater condition fails, that is,

$$g(x) \geq 0, \quad \forall\ x \in H^{-1}(d). \tag{23}$$

Under this assumption,

$$K = \{x \in H^{-1}(d) :\ g(x) = 0\} = \underset{H^{-1}(d)}{\text{argmin}}\ g, \tag{24}$$

provided $K \neq \emptyset$. By Corollary 4.3, for $\bar{x} \in H^{-1}(d)$,

$$\bar{x} \in \underset{H^{-1}(d)}{\text{argmin}}\ g \Longleftrightarrow \left\{ \begin{array}{c} B \text{ is positive semidefinite on } \ker H \text{ and} \\ \exists\ y \in \mathbb{R}^m,\ B\bar{x} + b + H^\top y = 0. \end{array} \right. \tag{25}$$

Therefore, if B is positive semidefinite on $\ker H$, then

$$\bar{x} \in \underset{K}{\text{argmin}}\ f \Longleftrightarrow \exists\ y \in \mathbb{R}^m,\ (\bar{x}, y) \in \underset{\tilde{K}}{\text{argmin}}\ \tilde{f}, \tag{26}$$

where

$$\tilde{K} = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m :\ \begin{pmatrix} B & H^\top \\ H & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -b \\ d \end{pmatrix} \right\} \text{ and}$$

$$\tilde{f}(x, y) = \frac{1}{2}(x \ \ y) \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (a \ \ 0) \begin{pmatrix} x \\ y \end{pmatrix} + \alpha = f(x).$$

Hence, an application of Corollary 4.3 to $\tilde{f}$ and $\tilde{K}$ instead of f and K again, leads to the following corollary.

Corollary 3.16. (*Slater condition fails*) *Let f, g be any quadratic functions with $\mu \in \mathbb{R}$ and $\bar{x} \in K$. Assume that* (23) *holds. Then the following statements are equivalent:*

(a) $\bar{x} \in \operatorname*{argmin}_K f$;

(b) *B is positive semidefinite on* $\ker H$*, A is positive semidefinite on* $\ker H \cap B^{-1}[(\ker H)^{\perp}]$*, and $\exists\ v \in \ker H$ and $\exists\ (y,z) \in \mathbb{R}^m \times \mathbb{R}^m$ such that*

$$A\bar{x} + a + Bv + H^{\top} z = 0, \;\; B\bar{x} + b + H^{\top} y = 0.$$

4 The Frank-Wolfe and Eaves theorem revisited

In this section motivated by the form of the Lagrangian introduced in the previous section, we revisited the Frank and Wolfe theorem [17], by providing several equivalences to the nonemptiness of the solution set, in contrast to the only equivalence between (a) and (d) (of Theorem 4.1) established by Frank and Wolfe, or Blum and Oettli (the latter authors use elementary analysis in their proof). We believe that our proof is still shorter than that by Blum and Oettli [5], and it is suitable for expository purposes. The original proof of Frank and Wolfe theorem requires a decomposition theorem for convex polyhedra. We recall that given a convex cone P, it is said that A is *copositive* on P if $x^{\top}Ax \geq 0$ for all $x \in P$. Furthermore, it said that a subset $K \subseteq \mathbb{R}^n$ is *asymptotically linear* [2, Definition 2.3.1] if for all $\rho > 0$ and all sequence $x_k \in K$, satisfying $\|x_k\| \to +\infty$, $\dfrac{x_k}{\|x_k\|} \to v \in K^{\infty}$, there exists $k_0 \in \mathbb{N}$, such that $x_k - \rho v \in K$ for all $k \geq k_0$.
Here, K^{∞} is the asymptotic cone of K defined as in Section 2.
Observe that polyhedral sets are asymptotically linear, but there are asymptotically linear sets that are not polyhedral, see after Definition 2.3.2 in [2]. For instance, convex sets without lines (see [2]).
Next theorem is a refinement of the Frank and Wolfe theorem when the constraints set is asymptotically linear, and likewise it improves some of the main results of Section 3 in [13].
Other extensions in different directions of the Frank-Wolfe theorem may be found in [28, 3].

Theorem 4.1. *Let $K \subseteq \mathbb{R}^n$ be closed, convex and asymptotically linear; $h(x) = \frac{1}{2}x^{\top}Ax + a^{\top}x + \alpha$ with $A \in S^n$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. The assertions* (a), (b), (c) *and* (d) *are equivalent, where*

(a) $-\infty < \nu \doteq \inf_{x \in K} h(x)$;

(*b*) *A is copositive on* K^∞ *and* $[v^\top Av = 0,\ v \in K^\infty \Longrightarrow (Ax+a)^\top v \geq 0 \ \ \forall\, x \in K]$;

(*c*) *A is copositive on* K^∞ *and* $[v^\top Av = 0,\ v \in K^\infty,\ (Ax+a)^\top v \leq 0,\ x \in K \Longrightarrow (Ax+a)^\top v = 0]$;

(*d*) $\operatorname*{argmin}_K h \neq \emptyset$.

Furthermore, we have (*e*) $\Longrightarrow$ (*f*) $\Longrightarrow$ (*g*) $\Longrightarrow$ (*h*), *where*

(*e*) *h is coercive, i.e.,* $\lim\limits_{\substack{\|x\|\to+\infty \\ x\in K}} h(x) = +\infty$;

(*f*) *A is copositive on* K^∞ *and* $[v^\top Av = 0,\ v \in K^\infty,\ (Ax+a)^\top v \leq 0,\ x \in K \Longrightarrow v = 0]$.

(*g*) $\operatorname*{argmin}_K h$ *is nonempty and bounded;*

(*h*) *A is copositive on* K^∞ *and* $[v^\top Av = 0,\ v \in K^\infty,\ (Ax+a)^\top v \leq 0,\ \forall\, x \in K \Longrightarrow v = 0]$.

It is worth noticing that under convexity on h, i.e., positive semidefiniteness of A (which infers that: $v^\top Av = 0$ *if and only if* $v \in \ker A$*), one obtains* (*h*) $\Longrightarrow$ (*f*)*, and therefore all of them are equivalent. In general,* (*h*) *does not imply* (*e*) *as Example 4.2 shows.*

Proof. (*a*) $\Longrightarrow$ (*b*): Let us prove first that A is copositive on K^∞. For $x_0 \in K$ and $v \in K^\infty$, we obtain by assumption,

$$h(x_0+tv) = h(x_0) + t\langle \nabla h(x_0), v\rangle + \frac{1}{2}t^2 v^\top Av \geq \nu \quad \forall\, t \in \mathbb{R}. \tag{27}$$

Thus,

$$\frac{1}{t^2}h(x_0) + \frac{1}{t}\langle \nabla h(x_0), v\rangle + \frac{1}{2}v^\top Av \geq \frac{\nu}{t^2} \quad \forall\, t \in \mathbb{R},\ t \neq 0.$$

Letting $t \to +\infty$, we get $v^\top Av \geq 0$ for all $v \in K^\infty$, proving that A is copositive on K^∞. Take $v \in K^\infty$ such that $v^\top Av = 0$, then from (27) we obtain, $(Ax_0+a)^\top v \geq 0$, concluding that (*b*) holds.

(*b*) $\Longrightarrow$ (*c*): It is straightforward.

(*b*) $\Longrightarrow$ (*d*): For every $k \in \mathbb{N}$, setting $B_k \doteq \{x \in K :\ \|x\| \leq k\}$, we may assume that $B_k \neq \emptyset$ for all $k \in \mathbb{N}$. Let us consider the problem

$$\inf_{x\in B_k} h(x), \tag{28}$$

which always has solution. Let x_k be such that

$$\|x_k\| = \min\{\|x\| :\ x \in \operatorname*{argmin}_{B_k} h\}.$$

Case 1. $\sup_{k\in\mathbb{N}} \|x_k\| < \infty$. One can check that any limit point of (x_k) belongs to $\operatorname{argmin}_K h$.

Case 2. $\sup_{k\in\mathbb{N}} \|x_k\| = +\infty$. We can assume that $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to v$ as $k \to +\infty$, thus $v \in K^\infty$.

Since K is asymptotically linear given $\rho > 0$ there exists k_0 such that $x_k - \rho v \in K$ for all $k \geq k_0$. We can also assume that $\left\|\dfrac{x_k}{\|x_k\|} - v\right\| < 1$ and $\dfrac{\rho}{\|x_k\|} < 1$ for all $k \geq k_0$. Then, by writing

$$x_k - \rho v = \Big(1 - \frac{\rho}{\|x_k\|}\Big)x_k + \frac{\rho}{\|x_k\|}\Big(x_k - \|x_k\|v\Big), \tag{29}$$

we get $\|x_k - \rho v\| < \|x_k\|$. On the other hand, given any $x \in K$, there exists $k_1 \in \mathbb{N}$ such that

$$h(x_k) = \frac{1}{2}x_k^\top A x_k + a^\top x_k + \alpha \leq h(x),\ \forall\ k \geq k_1.$$

It follows that $v^\top A v \leq 0$ and so by the copositive assumption $v^\top A v = 0$. Again by assumption we have $(Ax + a)^\top v \geq 0$ for all $x \in K$.

Set $u_k \doteq x_k - \rho v$. Then, for all $k \geq k_0$, $u_k \in K$, $\|u_k\| < \|x_k\|$ and

$$h(u_k) = h(x_k - \rho v) = h(x_k) - \rho(Ax_k + a)^\top v + \rho^2 v^\top A v \leq h(x_k).$$

This means that $u_k \in \operatorname{argmin}_{B_k} h$ for all k sufficiently large, contradicting the choice of x_k.

Consequently, Case 2 cannot happen, and hence $\operatorname{argmin}_K h \neq \emptyset$.

$(d) \implies (a)$: It is straightforward.

$(e) \implies (f)$: Evidently the coercive property of h implies the first part of (f), and the second part easily follows as well.

$(f) \implies (g)$: That $\operatorname{argmin}_K h \neq \emptyset$ follows from (c) implies (d). Suppose there exists a sequence of minimizers x_k such that $\|x_k\| \to +\infty$. Up to a subsequence we may assume that $\dfrac{x_k}{\|x_k\|} \to v \in K^\infty \setminus \{0\}$. From the equality $h(x_k) = \nu$ it follows that $v^\top A v = 0$. On the other hand, by the classical optimality condition, $\nabla h(x_k)^\top (x - x_k) \geq 0$ for all $x \in K$. Given $\rho > 0$, as above, we choose k sufficiently large such that $x_k - \rho v \in K$. Thus $(Ax_k + a)^\top v \leq 0$, which by assumption yields $v = 0$, giving a contradiction.

$(g) \implies (h)$: The first part of (h) is a consequence of (a) implies (b). Take $v \in K^\infty$ satisfying $v^\top A v = 0$ and $(Ax + a)^\top v \leq 0$ for all $x \in K$. We suppose on the contrary that $v \neq 0$. From the equality in (27) for x_0 to be a minimizer, we deduce that $h(x_0 + tv) \leq h(x_0)$ for all $t > 0$, which says that $x_0 + tv \in \operatorname{argmin}_K h$ for all $t > 0$, which is not possible if $\operatorname{argmin}_K h$ is bounded and $v \neq 0$. $\square$

The following example show that the reverse implications in the preceding theorem need not to be true in general.

Example 4.2. *This example shows, that in general* (h) *does not imply* (g)*. Take* $h_1(x_1, x_2) = x_1^2 - x_2^2$, $K_1 = \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - x_2| \leq 1\}$*. Thus* $K_1^\infty = \{(x, x) \in \mathbb{R}^2 : \ x \in \mathbb{R}\}$*. One can easily check that* (h) *holds but* $\operatorname*{argmin}_{K_1} h_1 = \emptyset$*.*

The case $K = H^{-1}(d) = \{x \in \mathbb{R}^n : \ Hx = d\}$ deserves a special attention, and it is in connection with the Lagrangian appeared in Section 3.

Corollary 4.3. *Let* $h(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha$ *with* $A \in S^n$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$*. The following assertions are equivalent:*

(a) $-\infty < \nu \doteq \inf_{x \in H^{-1}(d)} h(x)$;

(b) A *is positive semidefinite on* $\ker\, H$ *and* $[v^\top Av = 0, \ v \in \ker\, H \Longrightarrow (Ax + a)^\top v = 0 \ \ \forall \ x \in H^{-1}(d)]$;

(c) A *is positive semidefinite on* $\ker\, H$ *and* $[v^\top Av = 0, \ v \in \ker\, H \Longrightarrow \exists \ y \in \mathbb{R}^m : \ Av = H^\top y \text{ and } d^\top y + a^\top v = 0]$;

(d) $\operatorname*{argmin}_{H^{-1}(d)} h \neq \emptyset$;

(e) A *is positive semidefinite on* $\ker\, H$ *and there exist* $\bar{x} \in H^{-1}(d)$, $y \in \mathbb{R}^m$ *such that* $A\bar{x} + a + H^\top y = 0$*.*

Proof. By virtue of the previous theorem we need only to check $(b) \Longleftrightarrow (c)$ and $(d) \Longrightarrow (e) \Longrightarrow (a)$.
The equivalence between (b) and (c) follows as in (b) of Proposition 3.7.
$(d) \Longrightarrow (e)$: Let $\bar{x} \in \operatorname*{argmin}_{H^{-1}(d)} h$. Then by the usual necessary optimality condition, we have $\langle \nabla h(\bar{x}), x - \bar{x} \rangle \geq 0$ for all $x \in H^{-1}(d)$. Since $x - \bar{x} \in \ker\, H$ for all $x \in H^{-1}(d)$, we get $A\bar{x} + a = \nabla h(\bar{x}) \in (\ker\, H)^\perp = H^\top(\mathbb{R}^m)$. Hence there exists $y \in \mathbb{R}^m$ such that $A\bar{x} + a + H^\top y = 0$, which is the desired result.
$(e) \Longrightarrow (a)$: it is straightforward, once we notice that $H^{-1}(d) = \bar{x} + \ker\, H$ and

$$h(x + \bar{x}) = h(\bar{x}) + \langle \nabla h(\bar{x}), x \rangle + \frac{1}{2}x^\top Ax, \quad x \in \ker\, H.$$

□

When H is the null matrix and $d = 0$, the previous result admits a more precise formulation as expressed in the following corollary. Recall that when $A \succcurlyeq 0$, i.e., A is positive semidefinite (on $\mathbb{R}^n$), we have

$$v^\top Av = 0 \iff v \in \ker\, A.$$

Corollary 4.4. *Let* $h(x) = \dfrac{1}{2}x^\top Ax + a^\top x + \alpha$ *with* $A \in S^n$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. *The following assertions are equivalent:*

(a) $-\infty < \nu \doteq \inf_{x\in\mathbb{R}^n} h(x)$;

(b) $A \succcurlyeq 0$ *and* $[v \in \ker\ A \Longrightarrow a^\top v = 0]$;

(c) $\operatorname*{argmin}_{\mathbb{R}^n} h \neq \emptyset$;

(d) $A \succcurlyeq 0$ *and there exists* $\bar{x} \in \mathbb{R}^n$ *such that* $A\bar{x} + a = 0$.

5 Nonconvex quadratic objective with nonconvex constraints: necessary conditions for existence

We are now interested in establishing a necessary optimality conditions for the existence of solutions to problem (11) As before, $f(x) = \dfrac{1}{2}x^\top Ax + a^\top x + \alpha$, $g(x) = \dfrac{1}{2}x^\top Bx + b^\top x + \beta$. This implies that $K \doteq \{x \in H^{-1}(d) :\ g(x) \leq 0\}$ is closed.

Definition 5.1. [*10*] *A feasible point* $\bar{x} \in K$ *is said to be a local minimum of* (11) *if for any feasible direction* $v \neq 0$ *and some small enough* $t > 0$, *one has*

$$f(\bar{x}) \leq f(\bar{x} + tv).$$

The set of local minima of (11) is denoted by $\operatorname*{argminloc}_K f$.

Next result provides necessary conditions for the nonemptyness and boundedness of solutions to (11).

Lemma 5.2. *Let* f, g *and* $C = H^{-1}(d)$ *be as above.*

(a) *If* $\operatorname*{argminloc}_K f$ *is non-empty then,*

$$Hv = Av = Bv = 0,\ b^\top v \leq 0,\ a^\top v \leq 0 \Longrightarrow a^\top v = 0. \tag{30}$$

(b) *If* $\operatorname*{argmin}_K f$ *is non-empty and bounded, then*

$$Hv = Av = Bv = 0,\ b^\top v \leq 0,\ a^\top v \leq 0 \Longrightarrow v = 0. \tag{31}$$

Proof. (a): Let $v \in \mathbb{R}^n$ satisfy the left hand side of (30). Then v is a feasible direction of (11). Suppose, on the contrary, that $a^\top v < 0$. Then, for any $x \in K$ and $t > 0$, we have

$$f(x + tv) - f(x) = t\nabla f(x)^\top v + t^2 v^\top Av = t(x^\top Av + a^\top v) = ta^\top v < 0, \tag{32}$$

reaching a contradiction if x is a local minimum.
(b): Suppose that v satisfy the relation in the left hand side of (30). By (a), $a^\top v = 0$. It follows that, as above,

$$f(x + tv) = f(x), \ \forall\ x \in K, \ \forall\ t > 0.$$

If $x \in \operatorname*{argmin}_K f$, the previous equality gives a contradiction if $v \neq 0$. $\square$

When K is polyhedral, sufficient and necessary optimality conditions for a point to be a local minimum can be found in [29].

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