A new mixed finite element method for elastodynamics with weak symmetry*

Carlos García[†] Gabriel N. Gatica[‡] Salim Meddahi[§]

Abstract

We provide a new mixed finite element analysis for linear elastodynamics with reduced symmetry. The problem is formulated as a second order system in time by imposing only the Cauchy stress tensor and the rotation as primary and secondary variables, respectively. We prove that the resulting variational formulation is well-posed and provide a convergence analysis for a class of H(div)-conforming semi-discrete schemes. In addition, we use the Newmark trapezoidal rule to obtain a fully discrete version of the problem and carry out the corresponding convergence analysis. Finally, numerical tests illustrating the performance of the fully discrete scheme are presented.

Mathematics Subject Classification. 65N30, 65M12, 65M15, 74H15

Keywords. mixed finite elements, elastodynamics, error estimates

1 Introduction

We analyze a mixed finite element approximation of the linear elastodynamic problem with reduced symmetry. Mixed formulations in elasticity provide a direct finite element approximation of the Cauchy stress tensor and they are immune to the locking phenomenon that generally affects displacement based formulations in the nearly incompressible case. There are several families of mixed finite elements with weak symmetry for the steady elasticity problem, [4, 12, 18, 25]. Our aim here is to prove the stability of the corresponding Galerkin schemes for the elastodynamic problem.

Mixed methods for elastodynamics have already been studied in [5, 7, 10, 13, 21]. In contrast to the strong symmetry approach considered in [7, 13, 21] for the stress tensor, we are interested here in a weak imposition of this restriction, as in [5, 10]. The displacement-stress formulation method presented in [10] relies on a the dual hybrid method introduced in [15] for a two-dimensional problem. Here, we follow [5] and carry out a multi-dimensional error analysis for a class of mixed finite elements satisfying conditions that are known to hold true for the mixed families introduced in [4, 12, 18, 25]. More precisely, our present approach can be formally considered as the second order version of the first

*This work was partially supported by CONICYT-Chile through BASAL project CMM, Universidad de Chile; by project Anillo ACT1118 (ANANUM), Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción; and by the Ministry of Education of Spain through the project MTM2013-43671-P.

[†]CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: carlgarcia@udec.cl.

[‡]CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

[§]Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España, e-mail: salim@uniovi.es.

order hyperbolic system studied in [5], whose main variables are the stress tensor and the velocity. As a consequence, we only maintain the stress tensor as primary unknown (besides the rotation) and end up with a classical wave equation for a tensorial grad-div operator. The advantage of our formulation is that it naturally provides an a priori error bound for the stress variable in the H(div)-norm, which improves the L^2-error estimate obtained for this variable in [5]. Moreover, our error estimates are shown to be uniform with respect to compressibility at the semi-discrete and fully discrete levels. Finally, it is worthwhile to mention that, while the displacement is not explicitly involved in our formulation, it can be numerically post-processed by integrating twice the linear momentum equation.

The rest of the paper is organized as follows. In the preliminary Section 2 we fix some basic notations related with well-known Sobolev spaces. Then, in Section 3 we consider a tensorial wave equation for the grad-div operator and prove its well-posedness by using the classical Galerkin procedure. Next, in Section 4 we show that under suitable compatibility conditions on the initial data, the solution of the aforementioned wave equation is the Cauchy stress tensor corresponding to a properly defined elastodynamic problem. In Section 5 we introduce a semi-discretization of the problem relying on a family of finite dimensional subspaces satisfying standard hypotheses (for mixed finite elements in elasticity problems with reduced symmetry) and prove abstract error estimates. In turn, in Section 6 we use an implicit Newmark method to obtain a fully discrete version of the problem and carry out its convergence analysis. In Section 7 we derive asymptotic error estimates for an example based on the Arnold-Falk-Winther element. Finally, in Section 8 we present numerical results that confirm the theoretical convergence estimates.

2 Notations and preliminary results

We denote by $\boldsymbol{I}$ the identity matrix of $\mathbb{R}^{d\times d}$ $(d=2,3)$, and $\mathbf{0}$ represents the null vector in $\mathbb{R}^d$ or the null tensor in $\mathbb{R}^{d\times d}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{d\times d}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{d} \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{d}\left(\operatorname{tr}\boldsymbol{\tau}\right)\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{d} \tau_{ij}\sigma_{ij}$. Let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^d$ $(d=2,3)$, with boundary $\partial\Omega$. We denote by $\mathcal{D}(\Omega)$ the space of indefinitely differentiable function with compact support in Ω. For $s\in\mathbb{R}$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathbf{H}^s(\Omega) := \mathrm{H}^s(\Omega)^d$ or $\mathbb{H}^s(\Omega) := \mathrm{H}^s(\Omega)^{d\times d}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also denote by $(\cdot,\cdot)$ the inner product in $\mathrm{L}^2(\Omega)$, $\mathbf{L}^2(\Omega) := \mathrm{L}^2(\Omega)^d$ or $\mathbb{L}^2(\Omega) := \mathrm{L}^2(\Omega)^{d\times d}$. We introduce the Hilbert space

$$\mathbb{H}(\mathbf{div},\Omega) := \{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega) :\ \mathbf{div}\,\boldsymbol{\tau}\in\mathbf{L}^2(\Omega)\},$$

whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$. Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0,T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. For $1\le p\le\infty$, $\mathrm{L}^p(V)$ is the space of classes of functions $f:\ (0,T)\to V$ that are Böchner-measurable and such that $\|f\|_{\mathrm{L}^p(V)}<\infty$, with

$$\|f\|^p_{\mathrm{L}^p(V)} := \int_0^T \|f(t)\|_V^p\,\mathrm{d}t \quad (1\le p<\infty), \quad \|f\|_{\mathrm{L}^\infty(V)} := \operatorname*{ess\,sup}_{[0,T]}\|f(t)\|_V.$$

We use the notation $\mathcal{C}^0(V)$ for the Banach space consisting of all continuous functions $f:\ [0,T]\to V$. More generally, for any $k\in\mathbb{N}$, $\mathcal{C}^k(V)$ denotes the subspace of $\mathcal{C}^0(V)$ of all functions f with (strong) derivatives $\dfrac{\mathrm{d}^j f}{\mathrm{d}t^j}$ in $\mathcal{C}^0(V)$ for all $1 \le j \le k$. In what follows, we will use indistinctly the notations

$$\dot{f} := \frac{\mathrm{d}f}{\mathrm{d}t} \quad \text{and} \quad \ddot{f} := \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$$

to express the first and second derivatives with respect to the variable t. Furthermore, we will use the Sobolev space

$$\mathrm{W}^{1,p}(V) := \Big\{ f : \ \exists g \in \mathrm{L}^p(V) \text{ and } \exists f_0 \in V \text{ such that} \\ f(t) = f_0 + \int_0^t g(s)\,\mathrm{d}s \quad \forall t \in [0,T] \Big\},$$

and denote $\mathrm{H}^1(V) := \mathrm{W}^{1,2}(V)$. The space $\mathrm{W}^{k,p}(V)$ is defined recursively for all $k \in \mathbb{N}$.

On the other hand, given two Hilbert spaces S and Q and a bounded bilinear form $a : S \times Q \to \mathbb{R}$, we denote

$$\ker(a) := \{ s \in S : \ c(s,q) = 0 \quad \forall\, q \in Q \}.$$

We say that a satisfies the inf-sup condition for the pair $\{S, Q\}$, whenever there exists $\kappa > 0$ such that

$$\sup_{0 \ne s \in S} \frac{a(s,q)}{\|s\|_S} \ge \kappa \, \|q\|_Q \qquad \forall\, q \in Q\,.$$

We will repeatedly use the well-known fact (see [11]) that if a satisfies the inf-sup condition for the pair $\{S, Q\}$ and if ℓ belongs to the polar of $\ker(a)$ in S', defined by

$$\ker(a)^\circ := \{ \chi \in S'; \quad \chi(s) = 0 \quad \forall\, s \in \ker(a) \},$$

then there exists a unique $q \in Q$ such that

$$a(s,q) = \ell(s) \quad \forall\, s \in S\,.$$

Throughout this paper we use C (with or without subscripts) to denote generic constants independent of the parameters indicated at each instance. We point out that these constants may take different values at different places.

3 A wave equation in $\mathbb{H}(\mathbf{div}, \Omega)$

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded Lipschitz polygonal/polyhedral domain. We denote by $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$. We consider a subset $\emptyset \ne \Gamma \subset \partial\Omega$ and denote its complement $\Sigma := \partial\Omega \setminus \Gamma$. We consider the closed subspace of $\mathbb{H}(\mathbf{div}, \Omega)$ given by

$$\boldsymbol{\mathcal{W}} := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) : \quad \langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega} = 0 \quad \forall \boldsymbol{v} \in \mathbf{H}^{1/2}(\partial\Omega),\ \boldsymbol{v}|_\Gamma = \mathbf{0} \Big\},$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$ with respect to the $\mathbf{L}^2(\partial\Omega)$-inner product. Alternatively, recalling that the restriction of $\boldsymbol{\tau}\boldsymbol{n}$ to Σ belongs to $\mathbf{H}_{00}^{-1/2}(\Sigma) := \mathbf{H}_{00}^{1/2}(\Sigma)'$, where $\mathbf{H}_{00}^{1/2}(\Sigma)$ is the subspace of functions in $\mathbf{H}^{1/2}(\Sigma)$ whose extensions by zero on Γ are in $\mathbf{H}^{1/2}(\partial\Omega)$, we can also set

$$\boldsymbol{\mathcal{W}} := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) : \quad \boldsymbol{\tau}\boldsymbol{n} \, = \, 0 \quad \text{on} \quad \Sigma \Big\}.$$

Next, we assume that $\{\Omega_j, \quad j = 1 \cdots, J\}$ is a set of polygonal/polyhedral disjoint partition of $\bar{\Omega}$, i.e.,

$$\Omega_j \cap \Omega_i = \emptyset \quad \text{for all } 1 \le i \ne j \le J \quad \text{ and } \quad \bar{\Omega} = \cup_{j=1}^{J} \bar{\Omega}_j.$$

Then we consider piecewise constant functions $\mu(\boldsymbol{x})$, $\lambda(\boldsymbol{x})$, and $\rho(\boldsymbol{x})$ defined, for $j = 1, \ldots, J$, by $\mu|_{\Omega_j} := \mu_j > 0$, $\lambda|_{\Omega_j} := \lambda_j > 0$, and $\rho|_{\Omega_j} := \rho_j > 0$, and assume that there exist positive constants $\underline{\mu}$, $\bar{\mu}$, $\underline{\lambda}$, $\bar{\lambda}$, $\underline{\rho}$, and $\bar{\rho}$, such that

$$\underline{\mu} \le \mu_j \le \bar{\mu}, \quad \underline{\lambda} \le \lambda_j \le \bar{\lambda} \quad \text{and} \quad \underline{\rho} \le \rho_j \le \bar{\rho} \quad (1 \le j \le J).$$

In turn, we introduce on $\mathbf{L}^2(\Omega)$ the inner product $(\boldsymbol{u}, \boldsymbol{v})_\rho := (\rho^{-1}\boldsymbol{u}, \boldsymbol{v})$ and denote the corresponding norm

$$\|\boldsymbol{v}\|_\rho := \sqrt{(\boldsymbol{v}, \boldsymbol{v})_\rho}.$$

In addition, we consider the elasticity stiffness tensor $\mathcal{C}$ defined by $\mathcal{C}\boldsymbol{\tau} := \lambda\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau}$ and recall that its inverse (the compliance tensor) is given by

$$\mathcal{C}^{-1}\boldsymbol{\tau} = \frac{1}{2\mu}\left\{\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\}.$$

We endow $\mathbb{L}^2(\Omega)$ with the norm

$$\|\boldsymbol{\tau}\|_{\mathcal{C}}^2 := (\mathcal{C}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = \int_\Omega \frac{1}{2\mu}\boldsymbol{\tau}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \frac{1}{d(d\lambda + 2\mu)}\operatorname{tr}(\boldsymbol{\tau})^2. \tag{3.1}$$

The following result proves that

$$\|\boldsymbol{\tau}\|_{\mathcal{C},\mathrm{div}}^2 := \|\boldsymbol{\tau}\|_{\mathcal{C}}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_\rho^2$$

is a Hilbertian norm on $\boldsymbol{\mathcal{W}}$ that is equivalent to the $\mathbb{H}(\mathbf{div}, \Omega)$-norm uniformly in λ.

Lemma 3.1. *There exists a constant $\alpha > 0$, independent of λ, such that*

$$\alpha\,\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div},\Omega)}^2 \le \|\boldsymbol{\tau}\|_{\mathcal{C},\mathrm{div}}^2 \le \max\left(\frac{1}{2\underline{\mu}}, \frac{1}{\underline{\rho}}\right)\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div},\Omega)}^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}. \tag{3.2}$$

Proof. We deduce from (3.1) that

$$\frac{1}{2\bar{\mu}}\|\boldsymbol{\tau}^{\mathtt{D}}\|_{0,\Omega}^2 \le \|\boldsymbol{\tau}\|_{\mathcal{C}}^2 \le \frac{1}{2\underline{\mu}}\|\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall \boldsymbol{\tau} \in \mathbb{L}^2(\Omega), \tag{3.3}$$

which gives the upper bound of (3.2). Next, given $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega)$, we let $\boldsymbol{\tau}_0 := \boldsymbol{\tau} - \frac{1}{d|\Omega|}\left(\int_\Omega \operatorname{tr}\boldsymbol{\tau}\right)\boldsymbol{I}$. It is proved in [11, Proposition 9.1.1] that there exists $C_0 > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}_0\|_{0,\Omega}^2 \le C_0\left(\|\boldsymbol{\tau}^{\mathtt{D}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2\right) \qquad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega).$$

On the other hand, it is shown in [17, Lemma 2.5] (see also [16, Lemma 2.2]) that there exists $C_1 > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div},\Omega)}^2 \le C_1\|\boldsymbol{\tau}_0\|_{\mathbb{H}(\mathbf{div},\Omega)}^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

The lower bound of (3.2) follows now directly from (3.3), the last two inequalities, and the fact that $\mathbf{div}\,\boldsymbol{\tau}_0 = \mathbf{div}\,\boldsymbol{\tau}$ in Ω. $\square$

We now introduce the space of skew symmetric tensors

$$\boldsymbol{\mathcal{Q}} := \{\boldsymbol{s} \in \mathbb{L}^2(\Omega); \quad \boldsymbol{s} = -\boldsymbol{s}^{\mathtt{t}}\}\,,$$

and observe that the subspace $\boldsymbol{\mathcal{S}}$ of symmetric tensors in $\boldsymbol{\mathcal{W}}$ can be written, equivalently, as

$$\boldsymbol{\mathcal{S}} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}; \quad (\boldsymbol{\tau}, \boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}\}.$$

In addition, we notice that $\boldsymbol{\mathcal{S}}$ is closed in $\mathbb{H}(\mathbf{div}, \Omega)$, and hence, the density of $\mathbb{H}(\mathbf{div}, \Omega)$ in $\mathbb{L}^2(\Omega)$ proves that $\boldsymbol{\mathcal{S}}$ is also densely embedded in $\mathbb{L}^2_{\mathrm{sym}}(\Omega) := \{\boldsymbol{s} \in \mathbb{L}^2(\Omega); \quad \boldsymbol{s} = \boldsymbol{s}^{\mathtt{t}}\}$. We may then identify $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$ with its dual space and consider the Gelfand triple

$$\boldsymbol{\mathcal{S}} \hookrightarrow \mathbb{L}^2_{\mathrm{sym}}(\Omega) \hookrightarrow \boldsymbol{\mathcal{S}}',$$

where $\boldsymbol{\mathcal{S}}'$ is the dual space of $\boldsymbol{\mathcal{S}}$. The following inf-sup condition ([1, 8]) is essential in the forthcoming analysis: there exists $\beta > 0$ such that

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{(\boldsymbol{\tau}, \boldsymbol{s}) + (\mathbf{div}\, \boldsymbol{\tau}, \boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div},\Omega)}} \geq \beta(\|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}), \tag{3.4}$$

for all $(\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}} \times \mathbf{L}^2(\Omega)$.

Having established the above notations and preliminary results, we now introduce the wave equation in $\mathbb{H}(\mathbf{div}, \Omega)$. Indeed, given $\boldsymbol{f} \in \mathrm{L}^1(\mathbf{L}^2(\Omega))$, $\boldsymbol{\sigma}_0 \in \boldsymbol{\mathcal{S}}$, $\boldsymbol{\sigma}_1 \in \mathbb{L}^2_{\mathrm{sym}}(\Omega)$, and $\boldsymbol{r}_0, \boldsymbol{r}_1 \in \boldsymbol{\mathcal{Q}}$, we consider the problem:

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{W}}) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega)) \text{ and } \boldsymbol{r} \in \mathrm{W}^{1,\infty}(\boldsymbol{\mathcal{Q}}) \text{ such that}\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \;=\; -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\boldsymbol{\sigma}(t), \boldsymbol{s}) \;=\; 0\,,
\end{aligned} \tag{3.5}$$

for all $(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, and such that the following initial conditions are satisfied:

$$\begin{aligned}
\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0\,, \qquad \dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1,\\
\boldsymbol{r}(0) = \boldsymbol{r}_0\,, \qquad \dot{\boldsymbol{r}}(0) = \boldsymbol{r}_1.
\end{aligned} \tag{3.6}$$

We notice here that the second equation of (3.5) is the weak imposition of the symmetry of $\boldsymbol{\sigma}$, where $\boldsymbol{r}$ is the corresponding Lagrange multiplier. In this way, testing in particular the first equation of (3.5) with $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$, we arrive at the following reduced form of (3.5):

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{S}})) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2_{\mathrm{sym}}(\Omega)) \text{ such that}\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathcal{C}^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}\,,\\
&\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1.
\end{aligned} \tag{3.7}$$

In what follows it will be useful to consider the energy functional $\mathcal{E} : W^{1,\infty}(\mathbb{H}(\mathbf{div}, \Omega)) \to \mathrm{L}^\infty((0, T))$ defined by

$$\mathcal{E}(\boldsymbol{\tau})(t) := \frac{1}{2}\|\dot{\boldsymbol{\tau}}(t)\|^2_{\mathcal{C}} + \frac{1}{2}\|\mathbf{div}\, \boldsymbol{\tau}(t)\|^2_\rho \quad \forall \boldsymbol{\tau} \in W^{1,\infty}(\mathbb{H}(\mathbf{div}, \Omega))\,, \quad \forall\, t \in [0, T]\,. \tag{3.8}$$

Lemma 3.2. *Assume that* $\boldsymbol{f} \in \mathrm{W}^{1,1}(\mathbf{L}^2(\Omega))$. *Then, problem* (3.7) *admits at least a solution and there exists a constant* $C > 0$ *such that*

$$\operatorname*{ess\,sup}_{[0,T]} \mathcal{E}(\boldsymbol{\sigma})^{1/2}(t) \leq C\Big\{\|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\mathbf{L}^2(\Omega))} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C},\mathrm{div}} + \|\boldsymbol{\sigma}_1\|_{\mathcal{C}}\Big\}\,. \tag{3.9}$$

Proof. We only give a sketch of the proof since it follows by the classical Galerkin procedure (cf. [14, 23]). In fact, we first consider a family of finite dimensional subspaces $\{\boldsymbol{\mathcal{S}}_n\}$ of $\boldsymbol{\mathcal{S}}$ such that, for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$,

$$\lim_{n\to\infty} \inf_{\boldsymbol{\tau}_n \in \boldsymbol{\mathcal{S}}_n} \|\boldsymbol{\tau} - \boldsymbol{\tau}_n\|_{\mathrm{H}(\mathbf{div},\Omega)} = 0.$$

Then we denote by $\boldsymbol{\sigma}_{0,n}$ the $(\boldsymbol{\mathcal{S}}, \|\cdot\|_{\mathcal{C},\mathrm{div}})$-orthogonal projection of $\boldsymbol{\sigma}_0$ onto $\boldsymbol{\mathcal{S}}_n$ and by $\boldsymbol{\sigma}_{1,n}$ the $(\mathbb{L}^2_{\mathrm{sym}}(\Omega), \|\cdot\|_{\mathcal{C}})$-orthogonal projection of $\boldsymbol{\sigma}_1$ onto $\boldsymbol{\mathcal{S}}_n$. It is easy to show, by using the classical ODE theory, that the problem:

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma}_n \in \mathcal{C}^1(\boldsymbol{\mathcal{S}}_n) \text{ such that,}\\
&(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_n(t), \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}_n(t), \mathbf{div}\,\boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_n\,,\\
&\boldsymbol{\sigma}_n(0) = \boldsymbol{\sigma}_{0,n}\,, \quad \dot{\boldsymbol{\sigma}}_n(0) = \boldsymbol{\sigma}_{1,n}\,,
\end{aligned} \tag{3.10}$$

admits a unique solution. The first step of the proof reduces to deriving energy estimates for $\boldsymbol{\sigma}_n(t)$. To this end, we take $\boldsymbol{\tau} = \dot{\boldsymbol{\sigma}}_n(t)$ in (3.10) and integrate the resulting identity over $(0, t)$, which gives

$$\mathcal{E}(\boldsymbol{\sigma}_n)(t) - \mathcal{E}(\boldsymbol{\sigma}_n)(0) = -\int_0^t (\boldsymbol{f}(s), \mathbf{div}\,\dot{\boldsymbol{\sigma}}_n(s))_\rho \,\mathrm{d}s.$$

Next, integrating by parts the right-hand side yields

$$\mathcal{E}(\boldsymbol{\sigma}_n)(t) \;=\; \int_0^t (\dot{\boldsymbol{f}}(s), \mathbf{div}\,\boldsymbol{\sigma}_n(s))_\rho \,\mathrm{d}s - (\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\sigma}_n(t))_\rho \;+\; (\boldsymbol{f}(0), \mathbf{div}\,\boldsymbol{\sigma}_{0,n})_\rho + \mathcal{E}(\boldsymbol{\sigma}_n)(0)\,. \tag{3.11}$$

We now notice, according to the definition of $\boldsymbol{\sigma}_{0,n}$ and $\boldsymbol{\sigma}_{1,n}$, that

$$\mathcal{E}(\boldsymbol{\sigma}_n)(0) \leq \frac{1}{2}\|\boldsymbol{\sigma}_0\|^2_{\mathcal{C},\mathrm{div}} + \frac{1}{2}\|\boldsymbol{\sigma}_1\|^2_{\mathcal{C}}\,.$$

In turn, using the Sobolev embedding $\mathrm{W}^{1,1}(\mathbf{L}^2(\Omega)) \hookrightarrow \mathcal{C}^0(\mathbf{L}^2(\Omega))$ (see [24, Lemma 7.1]) and the Cauchy-Schwartz inequality, we deduce easily from (3.11) that there exists a constant $C > 0$ such that

$$\max_{[0,T]} \mathcal{E}(\boldsymbol{\sigma}_n)^{1/2}(t) \leq C\Big\{\|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\mathbf{L}^2(\Omega))} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C},\mathrm{div}} + \|\boldsymbol{\sigma}_1\|_{\mathcal{C}}\Big\}\,. \tag{3.12}$$

It follows from (3.12) that $(\dot{\boldsymbol{\sigma}}_n)_n$ is uniformly bounded in $\mathrm{L}^\infty(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ and $(\boldsymbol{\sigma}_n)_n$ is uniformly bounded in $\mathrm{L}^\infty(\boldsymbol{\mathcal{S}})$. We can then extract a weak-$*$ convergent subsequence (also denoted $(\boldsymbol{\sigma}_n)_n$) satisfying

$$\begin{aligned}
&\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_n(t), \boldsymbol{\tau})\dot{\psi}(t) \;+\; (\mathbf{div}\,\boldsymbol{\sigma}_n(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t\\
&\quad = -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \;+\; \psi(0)\,(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_n(0), \boldsymbol{\tau})
\end{aligned} \tag{3.13}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_n$ and for all $\psi \in \mathcal{C}^1([0, T])$ such that $\psi(T) = 0$. A classical procedure shows that the limit $\boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{S}}) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ of the subsequence $(\boldsymbol{\sigma}_n)_n$ satisfies

$$\begin{aligned}
&\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau})\dot{\psi}(t) \;+\; (\mathbf{div}\,\boldsymbol{\sigma}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t\\
&\quad = -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \;+\; \psi(0)\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_1, \boldsymbol{\tau})
\end{aligned} \tag{3.14}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$ and for all $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$. This proves that $\boldsymbol{\sigma}$ solves (3.7) if the time derivative is interpreted in the sense of distributions. In addition, we notice that $\boldsymbol{\sigma}_n$ also converges weakly to $\boldsymbol{\sigma}$ in $\mathrm{H}^1(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ and hence $\boldsymbol{\sigma}_n(0)$ converges weakly to $\boldsymbol{\sigma}(0)$ in $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$. Moreover, since $\boldsymbol{\sigma}_n(0) = \boldsymbol{\sigma}_{0,n}$ converges to $\boldsymbol{\sigma}_0$ in $\mathbb{L}^2(\Omega)$ as well, we conclude that the initial condition $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ is meaningful. Furthermore, it is clear from (3.14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}) = -(\mathbf{div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}\,, \tag{3.15}$$

from which it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t)$ belongs to $\mathrm{L}^1(\boldsymbol{\mathcal{S}}')$, and thus $\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t) \in \mathrm{W}^{1,1}(\boldsymbol{\mathcal{S}}') \hookrightarrow \mathcal{C}^0(\boldsymbol{\mathcal{S}}')$. Next, testing (3.15) with $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$ yields

$$\begin{aligned} &\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau})\dot{\psi}(t) \;+\; (\mathbf{div}\,\boldsymbol{\sigma}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \\ &= -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \;+\; \psi(0)\,\left\langle \mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(0), \boldsymbol{\tau}\right\rangle_{\boldsymbol{\mathcal{S}}} \end{aligned} \tag{3.16}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$, where $\langle \cdot,\cdot \rangle_{\boldsymbol{\mathcal{S}}}$ stands for the duality bracket between $\boldsymbol{\mathcal{S}}'$ and $\boldsymbol{\mathcal{S}}$ pivotal $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$, and hence, comparing (3.14) with (3.16) we deduce that $\dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1$ in $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$. Finally, the stability estimate (3.9) is obtained by taking the limit in (3.12). □

Lemma 3.3. *The solution of problem* (3.7) *is unique.*

Proof. Assume that $\boldsymbol{\sigma}$ is a solution of (3.7) with homogeneous data $\boldsymbol{f}(t) = \mathbf{0}$ and $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_1 = \mathbf{0}$. We proceed as in [14, 23] and consider with $s \in (0,T)$ fixed

$$\boldsymbol{w}(t) = \begin{cases} -\int_t^s \boldsymbol{\sigma}(z)\,\mathrm{d}z & t < s \\ 0 & t \geq s \end{cases} \;\in\; \mathrm{W}^{1,2}(\boldsymbol{\mathcal{S}})\,.$$

Then, testing (3.7) with $\boldsymbol{w}(t)$ and integrating by parts in the time variable, we obtain

$$\int_0^T (\mathbf{div}\,\boldsymbol{\sigma}(t), \mathbf{div}\,\boldsymbol{w}(t))_\rho - (\dot{\boldsymbol{\sigma}}(t), \dot{\boldsymbol{w}}(t))_{\mathcal{C}}\,\mathrm{d}t \;=\; 0\,,$$

which can be rewritten as

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{div}\,\boldsymbol{w}(t)\|_\rho^2 - \|\boldsymbol{\sigma}(t)\|_{\mathcal{C}}^2\Big)\mathrm{d}t = 0\,.$$

It follows that $\|\mathbf{div}\,\boldsymbol{w}(0)\|_\rho^2 + \|\boldsymbol{\sigma}(s)\|_{\mathcal{C}}^2 = 0$, and the proof is finished. □

It is important to remark that, following [23, Section 11.2.4], one can also show that the solution $\boldsymbol{\sigma}$ to problem (3.7) is actually in $\mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$.

Theorem 3.1. *Assume that* $\boldsymbol{f} \in \mathrm{W}^{1,1}(\mathbf{L}^2(\Omega))$. *Then problem* (3.5) *admits a unique solution. Moreover, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\max_{[0,T]}\|\boldsymbol{\sigma}(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} \;+\; \max_{[0,T]}\|\dot{\boldsymbol{\sigma}}(t)\|_{0,\Omega} \;+\; \|\boldsymbol{r}(t)\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} \\ &\quad\leq\; C\Big\{\|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\mathbf{L}^2(\Omega))} \;+\; \|\boldsymbol{\sigma}_0\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{\sigma}_1\|_{0,\Omega} \;+\; \|\boldsymbol{r}_0\|_{0,\Omega} + \|\boldsymbol{r}_1\|_{0,\Omega}\Big\}\,. \end{aligned} \tag{3.17}$$

Proof. We only have to prove the existence and uniqueness of the Lagrange multiplier $\boldsymbol{r}$. To this end, we consider $\mathcal{G} \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}')$ given by

$$\langle \mathcal{G}(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}} := (\mathcal{C}^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\tau}) + \int_0^t \Big(\int_0^s (\mathbf{div}\, \boldsymbol{\sigma}(z) + \boldsymbol{f}(z), \mathbf{div}\, \boldsymbol{\tau})_\rho \, \mathrm{d}z \Big) \mathrm{d}s$$
$$- t(\mathcal{C}^{-1}\boldsymbol{\sigma}_1 + \boldsymbol{r}_1, \boldsymbol{\tau}) - (\mathcal{C}^{-1}\boldsymbol{\sigma}_0 + \boldsymbol{r}_0, \boldsymbol{\tau}),$$

where $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{W}}}$ denotes the duality bracket between $\boldsymbol{\mathcal{W}}'$ and $\boldsymbol{\mathcal{W}}$ pivotal $\mathbb{L}^2(\Omega)$. Integrating (3.7) twice with respect to time yields

$$\langle \mathcal{G}(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}},$$

which means that $\mathcal{G}(t)$ belongs to the polar set of $\boldsymbol{\mathcal{S}}$ in $\boldsymbol{\mathcal{W}}'$. Moreover, $\boldsymbol{\mathcal{S}}$ is the kernel of $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \ni (\boldsymbol{\tau}, \boldsymbol{r}) \mapsto \int_\Omega \boldsymbol{\tau} : \boldsymbol{r}$ and (3.4) implies that this bilinear form satisfies the inf-sup condition for the pair $\{\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{Q}}\}$, which guarantees the existence of $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ such that

$$(\boldsymbol{r}(t), \boldsymbol{\tau}) = -\langle \mathcal{G}(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}. \tag{3.18}$$

We conclude that the pair $\{\boldsymbol{\sigma}, \boldsymbol{r}\}$ solves the first equation of (3.5) by differentiating twice the last identity in the sense of distributions with respect to t. Moreover, evaluating (3.18) and its time derivative at $t = 0$, we deduce that $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and $\dot{\boldsymbol{r}}(0) = \boldsymbol{r}_1$. Finally, using the inf-sup condition (3.4), the Cauchy-Schwarz inequality and (3.2), we deduce that there exists $C_1 > 0$ such that

$$\begin{aligned} \beta \, \|\dot{\boldsymbol{r}}(t)\|_{0,\Omega} &\le \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\int_\Omega \dot{\boldsymbol{r}}(t) : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} = \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\langle \dot{\mathcal{G}}(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &\le C_1 \left\{ \|\boldsymbol{\sigma}_1\|_{0,\Omega} + \|\boldsymbol{r}_1\|_{0,\Omega} + \max_{[0,T]} \mathcal{E}(\boldsymbol{\sigma})^{1/2}(t) + \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\mathrm{L}^2(\Omega)^n)} \right\}. \end{aligned} \tag{3.19}$$

Finally, we deduce from the fundamental theorem of calculus that

$$\|\boldsymbol{\sigma}(t)\|_{\mathcal{C}} \le T \max_{[0,T]} \|\dot{\boldsymbol{\sigma}}(t)\|_{\mathcal{C}} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C}} \quad \text{and} \quad \|\boldsymbol{r}(t)\|_{0,\Omega} \le T \max_{[0,T]} \|\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \|\boldsymbol{r}_0\|_{0,\Omega}, \tag{3.20}$$

so that (3.17) is obtained by combining (3.9), (3.19), and (3.20). □

4 Relationship with the elastodynamic problem

We assume that Ω represents an isotropic and linearly elastic body with mass density ρ and Lamé coefficients μ and λ. The solid is assumed to be fixed at Γ and free of stresses on Σ. The elastodynamic equations with body force $\boldsymbol{f} : \Omega \times [0, T] \to \mathbb{R}^d$ and initial data $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to \mathbb{R}$ are given by

$$\begin{aligned} \rho \ddot{\boldsymbol{u}} - \mathbf{div}\, \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) &= \boldsymbol{f}(t) && \text{in } \Omega \times (0, T], \\ \boldsymbol{u}(t) &= \mathbf{0} && \text{on } \Gamma \times (0, T], \\ \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\boldsymbol{n} &= \mathbf{0} && \text{on } \Sigma \times (0, T], \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega, \\ \dot{\boldsymbol{u}}(0) &= \boldsymbol{u}_1 && \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the displacement field and $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2} [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathtt{t}}]$ is the linearized strain tensor. In order to establish a relationship between problems (4.1) and (3.5) we need to introduce the subspace

$$\mathbb{V} := \Big\{ (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}; \quad (\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) + (\boldsymbol{\sigma}, \boldsymbol{s}) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \Big\}, \tag{4.2}$$

where

$$\boldsymbol{\mathcal{K}} := \Big\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}; \quad \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \Big\}.$$

Lemma 4.1. *The linear operator* $D : \mathbb{V} \to \mathbf{L}^2(\Omega)$ *uniquely characterized by*

$$(\mathbf{div}\,\boldsymbol{\tau}, D(\boldsymbol{\sigma}, \boldsymbol{r})) = -(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{4.3}$$

is well-defined and bounded.

Proof. We deduce from (3.4) that the bilinear form $(\boldsymbol{\tau}, \boldsymbol{v}) \mapsto \int_\Omega \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{v}$ satisfies the inf-sup condition for the pair $\{\boldsymbol{\mathcal{W}}, \mathbf{L}^2(\Omega)\}$. Moreover, by definition of $\mathbb{V}$, the linear form $\boldsymbol{\tau} \mapsto (\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau})$ vanishes identically on the kernel $\boldsymbol{\mathcal{K}}$ of this bilinear form. This proves the existence of a unique $D(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbf{L}^2(\Omega)$ satisfying (4.3). □

We are now ready to give the main result of this section.

Theorem 4.1. *We consider the same right hand side* $\boldsymbol{f}$ *in* (3.5) *and* (4.1)*, and assume that the initial data of these problems satisfy*

$$(\boldsymbol{\sigma}_0, \boldsymbol{r}_0)\,,\ (\boldsymbol{\sigma}_1, \boldsymbol{r}_1) \in \mathbb{V}\,, \quad \boldsymbol{u}_0 := D(\boldsymbol{\sigma}_0, \boldsymbol{r}_0)\,, \quad \textit{and} \quad \boldsymbol{u}_1 := D(\boldsymbol{\sigma}_1, \boldsymbol{r}_1)\,. \tag{4.4}$$

Then

$$\boldsymbol{u}(t) := \int_0^t \left\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z)\Big)\, dz \right\} ds + \boldsymbol{u}_0 + t\boldsymbol{u}_1 \tag{4.5}$$

solves the (primal) weak formulation of problem (4.1)*. Moreover, the solution* $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t))$ *of* (3.5) *coincides with the stress and rotation tensors associated with* $\boldsymbol{u}(t)$*, that is*

$$\boldsymbol{\sigma}(t) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \textit{and} \quad \boldsymbol{r}(t) = \frac{1}{2}\left[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^{\mathtt{t}}\right]. \tag{4.6}$$

Proof. We first notice that, testing (3.5) with $(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ and taking into account (4.4), we deduce that $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbb{V}$ for all $t \in [0, T]$. Hence, Lemma 4.1 ensures that there exists a unique $\boldsymbol{u}(t) := D(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbf{L}^2(\Omega)$ satisfying

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}(t)) = -(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t), \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \quad \forall t \in [0, T]. \tag{4.7}$$

On the other hand, integrating the first equation of (3.5) twice with respect to time yields,

$$(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t), \boldsymbol{\tau}) = (\mathcal{C}^{-1}\boldsymbol{\sigma}_0 + \boldsymbol{r}_0, \boldsymbol{\tau}) + t(\mathcal{C}^{-1}\boldsymbol{\sigma}_1 + \boldsymbol{r}_1, \boldsymbol{\tau}) - \int_0^t \Big(\int_0^s (\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau})_\rho \,\mathrm{d}z \Big) \mathrm{d}s.$$

Comparing the last identity with (4.7) we deduce that $\boldsymbol{u}$ is given by (4.5). On the other hand, testing (4.7) with $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{d\times d}$ yields

$$\nabla \boldsymbol{u}(t) = \mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t) \in \mathbb{L}^2(\Omega), \tag{4.8}$$

and considering the symmetric and skew symmetric parts of this identity gives (4.6). Then, integrating by parts the left-hand side of (4.7) and using the last identity yields

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, \boldsymbol{u}(t) \rangle_{\partial\Omega} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

Consequently, $\boldsymbol{u} \in \mathcal{C}^1(\mathbf{H}^1_\Gamma(\Omega))$ where $\mathbf{H}^1_\Gamma(\Omega) := \{\boldsymbol{v} \in \mathbf{H}^1(\Omega); \quad \boldsymbol{v}|_\Gamma = \mathbf{0}\}$. Multiplying (4.5) by ρ, testing with $\boldsymbol{v} \in \mathbf{H}^1_\Gamma(\Omega)$, integrating by parts in space and differentiating twice in time we find that $\boldsymbol{u} \in \mathcal{C}^1(\mathbf{H}^1_\Gamma(\Omega))$ satisfies the week displacement-based variational formulation of (4.1), namely,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho \boldsymbol{u}(t), \boldsymbol{v}) + (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{f}(t), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\Gamma(\Omega)\,,$$

and the result follows. □

Henceforth, we assume that condition (4.4) is satisfied, which will permit us to interpret the solution pair $(\boldsymbol{\sigma}, \boldsymbol{r})$ of (3.5) as the stress tensor and the rotation associated to the solution $\boldsymbol{u}$ of (4.1).

5 Semi-discretization in space

5.1 Finite element subspaces

We consider finite dimensional families of subspaces

$$\boldsymbol{\mathcal{W}}_h \subset \boldsymbol{\mathcal{W}} \qquad \boldsymbol{\mathcal{Q}}_h \subset \boldsymbol{\mathcal{Q}} \qquad \mathcal{U}_h \subset \mathbf{L}^2(\Omega)$$

indexed with a parameter $h \to 0$, and assume that there holds

$$\lim_{h\to 0} \left\{ \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div},\Omega)} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r} - \boldsymbol{s}_h\|_{0,\Omega} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{0,\Omega} \right\} = 0 \tag{5.1}$$

for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$, $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$ and $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$. Besides the approximation property (5.1) we need to impose conditions ensuring that the triple of spaces $\{\boldsymbol{\mathcal{W}}_h, \mathcal{U}_h, \boldsymbol{\mathcal{Q}}_h\}$ provides a stable Galerkin approximation method for the dual-mixed formulation of the (steady state) elasticity problem with weak symmetry. By virtue of the Babuška-Brezzi theory, such a stability is guaranteed by the following two hypotheses and Lemma 3.1 (see [4]).

Hypothesis 1. *There exists $\beta^* > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{\tau}, \boldsymbol{s}) + (\mathbf{div}\, \tau, \boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta^* (\|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}), \tag{5.2}$$

for all $(\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$.

Hypothesis 2. $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) = \mathcal{U}_h$ *and* $\rho^{-1}\, \mathbf{div}(\boldsymbol{\mathcal{W}}_h) = \mathcal{U}_h$.

We point out that in practice, as ρ is assumed to be a piecewise constant function, we will be able to choose the triangulations upon which the finite element spaces $\boldsymbol{\mathcal{W}}_h$ and $\mathcal{U}_h$ are constructed in such a way that the two conditions of Hypothesis 2 are equivalent.

Finally, we assume the existence of an operator satisfying the following stability and commuting diagram properties.

Hypothesis 3. *There exists a linear operator $\Pi_h : \mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega) \to \boldsymbol{\mathcal{W}}_h$, with $\epsilon > 0$, such that*

$$\|\Pi_h \boldsymbol{\tau}\|_{0,\Omega} \leq C \Big\{ \|\boldsymbol{\tau}\|_{\epsilon,\Omega} + \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} \Big\} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega) \tag{5.3}$$

for a constant $C > 0$ independent of h and

$$\mathbf{div}\, \Pi_h \boldsymbol{\tau} = U_h\, \mathbf{div}\, \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\, \Omega) \cap \mathbb{H}^\epsilon(\Omega)\,, \tag{5.4}$$

where U_h is the orthogonal projection from $(\mathbf{L}^2(\Omega), \|\cdot\|_{0,\Omega})$ onto $\mathcal{U}_h$.

We now introduce the discrete analogue of $\mathbb{V}$ (cf. (4.2)), that is

$$\mathbb{V}_h := \Big\{ (\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h; \quad (\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{r}_h, \boldsymbol{\tau}) + (\boldsymbol{\sigma}_h, \boldsymbol{s}) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h \Big\},$$

where

$$\boldsymbol{\mathcal{K}}_h := \Big\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h; \quad \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \Big\}.$$

Then, the discrete version of Lemma 4.1 reads as follows.

Lemma 5.1. *The linear operator $D_h : \mathbb{V}_h \to \mathcal{U}_h$ uniquely characterized by*

$$(\mathbf{div}\,\boldsymbol{\tau}, D_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) = -(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{r}_h, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h. \tag{5.5}$$

is well-defined and uniformly bounded.

Proof. The result is obtained by following the same steps given in the proof of its continuous counterpart and by using the discrete inf-sup condition (5.2). □

5.2 An auxiliary operator

In order to facilitate our analysis we now introduce an auxiliary operator Ξ and its discrete counterpart Ξ_h. More precisley, we define

$$\begin{aligned} \Xi : \boldsymbol{\mathcal{W}} &\to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \mathbf{L}^2(\Omega) \\ \boldsymbol{\sigma} &\mapsto \Xi\boldsymbol{\sigma} := (\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*), \end{aligned}$$

where $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \mathbf{L}^2(\Omega)$ is the solution of

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}^* + \boldsymbol{r}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \\ (\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v}) &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{L}^2(\Omega), \\ (\boldsymbol{\sigma}^*, \boldsymbol{s}) &= 0, \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \end{aligned} \tag{5.6}$$

It is easy to prove, using the continuous inf-sup condition (3.4), Lemma 3.1, and the Babuška-Brezzi theory, that $\Xi : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \mathbf{L}^2(\Omega)$ is well-defined and uniformly bounded in λ. In addition, we notice that $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \mathbb{V}$ for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ and $\boldsymbol{u}^* = D(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)$. Moreover, it is crucial for the forthcoming analysis to observe that

$$(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*) := \Xi\boldsymbol{\sigma} = (\boldsymbol{\sigma}, \boldsymbol{r}, D(\boldsymbol{\sigma}, \boldsymbol{r})) \qquad \forall (\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbb{V}. \tag{5.7}$$

Indeed, by virtue of Lemma 4.1, given $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbb{V}$ there exists a unique $\boldsymbol{u} := D(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbf{L}^2(\Omega)$ such that

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}) = -(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

from which it follows that $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \mathbf{L}^2(\Omega) \times \boldsymbol{\mathcal{Q}}$ is the unique solution to problem (5.6) with datum $\mathbf{div}\,\boldsymbol{\sigma}$.

In turn, the discrete counterpart of Ξ is given by

$$\begin{aligned} \Xi_h : \boldsymbol{\mathcal{W}} &\to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \\ \boldsymbol{\sigma} &\mapsto \Xi_h\boldsymbol{\sigma} := (\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*) \end{aligned}$$

where $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ is the solution of

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}_h^* + \boldsymbol{r}_h^*, \boldsymbol{\tau}) + (\boldsymbol{u}_h^*, \mathbf{div}\,\boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \\ (\mathbf{div}\,\boldsymbol{\sigma}_h^*, \boldsymbol{v}) &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{U}_h, \\ (\boldsymbol{\sigma}_h^*, \boldsymbol{s}) &= 0, \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h. \end{aligned} \tag{5.8}$$

Similarly to the continuous case, the discrete inf-sup condition given by Hypothesis 1, Lemma 3.1, the first condition of Hypothesis 2, and the Babuška-Brezzi theory imply that $\Xi_h : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ is well-defined and uniformly bounded in h and λ. In addition, there holds $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*) \in \mathbb{V}_h$ for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ and $\boldsymbol{u}_h^* = D_h(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*)$. Furthermore, we have the Céa estimate

$$\begin{aligned}\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{0,\Omega} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{0,\Omega} \\ \leq C\Big\{ \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div},\Omega)} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega} \Big\} \quad (5.9)\end{aligned}$$

with $C > 0$ independent of h and λ.

5.3 The semi-discrete problem

From now on, we assume that the discrete initial data are given by

$$(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h}, \boldsymbol{u}_{0,h}) := \Xi_h \boldsymbol{\sigma}_0 \qquad \text{and} \qquad (\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h}, \boldsymbol{u}_{1,h}) := \Xi_h \boldsymbol{\sigma}_1 \,, \tag{5.10}$$

which, according to a previous observation, yields $\boldsymbol{u}_{0,h} := D_h(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h})$ and $\boldsymbol{u}_{1,h} := D_h(\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h})$. Then, we consider the following semi-discrete counterpart of (3.5):

$$\begin{aligned}&\text{Find } \boldsymbol{\sigma}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}_h) \text{ and } \boldsymbol{r}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}}_h) \text{ such that}\\ &(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h(t) + \ddot{\boldsymbol{r}}_h(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}_h(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \;=\; -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h,\\ &\qquad\qquad (\boldsymbol{\sigma}_h(t), \boldsymbol{s}) \;=\; \mathbf{0} \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h,\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{0,h}\,, \qquad \dot{\boldsymbol{\sigma}}_h(0) = \boldsymbol{\sigma}_{1,h},\\ \boldsymbol{r}_h(0) = \boldsymbol{r}_{0,h}\,, \qquad \dot{\boldsymbol{r}}_h(0) = \boldsymbol{r}_{1,h}\,.\end{aligned} \tag{5.12}$$

The kernel of the bilinear form $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \ni (\boldsymbol{\tau}, \boldsymbol{s}) \mapsto \int_\Omega \boldsymbol{\tau} : \boldsymbol{s}$ is defined by

$$\boldsymbol{\mathcal{S}}_h := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h; \quad (\boldsymbol{\tau}, \boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\}\,,$$

which, being the subspace of $\boldsymbol{\mathcal{W}}_h$ whose elements are symmetric only in a discrete sense, is generally not contained in $\boldsymbol{\mathcal{S}}$. Then, as in the continuous case, we now introduce a reduced version of problem (5.11):

$$\begin{aligned}&\text{Find } \boldsymbol{\sigma}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{S}}_h) \text{ such that}\\ &(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}_h(t), \mathbf{div}\, \boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_h,\\ &\boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{0,h}, \quad \dot{\boldsymbol{\sigma}}_h(0) = \boldsymbol{\sigma}_{1,h}\,,\end{aligned} \tag{5.13}$$

whose unique solvability is ensured by classical ODE theory.

Next, we prove the existence of the Lagrange multiplier $\boldsymbol{r}_h(t)$ by proceeding as in the continuous case. To this end, we let $\mathcal{G}_h(t) \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}_h')$ be given by

$$\begin{aligned}\langle \mathcal{G}_h(t), \boldsymbol{\tau}\rangle \;:=\; &(\mathcal{C}^{-1}\boldsymbol{\sigma}_h(t), \boldsymbol{\tau}) + \int_0^t \Big\{ \int_0^s (\mathbf{div}\, \boldsymbol{\sigma}_h(z) + \boldsymbol{f}(z), \mathbf{div}\, \boldsymbol{\tau})_\rho \, \mathrm{d}z \Big\} \mathrm{d}s\\ &-(\mathcal{C}^{-1}\boldsymbol{\sigma}_{0,h} + \boldsymbol{r}_{0,h}, \boldsymbol{\tau}) - t(\mathcal{C}^{-1}\boldsymbol{\sigma}_{1,h} + \boldsymbol{r}_{1,h}, \boldsymbol{\tau})\,.\end{aligned}$$

Using the fact that $\boldsymbol{\sigma}_h(t)$ solves (5.13), we deduce that $\mathcal{G}_h(t)$ belongs to the polar set of $\boldsymbol{\mathcal{S}}_h$ in $\boldsymbol{\mathcal{W}}_h'$, and hence, by virtue of the discrete inf-sup condition (5.2), we deduce that there exists a unique $\boldsymbol{r}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}}_h)$ such that

$$(\boldsymbol{r}_h(t), \boldsymbol{\tau}) = -\langle \mathcal{G}_h(t), \boldsymbol{\tau}\rangle \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \tag{5.14}$$

which proves that $(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t))$ is the unique solution to (5.11). In turn, since by construction (see (5.10)) $(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h}) \in \mathbb{V}_h$ and $(\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h}) \in \mathbb{V}_h$, we find from (5.11) that $(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t)) \in \mathbb{V}_h$ for all $t \in [0,T]$. Consequently, we can propose the function $\boldsymbol{u}_h(t) := D_h(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t))$ as a semi-discrete approximation of the displacement $\boldsymbol{u}(t)$, which can be computed by solving a saddle point problem of the form (5.8) with datum $\mathbf{div}\,\boldsymbol{\sigma}_h(t)$. However, comparing (5.5) with the first equation of (5.11), we easily obtain the following explicit expression for the semi-discrete displacement field:

$$\boldsymbol{u}_h(t) = \int_0^t \Big\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}_h(z) + U_h \boldsymbol{f}(z) \Big)\,\mathrm{d}z \Big\}\,\mathrm{d}s + \boldsymbol{u}_{0,h} + t\boldsymbol{u}_{1,h}\,. \tag{5.15}$$

5.4 Convergence analysis

We begin by recalling from Section 5.2 (cf. (5.8)) that $(\boldsymbol{\sigma}_h^*(t), \boldsymbol{r}_h^*(t), \boldsymbol{u}_h^*(t)) = \Xi_h\,\boldsymbol{\sigma}(t)$. Then, we introduce

$$\boldsymbol{e}_{\sigma,h}(t) := \boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}_h(t) \quad \text{and} \quad \boldsymbol{e}_{r,h}(t) := \boldsymbol{r}_h^*(t) - \boldsymbol{r}_h(t)\,,$$

and deduce from (5.10) that

$$\boldsymbol{e}_{\sigma,h}(0) = \boldsymbol{e}_{r,h}(0) = \mathbf{0} \qquad \text{and} \qquad \dot{\boldsymbol{e}}_{\sigma,h}(0) = \dot{\boldsymbol{e}}_{r,h}(0) = \mathbf{0}. \tag{5.16}$$

Lemma 5.2. *Assume that the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem (3.5) satisfy the regularity assumptions $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega))$ for some $\epsilon > 0$ and $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{L}^2(\Omega))$. Then, there exists a constant $C > 0$ independent of λ and h such that*

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}} \\
&\quad \le\; C\left\{ \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \,+\, \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \right\}.
\end{aligned}$$

Proof. Let us first notice that, as $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbb{V}$ for all $t \in [0,T]$ (see the proof of Theorem 4.1), we deduce from (5.7) that

$$(\boldsymbol{\sigma}^*(t), \boldsymbol{r}^*(t), \boldsymbol{u}^*(t)) := \varXi\boldsymbol{\sigma}(t) = (\boldsymbol{\sigma}(t), \boldsymbol{r}(t), D(\boldsymbol{\sigma}(t), \boldsymbol{r}(t))) \qquad \forall\, t \in [0,T]\,, \tag{5.17}$$

and because of the regularity assumptions we also have

$$\begin{aligned}
\Big(\frac{\mathrm{d}^i\boldsymbol{\sigma}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i\boldsymbol{r}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i\boldsymbol{u}^*}{\mathrm{d}t^i}(t)\Big) &:= \frac{\mathrm{d}^i\varXi\boldsymbol{\sigma}(t)}{\mathrm{d}t^i} = \varXi\frac{\mathrm{d}^i\boldsymbol{\sigma}}{\mathrm{d}t^i}(t) \\
&= \Big(\frac{\mathrm{d}^i\boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i\boldsymbol{r}}{\mathrm{d}t^i}(t), D\big(\frac{\mathrm{d}^i\boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i\boldsymbol{r}}{\mathrm{d}t^i}(t)\big)\Big) \qquad \forall\, i \in \{1,2\}\,, \quad \forall\, t \in [0,T]\,.
\end{aligned} \tag{5.18}$$

Moreover, by virtue of (5.9), (5.18) and Hypothesis 3, there holds

$$\begin{aligned}
&\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega)} \\
&\quad \le C_0\Big\{ \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega)} \,+\, \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega)} \Big\},
\end{aligned} \tag{5.19}$$

with $C_0 > 0$ independent of h and λ. Next, it is straightforward to see that

$$\begin{aligned}
&(\mathcal{C}^{-1}\ddot{\boldsymbol{e}}_{\sigma,h}(t) + \ddot{\boldsymbol{e}}_{r,h}(t), \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{e}_{\sigma,h}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \\
&\quad =\; (\mathcal{C}^{-1}(\ddot{\boldsymbol{\sigma}}_h^*(t) - \ddot{\boldsymbol{\sigma}}(t)), \boldsymbol{\tau}) \,+\, (\ddot{\boldsymbol{r}}_h^*(t) - \ddot{\boldsymbol{r}}(t), \boldsymbol{\tau}) \,+\, (\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}(t)), \mathbf{div}\,\boldsymbol{\tau})_\rho
\end{aligned} \tag{5.20}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$, and, as a consequence of (5.18),

$$\begin{aligned}&(\mathcal{C}^{-1}\ddot{\boldsymbol{e}}_{\sigma,h}(t) + \ddot{\boldsymbol{e}}_{r,h}(t), \boldsymbol{\tau}) \,+\, (\mathbf{div}\,\boldsymbol{e}_{\sigma,h}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \\ &\quad = \,(\mathcal{C}^{-1}(\ddot{\boldsymbol{\sigma}}_h^*(t) - \ddot{\boldsymbol{\sigma}}^*(t)), \boldsymbol{\tau}) \,+\, (\ddot{\boldsymbol{r}}_h^*(t) - \ddot{\boldsymbol{r}}^*(t), \boldsymbol{\tau}) \,+\, (\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}^*(t)), \mathbf{div}\,\boldsymbol{\tau})_\rho\end{aligned} \tag{5.21}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$. Now, by definition of Ξ and Ξ_h, we have that

$$\mathbf{div}\,\boldsymbol{\sigma}_h^*(t) = U_h\,\mathbf{div}\,\boldsymbol{\sigma}^*(t) \qquad \forall\, t \in [0, T]\,, \tag{5.22}$$

and the second condition of Hypothesis 2 implies that

$$(\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}^*(t)), \mathbf{div}\,\boldsymbol{\tau})_\rho = 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,.$$

Consequently, $\boldsymbol{e}_{\sigma,h}(t) \in \boldsymbol{\mathcal{S}}_h$ and $\boldsymbol{e}_{r,h}(t) \in \boldsymbol{\mathcal{Q}}_h$ satisfy

$$\begin{aligned}(\mathcal{C}^{-1}\ddot{\boldsymbol{e}}_{\sigma,h}(t) + \ddot{\boldsymbol{e}}_{r,h}(t), \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{e}_{\sigma,h}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho =&\quad F(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \\ (\boldsymbol{e}_{\sigma,h}(t), \boldsymbol{s}) =&\quad \mathbf{0} \qquad\quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\,,\end{aligned} \tag{5.23}$$

with

$$F(\boldsymbol{\tau}) := \Big(\mathcal{C}^{-1}\big(\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}^*\big)(t) + \big(\ddot{\boldsymbol{r}}_h^* - \ddot{\boldsymbol{r}}^*\big)(t), \boldsymbol{\tau}\Big).$$

Taking $\boldsymbol{\tau} = \dot{\boldsymbol{e}}_{\sigma,h}(t)$ in the first equation of (5.23) and using the Cauchy-Schwarz inequality yields

$$\frac{\dot{\mathcal{E}}(\boldsymbol{e}_{\sigma,h})(t)}{2\sqrt{\mathcal{E}(\boldsymbol{e}_{\sigma,h})(t)}} \leq \frac{1}{\sqrt{2}}\left\{\big(\mathcal{C}(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t), (\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\big)^{1/2} + \|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{\mathcal{C}}\right\},$$

which, using (3.3) and the fact that

$$\mathcal{C}\boldsymbol{s} = 2\mu\boldsymbol{s} \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}, \tag{5.24}$$

implies that

$$\frac{\dot{\mathcal{E}}(\boldsymbol{e}_{\sigma,h})(t)}{2\sqrt{\mathcal{E}(\boldsymbol{e}_{\sigma,h})(t)}} \leq \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2\underline{\mu}}}\|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{0,\Omega} + \sqrt{2\overline{\mu}}\|(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\|_{0,\Omega}\right\}.$$

Integrating with respect to time gives

$$\max_{[0,T]} \mathcal{E}(\boldsymbol{e}_{\sigma,h})^{1/2}(t) \leq \int_0^T \left\{\frac{1}{2\sqrt{\underline{\mu}}}\|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{0,\Omega} + \sqrt{\overline{\mu}}\|(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\|_{0,\Omega}\right\} \mathrm{d}t. \tag{5.25}$$

On the other hand, we deduce easily from the identity $\boldsymbol{e}_{\sigma,h}(t) = \int_0^t \dot{\boldsymbol{e}}_{\sigma,h}(s)$ and (3.8) that there exists a constant $C_0 > 0$, independent of λ and h, such that

$$\|\boldsymbol{e}_{\sigma,h}(t)\|_{\mathcal{C},\mathrm{div}} + \|\dot{\boldsymbol{e}}_{\sigma,h}(t)\|_{\mathcal{C}} \leq C_0 \max_{[0,T]} \mathcal{E}(\boldsymbol{e}_{\sigma,h})^{1/2}(t) \qquad \forall\, t \in [0, T]. \tag{5.26}$$

In this way, combining (5.25) and (5.26) with the triangle inequality we arrive at

$$\begin{aligned}&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} \,+\, \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}} \\ &\quad \leq\, \|\boldsymbol{e}_{\sigma,h}(t)\|_{\mathcal{C},\mathrm{div}} \,+\, \|\dot{\boldsymbol{e}}_{\sigma,h}(t)\|_{\mathcal{C}} \,+\, \|(\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*)(t)\|_{\mathcal{C},\mathrm{div}} \,+\, \|(\dot{\boldsymbol{\sigma}}^* - \dot{\boldsymbol{\sigma}}_h^*)(t)\|_{\mathcal{C}}\,,\end{aligned} \tag{5.27}$$

which, together with (3.3) and (5.19), imply the existence of a constant $C_1 > 0$, independent of h and λ, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} \,+\, \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}} \\ &\quad \leq\, C_1\left\{\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega)} \,+\, \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega)}\right\}\end{aligned}$$

for all $t \in [0, T]$ and the result follows. □

Lemma 5.3. *Under the hypotheses of Lemma 5.2 there exists a constant $C > 0$, independent of λ and h, such that*

$$\begin{aligned} \|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} \leq C \Big\{ & \max_{[0,T]} \|(\dot{\boldsymbol{r}}(t) - Q_h \dot{\boldsymbol{r}})(t)\|_{0,\Omega} + \max_{[0,T]} \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega} \\ & + \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_0 - \boldsymbol{r}_{0,h}\|_{0,\Omega} \Big\}. \end{aligned} \tag{5.28}$$

Proof. By virtue of the inf-sup condition (5.2), and the identities provided by (3.18) and (5.14), we find that

$$\begin{aligned} \beta^* \|\dot{\boldsymbol{r}}_h(t) - Q_h \dot{\boldsymbol{r}}(t)\|_{0,\Omega} & \leq \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{\int_\Omega (\dot{\boldsymbol{r}}_h(t) - Q_h \dot{\boldsymbol{r}}(t)) : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ & \leq \|\dot{\boldsymbol{r}}(t) - Q_h \dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{|\int_\Omega (\dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)) : \boldsymbol{\tau}|}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ & = \|\dot{\boldsymbol{r}}(t) - Q_h \dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{|\langle \dot{\mathcal{G}}(t) - \dot{\mathcal{G}}_h(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}}|}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}. \end{aligned} \tag{5.29}$$

In turn, using the Cauchy-Schwarz inequality and (3.3) we have that

$$\begin{aligned} |\langle \dot{\mathcal{G}}(t) - \dot{\mathcal{G}}_h(t), \boldsymbol{\tau} \rangle_{\boldsymbol{\mathcal{W}}}| & \leq T \max_{[0,T]} \|\mathbf{div}\, \boldsymbol{\sigma}(t) - \mathbf{div}\, \boldsymbol{\sigma}_h(t)\|_\rho \|\mathbf{div}\, \boldsymbol{\tau}\|_\rho \\ & + \Big\{ \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \Big\} \|\boldsymbol{\tau}\|_{0,\Omega} \\ & \leq \Big\{ \frac{T}{\sqrt{\underline{\rho}}} \max_{[0,T]} \|\mathbf{div}\, \boldsymbol{\sigma}(t) - \mathbf{div}\, \boldsymbol{\sigma}_h(t)\|_\rho + \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} \\ & + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \Big\} \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}. \end{aligned} \tag{5.30}$$

In this way, combining (5.29) and (5.30) we deduce that

$$\begin{aligned} \|\dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)\|_{0,\Omega} & \leq \|\dot{\boldsymbol{r}}(t) - Q_h \dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \|Q_h \dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)\|_{0,\Omega} \\ & \leq \left(1 + \frac{1}{\beta^*}\right) \|\dot{\boldsymbol{r}}(t) - Q_h \dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \frac{1}{\beta^*} \Big\{ \frac{T}{\sqrt{\underline{\rho}}} \max_{[0,T]} \|\mathbf{div}\, \boldsymbol{\sigma}(t) - \mathbf{div}\, \boldsymbol{\sigma}_h(t)\|_\rho \\ & + \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \Big\}. \end{aligned}$$

Finally, the bound for $\|\boldsymbol{r}(t) - \boldsymbol{r}_h(t)\|_{0,\Omega}$ is obtained from the foregoing estimate and the identity $\boldsymbol{r}(t) - \boldsymbol{r}_h(t) = \boldsymbol{r}_0 - \boldsymbol{r}_{0,h} + \int_0^t (\dot{\boldsymbol{r}}(s) - \dot{\boldsymbol{r}}_h(s))\, \mathrm{d}s$, which completes the proof. □

Lemma 5.4. *Under the hypotheses of Lemma 5.2, there exists a constant $C > 0$, independent of λ and h, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \leq C \Big\{ \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + \max_{[0,T]} \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega} \Big\}.$$

Proof. According to (4.5) and (5.15) we have that $\ddot{\boldsymbol{u}}(t) := D(\ddot{\boldsymbol{\sigma}}(t), \ddot{\boldsymbol{r}}(t))$ and $\ddot{\boldsymbol{u}}_h(t) := D_h(\ddot{\boldsymbol{\sigma}}_h(t), \ddot{\boldsymbol{r}}_h(t))$, which satisfy

$$(\mathbf{div}\, \boldsymbol{\tau}, \ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)) = (\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, t \in [0, T]. \tag{5.31}$$

Then, it follows from (5.2), (5.31), (4.7), and Hypothesis 2, that

$$\begin{aligned}\beta^*\|\ddot{\boldsymbol{u}}_h(t) - U_h\ddot{\boldsymbol{u}}(t)\|_{0,\Omega} &\leq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}_h(t) - U_h\ddot{\boldsymbol{u}}(t), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &= \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}_h(t) - \ddot{\boldsymbol{u}}(t), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \leq \frac{1}{\underline{\rho}}\|\mathbf{div}\,\boldsymbol{\sigma}(t) - \mathbf{div}\,\boldsymbol{\sigma}_h(t)\|_{0,\Omega}\,,\end{aligned}$$

which, thanks to the triangle inequality, gives the estimate

$$\max_{[0,T]}\|\ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)\|_{0,\Omega} \leq \frac{1}{\beta^*}\max_{[0,T]}\|\ddot{\boldsymbol{u}}(t) - U_h\ddot{\boldsymbol{u}}(t)\|_{0,\Omega} + \frac{1}{\beta^*\underline{\rho}}\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega}.$$

The same estimates for $\|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_h(t)\|_{0,\Omega}$ and $\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{0,\Omega}$ are obtained after integrating, and the result follows. □

We conclude by providing the following convergence result.

Theorem 5.1. *Assume that the solutions* $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ *and* $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ *to problem* (3.5) *satisfy* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega))$ *for some* $\epsilon > 0$ *and* $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{L}^2(\Omega))$*. Then, there exists a constant* $C > 0$*, independent of* λ *and* h*, such that*

$$\begin{aligned}&\max_{[0,T]}\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \\ &\quad\leq C\left\{\|\boldsymbol{\sigma} - \mathit{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbb{H}(\mathbf{div},\Omega))} + \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))}\right\}.\end{aligned}$$

Proof. The required error estimate is a direct consequence of Lemmas 5.2, 5.3 and 5.4, and the norm equivalence provided by Lemma 3.1. □

Remark 5.1. *We notice that the uniformity of the error estimate provided by Theorem 5.1 with respect to the coefficient* λ *shows that the semi-discrete Galerkin scheme* (5.11) *is immune to locking phenomenon in the nearly incompressible case.*

Remark 5.2. *If the Lamé coefficients* λ *and* μ *are constant in* Ω*, it is shown in [22, Lemmas 3.2 and 3.4] that there exists an index* $\epsilon \in (0,1]$ *such that* $\mathbb{V} \subset \mathbb{H}^\epsilon(\Omega) \times \mathbb{H}^\epsilon(\Omega)$*. Hence, we only need to assume in Theorem 5.1 that the solution* $(\boldsymbol{\sigma}, \boldsymbol{r})$ *to problem* (3.5) *satisfies* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega))$*. Indeed, in such a case, the regularity* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega))$ *for some* $\epsilon > 0$ *is guaranteed by the fact that* $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbb{V}$*,* $\forall t \in [0,T]$*.*

6 Time-space discretization

6.1 The fully discrete scheme

Given $L \in \mathbb{N}$, we consider a uniform partition of the time interval $[0,T]$ with step size $\Delta t := T/L$. Then, for any continuous function $\phi : [0,T] \to \mathbb{R}$ and for each $k \in \{0, 1, \ldots, L\}$ we denote $\phi^k := \phi(t_k)$, where $t_k := k\,\Delta t$. In addition, we adopt the same notation for vector/tensor valued functions and consider $t_{k+\frac{1}{2}} := \frac{t_{k+1}+t_k}{2}$, $\phi^{k+\frac{1}{2}} := \frac{\phi^{k+1}+\phi^k}{2}$, $\phi^{k-\frac{1}{2}} := \frac{\phi^k+\phi^{k-1}}{2}$, and the discrete time derivatives

$$\partial_t\phi^k := \frac{\phi^{k+1} - \phi^k}{\Delta t} \quad \text{and} \quad \bar{\partial}_t\phi^k := \frac{\phi^k - \phi^{k-1}}{\Delta t}\,,$$

from which we notice that

$$\partial_t\bar{\partial}_t\phi^k = \frac{\bar{\partial}_t\phi^{k+1} - \bar{\partial}_t\phi^k}{\Delta t} = \frac{\partial_t\phi^k - \partial_t\phi^{k-1}}{\Delta t} = \frac{\phi^{k+1} - 2\phi^k + \phi^{k-1}}{\Delta t^2}.$$

In what follows we utilize the Newmark trapezoidal rule for the time discretization of (5.11): For $k = 1, \ldots, L-1$, we look for $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{r}_h^{k+1}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ solution of

$$\begin{aligned} \Big(\partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^k + \boldsymbol{r}_h^k), \boldsymbol{\tau}\Big) + \Big(\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{k+\frac{1}{2}} + \boldsymbol{\sigma}_h^{k-\frac{1}{2}}}{2}, \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho &= -\Big(\boldsymbol{f}(t_k), \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho \quad \forall\,\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \\ (\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{s}) &= 0 \quad \forall\,\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\,, \end{aligned} \tag{6.1}$$

where, for the sake of simplicity, we assume that the scheme (6.1) is started up with

$$(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0) := \varXi_h\boldsymbol{\sigma}_0, \quad \text{and} \quad (\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1) := \varXi_h\boldsymbol{\sigma}(t_1). \tag{6.2}$$

Then, we introduce the functions

$$\boldsymbol{e}_{\sigma,h}^k := \boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}_h^k \in \boldsymbol{\mathcal{S}}_h \quad \text{and} \quad \boldsymbol{e}_{r,h}^k := \boldsymbol{r}_h^*(t_k) - \boldsymbol{r}_h^k \in \boldsymbol{\mathcal{Q}}_h,$$

where, as usual, $(\boldsymbol{\sigma}_h^*(t_k), \boldsymbol{r}_h^*(t_k)) := \varXi_h\boldsymbol{\sigma}(t_k)$. We note here that (6.2) permits us to ignore the error at the first two initial steps since $\boldsymbol{e}_{\sigma,h}^0 = \boldsymbol{e}_{\sigma,h}^1 = \mathbf{0}$ and $\boldsymbol{e}_{\boldsymbol{r},h}^0 = \boldsymbol{e}_{\boldsymbol{r},h}^1 = \mathbf{0}$. Next, it is straightforward to see that

$$\begin{aligned} &\Big(\partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{e}_{\sigma,h}^k + \boldsymbol{e}_{r,h}^k), \boldsymbol{\tau}\Big) + \Big(\mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho \\ &\quad = (\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\chi}_2^k, \mathbf{div}\,\boldsymbol{\tau})_\rho \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \end{aligned} \tag{6.3}$$

where

$$\boldsymbol{\chi}_1^k := \partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{r}_h^*(t_k)) \;-\; (\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}^*(t_k) + \ddot{\boldsymbol{r}}^*(t_k))$$

and

$$\boldsymbol{\chi}_2^k := \frac{\boldsymbol{\sigma}_h^*(t_{k+1}) + 2\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{\sigma}_h^*(t_{k-1})}{4} - \boldsymbol{\sigma}^*(t_k)\,.$$

Moreover, thanks to the second condition of Hypothesis 2 there holds

$$(\mathbf{div}(\boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}^*(t_k)), \mathbf{div}\,\boldsymbol{\tau})_\rho \;=\; 0 \qquad \forall\,\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,,$$

and hence the consistency term $\boldsymbol{\chi}_2^k$ can be substituted in the error equation (6.3) by

$$\bar{\boldsymbol{\chi}}_2^k \;=\; \boldsymbol{\chi}_2^k \;-\; (\boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}^*(t_k)) \;=\; \frac{\boldsymbol{\sigma}_h^*(t_{k+1}) - 2\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{\sigma}_h^*(t_{k-1})}{4}\,.$$

6.2 Convergence results

We begin the analysis with the following stability result for the main variable $\boldsymbol{\sigma}$.

Lemma 6.1. *There exists a constant $C > 0$, independent of λ, h and Δt, such that for each n there holds*

$$\begin{aligned} &\max_n \|\partial_t\boldsymbol{e}_{\sigma,h}^n\|_{\mathcal{C}} \;+\; \max_n \|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\|_\rho \\ &\quad\leq\; C\Big\{\max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} \;+\; \max_n \|\mathbf{div}\,\partial_t\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \;+\; \max_n \|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega}\Big\}\,. \end{aligned} \tag{6.4}$$

Proof. Taking $\boldsymbol{\tau} = \dfrac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t}$ in (6.3) and using that

$$\frac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t} = \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t} = \frac{\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2},$$

we find that

$$\frac{1}{2\Delta t}\Big(\mathcal{C}^{-1}(\partial_t \boldsymbol{e}_{\sigma,h}^{k} - \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}), (\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1})\Big) + \frac{1}{2\Delta t}\Big(\mathbf{div}(\boldsymbol{e}_{\sigma,h}^{k+\frac12} + \boldsymbol{e}_{\sigma,h}^{k-\frac12}), \mathbf{div}(\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12})\Big)_\rho$$
$$= \Big(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\Big) + \Big(\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}\Big)_\rho,$$

which can also be written as

$$\frac{1}{2\Delta t}\Big(\|\partial_t \boldsymbol{e}_{\sigma,h}^{k}\|_{\mathcal{C}}^2 - \|\partial_t \boldsymbol{e}_{\sigma,h}^{k-1}\|_{\mathcal{C}}^2\Big) + \frac{1}{2\Delta t}\Big(\|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{k+\frac12}\|_\rho^2 - \|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{k-\frac12}\|_\rho^2\Big)$$
$$= \Big(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\Big) + \Big(\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}\Big)_\rho.$$

In this way, summing up the foregoing identity over $k = 1, \ldots, n$, gives

$$\|\partial_t \boldsymbol{e}_{\sigma,h}^{n}\|_{\mathcal{C}}^2 + \|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac12}\|_\rho^2 = 2\Delta t \sum_{k=1}^{n} \Big(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\Big) + 2\Delta t \sum_{k=1}^{n} \Big(\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}\Big)_\rho$$
$$= 2\Delta t \sum_{k=1}^{n} \Big(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^{k} + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\Big) - 2\,\Delta t \sum_{k=1}^{n-1} (\mathbf{div}\,\partial_t \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{k+\frac12})_\rho + 2(\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n, \mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac12})_\rho.$$

It is now straightforward to deduce from the last identity, the Cauchy-Schwarz inequality, and (3.3) that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\max_n \|\partial_t \boldsymbol{e}_{\sigma,h}^{n}\|_{\mathcal{C}} + \max_n \|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac12}\|_\rho \leq C_0 \left\{ \Delta t \sum_{k=1}^{L} \|\mathcal{C}\boldsymbol{\chi}_1^k\|_{\mathcal{C}} + \frac{\Delta t}{\sqrt{\rho}} \sum_{k=1}^{L} \|\mathbf{div}\,\partial_t \bar{\boldsymbol{\chi}}_2^k\|_{0,\Omega} + \frac{1}{\sqrt{\rho}} \max_n \|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \right\}, \tag{6.5}$$

and the result follows. □

We now turn to prove stability estimates for the Lagrange multiplier $\boldsymbol{r}$.

Lemma 6.2. *There exists a constant $C > 0$, independent of h, such that for each n there holds*

$$\max_n \|\partial_t \boldsymbol{e}_{r,h}^{n}\|_{0,\Omega} \leq C \Big\{ \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \max_n \|\mathbf{div}\,\partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} + \max_n \|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \Big\}. \tag{6.6}$$

Proof. Given $k \geq 1$ we deduce from the error equation (6.3) that

$$(\bar{\partial}_t \boldsymbol{e}_{r,h}^{k+1} - \bar{\partial}_t \boldsymbol{e}_{r,h}^{k}, \boldsymbol{\tau}) = -(\mathcal{C}^{-1}(\bar{\partial}_t \boldsymbol{e}_{\sigma,h}^{k+1} - \bar{\partial}_t \boldsymbol{e}_{\sigma,h}^{k}), \boldsymbol{\tau}) - \Delta t \Big(\mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} + \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{2}, \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho$$
$$+ \Delta t(\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) + \Delta t(\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\boldsymbol{\tau})_\rho,$$

which, summing over $k = 1, \ldots, n$, yields

$$(\partial_t \boldsymbol{e}_{r,h}^{n+1}, \boldsymbol{\tau}) \,=\, -(\mathcal{C}^{-1}\partial_t \boldsymbol{e}_{\sigma,h}^{n+1}, \boldsymbol{\tau}) \,-\, \Delta t \sum_{k=1}^{n} \Big(\mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \mathbf{div}\,\boldsymbol{\tau} \Big)_\rho$$
$$+\, \Delta t \sum_{k=1}^{n} (\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) \,+\, \Delta t \sum_{k=1}^{n} (\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\,\boldsymbol{\tau})_\rho \,.$$

It follows from the inf-sup condition (5.2), the Cauchy-Schwarz inequality, and (3.3) that there exists a constant $C_1 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned} \beta^* \|\partial_t \boldsymbol{e}_{r,h}^{n+1}\|_{0,\Omega} \,&\leq\, \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\partial_t \boldsymbol{e}_{r,h}^{n+1}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &\leq\, C_1 \left\{ \max_n (\|\partial_t \boldsymbol{e}_{\sigma,h}^{n+1}\|_{\mathcal{C}} \,+\, \frac{1}{\sqrt{\underline{\rho}}} \max_n \|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\|_\rho \,+\, \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} \,+\, \frac{1}{\underline{\rho}} \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \right\}, \end{aligned}$$

and the result follows from Lemma 6.1. □

Lemma 6.3. *Assume that the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem* (3.5) *satisfy $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega)) \cap \mathcal{C}^4(\mathbb{H}(\mathbf{div},\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega))$. Then, there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\begin{aligned} &\max_n \|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n\|_{\mathcal{C}} \,+\, \max_n \|\mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}})\|_\rho \,+\, \max_n \|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n\|_{0,\Omega} \\ &\qquad \leq\, C \Big\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \\ &\qquad\quad +\, \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \,+\, (\Delta t)^2 \, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \Big\}. \end{aligned} \tag{6.7}$$

Proof. It follows from the triangle inequality and the stability estimates (6.4) and (6.6) that

$$\begin{aligned} &\max_n \|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n\|_{\mathcal{C}} \,+\, \max_n \|\mathbf{div}\,\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_\rho \,+\, \max_n \|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n\|_{0,\Omega} \\ &\leq\, \max_n \|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^*(t_n)\|_{\mathcal{C}} \,+\, \max_n \Big\| \,\mathbf{div}\, \big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\big) \Big\|_\rho \\ &+\, \max_n \|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^*(t_n)\|_{0,\Omega} \,+\, \max_n \|\partial_t \boldsymbol{e}_{\sigma,h}^n\|_{\mathcal{C}} + \max_n \|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\|_\rho \,+\, \max_n \|\partial_t \boldsymbol{e}_{r,h}^n\|_{\mathcal{C}} \\ &\leq\, \max_n \|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^*(t_n)\|_{\mathcal{C}} \,+\, \max_n \|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^*(t_n)\|_{0,\Omega} \\ &+\, \max_n \Big\| \,\mathbf{div}\, \big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\big) \Big\|_\rho \\ &+\, C \Big\{ \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} \,+\, \max_n \|\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \,+\, \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \Big\}. \end{aligned} \tag{6.8}$$

Then, using Taylor expansions centered at $t = t_n$ with integral remainder and keeping in mind (5.24) we have that

$$\begin{aligned} \mathcal{C}\boldsymbol{\chi}_1^n &= \ddot{\boldsymbol{\sigma}}_h^*(t_n) - \ddot{\boldsymbol{\sigma}}^*(t_n) \,+\, 2\mu(\ddot{\boldsymbol{r}}_h^*(t_n) - \ddot{\boldsymbol{r}}^*(t_n)) \\ &\quad +\, \frac{1}{6(\Delta t)^2} \int_{t_{n-1}}^{t_{n+1}} (\frac{\mathrm{d}^4 \boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^4} \,+\, 2\mu \frac{\mathrm{d}^4 \boldsymbol{r}_h^*(t)}{\mathrm{d}t^4}) \, (\Delta t - |t - t_n|)^3 \, \mathrm{d}t \,, \end{aligned} \tag{6.9}$$

$$\bar{\boldsymbol{\chi}}_2^n = \frac{1}{4}\int_{t_{n-1}}^{t_{n+1}} \ddot{\boldsymbol{\sigma}}_h^*(t)(\Delta t - |t-t_n|)\,\mathrm{d}t\,, \tag{6.10}$$

and

$$\begin{aligned}\partial_t\bar{\boldsymbol{\chi}}_2^k = \frac{\boldsymbol{\sigma}_h^*(t_{n+2}) - 3\boldsymbol{\sigma}_h^*(t_{n+1}) + 3\boldsymbol{\sigma}_h^*(t_n) - \boldsymbol{\sigma}_h^*(t_{n-1})}{4\Delta t} &= \frac{1}{8\Delta t}\left\{\int_{t_n}^{t_{n+2}} \frac{\mathrm{d}^3\boldsymbol{\sigma}_h^*}{\mathrm{d}t^3}(t)(t_{n+2}-t)^2\,\mathrm{d}t\right. \\ \left. - 3\int_{t_n}^{t_{n+1}} \frac{\mathrm{d}^3\boldsymbol{\sigma}_h^*}{\mathrm{d}t^3}(t)(t_{n+1}-t)^2\,\mathrm{d}t + \int_{t_{n-1}}^{t_n} \frac{\mathrm{d}^3\boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_{n-1}-t)^2\,\mathrm{d}t\right\}.\end{aligned} \tag{6.11}$$

In turn, Taylor expansions centered this time at $t = t_{n+\frac{1}{2}}$ give

$$\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2} = \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2}\int_{t_n}^{t_{n+1}} \ddot{\boldsymbol{\sigma}}_h^*(t)(\frac{\Delta t}{2} - |t - t_{n+\frac{1}{2}}|)\,\mathrm{d}t, \tag{6.12}$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{\sigma}_h^*(t_n) &= \dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \dot{\boldsymbol{\sigma}}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2\Delta t}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \frac{\mathrm{d}^3\boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_{n+1}-t)^2\,\mathrm{d}t \\ &\quad - \frac{1}{2\Delta t}\int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_n-t)^2\,\mathrm{d}t\,.\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{r}_h^*(t_n) &= \dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \dot{\boldsymbol{r}}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2\Delta t}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \frac{\mathrm{d}^3\boldsymbol{r}_h^*(t)}{\mathrm{d}t^3}(t_{n+1}-t)^2\,\mathrm{d}t \\ &\quad - \frac{1}{2\Delta t}\int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{r}_h^*(t)}{\mathrm{d}t^3}(t_n-t)^2\,\mathrm{d}t\,.\end{aligned} \tag{6.14}$$

Having established the above estimates, we now deduce from (6.9), (6.10) and (6.11) that there exists a constant $C_1 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}&\max_n\|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \max_n\|\mathbf{div}\,\partial_t\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} + \max_n\|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \le C_1\Big\{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ &\quad + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + (\Delta t)^2\big(\|\boldsymbol{r}_h^*\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))}\big)\Big\},\end{aligned} \tag{6.15}$$

whereas (6.12), (6.13) and (6.14) yield the existence of a constant $C_2 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}&\max_n\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{\sigma}_h^*(t_n)\|_{\mathcal{C}} + \max_n\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{r}_h^*(t_n)\|_{0,\Omega} \\ &\quad + \max_n\Big\|\mathbf{div}\,\big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\big)\Big\|_\rho \le C_2\Big\{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{1,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ &\quad + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{1,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + (\Delta t)^2\big(\|\boldsymbol{r}_h^*\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{3,\infty}(\mathrm{H}(\mathbf{div},\Omega))}\big)\Big\}.\end{aligned} \tag{6.16}$$

Finally, we deduce from the uniform boundedness of $\Xi_h : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ with respect to h and λ, and from our regularity assumptions, that there exists a constant $C_3 > 0$, independent of h and λ, such that

$$\|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}_h^*\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \le C_3\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))}\,, \tag{6.17}$$

and thus, combining (6.15), (6.16), and (6.17) with (6.8), we conclude that

$$\begin{aligned}&\max_n\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{\sigma}_h^n\|_{\mathcal{C}} + \max_n\|\mathbf{div}\,\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_\rho + \max_n\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t\boldsymbol{r}_h^n\|_{0,\Omega} \\ &\le C_4\Big\{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + (\Delta t)^2\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))}\Big\},\end{aligned} \tag{6.18}$$

and the result follows from (5.18) and (5.19). □

Lemma 6.4. *Under the hypotheses of Lemma 6.3 there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\begin{aligned}\max_n \|\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_{\mathcal{C}} \,+\, \max_n \|\boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}}\|_{0,\Omega} \,\leq\, C \Big\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \,+\, \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \\ + (\Delta t)^2 \left(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r}\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \right) \Big\} .\end{aligned} \tag{6.19}$$

Proof. We first notice that

$$\begin{aligned}(\boldsymbol{\sigma}(t_{k+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{k+\frac{1}{2}}) - (\boldsymbol{\sigma}(t_{k-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{k-\frac{1}{2}}) = \boldsymbol{\sigma}(t_{k+\frac{1}{2}}) - \boldsymbol{\sigma}(t_{k-\frac{1}{2}}) \\ - \frac{\Delta t}{2}(\dot{\boldsymbol{\sigma}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{\sigma}}(t_{k-\frac{1}{2}})) + \frac{\Delta t}{2}(\dot{\boldsymbol{\sigma}}(t_{k+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^k) + \frac{\Delta t}{2}(\dot{\boldsymbol{\sigma}}(t_{k-\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^{k-1})\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}(\boldsymbol{r}(t_{k+\frac{1}{2}}) - \boldsymbol{r}_h^{k+\frac{1}{2}}) - (\boldsymbol{r}(t_{k-\frac{1}{2}}) - \boldsymbol{r}_h^{k-\frac{1}{2}}) = \boldsymbol{r}(t_{k+\frac{1}{2}}) - \boldsymbol{r}(t_{k-\frac{1}{2}}) \\ - \frac{\Delta t}{2}(\dot{\boldsymbol{r}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{r}}(t_{k-\frac{1}{2}})) + \frac{\Delta t}{2}(\dot{\boldsymbol{r}}(t_{k+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^k) + \frac{\Delta t}{2}(\dot{\boldsymbol{r}}(t_{k-\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^{k-1}) .\end{aligned} \tag{6.21}$$

Then, using a Taylor expansion centered at $t = t_k$, we find that

$$\begin{aligned}\boldsymbol{\sigma}(t_{k+\frac{1}{2}}) - \boldsymbol{\sigma}(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\boldsymbol{\sigma}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{\sigma}}(t_{k-\frac{1}{2}})) = \frac{1}{2}\int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(t_{k+\frac{1}{2}} - t)^2 \,\mathrm{d}t \\ + \frac{1}{2}\int_{t_{k-\frac{1}{2}}}^{t_k} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(t_{k-\frac{1}{2}} - t)^2 \,\mathrm{d}t \,-\, \frac{\Delta t}{2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(\frac{\Delta t}{2} - |t - t_k|)\,\mathrm{d}t\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}\boldsymbol{r}(t_{k+\frac{1}{2}}) - \boldsymbol{r}(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\boldsymbol{r}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{r}}(t_{k-\frac{1}{2}})) = \frac{1}{2}\int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(t_{k+\frac{1}{2}} - t)^2 \,\mathrm{d}t \\ + \frac{1}{2}\int_{t_{k-\frac{1}{2}}}^{t_k} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(t_{k-\frac{1}{2}} - t)^2 \,\mathrm{d}t - \frac{\Delta t}{2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(\frac{\Delta t}{2} - |t - t_k|)\,\mathrm{d}t .\end{aligned} \tag{6.23}$$

In this way, substituting (6.22) in (6.20) and (6.23) in (6.21), and summing the resulting identities over $k = 1, \ldots, n$, we deduce that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}\max_n \|\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_{\mathcal{C}} \,+\, \max_n \|\boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}}\|_{0,\Omega} \,\leq\, C_0 \Big\{ (\Delta t)^2 \big(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} \\ + \|\boldsymbol{r}\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} \big) \,+\, \max_n \|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n\|_{\mathcal{C}} \,+\, \max_n \|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n\|_{0,\Omega} \Big\} .\end{aligned}$$

Finally, (6.19) is a direct consequence of the foregoing estimate and Lemma 6.3. □

It follows from (6.1) and the fact that $(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0)$ and $(\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1)$ belong to $\mathbb{V}_h$ that for each $n \in \{2, \ldots, L\}$, $(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n)$ belongs to $\mathbb{V}_h$ as well. Hence, we may define $\boldsymbol{u}_h^n := D_h(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n) \in \mathcal{U}_h$, which is characterized by

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}_h^n) \,=\, -(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^n + \boldsymbol{r}_h^n, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, n \in \{0, \ldots, L\} . \tag{6.24}$$

Moreover, we propose $\boldsymbol{u}_h^{n+\frac{1}{2}}$ and $\boldsymbol{a}_h^n := \bar{\partial}_t\partial_t\boldsymbol{u}^n$ as suitable approximations of the displacement field $u(t_{n+\frac{1}{2}})$ and the acceleration $\ddot{\boldsymbol{u}}(t_n)$, respectively. To this regard, we remark again that, one can compute $\boldsymbol{u}_h^{n+\frac{1}{2}}$ by solving a saddle point problem of the form (5.8) with right-hand side $\mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac{1}{2}}$. However, a better option consists in using an explicit representation of the fully discrete displacement field $\boldsymbol{u}_h^{n+\frac{1}{2}}$, which is obtained as follows. We first notice from the characterization (6.24) of the operator D_h that

$$\Big(\mathbf{div}\,\boldsymbol{\tau}, \bar{\partial}_t\partial_t\boldsymbol{u}_h^n\Big) = -\Big(\partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^n + \boldsymbol{r}_h^n), \boldsymbol{\tau}\Big) \qquad \forall\,\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h, \quad \forall\, n\in\{1,\dots,L-1\}\,,$$

whereas from the first equation of (6.1) we have that

$$\Big(\mathbf{div}\,\boldsymbol{\tau}, \bar{\partial}_t\partial_t\boldsymbol{u}_h^n\Big) = \Big(\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} + \boldsymbol{f}(t_n), \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h, \quad \forall\, n\in\{1,\dots,L-1\}\,.$$

It follows from the foregoing equation that

$$\boldsymbol{a}_h^n := \bar{\partial}_t\partial_t\boldsymbol{u}^n = \rho^{-1}\left\{\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} + U_h\boldsymbol{f}(t_n)\right\}, \tag{6.25}$$

and summing twice the last identity we obtain

$$\boldsymbol{u}_h^n = (\Delta t)^2\sum_{l=1}^{n-1}\sum_{k=1}^{l}\rho^{-1}\left\{\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{k+\frac{1}{2}} + \boldsymbol{\sigma}_h^{k-\frac{1}{2}}}{2} + U_h\boldsymbol{f}(t_k)\right\} + \boldsymbol{u}_h^0 + t_n\bar{\partial}_t\boldsymbol{u}_h^1 \quad \forall\, n\in\{2,\dots,L\}\,, \tag{6.26}$$

with $\boldsymbol{u}_h^0 := D_h(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0)$ and $\boldsymbol{u}_h^1 := D_h(\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1)$.

Lemma 6.5. *Under the hypotheses of Lemma 6.3 there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\begin{aligned}\max_n\|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} + \max_n\|\boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}}\|_{0,\Omega} \leq C\Big\{ &(\Delta t)^2\|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ + \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + \max_n\|\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_{\mathrm{H}(\mathbf{div},\Omega)} &+ \max_n\|\boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}}\|_{0,\Omega}\Big\}.\end{aligned}$$

Proof. We begin by observing, thanks to the inf-sup condition (5.2) and Hypothesis 2, that

$$\beta^*\|U_h\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \leq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h}\frac{(U_h\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} = \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h}\frac{(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}. \tag{6.27}$$

Next, we notice that by definition of $\ddot{\boldsymbol{u}}(t) = D(\ddot{\boldsymbol{\sigma}}(t), \ddot{\boldsymbol{r}}(t))$ and $\boldsymbol{a}_h^n$, it holds that

$$(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\,\boldsymbol{\tau}) = \Big(\mathbf{div}\,\big(\boldsymbol{\sigma}(t_n) - \frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2}\big), \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho. \tag{6.28}$$

In this way, writing

$$\boldsymbol{\sigma}(t_n) - \frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} = -\frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\ddot{\boldsymbol{\sigma}}(t)(\frac{\Delta t}{2} - |t - t_n|)\,\mathrm{d}t + \frac{1}{2}(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}) + \frac{1}{2}(\boldsymbol{\sigma}(t_{n-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n-\frac{1}{2}})\,,$$

we deduce from (6.27) and (6.28) that

$$\beta^* \max_n \|U_h \ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \leq \max_n \sup_{\boldsymbol{\tau}\in \boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \leq \frac{(\Delta t)^2}{4\underline{\rho}} \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))}$$
$$+ \frac{1}{2\underline{\rho}} \max_n \|\mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12})\|_{0,\Omega} + \frac{1}{2\underline{\rho}} \max_n \|\mathbf{div}(\boldsymbol{\sigma}(t_{n-\frac12}) - \boldsymbol{\sigma}_h^{n-\frac12})\|_{0,\Omega}\,,$$

which, combined with the triangle inequality, gives

$$\max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \leq \max_n \|\ddot{\boldsymbol{u}}(t_n) - U_h\ddot{\boldsymbol{u}}(t_n)\|_{0,\Omega} + \max_n \|U_h\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega}$$
$$\leq \max_n \|\ddot{\boldsymbol{u}}(t_n) - U_h\ddot{\boldsymbol{u}}(t_n)\|_{0,\Omega} + \frac{(\Delta t)^2}{4\underline{\rho}\beta^*} \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \frac{1}{2\underline{\rho}\beta^*} \max_n \|\mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12})\|_{0,\Omega}\,.$$

On the other hand, in order prove error estimates for the displacement, we use again the inf-sup condition (5.2) and the identities (4.7) and (6.24) to obtain

$$\begin{aligned}
\beta^* \|\boldsymbol{u}_h^{n+\frac12} - U_h\boldsymbol{u}(t_{n+\frac12})\|_{0,\Omega} &\leq \sup_{\boldsymbol{\tau}\in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{u}_h^{n+\frac12} - U_h\boldsymbol{u}(t_{n+\frac12}), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\\
&= \sup_{\boldsymbol{\tau}\in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{u}_h^{n+\frac12} - \boldsymbol{u}(t_{n+\frac12}), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\\
&= \sup_{\boldsymbol{\tau}\in \boldsymbol{\mathcal{W}}_h} \frac{(\mathcal{C}^{-1}\boldsymbol{\sigma}(t_{n+\frac12}) + \boldsymbol{r}(t_{n+\frac12}) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h^{n+\frac12} - \boldsymbol{r}_h^{n+\frac12}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\\
&\leq \|\boldsymbol{r}(t_{n+\frac12}) - \boldsymbol{r}_h^{n+\frac12}\|_{0,\Omega} + \frac{1}{2\mu}\|\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12}\|_{0,\Omega}\,.
\end{aligned}$$

Finally, the triangle inequality gives the estimate

$$\max_n \|\boldsymbol{u}(t_{n+\frac12}) - \boldsymbol{u}_h^{n+\frac12}\|_{0,\Omega} \leq \max_n \|\boldsymbol{u}(t_{n+\frac12}) - U_h\boldsymbol{u}(t_{n+\frac12})\|_{0,\Omega}$$
$$+ \tfrac{1}{\beta^*} \max_{[0,T]} \|\boldsymbol{r}(t_{n+\frac12}) - \boldsymbol{r}_h^{n+\frac12}\|_{0,\Omega} + \tfrac{1}{2\mu\beta^*} \max_n \|\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12}\|_{0,\Omega}\,,$$

and the result follows. □

Theorem 6.1. *Assume that the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem (3.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega)) \cap \mathcal{C}^4(\mathbb{H}(\mathbf{div},\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega))$. Then, there exists a constant $C > 0$ independent of λ, h and Δt such that*

$$\max_n \|\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12}\|_{\mathrm{H}(\mathbf{div},\Omega)} + \max_n \|\boldsymbol{r}(t_{n+\frac12}) - \boldsymbol{r}_h^{n+\frac12}\|_{0,\Omega} + \max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega}$$
$$+ \max_n \|\boldsymbol{u}(t_{n+\frac12}) - \boldsymbol{u}_h^{n+\frac12}\|_{0,\Omega} \leq C \Big\{ \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))}$$
$$+ \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + (\Delta t)^2 \big(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))}\big) \Big\}.$$

Proof. The result is a direct consequence of Lemmas 6.3, 6.4 and 6.5 and the norm equivalence provided by Lemma 3.1. □

Remark 6.1. *We end this section by remarking, as shown by Theorem 6.1, that the fully discrete scheme maintains the convergence properties obtained in Theorem 5.1 for the semidiscrete Galerkin scheme as discussed at the end of Section 5. Indeed,* (6.7) *shows that the fully discrete scheme can deal safely with nearly incompressible materials. Finally, we notice from* (6.26) *that the displacement field can also be post-processed at the fully discrete level.*

7 Asymptotic error estimates for the AFW element

It is important to notice that Hypotheses 1, 2 and 3 are satisfied for most known mixed finite elements [4, 12, 18, 25] for the steady elasticity problem with reduced symmetry (see [5] for more details). However, for the sake of brevity we restrict our choice of finite element examples to the Arnold-Falk-Winther (AFW) family [4]. We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega}$ into triangles/tetrahedra K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. In what follows, we assume that $\mathcal{T}_h$ is compatible with the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$, i.e.,

$$\{K \in \mathcal{T}_h, \quad K \subset \bar{\Omega}_j\} = \bar{\Omega}_j \quad \forall j = 1, \cdots, J.$$

Hereafter, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^d$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D. The space of piecewise polynomial functions of degree at most m relatively to $\mathcal{T}_h$ is denoted by

$$\mathcal{P}_m(\mathcal{T}_h) := \{v \in L^2(\Omega); \quad v|_K \in \mathcal{P}_m(K), \quad \forall K \in \mathcal{T}_h\}.$$

For $k \geq 1$, the finite element spaces

$$\boldsymbol{\mathcal{W}}_h := \mathcal{P}_k(\mathcal{T}_h)^{d\times d} \cap \boldsymbol{\mathcal{W}}, \qquad \boldsymbol{\mathcal{Q}}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^{d\times d} \cap \boldsymbol{\mathcal{Q}} \qquad \text{and} \qquad \mathcal{U}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^d$$

correspond to the Arnold-Falk-Winther (AFW) family introduced in [4] for the steady elasticity problem. It is shown in [4] that Hypothesis 1 and the first condition of Hypothesis 2 hold true. Moreover, the fact that $\mathcal{T}_h$ is compatible with the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$ implies that the second condition of Hypothesis 2 follows from the first one.

We also let $\Pi_h : \mathbb{H}^1(\Omega) \to \boldsymbol{\mathcal{W}}_h$ be the tensorial version of the BDM-interpolation operator and recall the following classical error estimate, see [11, Proposition 2.5.4],

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^m \|\boldsymbol{\tau}\|_{m,\Omega} \qquad \forall \boldsymbol{\tau} \in \mathbb{H}^m(\Omega) \quad \text{with } 1 \leq m \leq k+1. \tag{7.1}$$

Moreover, thanks to the commutativity property, if $\operatorname{div}\boldsymbol{\tau} \in \mathbf{H}^k(\Omega)$, then

$$\|\operatorname{div}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\|_{0,\Omega} = \|\operatorname{div}\boldsymbol{\tau} - U_h\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^m \|\mathbf{div}\,\boldsymbol{\tau}\|_{m,\Omega} \quad \text{for } 0 \leq m \leq k. \tag{7.2}$$

In addition, it is well known (see, e.g. [19, Theorem 3.16]) that Π_h is defined on $\mathbb{H}^\epsilon(\Omega) \cap \mathbb{H}(\mathbf{div}, \Omega)$ for any $\epsilon > 0$ and there exists $C > 0$, independent of h, such that

$$\|\Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq C \left\{ \|\boldsymbol{\tau}\|_{\epsilon,\Omega} + \|\,\mathbf{div}\ \boldsymbol{\tau}\|_{0,\Omega} \right\}, \tag{7.3}$$

which proves that Hypothesis 3 is satisfied.

We deduce from (7.1), (7.2) and Theorem 5.1 that if the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{\text{sym}}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem (3.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}^k(\Omega))$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbf{H}^k(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{H}^k(\Omega))$, then there exists a constant $C > 0$ independent of h such that

$$\max_{[0,T]} \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \leq Ch^k. \tag{7.4}$$

Similarly, it follows from Theorem 6.1 that if the solutions to problem (3.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^4(\mathbb{H}(\mathbf{div}, \Omega)) \cap \mathcal{C}^2(\mathbb{H}^k(\Omega))$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbf{H}^k(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega)) \cap \mathcal{C}^2(\mathbb{H}^k(\Omega))$ then, there exists a constant $C > 0$ independent of h, Δt and λ such that

$$\begin{aligned} &\max_n \|\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\|_{\mathrm{H}(\mathbf{div},\Omega)} + \max_n \|\boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}}\|_{0,\Omega} \\ &+ \max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} + \max_n \|\boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}}\|_{0,\Omega} \leq C \left\{ h^k + (\Delta t)^2) \right\}. \end{aligned} \tag{7.5}$$

$h = \Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	4.65e−01	–	3.18e−02	–	9.53e−00	–	1.23e−01	–
1/16	1.08e−01	2.11	9.03e−03	1.82	2.27e−00	2.07	3.05e−02	2.01
1/32	2.65e−02	2.02	2.47e−03	1.87	5.59e−01	2.02	7.56e−03	2.01
1/64	6.63e−03	2.00	6.47e−04	1.93	1.39e−01	2.01	1.89e−03	2.00
1/128	1.65e−03	2.01	1.66e−04	1.96	3.47e−02	2.00	4.72e−04	2.00
1/256	4.10e−04	2.01	4.19e−05	1.99	8.67e−03	2.00	1.18e−04	2.00

Table 8.1: Convergence history in the case $\lambda = \mu = \omega = 1$ and $k = 2$.

8 A mixed FEM example and numerical results

We present a series of numerical experiments confirming the good performance of the fully discrete Galerkin scheme (6.1). For simplicity we consider a two-dimensional model problem and the AFW element element for the spatial discretization. All the numerical results have been obtained by using FEniCS [20].

We choose $\Omega = (0,1)\times(0,1)$, $T = 1$, $\rho = 1$ and select the data $\boldsymbol{f}$ so that the exact solution is given by

$$\boldsymbol{u}(x_1, x_2) = \sin(2\pi\omega x_1)\sin(2\pi\omega x_2)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}. \tag{8.1}$$

We also assume that the body is fixed on the whole boundary, i.e., we take $\Gamma = \partial\Omega$. The numerical results have been obtained by considering nested sequences of uniform triangular meshes $\mathcal{T}_h$ of the unit square Ω. For each mesh size h, we take $\Delta t = h$ and the individual relative errors produced by the fully discrete Galerkin method (6.1) are measured at the final time step as follows:

$$\mathtt{e}_h(\boldsymbol{\sigma}) := \frac{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{L-\frac{1}{2}}\|_{\mathrm{H}(\mathbf{div},\Omega)}}{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}})\|_{\mathrm{H}(\mathbf{div},\Omega)}}, \qquad \mathtt{e}_h(\boldsymbol{r}) := \frac{\|\boldsymbol{r}(t_{L-\frac{1}{2}}) - \boldsymbol{r}_h^{L-\frac{1}{2}}\|_{0,\Omega}}{\|\boldsymbol{r}(t_{L-\frac{1}{2}})\|_{0,\Omega}},$$

$$\mathtt{e}_h(\boldsymbol{u}) := \frac{\|\boldsymbol{u}(t_{L-\frac{1}{2}}) - \boldsymbol{u}_h^{L-\frac{1}{2}}\|_{0,\Omega}}{\|\boldsymbol{u}(t_{L-\frac{1}{2}})\|_{0,\Omega}} \qquad \ddot{\mathtt{e}}_h(\boldsymbol{u}) := \frac{\|\ddot{\boldsymbol{u}}(t_{L-1}) - \boldsymbol{a}_h^{L-1}\|_{0,\Omega}}{\|\ddot{\boldsymbol{u}}(t_{L-1})\|_{0,\Omega}},$$

where $(\boldsymbol{\sigma}, \boldsymbol{r})$ and $\{(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n),\ n = 0,\dots,L\}$ are the solutions of (3.5) and (6.1) respectively and $\boldsymbol{a}_h^{L-1}$ is obtained from (6.25). We introduce the experimental rates of convergence

$$\mathtt{r}_h(\boldsymbol{\sigma}) := \frac{\log(\mathtt{e}_h(\boldsymbol{\sigma})/\mathtt{e}_{\hat{h}}(\boldsymbol{\sigma}))}{\log(h/\hat{h})}, \qquad \mathtt{r}_h(\boldsymbol{r}) := \frac{\log(\mathtt{e}_h(\boldsymbol{r})/\mathtt{e}_{\hat{h}}(\boldsymbol{r}))}{\log(h/\hat{h})},$$

$$\mathtt{r}_h(\boldsymbol{u}) := \frac{\log(\mathtt{e}_h(\boldsymbol{u})/\mathtt{e}_{\hat{h}}(\boldsymbol{u}))}{\log(h/\hat{h})}, \qquad \ddot{\mathtt{r}}_h(\boldsymbol{u}) := \frac{\log(\ddot{\mathtt{e}}_h(\boldsymbol{u})/\ddot{\mathtt{e}}_{\hat{h}}(\boldsymbol{u}))}{\log(h/\hat{h})},$$

where $\mathtt{e}_h$ and $\hat{\mathtt{e}}_{\hat{h}}$ are the errors corresponding to two consecutive triangulations with mesh sizes h and $\hat{h}$, respectively.

We report in Table 8.1 the relative errors and the convergence orders obtained for the AFW element of order $k = 2$ (AFW(2)) and with an exact solution defined as in (8.1) with $\lambda = \mu = \omega = 1$. It is clear that the correct quadratic convergence rate of the error (see (7.5)) is attained in each variable. To test the locking-free character of the method in the nearly incompressible case, we consider now Lamé coefficients λ and μ corresponding to a Poisson ratio $\nu = 0.499$ and a Young modulus $E = 10$. We fix the polynomial degree to $k = 2$, take $\omega = 1$ and report in Table 8.2 the experimental rates of

$h = \Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	4.18e−01	–	4.43e−00	–	5.28e+03	–	3.53e−01	–
1/16	9.70e−02	2.11	5.69e−01	2.96	1.25e+03	2.07	3.67e−02	3.26
1/32	2.23e−02	2.12	7.80e−02	2.87	3.09e+02	2.02	7.73e−03	2.25
1/64	6.12e−03	1.87	9.50e−03	3.04	7.67e+01	2.01	1.88e−03	2.04
1/128	1.48e−03	2.05	1.14e−03	3.06	1.91e+01	2.00	4.72e−04	2.00
1/256	3.43e−04	2.11	1.53e−04	2.90	4.79e+00	2.00	1.18e−04	2.00

Table 8.2: Convergence history in a nearly incompressible case: $\nu = 0.499$, $\omega = 1$, $k = 2$.

$h = \Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	2.25e+02	–	1.22e+01	–	3.16e+04	–	4.39e+02	–
1/16	1.59e+02	0.50	1.68e−00	2.86	3.08e+04	0.04	2.29e+01	4.26
1/32	2.44e+01	2.70	1.78e−01	3.24	5.18e+03	2.57	1.21e−01	7.56
1/64	1.28e+00	4.25	4.88e−03	5.19	3.35e+02	3.95	7.75e−03	3.97
1/128	7.82e−02	4.03	2.29e−04	4.41	2.24e+01	3.90	5.31e−04	3.87
1/256	5.54e−03	3.82	1.25e−05	4.19	1.43e+00	3.98	3.37e−05	3.98

Table 8.3: Convergence history in the case $\lambda = \mu = 1$, $\omega = 16$ and $k = 4$.

convergence. We observe that the method is thoroughly robust for nearly incompressible materials. Finally, we notice that the higher ω is in (8.1), the smaller is the mesh size h needed to reduce the predominance of the spatial component of the error. In such a case, it is meaningful to use a polynomial degree $k > 2$ in order to make the error reach its asymptotic behavior without using too small mesh sizes h. This is illustrated in Table 8.3 where AFW(4) is used with the choice $\omega = 16$.

References

[1] D. N. Arnold, F. Brezzi, and J. Douglas, *PEERS: A new mixed finite element method for plane elasticity*. Japan J. Appl. Math., 1 (1984), pp. 347–367.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Differential complexes and stability of finite element methods. II: The elasticity complex*, in: Compatible Spatial Discretizations, D. N. Arnold, et al., eds., IMA Volumes in Mathematics and its Applications, Vol. 142, Springer Verlag 2005, pp. 47–67.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*. Acta Numerica, 15 (2006), pp. 1–155.

[4] D. N. Arnold, R. S. Falk, and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*. Math. Comp., 76 (2007), pp. 1699–1723.

[5] D.N. Arnold and J.J. Lee, *Mixed methods for elastodynamics with weak symmetry*. SIAM J. Numer. Anal., 52 (2014), pp. 274–2769.

[6] D. N. Arnold and R. Winther, *Mixed finite elements for elasticity*. Numer. Math., 92 (2002), pp. 401–419.

[7] E. Bécache, P. Joly,and C. Tsogka, *A new family of mixed finite elements for the linear elastodynamic problem*. SIAM J. Numer. Anal. 39 (2002), pp. 2109–2132.

[8] D. Boffi, F. Brezzi, and M. Fortin, *Reduced symmetry elements in linear elasticity.* Comm. Pure Appl. Anal., 8 (2009), pp. 1–28.

[9] D. Boffi, A. Buffa and L. Gastaldi, *Convergence analysis for hyperbolic evolution problems in mixed form.* Numer. Linear Algebra Appl. 20 (2013), pp. 541–556.

[10] L. Boulaajine, M. Farhloul and L. Paquet, *A priori error estimation for the dual mixed finite element method of the elastodynamic problem in a polygonal domain, II.* J. Comput. Appl. Math. 235 (2011), no. 5, pp. 1288–1310.

[11] D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics, 44. Springer, Heidelberg, 2013.

[12] B. Cockburn, J. Gopalakrishnan, and J. Guzmán, *A new elasticity element made for enforcing weak stress symmetry.* Math. Comp. 79 (2010), pp. 1331–1349.

[13] J. Jr. Douglas and C. P.Gupta, *Superconvergence for a mixed finite element method for elastic wave propagation in a plane domain.* Numer. Math. 49 (1986), pp. 189–202.

[14] L.C. Evans, Partial Differential Equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[15] M. Farhloul and M. Fortin, *Dual hybrid methods for the elasticity and the Stokes problems: a unified approach.* Numer. Math. 76 (1997), pp. 419–440.

[16] G. N. Gatica, *Analysis of a new augmented mixed finite element method for linear elasticity allowing* $\mathbb{RT}_0 - \mathbb{P}_1 - \mathbb{P}_0$ *approximations.* M2AN Math. Model. Numer. Anal., 40 (2006), pp. 1–28.

[17] G. N. Gatica, A Simple Introduction to the Mixed Finite Element Method. Theory and Applications. Springer Briefs in Mathematics. Springer, Cham, 2014.

[18] J. Gopalakrishnan and J. Guzmán, *A second elasticity element using the matrix bubble.* IMA J. Numer. Anal. 32 (2012), pp. 352–372.

[19] R. Hiptmair, *Finite elements in computational electromagnetism.* Acta Numerica, vol. 11, pp. 237-339, (2002).

[20] A. Logg, K.-A. Mardal, G. N. Wells et al., Automated Solution of Differential Equations by the Finite Element Method. Springer 2012.

[21] Ch. G. Makridakis, *On mixed finite element methods for linear elastodynamics.* Numer. Math. 61 (1992), pp. 235–260.

[22] S. Meddahi, D, Mora and R. Rodríguez, *Finite element spectral analysis for the mixed formulation of the elasticity equations.* SIAM J. Numer. Anal., 51 (2013), pp. 1041–1063.

[23] M. Renardy and R. Rogers, An introduction to Partial Differential Equations. Texts in Applied Mathematics, 13. Springer-Verlag, New York, 2004.

[24] T. Roubíček, Nonlinear Partial Differential Equations with Applications. Second edition. International Series of Numerical Mathematics, 153. Birkhuser/Springer Basel AG, Basel, 2013.

[25] R. Stenberg, *A family of mixed finite elements for the elasticity problem.* Numer. Math. 53 (1988), pp. 513–538.