Analysis of an augmented fully-mixed formulation for the non-isothermal Oldroyd–Stokes problem*

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Abstract

We introduce and analyse an augmented mixed variational formulation for the non-isothermal Oldroyd-Stokes problem. More precisely, the underlying model consists of the Stokes-type equation for Oldroyd viscoelasticity, coupled with the heat equation through a convective term and the viscosity of the fluid. The original unknowns are the polymeric part of the extra-stress tensor, the velocity, the pressure, and the temperature of the fluid. In turn, for convenience of the analysis, the strain tensor, the vorticity, and an auxiliary symmetric tensor are introduced as further unknowns. This allows to join the polymeric and solvent viscosities in an adimensional viscosity, and to eliminate the polymeric part of the extra-stress tensor and the pressure from the system, which, together with the solvent part of the extra-stress tensor, are easily recovered later on through suitable post-processing formulae. In this way, a fully mixed approach is applied, in which the heat flux vector is incorporated as an additional unknown as well. Furthermore, since the convective term in the heat equation forces both the velocity and the temperature to live in a smaller space than usual, we augment the variational formulation by using the constitutive and equilibrium equations, the relation defining the strain and vorticity tensors, and the Dirichlet boundary condition on the temperature. The resulting augmented scheme is then written equivalently as a fixed-point equation, so that the well-known Schauder and Banach theorems, combined with the Lax-Milgram theorem and certain regularity assumptions, are applied to prove the unique solvability of the continuous system. As for the associated Galerkin scheme, whose solvability is established similarly to the continuous case by using the Brouwer fixed-point and Lax-Milgram theorems, we employ Raviart–Thomas approximations of order k for the stress tensor and the heat flux vector, continuous piecewise polynomials of order $\leq k+1$ for velocity and temperature, and piecewise polynomials of order $\leq k$ for the strain tensor and the vorticity. Finally, we derive optimal a priori error estimates and provide several numerical results illustrating the good performance of the scheme and confirming the theoretical rates of convergence.

Key words: Oldroyd–Stokes problem, non-isothermal, stress-velocity formulation, fixed-point theory, augmented fully-mixed formulation, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

*This work was partially supported by CONICYT-Chile through BASAL project CMM, Universidad de Chile, project Fondecyt 1161325, and the Becas-Chile Programme for Chilean students; by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción; and by Universidad del Bío-Bío through DIUBB project GI 171508/VC.

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1 Introduction

The numerical simulation of viscoelastic fluid flows has become increasingly important for a variety of research areas in the fields of the natural sciences and engineering branches. This fact has been motivated by its diverse applications in industry such as design of heat exchangers and chemical reactors, cooling processes, and polymer processing (see, e.g., [22, 10, 25, 29]), to name a few. The complexity of the governing equations and the physical domains makes analysis of the mathematical models and the associated numerical methods especially difficult. Current efforts to model isothermal viscoelastic flows often revolve around the solution of the Stokes problem for the Oldroyd viscoelastic model (see, e.g., [5, 4, 8, 3], and the references therein). In particular, in [3] the authors analysed an extra stress-vorticity formulation and proved that this formulation satisfies an inf-sup condition and consequently, classical finite element spaces can be used for its approximation. We remark that, although most of the research on the viscoelastic fluid flows concerns isothermal cases, many flows of practical interest in polymeric melt processing are non-isothermal (see, e.g., [31, 27, 15, 24]). The combination of high viscosities of polymeric melts and high deformation rates results in the transformation of large amounts of mechanical energy into heat, and therefore in a temperature rise of the material. This phenomenon is, for instance, used in extruders where viscous dissipation is employed to enhance melting of the material (see [31] for details). This kind of fluid flows has motivated the introduction of the coupled problem between the Stokes equation for the Oldroyd viscoelastic model and the heat equation through a convective term and the viscosity of the fluid, thus arising the so called non-isothermal Oldroyd–Stokes problem.

Up to the authors' knowledge, [14] constitutes one of the first works in analysing a finite element discretization for the non-isothermal Oldroyd–Stokes equations. In that work, the authors provide a complete analysis of a mixed-primal formulation for the coupled problem, in which the main unknowns are the polymeric part of the extra-stress tensor, the velocity, the pressure and the temperature of the fluid. The focus of this work is the discrete scheme, where by considering piecewise quadratic elements for the velocity and the temperature, continuous piecewise linear elements for the pressure, and discontinuous piecewise linear elements for the polymeric part of the extra-stress tensor, it is proved existence of at least one solution by using inverse inequalities of L^∞ into L^2 and the Schaefer fixed-point theorem. In addition, the Galerkin scheme has optimal rates of convergence under a smallness assumption on the data. Later on, a new dual-mixed formulation was introduced and analysed in [17], where the solvent part of the extra-stress tensor, the vorticity, and the heat flux vector are set as further unknowns (besides the polymeric part of the extra-stress tensor, the velocity, the pressure and the temperature). The corresponding mixed finite element scheme employs Raviart–Thomas elements of lowest order plus bubble function for the solvent part of the extra-stress tensor, Raviart–Thomas elements of lowest order for the heat flux vector, continuous piecewise linear elements for the vorticity, and piecewise constants for the polymeric part of the extra-stress tensor, velocity, pressure and the temperature of the fluid. Existence of solution and convergence of the numerical scheme are proved and optimal error estimates are also provided by using inverse inequalities of L^∞ into L^2, smallnes assumption on the data and the Schaefer fixed-point theorem. We remark that this formulation has properties analogous to finite volume methods, namely local conservation of momentum and mass.

The purpose of the present paper is to contribute in the development of new numerical methods for the non-isothermal Oldroyd–Stokes problem. To that end, unlike to [14] and [17], and in order to obtain a new fully-mixed formulation of this coupled problem, we first introduce the strain tensor as a new unknown, which allows us, on one hand, to eliminate the polymeric part of the extra-stress tensor from the system and compute it as a simple post-process of the solution, and on the other hand, to join the polymeric and solvent viscosities in an adimensional viscosity. In addition, for convenience of the analysis we also consider the stress and vorticity tensors as auxiliary unknowns, thanks to which the pressure can be eliminated from the system and approximated later on by a postprocessing formula. In turn, for deriving the mixed formulation of the heat equation we proceed similarly to [17] (see also [12, 13]) and set the heat-flux vector as a further unknown. Furthermore, the difficulty given by the fact that the fluid velocity and the temperature lives in H^1 instead of L^2 as usual, is resolved as in [12, 13] by augmenting the variational formulation with suitable Galerkin type expressions arising from the constitutive and equilibrium equations, the relation defining the strain and vorticity tensors, and the Dirichlet boundary condition on the temperature. Then, following [12] and [2], we combine classical fixed-point arguments, suitable regularity assumptions on the decoupled problems, the Lax–Milgram lemma, the Sobolev embedding and Rellich–Kondrachov theorems, and sufficiently small data assumptions to establish existence and uniqueness of solution of the continuous problem. Similarly, the existence of solution of the discrete problem relies on the Brouwer fixed-point theorem and analogous arguments to those employed in the continuous analysis. Moreover, applying a Strang-type lemma valid for linear problems, we are able to derive the corresponding Céa estimate and to provide optimal a priori error bounds for the Galerkin solution. Finally, we point out that the main advantages of approximating the solution of the coupled system through this new approach include, on one hand, the fact that no discrete inf-sup conditions are required for the discrete analysis, and therefore arbitrary finite element subspaces can be employed, and on the other hand, the possibility of recovering by post-processing formulae the pressure, the polymeric part and solvent part of the extra-stress tensor in terms of the solution, conserving the same rates of convergence.

The rest of this work is organised as follows. The remainder of this section describes standard notations and functional spaces to be employed along the paper. In Section 2 we introduce the model problem and derive the augmented fully-mixed variational formulation. Next, in Section 3 we establish the well-posedness of this continuous scheme by means of a fixed-point strategy and the Schauder and Banach fixed point theorems. The corresponding Galerkin system is introduced and analysed in Section 4, where the discrete analogue of the theory used in the continuous case is employed to prove existence of solution. In addition, a suitable Strang-type lemma is utilized here to derive the corresponding a priori error estimate and the resulting rates of convergence. Finally, in Section 5 we report several numerical essays illustrating the accuracy of our augmented fully-mixed finite element method.

Preliminary notations

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \{2,3\}$, denote a bounded domain with Lipschitz boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ and $|\Gamma_{\mathrm{D}}|, |\Gamma_{\mathrm{N}}| > 0$, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. For $s \geq 0$ and $p \in [1, +\infty]$, we define by $\mathrm{L}^p(\Omega)$ and $\mathrm{W}^{s,p}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{s,p}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$. If $p = 2$, we write $\mathrm{H}^s(\Omega)$ in place of $\mathrm{W}^{s,2}(\Omega)$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{s,\Omega}$, respectively, and the seminorm by $|\cdot|_{s,\Omega}$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$, we set the gradient, and divergence operator, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}.$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}\,(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. In addition, we recall that

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\Big\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\Omega},$$

is a standard Hilbert space in the realm of mixed problems. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^n$, $\mathrm{P}_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, we set $\mathbf{P}_k(S) := [\mathrm{P}_k(S)]^n$ and $\mathbb{P}_k(S) := [\mathrm{P}_k(S)]^{n\times n}$. Finally, we end this section by mentioning that, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The continuous formulation

In this section we introduce the model problem and derive the corresponding weak formulation.

2.1 The model problem

The non-isothermal Oldroyd–Stokes problem consists of a system of equations where the Stokes equation for the Oldroyd viscoelastic model introduced in [5], is coupled with the heat equation through a convective term and the viscosity of the fluid (cf. [14, 17]). More precisely, given a body force $\mathbf{f}$, and a heat source g, the aforementioned system of equations is given by

$$\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{P}} - 2\mu_{\mathrm{P}}(\theta)\mathbf{e}(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega,\\
-\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{P}} + 2\epsilon\mu_{\mathrm{N}}(\theta)\mathbf{e}(\mathbf{u})) + \nabla p &= \mathbf{f} && \text{in } \Omega,\\
\mathrm{div}\,\mathbf{u} &= 0 && \text{in } \Omega,\\
-\mathrm{div}\,(\kappa\nabla\theta) + \mathbf{u}\cdot\nabla\theta &= g && \text{in } \Omega,\\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma,\\
\theta &= \theta_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}},\\
\kappa\nabla\theta\cdot\mathbf{n} &= 0 && \text{on } \Gamma_{\mathrm{N}},
\end{aligned} \tag{2.1}$$

where the unknowns are the polymeric part of the extra-stress tensor $\boldsymbol{\sigma}_{\mathrm{P}}$, the velocity $\mathbf{u}$, the pressure p, and the temperature θ of a fluid occupying the region Ω. In addition, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\Big\{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{t}}\Big\}$ stands for the strain tensor of small deformations, κ is the thermal conductivity coefficient, μ_{P} and μ_{N} are the polymeric and solvent (or newtonian) viscosities, respectively, which are given by the following Arrhenius relationship:

$$\mu_{\mathrm{P}}(\theta) = a_1 \exp\left(\frac{b_1}{\theta}\right), \quad \mu_{\mathrm{N}}(\theta) = a_2 \exp\left(\frac{b_2}{\theta}\right), \tag{2.2}$$

where the coefficients a_1, b_1, a_2, and b_2 are defined so that

$$0 < \mu_{\mathrm{P}}(s) \le 1, \quad 0 < \mu_{\mathrm{N}}(s) \le 1 \quad \forall s \ge 0. \tag{2.3}$$

Furthermore, we assume that both the polymeric and solvent viscosities are Lipschitz continuous and bounded from above and from below, that is,

$$|\mu_{\mathrm{P}}(s) - \mu_{\mathrm{P}}(t)| \le L_{\mu_{\mathrm{P}}}|s-t|, \quad |\mu_{\mathrm{N}}(s) - \mu_{\mathrm{N}}(t)| \le L_{\mu_{\mathrm{N}}}|s-t| \quad \forall s,t \ge 0, \tag{2.4}$$

and

$$\mu_{1,\mathrm{P}} \le \mu_{\mathrm{P}}(s) \le \mu_{2,\mathrm{P}}, \quad \mu_{1,\mathrm{N}} \le \mu_{\mathrm{N}}(s) \le \mu_{2,\mathrm{N}} \quad \forall s \ge 0. \tag{2.5}$$

Note that a small real parameter $\epsilon > 0$ on the second equation of (2.1) is introduced to make the effect of the solvent viscosity much smaller than that of the polymeric part. Moreover, it is well known that uniqueness of a pressure solution of (2.1) (see, e.g., [30]) is ensured in the space

$$\mathrm{L}_0^2(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

Now, in order to derive our augmented fully-mixed formulation we first need to rewrite (2.1) as a first-order system of equations. To this end, unlike to [14] and [17], we begin by introducing the strain tensor as an additional unknown $\mathbf{t} := \mathbf{e}(\mathbf{u})$, whence the polymeric and solvent parts of the extra-stress tensor can be written, respectively, as

$$\boldsymbol{\sigma}_{\mathrm{P}} \,=\, 2\mu_{\mathrm{P}}(\theta)\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{N}} \,=\, 2\epsilon\mu_{\mathrm{N}}(\theta)\mathbf{t} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

Next, defining the dimensionless effective viscosity as in [17], that is

$$\mu(\theta) \,:=\, 2\mu_{\mathrm{P}}(\theta) + 2\epsilon\mu_{\mathrm{N}}(\theta)\,, \tag{2.7}$$

and adopting the approach from [20] (see also [19, 9, 7]), we introduce the auxiliary unknowns

$$\boldsymbol{\rho} := \nabla\mathbf{u} - \mathbf{e}(\mathbf{u}) \quad \text{and} \quad \boldsymbol{\sigma} := \mu(\theta)\mathbf{t} - p\mathbb{I} \quad \text{in} \quad \Omega,$$

where $\boldsymbol{\rho}$ is the vorticity (or skew-symmetric part of the velocity gradient). In this way, utilising the incompressibility condition $\operatorname{div}\mathbf{u} = \operatorname{tr}(\mathbf{e}(\mathbf{u})) = 0$, we find that the equations modelling the fluid in (2.1) can be rewritten, equivalently, as the set of equations with unknowns $\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\rho}$ and $\mathbf{u}$, given by

$$\begin{gathered} \mathbf{t} + \boldsymbol{\rho} = \nabla\mathbf{u} \quad \text{in} \quad \Omega, \quad \boldsymbol{\sigma}^{\mathrm{d}} = \mu(\theta)\mathbf{t} \quad \text{in} \quad \Omega, \quad -\mathbf{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad p = -\frac{1}{n}\operatorname{tr}\boldsymbol{\sigma} \quad \text{in} \quad \Omega, \quad \int_\Omega \operatorname{tr}\boldsymbol{\sigma} = 0, \end{gathered} \tag{2.8}$$

where both $\mathbf{t}$ and $\boldsymbol{\sigma}$ are symmetric tensors, and $\operatorname{tr}\mathbf{t} = 0$ holds in Ω. Note that the fifth equation in (2.8) allows us to eliminate the pressure p from the system (which anyway can be approximated later on through a post–processing procedure), whereas the last equation takes care of the requirement that $p \in \mathrm{L}_0^2(\Omega)$. In addition, it easy to see from (2.4) and (2.5) that the fluid viscosity $\mu(\cdot)$ is Lipschitz continuous and bounded from above and from below, that is, there exist constants $L_\mu > 0$ and $\mu_1, \mu_2 > 0$, such that

$$|\mu(s) - \mu(t)| \le L_\mu|s-t| \quad \forall s,t \ge 0, \tag{2.9}$$

and

$$\mu_1 \le \mu(s) \le \mu_2 \quad \forall s \ge 0. \tag{2.10}$$

Similarly, for the convection-diffusion equation modelling the temperature of the fluid in (2.1), we adopt the approach from [17] (see also [12, 13]) and introduce as a further unknown the heat flux vector

$$\mathbf{p} := \kappa \nabla\theta - \theta\,\mathbf{u} \quad \text{in} \quad \Omega,$$

so that, utilising the incompressibility condition $\operatorname{div}\mathbf{u} = 0$ in Ω and the homogenous Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on Γ, the remaining equations in the system (2.1) can be rewritten, equivalently, as

$$\begin{aligned} \kappa^{-1}\mathbf{p} + \kappa^{-1}\theta\,\mathbf{u} = \nabla\theta \quad \text{in} \quad \Omega, \qquad -\operatorname{div}\mathbf{p} = g \quad \text{in} \quad \Omega, \\ \theta = \theta_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \qquad \mathbf{p}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}. \end{aligned} \tag{2.11}$$

We end this section emphasizing from (2.6) that we can recover the polymeric and solvent parts of the extra-stress tensor in terms of θ and $\mathbf{t}$, whereas from the fifth equation of (2.8) we obtain the pressure in terms of $\boldsymbol{\sigma}$. Alternatively, from (2.6), (2.7), and the second equation of (2.8), we arrive at the identity

$$\boldsymbol{\sigma}_{\mathrm{P}} + \boldsymbol{\sigma}_{\mathrm{N}} = \boldsymbol{\sigma}^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \tag{2.12}$$

from which each part of the extra stress can be computed in terms of $\boldsymbol{\sigma}^{\mathrm{d}}$ and the other part. The formulae provided by (2.6), (2.12), and the fifth equation of (2.8), will suggest in Section 5 suitable approximations of the polymeric and solvent parts of the extra-stress tensor, and the pressure (cf. (4.25)). They will all depend on the unique finite element solution of a Galerkin scheme to be introduced below (cf. (4.2)), and hence the same rates of convergence will be obtained.

2.2 The augmented fully-mixed variational formulation

In this section we derive the weak formulation of the coupled system (2.8)–(2.11). We begin by recalling (see, e.g., [6, 18, 21]) that there holds

$$\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathbb{R}\mathbb{I}, \tag{2.13}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \operatorname{tr}\boldsymbol{\tau} = 0 \right\}.$$

In this way, decomposing $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$ as $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I}$, with $\boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$ and $c \in \mathbb{R}$, noticing that $\boldsymbol{\tau}^{\mathrm{d}} = \boldsymbol{\tau}_0^{\mathrm{d}}$ and $\mathbf{div}\boldsymbol{\tau} = \mathbf{div}\boldsymbol{\tau}_0$, and using the last equation of (2.8), we deduce that both $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can be considered hereafter in $\mathbb{H}_0(\mathbf{div};\Omega)$. In addition, thanks to the incompressibility condition and the first equation of (2.8), we can look for the strain tensor $\mathbf{t}$ in the space

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \left\{ \mathbf{r} \in \mathbb{L}^2(\Omega) : \quad \mathbf{r}^{\mathrm{t}} = \mathbf{r} \quad \text{and} \quad \operatorname{tr}\mathbf{r} = 0 \right\},$$

whereas the vorticity $\boldsymbol{\rho}$ lives in

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega) := \left\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\eta}^{\mathrm{t}} = -\boldsymbol{\eta} \right\}.$$

In turn, the homogeneous Neumann boundary condition for $\mathbf{p}$ on Γ_{N} (cf. fourth equation in (2.11)) suggests the introduction of the functional space

$$\mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega) := \left\{ \mathbf{q} \in \mathbf{H}(\operatorname{div};\Omega) : \quad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}} \right\}.$$

Hence, we begin the derivation of our weak formulation by testing the first equations of (2.8) and (2.11) with arbitrary $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$ and $\mathbf{q} \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega)$, respectively. Then, integrating by parts, utilising the identity $\mathbf{t} : \boldsymbol{\tau} = \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}}$ (which follows from the fact that $\mathbf{t} : \mathbb{I} = \operatorname{tr}\mathbf{t} = 0$), and imposing the remaining equations weakly, which includes the symmetry of $\boldsymbol{\sigma}$, we arrive at the variational problem: Find $\mathbf{t} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$, $\boldsymbol{\rho} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, $\mathbf{p} \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega)$, and $\mathbf{u}$, θ in suitable spaces to be defined, such that

$$\begin{aligned}
\int_\Omega \mu(\theta)\mathbf{t} : \mathbf{r} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{r} &= 0 && \forall \mathbf{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega),\\
\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\boldsymbol{\tau} + \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega),\\
-\int_\Omega \mathbf{v}\cdot\mathbf{div}\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} &= \int_\Omega \mathbf{f}\cdot\mathbf{v} && \forall (\mathbf{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega)\times\mathbb{L}^2_{\mathrm{skew}}(\Omega),\\
\kappa^{-1}\int_\Omega \mathbf{p}\cdot\mathbf{q} + \int_\Omega \theta\operatorname{div}\mathbf{q} + \kappa^{-1}\int_\Omega \theta\,\mathbf{u}\cdot\mathbf{q} &= \langle \mathbf{q}\cdot\mathbf{n}, \theta_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}} && \forall \mathbf{q} \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega),\\
-\int_\Omega \psi\operatorname{div}\mathbf{p} &= \int_\Omega g\,\psi && \forall \psi \in \mathrm{L}^2(\Omega).
\end{aligned} \tag{2.14}$$

Before continuing we observe that the third term on the left-hand side of the fourth equation in (2.14) requires a suitable regularity for both unknowns $\mathbf{u}$ and θ. Indeed, by applying Cauchy–Schwarz and Hölder's inequalities, and then the continuous injection $\mathbf{i}$ of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$ (see, e.g., [1, Theorem 6.3] or [32, Theorem 1.3.5]), we find that there exist a positive constant $c(\Omega) := \|\mathbf{i}\|^2$, such that

$$\left|\int_\Omega \theta\,\mathbf{u}\cdot\mathbf{q}\right| \le c(\Omega)\|\theta\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{q}\|_{0,\Omega} \qquad \forall\,\theta \in \mathrm{H}^1(\Omega)\ \forall\,\mathbf{u} \in \mathbf{H}^1(\Omega)\ \forall\,\mathbf{q} \in \mathbf{L}^2(\Omega). \tag{2.15}$$

According to the above, and in order to be able to analyse the present variational formulation of the coupled system (2.8)–(2.11), we propose to seek $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ and $\theta \in \mathrm{H}^1(\Omega)$, and to restrict the set of corresponding test functions $\mathbf{v}$ and ψ to the same spaces, respectively. In this way, similarly as in [12] (see also [13]), we augment (2.14) through the following redundant Galerkin terms arising from the constitutive and equilibrium equations, the relation between the strain tensor and $\mathbf{t}$, the definition of the vorticity in terms of the velocity gradient, and the Dirichlet boundary condition on the temperature:

$$\begin{aligned}
\kappa_1\int_\Omega \Big\{\boldsymbol{\sigma}^{\mathrm{d}} - \mu(\theta)\mathbf{t}\Big\} : \boldsymbol{\tau}^{\mathrm{d}} = 0 &\qquad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_2\int_\Omega \mathbf{div}\boldsymbol{\sigma}\cdot\mathbf{div}\boldsymbol{\tau} = -\kappa_2\int_\Omega \mathbf{f}\cdot\mathbf{div}\boldsymbol{\tau} &\qquad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_3\int_\Omega \Big\{\mathbf{e}(\mathbf{u}) - \mathbf{t}\Big\} : \mathbf{e}(\mathbf{v}) = 0 &\qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega),\\
\kappa_4\int_\Omega \Big(\boldsymbol{\rho} - \Big\{\nabla\mathbf{u} - \mathbf{e}(\mathbf{u})\Big\}\Big) : \boldsymbol{\eta} = 0 &\qquad \forall \boldsymbol{\eta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega),
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\kappa_5\int_\Omega \Big\{\nabla\theta - \kappa^{-1}\mathbf{p} - \kappa^{-1}\theta\,\mathbf{u}\Big\}\cdot\nabla\psi = 0 &\qquad \forall \psi \in \mathrm{H}^1(\Omega),\\
\kappa_6\int_\Omega \operatorname{div}\mathbf{p}\operatorname{div}\mathbf{q} = -\kappa_6\int_\Omega g\operatorname{div}\mathbf{q} &\qquad \forall \mathbf{q} \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega),\\
\kappa_7\int_{\Gamma_{\mathrm{D}}} \theta\,\psi = \kappa_7\int_{\Gamma_{\mathrm{D}}} \theta_{\mathrm{D}}\,\psi &\qquad \forall \psi \in \mathrm{H}^1(\Omega),
\end{aligned} \tag{2.17}$$

where $(\kappa_1, \ldots, \kappa_7)$ is a vector of positive parameters to be specified later.

At this point we remark that there are many different ways of ordering the augmented fully-mixed variational formulation described above, but for the sake of the subsequent analysis we proceed as in [12, Section 3.1], and adopt one leading to an uncoupled structure. To that end, we start by grouping appropriately some of the unknowns and spaces as follows:

$$\underline{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\rho}) \in \mathbb{H} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega),$$

where $\mathbb{H}$ is endowed with the norm

$$\|\underline{\mathbf{r}}\|^2_{\mathbb{H}} := \|\mathbf{r}\|^2_{0,\Omega} + \|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} + \|\boldsymbol{\eta}\|^2_{0,\Omega} \quad \forall \underline{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\tau}, \boldsymbol{\eta}) \in \mathbb{H}.$$

Hence, the augmented fully-mixed variational formulation for the non-isothermal Oldroyd–Stokes problem reads: Find $(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega) \times \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathbf{A}_\theta((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v})) &= \mathbf{F}(\underline{\mathbf{r}}, \mathbf{v}) \quad \forall (\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega), \\ \widetilde{\mathbf{A}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) + \widetilde{\mathbf{B}}_{\mathbf{u}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) &= \widetilde{\mathbf{F}}(\mathbf{q}, \psi) \quad \forall (\mathbf{q}, \psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega), \end{aligned} \tag{2.18}$$

where, given $\phi \in \mathrm{H}^1(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$, $\mathbf{A}_\phi$, $\widetilde{\mathbf{A}}$, and $\widetilde{\mathbf{B}}_{\mathbf{w}}$ are the bilinear forms defined, respectively, as

$$\begin{aligned} \mathbf{A}_\phi((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v})) &:= \int_\Omega \mu(\phi)\mathbf{t} : \Big\{\mathbf{r} - \kappa_1 \boldsymbol{\tau}^{\mathrm{d}}\Big\} + \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \Big\{\kappa_1 \boldsymbol{\tau}^{\mathrm{d}} - \mathbf{r}\Big\} + \int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}} \\ &+ \int_\Omega \Big\{\mathbf{u} + \kappa_2 \mathbf{div}\boldsymbol{\sigma}\Big\} \cdot \mathbf{div}\boldsymbol{\tau} - \int_\Omega \mathbf{v} \cdot \mathbf{div}\boldsymbol{\sigma} + \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} \\ &+ \kappa_3 \int_\Omega \Big\{\mathbf{e}(\mathbf{u}) - \mathbf{t}\Big\} : \mathbf{e}(\mathbf{v}) + \kappa_4 \int_\Omega \Big(\boldsymbol{\rho} - \Big\{\nabla \mathbf{u} - \mathbf{e}(\mathbf{u})\Big\}\Big) : \boldsymbol{\eta}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \widetilde{\mathbf{A}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) &:= \kappa^{-1} \int_\Omega \mathbf{p} \cdot \Big\{\mathbf{q} - \kappa_5 \nabla \psi\Big\} + \int_\Omega \Big\{\theta + \kappa_6 \mathrm{div}\, \mathbf{p}\Big\} \mathrm{div}\, \mathbf{q} - \int_\Omega \psi\, \mathrm{div}\, \mathbf{p} \\ &+ \kappa_5 \int_\Omega \nabla\theta \cdot \nabla\psi + \kappa_7 \int_{\Gamma_{\mathrm{D}}} \theta\, \psi, \end{aligned} \tag{2.20}$$

and

$$\widetilde{\mathbf{B}}_{\mathbf{w}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) := \kappa^{-1} \int_\Omega \theta\, \mathbf{w} \cdot \Big\{\mathbf{q} - \kappa_5 \nabla\psi\Big\}, \tag{2.21}$$

for all $(\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega)$ and for all $(\mathbf{p}, \theta), (\mathbf{q}, \psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$. In turn, $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ are the bounded linear functionals given by

$$\mathbf{F}(\underline{\mathbf{r}}, \mathbf{v}) := \int_\Omega \mathbf{f} \cdot \Big\{\mathbf{v} - \kappa_2 \mathbf{div}\boldsymbol{\tau}\Big\}, \tag{2.22}$$

for all $(\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega)$ and

$$\widetilde{\mathbf{F}}(\mathbf{q}, \psi) := \langle \mathbf{q} \cdot \mathbf{n}, \theta_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} + \int_\Omega g \Big\{\psi - \kappa_6 \mathrm{div}\, \mathbf{q}\Big\} + \kappa_7 \int_{\Gamma_{\mathrm{D}}} \theta_{\mathrm{D}} \psi, \tag{2.23}$$

for all $(\mathbf{q}, \psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$.

3 Analysis of the continuous formulation

In this section we proceed similarly as in [12] (see also [13, 2]) and utilise a fixed-point strategy to prove that problem (2.18) is well posed. More precisely, in Section 3.1 we rewrite (2.18) as an equivalent fixed-point equation in terms of an operator $\mathbf{T}$. Next in Section 3.2 we show that $\mathbf{T}$ is well defined, and finally in Section 3.3 we apply the well known Schauder and Banach fixed–point theorems to conclude that $\mathbf{T}$ has a unique fixed point.

3.1 The fixed-point approach

We start by defining the operator $\mathbf{S} : \mathrm{H}^1(\Omega) \to \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ by

$$\mathbf{S}(\phi) := (\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) = (\underline{\mathbf{t}}, \mathbf{u}) \quad \forall \phi \in \mathrm{H}^1(\Omega), \tag{3.1}$$

where $\mathbf{S}_1(\phi) := (\mathbf{S}_1^{\mathbf{t}}(\phi), \mathbf{S}_1^{\boldsymbol{\sigma}}(\phi), \mathbf{S}_1^{\boldsymbol{\rho}}(\phi))$ and $(\underline{\mathbf{t}}, \mathbf{u})$ is the unique solution of the problem: Find $(\underline{\mathbf{t}}, \mathbf{u}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ such that

$$\mathbf{A}_\phi((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v})) = \mathbf{F}(\underline{\mathbf{r}}, \mathbf{v}) \quad \forall (\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega), \tag{3.2}$$

where the bilinear form $\mathbf{A}_\phi$ is given by (2.19). In turn, the functional $\mathbf{F}$ is defined exactly as in (2.22). In addition, we also introduce the operator $\widetilde{\mathbf{S}} : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_{\Gamma_\mathrm{N}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$ defined as

$$\widetilde{\mathbf{S}}(\mathbf{w}) := (\widetilde{\mathbf{S}}_1(\mathbf{w}), \widetilde{\mathbf{S}}_2(\mathbf{w})) = (\mathbf{p}, \theta) \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{3.3}$$

where $(\mathbf{p}, \theta)$ is the unique solution of the problem: Find $(\mathbf{p}, \theta) \in \mathbf{H}_{\Gamma_\mathrm{N}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$ such that

$$\widetilde{\mathbf{A}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) + \widetilde{\mathbf{B}}_{\mathbf{w}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) = \widetilde{\mathbf{F}}(\mathbf{q}, \psi) \quad \forall (\mathbf{q}, \psi) \in \mathbf{H}_{\Gamma_\mathrm{N}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)\,. \tag{3.4}$$

Here the bilinear form $\widetilde{\mathbf{A}}$ and the functional $\widetilde{\mathbf{F}}$ are defined exactly as in (2.20) and (2.23), respectively. In turn, the bilinear form $\widetilde{\mathbf{B}}_{\mathbf{w}}$ is given by (2.21). In this way, we define the operator $\mathbf{T} : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\Omega)$ as

$$\mathbf{T}(\phi) := \widetilde{\mathbf{S}}_2(\mathbf{S}_2(\phi)) \quad \forall \phi \in \mathrm{H}^1(\Omega), \tag{3.5}$$

and realise that (2.18) can be rewritten as the fixed-point problem: Find $\theta \in \mathrm{H}^1(\Omega)$ such that

$$\mathbf{T}(\theta) = \theta. \tag{3.6}$$

This fact certainly requires both operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ to be well defined. In other words, we first need to analyse the well-posedness of the uncoupled problems (3.2) and (3.4). The next section is devoted to this matter.

We end this section by recalling, for later use, that there exist positive constants $c_1(\Omega)$ and $c_2(\Omega)$, such that (see [18, Lemma 2.3] and [26, Theorem 5.11.2], respectively, for details)

$$c_1(\Omega)\|\boldsymbol{\tau}\|_{0,\Omega}^2 \le \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}; \Omega), \tag{3.7}$$

$$|\psi|_{1,\Omega} + \|\psi\|_{0,\Gamma_\mathrm{D}} \ge c_2(\Omega)\|\psi\|_{1,\Omega} \quad \forall\, \psi \in \mathrm{H}^1(\Omega), \tag{3.8}$$

and

$$\|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 \ge \frac{1}{2}|\mathbf{v}|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.9}$$

where (3.9) is the well known Korn inequality (see [28, Theorem 10.1]).

3.2 Well-posedness of the uncoupled problems

We begin by establishing a result that provides conditions under which the operator $\mathbf{S}$ in (3.1) is well-defined, or equivalently, the problem (3.2) is well-posed.

Lemma 3.1 *Assume that*

$$\kappa_1 \in \left(0, \frac{2\delta_1\mu_1}{\mu_2}\right), \quad \kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right), \quad \kappa_4 \in \left(0, 2\delta_3\kappa_3\left(1 - \frac{\delta_2}{2}\right)\right), \quad \text{and} \quad \kappa_2 > 0,$$

with $\delta_1 \in \left(0, \frac{2}{\mu_2}\right)$, *and* $\delta_2, \delta_3 \in (0,2)$. *Then, for each* $\phi \in \mathrm{H}^1(\Omega)$, *the problem* (3.2) *has a unique solution* $(\underline{\mathbf{t}}, \mathbf{u}) := \mathbf{S}(\phi) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$. *Moreover, there exists a constant* $c_{\mathbf{S}} > 0$, *independent of* ϕ, *such that there holds*

$$\|\mathbf{S}(\phi)\| = \|(\underline{\mathbf{t}}, \mathbf{u})\| \le c_{\mathbf{S}}\|\mathbf{f}\|_{0,\Omega}. \tag{3.10}$$

Proof. For a given $\phi \in \mathrm{H}^1(\Omega)$, we observe from (2.19) that $\mathbf{A}_\phi$ is clearly a bilinear form. Also, from Cauchy–Schwarz inequality we deduce that there exists a positive constant, which we denote by $\|\mathbf{A}_\phi\|$, only depending on $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, and μ_2 (cf. (2.10)), such that

$$\Big|\mathbf{A}_\phi((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v}))\Big| \le \|\mathbf{A}_\phi\|\|(\underline{\mathbf{t}}, \mathbf{u})\|\|(\underline{\mathbf{r}}, \mathbf{v})\|, \tag{3.11}$$

for all $(\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$. It turn, we have from (2.19) that

$$\begin{aligned}\mathbf{A}_\phi((\underline{\mathbf{r}}, \mathbf{v}), (\underline{\mathbf{r}}, \mathbf{v})) = &\int_\Omega \mu(\phi)\mathbf{r} : \mathbf{r} - \kappa_1 \int_\Omega \mu(\phi)\mathbf{r} : \boldsymbol{\tau}^{\mathrm{d}} + \kappa_1\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \kappa_2\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}^2 + \kappa_3\|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 \\ &- \kappa_3 \int_\Omega \mathbf{r} : \mathbf{e}(\mathbf{v}) + \kappa_4\|\boldsymbol{\eta}\|_{0,\Omega}^2 - \kappa_4 \int_\Omega \Big\{\nabla\mathbf{v} - \mathbf{e}(\mathbf{v})\Big\} : \boldsymbol{\eta}.\end{aligned}$$

Hence, we proceed similarly to the proof of [7, Lemma 3.6], utilise the Cauchy–Schwarz and Young inequalities, apply the boundedness of μ (cf. (2.10)), and the fact that

$$\|\nabla\mathbf{v} - \mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 = |\mathbf{v}|_{1,\Omega}^2 - \|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2,$$

to obtain that for any $\delta_1, \delta_2, \delta_3 > 0$, and for all $(\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$, there holds

$$\begin{aligned}\mathbf{A}_\phi((\underline{\mathbf{r}}, \mathbf{v}), (\underline{\mathbf{r}}, \mathbf{v})) \ge &\left\{\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right) - \frac{\kappa_3}{2\delta_2}\right\}\|\mathbf{r}\|_{0,\Omega}^2 + \kappa_1\left(1 - \frac{\mu_2\delta_1}{2}\right)\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \kappa_2\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}^2 \\ &+ \left\{\kappa_3\left(1 - \frac{\delta_2}{2}\right) + \frac{\kappa_4}{2\delta_3}\right\}\|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 - \frac{\kappa_4}{2\delta_3}|\mathbf{v}|_{1,\Omega}^2 + \kappa_4\left(1 - \frac{\delta_3}{2}\right)\|\boldsymbol{\eta}\|_{0,\Omega}^2,\end{aligned}$$

which, together with the Korn inequality (3.9), implies

$$\begin{aligned}\mathbf{A}_\phi((\underline{\mathbf{r}}, \mathbf{v}), (\underline{\mathbf{r}}, \mathbf{v})) \ge &\left\{\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right) - \frac{\kappa_3}{2\delta_2}\right\}\|\mathbf{r}\|_{0,\Omega}^2 + \kappa_1\left(1 - \frac{\mu_2\delta_1}{2}\right)\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \kappa_2\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}^2 \\ &+ \left\{\frac{\kappa_3}{2}\left(1 - \frac{\delta_2}{2}\right) - \frac{\kappa_4}{4\delta_3}\right\}|\mathbf{v}|_{1,\Omega}^2 + \kappa_4\left(1 - \frac{\delta_3}{2}\right)\|\boldsymbol{\eta}\|_{0,\Omega}^2.\end{aligned} \tag{3.12}$$

Then, assuming the stipulated hypotheses on $\delta_1, \kappa_1, \kappa_3, \delta_2, \delta_3, \kappa_4$, and κ_2, and applying the inequality (3.7), we can define the positive constants

$$\alpha_1(\Omega) := \left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right) - \frac{\kappa_3}{2\delta_2}, \quad \alpha_2(\Omega) := \min\left\{\kappa_1\left(1 - \frac{\mu_2\delta_1}{2}\right), \frac{\kappa_2}{2}\right\},$$

$$\alpha_3(\Omega) := \min\left\{c_1(\Omega)\alpha_2(\Omega), \frac{\kappa_2}{2}\right\}, \quad \alpha_4(\Omega) := \frac{\kappa_3}{2}\left(1 - \frac{\delta_2}{2}\right) - \frac{\kappa_4}{4\delta_3}, \quad \text{and} \quad \alpha_5(\Omega) := \kappa_4\left(1 - \frac{\delta_3}{2}\right),$$

which allow us to deduce from (3.12) that

$$\mathbf{A}_\phi((\underline{\mathbf{r}},\mathbf{v}),(\underline{\mathbf{r}},\mathbf{v})) \geq \alpha(\Omega)\|(\underline{\mathbf{r}},\mathbf{v})\|^2 \quad \forall(\underline{\mathbf{r}},\mathbf{v}) \in \mathbb{H}\times\mathbf{H}_0^1(\Omega), \tag{3.13}$$

where

$$\alpha(\Omega) := \min\Big\{\alpha_1(\Omega),\alpha_3(\Omega),c_p\alpha_4(\Omega),\alpha_5(\Omega)\Big\},$$

and c_p is the positive constant provided by Poincaré's inequality (see [33, Théorème 1.2-5]). In turn, concerning the linear functional $\mathbf{F}$ and using the Cauchy–Schwarz inequality, we find that

$$\|\mathbf{F}\| \leq M_{\mathbf{S}}\|\mathbf{f}\|_{0,\Omega}, \tag{3.14}$$

where $M_{\mathbf{S}} := (1+\kappa_2^2)^{1/2}$. We conclude by Lax–Milgram theorem (see, e.g., [18, Theorem 1.1]) that there is a unique solution $(\underline{\mathbf{t}},\mathbf{u}) := \mathbf{S}(\phi) \in \mathbb{H}\times\mathbf{H}_0^1(\Omega)$ of (3.2), and the corresponding continuous dependence result together with the ellipticity constant $\alpha(\Omega)$ and the estimate (3.14) imply (3.10) with the positive constant $c_{\mathbf{S}} := M_{\mathbf{S}}/\alpha(\Omega)$, which is clearly independent of ϕ. □

On the other hand, again we use the Lax–Milgram theorem to establishes the well-posedness of problem (3.4), or equivalently, that the operator $\widetilde{\mathbf{S}}$ (cf. (3.3)) is well-defined.

Lemma 3.2 *Assume that $\kappa_5 \in (0,2\widetilde{\delta})$, with $\widetilde{\delta} \in (0,2\kappa)$, and $\kappa_6,\kappa_7 > 0$. Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that $\|\mathbf{w}\|_{1,\Omega} \leq \dfrac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)}$, where $c(\Omega)$ is the constant in (2.15) and $\widetilde{\alpha}(\Omega)$ is the ellipticity constant of the bilinear form $\widetilde{\mathbf{A}}$ given below in (3.17). Then, there exists a unique $(\mathbf{p},\theta) := \widetilde{\mathbf{S}}(\mathbf{w}) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times\mathrm{H}^1(\Omega)$ solution of (3.4). Moreover, there exists a constant $c_{\widetilde{\mathbf{S}}} > 0$, independent of $\mathbf{w}$, such that there holds*

$$\|\widetilde{\mathbf{S}}(\mathbf{w})\| = \|(\mathbf{p},\theta)\| \leq c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}. \tag{3.15}$$

Proof. For a given $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ as stated, we observe from (2.20) and (2.21) that $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$ is clearly a bilinear form. Now, applying the Cauchy–Schwarz inequality and the estimate (2.15), we deduce that

$$\Big|\widetilde{\mathbf{A}}((\mathbf{p},\theta),(\mathbf{q},\psi))\Big| \leq \|\widetilde{\mathbf{A}}\|\|(\mathbf{p},\theta)\|\|(\mathbf{q},\psi)\|$$

and

$$\Big|\widetilde{\mathbf{B}}_{\mathbf{w}}((\mathbf{p},\theta),(\mathbf{q},\psi))\Big| \leq \kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\mathbf{w}\|_{1,\Omega}\|\theta\|_{1,\Omega}\|(\mathbf{q},\psi)\|, \tag{3.16}$$

for all $(\mathbf{p},\theta),(\mathbf{q},\psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times\mathrm{H}^1(\Omega)$. Then, by gathering the foregoing inequalities, we find that there exists a positive constant, which we denote by $\|\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}\|$, only depending on $\kappa,\kappa_5,\kappa_6,\kappa_7,c(\Omega)$, and the bound for $\|\mathbf{w}\|_{1,\Omega}$ assumed here, such that

$$\Big|(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}})((\mathbf{p},\theta),(\mathbf{q},\psi))\Big| \leq \|\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}\|\|(\mathbf{p},\theta)\|\|(\mathbf{q},\psi)\|,$$

for all $(\mathbf{p},\theta),(\mathbf{q},\psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times\mathrm{H}^1(\Omega)$. In turn, from (2.20) we have that

$$\widetilde{\mathbf{A}}((\mathbf{q},\psi),(\mathbf{q},\psi)) = \kappa^{-1}\|\mathbf{q}\|_{0,\Omega}^2 + \kappa_6\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega}^2 - \kappa^{-1}\kappa_5\int_\Omega \mathbf{q}\cdot\nabla\psi + \kappa_5|\psi|_{1,\Omega}^2 + \kappa_7\|\psi\|_{0,\Gamma_{\mathrm{D}}}\,,$$

and hence, using the Cauchy–Schwarz and Young inequalities, we obtain that for any $\widetilde{\delta} > 0$ and for all $(\mathbf{q},\psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times\mathrm{H}^1(\Omega)$, there holds

$$\widetilde{\mathbf{A}}((\mathbf{q},\psi),(\mathbf{q},\psi)) \geq \kappa^{-1}\left(1-\frac{\kappa_5}{2\widetilde{\delta}}\right)\|\mathbf{q}\|_{0,\Omega}^2 + \kappa_6\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega}^2 + \kappa_5\left(1-\frac{\kappa^{-1}}{2}\widetilde{\delta}\right)|\psi|_{1,\Omega}^2 + \kappa_7\|\psi\|_{0,\Gamma_{\mathrm{D}}}.$$

In this way, applying the inequality (3.8), we can define the constants

$$\widetilde{\alpha}_1(\Omega) := \min\left\{\kappa^{-1}\left(1-\frac{\kappa_5}{2\widetilde{\delta}}\right), \kappa_6\right\} \quad \text{and} \quad \widetilde{\alpha}_2(\Omega) := c_2(\Omega)\min\left\{\kappa_5\left(1-\frac{\kappa^{-1}}{2}\widetilde{\delta}\right), \kappa_7\right\},$$

which are positive thanks to the hypotheses on $\widetilde{\delta}$, κ_5, κ_6, and κ_7. In this way, it follows that

$$\widetilde{\mathbf{A}}((\mathbf{q},\psi),(\mathbf{q},\psi)) \geq \widetilde{\alpha}(\Omega)\|(\mathbf{q},\psi)\|^2 \quad \forall (\mathbf{q},\psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times \mathrm{H}^1(\Omega), \tag{3.17}$$

with $\widetilde{\alpha}(\Omega) := \min\left\{\widetilde{\alpha}_1(\Omega), \widetilde{\alpha}_2(\Omega)\right\}$, which shows that $\widetilde{\mathbf{A}}$ is elliptic. Therefore, combining now (3.16), (3.17), and the bound for $\|\mathbf{w}\|_{1,\Omega}$ assumed here, we deduce that for all $(\mathbf{q},\psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega)\times \mathrm{H}^1(\Omega)$, there holds

$$(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}})((\mathbf{q},\psi),(\mathbf{q},\psi)) \geq \left\{\widetilde{\alpha}(\Omega) - \kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\mathbf{w}\|_{1,\Omega}\right\}\|(\mathbf{q},\psi)\|^2 \geq \frac{\widetilde{\alpha}(\Omega)}{2}\|(\mathbf{q},\psi)\|^2, \tag{3.18}$$

which proves the ellipticity of $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$, with constant $\dfrac{\widetilde{\alpha}(\Omega)}{2}$, independent of $\mathbf{w}$. On the other hand, it is easy to see from (2.23), by using Cauchy–Schwarz's inequality and the trace theorems in $\mathbf{H}(\mathrm{div}\,;\Omega)$ and $\mathrm{H}^1(\Omega)$, whose boundedness constants are given by 1 and $\|\gamma_0\|$, respectively, that the functional $\widetilde{\mathbf{F}}$ is bounded with

$$\|\widetilde{\mathbf{F}}\| \leq M_{\widetilde{\mathbf{S}}}\left\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}, \tag{3.19}$$

where $M_{\widetilde{\mathbf{S}}} := \max\left\{(1+\kappa_6^2)^{1/2}, \kappa_7\|\gamma_0\|\right\}$. Summing up, and owing to the hypotheses on κ_5, κ_6 and κ_7, we have proved that for any sufficiently small $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, the bilinear form $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$ and the functional $\widetilde{\mathbf{F}}$ satisfy the hypotheses of the Lax–Milgram theorem (see, e.g., [18, Theorem 1.1]), which guarantees the well-posedness of (3.4) and the continuous dependence estimate (3.15) with $c_{\widetilde{\mathbf{S}}} := 2M_{\widetilde{\mathbf{S}}}/\widetilde{\alpha}(\Omega)$. $\square$

At this point we remark that the restriction on $\|\mathbf{w}\|_{1,\Omega}$ in Lemma 3.2 could also have been taken as $\|\mathbf{w}\|_{1,\Omega} \leq \omega\dfrac{\widetilde{\alpha}(\Omega)}{\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)}$ with any $\omega \in (0,1)$. However, we have chosen $\omega = \dfrac{1}{2}$ for simplicity and because it yields a joint maximization of the ellipticity constant of $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$ and the upper bound for $\|\mathbf{w}\|_{1,\Omega}$. In addition, we also remark that the constants $\alpha(\Omega)$ and $\widetilde{\alpha}(\Omega)$ yielding the ellipticity of $\mathbf{A}_\phi$ and $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$, respectively, can be maximized by taking the parameters $\delta_1, \kappa_1, \delta_2, \kappa_3, \delta_3, \kappa_4, \widetilde{\delta}$, and κ_5 as the middle points of their feasible ranges, and by choosing κ_2, κ_6 and κ_7 so that they maximize the minima defining $\alpha_2(\Omega)$, $\widetilde{\alpha}_1(\Omega)$, and $\widetilde{\alpha}_2(\Omega)$, respectively. More precisely, we simply take

$$\begin{gathered}\delta_1 = \frac{1}{\mu_2}, \quad \kappa_1 = \frac{\delta_1\mu_1}{\mu_2} = \frac{\mu_1}{\mu_2^2}, \quad \delta_2 = 1, \quad \kappa_3 = \delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right) = \frac{\mu_1}{2}, \quad \delta_3 = 1,\\ \kappa_4 = \delta_3\kappa_3\left(1-\frac{\delta_2}{2}\right) = \frac{\mu_1}{4}, \quad \kappa_2 = 2\kappa_1\left(1-\frac{\mu_2\delta_1}{2}\right) = \frac{\mu_1}{\mu_2^2}, \quad \widetilde{\delta} = \kappa,\\ \kappa_5 = \widetilde{\delta} = \kappa, \quad \kappa_6 = \kappa^{-1}\left(1-\frac{\kappa_5}{2\widetilde{\delta}}\right) = \frac{\kappa^{-1}}{2}, \quad \kappa_7 = \kappa_5\left(1-\frac{\kappa^{-1}}{2}\widetilde{\delta}\right) = \frac{\kappa}{2},\end{gathered} \tag{3.20}$$

which yields

$$\begin{gathered}\alpha_1(\Omega) = \frac{\mu_1}{4}, \quad \alpha_2(\Omega) = \frac{\mu_1}{2\mu_2^2}, \quad \alpha_3(\Omega) = \min\left\{c_1(\Omega), 1\right\}\frac{\mu_1}{2\mu_2^2},\\ \alpha_4(\Omega) = \frac{\mu_1}{16}, \quad \alpha_5(\Omega) = \frac{\mu_1}{8}, \quad \widetilde{\alpha}_1(\Omega) = \frac{\kappa^{-1}}{2}, \quad \widetilde{\alpha}_2(\Omega) = c_2(\Omega)\frac{\kappa}{2},\end{gathered}$$

and hence

$$\alpha(\Omega) = \min\left\{\min\left\{c_1(\Omega),1\right\}\frac{\mu_1}{2\mu_2^2}, c_p\frac{\mu_1}{16}, \frac{\mu_1}{8}\right\}, \quad \text{and} \quad \widetilde{\alpha}(\Omega) = \frac{1}{2}\min\left\{\kappa^{-1}, c_2(\Omega)\kappa\right\}.$$

The explicit values of the stabilization parameters κ_i, $i \in \{1,\dots,7\}$, given in (3.20), will be employed in Section 5 for the corresponding numerical experiments.

3.3 Solvability analysis of the fixed-point equation

Having proved the well-posedness of the uncoupled problems (3.2) and (3.4), which ensures that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\mathbf{T}$ are well defined, we now aim to establish the existence of a unique fixed point of the operator $\mathbf{T}$. For this purpose, in what follows we verify the hypothesis of the Schauder and Banach fixed-point theorems. We begin the analysis with the following straightforward consequence of Lemmas 3.1 and 3.2.

Lemma 3.3 *Suppose that the parameters κ_i, $i \in \{1,\dots,7\}$, satisfy the conditions required by Lemmas* 3.1 *and* 3.2. *Let $\mathcal{W}$ be the closed and convex subset of* $\mathrm{H}^1(\Omega)$ *defined by*

$$\mathcal{W} := \left\{\phi \in \mathrm{H}^1(\Omega) : \quad \|\phi\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\left\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}\right\},$$

where $c_{\widetilde{\mathbf{S}}}$ is the constant given by (3.15). *In addition, assume that the datum $\mathbf{f}$ satisfy*

$$c_{\mathbf{S}}\|\mathbf{f}\|_{0,\Omega} \le \frac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)}, \tag{3.21}$$

where $c_{\mathbf{S}}$ is the constant given by (3.10). *Then $\mathbf{T}(\mathcal{W}) \subseteq \mathcal{W}$.*

Proof. Given $\phi \in \mathcal{W}$, we get from (3.10) (cf. Lemma 3.1) that

$$\|\mathbf{S}(\phi)\| = \|(\underline{\mathbf{t}}, \mathbf{u})\| \le c_{\mathbf{S}}\|\mathbf{f}\|_{0,\Omega},$$

and hence, thanks to the constraint (3.21), we observe that $\mathbf{u} = \mathbf{S}_2(\phi)$ satisfies the hypotheses of Lemma 3.2. Moreover, the corresponding estimate (3.15) gives

$$\|\mathbf{T}(\phi)\|_{1,\Omega} = \|\widetilde{\mathbf{S}}_2(\mathbf{u})\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\left\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\},$$

which implies that $\mathbf{T}(\phi) \in \mathcal{W}$, thus finishing the proof. □

Next, we establish two lemmas that will be useful to derive conditions under which the operator $\mathbf{T}$ is continuous and compact. To that end, and similarly as in [2, Section 3.3], we first introduce suitable regularity hypotheses on the operator $\mathbf{S}$, which will be employed later on. In fact, for the remainder of this paper we proceed as in [2, eq. (3.22)], and suppose that $\mathbf{f} \in \mathbf{H}^\delta(\Omega)$, for some $\delta \in (0,1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$). Then, we assume that for each $\phi \in \mathrm{H}^1(\Omega)$ there holds $\mathbf{S}(\phi) \in \big(\mathbb{H}^\delta(\Omega) \times (\mathbb{H}_0(\mathbf{div};\Omega) \cap \mathbb{H}^\delta(\Omega)) \times \mathbb{H}^\delta(\Omega)\big) \times \mathbf{H}^{1+\delta}(\Omega)$, with

$$\|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\delta,\Omega} + \|\mathbf{S}_1^{\boldsymbol{\sigma}}(\phi)\|_{\delta,\Omega} + \|\mathbf{S}_1^{\boldsymbol{\rho}}(\phi)\|_{\delta,\Omega} + \|\mathbf{S}_2(\phi)\|_{1+\delta,\Omega} \le \widehat{C}_{\mathbf{S}}\|\mathbf{f}\|_{\delta,\Omega}, \tag{3.22}$$

where $\widehat{C}_{\mathbf{S}}$ is a positive constant independent of ϕ. The reason of the stipulated ranges for δ will be clarified in the forthcoming analysis (see below proof of Lemmas 3.4 and 3.7). More precisely, we remark in advance that the regularity estimate (3.22) is needed in the proof of Lemmas 3.4 and 3.7 to bound an expression of the form $\|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\mathbf{L}^{2p}(\Omega)}$ in terms of $\|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\delta,\Omega}$, and hence of the data at the right-hand side of (3.22).

Lemma 3.4 *There exists a positive constant $C_{\mathbf{S}}$, depending on L_μ, the parameter κ_1, the ellipticity constant $\alpha(\Omega)$ of the bilinear form $\mathbf{A}_\phi$ (cf.* (3.2)*), and δ (cf.* (3.22)*), such that*

$$\|\mathbf{S}(\phi) - \mathbf{S}(\widetilde{\phi})\| \le C_{\mathbf{S}} \|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\delta,\Omega} \|\phi - \widetilde{\phi}\|_{\mathrm{L}^{n/\delta}(\Omega)} \quad \forall \phi, \widetilde{\phi} \in \mathrm{H}^1(\Omega). \tag{3.23}$$

Proof. We proceed as in [2, Lemma 3.9]. In fact, given $\phi, \widetilde{\phi} \in \mathrm{H}^1(\Omega)$, we let $(\underline{\mathbf{t}}, \mathbf{u}) := \mathbf{S}(\phi)$ and $(\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}}) := \mathbf{S}(\widetilde{\phi})$ be the corresponding solutions of problem (3.2). Then, using the bilinearity of $\mathbf{A}_\phi$ for any ϕ, it follows easily from (3.2) that

$$\mathbf{A}_{\widetilde{\phi}}((\underline{\mathbf{t}}, \mathbf{u}) - (\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}}), (\underline{\mathbf{r}}, \mathbf{v})) = -\int_\Omega \Big\{\mu(\phi) - \mu(\widetilde{\phi})\Big\} \mathbf{t} : \Big\{\mathbf{r} - \kappa_1 \boldsymbol{\tau}^{\mathrm{d}}\Big\},$$

for all $(\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$. Hence, applying the ellipticity of $\mathbf{A}_\phi$ (cf. (2.19)), Cauchy–Schwarz inequality, the Lipschitz-continuity assumption (2.9), and then Hölder inequality, we find that

$$\begin{aligned} \alpha(\Omega)\|(\underline{\mathbf{t}}, \mathbf{u}) - (\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}})\|^2 &\le \mathbf{A}_{\widetilde{\phi}}((\underline{\mathbf{t}}, \mathbf{u}) - (\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}}), (\underline{\mathbf{t}}, \mathbf{u}) - (\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}})) \\ &= -\int_\Omega \Big\{\mu(\phi) - \mu(\widetilde{\phi})\Big\} \mathbf{t} : \Big\{\big(\mathbf{t} - \widetilde{\mathbf{t}}\big) - \kappa_1\big(\boldsymbol{\sigma}^{\mathrm{d}} - \widetilde{\boldsymbol{\sigma}}^{\mathrm{d}}\big)\Big\} \\ &\le L_\mu (1 + \kappa_1^2)^{1/2} \|\mathbf{t}\|_{\mathbf{L}^{2p}(\Omega)} \|\phi - \widetilde{\phi}\|_{\mathrm{L}^{2q}(\Omega)} \|(\underline{\mathbf{t}}, \mathbf{u}) - (\widetilde{\underline{\mathbf{t}}}, \widetilde{\mathbf{u}})\|, \end{aligned} \tag{3.24}$$

where $p, q \in [1, +\infty)$ are such that $1/p + 1/q = 1$. Next, given the further regularity δ assumed in (3.22), we recall that the Sobolev embedding theorem (cf. [1, Theorem 4.12], [32, Theorem 1.3.4]) establishes the continuous injection $i_\delta : \mathrm{H}^\delta(\Omega) \to \mathrm{L}^{\delta^*}(\Omega)$ with boundedness constant $C_\delta > 0$, where

$$\delta^* := \begin{cases} \dfrac{2}{1-\delta} & \text{if } n = 2, \\ \dfrac{6}{3 - 2\delta} & \text{if } n = 3. \end{cases}$$

Thus, choosing p such that $2p = \delta^*$ and recalling that $\mathbf{t} := \mathbf{S}_1^{\mathbf{t}}(\phi)$, we find that

$$\|\mathbf{t}\|_{\mathbf{L}^{2p}(\Omega)} = \|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\mathbf{L}^{2p}(\Omega)} \le C_\delta \|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\delta,\Omega}. \tag{3.25}$$

In turn, according to the above choice of p, that is $p = \delta^*/2$, it readily follows that

$$2q := \frac{2p}{p-1} = \begin{cases} \dfrac{2}{\delta} & \text{if } n = 2 \\ \dfrac{3}{\delta} & \text{if } n = 3 \end{cases} = \frac{n}{\delta}. \tag{3.26}$$

Therefore, inequalities (3.24) and (3.25) together with identity (3.26) conclude (3.23) with constant $C_{\mathbf{S}} := L_\mu (1 + \kappa_1^2)^{1/2} C_\delta / \alpha(\Omega)$. $\square$

In turn, the following result establishes the Lipschitz-continuity of the operator $\widetilde{\mathbf{S}}$.

Lemma 3.5 *There exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending on κ, the parameter κ_5, the ellipticity constant $\widetilde{\alpha}(\Omega)$ of the bilinear form $\widetilde{\mathbf{A}}$ (cf.* (3.17)*), and the constant $c(\Omega)$ (cf.* (2.15)*), such that for all $\mathbf{w}, \widetilde{\mathbf{w}} \in \mathbf{H}_0^1(\Omega)$ with $\|\mathbf{w}\|_{1,\Omega}, \|\widetilde{\mathbf{w}}\|_{1,\Omega} \le \dfrac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1 + \kappa_5^2)^{1/2} c(\Omega)}$, there holds*

$$\|\widetilde{\mathbf{S}}(\mathbf{w}) - \widetilde{\mathbf{S}}(\widetilde{\mathbf{w}})\| \le C_{\widetilde{\mathbf{S}}} \|\widetilde{\mathbf{S}}_2(\mathbf{w})\|_{1,\Omega} \|\mathbf{w} - \widetilde{\mathbf{w}}\|_{1,\Omega}. \tag{3.27}$$

Proof. It follows almost straightforwardly from a slight modification of the proof of [13, Lemma 3.7]. We omit further details. $\square$

As a consequence of the previous lemmas we establish the following result providing an estimate needed to derive next the required continuity and compactness properties of the operator $\mathbf{T}$.

Lemma 3.6 *Let* $\mathcal{W} := \Big\{\phi \in \mathrm{H}^1(\Omega) : \quad \|\phi\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}\Big\}$*, and assume that the datum* $\mathbf{f}$ *satisfies* (3.21)*. Then, there holds*

$$\|\mathbf{T}(\phi) - \mathbf{T}(\widetilde{\phi})\|_{1,\Omega} \le C_{\mathbf{S}} C_{\widetilde{\mathbf{S}}} \|\mathbf{T}(\phi)\|_{1,\Omega} \|\mathbf{S}_1^{\mathbf{t}}(\phi)\|_{\delta,\Omega} \|\phi - \widetilde{\phi}\|_{\mathrm{L}^{n/\delta}(\Omega)}, \tag{3.28}$$

where $C_{\mathbf{S}}$ *and* $C_{\widetilde{\mathbf{S}}}$ *are the constants given by* (3.23) *and* (3.27)*, respectively.*

Proof. It suffices to recall that $\mathbf{T}(\phi) = \widetilde{\mathbf{S}}_2(\mathbf{S}_2(\phi)) \quad \forall \phi \in \mathrm{H}^1(\Omega)$ (cf. (3.5)), and then apply Lemmas 3.3, 3.4 and 3.5. $\square$

Owing to the above analysis, we establish now the announced properties of the operator $\mathbf{T}$.

Lemma 3.7 *Let* $\mathcal{W} := \Big\{\phi \in \mathrm{H}^1(\Omega) : \quad \|\phi\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}\Big\}$*, and assume that the datum* $\mathbf{f}$ *satisfies* (3.21)*. Then,* $\mathbf{T} : \mathcal{W} \to \mathcal{W}$ *is continuous and* $\overline{\mathbf{T}(\mathcal{W})}$ *is compact.*

Proof. The required result follows basically from (3.28), the Rellich–Kondrachov compactness theorem (cf. [1, Theorem 6.3], [32, Theorem 1.3.5]), the specified range of the constant δ involved in the further regularity assumptions given by (3.22), and the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence. We omit further details and refer to [2, Lemma 3.12]. $\square$

Finally, the main result of this section is given as follows.

Theorem 3.8 *Suppose that the parameters* κ_i*,* $i \in \{1, \ldots, 7\}$*, satisfy the conditions required by Lemmas* 3.1 *and* 3.2*. Let* $\mathcal{W} := \Big\{\phi \in \mathrm{H}^1(\Omega) : \quad \|\phi\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}\Big\}$*, and assume that the datum* $\mathbf{f}$ *satisfy* (3.21)*. Then the augmented fully-mixed problem* (2.18) *has at least one solution* $(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,; \Omega) \times \mathrm{H}^1(\Omega)$ *with* $\theta \in \mathcal{W}$*, and there holds*

$$\|(\underline{\mathbf{t}}, \mathbf{u})\| \le c_{\mathbf{S}} \|\mathbf{f}\|_{0,\Omega}, \tag{3.29}$$

and

$$\|(\mathbf{p}, \theta)\| \le c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}, \tag{3.30}$$

where $c_{\mathbf{S}}$ *and* $c_{\widetilde{\mathbf{S}}}$ *are the constants specified in Lemmas* 3.1 *and* 3.2*, respectively. Moreover, assume that the data* $\mathbf{f}$*,* g *and* θ_{D} *are sufficiently small so that, with the constants* $C_{\mathbf{S}}$*,* $C_{\widetilde{\mathbf{S}}}$ *and* $\widehat{C}_{\mathbf{S}}$ *from Lemmas* 3.4 *and* 3.5*, and estimate* (3.22)*, respectively, and denoting by* $\widetilde{C}_\delta$ *the boundedness constant of the continuous injection of* $\mathrm{H}^1(\Omega)$ *into* $\mathrm{L}^{n/\delta}(\Omega)$*, there holds*

$$\widetilde{C}_\delta \widehat{C}_{\mathbf{S}} C_{\mathbf{S}} C_{\widetilde{\mathbf{S}}} c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}\|\mathbf{f}\|_{\delta,\Omega} < 1. \tag{3.31}$$

Then the solution θ *is unique in* $\mathcal{W}$*.*

Proof. The equivalence between (2.18) and the fixed-point equation (3.6), together with Lemmas 3.3 and 3.7, confirm the existence of solution of (2.18) as a direct application of the Schauder fixed-point theorem [11, Theorem 9.12-1(b)]. In addition, it is clear that the estimates (3.29) and (3.30) follow straightforwardly from (3.10) and (3.15), respectively. Furthermore, given another solution $\widetilde{\theta} \in \mathcal{W}$ of (3.6), the estimates $\|\mathbf{T}(\theta)\|_{1,\Omega} = \|\theta\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}$,

$$\|\mathbf{S}_1^{\mathbf{t}}(\theta)\|_{\delta,\Omega} \le \widehat{C}_{\mathbf{S}}\|\mathbf{f}\|_{\delta,\Omega},$$

and

$$\|\phi\|_{\mathrm{L}^{n/\delta}(\Omega)} \le \widetilde{C}_{\delta}\|\phi\|_{1,\Omega} \quad \forall \phi \in \mathrm{H}^1(\Omega), \tag{3.32}$$

confirm (3.31) as a sufficient condition for concluding, together with (3.28), that $\theta = \widetilde{\theta}$. In other words, (3.31) constitutes the condition that makes the operator $\mathbf{T}$ to become a contraction, thus yielding, thanks to the Banach fixed-point theorem, the existence of a unique fixed point of $\mathbf{T}$ in $\mathcal{W}$. □

4 The Galerkin scheme

In this section we introduce and analyse the Galerkin scheme of the augmented fully-mixed formulation (2.18). We analyse its solvability by employing a discrete version of the fixed-point strategy developed in Sections 3.1 and 3.2. Finally, we derive the corresponding Céa estimate and rates of convergence of our Galerkin scheme.

4.1 Discrete setting

Let $\mathcal{T}_h$ be a regular triangulation of Ω made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and define the meshsize $h := \max\Big\{h_K : \quad K \in \mathcal{T}_h\Big\}$. Then, for each $K \in \mathcal{T}_h$ we set the local Raviart–Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_n)^{\mathrm{t}}$ is a generic vector of $\mathbb{R}^n$. Then, we introduce the finite element subspaces approximating the unknowns $\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \mathbf{u}, \mathbf{p}$ and θ as follows

$$\begin{aligned}
\mathbb{H}_h^{\mathbf{t}} &:= \Big\{\mathbf{r}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \mathbf{r}_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\Big\},\\
\mathbb{H}_h^{\boldsymbol{\sigma}} &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall \mathbf{c} \in \mathbb{R}^n \quad \forall K \in \mathcal{T}_h\Big\},\\
\mathbb{H}_h^{\boldsymbol{\rho}} &:= \Big\{\boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) : \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\Big\},\\
\mathbf{H}_h^{\mathbf{u}} &:= \Big\{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) : \quad \mathbf{v}_h|_K \in \mathbf{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h, \quad \mathbf{v}_h = 0 \text{ on } \Gamma\Big\},\\
\mathbf{H}_h^{\mathbf{p}} &:= \Big\{\mathbf{q}_h \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega) : \quad \mathbf{q}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\Big\},\\
\mathrm{H}_h^{\theta} &:= \Big\{\psi_h \in \mathrm{C}(\overline{\Omega}) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h\Big\}.
\end{aligned} \tag{4.1}$$

In this way, by defining $\underline{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)$, $\underline{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\tau}_h, \boldsymbol{\eta}_h) \in \mathbb{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$, the Galerkin scheme of (2.18) reads: Find $(\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ such that

$$\begin{aligned}
\mathbf{A}_{\theta_h}((\underline{\mathbf{t}}_h, \mathbf{u}_h), (\underline{\mathbf{r}}_h, \mathbf{v}_h)) &= \mathbf{F}(\underline{\mathbf{r}}_h, \mathbf{v}_h) \quad \forall (\underline{\mathbf{r}}_h, \mathbf{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}},\\
\widetilde{\mathbf{A}}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\mathbf{u}_h}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) &= \widetilde{\mathbf{F}}(\mathbf{q}_h, \psi_h) \quad \forall (\mathbf{q}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}.
\end{aligned} \tag{4.2}$$

Similarly to the continuous context, in order to analyse problem (4.2) we rewrite it equivalently as a fixed-point problem. Indeed, we firstly define $\mathbf{S}_h : \mathrm{H}_h^{\theta} \to \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}$ by

$$\mathbf{S}_h(\phi_h) := (\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h)) = (\underline{\mathbf{t}}_h, \mathbf{u}_h) \quad \forall \phi_h \in \mathrm{H}_h^{\theta}, \tag{4.3}$$

where $\mathbf{S}_{1,h}(\phi_h) := (\mathbf{S}_{1,h}^{\mathbf{t}}(\phi_h), \mathbf{S}_{1,h}^{\boldsymbol{\sigma}}(\phi_h), \mathbf{S}_{1,h}^{\boldsymbol{\rho}}(\phi_h))$ and $(\underline{\mathbf{t}}_h, \mathbf{u}_h)$ is the unique solution of the discrete version of the problem (3.2): Find $(\underline{\mathbf{t}}_h, \mathbf{u}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{A}_{\phi_h}((\underline{\mathbf{t}}_h, \mathbf{u}_h), (\underline{\mathbf{r}}_h, \mathbf{v}_h)) = \mathbf{F}(\underline{\mathbf{r}}_h, \mathbf{v}_h) \quad \forall (\underline{\mathbf{r}}_h, \mathbf{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}, \tag{4.4}$$

where the bilinear form $\mathbf{A}_{\phi_h}$ (with ϕ_h in place of ϕ) and the functional $\mathbf{F}$ are defined as in (2.19) and (2.22), respectively. Secondly, we define the operator $\widetilde{\mathbf{S}}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ as

$$\widetilde{\mathbf{S}}_h(\mathbf{w}_h) := (\widetilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \widetilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)) = (\mathbf{p}_h, \theta_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{4.5}$$

where $(\mathbf{p}_h, \theta_h)$ is the unique solution of the discrete version of the problem (3.4): Find $(\mathbf{p}_h, \theta_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ such that

$$\widetilde{\mathbf{A}}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\mathbf{w}_h}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) = \widetilde{\mathbf{F}}(\mathbf{q}_h, \psi_h) \quad \forall (\mathbf{q}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}, \tag{4.6}$$

where the bilinear form $\widetilde{\mathbf{A}}$ and the functional $\widetilde{\mathbf{F}}$ are defined as in (2.20) and (2.23), respectively, whereas, $\widetilde{\mathbf{B}}_{\mathbf{w}_h}$ is the bilinear form given by (2.21) (with $\mathbf{w}_h$ instead of $\mathbf{w}$). Finally, we introduce the operator $\mathbf{T}_h : \mathrm{H}_h^{\theta} \to \mathrm{H}_h^{\theta}$ as

$$\mathbf{T}_h(\phi_h) := \widetilde{\mathbf{S}}_{2,h}(\mathbf{S}_{2,h}(\phi_h)) \quad \forall \phi_h \in \mathrm{H}_h^{\theta}, \tag{4.7}$$

and realise that solving (4.2) is equivalent to seeking a fixed point of the operator $\mathbf{T}_h$, that is: Find $\theta_h \in \mathrm{H}_h^{\theta}$ such that

$$\mathbf{T}_h(\theta_h) = \theta_h. \tag{4.8}$$

4.2 Solvability analysis

Now we establish the solvability of problem (4.2) by studying the equivalent fixed-point problem (4.8). To that end, first we guarantee that the discrete problems (4.4) and (4.6) are well-posed. Indeed, it is easy to see that the respective proofs are almost verbatim of the continuous analogues provided in Section 3.2, and hence we simply state the corresponding results as follows.

Lemma 4.1 *Assume that κ_i, $i \in \{1, \ldots, 4\}$, satisfy the conditions required by Lemma* 3.1. *Then, for each $\phi_h \in \mathrm{H}_h^{\theta}$, the problem* (4.4) *has a unique solution $(\underline{\mathbf{t}}_h, \mathbf{u}_h) := \mathbf{S}_h(\phi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}$. Moreover, with the same constant $c_{\mathbf{S}} > 0$ from* (3.10)*, which is independent of ϕ_h, there holds*

$$\|\mathbf{S}_h(\phi_h)\| = \|(\underline{\mathbf{t}}_h, \mathbf{u}_h)\| \leq c_{\mathbf{S}} \|\mathbf{f}\|_{0,\Omega}. \tag{4.9}$$

Lemma 4.2 *Assume that κ_i, $i \in \{5, 6, 7\}$, satisfy the conditions required by Lemma* 3.2. *Let $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_h\|_{1,\Omega} \leq \dfrac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1+\kappa_5^2)^{1/2} c(\Omega)}$, where $c(\Omega)$ and $\widetilde{\alpha}(\Omega)$ are the positive constants provided by* (2.15) *and* (3.17)*, respectively. Then, there exist a unique $(\mathbf{p}_h, \theta_h) := \widetilde{\mathbf{S}}_h(\mathbf{w}_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ solution of* (4.6)*. Moreover, with the same constant $c_{\widetilde{\mathbf{S}}} > 0$ from* (3.15)*, which is independent of $\mathbf{w}_h$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h)\| = \|(\mathbf{p}_h, \theta_h)\| \leq c_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \Big\}. \tag{4.10}$$

We now proceed to analyse the fixed-point equation (4.8). More precisely, in what follows we verify the hypotheses of the Brouwer fixed-point theorem (cf. [11, Theorem 9.9-2]). We begin with the discrete version of Lemma 3.3. Its proof, being a simple translation of the arguments proving that lemma, is omitted.

Lemma 4.3 *Let* $\mathcal{W}_h := \Big\{ \phi_h \in \mathrm{H}_h^\theta : \quad \|\phi_h\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \Big\} \Big\}$, *and assume that the datum* $\mathbf{f}$ *satisfies* (3.21). *Then* $\mathbf{T}(\mathcal{W}_h) \subseteq \mathcal{W}_h$.

The discrete analogue of Lemma 3.4 is provided next. We notice in advance that, instead of the regularity assumptions employed in the proof of that result, which actually are not needed nor could be applied in the present discrete case, we simply utilise a $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument.

Lemma 4.4 *There exists a positive constant* $C_{\mathbf{S}_h}$, *depending on* L_μ, κ_1, *and* $\alpha(\Omega)$, *such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\widetilde{\phi}_h)\| \le C_{\mathbf{S}_h} \|\mathbf{S}^{\mathbf{t}}_{1,h}(\phi_h)\|_{\mathbf{L}^4(\Omega)} \|\phi_h - \widetilde{\phi}_h\|_{\mathrm{L}^4(\Omega)} \quad \forall \phi_h, \widetilde{\phi}_h \in \mathrm{H}_h^\theta. \tag{4.11}$$

Proof. Given $\phi_h, \widetilde{\phi}_h \in \mathrm{H}_h^\theta$, we first let $(\underline{\mathbf{t}}_h, \mathbf{u}_h) := \mathbf{S}_h(\phi_h)$ and $(\widetilde{\underline{\mathbf{t}}}_h, \widetilde{\mathbf{u}}_h) := \mathbf{S}_h(\widetilde{\phi}_h)$ be the corresponding solutions of problem (4.4). Next, we proceed analogouly as in the proof of Lemma 3.4, except for the derivation of the discrete analogue of the right-hand side of (3.24), where, instead of choosing the values of p and q determined by the regularity parameter δ, it suffices to take $p = q = 2$ (see [2]), thus obtaining

$$\alpha(\Omega) \|(\underline{\mathbf{t}}_h, \mathbf{u}_h) - (\widetilde{\underline{\mathbf{t}}}_h, \widetilde{\mathbf{u}}_h)\|^2 \le L_\mu (1 + \kappa_1^2)^{1/2} \|\mathbf{t}_h\|_{\mathbf{L}^4(\Omega)} \|\phi_h - \widetilde{\phi}_h\|_{\mathrm{L}^4(\Omega)} \|(\underline{\mathbf{t}}_h, \mathbf{u}_h) - (\widetilde{\underline{\mathbf{t}}}_h, \widetilde{\mathbf{u}}_h)\|.$$

Then, the fact that the elements of $\mathbb{H}_h^{\mathbf{t}}$ are piecewise polynomials insures that $\|\mathbf{t}_h\|_{\mathbf{L}^4(\Omega)} < +\infty$, and hence the foregoing equation yields (4.11) with $C_{\mathbf{S}_h} := L_\mu (1 + \kappa_1^2)^{1/2} / \alpha(\Omega)$. Further details are omitted. □

Next, we address the Lipschitz-continuity of $\widetilde{\mathbf{S}}_h$, its proof is omitted since it is almost verbatim as that of the corresponding continuous estimate provided by Lemma 3.5.

Lemma 4.5 *Let* $C_{\widetilde{\mathbf{S}}}$ *be the constant provided by Lemma* 3.5. *Then, given* $\mathbf{w}_h, \widetilde{\mathbf{w}}_h \in \mathbf{H}_h^{\mathbf{u}}$ *such that* $\|\mathbf{w}_h\|_{1,\Omega}, \|\widetilde{\mathbf{w}}_h\|_{1,\Omega} \le \dfrac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1 + \kappa_5^2)^{1/2} c(\Omega)}$, *there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h) - \widetilde{\mathbf{S}}_h(\widetilde{\mathbf{w}}_h)\| \le C_{\widetilde{\mathbf{S}}} \|\widetilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)\|_{1,\Omega} \|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{1,\Omega}. \tag{4.12}$$

Now, utilising Lemmas 4.4 and 4.5, we can prove the discrete version of Lemma 3.6.

Lemma 4.6 *Let* $\mathcal{W}_h := \Big\{ \phi_h \in \mathrm{H}_h^\theta : \quad \|\phi_h\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \Big\} \Big\}$, *and assume that the datum* $\mathbf{f}$ *satisfies* (3.21). *Then, there holds*

$$\|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\widetilde{\phi}_h)\|_{1,\Omega} \le C_{\mathbf{S}_h} C_{\widetilde{\mathbf{S}}} \|\mathbf{T}(\phi_h)\|_{1,\Omega} \|\mathbf{S}^{\mathbf{t}}_{1,h}(\phi_h)\|_{\mathbf{L}^4(\Omega)} \|\phi_h - \widetilde{\phi}_h\|_{\mathrm{L}^4(\Omega)}, \tag{4.13}$$

where $C_{\widetilde{\mathbf{S}}}$ *and* $C_{\mathbf{S}_h}$ *are the constants provided by Lemmas 3.5 and 4.4, respectively.*

Consequently, since the foregoing lemma and the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$ confirm the continuity of $\mathbf{T}_h$, we conclude, thanks to the Brouwer fixed-point theorem (cf. [11, Theorem 9.9-2]) and Lemmas 4.3 and 4.6, the main result of this section.

Theorem 4.7 *Suppose that the parameters κ_i, $i \in \{1, \ldots, 7\}$, satisfy the conditions required by Lemmas* 3.1 *and* 3.2. *Let* $\mathcal{W}_h := \Big\{ \phi_h \in \mathrm{H}_h^\theta : \quad \|\phi_h\|_{1,\Omega} \le c_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \Big\} \Big\}$, *and assume that the datum* $\mathbf{f}$ *satisfies* (3.21). *Then the Galerkin scheme* (4.2) *has at least one solution* $(\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^\theta$ *with* $\theta_h \in \mathcal{W}_h$, *and there holds*

$$\|(\underline{\mathbf{t}}_h, \mathbf{u}_h)\| \le c_{\mathbf{S}} \|\mathbf{f}\|_{0,\Omega}, \tag{4.14}$$

and

$$\|(\mathbf{p}_h, \theta_h)\| \le c_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}} + \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \Big\}, \tag{4.15}$$

where $c_{\mathbf{S}}$ *and* $c_{\widetilde{\mathbf{S}}}$ *are the constants provided by Lemmas* 3.1 *and* 3.2, *respectively.*

We end this section by remarking that the lack of suitable estimates for $\|\mathbf{S}^{\mathbf{t}}_{1,h}(\phi_h)\|_{\mathbf{L}^4(\Omega)}$ stops us of trying to use (4.13) to derive a contraction estimate for $\mathbf{T}_h$. This is the reason why in the foregoing Theorem 4.7 we are able only to guarantee existence, but no uniqueness, of a discrete solution.

4.3 Convergence of the Galerkin scheme

Given $(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega) \times \mathrm{H}^1(\Omega)$ with $\theta \in \mathcal{W}$, and $(\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^\theta$ with $\theta_h \in \mathcal{W}_h$, solutions of (2.18) and (4.2), respectively, we now aim to derive a corresponding a priori error estimate. For this purpose, we first observe from (2.18) and (4.2) that the above problems can be rewritten as two pairs of corresponding continuous and discrete formulations, namely

$$\begin{aligned} \mathbf{A}_\theta((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{r}}, \mathbf{v})) &= \mathbf{F}(\underline{\mathbf{r}}, \mathbf{v}) \qquad \forall (\underline{\mathbf{r}}, \mathbf{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega), \\ \mathbf{A}_{\theta_h}((\underline{\mathbf{t}}_h, \mathbf{u}_h), (\underline{\mathbf{r}}_h, \mathbf{v}_h)) &= \mathbf{F}(\underline{\mathbf{r}}_h, \mathbf{v}_h) \quad \forall (\underline{\mathbf{r}}_h, \mathbf{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \widetilde{\mathbf{A}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) + \widetilde{\mathbf{B}}_{\mathbf{u}}((\mathbf{p}, \theta), (\mathbf{q}, \psi)) &= \widetilde{\mathbf{F}}(\mathbf{q}, \psi) \qquad \forall (\mathbf{q}, \psi) \in \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega) \times \mathrm{H}^1(\Omega), \\ \widetilde{\mathbf{A}}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\mathbf{u}_h}((\mathbf{p}_h, \theta_h), (\mathbf{q}_h, \psi_h)) &= \widetilde{\mathbf{F}}(\mathbf{q}_h, \psi_h) \quad \forall (\mathbf{q}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^\theta. \end{aligned} \tag{4.17}$$

Then, as suggested by the structure of the foregoing systems, in what follows we apply the well-known Strang lemma for elliptic variational problems (see, e.g., [34, Theorem 11.1]) to (4.16) and (4.17). This auxiliary result is stated first.

Lemma 4.8 *Let V be a Hilbert space, $F \in V'$, and $A : V \times V \to \mathbb{R}$ be a bounded and V-elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite dimensional subspaces of V, and for each $h > 0$ consider a bounded bilinear form $A_h : V_h \times V_h \to \mathbb{R}$ and a functional $F_h \in V_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic, that is, there exists a constant $\widetilde{\alpha} > 0$, independent of h, such that*

$$A_h(v_h, v_h) \ge \widetilde{\alpha} \|v_h\|_V^2 \quad \forall v_h \in V_h, \quad \forall h > 0.$$

In turn, let $u \in V$ and $u_h \in V_h$ such that

$$A(u, v) = F(v) \quad \forall v \in V \quad \textit{and} \quad A_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

Then, for each $h > 0$ there holds

$$\begin{aligned}\|u - u_h\|_V \quad &\le \quad C_{\mathrm{ST}} \left\{ \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{\left| F(w_h) - F_h(w_h) \right|}{\|w_h\|_V} \right. \\ &\left. + \inf_{\substack{v_h \in V_h \\ v_h \neq 0}} \left(\|u - v_h\|_V + \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{\left| A(v_h, w_h) - A_h(v_h, w_h) \right|}{\|w_h\|_V} \right) \right\},\end{aligned}$$

where $C_{\mathrm{ST}} := \widetilde{\alpha}^{-1} \max \left\{ 1, \|A\| \right\}$.

In the sequel, for the sake of simplicity, we denote as usual

$$\operatorname{dist}\left((\underline{\mathbf{t}}, \mathbf{u}), \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \right) := \inf_{(\underline{\mathbf{r}}_h, \mathbf{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}} \|(\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{r}}_h, \mathbf{v}_h)\|$$

and

$$\operatorname{dist}\left((\mathbf{p}, \theta), \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta} \right) := \inf_{(\mathbf{q}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}} \|(\mathbf{p}, \theta) - (\mathbf{q}_h, \psi_h)\|.$$

The following Lemma provides a preliminary estimate for the error $\|(\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{t}}_h, \mathbf{u}_h)\|$.

Lemma 4.9 *Let* $C_{\mathrm{ST}} := \dfrac{1}{\alpha(\Omega)} \max \left\{ 1, \|\mathbf{A}_\theta\| \right\}$*, where $\alpha(\Omega)$ is the constant yielding the ellipticity of $\mathbf{A}_\phi$ for any $\phi \in \mathrm{H}^1(\Omega)$ (cf. (3.13)). Then, there holds*

$$\begin{aligned}\|(\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{t}}_h, \mathbf{u}_h)\| \quad &\le \quad C_{\mathrm{ST}} \Big\{ L_\mu (1 + \kappa_1^2)^{1/2} C_\delta \|\mathbf{t}\|_{\delta, \Omega} \|\theta - \theta_h\|_{\mathrm{L}^{n/\delta}(\Omega)} \\ &+ (1 + 2\|\mathbf{A}_\theta\|) \operatorname{dist}\left((\underline{\mathbf{t}}, \mathbf{u}), \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \right) \Big\}\end{aligned} \tag{4.18}$$

Proof. We proceed similarly as in [2, Lemma 5.3]. In fact, from Lemmas 3.1 and 4.1, we have that the bilinear forms $\mathbf{A}_\theta$ and $\mathbf{A}_{\theta_h}$ are both bounded and elliptic with the same constants $\|\mathbf{A}_\theta\|$ and $\alpha(\Omega)$, respectively. In addition, $\mathbf{F}$ is a linear and bounded functional in $\mathbb{H} \times \mathbf{H}_0^1(\Omega)$ and, in particular, in $\mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}$. Then, by applying Lemma 4.8 to the context (4.16), we obtain

$$\begin{aligned}\|(\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{t}}_h, \mathbf{u}_h)\| \le C_{\mathrm{ST}} &\inf_{\substack{(\underline{\mathbf{r}}_h, \mathbf{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \\ (\underline{\mathbf{r}}_h, \mathbf{v}_h) \neq \mathbf{0}}} \left\{ \|(\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{r}}_h, \mathbf{v}_h)\| \right. \\ + &\left. \sup_{\substack{(\underline{\mathbf{s}}_h, \mathbf{w}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \\ (\underline{\mathbf{s}}_h, \mathbf{w}_h) \neq \mathbf{0}}} \frac{\left| \mathbf{A}_\theta((\underline{\mathbf{r}}_h, \mathbf{v}_h), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) - \mathbf{A}_{\theta_h}((\underline{\mathbf{r}}_h, \mathbf{v}_h), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) \right|}{\|(\underline{\mathbf{s}}_h, \mathbf{w}_h)\|} \right\}.\end{aligned} \tag{4.19}$$

In turn, in order to estimate the supremum in (4.19), we add and subtract suitable terms to write

$$\begin{aligned}&\mathbf{A}_\theta((\underline{\mathbf{r}}_h, \mathbf{v}_h), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) - \mathbf{A}_{\theta_h}((\underline{\mathbf{r}}_h, \mathbf{v}_h), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) = \mathbf{A}_\theta((\underline{\mathbf{r}}_h, \mathbf{v}_h) - (\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) \\ &+ (\mathbf{A}_\theta - \mathbf{A}_{\theta_h})((\underline{\mathbf{t}}, \mathbf{u}), (\underline{\mathbf{s}}_h, \mathbf{w}_h)) + \mathbf{A}_{\theta_h}((\underline{\mathbf{t}}, \mathbf{u}) - (\underline{\mathbf{r}}_h, \mathbf{v}_h), (\underline{\mathbf{s}}_h, \mathbf{w}_h)),\end{aligned}$$

whence, applying the boundedness (3.11) to the first and third terms on the right-hand side of the foregoing equation, and proceeding analogously as for the derivation of (3.24) with the second one, we find that

$$\begin{aligned}
\sup_{\substack{(\underline{\mathbf{s}}_h,\mathbf{w}_h)\in\mathbb{H}_h\times\mathbf{H}_h^{\mathbf{u}}\\(\underline{\mathbf{s}}_h,\mathbf{w}_h)\neq\mathbf{0}}} & \frac{\Big|\mathbf{A}_\theta((\underline{\mathbf{r}}_h,\mathbf{v}_h),(\underline{\mathbf{s}}_h,\mathbf{w}_h))-\mathbf{A}_{\theta_h}((\underline{\mathbf{r}}_h,\mathbf{v}_h),(\underline{\mathbf{s}}_h,\mathbf{w}_h))\Big|}{\|(\underline{\mathbf{s}}_h,\mathbf{w}_h)\|} \\
& \leq\; L_\mu(1+\kappa_1^2)^{1/2}C_\delta\|\mathbf{t}\|_{\delta,\Omega}\|\theta-\theta_h\|_{\mathrm{L}^{n/\delta}(\Omega)}+2\|\mathbf{A}_\theta\|\|(\underline{\mathbf{t}},\mathbf{u})-(\underline{\mathbf{r}}_h,\mathbf{v}_h)\|.
\end{aligned}\tag{4.20}$$

Finally, by replacing the inequality (4.20) into (4.19), we get (4.18), which ends the proof. □

Next, we have the following result concerning $\|(\mathbf{p},\theta)-(\mathbf{p}_h,\theta_h)\|$.

Lemma 4.10 *Let* $\widetilde{C}_{\mathrm{ST}}:=\dfrac{2}{\widetilde{\alpha}(\Omega)}\max\Big\{1,\|\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{u}}\|\Big\}$, *where* $\widetilde{\alpha}(\Omega)$ *is the constant yielding the ellipticity of both* $\widetilde{\mathbf{A}}$ *and* $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\mathbf{w}}$, *for any* $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$ *(cf. (3.17) and (3.18)). Then, there holds*

$$\begin{aligned}
\|(\mathbf{p},\theta)-(\mathbf{p}_h,\theta_h)\| \;&\leq\; \widetilde{C}_{\mathrm{ST}}\Big\{\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\theta\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\\
&+\quad\Big(1+\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\,\Big((\mathbf{p},\theta),\mathbf{H}_h^{\mathbf{p}}\times\mathrm{H}_h^\theta\Big)\Big\}.
\end{aligned}\tag{4.21}$$

Proof. It follows almost straightforwardly from a slight modification of the proof of [13, Lemma 5.3]. We omit further details. □

We now combine the inequalities provided by Lemmas 4.9 and 4.10 to derive the a priori estimate for the total error $\|(\underline{\mathbf{t}},\mathbf{u},\mathbf{p},\theta)-(\underline{\mathbf{t}}_h,\mathbf{u}_h,\mathbf{p}_h,\theta_h)\|$. Indeed, by gathering together the estimates (4.18) and (4.21), it follows that

$$\begin{aligned}
&\|(\underline{\mathbf{t}},\mathbf{u},\mathbf{p},\theta)-(\underline{\mathbf{t}}_h,\mathbf{u}_h,\mathbf{p}_h,\theta_h)\|\leq\widetilde{C}_{\mathrm{ST}}\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\theta\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\\
&+C_{\mathrm{ST}}L_\mu(1+\kappa_1^2)^{1/2}C_\delta\|\mathbf{t}\|_{\delta,\Omega}\|\theta-\theta_h\|_{\mathrm{L}^{n/\delta}(\Omega)}\;+\;C_{\mathrm{ST}}(1+2\|\mathbf{A}_\theta\|)\mathrm{dist}\Big((\underline{\mathbf{t}},\mathbf{u}),\mathbb{H}_h\times\mathbf{H}_h^{\mathbf{u}}\Big)\\
&+\widetilde{C}_{\mathrm{ST}}\Big(1+\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\,\Big((\mathbf{p},\theta),\mathbf{H}_h^{\mathbf{p}}\times\mathrm{H}_h^\theta\Big).
\end{aligned}$$

Then, by noting that $\theta\in\mathcal{W}$, using the estimate (3.22) to bound $\|\mathbf{t}\|_{\delta,\Omega}$, and recalling that $\widetilde{C}_\delta$ is the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\delta}(\Omega)$ (cf. (3.32)), from the latter inequality we find that

$$\begin{aligned}
&\|(\underline{\mathbf{t}},\mathbf{u},\mathbf{p},\theta)-(\underline{\mathbf{t}}_h,\mathbf{u}_h,\mathbf{p}_h,\theta_h)\|\leq\mathbf{C}(\mathbf{f},g,\theta_{\mathrm{D}})\|(\underline{\mathbf{t}},\mathbf{u},\mathbf{p},\theta)-(\underline{\mathbf{t}}_h,\mathbf{u}_h,\mathbf{p}_h,\theta_h)\|\\
&+C_{\mathrm{ST}}(1+2\|\mathbf{A}_\theta\|)\mathrm{dist}\Big((\underline{\mathbf{t}},\mathbf{u}),\mathbb{H}_h\times\mathbf{H}_h^{\mathbf{u}}\Big)\\
&+\widetilde{C}_{\mathrm{ST}}\Big(1+\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\,\Big((\mathbf{p},\theta),\mathbf{H}_h^{\mathbf{p}}\times\mathrm{H}_h^\theta\Big)\,,
\end{aligned}\tag{4.22}$$

where

$$\mathbf{C}(\mathbf{f},g,\theta_{\mathrm{D}}):=\max\Big\{\mathbf{C}_1(\mathbf{f},g,\theta_{\mathrm{D}}),\mathbf{C}_2(\mathbf{f},g,\theta_{\mathrm{D}})\Big\},$$

with

$$\mathbf{C}_1(\mathbf{f},g,\theta_{\mathrm{D}}):=\widetilde{C}_{\mathrm{ST}}\kappa^{-1}(1+\kappa_5^2)^{1/2}c(\Omega)c_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega}+\|\theta_{\mathrm{D}}\|_{0,\Gamma_{\mathrm{D}}}+\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}$$

and

$$\mathbf{C}_2(\mathbf{f},g,\theta_{\mathrm{D}}):=C_{\mathrm{ST}}L_\mu(1+\kappa_1^2)^{1/2}C_\delta\widehat{C}_{\mathbf{S}}\widetilde{C}_\delta\|\mathbf{f}\|_{\delta,\Omega}.$$

Consequently, we can establish the following result providing the complete Céa estimate.

Theorem 4.11 *Assume that the data* $\mathbf{f}, g$ *and* θ_{D} *satisfy:*

$$\mathbf{C}_i(\mathbf{f}, g, \theta_{\mathrm{D}}) \leq \frac{1}{2} \quad \forall i \in \{1,2\}. \tag{4.23}$$

Then, there exists a positive constant C*, depending only on parameters, data and other constants, all of them independent of* h*, such that*

$$\|(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) - (\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h)\| \leq C\Big\{\mathrm{dist}\Big((\underline{\mathbf{t}}, \mathbf{u}), \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}\Big) + \mathrm{dist}\Big((\mathbf{p}, \theta), \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}\Big)\Big\}. \tag{4.24}$$

Proof. From (4.22) and (4.23), it follows that

$$\begin{aligned}\|(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) - (\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h)\| &\leq 2C_{\mathrm{ST}}(1 + 2\|\mathbf{A}_\theta\|)\mathrm{dist}\Big((\underline{\mathbf{t}}, \mathbf{u}), \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}\Big)\\ &+ 2\widetilde{C}_{\mathrm{ST}}\Big(1 + \kappa^{-1}(1+\kappa_5^2)^{1/2} c(\Omega)\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\Big((\mathbf{p}, \theta), \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}\Big),\end{aligned}$$

and then, the rest of the proof reduces to employ the triangle inequality on the term $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$ and use that both $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{u}_h\|_{1,\Omega}$ are bounded by $c_{\mathbf{S}}\|\mathbf{f}\|_{0,\Omega}$ (cf. Lemmas 3.1 and 4.1). $\square$

Now, in order to approximate the polymeric and solvent parts of the extra-stress tensor, as well as the pressure, we propose, motivated by (2.6), (2.12), and the fifth equation of (2.8), the expressions

$$\widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h} = 2\mu_{\mathrm{P}}(\theta_h)\mathbf{t}_h, \quad \boldsymbol{\sigma}_{\mathrm{N},h} = 2\epsilon\mu_{\mathrm{N}}(\theta_h)\mathbf{t}_h, \quad \widehat{\boldsymbol{\sigma}}_{\mathrm{P},h} = \boldsymbol{\sigma}_h^{\mathrm{d}} - \boldsymbol{\sigma}_{\mathrm{N},h}, \quad \text{and} \quad p_h = -\frac{1}{n}\mathrm{tr}\,\boldsymbol{\sigma}_h\,, \tag{4.25}$$

respectively, with $(\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ being the unique solution of the discrete problem (4.2). The corresponding error estimates are established in the following lemma.

Lemma 4.12 *Assume that the hypotheses of Theorem* 4.11 *hold. Let* $(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_{\Gamma_{\mathrm{N}}}(\mathrm{div}\,;\Omega) \times \mathrm{H}^1(\Omega)$ *and* $(\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}$ *be the unique solutions of the continuous and discrete problems* (2.18) *and* (4.2)*, respectively. Then, there exists a positive constant* C*, depending only on parameters, data and other constants, all of them independent of* h*, such that*

$$\|p - p_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{N}} - \boldsymbol{\sigma}_{\mathrm{N},h}\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega} \leq C\Big\{\mathrm{dist}\Big((\underline{\mathbf{t}}, \mathbf{u}), \mathbb{H}_h \times \mathbf{H}_h^{\mathbf{u}}\Big) + \mathrm{dist}\Big((\mathbf{p}, \theta), \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\theta}\Big)\Big\}.$$

Proof. From (2.6) and (4.25), adding and subtracting $2\mu_{\mathrm{P}}(\theta_h)\mathbf{t}$, it is clear that

$$\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h} = 2(\mu_{\mathrm{P}}(\theta) - \mu_{\mathrm{P}}(\theta_h))\mathbf{t} + 2\mu_{\mathrm{P}}(\theta_h)(\mathbf{t} - \mathbf{t}_h).$$

Next, employing the triangle and Hölder inequalities, the estimate (3.22) to bound $\|\mathbf{t}\|_{\delta,\Omega}$, the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\delta}(\Omega)$, and the Lipschitz-continuity assumption (2.4), it is not difficult to see that there exists a positive constant c, depending only on data and other constants, all of them independent of h, such that

$$\|\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega} \leq c\Big\{\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\theta - \theta_h\|_{1,\Omega}\Big\}.$$

In this way, following similar arguments for the solvent part of the extra-stress tensor $\boldsymbol{\sigma}_{\mathrm{N}}$ (cf. (4.25)), we obtain

$$\|p - p_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{N}} - \boldsymbol{\sigma}_{\mathrm{N},h}\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega} \leq C\Big\{\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\theta - \theta_h\|_{1,\Omega}\Big\}.$$

Then, the result is a direct application of Theorem 4.11. Observe that the proof is also valid if we consider $\widehat{\boldsymbol{\sigma}}_{\mathrm{P},h}$ in place of $\widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}$. $\square$

Finally, we complete our a priori error analysis with the following results which provides the corresponding rate of convergence of our Galerkin scheme (4.2).

Theorem 4.13 *In addition to the hypotheses of Theorems* 3.8, 4.7 *and* 4.11*, assume that there exists* $s > 0$ *such that* $\mathbf{t} \in \mathbb{H}^s(\Omega)$*,* $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$*,* $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$*,* $\boldsymbol{\rho} \in \mathbb{H}^s(\Omega)$*,* $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$*,* $\mathbf{p} \in \mathbf{H}^s(\Omega)$*,* $\operatorname{div}\mathbf{p} \in \mathrm{H}^s(\Omega)$*, and* $\theta \in \mathrm{H}^{s+1}(\Omega)$*. Then, there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned}\|(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) - (\underline{\mathbf{t}}_h, \mathbf{u}_h, \mathbf{p}_h, \theta_h)\| \leq C h^{\min\{s,k+1\}} \Big\{ &\|\mathbf{t}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega} + \|\boldsymbol{\rho}\|_{s,\Omega} \\ &+ \|\mathbf{u}\|_{s+1,\Omega} + \|\mathbf{p}\|_{s,\Omega} + \|\operatorname{div}\mathbf{p}\|_{s,\Omega} + \|\theta\|_{s+1,\Omega} \Big\}.\end{aligned} \tag{4.26}$$

Proof. It follows directly from the Céa estimate (4.24) and the well-known approximation properties of the discrete spaces $\mathbb{H}_h^{\mathbf{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\rho}}, \mathbf{H}_h^{\mathbf{u}}, \mathbf{H}_h^{\mathbf{p}}$, and H_h^{θ} (cf. [6, 11]). □

Consequently, from Lemma 4.12 and Theorem 4.13 we obtain the optimal convergence of the post-processed unknowns introduced in (4.25).

Lemma 4.14 *Let* $(\underline{\mathbf{t}}, \mathbf{u}, \mathbf{p}, \theta) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_{\Gamma_{\mathrm{N}}}(\operatorname{div};\Omega) \times \mathrm{H}^1(\Omega)$ *be the unique solutions of the continuous problem* (2.18)*, and let* $\boldsymbol{\sigma}_{\mathrm{P}}$*,* $\boldsymbol{\sigma}_{\mathrm{N}}$*, and* p *given by* (2.6) *and the fifth equation of* (2.8) *In addition, let* $\widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}$ *(or* $\widehat{\boldsymbol{\sigma}}_{\mathrm{P},h}$*),* $\boldsymbol{\sigma}_{\mathrm{N},h}$*, and* p_h *be the discrete counterparts introduced in* (4.25)*. Assume that hypotheses of Theorem 4.13 hold. Then, there exist* $C > 0$*, independent of* h*, such that*

$$\begin{aligned}\|p - p_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{N}} - \boldsymbol{\sigma}_{\mathrm{N},h}\|_{0,\Omega} + \|\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega} \leq C h^{\min\{s,k+1\}} \Big\{ &\|\mathbf{t}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega} \\ &+ \|\boldsymbol{\rho}\|_{s,\Omega} + \|\mathbf{u}\|_{s+1,\Omega} + \|\mathbf{p}\|_{s,\Omega} + \|\operatorname{div}\mathbf{p}\|_{s,\Omega} + \|\theta\|_{s+1,\Omega} \Big\}.\end{aligned}$$

5 Numerical results

In this section we present some examples illustrating the performance of our augmented fully-mixed finite element scheme (4.2), and confirming the rates of convergence provided by Theorem 4.13 and Lemma 4.14. Our implementation is based on a *FreeFem++* code [23], in conjunction with the direct linear solver UMFPACK [16]. A Picard algorithm with a fixed tolerance $tol = 1E-8$ has been used for the corresponding fixed-point problem (4.8) and the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in R^N, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h^{\mathbf{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\rho}}, \mathbf{H}_h^{\mathbf{u}}, \mathbf{H}_h^{\mathbf{p}}$, and H_h^{θ}. As usual, the individual errors are denoted by:

$$\mathrm{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad \mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}, \quad \mathrm{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega}, \quad \mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega},$$

$$\mathrm{e}(\mathbf{p}) := \|\mathbf{p} - \mathbf{p}_h\|_{\operatorname{div};\Omega}, \quad \mathrm{e}(\theta) := \|\theta - \theta_h\|_{1,\Omega}, \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega},$$

$$\mathrm{e}(\boldsymbol{\sigma}_{\mathrm{N}}) := \|\boldsymbol{\sigma}_{\mathrm{N}} - \boldsymbol{\sigma}_{\mathrm{N},h}\|_{0,\Omega}, \quad \mathrm{e}(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}) := \|\boldsymbol{\sigma}_{\mathrm{P}} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega}, \quad \mathrm{e}(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}) := \|\boldsymbol{\sigma}_{\mathrm{P}} - \widehat{\boldsymbol{\sigma}}_{\mathrm{P},h}\|_{0,\Omega}.$$

In addition, we let $r(\cdot)$ be the experimental rate of convergence given by

$$r(\%) := \frac{\log(\mathrm{e}(\%)/\mathrm{e}'(\%))}{\log(h/h')} \qquad \text{for each } \% \in \Big\{\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \mathbf{u}, \mathbf{p}, \theta, p, \boldsymbol{\sigma}_{\mathrm{N}}, \widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}, \widehat{\boldsymbol{\sigma}}_{\mathrm{P}}\Big\},$$

where e and e′ denote errors computed on two consecutive meshes of sizes h and h', respectively.

The examples to be considered in this section are described next. In all of them, as in [14, Section 2], we choose the coefficients of the polymer and solvent viscosity a_1, b_1, a_2 and b_2 (cf. (2.2)) as follow:

$$b_1 = b_2 = \frac{\Delta E}{R}, \quad a_2 = \exp\left(\frac{-\Delta E}{R\theta_R}\right), \quad \text{and} \quad a_1 = (1-\epsilon)a_2,$$

where ΔE is the activation energy, R is the ideal gas constant, and θ_R is a reference temperature of the fluid. Note that the constraint (2.3) will be satisfied as long as the temperature of the system stays above θ_R. In turn, we consider $\kappa = 1$, $\epsilon = 0.01$, and according to (3.20), the stabilization parameters are taken as $\kappa_1 = \mu_1/\mu_2^2$, $\kappa_2 = \kappa_1$, $\kappa_3 = \mu_1/2$, $\kappa_4 = \mu_1/4$, $\kappa_5 = \kappa$, $\kappa_6 = \kappa^{-1}/2$, and $\kappa_7 = \kappa/2$. In addition, the conditions $\int_\Omega \operatorname{tr} \boldsymbol{\sigma}_h = 0$ is imposed via a penalization strategy.

In our first example we illustrate the accuracy of our method in 2D by considering the square domain $\Omega := (0,1)^2$, the boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} := \{0\} \times (0,1)$ and $\Gamma_{\mathrm{N}} := \Gamma \setminus \overline{\Gamma}_{\mathrm{D}}$. The following viscosity parameters correspond to polystyrene [24, Section 4.2]:

$$\frac{\Delta E}{R} = 14500, \quad \theta_R = 538.$$

The data $\mathbf{f}$, g, and θ_{D} are chosen so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &:= \begin{pmatrix} \pi x_1^2 (x_1-1)^2 \sin(2\pi x_2) \\ -2x_1(x_1-1)(2x_1-1)\sin(\pi x_2)^2 \end{pmatrix}, \\
p(\mathbf{x}) &:= \cos(\pi x_1)\sin(\pi x_2), \\
\theta(\mathbf{x}) &:= 10(x_1-1)^2 \sin(\pi x_2)^2 + 540 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega.
\end{aligned}$$

In our second example we consider a four-to-one contraction domain $\Omega := (0,2)\times(0,1) \setminus (1,2)\times(0.25,1)$, the boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} := \{0\} \times (0,1)$ and $\Gamma_{\mathrm{N}} := \Gamma \setminus \overline{\Gamma}_{\mathrm{D}}$. The following viscosity parameters correspond to Nylon-6,6 [24, Section 4.2]:

$$\frac{\Delta E}{R} = 6600, \quad \theta_R = 563.$$

The data $\mathbf{f}$, g, and θ_{D} are chosen so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &:= \begin{pmatrix} 2x_1^2 x_2 (x_1-1)^2 (x_1-2)^2 (x_2-1)(4x_2-1)(12x_2^2 - 10x_2 + 1) \\ -2x_1 x_2^2 (x_1-1)(x_1-2)(3x_1^2 - 6x_1 + 2)(x_2-1)^2 (4x_2-1)^2 \end{pmatrix}, \\
p(\mathbf{x}) &:= (x_1 - 0.5)\cos(4\pi x_2), \\
\theta(\mathbf{x}) &:= x_1(2x_1^2 - 9x_1 + 12)\sin(2\pi x_2)^2 + 580 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega.
\end{aligned}$$

In our third example we illustrate the accuracy of our method in 3D by considering the cube domain $\Omega := (0,1)^3$, the boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} := (0,1)^2 \times \{0\}$ and $\Gamma_{\mathrm{N}} := \Gamma \setminus \overline{\Gamma}_{\mathrm{D}}$. The viscosity parameters are the same as in the first example and the data $\mathbf{f}$, g, and θ_{D} are chosen so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &:= \begin{pmatrix} 8x_1^2 x_2 x_3 (x_1-1)^2 (x_2-1)(x_3-1)(x_2-x_3) \\ -8x_1 x_2^2 x_3 (x_1-1)(x_2-1)^2 (x_3-1)(x_1-x_3) \\ 8x_1 x_2 x_3^2 (x_1-1)(x_2-1)(x_3-1)^2 (x_1-x_2) \end{pmatrix}, \\
p(\mathbf{x}) &:= (x_1 - 0.5)^3 \sin(x_2 + x_3), \\
\theta(\mathbf{x}) &:= 10\sin(\pi x_1)^2 \sin(\pi x_2)^2 (x_3-1)^2 + 540 \quad \forall\, \mathbf{x} := (x_1, x_2, x_3) \in \Omega.
\end{aligned}$$

Finally, in our fourth example we illustrate the accuracy of the 3D version of the four-to-one domain $\Omega := (0,2)\times(0,1)^2 \setminus (1,2)\times(0.25,1)^2$, the boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} := \{0\}\times(0,1)^2$ and $\Gamma_{\mathrm{N}} := \Gamma \setminus \overline{\Gamma}_{\mathrm{D}}$. The viscosity parameters are the same as in the second example and the data $\mathbf{f}$, g, and θ_{D} are chosen so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := \begin{pmatrix} 4x_1^2(x_1-1)^2(x_1-2)^2x_2(x_2-1)(2x_2-1)x_3(x_3-1)(4x_3-1)(12x_3^2-10x_3+1) \\ 4x_1(x_1-1)(x_1-2)(3x_1^2-6x_1+2)x_2^2(x_2-1)^2x_3(x_3-1)(4x_3-1)(12x_3^2-10x_3+1) \\ -8x_1(x_1-1)(x_1-2)(3x_1^2-6x_1+2)x_2(x_2-1)(2x_2-1)x_3^2(4x_3-1)^2(x_3-1)^2 \end{pmatrix},$$

$$p(\mathbf{x}) := (x_1-0.5)(x_2-0.5)\cos(4\pi x_3),$$

$$\theta(\mathbf{x}) := x_1(2x_1^2-9x_1+12)\sin(\pi x_2)^2\sin(2\pi x_3)^2 + 570 \quad \forall\, \mathbf{x} := (x_1,x_2,x_3) \in \Omega.$$

We remark that in all the examples, the temperature is given as a function $\widehat{\theta}(\mathbf{x})$ plus a big constant chosen such that $c > \theta_R$, that is, $\theta(\mathbf{x}) := \widehat{\theta}(\mathbf{x}) + c$. Then, the heat-flux vector is compute as:

$$\mathbf{p}(\mathbf{x}) = \kappa\nabla\widehat{\theta}(\mathbf{x}) - \widehat{\theta}(\mathbf{x})\mathbf{u}(\mathbf{x}) - c\mathbf{u}(\mathbf{x}),$$

which implies that the errors of $\mathbf{p}$ are influenced for c, and then they are higher than in the other unknowns as we will see below.

In Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6, we summarise the convergence history for a sequence of quasi-uniform triangulations, which required around four fixed-point iterations. In particular, for the 2D examples in Tables 5.1, 5.2, 5.3, and 5.4, we observe that the rate of convergence $O(h^{k+1})$ predicted by Theorem 4.13 and Lemma 4.14 (when $s = k+1$) is attained in all the variables (with $k=0$ and $k=1$). Notice that the higher the order of the finite element chosen the lower the number of iterations. In turn, in Tables 5.5 and 5.6 we observe that optimal rates of convergence are also obtained (with $k=0$) for our 3D examples. On the other hand, some components of the approximate solutions for the four examples are displayed in Figures 5.1, 5.2, 5.3, and 5.4. All the figures were built using the $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation with 353853, 430221, 3314052, and 4148740 degrees of freedom for the Examples 1, 2, 3, and 4, respectively. In particular, we can observe in Figure 5.1 that the temperature is higher in the left side and then it dissipates to the others sides meanwhile in Figure 5.2 the temperature is lightly higher in the right side. Next, analogously to Figures 5.1 and 5.2, in Figures 5.3 and 5.4 we can observe that the temperature is higher at the bottom of the cube and at the left side of the four-to-one domain and then it dissipates at the others sides, respectively. Moreover, it can be seen that the velocity streamlines of the fluid are higher inside of the domain and lower close to the boundary as expected.

References

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[2] M. Alvarez, G.N. Gatica and R. Ruiz-Baier, *An augmented mixed-primal finite element method for a coupled flow-transport problem.* ESAIM Math. Model. Numer. Anal. 49 (2015), no. 5, 1399–1427.

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
1467	0.196	0.1540	–	1.2323	–	0.2549	–	0.2609	–	18.7854	–
5631	0.097	0.0759	1.002	0.6258	0.961	0.1452	0.784	0.1266	1.025	9.6388	0.946
22131	0.048	0.0376	0.995	0.3099	0.993	0.0799	0.844	0.0618	1.014	4.7401	1.003
87837	0.025	0.0189	1.031	0.1564	1.024	0.0396	1.052	0.0311	1.026	2.4056	1.015
353853	0.013	0.0092	1.096	0.0768	1.090	0.0193	1.103	0.0155	1.072	1.1875	1.082

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
3.6159	–	0.1322	–	0.3557	–	0.3521	–	0.3095	–	5
1.4896	1.257	0.0677	0.949	0.1717	1.033	0.1700	1.033	0.1493	1.033	5
0.6674	1.135	0.0325	1.039	0.0830	1.026	0.0822	1.026	0.0727	1.018	4
0.3326	1.042	0.0150	1.154	0.0417	1.031	0.0413	1.031	0.0361	1.047	4
0.1631	1.093	0.0073	1.105	0.0201	1.119	0.0199	1.119	0.0175	1.111	4

Table 5.1: EXAMPLE 1, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
3666	0.196	0.0264	–	0.1535	–	0.0227	–	0.0370	–	2.4423	–
14076	0.097	0.0063	2.037	0.0374	2.002	0.0056	1.987	0.0086	2.073	0.5847	2.027
55326	0.048	0.0015	2.045	0.0089	2.024	0.0013	2.048	0.0020	2.065	0.1379	2.041
219591	0.025	0.0004	1.989	0.0023	2.037	0.0003	1.998	0.0005	1.986	0.0357	2.024
884631	0.013	0.0001	2.187	0.0006	2.153	0.0001	2.196	0.0001	2.195	0.0088	2.148

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
0.2957	–	0.0155	–	0.0455	–	0.0450	–	0.0861	–	4
0.0692	2.060	0.0041	1.899	0.0107	2.049	0.0106	2.049	0.0177	2.246	4
0.0154	2.119	0.0010	1.965	0.0025	2.039	0.0025	2.039	0.0041	2.063	4
0.0039	2.047	0.0003	2.074	0.0007	2.006	0.0007	2.006	0.0011	2.018	4
0.0010	2.133	0.0001	2.138	0.0002	2.185	0.0002	2.185	0.0003	2.198	4

Table 5.2: EXAMPLE 1, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbb{P}_1 - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_2$ approximation of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
1803	0.190	0.1627	–	2.3476	–	0.1990	–	0.2346	–	91.0099	–
6987	0.103	0.0872	1.017	1.1683	1.139	0.1209	0.814	0.1138	1.181	43.1779	1.217
27345	0.049	0.0432	0.953	0.5758	0.959	0.0650	0.841	0.0553	0.978	21.7689	0.929
107985	0.026	0.0219	1.052	0.2936	1.040	0.0326	1.066	0.0279	1.058	10.9728	1.059
430221	0.013	0.0108	1.062	0.1449	1.062	0.0168	0.996	0.0136	1.076	5.4528	1.051

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
10.2650	–	0.2544	–	0.2532	–	0.2507	–	0.2724	–	5
4.4925	1.348	0.1108	1.356	0.1339	1.040	0.1325	1.040	0.1387	1.101	4
2.1518	0.998	0.0495	1.092	0.0668	0.942	0.0662	0.942	0.0702	0.924	4
1.0794	1.066	0.0230	1.185	0.0339	1.051	0.0335	1.051	0.0349	1.078	3
0.5271	1.077	0.0112	1.085	0.0167	1.060	0.0166	1.060	0.0171	1.072	3

Table 5.3: Example 2, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
4506	0.190	0.0357	–	0.4304	–	0.0318	–	0.0504	–	16.7122	–
17466	0.103	0.0085	2.334	0.1039	2.319	0.0073	2.397	0.0117	2.386	4.0081	2.330
68361	0.049	0.0021	1.906	0.0247	1.946	0.0018	1.927	0.0028	1.938	1.0104	1.868
269961	0.026	0.0005	2.096	0.0064	2.087	0.0005	2.097	0.0007	2.095	0.2605	2.095
1075551	0.013	0.0001	2.131	0.0016	2.117	0.0001	2.125	0.0002	2.136	0.0627	2.141

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
1.9318	–	0.0741	–	0.0509	–	0.0504	–	0.0911	–	4
0.4124	2.520	0.0152	2.582	0.0122	2.325	0.0121	2.325	0.0238	2.191	3
0.1039	1.869	0.0033	2.055	0.0030	1.909	0.0030	1.909	0.0056	1.952	3
0.0264	2.115	0.0087	2.087	0.0008	2.097	0.0008	2.097	0.0015	2.089	3
0.0065	2.097	0.0002	2.148	0.0002	2.131	0.0002	2.131	0.0003	2.156	3

Table 5.4: Example 2, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbb{P}_1 - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_2$ approximation of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
7028	0.354	0.0149	–	0.1252	–	0.0181	–	0.0259	–	23.7426	–
53604	0.177	0.0082	0.862	0.0644	0.959	0.0119	0.606	0.0139	0.899	12.3360	0.945
419012	0.088	0.0042	0.970	0.0324	0.994	0.0068	0.812	0.0070	0.979	6.2286	0.986
3314052	0.044	0.0021	0.995	0.0162	1.001	0.0036	0.927	0.0035	0.998	3.1220	0.997

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
5.1532	–	0.0170	–	0.0328	–	0.0325	–	0.0312	–	4
2.8687	0.845	0.0096	0.817	0.0194	0.758	0.0192	0.758	0.0187	0.736	3
1.4810	0.954	0.0046	1.055	0.0103	0.908	0.0102	0.908	0.0103	0.868	3
0.7470	0.987	0.0022	1.076	0.0053	0.966	0.0052	0.966	0.0054	0.938	3

Table 5.5: Example 3, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximations of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

N	h	e($\mathbf{t}$)	$r(\mathbf{t})$	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e($\mathbf{u}$)	$r(\mathbf{u})$	e($\mathbf{p}$)	$r(\mathbf{p})$
8884	0.354	0.0657	–	1.0895	–	0.0705	–	0.1070	–	120.9619	–
67396	0.177	0.0414	0.667	0.6751	0.691	0.0478	0.559	0.0711	0.590	55.2472	1.131
525316	0.088	0.0227	0.865	0.3443	0.971	0.0290	0.723	0.0376	0.917	28.2498	0.968
4148740	0.044	0.0116	0.966	0.1727	0.995	0.0157	0.882	0.0189	0.992	14.2047	0.992

e(θ)	$r(\theta)$	e(p)	$r(p)$	e($\boldsymbol{\sigma}_{\mathrm{N}}$)*	$r(\boldsymbol{\sigma}_{\mathrm{N}})$	e($\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widetilde{\boldsymbol{\sigma}}_{\mathrm{P}})$	e($\widehat{\boldsymbol{\sigma}}_{\mathrm{P}}$)	$r(\widehat{\boldsymbol{\sigma}}_{\mathrm{P}})$	iter
7.6590	–	0.1087	–	0.1318	–	0.1305	–	0.1300	–	3
6.1383	0.319	0.0755	0.525	0.0793	0.732	0.0785	0.732	0.0810	0.683	3
3.2313	0.926	0.0337	1.165	0.0443	0.841	0.0439	0.841	0.0463	0.806	3
1.6359	0.982	0.0144	1.224	0.0233	0.926	0.0231	0.926	0.0244	0.926	3

Table 5.6: Example 4, Degrees of freedom, mesh sizes, errors, rates of convergence, and number of iterations for the fully-mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximations of the non-isothermal Oldroyd–Stokes equations (* errors divided by $\epsilon = 0.01$).

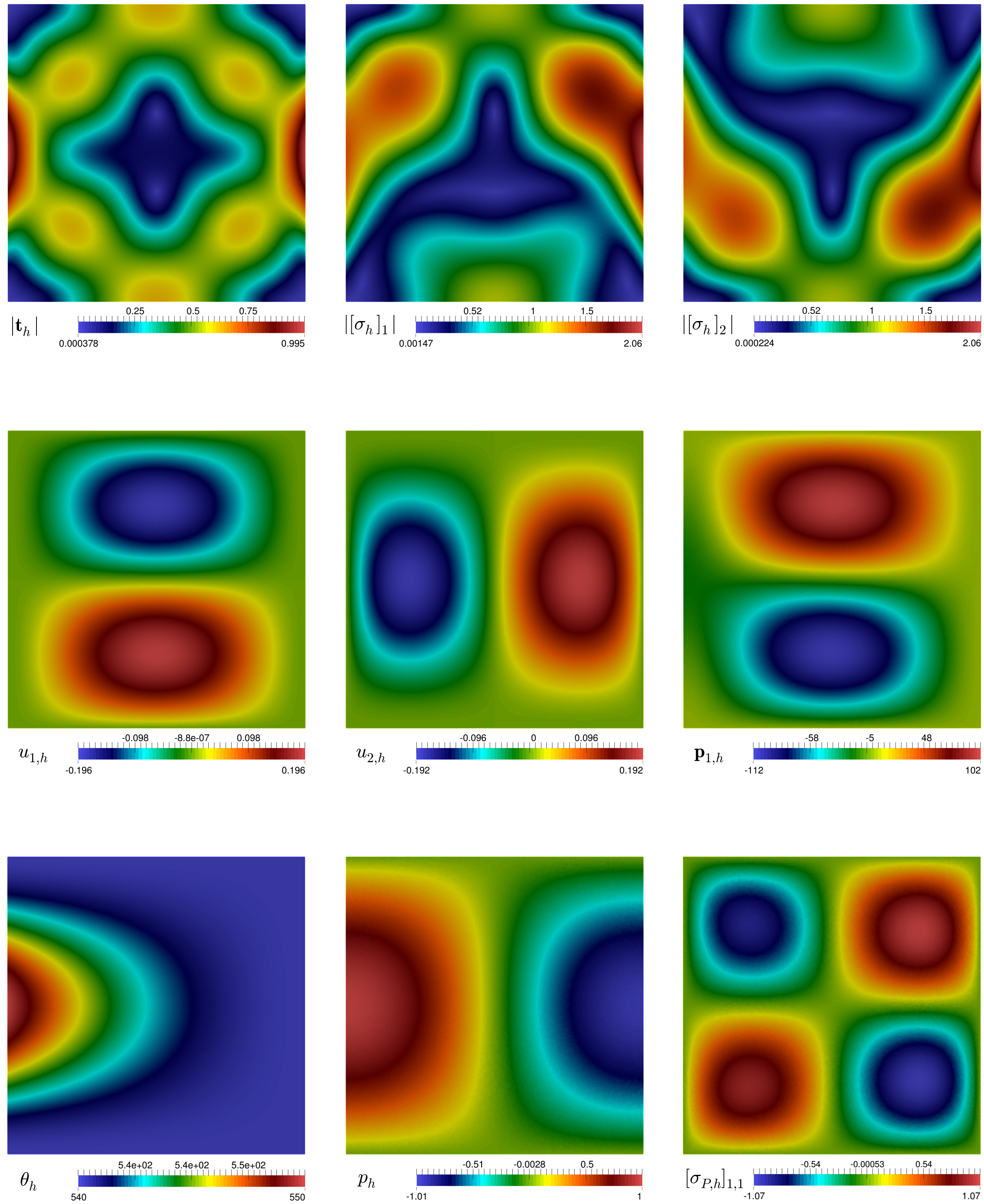

Figure 5.1: Example 1: $\mathbb{P}_0-\mathbb{RT}_0-\mathbb{P}_0-\mathbf{P}_1-\mathbf{RT}_0-\mathrm{P}_1$ approximated spectral norm of strain tensor and the stress tensor components (top panels), velocity and heat flux vector components (centre panels), and temperature and pressure fields, and polymeric part of the extra-stress tensor component (bottom row).

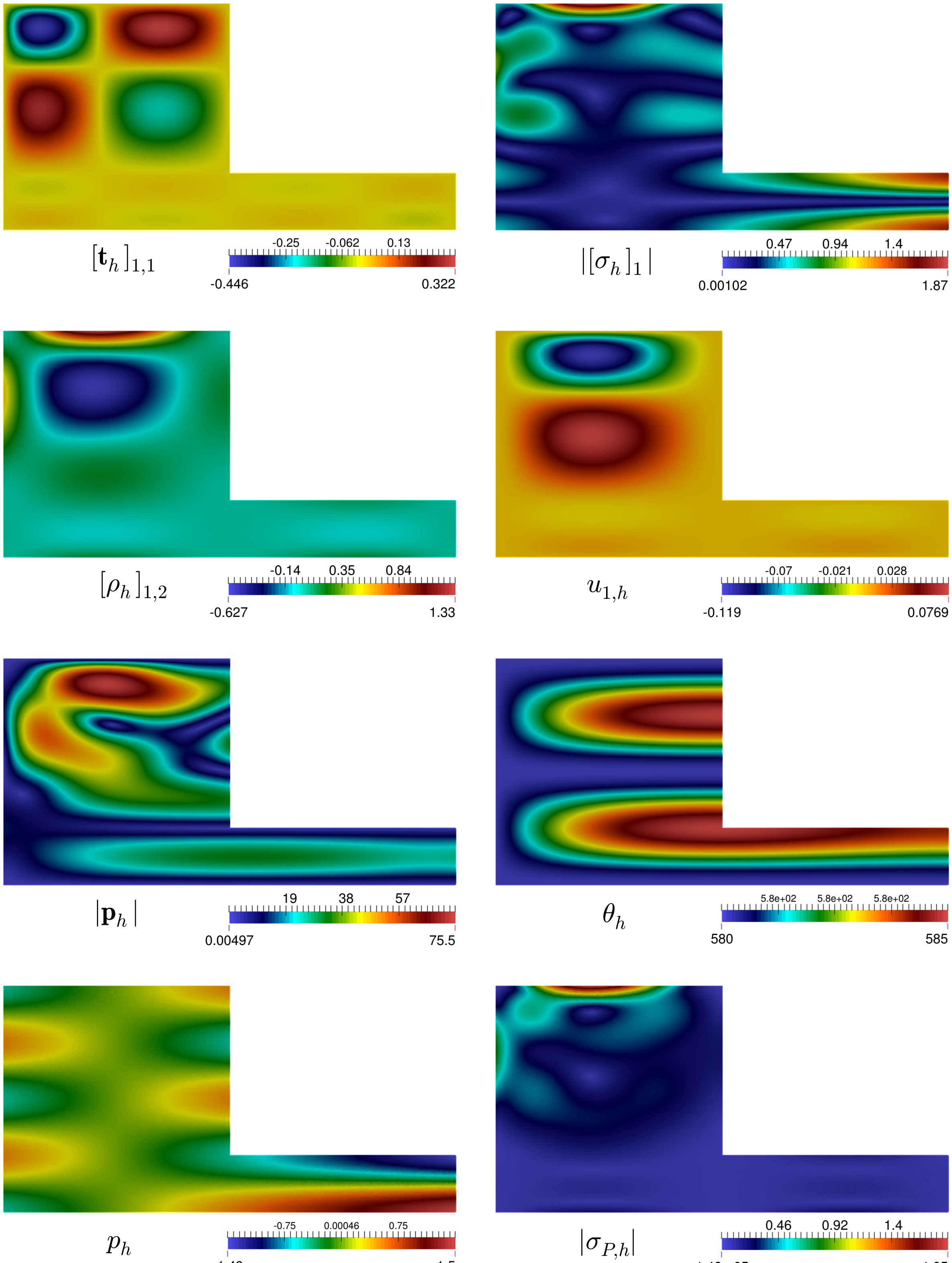

Figure 5.2: Example 2: $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation of some components of the approximate solutions.

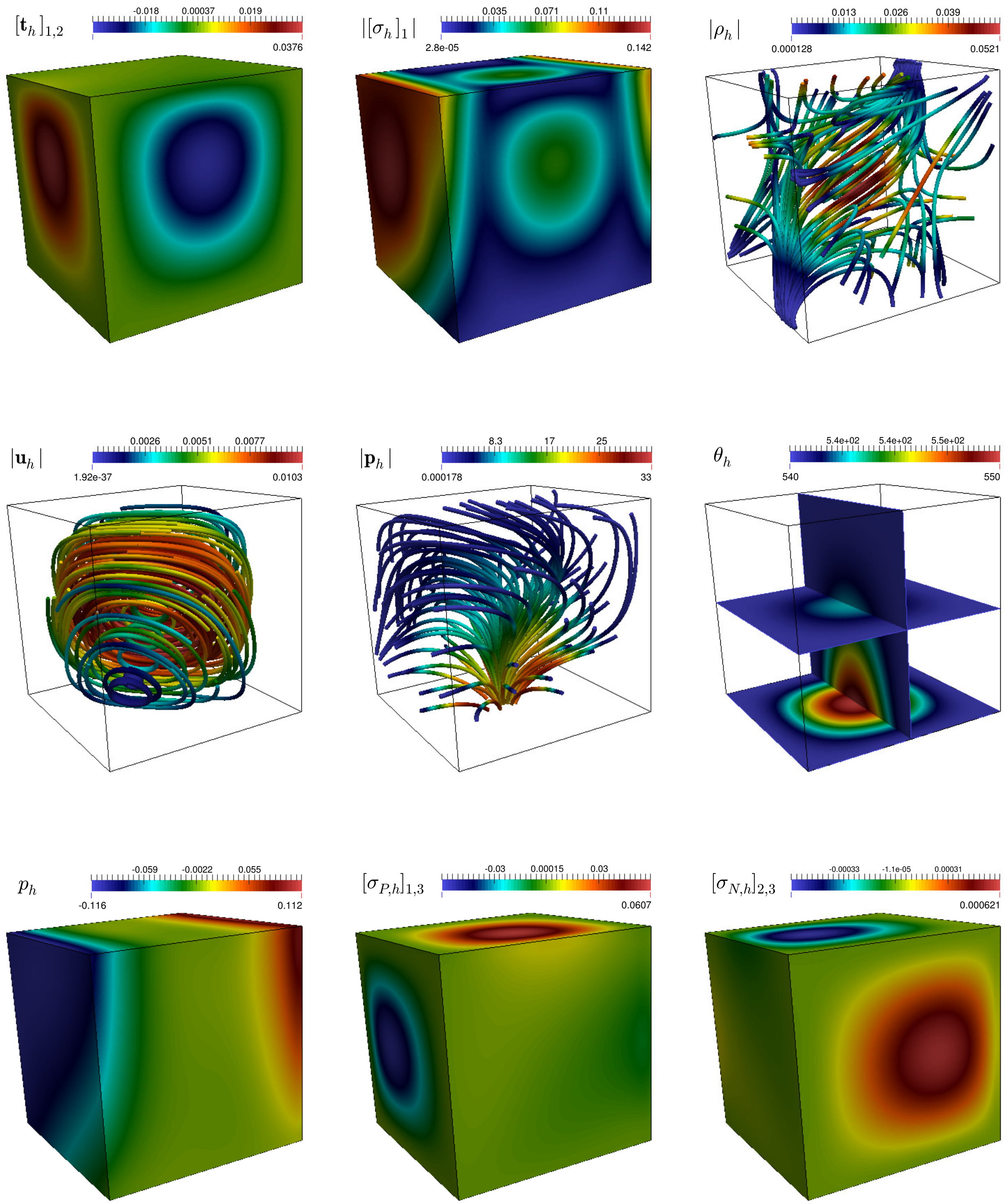

Figure 5.3: Example 3: $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation of the strain tensor component, approximated spectral norm of the stress tensor component, and vorticity streamlines (top panels), velocity streamlines, heat flux streamlines, and temperature field (centre panels), and pressure field, polymeric part and solvent part of the extra-stress tensor component (bottom row).

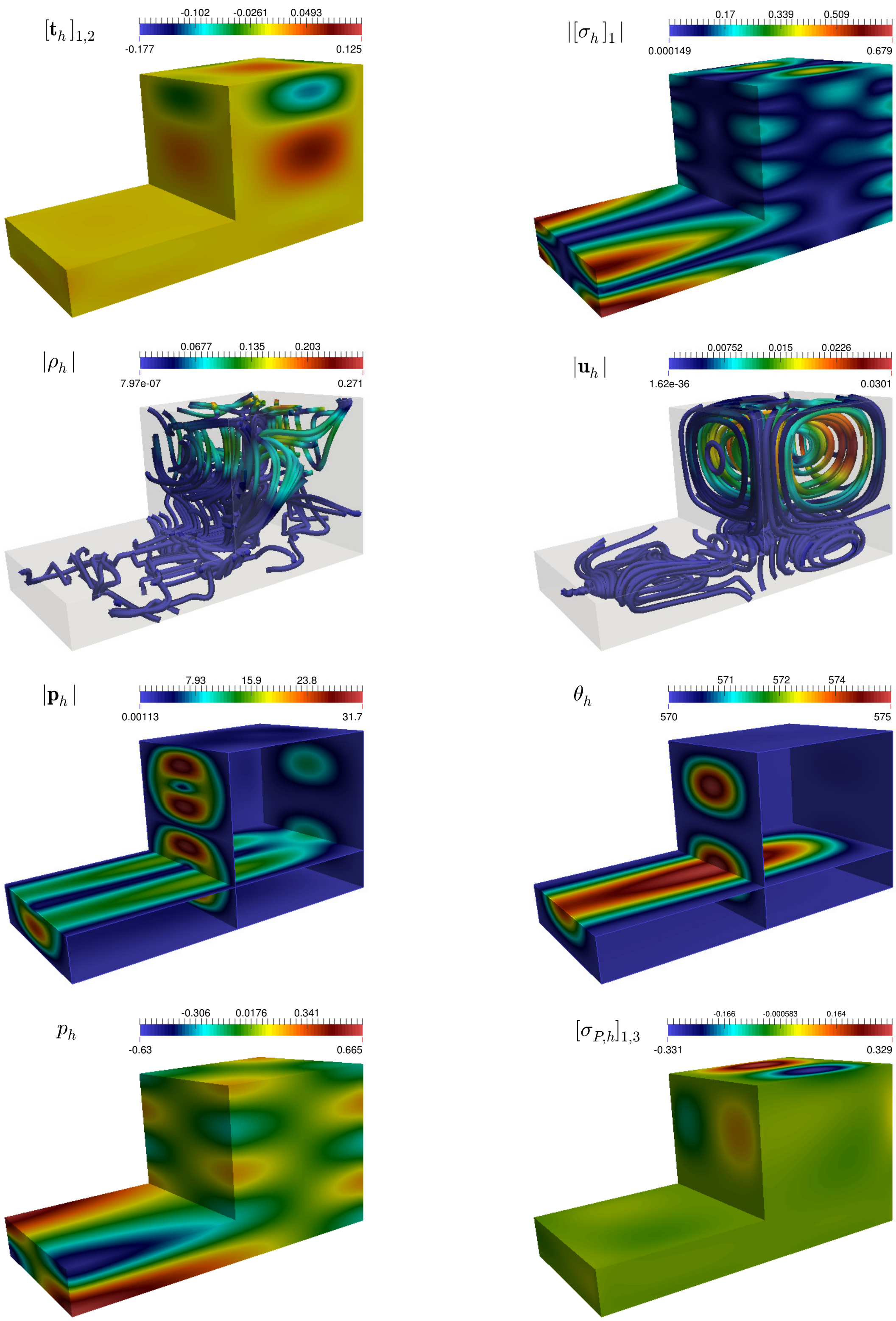

Figure 5.4: Example 4: $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ approximation of some components of the approximate solutions.

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