Second-order schemes for axisymmetric Navier-Stokes-Brinkman and transport equations modelling water filters

Graham Baird · Raimund Bürger · Paul E. Méndez · Ricardo Ruiz-Baier

June 24, 2019

Abstract Soil-based water filtering devices can be described by models of viscous flow in porous media coupled with equations for the transport of distinct contaminant species within water, and being susceptible to adsorption in the medium that represents soil. Such models are analysed mathematically, and suitable numerical methods for their approximate solution are designed. The governing equations are the Navier-Stokes-Brinkman equations for the flow of the fluid through a porous medium coupled with a convection-diffusion equation for the transport of the contaminants plus a system of ordinary differential equations accounting for the degradation of the adsorption properties of each contaminant. These equations are written in meridional axisymmetric form and the corresponding weak formulation adopts a mixed-primal structure. A second-order, (axisymmetric) divergence-conforming discretisation of this problem is introduced and the solvability, stability, and spatio-temporal convergence of the numerical method are analysed. Some numerical examples illustrate the main features of the problem and the properties of the numerical scheme.

Keywords Viscous flow in porous media · advection-diffusion-reaction equations · PDE-ODE coupling · mixed finite element methods · a priori error estimation

Mathematics Subject Classification (2010) 65M30 · 76S05 · 76R50.

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1 Introduction

1.1 Scope

We are interested in the analysis and numerical approximation of the flow of a viscous fluid through a porous medium, where it is assumed that the fluid carries a number m of components that are adsorbed by the porous medium. While viscous flow in porous media with adsorption arises in several applications including polymer flooding as part of the process of enhanced oil recovery in petroleum engineering [17], chromatography [34], or water decontamination and removal of pollutants such as heavy metals or radioactive ions [38], the particular formulation in the present work is motivated by a model of a soil-based water filtering device designed to remove contaminants from water by adsorption [31].

The governing equations for this process can be formulated as follows. We assume that the porous medium is represented by a simply connected spatial domain $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is split into three disjoints parts Γ^{in}, Γ^{wall} and Γ^{out} representing the inlet, walls, and outlet boundaries. For all times $0 < t \leq T$, we consider the Navier-Stokes-Brinkman equations written in terms of the volume average flow velocity $\boldsymbol{u}(t) : \Omega \to \mathbb{R}^3$ and the fluid pressure $p(t) : \Omega \to \mathbb{R}$; as well as the balances for contaminant concentration possessing sink terms that depend on the rate of degradation of the adsorption properties of each material, described in terms of the vector of concentrations of $m \geq 2$ distinct types of contaminants $\vec{\theta}(t) = (\theta_1(t), \dots, \theta_m(t)) : \Omega \to \mathbb{R}^m$ and of the adsorption capacity relative to each contaminant $\vec{s}(t) = (s_1(t), \dots, s_m(t)) : \Omega \to \mathbb{R}^m$. The coupled set of governing equations (three partial differential equations (PDEs) and one ordinary differential equation (ODE)) adopts the form

$$\rho_{\mathrm{f}}(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \mathbb{K}^{-1} \nu \boldsymbol{u} - \mathbf{div}(\nu \boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I}) = \boldsymbol{F}(\vec{\theta}), \tag{1.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.1b}$$

$$\phi \partial_t \vec{\theta} - \operatorname{div}(\mathbb{D} \nabla \vec{\theta}) + (\boldsymbol{u} \cdot \nabla)\vec{\theta} = -\rho_{\mathrm{b}} \partial_t \vec{s}, \tag{1.1c}$$

$$\partial_t \vec{s} = \boldsymbol{G}(\vec{s}, \vec{\theta}) \qquad \text{in } \Omega \times (0, T], \tag{1.1d}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the strain rate tensor, $\mathbb{D} = \operatorname{diag}(D_1(\boldsymbol{x}), \dots, D_m(\boldsymbol{x}))$ denotes a space-dependent and positive definite matrix containing diffusivity coefficients, $\nu > 0$ is the constant fluid viscosity, $\rho_{\mathrm{f}}, \rho_{\mathrm{b}}$ are the constant densities of the fluid phases and of the bulk filter medium, $\phi(\boldsymbol{x})$ is the porosity of the soil constituting the porous medium, and $\mathbb{K}(\boldsymbol{x}) > 0$ is the permeability tensor (assumed symmetric and uniformly positive definite). The source and reaction terms are

$$\boldsymbol{F}(\vec{\theta}) = \boldsymbol{g} \sum_{i=1}^{m} \theta_i; \quad G_i(s_i, \theta_i) = k_i^+(\boldsymbol{x})\big(s_i^{\max} - s_i\big)\theta_i, \quad i = 1, \dots, m, \tag{1.2}$$

where $\boldsymbol{G} = (G_1, \dots, G_m)^{\mathrm{T}}$, $\boldsymbol{g}$ is the gravity acceleration, $s_i^{\max}$ is a constant representing the maximum amount of contaminant i that can be absorbed at a given point, and $k_i^+(\boldsymbol{x})$ is a spatially-dependent modulation coefficient accounting for the forward adsorption rate related to the loss of contaminant i due to the filtering process (boundary conditions and further assumptions will be specified in later parts of the paper).

Thus, the flow of the incompressible fluid through Ω is modelled by the Navier-Stokes-Brinkman equation (1.1a) and the continuity equation (1.1b), which express the conservation of momentum and mass respectively. Equation (1.1c) describes the evolution of $\vec{\theta}$

within Ω, under the effects of advection and diffusion, in addition to adsorption by the filter media. Given the typical operating conditions within the filter, we would expect the effects of advection to dominate those from diffusion, as noted in [31]. The sink term $-\rho_{\mathrm{b}}\partial_t\vec{s}$ in (1.1c) accounts for the net and local removal of each contaminant type due to the filtration process. This adsorption process is described by a multicomponent Langmuir-type model, as given by (1.1d) and (1.2). Under this model, it is assumed that each site has a maximum capacity for each individual contaminant, which we take to be uniform across the two layers of filter media. In this way, the adsorption is noncompetitive and the saturation of a site by one contaminant does not prevent adsorption of the other contaminants at the same site. It is also assumed that the adsorption process is irreversible for all contaminants and all filter layers, so that once adsorbed the contaminants remain attached to the filter media with no desorption back into the fluid. As described previously, for each contaminant we ascribe a spatially dependent adsorption rate $k_i^+(\boldsymbol{x})$, so (1.2) stipulates that the rate of removal of a contaminant at a site is proportional to the concentration of the contaminant present in the fluid at the site, the remaining capacity of the filter media at the site and the adsorption rate.

While the modelling of a filter calls for a three-dimensional domain, in practice most filter designs display rotational symmetry around their central axis, with the flow also expected to exhibit such symmetry. This property motivates an axisymmetric formulation of the problem, allowing for the reduction from three to two spatial dimensions, which evidently reduces the computational cost associated with its solution. Thus, the model which is eventually analysed herein is a reformulation of (1.1) along with suitable initial and boundary conditions as a meridional axisymmetric PDE-ODE initial-boundary value problem. It is the purpose of this paper to advance a second-order divergence-conforming discretisation for this problem. Specifically, we introduce an axisymmetric $\boldsymbol{H}(\mathrm{div})$-conforming method based on two-dimensional Brezzi-Douglas-Marini (BDM) spaces [14] combined with an implicit, second-order backward differentiation formula (BDF2) for time discretization. Based on discrete stability properties, we prove that the discrete problem has at least one solution. At the core of this paper is the derivation of an optimal a priori error estimate for the numerical scheme, where the main difficulty is the fully discrete analysis verifying that each of the terms is bounded optimally in the corresponding weighted spaces. Numerical examples illustrate the model and reconfirm the theoretical order of accuracy.

1.2 Related work

To put the present work into the proper perspective, we mention that several studies treat the axisymmetric formulation of the Stokes and Navier-Stokes flows, including the discretisation employing spectral, mortar, and stabilized finite elements (see e.g. [7, 10, 11, 13, 23], and references cited in these works). More recently, mixed formulations of Brinkman flow including the numerical analysis of finite element (FE) approximations were studied. Anaya et al. [4] presented an augmented finite element approximation based on an extension to the vorticity-based Stokes problem was. A related recent model in [5] incorporates a stream function and vorticity formulation of axisymmetric Brinkman flow, for which a conforming mixed FE approximation is employed.

Papers concerning the coupling of flow and transport problems in a stationary and non-stationary setting include [19, 29, 39]. In [19] the authors present a FE method with projection-based stabilization for the double-diffusive convection in Darcy-Brinkman flow, and a FE error analysis and a convergence analysis are performed for the time-dependent case.

A time dependent Boussinesq model with nonlinear viscosity depending on the temperature is proposed in [2]. The authors analyze first and second order numerical schemes based on finite element methods and derive an optimal a priori error estimate for each numerical scheme. A related non-stationary phase-change Boussinesq model is presented in [39], where a second order finite element method for the primal formulation of the problem in terms of velocity, temperature, and pressure is constructed, and conditions for its stability are provided.

The coupling of advection-diffusion-reaction systems with Brinkman equations in their velocity-vorticity-pressure formulation, is studied in [29]. The equations are discretised in space using mixed FE methods on unstructured meshes, whereas the time integration hinges on an operator splitting strategy that uses the differences in scales between the reaction, advection, and diffusion processes. The authors compare several coupling strategies in terms of memory usage, iteration count, speed of calculation, and dynamics of the energy norm.

With respect to axisymmetric formulations, we mention that the numerical analysis of the axisymmetric Darcy and Stokes-Darcy flow using Raviar-Thomas (RT) and Brezzi-Douglas-Marini (BDM) finite elements was presented in [22,23]. In [22], the authors established the stability of the RT and BDM approximations for an axisymmetric Darcy flow problem by extending the Stenberg criteria, and they also derive a priori error estimates. A similar problem, addressing Brinkman flows coupled with a first-order transport-adsorption PDE, is approximated numerically in [17] by an $\boldsymbol{H}(\mathrm{div})$-conforming scheme in combination with DG method specifically tailored for discontinuous fluxes.

Other contributions to the design of numerical methods for axisymmetric formulations of coupled flow and transport problems include [3,15]. Furthermore, in [16] a semi-discrete discontinuous finite volume element (FVE) scheme is proposed and the unique solvability of both the nonlinear continuous problem and the semi-discrete counterpart is discussed. An FVE method is also proposed in [15] to discretise a Stokes equation for flow coupled with a parabolic equation modelling sedimentation. The method is based on a stabilized discontinuous Galerkin formulation for the concentration field, and a multiscale stabilized pair of $\mathcal{P}_1$-$\mathcal{P}_1$ elements for velocity and pressure, respectively. A mixed variational formulation of a Darcy-Forchheimer flow coupled with a energy equation is semi-discretised in [3] using Raviart-Thomas elements for fluxes and piecewise constant elements for the pressure, a posteriori error estimates are also established.

The technological application behind the water filter model goes back to the observation that it is possible to remove arsenic from water by passing it through iron-rich laterite soil [41,42]. The arsenic is removed through an adsorption process, which may be enhanced by chemically treating the laterite to increase its porosity and surface area, improving the adsorption efficiency [43]. Clearly, the formulation of accurate mathematical models of these filters, in addition to their efficient computational solution, would greatly aid in the development of improved filters and guidelines for their safe operation. The development and analysis of such a model forms the basis of the work [31], where the authors examined the removal of a single contaminant (arsenic; case $m = 1$ in our notation) in a cylindrical filter of uniform media. The authors utilised a Darcy-Brinkman equation, coupled with an advection-diffusion-adsorption equation to model the flow of the contaminated water through the filter and the removal of the arsenic through adsorption. In practice, however, there are likely $m > 1$ contaminants present, which calls for a filter consisting of multiple (up to m) layers in order to allow for their removal. In this work we attempt to study the filtration process in a soil-based water filter consisting of two distinct layers of differing media, in the presence of multiple contaminant species.

Problems of a similar nature abound in the literature. For example, [25] considers the numerical solution, via a finite volume method, of a double diffusive problem within a porous medium. The problem in question concerns the flow of a fluid within the porous medium, and the transport within that fluid of both heat and some particulate species (or secondary fluid constituent). The paper [35] considers a similar double diffusive problem, however, much like our proposed layered filter, the authors allow for the possibility of heterogeneous stratified porous media. While many of the studies concerning double diffusive problems consider entirely closed domains filled with porous media, a large number of application cases, such as our filter, feature partial enclosures with openings or infiltrations. The article [36] introduces such a feature, with the addition of 'free ports' to their model domain. Considering other potential variants, the authors of [40] extend the usual double diffusive problem by a first-order reaction process between the diffusing species and the fluid. This reaction process necessitates the addition of a sink term to the equation governing the species concentration that plays a role similar to that on the right-hand side of (1.1c).

1.3 Outline of the paper

The remainder of this paper is organized as follows. In Section 2 we introduce the model problem and state some preliminaries for its analysis, starting with a description of the initial and boundary conditions for (1.1) that correspond to the filter model (Section 2.1). Next, in Section 2.2, we reformulate (1.1) and the corresponding initial and boundary conditions in meridional axisymmetric form, which under suitable assumptions leads to model in two (namely, radial and vertical) space dimensions. We provide in Section 2.3 some preliminaries on functional spaces associated with radially symmetric functions. The weak (variational) formulation of the axisymmetric problem is stated in Section 2.4. Further assumptions on the model coefficients, as well as a number of inequalities related to the bilinear and trilinear forms involved in the weak formulation, are stated in Section 2.5. Section 3 outlines the well-posedness analysis (proof of existence and uniqueness of a weak solution) of the axisymmetric problem derived in Section 2.4. Section 4 is devoted to the description of the spatio-temporal discretisation of the axisymmetric model, starting by Section 4.1, where we introduce the basic triangulation of the computational domain and some notation. We then proceed to specify, in Section 4.2, the axisymmetric $\boldsymbol{H}(\mathrm{div})$-conforming method, where we first derive a semi-discrete (continuous in time) Galerkin formulation for the model problem, based on two-dimensional BDM spaces adapted to the axisymmtric setting, and then pass to a fully discrete scheme by applying a second-order time discretization through an implicit backward differentiation formula (BDF2). Next, in Section 4.3, we establish discrete stability properties of the bilinear and trilinear forms involved in the method. These properties allow us to prove (in Section 4.4) the existence of a discrete solution. Then, in Section 5, we prove an optimal a priori error estimate for the numerical scheme, where we verify that each of the terms is bounded optimally in the corresponding weighted space. Finally, in Section 6 we present numerical examples generated by the method introduced. Example 1 (Section 6.1) is an accuracy test with a manufactured known exact solution of (1.1) equipped with initial and boundary conditions. Results confirm that the method converges to the exact solution with the expected second-order rate. Next, in Example 2 (Section 6.2), numerical results are validated against experimen-

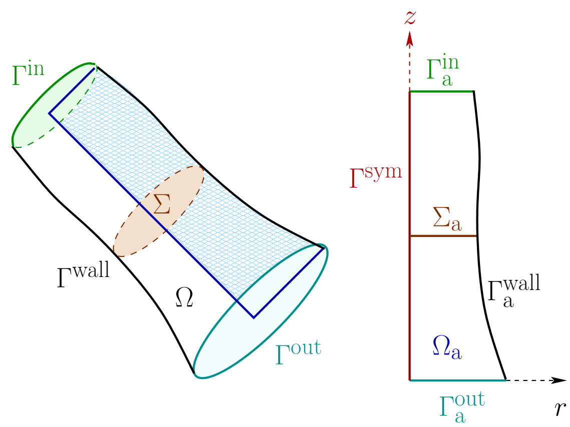

Fig. 1 Left: schematic representation of the domain Ω, its various boundaries Γ^{in}, Γ^{wall} and Γ^{out}, and the material interface Σ. Right: reduction to the axisymmetric configuration

tal data, and in Example 3 (Section 6.3) we solve the full two-layer, two-contaminant filter model.

2 Model problem and preliminaries

2.1 Initial and boundary conditions

Let us consider a porous skeleton consisting of two different materials separated by an interface, where the matrix is saturated with an incompressible interstitial fluid (see a diagrammatic representation on the left part of Figure 1). The coupled set of governing equations (1.1) is posed along with the initial and boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}^{\text{in}}, \quad \vec{\theta} = \vec{\theta}^{\text{in}} \qquad \text{on } \Gamma^{\text{in}} \times (0,T], \tag{2.1a}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \mathbb{D}\nabla\vec{\theta}\cdot\boldsymbol{n} = \vec{0} \qquad \text{on } \Gamma^{\text{wall}} \times (0,T], \tag{2.1b}$$

$$(\nu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I})\boldsymbol{n} = \boldsymbol{0}, \quad \mathbb{D}\nabla\vec{\theta}\cdot\boldsymbol{n} = \vec{0} \qquad \text{on } \Gamma^{\text{out}} \times (0,T], \tag{2.1c}$$

$$\vec{\theta}(0) = \vec{0}, \quad \boldsymbol{u}(0) = \vec{0}, \quad \vec{s}(0) = \vec{0} \qquad \text{in } \Omega. \tag{2.1d}$$

Condition (2.1a) indicates that the contaminated water enters the filter at Γ^{in} with a constant influx velocity, and each contaminant θ_i, $1 \le i \le m$ present at a fixed concentration θ_i^{in}; while condition (2.1c) accounts for zero normal stress and zero contaminant flux at the outlet. The system is preliminarily flushed with clean water and so there are no contaminants in the filter. Once the flow is at rest, we consider the initial conditions (2.1d).

The two distinct materials that compose the porous domain will have different permeability, porosity, as well as adsorption rate. Moreover, the diffusivities of the contaminants will vary from one type of porous structure to another. However it is important to remark that these differences in material properties, at least in the applications we address here, are not large enough to modify the flow regime between the two subdomains and this explains why (1.1a)–(1.1d) are defined on the whole domain Ω. Should this not be the case, one needs to solve explicitly for the coupling of Navier-Stokes/Brinkman or Brinkman/Darcy equations including suitable transmission conditions at the interface (see for instance [6,23] for formulations tailored to axisymmetric domains).

2.2 An axisymmetric formulation

Assuming that the data, the domain and the expected flow properties are all symmetric with respect to a given axis of symmetry denoted Γ^{sym}, we may rewrite the model equations in the *meridional* domain Ω_{a} (see the right part of Figure 1). In this case the velocity only possess radial and vertical components and we recall that the divergence operator in axisymmetric coordinates (in radial and height variables r, z) is

$$\mathrm{div}_{\mathrm{a}}\, \boldsymbol{v} := \partial_z v_z + \frac{1}{r}\partial_r(r v_r).$$

Then, making abuse of notation, we may rewrite system of PDES (1.1) as

$$\rho_{\mathrm{f}}(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \mathbb{K}^{-1}\nu\boldsymbol{u} - \mathbf{div}_{\mathrm{a}}(\nu\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p + \nu(\boldsymbol{u}_r/r^2)\boldsymbol{e}_1 = \boldsymbol{F}(\vec{\theta}), \tag{2.2a}$$

$$\mathrm{div}_{\mathrm{a}}\, \boldsymbol{u} = 0, \tag{2.2b}$$

$$\phi\partial_t\vec{\theta} - \mathrm{div}_{\mathrm{a}}(\mathbb{D}\nabla\vec{\theta}) + (\boldsymbol{u}\cdot\nabla)\vec{\theta} = -\rho_{\mathrm{b}}\partial_t\vec{s}, \tag{2.2c}$$

$$\partial_t\vec{s} = \boldsymbol{G}(\vec{s},\vec{\theta}) \qquad \text{for } (r,z,t)\in\Omega_{\mathrm{a}}\times(0,T], \tag{2.2d}$$

while the corresponding initial and boundary conditions (2.1) take the form

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{in}}, \quad \vec{\theta} = \vec{\theta}^{\mathrm{in}} \qquad \text{on } \Gamma^{\mathrm{in}}_{\mathrm{a}}\times(0,T], \tag{2.3a}$$

$$\boldsymbol{u} = \mathbf{0}, \quad \mathbb{D}\nabla\vec{\theta}\cdot\boldsymbol{n} = \vec{0} \qquad \text{on } \Gamma^{\mathrm{wall}}_{\mathrm{a}}\times(0,T], \tag{2.3b}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbb{D}\nabla\vec{\theta}\cdot\boldsymbol{n} = \vec{0} \qquad \text{on } \Gamma^{\mathrm{sym}}\times(0,T], \tag{2.3c}$$

$$(\nu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbb{I})\boldsymbol{n} = \mathbf{0}, \quad \mathbb{D}\nabla\vec{\theta}\cdot\boldsymbol{n} = \vec{0}, \qquad \text{on } \Gamma^{\mathrm{out}}_{\mathrm{a}}\times(0,T], \tag{2.3d}$$

$$\vec{\theta}(0) = \vec{0}, \quad \boldsymbol{u}(0) = \mathbf{0}, \quad \vec{s}(0) = \vec{0} \qquad \text{in } \Omega_{\mathrm{a}}, \tag{2.3e}$$

where the condition (2.3c) at the symmetry axis indicates slip velocity and zero normal fluxes.

2.3 Preliminaries on spaces of radially symmetric functions

For $\alpha\in\mathbb{R}$ and $1\le p<\infty$, let $L^p_\alpha(\Omega_{\mathrm{a}})$ denote the space of measurable functions v on Ω_{a} such that

$$\|v\|^p_{L^p_\alpha(\Omega_{\mathrm{a}})} := \int_{\Omega_{\mathrm{a}}} |v|^p r^\alpha \,\mathrm{d}r\,\mathrm{d}z \le \infty,$$

and let us denote the scalar product in $L^2_\alpha(\Omega_{\mathrm{a}})$ by $(\cdot,\cdot)_{\alpha,\Omega_{\mathrm{a}}}$. Moreover we introduce $H^q_\alpha(\Omega_{\mathrm{a}})$ as the space of functions in $L^p_\alpha(\Omega_{\mathrm{a}})$ whose derivatives up to order q are also in $L^p_\alpha(\Omega_{\mathrm{a}})$, and we denote by $H^q_{\alpha,j}(\Omega_{\mathrm{a}})$ its restriction to functions with null trace on a given portion Γ^j_{a} of the boundary. By $\boldsymbol{L}$ and $\mathbb{L}$ we denote the corresponding vectorial and tensorial counterparts of the scalar functional space L, we also will use $\vec{L}$ when the number of components of the vectorial space depends on m. Furthermore, the space $V_1^1(\Omega_\mathrm{a}) := H_1^1(\Omega_\mathrm{a}) \cap L^2_{-1}(\Omega_\mathrm{a})$ is endowed with the following norm and seminorm:

$$\|v\|_{V_1^1(\Omega_\mathrm{a})} := \left(\|v\|^2_{L_1^2(\Omega_\mathrm{a})} + |v|^2_{H_1^1(\Omega_\mathrm{a})} + \|v\|^2_{L^2_{-1}(\Omega_\mathrm{a})}\right)^{1/2},$$
$$|v|_{V_1^1(\Omega_\mathrm{a})} := \left(|v|^2_{H_1^1(\Omega_\mathrm{a})} + \|v\|^2_{L^2_{-1}(\Omega_\mathrm{a})}\right)^{1/2}.$$

Let us define the space

$$\boldsymbol{H}_0(\mathrm{div_a};\Omega_\mathrm{a}) := \left\{\boldsymbol{v} \in \boldsymbol{L}_1^2(\Omega_\mathrm{a}) : \mathrm{div_a}\,\boldsymbol{v} \in L_1^2(\Omega_\mathrm{a}) \quad \text{and} \quad \boldsymbol{v}|_{\partial\Omega_\mathrm{a}} \cdot \boldsymbol{n} = 0\right\},$$

endowed with the following norm

$$\|\boldsymbol{v}\|_{\mathrm{div_a},\Omega_\mathrm{a}} = \left(\|\boldsymbol{v}\|_{\boldsymbol{L}_1^2(\Omega_\mathrm{a})} + \|\mathrm{div_a}(\boldsymbol{v})\|_{L_1^2(\Omega_\mathrm{a})}\right)^{1/2}.$$

The essential boundary conditions (2.3a), (2.3b)$_1$, (2.3c)$_1$ suggest to employ the functional spaces

$$\boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_\mathrm{a}) := \left\{\boldsymbol{v} \in V_1^1(\Omega_\mathrm{a}) \times H_1^1(\Omega_\mathrm{a}) : \boldsymbol{v}|_{\Gamma_\mathrm{a}^\mathrm{in} \cup \Gamma_\mathrm{a}^\mathrm{wall}} = \boldsymbol{0} \ \text{ and } \ \boldsymbol{v}|_{\Gamma^\mathrm{sym}} \cdot \boldsymbol{n} = 0\right\},$$
$$\vec{H}^1_{1,\mathrm{in}}(\Omega_\mathrm{a}) = \left\{\vec{\psi} \in \vec{H}_1^1(\Omega_\mathrm{a}) : \vec{\psi}|_{\Gamma_\mathrm{a}^\mathrm{in}} = \vec{0}\right\}.$$

In what follows, to make notation more concise, we write L_1^2 instead of $L_1^2(\Omega_\mathrm{a})$, and proceed similarly for $\boldsymbol{V}_1^1(\Omega_\mathrm{a})$, $\vec{L}_1^2(\Omega_\mathrm{a})$, $\vec{H}_1^1(\Omega_\mathrm{a})$, and other spaces of functions defined on Ω_a as well as their corresponding norms. That is, in the remainder any space of functions and corresponding norm whose domain is not specified is understood to refer to functions defined on Ω_a.

2.4 Weak formulation of the axisymmetric problem

For a fixed $t > 0$, the weak (variational) formulation of problem (2.2), (2.3) is obtained after testing against suitable functions and applying integration by parts in axisymmetric coordinates; and it can be formulated as follows:

Find $(\boldsymbol{u}(t), p(t), \vec{\theta}(t), s(t)) \in \boldsymbol{V}_1^1 \times L_1^2 \times \vec{H}_1^1 \times L_1^2$ such that (2.3a) holds, and

$$\left(\rho_\mathrm{f}\partial_t\boldsymbol{u}(t),\boldsymbol{v}\right)_{1,\Omega_\mathrm{a}} + a_1\big(\boldsymbol{u}(t),\boldsymbol{v}\big) + c_1\big(\boldsymbol{u}(t);\boldsymbol{u}(t),\boldsymbol{v}\big) + b\big(\boldsymbol{v},p(t)\big) = d_1(\vec{\theta},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_\mathrm{a}), \tag{2.4a}$$

$$b\big(\boldsymbol{u}(t),q\big) = 0 \quad \text{for all } q \in L_1^2, \tag{2.4b}$$

$$\left(\phi\partial_t\vec{\theta}(t),\vec{\psi}\right)_{1,\Omega_\mathrm{a}} + a_2\big(\vec{\theta}(t),\vec{\psi}\big) + c_2\big(\boldsymbol{u}(t);\vec{\theta}(t),\vec{\psi}\big) + d_2\big(s(t);\vec{\theta}(t),\vec{\psi}\big) = 0 \quad \text{for all } \vec{\psi} \in \vec{H}^1_{1,\mathrm{in}}(\Omega_\mathrm{a}), \tag{2.4c}$$

$$\left(\partial_t\vec{s}(t),\vec{l}\right)_{1,\Omega_\mathrm{a}} + d_3\big(\vec{\theta}(t);\vec{s}(t),\vec{l}\big) - d_4\big(\vec{\theta}(t),\vec{l}\big) = 0 \quad \text{for all } l \in \vec{L}_1^2, \tag{2.4d}$$

where the bilinear, trilinear, and nonlinear forms are defined as follows for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}_1^1$, $q \in L_1^2$, $\vec{s},\vec{l} \in \vec{L}_1^2$, and $\vec{\theta},\vec{\psi} \in \vec{H}_1^1$:

$$a_1(\boldsymbol{u},\boldsymbol{v}) := \int_{\Omega_\mathrm{a}} \mathbb{K}^{-1}\nu\boldsymbol{u}\cdot\boldsymbol{v}r\,\mathrm{d}r\,\mathrm{d}z + \int_{\Omega_\mathrm{a}} \nu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})r\,\mathrm{d}r\,\mathrm{d}z + \int_{\Omega_\mathrm{a}} \frac{\nu}{r}u_r v_r\,\mathrm{d}r\,\mathrm{d}z,$$

$$a_2(\vec{\theta},\vec{\psi}) := \int_{\Omega_\mathrm{a}} \mathbb{D}\nabla\vec{\theta} : \nabla\vec{\psi} r \,\mathrm{d}r\,\mathrm{d}z, \quad b(\boldsymbol{v},q) := -\int_{\Omega_\mathrm{a}} q \,\mathrm{div}_\mathrm{a}\,\boldsymbol{v} r\,\mathrm{d}r\,\mathrm{d}z,$$
$$c_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) := \int_{\Omega_\mathrm{a}} \rho_\mathrm{f}(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} r\,\mathrm{d}r\,\mathrm{d}z, \quad c_2(\boldsymbol{v};\vec{\theta},\vec{\psi}) := \int_{\Omega_\mathrm{a}} (\boldsymbol{v}\cdot\nabla)\vec{\theta}\cdot\vec{\psi} r\,\mathrm{d}r\,\mathrm{d}z,$$
$$d_1(\vec{\psi},\boldsymbol{v}) := \int_{\Omega_\mathrm{a}} \boldsymbol{F}(\vec{\psi})\cdot\boldsymbol{v} r\,\mathrm{d}r\,\mathrm{d}z, \quad d_2(\vec{s};\vec{\theta},\vec{\psi}) := \int_{\Omega_\mathrm{a}} \sum_{i=1}^m (f(\boldsymbol{x},s_i)\theta_i\psi_i) r\,\mathrm{d}r\,\mathrm{d}z,$$
$$d_3(\vec{\psi};\vec{s},\vec{l}) := \int_{\Omega_\mathrm{a}} \sum_{i=1}^m g(\boldsymbol{x},\psi_i) s_i l_i r\,\mathrm{d}r\,\mathrm{d}z,$$
$$d_4(\vec{\psi},\vec{l}) := \int_{\Omega_\mathrm{a}} \sum_{i=1}^m g(\boldsymbol{x},\psi_i) s_i^{\max} l_i r\,\mathrm{d}r\,\mathrm{d}z.$$

2.5 Further assumptions and preliminaries

The permeability tensor $\mathbb{K} \in [C(\overline{\Omega_\mathrm{a}})]^{d\times d}$ is assumed symmetric and uniformly positive definite, hence its inverse satisfies

$$\boldsymbol{v}^\mathrm{T}\mathbb{K}^{-1}(\boldsymbol{x})\boldsymbol{v} \geq \alpha_1|\boldsymbol{v}|^2 \quad \text{for all } \boldsymbol{v}\in\mathbb{R}^d \text{ and } \boldsymbol{x}\in\Omega_\mathrm{a}, \text{ for a constant } \alpha_1>0.$$

We also require $\mathbb{D}$ to be positive definite, i.e.,

$$\vec{\psi}^\mathrm{T}\mathbb{D}\vec{\psi} \geq \alpha_2|\vec{\psi}|^2 \quad \text{for all } \vec{\psi}\in\mathbb{R}^m, \text{ for a constant } \alpha_2>0.$$

We assume there exist constants $f_1, f_2, g_1, g_2 > 0$ such that $f_1 \leq f(\boldsymbol{x},s) \leq f_2$, $g_1 \leq g(\boldsymbol{x},\theta) \leq g_2$, and that f and g are Lipschitz continuous and satisfy

$$\big|f(s_1)-f(s_2) \leq |f|_\mathrm{Lip}|s_1-s_2|, \quad \big|g(\theta_1)-g(\theta_2)\big| \leq |g|_\mathrm{Lip}|\theta_1-\theta_2|.$$

These assumptions imply that for all $\vec{s}_1,\vec{s}_2,\vec{s},\vec{l}\in\vec{L}_1^2$ and $\vec{\theta},\vec{\psi}\in\vec{H}_1^1$ such that $s_i^{\max}\leq s^{\max}$, there hold

$$d_2(\vec{s};\vec{\theta},\vec{\theta}) \geq f_1\|\vec{\theta}\|^2_{\vec{L}_1^2}, \tag{2.5}$$
$$d_2(\vec{s};\vec{\theta},\vec{\psi}) \leq f_2\|\vec{\theta}\|_{\vec{L}_1^2}\|\vec{\psi}\|_{\vec{L}_1^2}, \tag{2.6}$$
$$d_2(\vec{s}_2;\vec{\theta},\vec{\psi}) - d_2(\vec{s}_1;\vec{\theta},\vec{\psi}) \leq |f|_\mathrm{Lip}\|\vec{s}_2-\vec{s}_1\|_{\vec{L}_1^2}\|\vec{\theta}\|_{\vec{H}_1^1}\|\vec{\psi}\|_{\vec{H}_1^1}, \tag{2.7}$$
$$d_3(\vec{\psi};\vec{s},\vec{s}) \geq g_1\|\vec{s}\|^2_{\vec{L}_1^2}, \tag{2.8}$$
$$d_3(\vec{\psi};\vec{s},\vec{l}) \leq g_2\|\vec{s}\|_{\vec{L}_1^2}\|\vec{l}\|_{\vec{L}_1^2}, \tag{2.9}$$
$$d_4(\vec{\psi},\vec{l}) \leq g_2 s^{\max}\|\vec{l}\|_{\vec{L}_1^2} \leq C_d\|\vec{l}\|_{\vec{L}_1^2}. \tag{2.10}$$

If in addition $\vec{s}\in\vec{H}_1^1$, we also get

$$d_3(\vec{\theta}_2;\vec{s},\vec{l}) - d_3(\vec{\theta}_1;\vec{s},\vec{l}) \leq |g|_\mathrm{Lip}\|\vec{\theta}_2-\vec{\theta}_1\|_{\vec{H}_1^1}\|\vec{s}\|_{\vec{H}_1^1}\|\vec{l}\|_{\vec{L}_1^2}. \tag{2.11}$$

Due to the uniform boundedness of $\mathbb{K}^{-1}$ and $\mathbb{D}$, one can easily establish the following properties for all $\boldsymbol{u},\boldsymbol{v},\in\boldsymbol{V}_1^1$, $q\in L_1^2$, and $\vec{\theta},\vec{\psi}\in\vec{H}^1$:

$$\big|a_1(\boldsymbol{u},\boldsymbol{v})\big| \leq C_a\|\boldsymbol{u}\|_{\boldsymbol{V}_1^1}\|\boldsymbol{v}\|_{\boldsymbol{V}_1^1}, \tag{2.12a}$$

$$\left|a_2(\vec{\theta},\vec{\psi})\right| \le \hat{C}_a \|\vec{\theta}\|_{\vec{H}^1_1} \|\vec{\psi}\|_{\vec{H}^1_1}, \tag{2.12b}$$

$$\left|b(\boldsymbol{v},q)\right| \le \|\boldsymbol{v}\|_{\boldsymbol{V}^1_1} \|q\|_{L^2_1}, \tag{2.12c}$$

$$\left|d_1(\vec{\theta},\boldsymbol{v})\right| \le C_F \|\vec{\theta}\|_{\vec{H}^1_1} \|\boldsymbol{v}\|_{\boldsymbol{V}^1_1}. \tag{2.12d}$$

Moreover, thanks to the axisymmetric version of the well-known Sobolev embeddings (see [9, 30]), we have that for $\hat{p} \ge 1$,

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^{\hat{p}}_1} \le C^*_{\hat{p}} \|\boldsymbol{w}\|_{\boldsymbol{V}^1_1} \quad \text{for all } \boldsymbol{w} \in \mathbf{V}^1_1, \tag{2.13}$$

where the constant $C^*_{\hat{p}} > 0$ depends only upon $|\Omega_{\mathrm{a}}|$ and $\hat{p}$. Also, for $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1_1$ and $\vec{\theta},\vec{\psi} \in \vec{H}^1_1$, Hölder's inequality and (2.13) with $\frac{1}{\hat{p}} + \frac{1}{\hat{p}^*} = \frac{1}{2}$ imply that (see [18])

$$\begin{aligned}
\left|c_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\right| &\le C_v \|\boldsymbol{w}\|_{\boldsymbol{H}^1_1} \|\boldsymbol{u}\|_{\boldsymbol{H}^1_1} \|\boldsymbol{v}\|_{\boldsymbol{H}^1_1}, \\
\left|c_2(\boldsymbol{w};\vec{\theta},\vec{\psi})\right| &\le \bar{C}_v \|\boldsymbol{w}\|_{\boldsymbol{H}^1_1} \|\vec{\theta}\|_{\boldsymbol{H}^1_1} \|\vec{\psi}\|_{\vec{L}^3_1}, \\
\left|c_2(\boldsymbol{w};\vec{\theta},\vec{\psi})\right| &\le \hat{C}_v \|\boldsymbol{w}\|_{\boldsymbol{H}^1_1} \|\vec{\theta}\|_{\vec{H}^1_1} \|\vec{\psi}\|_{\vec{H}^1_1}.
\end{aligned}$$

Next, Poincaré's inequality and the positive definiteness of $\mathbb{D}$ readily imply the following coercivities (see [12, Chapter IX]):

$$a_1(\boldsymbol{v},\boldsymbol{v}) \ge \alpha_a \|\boldsymbol{v}\|^2_{\boldsymbol{V}^1_1} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}^1_{\mathrm{in,wall}}(\Omega_{\mathrm{a}}), \tag{2.14}$$

$$a_2(\vec{\psi},\vec{\psi}) \ge \hat{\alpha}_a \|\vec{\psi}\|^2_{\vec{H}^1_1} \qquad \text{for all } \vec{\psi} \in \vec{H}^1_{1,\mathrm{in}}(\Omega_{\mathrm{a}}). \tag{2.15}$$

We then proceed to characterise the kernel of the bilinear form $b(\cdot,\cdot)$ as

$$\begin{aligned}
\boldsymbol{X} :=& \left\{\boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}) : b(\boldsymbol{v},q) = 0 \text{ for all } q \in L^2_1\right\} \\
=& \left\{\boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}) : \mathrm{div}_{\mathrm{a}}\, \boldsymbol{v} = 0 \text{ a.e. in } \Omega_{\mathrm{a}}\right\},
\end{aligned}$$

and using integration by parts directly implies the relations (see [12, Section IX.2])

$$\begin{aligned}
&c_1(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) = 0 \quad \text{and} \quad c_2(\boldsymbol{w};\vec{\psi},\vec{\psi}) = 0 \\
&\text{for all } \boldsymbol{w} \in \boldsymbol{X},\ \boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}), \text{ and } \vec{\psi} \in \vec{H}^1_{1,\mathrm{in}}(\Omega_{\mathrm{a}}).
\end{aligned} \tag{2.16}$$

Note that for a given $\boldsymbol{w} \in \boldsymbol{X}$, property (2.14) together with (2.16) readily lead to the ellipticity of the bilinear form

$$a_1(\cdot,\cdot) + c_1(\boldsymbol{w},\cdot,\cdot) : \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}) \times \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}) \to \mathbb{R}.$$

Moreover, it is well known (i.e. [12, Proposition IX.1.1]) that an inf-sup condition holds for $b(\cdot,\cdot)$ in the following sense:

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}) \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}^1_1}} \ge \beta \|q\|_{L^2_1} \quad \text{for all } q \in L^2_1.$$

3 Well-posedness analysis of the continuous problem

This part of our analysis will be restricted to the case of no-slip velocity boundary conditions on the whole boundary. Then we introduce the spaces

$$H^1_{1,\diamond} := \{w \in H^1_1 : w = 0 \text{ on } \partial\Omega_{\mathrm{a}}\}, \quad V^1_{1,\diamond} := \{w \in V^1_1 : w = 0 \text{ on } \partial\Omega_{\mathrm{a}}\},$$

and $\boldsymbol{V}^1_{1,\diamond} := V^1_{1,\diamond} \times H^1_{1,\diamond}$.

From [28], we recall the weighted Sobolev inequality:

Lemma 3.1 *For all $v \in \boldsymbol{H}^1_1$ there holds*

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^4_1} \leq \hat{C}\|\boldsymbol{v}\|_{\boldsymbol{L}^2_1}|\boldsymbol{v}|_{\boldsymbol{H}^1_1}.$$

We will also use the following lemma (for its proof in the axysimmetric case we refer the reader to [12, Chapter IX]):

Lemma 3.2 *If $(\boldsymbol{u}, p, \vec{\theta}, \vec{s}) \in \boldsymbol{V}^1_{1,\diamond} \times L^2_1 \times \vec{H}^1_{1,\diamond} \times \vec{L}^2_1$ solves (2.4), then $\boldsymbol{u} \in \boldsymbol{X}$ is a solution of the following reduced problem:*

For all $t \in (0, T]$, find $(\boldsymbol{u}, \vec{\theta}, s) \in \boldsymbol{X} \times \vec{H}^1_{1,\diamond} \times \vec{L}^2_1$ such that

$$\begin{aligned}
&\big(\rho_{\mathrm{f}}\partial_t \boldsymbol{u}(t), \boldsymbol{v}\big)_{1,\Omega_{\mathrm{a}}} + a_1\big(\boldsymbol{u}(t), \boldsymbol{v}\big) \\
&\qquad + c_1\big(\boldsymbol{u}(t); \boldsymbol{u}(t), \boldsymbol{v}\big) = d_1(\vec{\theta}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}^1_{1,\mathrm{in,wall}}(\Omega_{\mathrm{a}}), && \text{(3.1a)}\\
&\big(\phi\partial_t \vec{\theta}(t), \vec{\psi}\big)_{1,\Omega_{\mathrm{a}}} + a_2\big(\vec{\theta}(t), \vec{\psi}\big) \\
&\qquad + c_2\big(\boldsymbol{u}(t); \vec{\theta}(t), \vec{\psi}\big) + d_2\big(\vec{s}(t); \vec{\theta}(t), \vec{\psi}\big) = 0 \quad \text{for all } \vec{\psi} \in \vec{H}^1_{1,\mathrm{in}}(\Omega_{\mathrm{a}}), && \text{(3.1b)}\\
&\big(\partial_t \vec{s}(t), \vec{l}\big)_{1,\Omega_{\mathrm{a}}} + d_3\big(\vec{\theta}(t); \vec{s}(t), \vec{l}\big) - d_4\big(\vec{\theta}(t), \vec{l}\big) = 0 \quad \text{for all } \vec{l} \in \vec{L}^2_1. && \text{(3.1c)}
\end{aligned}$$

Conversely, if $(\boldsymbol{u}, \vec{\theta}, \vec{s}) \in \boldsymbol{X} \times \vec{H}^1_{1,\diamond} \times \vec{L}^2_1$ is a solution of (3.1), then there exists a pressure $p \in L^2_1$ such that $(\boldsymbol{u}, p, \vec{\theta}, \vec{s})$ is a solution of (2.4).

A problem very similar to (2.4) but in Cartesian coordinates has been studied in [1]. There the authors establish existence of solution by the Galerkin method in combination with the Cauchy-Lipschitz theorem. The same ideas carry over to our case. For this we have to take into account that $\boldsymbol{F}$ is a Lipschitz-continuous function, and we also require to adapt the approach incorporating equivalent embedding theorems stated for weighted Sobolev spaces in [28], as well as weighted Poincaré-type inequalities available from [37, Section 4.3].

Theorem 3.1 *Assume that for $r \geq 4$,*

$$(\boldsymbol{u}, \vec{\theta}, s) \in L^2\big(0, T; \boldsymbol{X} \cap W^{1,r}_1(\Omega_{\mathrm{a}})\big) \times L^2(0, T; \vec{H}^1_{1,\diamond}) \times L^2\big(0, T; \vec{H}^1_1\big)$$

is a solution to problem (3.1). Then such solution is unique.

Proof. Throughout the proof, and for simplicity of the presentation, we assume that the model constants are scaled as $\phi, \rho_{\mathrm{b}}, \rho_{\mathrm{f}} = 1$. Let $(\boldsymbol{u}_1, \vec{\theta}_1, s_1)$ and $(\boldsymbol{u}_2, \vec{\theta}_2, s_2)$ be two solutions of (3.1). We denote

$$\boldsymbol{U} := \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad \vec{\Theta} := \vec{\theta}_1 - \vec{\theta}_2, \quad \text{and} \quad \vec{S} := \vec{s}_1 - \vec{s}_2.$$

Now, from (3.1b), by adding and subtracting $c_2(\boldsymbol{u}_2, \vec{\theta}_1, \vec{\Theta})$ and $d_2(\vec{s}_2, \vec{\theta}_1, \vec{\Theta})$; and using properties (2.16) and (2.7) we obtain

$$
\begin{aligned}
&(\partial_t \vec{\Theta}, \vec{\Theta})_{1,\Omega_\mathrm{a}} + a_2(\vec{\Theta}, \vec{\Theta}) \\
&= -c_2(\boldsymbol{U}; \vec{\theta}_1, \vec{\Theta}) - d_2(\vec{s}_2; \vec{\Theta}, \vec{\Theta}) - d_2(\vec{s}_1; \vec{\theta}_1, \vec{\Theta}) + d_2(\vec{s}_2, \vec{\theta}_1, \vec{\Theta}), \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\Theta}\|^2_{\vec{L}^2_1} + \alpha_2|\vec{\Theta}|^2_{\vec{H}^1_1} \leq \|\boldsymbol{U}\|^2_{\vec{L}^4_1}|\vec{\theta}_1|_{\vec{H}^1_1}\|\vec{\Theta}\|_{\vec{L}^4} + |f|_{\mathrm{Lip}}\|\vec{S}\|_{\vec{L}^2_1}\|\vec{\theta}_1\|_{\vec{H}^1_1}\|\vec{\Theta}\|_{\vec{L}^4_1}.
\end{aligned}
$$

By Lemma 3.1 and Young's inequality it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\Theta}\|^2_{\vec{L}^2_1} + \alpha_2|\vec{\Theta}|^2_{\vec{H}^1_1} \\
&\leq \frac{\hat{C}}{4}\left(\varepsilon_1|\boldsymbol{U}|^2_{\boldsymbol{H}^1_1} + \frac{1}{\varepsilon_1}|\vec{\theta}_1|_{\vec{H}^1_1}\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \varepsilon_2|\vec{\Theta}|^2_{\vec{H}^1_1} + \frac{1}{\varepsilon_2}|\vec{\theta}_1|^2_{\vec{H}^1_1}\|\vec{\Theta}\|^2_{\vec{L}^2_1}\right).
\end{aligned}
\tag{3.2}
$$

Now, selecting $\boldsymbol{v} = \boldsymbol{U}$ in (3.1a), adding and subtracting $c_1(\boldsymbol{u}_2; \boldsymbol{u}_1; \boldsymbol{U})$, and employing properties (2.16) and (2.12d), we can readily see that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \nu\|\boldsymbol{\varepsilon}(\boldsymbol{U})\|^2_{\mathbb{L}^2_1} + \nu\|\boldsymbol{U}_r\|^2_{L^2_{-1}} \leq \|\boldsymbol{U}\|^2_{\boldsymbol{L}^4_1}|\boldsymbol{u}_1|_{\boldsymbol{H}^1_1} + C_{\boldsymbol{F}}|\vec{\Theta}|_{\vec{L}^2_1}\|\boldsymbol{U}\|_{L^2_1}.
$$

Applying Lemma 3.1 and Young's inequality we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \alpha_a\|\boldsymbol{U}\|^2_{\boldsymbol{V}^1_1} \\
&\leq \frac{\hat{C}\varepsilon_3}{2}|\boldsymbol{U}|^2_{\boldsymbol{V}^1_1} + \frac{\hat{C}}{2\varepsilon_3}\|\boldsymbol{U}\|^2_{L^2_1}|\boldsymbol{u}_1|_{\boldsymbol{H}^1_1} + \frac{C_{\boldsymbol{F}}}{2}\big(|\vec{\Theta}|^2_{\vec{L}^2_1} + \|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1}\big).
\end{aligned}
\tag{3.3}
$$

In the same manner, from (3.1c), after adding and subtracting $d_3(\vec{\theta}_2; \vec{s}_1, \vec{S})$, using (2.8), (2.9), (2.10) and (2.11), we can assert that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{S}\|_{\vec{L}^2_1} + g_1\|\vec{S}\|_{\vec{L}^2_1} \\
&\leq \frac{|g|_{\mathrm{Lip}}}{2}\big(\|\vec{\Theta}\|^2_{L^2_1} + \|\vec{s}_1\|^2_{\vec{H}^1_1}\|\vec{S}\|^2_{L^2_1}\big) + \frac{|g|_{\mathrm{Lip}}s^{\max}}{2}\big(\|\vec{\Theta}\|^2_{\vec{L}^2_1} + \|\vec{S}\|^2_{\vec{L}^2_1}\big),
\end{aligned}
\tag{3.4}
$$

and choosing $\varepsilon_1 = 2\nu\alpha_a/\hat{C}$, $\varepsilon_2 = 2\alpha_2/\hat{C}$ and $\varepsilon_3 = \nu\alpha_a/\hat{C}$, we obtain from (3.2), (3.3) and (3.4) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \|\vec{\Theta}\|^2_{\vec{L}^2_1} + \|\vec{S}\|^2_{\vec{L}^2_1}\big) \\
&\leq C\big(|\boldsymbol{u}_1|^2_{\boldsymbol{H}^1_1} + |\vec{\theta}_1|^2_{\vec{H}^1_1} + \|\vec{\theta}_1\|^2_{\vec{L}^1_2} + \|\vec{s}_1\|^2_{\vec{H}^1_1} + 1\big)\big(\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \|\vec{\Theta}\|^2_{\vec{L}^2_1} + \|\vec{S}\|^2_{\vec{L}^2_1}\big).
\end{aligned}
$$

We may now integrate from $\tau = 0$ to $\tau = t$ to infer the bound

$$
\begin{aligned}
&\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \|\vec{\Theta}\|^2_{\vec{L}^2_1} + \|\vec{S}\|^2_{\vec{L}^2_1} \\
&\leq \int_0^t C\big(|\boldsymbol{u}_1|^2_{\boldsymbol{H}^1_1} + |\vec{\theta}_1|^2_{\vec{H}^1_1} + \|\vec{\theta}_1\|^2_{\vec{L}^1_2} + \|\vec{s}_1\|^2_{\vec{H}^1_1} + 1\big)\big(\|\boldsymbol{U}\|^2_{\boldsymbol{L}^2_1} + \|\vec{\Theta}\|^2_{\vec{L}^2_1} + \|\vec{S}\|^2_{\vec{L}^2_1}\big)\,\mathrm{d}\tau.
\end{aligned}
$$

Applying Gronwall's lemma, we now conclude that $\boldsymbol{U} = \boldsymbol{0}$, $\vec{\Theta} = \vec{0}$ and $\vec{S} = \vec{0}$. □

4 Spatio-temporal discretisation

4.1 Preliminaries

Let us denote by $\mathcal{T}_h$ a regular partition of Ω_a composed by triangular elements K of diameter h_K. The mesh size will be denoted by $h = \max\{h_K,\ K \in \mathcal{T}_h\}$, and for any interior edge e in $\mathcal{E}_h$, we will label K^- and K^+ the elements adjacent to it, while h_e will stand for the maximum diameter of the edge. We suppose that $\boldsymbol{v}, w$ are smooth vector and scalar fields defined over $\mathcal{T}_h$. Then, by $(\boldsymbol{v}^\pm, w^\pm)$ we will denote the traces of $(\boldsymbol{v}, w)$ on e being the extensions from the interiors of the elements K^+ and K^-, respectively. Let $\boldsymbol{n}_e$ denote the outward unit normal vector to e on K, we define the tangential component of $\boldsymbol{u}$ on each face e as $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n}_e)\boldsymbol{n}_e$. We introduce the average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators as follows:

$$\{\!\{\boldsymbol{v}\}\!\} = (\boldsymbol{v}^- + \boldsymbol{v}^+)/2, \quad \{\!\{w\}\!\} = (w^- + w^+)/2,$$
$$[\![\boldsymbol{v}]\!] = (\boldsymbol{v}^- - \boldsymbol{v}^+), \quad [\![w]\!] = (w^- - w^+),$$

whereas for boundary jumps and averages we adopt the convention that $\{\!\{\boldsymbol{v}\}\!\} = [\![\boldsymbol{v}]\!] = \boldsymbol{v}$, and $\{\!\{w\}\!\} = [\![w]\!] = w$. In addition, we will use the symbol ∇_h to denote the broken gradient operator and $\boldsymbol{\varepsilon}_h$ to denote its symmetrised counterpart.

4.2 An axisymmetric $\boldsymbol{H}(\mathrm{div})$-conforming method

First, we recall the definition of the two-dimensional Brezzi-Douglas-Marini (BDM) spaces (see e.g. [14]) locally on an element $K \in \mathcal{T}_h$, $\mathrm{BDM}_k(K) := (\mathcal{P}_k(K))^2$, where $\mathcal{P}_k(K)$ denotes the local space spanned by polynomials of degree up to k. In turn, related to the axisymmetric setting, as in [22] we define

$$\begin{aligned}\mathrm{BDM}_k^{\mathrm{axi}}(K) :=& \big\{\boldsymbol{v} \in \mathrm{BDM}_k(K) : \boldsymbol{v}\cdot\boldsymbol{n}_K|_{\Gamma^{\mathrm{sym}}} = 0\big\}\\ =& \big\{(v_r, v_z)^{\mathrm{T}} \in \mathrm{BDM}_k : v_r|_{\Gamma^{\mathrm{sym}}} = 0\big\},\end{aligned}$$

where the associated degrees of freedom are given by

$$\begin{aligned}\int_{\mathcal{E}_h} \boldsymbol{v}\cdot\boldsymbol{n}_K p r\,\mathrm{d}s, &\quad p \in R_k(\partial K) \quad \text{for } k \geq 0,\\ \int_K \boldsymbol{v}\cdot\nabla p r\,\mathrm{d}r\,\mathrm{d}z, &\quad p \in \mathcal{P}_{k-1}(K) \quad \text{for } k \geq 1,\\ \int_K \boldsymbol{v}\cdot\mathbf{curl}(b_K p) r\,\mathrm{d}r\,\mathrm{d}z, &\quad p \in \mathcal{P}_{k-2}(K) \quad \text{for } k \geq 2,\end{aligned}$$

where b_K denotes a bubble function on the element K and

$$R_k(\partial K) := \big\{\phi \in L^2(\partial K) : \phi|_e \in \mathcal{P}_k(e), e \in \mathcal{E}_h(K)\big\}.$$

Then, globally, for an integer k and a mesh $\mathcal{T}_h$ on Ω, we utilize the discrete spaces

$$\begin{aligned}\boldsymbol{\mathcal{H}}_h^k &:= \big\{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div}_\mathrm{a};\Omega_\mathrm{a}) : \boldsymbol{v}_h|_K \in \mathrm{BDM}_k^{\mathrm{axi}}(K) \text{ for all } K \in \mathcal{T}_h\big\},\\ \mathcal{Y}_h^k &:= \big\{q_h \in L^2_1(\Omega_\mathrm{a}) : q_h|_K \in \mathcal{P}_k(K) \text{ for all } K \in \mathcal{T}_h\big\},\\ \mathcal{M}_h^k &:= \big\{\psi_h \in C(\overline{\Omega_\mathrm{a}}) : \psi_h|_K \in \mathcal{P}_k(K) \text{ for all } K \in \mathcal{T}_h\big\},\end{aligned}$$

to define the following finite element subspaces for the approximation of the unknowns $\boldsymbol{u}, p, \vec{\theta}$ and $\vec{s}$, respectively, where the polynomial degree is $k \geq 1$:

$$\boldsymbol{V}_h := \boldsymbol{\mathcal{H}}_h^k \cap \boldsymbol{H}_0(\mathrm{div}_\mathrm{a}; \Omega_\mathrm{a}), \quad \mathcal{Q}_h := \mathcal{Y}_h^{k-1},$$
$$\vec{\mathcal{M}}_{h,0} := \vec{\mathcal{M}}_h^k \cap \vec{H}^1_{1,\mathrm{in}}(\Omega), \quad \vec{\mathcal{S}}_h := \vec{\mathcal{Y}}_h^{k-1}.$$

Let us recall that for axisymmetric cases the property $\mathrm{div}_\mathrm{a}\, \boldsymbol{V}_h \subseteq \mathcal{Q}_h$ is not preserved [23], and let us also recall from [22] the following discrete inf-sup condition for $b(\cdot,\cdot)$, where $\tilde{\beta}$ is independent of h:

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathcal{T}_h^1}} \geq \tilde{\beta} \|q_h\|_{L^2_{1,0}(\Omega_\mathrm{a})} \quad \text{for all } q_h \in \mathcal{Q}_h. \tag{4.1}$$

Associated with these finite-dimensional spaces, we state the following semi-discrete Galerkin formulation for problem (1.1), (2.1):

$$\begin{aligned}
&\text{For a fixed } t > 0, \text{ find } (\boldsymbol{u}_h(t), p_h(t), \vec{\theta}_h(t), \vec{s}_h(t)) \in \boldsymbol{V}_h \times \mathcal{Q}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h \\
&\text{such that for all } (\boldsymbol{v}_h, q_h, \vec{\psi}_h, \vec{l}_h) \in \boldsymbol{V}_h \times \mathcal{Q}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h: \\
&\big(\rho_\mathrm{f} \partial_t \boldsymbol{u}_h(t), \boldsymbol{v}\big)_{1,\Omega_\mathrm{a}} + a_1^h\big(\boldsymbol{u}_h(t), \boldsymbol{v}_h\big) \\
&\qquad + c_1^h\big(\boldsymbol{u}_h(t); \boldsymbol{u}_h(t), \boldsymbol{v}_h\big) + b\big(\boldsymbol{v}_h, p_h(t)\big) = d_1\big(\vec{\theta}_h(t), \boldsymbol{v}_h\big), \\
&b\big(\boldsymbol{u}_h(t), q_h\big) = 0, \\
&\big(\phi \partial_t \vec{\theta}_h(t), \vec{\psi}\big)_{1,\Omega_\mathrm{a}} + a_2\big(\vec{\theta}_h(t), \vec{\psi}_h\big) \\
&\qquad + c_2^h\big(\boldsymbol{u}_h(t); \vec{\theta}_h(t), \vec{\psi}_h\big) = d_2\big(\vec{s}_h(t); \vec{\theta}_h(t), \vec{\psi}_h\big), \\
&\big(\partial_t \vec{s}_h(t), \vec{l}_h\big)_{1,\Omega_\mathrm{a}} + d_3\big(\vec{\theta}_h(t); \vec{s}_h(t), \vec{l}_h\big) = d_4\big(\vec{\theta}_h(t), \vec{l}_h\big).
\end{aligned} \tag{4.2}$$

Here the discrete versions of the trilinear forms $a_1(\cdot,\cdot)$, $c_1(\cdot;\cdot,\cdot)$ and $c_2(\cdot;\cdot,\cdot)$ are defined using a symmetric interior penalty, an upwind approach and a skew-symmetric form, respectively (see e.g. [17,26,27]):

$$\begin{aligned}
a_1^h(\boldsymbol{u}, \boldsymbol{v}) &:= \int_{\Omega_\mathrm{a}} \Big(\mathbb{K}^{-1} \boldsymbol{u} \cdot \boldsymbol{v} + \nu \boldsymbol{\varepsilon}_h(\boldsymbol{u}) : \boldsymbol{\varepsilon}_h(\boldsymbol{v}) + \nu \frac{\boldsymbol{u}_r}{r} \frac{\boldsymbol{v}_r}{r}\Big)\, r \,\mathrm{d}r\,\mathrm{d}z \\
&\quad - \sum_{e \in \mathcal{E}_h} \int_e \Big(\{\!\{\nu \boldsymbol{\varepsilon}_h(\boldsymbol{u}) \boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{v}_{\boldsymbol{\tau}}]\!] - \{\!\{\nu \boldsymbol{\varepsilon}_h(\boldsymbol{v}) \boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!] \\
&\qquad\qquad + \frac{a_0}{h_e} \nu [\![\boldsymbol{u}_{\boldsymbol{\tau}}]\!] \cdot [\![\boldsymbol{v}_{\boldsymbol{\tau}}]\!] \Big) r \,\mathrm{d}s, \\
c_1^h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) &:= \frac{1}{2} \int_{\Omega_\mathrm{a}} \big((\boldsymbol{w} \cdot \nabla_h) \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{w} \cdot \nabla_h) \boldsymbol{v} \cdot \boldsymbol{u}\big)\, r \,\mathrm{d}r\,\mathrm{d}z \\
&\quad + \sum_{e \in \mathcal{E}_h} \int_e \hat{\boldsymbol{w}}^{\mathrm{up}}(\boldsymbol{u}) \cdot \boldsymbol{v} r \,\mathrm{d}s, \\
c_2^h(\boldsymbol{u}_h; \vec{\theta}_h, \vec{\psi}_h) &:= \frac{1}{2} \left(\int_{\Omega_\mathrm{a}} (\boldsymbol{v} \cdot \nabla_h) \vec{\theta} \cdot \vec{\psi} r \,\mathrm{d}r\,\mathrm{d}z - \int_{\Omega_\mathrm{a}} (\boldsymbol{v} \cdot \nabla_h) \vec{\psi} \cdot \vec{\theta} r \,\mathrm{d}r\,\mathrm{d}z \right),
\end{aligned}$$

where the fluxes are defined as

$$\hat{\boldsymbol{w}}^{\mathrm{up}}(\boldsymbol{u}) := \frac{1}{2}\big(\boldsymbol{w} \cdot \boldsymbol{n}_K - |\boldsymbol{w} \cdot \boldsymbol{n}_K|\big)(\boldsymbol{u}^e - \boldsymbol{u}),$$

and $\boldsymbol{u}^e$ denotes the trace of $\boldsymbol{u}$ taken from within the exterior of K.

We then proceed with the method of lines, and for the time discretisation we partition the interval $[0,T]$ into N subintervals $[t_{n-1},t_n]$ of length δ_t. We will use an implicit, second-order backward differentiation formula (BDF2). Starting from the interpolates $\boldsymbol{u}_h^0$, $\vec{\theta}_h^0$ and $\vec{s}_h^0$ of the initial data on$\boldsymbol{V}_h$, $\vec{\mathcal{M}}_{h,0}$ and $\vec{\mathcal{S}}_h$, respectively, we solve for $n=1,\dots,N-1$ the nonlinear system

$$\begin{aligned}
&\left(\boldsymbol{u}_h^{n+1}-\frac{4}{3}\boldsymbol{u}_h^n+\frac{1}{3}\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(d_1(\vec{\theta}_h^{n+1},\boldsymbol{v}_h)-a_1^h(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)-c_1^h(\boldsymbol{u}_h^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)-b(\boldsymbol{v}_h,p_h^{n+1})\big),\\
&b(\boldsymbol{u}_h^{n+1},q_h)=0,\\
&\left(\vec{\theta}_h^{n+1}-\frac{4}{3}\vec{\theta}_h^n+\frac{1}{3}\vec{\theta}_h^{n-1},\vec{\psi}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(-d_2(s_h^{n+1};\vec{\theta}_h^{n+1},\vec{\psi}_h)-a_2(\vec{\theta}_h^{n+1},\vec{\psi}_h)-c_2^h(\boldsymbol{u}_h^{n+1};\vec{\theta}_h^{n+1},\vec{\psi}_h)\big),\\
&\left(\vec{s}_h^{n+1}-\frac{4}{3}\vec{s}_h^n+\frac{1}{3}\vec{s}_h^{n-1},\vec{l}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(-d_3(\vec{\theta}_h^{n+1};\vec{s}_h^{n+1},\vec{l}_h)+d_4(\vec{\theta}_h^{n+1},\vec{l}_h)\big)
\end{aligned}\tag{4.3}$$

for all $\boldsymbol{v}_h\in\boldsymbol{V}_h, q_h\in\mathcal{Q}_h,\vec{\psi}_h\in\vec{\mathcal{M}}_h$ and $\vec{s}_h\in\vec{\mathcal{S}}_h$.

Then, in a way analogous to the continuous case, we define the discrete kernel

$$\boldsymbol{X}_h:=\big\{\boldsymbol{v}_h\in\boldsymbol{V}_h:b(\boldsymbol{v}_h,q_h)=0\text{ for all }q_h\in\mathcal{Q}_h\big\},$$

however we cannot obtain a characterisation analogous to the discrete case. Nevertheless, owing to the inf-sup condition (4.1), we can consider the following equivalent reduced problem:

$$\begin{aligned}
&\text{Find }(\boldsymbol{u}_h^{n+1},\vec{\theta}_h^{n+1},\vec{s}_h^{n+1})\in\boldsymbol{X}_h\times\mathcal{M}_{h,0}\times\vec{\mathcal{S}}_h\\
&\text{such that for all }\boldsymbol{v}_h\in\boldsymbol{V}_h,\vec{\psi}_h\in\vec{\mathcal{M}}_h\text{ and }\vec{l}_h\in\vec{\mathcal{S}}_h,\\
&\left(\boldsymbol{u}_h^{n+1}-\frac{4}{3}\boldsymbol{u}_h^n+\frac{1}{3}\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(d_1(\vec{\theta}_h^{n+1},\boldsymbol{v}_h)-a_1^h(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)-c_1^h(\boldsymbol{u}_h^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)\big),\\
&\left(\vec{\theta}_h^{n+1}-\frac{4}{3}\vec{\theta}_h^n+\frac{1}{3}\vec{\theta}_h^{n-1},\vec{\psi}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(-d_2(s_h^{n+1};\vec{\theta}_h^{n+1},\vec{\psi}_h)-a_2(\vec{\theta}_h^{n+1},\vec{\psi}_h)-c_2^h(\boldsymbol{u}_h^{n+1};\vec{\theta}_h^{n+1},\vec{\psi}_h)\big),\\
&\left(\vec{s}_h^{n+1}-\frac{4}{3}\vec{s}_h^n+\frac{1}{3}\vec{s}_h^{n-1},\vec{l}_h\right)_{1,\Omega_\mathrm{a}}\\
&=\frac{2}{3}\delta_t\big(-d_3(\vec{\theta}_h^{n+1};\vec{s}_h^{n+1},\vec{l}_h)+d_4(\vec{\theta}_h^{n+1},\vec{l}_h)\big).
\end{aligned}\tag{4.4}$$

4.3 Discrete stability properties

For the subsequent analysis, we introduce for $r \geq 0$ the broken $\boldsymbol{H}^r_\alpha(\mathcal{T}_h)$ space

$$\boldsymbol{H}^r_\alpha(\mathcal{T}_h) = \big\{\boldsymbol{v} \in \boldsymbol{L}^2_\alpha : \boldsymbol{v}|_K \in \boldsymbol{H}^r_\alpha(K), K \in \mathcal{T}_h\big\},$$

as well as the following parameter- and mesh- dependent broken norms

$$\begin{aligned}
\|\boldsymbol{v}\|^2_{*,\mathcal{T}_h} &:= \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\varepsilon}_h(\boldsymbol{v})\|^2_{\mathbb{L}^2_1(K)} + \sum_{K\in\mathcal{T}_h} \|v_r\|_{L^2_{-1}(K)} + \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \|[\![\boldsymbol{v}_{\boldsymbol{\tau}}]\!]\|^2_{\boldsymbol{L}^2_1(e)}, \\
\|\boldsymbol{v}\|^2_{\mathcal{T}^1_h} &:= \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_1(\Omega_\mathrm{a})} + \nu\|\boldsymbol{v}\|^2_{*,\mathcal{T}_h} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1_1(\mathcal{T}_h), \\
\|\boldsymbol{v}\|^2_{\mathcal{T}^2_h} &:= \|\boldsymbol{v}\|^2_{\mathcal{T}^1_h} + \sum_{K\in\mathcal{T}_h} h^2_K |\boldsymbol{v}|^2_{H^2_1(K)} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^2_1(\mathcal{T}_h),
\end{aligned}$$

where the stronger norm $\|\cdot\|^2_{\mathcal{T}^2_h}$ is used to show continuity. It can be proven that this norm is equivalent to $\|\cdot\|_{\mathcal{T}^1_h}$ on $\boldsymbol{H}^1_1(\mathcal{T}_h)$ (see [21] and [8]). Finally, adapting the argument used in [26, Proposition 4.5] and relying on the equivalent weighted Sobolev embeddings in [28] we have the following discrete Sobolev embedding: for $r = 2, 4$ there exists a constant $C_\mathrm{emb} > 0$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^r_1} \leq C_\mathrm{emb}\|\boldsymbol{v}\|_{\mathcal{T}^1_h} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1_1(\mathcal{T}_h). \tag{4.5}$$

Using these norms, we can establish continuity of the trilinear and bilinear forms involved, stated in the following lemma that can be proved following [32, Section 3.3.2], [22, Section 3] and [8, Section 4].

Lemma 4.1 *The following properties hold:*

$$\begin{aligned}
\big|a^h_1(\boldsymbol{u},\boldsymbol{v})\big| &\leq C\|\boldsymbol{u}\|_{\mathcal{T}^2_h}\|\boldsymbol{v}\|_{\mathcal{T}^1_h} && \text{for all } \boldsymbol{u} \in \boldsymbol{H}^2_1(\mathcal{T}_h),\ \boldsymbol{v} \in \boldsymbol{V}_h, \\
\big|a^h_1(\boldsymbol{u},\boldsymbol{v})\big| &\leq \tilde{C}_a\|\boldsymbol{u}\|_{\mathcal{T}^1_h}\|\boldsymbol{v}\|_{\mathcal{T}^1_h} && \text{for all } \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}_h, \\
\big|b(\boldsymbol{v},q)\big| &\leq \|\boldsymbol{v}\|_{\mathcal{T}^1_h}\|q\|_{L^2_1(\Omega_\mathrm{a})} && \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1_1(\mathcal{T}_h),\ q \in L^2_1(\Omega),
\end{aligned}$$

and for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1_1(\mathcal{T}_h)$ *and* $\vec{\psi},\vec{\theta} \in [H^1_1(\Omega)]^m$*, there holds*

$$\big|d_1(\vec{\theta},\boldsymbol{v})\big| \leq C_{\boldsymbol{F}}\|\vec{\theta}\|_{\vec{H}^1_1}\|\boldsymbol{v}\|_{\mathcal{T}^1_h}, \tag{4.6a}$$

$$\big|c^h_2(\boldsymbol{w};\vec{\theta},\vec{\psi})\big| \leq \tilde{C}\|\boldsymbol{w}\|_{\mathcal{T}^1_h}\|\vec{\psi}\|_{\vec{H}^1_1}\|\vec{\theta}\|_{\vec{H}^1_1}. \tag{4.6b}$$

Note that while the coercivity of the form $a_2(\cdot,\cdot)$ in the discrete setting is readily implied by (2.15), there also holds (cf. [27, Lemma 3.2])

$$a^h_1(\boldsymbol{v},\boldsymbol{v}) \geq \tilde{\alpha}_a\|\boldsymbol{v}\|^2_{\mathcal{T}^1_h} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_h, \tag{4.7}$$

provided that $a_0 > 0$ is sufficiently large and independent of the mesh size.

Let $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div}^0;\Omega)$, due to the skew-symmetric form of the operators c^h_1 and c^h_2, and the positivity of the non-linear upwind term of c^h_1 (see e.g. [33]), we can write

$$c^h_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) \geq 0 \quad \text{for all } \boldsymbol{u} \in \boldsymbol{V}_h, \tag{4.8}$$

$$c^h_2(\boldsymbol{w};\vec{\psi}_h,\vec{\psi}_h) = 0 \quad \text{for all } \vec{\psi}_h \in \mathcal{M}_h, \tag{4.9}$$

as well as the following relation (which is based on (4.5) and follows by the same method as in [20, 26]):

$$\begin{aligned}
&\text{For any } \boldsymbol{w}_1,\boldsymbol{w}_2,\boldsymbol{u} \in \boldsymbol{H}^2_1(\mathcal{T}_h) \text{ there holds for all } \boldsymbol{v} \in \boldsymbol{V}_h \\
&\big|c^h_1(\boldsymbol{w}_1;\boldsymbol{u},\boldsymbol{v})\big| - \big|c^h_1(\boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})\big| \leq \tilde{C}_c\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\mathcal{T}^1_h}\|\boldsymbol{v}\|_{\mathcal{T}^1_h}\|\boldsymbol{u}\|_{\mathcal{T}^1_h}.
\end{aligned} \tag{4.10}$$

4.4 Existence of discrete solutions

In what follows we will use the following algebraic relation: for any real numbers a^{n+1}, a^n, a^{n-1} and defining $\Lambda a^n := a^{n+1} - 2a^n + a^{n-1}$, we have

$$\begin{aligned} 2(3a^{n+1} - 4a^n + a^{n-1}, a^n) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\Lambda a^n|^2 \\ - |a^n|^2 - |2a^n - a^{n-1}|^2. \end{aligned} \tag{4.11}$$

Theorem 4.1 *Let $(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}) \in \boldsymbol{X}_h \times \vec{\mathcal{M}}_{h,0} \times \mathcal{S}_h$ be a solution of problem (4.4). Then the following bounds are satisfied, where C_1, C_2 and C_3 are constants independent of h and δ_t:*

$$\begin{aligned} &\|\boldsymbol{u}_h^{n+1}\|^2_{\boldsymbol{L}^2_1} + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2_1} + \sum_{j=1}^n \|\Lambda \boldsymbol{u}_h^j\|^2_{\boldsymbol{L}^2_1} + \sum_{j=1}^n \delta_t \|\boldsymbol{u}_h^{j+1}\|^2_{\mathcal{T}_h^1} \\ &\le C_1\big(\|\vec{\theta}_h^1\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|^2_{\vec{L}^2_1} + \|\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2_1} + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|^2_{\boldsymbol{L}^2_1}\big), \\ &\|\vec{\theta}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^{n+1} - \vec{\theta}_h^n\|^2_{\vec{L}^2_1} + \sum_{j=1}^n \|\Lambda \vec{\theta}_h^j\|^2_{\vec{L}^2_1} + \sum_{j=1}^n \delta_t |\vec{\theta}_h^{j+1}|^2_{\vec{L}^2_1} \\ &\le C_2\big(\|\vec{\theta}_h^1\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|^2_{\vec{L}^2_1}\big), \\ &\|\vec{s}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^{n+1} - \vec{s}_h^n\|^2_{\vec{L}^2_1} + \sum_{j=1}^n \|\Lambda \vec{s}_h^j\|^2_{\vec{L}^2_1} \\ &\le C_3\big(\|\vec{\theta}_h^1\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|^2_{\vec{L}^2_1} + \|\vec{s}_h^1\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^1 - \vec{s}_h^0\|^2_{\vec{L}^2_1} + n\delta_t C_d^2\big). \end{aligned} \tag{4.12}$$

Proof. First we take $\vec{\psi}_h = 4\vec{\theta}_h^{n+1}$ in the second equation of (4.4) and use properties (2.5), (4.9) and relation (4.11) to deduce the inequality

$$\begin{aligned} &\|\vec{\theta}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^{n+1} - \vec{\theta}_h^n\|^2_{\vec{L}^2_1} + \|\Lambda \vec{\theta}_h^n\|^2_{\vec{L}^2_1} + 4\alpha_2 \delta_t |\vec{\theta}_h^{n+1}|^2_{H^1_1} \\ &\le \|\vec{\theta}_h^n\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^n - \vec{\theta}_h^{n-1}\|^2_{\vec{L}^2_1}. \end{aligned}$$

Hence, summing over n, we get

$$\begin{aligned} &\|\vec{\theta}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^{n+1} - \vec{\theta}_h^n\|^2_{\vec{L}^2_1} + \sum_{j=1}^n \|\Lambda \vec{\theta}_h^j\|^2_{\vec{L}^2_1} + 4\alpha_2 \sum_{j=1}^n \delta_t |\vec{\theta}_h^{j+1}|^2_{H^1_1} \\ &\le \|\vec{\theta}_h^1\|^2_{\vec{L}^2_1} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|^2_{\vec{L}^2_1}. \end{aligned} \tag{4.13}$$

Similarly, in the third equation of (4.4), we take $\vec{l}_h = 4\vec{s}_h^{n+1}$ and apply (2.10), (2.8) together with Young's inequality to get

$$\begin{aligned} &\|\vec{s}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^{n+1} - \vec{s}_h^n\|^2_{\vec{L}^2_1} + \|\Lambda \vec{s}_h^n\|^2_{\vec{L}^2_1} \\ &\le 4\delta_t C_d \|\vec{\theta}_h^{n+1}\|_{\vec{L}^2_1} + \|\vec{s}_h^n\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^n - \vec{s}_h^{n-1}\|^2_{\vec{L}^2_1} \\ &\le 2\delta_t C_p |\vec{\theta}_h^{n+1}|^2_{\vec{H}^1_1} + 2C_d^2 \delta_t + \|\vec{s}_h^n\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^n - \vec{s}_h^{n-1}\|^2_{\vec{L}^2_1}. \end{aligned}$$

Summing over n we therefore obtain

$$\|\vec{s}_h^{n+1}\|^2_{\vec{L}^2_1} + \|2\vec{s}_h^{n+1} - \vec{s}_h^n\|^2_{\vec{L}^2_1} + \sum_{j=1}^n \|\Lambda \vec{s}_h^j\|^2_{\vec{L}^2_1}$$

$$\leq 2C_p \sum_{j=1}^{n} \delta_t |\vec{\theta}_h^{j+1}|^2_{\vec{H}_1^1} + 2n\delta_t C_d^2 + \|\vec{s}_h^1\|^2_{\vec{L}_1^2} + \|2\vec{s}_h^1 - \vec{s}_h^0\|^2_{\vec{L}_1^2}. \tag{4.14}$$

We get the second result of (4.12) by replacing (4.13) in (4.14). Finally we take $\boldsymbol{v}_h = 4\boldsymbol{u}_h^{n+1}$ in the first equation of (4.4) and apply (4.11), (4.6a), (4.7) and (4.8) to deduce the estimate

$$\begin{aligned}&\|\boldsymbol{u}_h^{n+1}\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \|\Lambda \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + 4\delta_t \tilde{\alpha}_a \|\boldsymbol{u}_h^{n+1}\|^2_{\mathcal{T}_h^1}\\ &\leq 4\delta_t C_{\boldsymbol{F}} \|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2} \|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2} + \|\boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{\boldsymbol{L}_1^2}.\end{aligned}$$

Now we use Young's inequality with $\varepsilon = \tilde{\alpha}_a$ to arrive at

$$\begin{aligned}&\|\boldsymbol{u}_h^{n+1}\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \|\Lambda \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \delta_t 2\tilde{\alpha}_a \|\boldsymbol{u}_h^{n+1}\|^2_{\mathcal{T}_h^1}\\ &\leq 2\frac{C_{\boldsymbol{F}}^2 C_p}{\tilde{\alpha}_a} \delta_t |\vec{\theta}_h^{n+1}|^2_{\vec{H}_1^1} + \|\boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2_{\boldsymbol{L}_1^2},\end{aligned}$$

and summing over n we can assert that

$$\begin{aligned}&\|\boldsymbol{u}_h^{n+1}\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}_1^2} + \sum_{j=1}^{n} \|\Lambda \boldsymbol{u}_h^j\|^2_{\boldsymbol{L}_1^2} + 2\tilde{\alpha}_a \sum_{j=1}^{n} \delta_t \|\boldsymbol{u}_h^{j+1}\|^2_{\mathcal{T}_h^1}\\ &\leq \frac{C_{\boldsymbol{F}}^2 C_p}{2} \sum_{j=1}^{n} \delta_t |\vec{\theta}_h^{j+1}|^2_{\vec{H}_1^1} + \|\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|^2_{\boldsymbol{L}_1^2}.\end{aligned} \tag{4.15}$$

Finally we get the first result in (4.12) from the bounds (4.13) and (4.15). □

Theorem 4.2 *Assume that*

$$\frac{C_{\boldsymbol{F}}}{\tilde{\alpha}_a} \leq \frac{\alpha_2}{C_p}. \tag{4.16}$$

Then problem (4.3) admits at least one solution

$$(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}) \in \boldsymbol{V}_h \times \mathcal{Q}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h.$$

Proof. To simplify the proof we introduce the following constants:

$$\begin{aligned}C_u &:= C_1\big(\|\vec{\theta}_h^1\|_{\vec{L}_1^2} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|_{\vec{L}_1^2} + \|\boldsymbol{u}_h^1\|_{\boldsymbol{L}_1^2} + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{\boldsymbol{L}_1^2}\big),\\ C_\theta &:= C_2\big(\|\vec{\theta}_h^1\|_{\vec{L}_1^2} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|_{\vec{L}_1^2}\big),\\ C_s &:= C_3\big(\|\vec{\theta}_h^1\|_{\vec{L}_1^2} + \|2\vec{\theta}_h^1 - \vec{\theta}_h^0\|_{\vec{L}_1^2} + \|\vec{s}_h^1\|_{\vec{L}_1^2} + \|2\vec{s}_h^1 - \vec{s}_h^0\|_{\vec{L}_1^2} + n\delta_t C_d^2\big).\end{aligned}$$

We shall make use of Brouwer's fixed-point theorem in the form given by [24, Corollary 1.1, Chapter IV]:

Theorem 4.3 (Brouwer's fixed-point theorem) *Let H be a finite-dimensional Hilbert space with scalar product denoted by $(,)_H$ and corresponding norm $\|\cdot\|_H$. Let $\Phi : H \to H$ be a continuous mapping for which there exists $\mu > 0$ such that $(\Phi(u), u)_H \geq 0$ for all $u \in H$ with $\|u\|_H = \mu$. Then there exists an element $u \in H$ such that $\Phi(u) = 0, \|u\|_H \leq \mu$.*

We proceed by induction on $n \geq 2$. We define the mapping

$$\varPhi : \boldsymbol{X}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h \to \boldsymbol{X}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h$$

using the relation

$$\begin{aligned}
&\big(\varPhi(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}), (\boldsymbol{v}_h, \vec{\psi}_h, \vec{l}_h)\big)_{1,\Omega_\mathrm{a}} \\
&= \frac{1}{2\delta_t}\big(3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^{n} + \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h\big)_{1,\Omega_\mathrm{a}} + a_1^h\big(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) + c_1^h\big(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) \\
&\quad - \big(\boldsymbol{F}\big(\vec{\theta}_h^{n+1}\big), \boldsymbol{v}_h\big)_{1,\Omega_\mathrm{a}} + \frac{1}{2\delta_t}\big(3\vec{\theta}_h^{n+1} - 4\vec{\theta}_h^{n} + \vec{\theta}_h^{n-1}, \vec{\psi}_h\big)_{1,\Omega_\mathrm{a}} + a_2\big(\vec{\theta}_h^{n+1}, \vec{\psi}_h\big) \\
&\quad + c_2^h\big(\boldsymbol{u}_h^{n+1}; \vec{\theta}_h^{n+1}, \vec{\psi}_h\big) + d_2\big(\vec{s}_h^{n+1}; \vec{\theta}_h^{n+1}, \vec{\psi}_h\big) \\
&\quad + \frac{1}{2\delta_t}\big(3\vec{s}_h^{n+1} - 4\vec{s}_h^{n} + \vec{s}_h^{n-1}, \vec{l}_h\big)_{1,\Omega_\mathrm{a}} + d_3\big(\vec{\theta}_h^{n+1}; \vec{s}_h^{n+1}, \vec{l}_h\big) - d_4\big(\vec{\theta}_h^{n+1}, \vec{l}_h\big).
\end{aligned}$$

Note this map is well-defined and continuous on $\boldsymbol{X}_h \times \vec{\mathcal{M}}_{h,0} \times \vec{\mathcal{S}}_h$. On the other hand, if we take

$$(\boldsymbol{v}_h, \vec{\psi}_h, \vec{l}_h) = (\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}),$$

and employ (4.8), (4.9), (4.6a) and (4.7), we obtain

$$\begin{aligned}
&\big(\varPhi(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}), (\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1})\big)_{1,\Omega_\mathrm{a}} \\
&\geq -\frac{1}{2\delta_t}\|4\boldsymbol{u}_h^{n} - \boldsymbol{u}_h^{n-1}\|_{\boldsymbol{L}_1^2}\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2} + \tilde{\alpha}_a\|\boldsymbol{u}_h^{n+1}\|_{\mathcal{T}_h^1}^2 - C_{\boldsymbol{F}}\|\theta_h^{n+1}\|_{\vec{L}_1^2}\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2} \\
&\quad - \frac{1}{2\delta_t}\|4\vec{\theta}_h^{n} - \vec{\theta}_h^{n-1}\|_{\vec{L}_1^2}\|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2} + \alpha_2|\vec{\theta}^{n+1}|_{\vec{H}_1^1}^2 + \frac{3}{2\delta_t}\|\vec{s}_h^{n+1}\|_{L_2^1}^2 \\
&\quad - \frac{1}{2\delta_t}\|4\vec{s}_h^{n} - \vec{s}_h^{n-1}\|_{\vec{L}_1^2}\|\vec{s}_h^{n+1}\|_{\vec{L}_1^2} - C_d\|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2}.
\end{aligned}$$

Next, using (4.12), inequality (4.16) and Young's inequality with constant $\varepsilon_1 = \tilde{\alpha}_a/C_{\boldsymbol{F}}$, we deduce that

$$\begin{aligned}
&\big(\varPhi(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}), (\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1})\big)_{1,\Omega_\mathrm{a}} \\
&\geq \frac{\tilde{\alpha}_a}{2}\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2}^2 + \frac{3}{2\delta_t}\|\vec{s}_h^{n+1}\|_{\vec{L}_1^2}^2 + \frac{\alpha_2}{2Cp}\|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2}^2 - \frac{5}{2\delta_t}C_u\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2(\Omega_\mathrm{a})} \\
&\quad - \left(\frac{5}{2\delta_t}C_\theta + C_d\right)\|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2} - \frac{5}{2\delta_t}C_s\|\vec{s}_h^{n+1}\|_{\vec{L}_1^2}.
\end{aligned}$$

Then, setting

$$C_R = \min\left\{\frac{\tilde{\alpha}_a}{2}, \frac{3}{2\delta_t}, \frac{\alpha_2}{2C_p}\right\}, \quad C_r = 2\max\left\{\frac{5}{2\delta_t}C_u, \frac{5}{2\delta_t}C_\theta + C_d, \frac{5}{2\delta_t}C_s\right\},$$

we may apply the inequality $a + b \leq \sqrt{2}(a^2+b^2)^{1/2}$, valid for all $a, b \in \mathbb{R}$, to obtain

$$\begin{aligned}
&\big(\varPhi(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}), (\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1})\big)_{1,\Omega_\mathrm{a}} \\
&\geq C_R\big(\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2}^2 + \|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2}^2 + \|\vec{s}_h^{n+1}\|_{\vec{L}_1^2}^2\big) \\
&\quad - C_r\big(\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{L}_1^2}^2 + \|\vec{\theta}_h^{n+1}\|_{\vec{L}_1^2}^2 + \|\vec{s}_h^{n+1}\|_{\vec{L}_1^2}^2\big)^{1/2}.
\end{aligned}$$

Hence, the right-hand side is nonnegative on a sphere of radius $r := C_r/C_R$. Consequently, by Theorem 4.3, there exists a solution to the fixed-point problem

$$\Phi(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \vec{s}_h^{n+1}) = 0.$$

The existence of p_h^{n+1} satisfying (4.3) is guaranteed by (4.1). □

Note that unlike conforming discretisations, one cannot directly establish a discrete version of Theorem 3.1. In fact we were not able to control the discrete norms arising from (4.10), which would be necessary to establish a discrete counterpart of (3.3). However, even when uniqueness of the discrete counterpart remains an open problem, our non-exhaustive selection of numerical examples did not present any difficulties in this regard.

5 A priori error analysis

The following development follows the structure adopted in [2]. We start by recalling some interpolation results from [10] and [22].

Lemma 5.1 *Let $\mathcal{L}_h$ be the Lagrange interpolation operator $\mathcal{L}_h : C^0(\Omega_{\mathrm{a}}) \to V_h$, where V_h denotes the space of Lagrange finite elements of order k. We also consider its vectorial counterpart, keeping the same notation. Then for all l and for all p such that $1 \leq l \leq k+1$, $1 \leq p \leq +\infty$, $l > \frac{3}{p}$ or $p = 1, l = 3$ there exists a constant $C^* > 0$ independent of h, such that for all $v \in W_1^{l,p}(\Omega_{\mathrm{a}})$, the following inequalities hold;*

$$\|v - \mathcal{L}_h\, v\|_{L_1^p(\Omega_{\mathrm{a}})} \leq C^* h^l |v|_{W_1^{l,p}(\Omega_{\mathrm{a}})}, \qquad |v - \mathcal{L}_h\, v|_{L_1^p(\Omega_{\mathrm{a}})} \leq C^* h^{l-1} |v|_{W_1^{l,p}(\Omega_{\mathrm{a}})}.$$

Lemma 5.2 *Let Π_h be the $\mathrm{BDM}_k^{\mathrm{axi}}$ interpolation operator $\Pi_h : C^0(\Omega_{\mathrm{a}}) \to \boldsymbol{\mathcal{H}}_h^k$. Then for all $v \in H_1^{k+1}(\Omega_{\mathrm{a}})$, the following inequalities hold:*

$$\|v - \Pi_h\, v\|_{L_1^2(\Omega_{\mathrm{a}})} \leq C^* h^{k+1} |v|_{H_1^{k+1}(\Omega_{\mathrm{a}})}, \qquad \|v - \Pi_h\, v\|_{\mathcal{T}_h^1} \leq C^* h^k \|v\|_{H_1^{k+1}(\Omega_{\mathrm{a}})}.$$

Proof. The first result comes from [22, Corollary A.7]. The proof of the second result comes much in the same way as in the Cartesian case, by making use of the equivalent weighted inverse inequalities and weighted approximation properties proved in [10], see [23, Section 3.1] and [8]. □

Lemma 5.3 *Let $\mathcal{I}_h$ denote the modified Clément interpolation operator*

$$\mathcal{I}_h : H_{0,1}^1(\Omega_{\mathrm{a}}) \to \mathcal{M}_h^k,$$

and the same notation is kept for its vectorial counterpart. Then for all l and for all p such that $1 \leq l \leq k+1$, $1 \leq p \leq +\infty$ there exists a constant $C^* > 0$ independently of h such that for any function $v \in W_1^{l,p}(\Omega_{\mathrm{a}})$,

$$\|v - \mathcal{I}_h\, v\|_{L_1^p(\Omega_{\mathrm{a}})} \leq C^* h^l |v|_{W_1^{l,p}(\Omega_{\mathrm{a}})}.$$

Lemma 5.4 *Assume that $\boldsymbol{u} \in \boldsymbol{H}_1^2$ and $\vec{\theta} \in \vec{H}_1^1$. Then*

$$\begin{aligned}
\big(\partial_t \boldsymbol{u}(t), \boldsymbol{v}\big)_{1,\Omega_{\mathrm{a}}} + a_1^h\big(\boldsymbol{u}(t), \boldsymbol{v}\big) + c_1^h\big(\boldsymbol{u}(t); \boldsymbol{u}(t), \boldsymbol{v}\big) + b(\boldsymbol{v}, p) - d_1\big(\vec{\theta}(t), \boldsymbol{v}\big) &= 0, \\
\big(\partial_t \vec{\theta}(t), \vec{\psi}\big)_{1,\Omega_{\mathrm{a}}} + a_2\big(\vec{\theta}(t), \vec{\psi}\big) + c_2^h\big(\vec{u}(t); \vec{\theta}(t), \vec{\psi}\big) + d_2\big(\vec{s}(t); \vec{\theta}(t), \vec{\psi}\big) &= 0
\end{aligned}$$

for all $(\boldsymbol{v}, \vec{\psi}) \in \boldsymbol{V}_h \times \mathcal{M}_{h,0}$. A similar result also holds for the fourth equation in (4.2).

Proof. Since we assume $\boldsymbol{u} \in \boldsymbol{H}_1^2(\Omega_{\rm a})$, integration by parts yields the required result. See also [8]. □

Now we decompose the errors as follows:

$$\begin{aligned}
\boldsymbol{u} - \boldsymbol{u}_h &= E_{\boldsymbol{u}} + \xi_{\boldsymbol{u}} = (\boldsymbol{u} - \Pi_h\,\boldsymbol{u}) + (\Pi_h\,\boldsymbol{u} - \boldsymbol{u}_h),\\
p - p_h &= E_p + \xi_p = (p - \mathcal{L}_h\,p) + (\mathcal{L}_h\,p - p_h),\\
\vec{\theta} - \vec{\theta}_h &= E_{\vec{\theta}} + \xi_{\vec{\theta}} = (\vec{\theta} - \mathcal{I}_h\,\vec{\theta}) + (\mathcal{I}_h\,\vec{\theta} - \vec{\theta}_h),\\
\vec{s} - \vec{s}_h &= E_{\vec{s}} + \xi_{\vec{s}} = (\vec{s} - \mathcal{L}_h\,\vec{s}) + (\mathcal{L}_h\,\vec{s} - \vec{s}_h).
\end{aligned}$$

Assuming that $\boldsymbol{u}_h^0 = \Pi_h\,\boldsymbol{u}(0)$, $\vec{\theta}_h^0 = \mathcal{I}_h\,\vec{\theta}(0)$ and $\vec{s}_h^0 = \mathcal{L}_h\,\vec{s}(0)$, we will use also the notation $E_{\boldsymbol{u}}^n = \boldsymbol{u}(t_n) - \Pi_h\,\boldsymbol{u}(t_n)$ and $\xi_{\boldsymbol{u}}^n = \Pi_h\,\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$, and the corresponding notation for other variables. Since for the first time iteration of system (4.3) we adopt a backward Euler scheme, we require error estimates for this step.

Theorem 5.1 *Let us assume that*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;H_1^3) \cap L^\infty(0,T;\boldsymbol{V}_{1,\diamond}^1(\Omega_{\rm a})), \quad \boldsymbol{u}' \in L^\infty(0,T;\boldsymbol{H}_1^1),\\
&\boldsymbol{u}'' \in L^\infty(0,T;\boldsymbol{L}_1^2), \quad p \in L^\infty(0,T;H_1^2), \quad \vec{\theta} \in L^\infty(0,T;\vec{H}_{1,\diamond}^3(\Omega_{\rm a})),\\
&\vec{\theta}' \in L^\infty(0,T;\vec{H}_1^2), \quad \vec{\theta}'' \in L^\infty(0,T;\vec{L}_1^2), \quad \vec{s} \in L^\infty(0,T;\vec{H}_1^3),\\
&\vec{s}' \in L^\infty(0,T;\vec{H}_1^2), \quad \vec{s}'' \in L^\infty(0,T;\vec{H}_1^1),
\end{aligned}$$

and also that $\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)} < M$ *for a sufficiently small constant* $M > 0$ *(a precise condition for M, can be found in Theorem 5.2). Then there exist positive constants* C_u^1*,* C_θ^1*,* C_s^1*, independently of* h *and* δ_t*, such that*

$$\begin{aligned}
\|\xi_{\boldsymbol{u}}^1\|_{\boldsymbol{L}_1^2}^2 + \frac{1}{2}\delta_t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}\|_{\mathcal{T}_h^1}^2 &\le C_u^1(h^{2k} + \delta_t^4),\\
\frac{1}{4}\|\xi_{\vec{\theta}}^1\|_{\vec{L}_1^2}^2 + \frac{1}{2}\delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}\|_{\vec{H}_1^1}^2 &\le C_\theta^1(h^{2k} + \delta_t^4),\\
\frac{1}{2}\|\xi_{\vec{s}}^1\|_{\vec{L}_1^2}^2 + \frac{1}{2}\delta_t g_1\|\xi_{\vec{s}}^1\|_{\vec{L}_1^2}^2 &\le C_s^1(h^{2k} + \delta_t^4).
\end{aligned}$$

Proof. Since these bounds are similar to those used in Theorems 5.2–5.4, we postpone some details until the proof of those theorems. First, based on the regularity assumptions for $\boldsymbol{u}$, for all $\boldsymbol{x}$ there exists $\gamma \in (0,1)$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}(\delta_t) - \delta_t\boldsymbol{u}'(\delta_t) + \frac{1}{2}\delta_t^2\boldsymbol{u}''(\delta_t\gamma),$$

where $\boldsymbol{u}$ satisfies the error inequality

$$\begin{aligned}
&\|\xi_{\boldsymbol{u}}^1\|_{\boldsymbol{L}_1^2}^2 + \delta_t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^1\|_{\mathcal{T}_h^1}^2\\
&\le -\big(\Pi_h\,\boldsymbol{u}(\delta_t) - \boldsymbol{u}(\delta_t) - (\boldsymbol{u}_h^0 - \boldsymbol{u}(0)), \xi_{\boldsymbol{u}}^1\big)_{1,\Omega_{\rm a}} + \delta_t b\big(\mathcal{L}_h\,p(\delta_t) - p(\delta_t), \xi_{\boldsymbol{u}}^1\big)\\
&\quad + \delta_t a_1^h\big(\Pi_h\,\boldsymbol{u}(\delta_t), \xi_{\boldsymbol{u}}^1\big) - \delta_t\big(c_1^h(\boldsymbol{u}_h^1;\boldsymbol{u}_h^1,\xi_{\boldsymbol{u}}^1) - c_1^h(\boldsymbol{u}(\delta_t),\boldsymbol{u}(\delta_t),\xi_{\boldsymbol{u}}^1)\big)\\
&\quad - \delta_t d_1\big(\vec{\theta}_h^1 - \vec{\theta}(\delta_t), \xi_{\boldsymbol{u}}^1\big) - \frac{\delta_t^2}{2}\big(\boldsymbol{u}''(\delta_t\gamma), \xi_{\boldsymbol{u}}^1\big),
\end{aligned}$$

which follows by choosing $\xi_{\boldsymbol{u}}^1$ as test function in the first equation of Lemma 5.4 and system (4.2), performing an Euler scheme step, subtracting both equations and adding $\pm a_1^h(\Pi_h\,\boldsymbol{u}(\delta_t), \xi_{\boldsymbol{u}}^1)$.

Now by applying the error approximation results from Lemmas 5.1 to 5.3, Young's inequality and the stability properties from Section 4.3, we get

$$\begin{aligned}&\|\xi^1_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2_1}+\frac{1}{4}\delta_t\tilde{\alpha}_a\|\xi^1_{\boldsymbol{u}}\|^2_{\mathcal{T}^1_h}\\&\le Ch^{2k}\delta_t\big(\|\boldsymbol{u}(\delta_t)\|^2_{\boldsymbol{H}^{k+1}_1}+\|\boldsymbol{u}(0)\|^2_{\boldsymbol{H}^{k+1}_1}+\|\vec{\theta}(\delta_t)\|^2_{H^{k+1}_1}+\|p(\delta_t)\|^2_{H^k_1}\big)\\&\quad+C\delta_t^4\|\boldsymbol{u}''\|^2_{L^\infty(0,\delta_t;\boldsymbol{L}^2_1)}+48C^2_{\boldsymbol{F}}\delta_t\|\xi^1_{\vec{\theta}}\|^2_{\vec{L}^2_1}.\end{aligned}\tag{5.1}$$

Next we follow the same steps to obtain for $\vec{\theta}$

$$\begin{aligned}&\frac{1}{2}\|\xi^1_{\vec{\theta}}\|^2_{\vec{L}^2_1}+\frac{1}{2}\delta_t\hat{\alpha}_a\|\xi^1_{\vec{\theta}}\|^2_{\vec{H}^1_1}\\&\le C\delta_t h^{2k}\big(\|\boldsymbol{u}(\delta_t)\|^2_{\boldsymbol{H}^{k+1}_1}+\|\vec{\theta}(\delta_t)\|^2_{\vec{H}^{k+1}_1}+\|\vec{\theta}(0)\|^2_{\vec{H}^{k+1}_1}\big)\\&\quad+C\delta_t^4\|T''\|^2_{L^\infty(0,\delta_t;L^2_1)}+\frac{3\tilde{C}C^*\delta_t}{2\hat{\alpha}_a}\|\xi_{\boldsymbol{u}}\|^2_{\mathcal{T}^1_h}+\frac{5\delta_t|f|^2_{\mathrm{Lip}}C^*}{\hat{\alpha}_a}\|\vec{\theta}(\delta_t)\|^2\|\xi^1_{\vec{s}}\|^2_{\vec{L}^2_1},\end{aligned}\tag{5.2}$$

and analogously for $\vec{s}$

$$\begin{aligned}&\frac{1}{2}\|\xi_{\vec{s}}\|^2_{\vec{L}^2_1}+\frac{1}{2}\delta_t g_1\|\xi_{\vec{s}}\|^2_{\vec{L}^2_1}\\&\le Ch^{2k}\delta_t^2\big(\|\vec{s}(\delta_t)\|^2_{\vec{H}^k_1}+\|\vec{s}(0)\|^2_{\vec{H}^k_1}+\|\vec{\theta}(\delta_t)\|^2_{\vec{H}^{k+1}_1}\big)\\&\quad+C\delta_t^4\|\vec{s}''\|^2_{L^\infty(0,\delta_t;\vec{L}^2_1)}+\frac{5|g|^2_{\mathrm{Lip}}\delta_t}{2g_1}\big(1+\|\vec{s}(\delta_t)\|^2_{\vec{H}^1_1}\big)\|\xi_{\vec{\theta}}\|^2_{\vec{H}^1_1}.\end{aligned}\tag{5.3}$$

In this way, from (5.1) and (5.3) we have that

$$\begin{aligned}&\frac{3\tilde{C}C^*\epsilon_2\delta_t}{2\hat{\alpha}_a}\|\xi_{\boldsymbol{u}}\|^2_{\mathcal{T}^1_h}\le C(h^{2k}+\delta_t^4)+\frac{144\tilde{C}C^*C^2_{\boldsymbol{F}}\delta_t}{\tilde{\alpha}_a\hat{\alpha}}\|\xi^1_{\vec{\theta}}\|^2_{\vec{L}^2_1},\\&\frac{5\delta_t|f|^2_{\mathrm{Lip}}C^*}{\hat{\alpha}_a}\|\vec{\theta}(\delta_t)\|^2\|\xi^1_{\vec{s}}\|^2_{\vec{L}^2_1}\\&\le C(h^{2k}+\delta_t^4)+\frac{25\delta_t|f|^2_{\mathrm{Lip}}C^*|g|^2_{\mathrm{Lip}}}{\hat{\alpha}_a g_1^2}\big(1+\|\vec{s}(\delta_t)\|^2_{\vec{H}^1_1}\big)\|\vec{\theta}(\delta_t)\|^2\|\xi_{\vec{\theta}}\|^2_{\vec{L}^2_1}.\end{aligned}$$

We substitute these inequalities into (5.2) and consider δ_t sufficiently small such that the terms multiplying $\|\xi_{\vec{\theta}}\|^2_{\vec{L}^2_1}$ can be absorbed into the left-hand side of the inequality to get

$$\frac{1}{4}\|\xi^1_{\vec{\theta}}\|^2_{\vec{L}^2_1}+\frac{1}{2}\delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}\|^2_{\vec{H}^1_1}\le C^1_\theta(h^{2k}+\delta_t^4).\tag{5.4}$$

Finally we deduce the first and third desired estimates by directly substituting (5.4) on (5.1) and (5.3). □

Theorem 5.2 *Let* $(\boldsymbol{u},p,\vec{\theta},\vec{s})$ *be the solution of* (2.2), (2.3) *under the assumptions of Section 3, and* $(\boldsymbol{u}_h,p_h,\vec{\theta}_h,\vec{s}_h)$ *be the solution of* (4.3). *Suppose that*

$$\begin{aligned}&\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^{k+1}_1)\cap L^\infty(0,T;\boldsymbol{V}^1_{1,\diamond}(\Omega_{\mathrm{a}})),\\&\vec{\theta}\in L^\infty(0,T;\vec{H}^{k+1}_{1,\diamond}(\Omega_{\mathrm{a}})),\quad \boldsymbol{u}'\in L^\infty(0,T;\boldsymbol{H}^k_1),\quad \boldsymbol{u}^{(3)}\in L^2(0,T;\boldsymbol{L}^2_1)\end{aligned}$$

and $\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)} < M$ *for a sufficiently small constant* $M > 0$. *Then there exist positive constants* $C, \gamma_1 \geq 0$ *independent of* h *and* δ_t *such that for all* $m + 1 \leq N$,

$$\begin{aligned}&\|\xi_{\boldsymbol{u}}^{m+1}\|_{\boldsymbol{L}_1^2}^2 + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|_{\boldsymbol{L}_1^2}^2 + \sum_{n=1}^m \|\Lambda\xi_{\boldsymbol{u}}^n\|_{\boldsymbol{L}_1^2}^2 + \sum_{n=1}^m \delta_t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \\ &\leq C(\delta_t^4 + h^{2k}) + \sum_{n=1}^m \gamma_1\delta_t\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2.\end{aligned}$$

Proof. We choose as test function $\boldsymbol{v}_h = \xi_{\boldsymbol{u}}^{n+1}$ in the first equation of (4.3) and insert the terms

$$\begin{aligned}&\pm\frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1}), \xi_{\boldsymbol{u}}^{n+1}\big),\\ &\pm\frac{1}{2\delta_t}\big(3\Pi_h\,\boldsymbol{u}(t_{n+1}) - 4\Pi_h\,\boldsymbol{u}(t_n) + \Pi_h\,\boldsymbol{u}(t_{n-1}), \xi_{\boldsymbol{u}}^{n+1}\big), \quad \pm a_1^h\big(\Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big).\end{aligned}$$

Hence we get

$$\begin{aligned}&\frac{1}{2\delta_t}\big(3\xi_{\boldsymbol{u}}^{n+1} - 4\xi_{\boldsymbol{u}}^n + \xi_{\boldsymbol{u}}^{n-1}, \xi_{\boldsymbol{u}}^{n+1}\big)_{1,\Omega_\mathrm{a}} + \frac{1}{2\delta_t}\big(3E_{\boldsymbol{u}}^{n+1} - 4E_{\boldsymbol{u}}^n + E_{\boldsymbol{u}}^{n-1}, \xi_{\boldsymbol{u}}^{n+1}\big)_{1,\Omega_\mathrm{a}}\\ &+ \frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1}), \xi_{\boldsymbol{u}}^{n+1}\big)_{1,\Omega_\mathrm{a}} + a_1^h(\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\\ &+ a_1^h(\Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) + c_1^h(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}) + b(\xi_{\boldsymbol{u}}^{n+1}, p_h^{n+1})\\ &= d_1(\vec{\theta}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}).\end{aligned} \tag{5.5}$$

Considering the first equation on Lemma 5.4 at $t = t_{n+1}$ with $\boldsymbol{v} = \xi_{\boldsymbol{u}}^{n+1}$, and after inserting the term

$$\pm\frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1}), \xi_{\boldsymbol{u}}^{n+1}\big)_{1,\Omega_\mathrm{a}},$$

we readily deduce the identity

$$\begin{aligned}&\frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1}), \xi_{\boldsymbol{u}}^{n+1}\big)_{1,\Omega_\mathrm{a}} + a_1^h\big(\boldsymbol{u}(t_n), \xi_{\boldsymbol{u}}^{n+1}\big)\\ &+ c_1^h\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big) + b(\xi_{\boldsymbol{u}}^{n+1}, p(t_{n+1}))\\ &= d_1(\vec{\theta}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})\\ &- \left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1})\big), \xi_{\boldsymbol{u}}^{n+1}\right)_{1,\Omega_\mathrm{a}}.\end{aligned} \tag{5.6}$$

We can then subtract (5.6) from (5.5) and multiply both sides by $4\delta_t$ to obtain an identity $I_1 + I_2 + \cdots + I_8 = 0$, where

$$\begin{aligned}&I_1 := 2\big(3\xi_{\boldsymbol{u}}^{n+1} - 4\xi_{\boldsymbol{u}}^n + \xi_{\boldsymbol{u}}^{n-1}, \xi_{\boldsymbol{u}}^{n+1}\big), \quad I_2 := 4\delta_t a_1^h(\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_{1,\Omega_\mathrm{a}},\\ &I_3 := 4\delta_t\left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\delta_t}\big(3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1})\big), \xi_{\boldsymbol{u}}^{n+1}\right)_{1,\Omega_\mathrm{a}},\\ &I_4 := 2\big(3E_{\boldsymbol{u}}^{n+1} - 4E_{\boldsymbol{u}}^n + E_{\boldsymbol{u}}^{n-1}, \xi_{\boldsymbol{u}}^{n+1}\big), \quad I_5 := -4\delta_t d_1(\vec{\theta}_h^{n+1} - \vec{\theta}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_{1,\Omega_\mathrm{a}},\\ &I_6 := 4\delta_t a_1^h\big(E_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big),\\ &I_7 := 4\delta_t\big(c_1^h(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}) - c_1^h\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)\big),\end{aligned}$$

$$I_8 := 4\delta_t b(\xi_{\boldsymbol{u}}^{n+1}, p_h^{n+1} - p(t_{n+1})).$$

For the first term, using (4.11) we can assert that

$$I_1 = \|\xi_{\boldsymbol{u}}^{n+1}\|_{\boldsymbol{L}_1^2}^2 + \|2\xi_{\boldsymbol{u}}^{n+1} - \xi_{\boldsymbol{u}}^n\|_{\boldsymbol{L}_1^2}^2 + \|\Lambda\xi_{\boldsymbol{u}}^{n+1}\|_{\boldsymbol{L}_1^2}^2 - \|\xi_{\boldsymbol{u}}^n\|_{\boldsymbol{L}_1^2}^2 - \|2\xi_{\boldsymbol{u}}^n - \xi_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}_1^2}^2.$$

Using the ellipticity stated in (4.7), we readily get

$$I_2 \geq 4\delta_t\tilde{\alpha_a}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2.$$

By using Taylor's formula with integral remainder we have

$$\left|\boldsymbol{u}'(t_{n+1}) - \frac{3\boldsymbol{u}(t_{n+1}) - 4\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1})}{2\delta_t}\right| = \frac{\delta_t^{3/2}}{2\sqrt{3}}\|\boldsymbol{u}^{(3)}\|_{L^2(t^{n-1},t^{n+1};\boldsymbol{L}_1^2)},$$

then by combining Cauchy-Schwarz and Young's inequality, we obtain the bound

$$|I_3| \leq \frac{\delta_t^4}{24\varepsilon_1}\|\boldsymbol{u}^{(3)}\|_{L^2(t_{n-1},t_{n+1};\boldsymbol{L}_1^2)}^2 + \frac{\delta_t\varepsilon_1}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2.$$

Now we insert $\pm 4\delta_t E'_{\boldsymbol{u}}(t_{n+1})$ into the fourth term, which leads to

$$\begin{aligned} I_4 = &-4\delta_t(E'_{\boldsymbol{u}}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_{1,\Omega_{\mathrm{a}}} \\ &+ \left(E'_{\boldsymbol{u}}(t_{n+1}) - \frac{3E_{\boldsymbol{u}}^{n+1} - 4E_{\boldsymbol{u}}^n + E_{\boldsymbol{u}}^{n-1}}{2\delta_t}, \xi_{\boldsymbol{u}}^{n+1}\right)_{1,\Omega_{\mathrm{a}}}. \end{aligned}$$

Proceeding as before and using Lemma 5.2 on the first term of I_4, we get

$$\begin{aligned} |I_4| \leq &\frac{C}{2\varepsilon_2}h^{2k}\|\boldsymbol{u}'\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}_1^k)}^2 + \frac{\delta_t\varepsilon_2}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \\ &+ \frac{\delta_t^4 C}{2\varepsilon_3}\|\boldsymbol{u}^{(3)}\|_{L^2(0,T;\boldsymbol{L}_1^2)}^2 + \frac{\delta_t\varepsilon_3}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2. \end{aligned}$$

Now by (4.6a), appealing to Lemma 5.3, and inserting $\pm 4\delta_t d_1(\mathcal{I}_h\vec{\theta}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})$, we are left with

$$\begin{aligned} |I_5| &\leq 4\delta_t C_{\boldsymbol{F}}\|\xi_{\vec{\theta}}^{n+1} + E_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1} \\ &\leq \frac{16C_{\boldsymbol{F}}^2\delta_t}{2\varepsilon_4}\left(Ch^{2k}\|\vec{\theta}\|_{L^\infty(0,T;\vec{H}_1^{k+1})}^2 + \|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2\right) + \frac{\delta_t\varepsilon_4}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2. \end{aligned}$$

And again by Lemmas 5.2 and 4.1 we immediately have

$$|I_6| \leq 4\delta_t\tilde{C}_a\|E_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1} \leq \frac{2\tilde{C}_a^2\delta_t h^{2k}}{\varepsilon_5}\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}^2 + \frac{\delta_t\varepsilon_5}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2.$$

Adding and subtracting suitable terms within I_7 yields

$$I_7 = \tilde{I}_7 - 4\delta t c_1^h(\boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}),$$

where we define

$$\begin{aligned} \tilde{I}_7 := -4\delta_t\big(&c_1^h(\boldsymbol{u}(t_{n+1}), \Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\Pi_h\,\boldsymbol{u}(t_{n+1}), \Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) \\ &+ c_1^h(\Pi_h\,\boldsymbol{u}(t_{n+1}), \Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\Pi_h\,\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1} \end{aligned}$$

$$+ c_1^h(\Pi_h\, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}(t_{n+1})\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})\big).$$

The bound (4.10) and Lemma 5.2 imply that

$$\begin{aligned}
|\tilde{I}_7| \le\ & 4\delta_t \tilde{C}_c\big(\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \|\Pi_h\, \boldsymbol{u}(t_{n+1})\|_{\mathcal{T}_h^1} + \|\Pi_h\, \boldsymbol{u}(t_{n+1})\|_{\mathcal{T}_h^1}\|E_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1} \\
& + \|E_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}\|\boldsymbol{u}(t_{n+1})\|_{\mathcal{T}_h^1}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}\big) \\
\le\ & 4\delta_t \Bigg(\tilde{C}_c C^* \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)}^2 \\
& + \frac{h^{2k} C \tilde{C}_c^2}{2\varepsilon_6} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}^2 + \frac{\varepsilon_6}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \\
& + \frac{C h^{2k} \tilde{C}_c^2}{2\varepsilon_7} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)}^2 + \frac{\varepsilon_7}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \Bigg) \\
\le\ & 4\delta_t \Bigg(C^* \tilde{C}_c M \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 + \frac{h^{2k} C}{2\varepsilon_6} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}^2 \\
& + \frac{\varepsilon_6}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 + \frac{C h^{2k}}{2\varepsilon_7} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)}^2 \\
& + \frac{\varepsilon_7}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \Bigg),
\end{aligned}$$

where C^* is a positive constant coming from Lemma 5.2. Finally, using Lemmas 4.1 and 5.1 we obtain

$$|I_8| \le \frac{8\delta_t C h^{2k}}{\varepsilon_8} \|p\|_{L^\infty(0,T;H^k(\Omega_\mathrm{a}))}^2 + \frac{\delta_t \varepsilon_8}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2.$$

Hence, by choosing $\varepsilon_i = \tilde{\alpha}_a/3$ for $i = \{1,2,3,4,5,8\}$, $\varepsilon_6 = \varepsilon_7 = 7\tilde{\alpha}_a/16$, collecting the above estimates, and summing over $1 \le n \le m$ for all $m+1 \le N$; we get

$$\begin{aligned}
&\|\xi_{\boldsymbol{u}}^{m+1}\|_{\boldsymbol{L}_1^2}^2 + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|_{\boldsymbol{L}_1^2}^2 + \sum_{n=1}^m \|\Lambda \xi_{\boldsymbol{u}}^n\|_{\boldsymbol{L}_1^2}^2 - 3\|\xi_{\boldsymbol{u}}^1\|_{\boldsymbol{L}_1^2}^2 \\
&+ \sum_{n=1}^m \delta_t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \le C(\delta_t^4 + h^{2k}) + \frac{24 C_{\boldsymbol{F}}^2 \delta_t}{\tilde{\alpha}_a} \sum_{n=1}^m \|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2.
\end{aligned}$$

where $4\tilde{C}_c C^* M \le \tilde{\alpha}_a/4$ and $\gamma_1 = 24C_{\boldsymbol{F}}^2/\tilde{\alpha}_a$. Finally, using Theorem 5.1, we get the desired result. □

Theorem 5.3 *Let $(\boldsymbol{u}, p, \vec{\theta}, \vec{s})$ be the solution of (2.2), (2.3) under the assumptions of Section 3 and $(\boldsymbol{u}_h, p_h, \vec{\theta}_h, \vec{s}_h)$ be the solution of (4.3). If*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}_1^{k+1}) \cap L^\infty(0,T;\boldsymbol{V}_{1,\diamond}^1(\Omega_\mathrm{a})), \quad \vec{\theta} \in L^\infty(0,T;\vec{H}_{1,\diamond}^{k+1}(\Omega_\mathrm{a})), \\
&\vec{\theta}' \in L^\infty(0,T;\vec{H}_1^k), \quad \vec{\theta}^{(3)} \in L^2(0,T;\vec{L}_1^2), \quad \vec{s} \in L^\infty(0,T;\vec{H}_1^k),
\end{aligned}$$

then there exist constants $C, \gamma_s, \gamma_u > 0$, independent of h and δ_t, such that for all $m+1 \le N$

$$\begin{aligned}
&\|\xi_{\vec{\theta}}^{m+1}\|_{\vec{L}_1^2}^2 + \|2\xi_{\vec{\theta}}^{m+1} - \xi_{\vec{\theta}}^m\|_{\vec{L}_1^2}^2 + \sum_{n=1}^m \|\Lambda \xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2 + \sum_{n=1}^m \delta_t \hat{\alpha}_a \|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2 \\
&\qquad \le C(\delta_t^4 + h^{2k}) + \sum_{n=1}^m \gamma_s \delta_t \|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2 + \sum_{n=1}^m \gamma_u \delta_t \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2.
\end{aligned}$$

Proof. Proceeding similarly as in the proof of Theorem 5.2, we choose as test function $\vec{\psi}_h = \xi_{\vec{\theta}}^{n+1}$ in the second equation of (4.3) and insert suitable additional terms to obtain the following identity, which is analogous to (5.5):

$$\begin{aligned}
&\frac{1}{2\delta_t}\big(3\xi_{\vec{\theta}}^{n+1} - 4\xi_{\vec{\theta}}^{n} + \xi_{\vec{\theta}}^{n-1}, \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}} + \frac{1}{2\delta_t}\big(3E_{\vec{\theta}}^{n+1} - 4E_{\vec{\theta}}^{n} + E_{\vec{\theta}}^{n-1}, \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}} \\
&+ \frac{1}{2\delta_t}\big(3\vec{\theta}(t_{n+1}) - 4\vec{\theta}(t_n) + \vec{\theta}(t_{n-1}), \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}} \\
&+ a_2^h(\xi_{\vec{\theta}}^{n+1}, \xi_{\vec{\theta}}^{n+1}) + a_2^h(\mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}) + c_2^h(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \xi_{\vec{\theta}}^{n+1}) \\
&= -d_2(\vec{s}_h^{n+1}; \vec{\theta}_h^{n+1}, \xi_{\vec{\theta}}^{n+1}).
\end{aligned} \tag{5.7}$$

Starting from the second equation in Lemma 5.4, focusing on $t = t_{n+1}$, using $\vec{\psi} = \xi_{\vec{\theta}}^{n+1}$ and proceeding as in the derivation of (5.6), we obtain

$$\begin{aligned}
&\frac{1}{2\delta_t}\big(3\vec{\theta}(t_{n+1}) - 4\vec{\theta}(t_n) + \vec{\theta}(t_{n-1}), \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}} + a_2^h\big(\vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) \\
&+ c_2^h\big(\boldsymbol{u}(t_{n+1}), \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) \\
&= d_2\big(\vec{s}(t_{n+1}); \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) \\
&\quad - \left(\vec{\theta}'(t_{n+1}) - \frac{3\vec{\theta}(t_{n+1}) - 4\vec{\theta}(t_n) + \vec{\theta}(t_{n-1})}{2\delta_t}, \xi_{\vec{\theta}}^{n+1}\right)_{1,\Omega_{\rm a}}.
\end{aligned} \tag{5.8}$$

Next we proceed to subtract (5.8) from (5.7), and to multiply both sides by $4\delta_t$. This leads to an identity $\hat{I}_1 + \hat{I}_2 + \cdots + \hat{I}_7 = 0$, where

$$\begin{aligned}
&\hat{I}_1 := 2\big(3\xi_{\vec{\theta}}^{n+1} - 4\xi_{\vec{\theta}}^{n} + \xi_{\vec{\theta}}^{n-1}, \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}}, \quad \hat{I}_2 := 4\delta_t a_2^h\big(\xi_{\vec{\theta}}^{n+1}, \xi_{\vec{\theta}}^{n+1}\big), \\
&\hat{I}_3 := 4\delta_t\left(\vec{\theta}'(t_{n+1}) - \frac{3\vec{\theta}(t_{n+1}) - 4\vec{\theta}(t_n) + \vec{\theta}(t_{n-1})}{2\delta_t}, \xi_{\vec{\theta}}^{n+1}\right)_{1,\Omega_{\rm a}}, \\
&\hat{I}_4 := 2\big(3E_{\vec{\theta}}^{n+1} - 4E_{\vec{\theta}}^{n} + E_{\vec{\theta}}^{n-1}, \xi_{\vec{\theta}}^{n+1}\big)_{1,\Omega_{\rm a}}, \quad \hat{I}_5 := 4\delta_t a_2^h\big(E_{\vec{\theta}}^{n+1}, \xi_{\vec{\theta}}^{n+1}\big), \\
&\hat{I}_6 := 4\delta_t\big(c_2^h\big(\boldsymbol{u}_h^{n+1}, \vec{\theta}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) - c_1^h\big(\boldsymbol{u}(t_{n+1}), \vec{\theta}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)\big), \\
&\hat{I}_7 := 4\delta_t\big(d_2\big(\vec{s}_h^{n+1}, \vec{\theta}_h^{n+1}, \xi_{\vec{\theta}}^{n+1}\big) - d_2\big(\vec{s}(t_{n+1}); \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big)\big).
\end{aligned}$$

For the first, second, and third terms, we use (4.11), (2.15), and Taylor expansion together with Young's inequality, respectively, to obtain

$$\begin{aligned}
\hat{I}_1 &= \|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2 + \|2\xi_{\vec{\theta}}^{n+1} - \xi_{\vec{\theta}}^{n}\|_{\vec{L}_1^2}^2 + \|\Lambda\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2 - \|\xi_{\vec{\theta}}^{n}\|_{\vec{L}_1^2}^2 - \|2\xi_{\vec{\theta}}^{n} - \xi_{\vec{\theta}}^{n-1}\|_{\vec{L}_1^2}^2, \\
\hat{I}_2 &\geq 4\delta_t\hat{\alpha}_{\rm a}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2, \\
|\hat{I}_3| &\leq \frac{\delta_t^4}{24\varepsilon_1}\|\vec{\theta}^{(3)}\|_{L^2(t_{n-1},t_{n+1};\vec{L}_1^2)}^2 + \frac{\delta_t\varepsilon_1}{2}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2.
\end{aligned}$$

Inserting $\pm 4\delta_t E_{\vec{\theta}}'(t_{n+1})$ into $\hat{I}_4$ and using Lemma 5.3 leads to the bound

$$\begin{aligned}
|\hat{I}_4| \leq{}& \frac{C}{2\varepsilon_2}h^{2k}\|\vec{\theta}'\|_{L^\infty(0,T;\vec{H}_1^k)}^2 + \frac{\delta_t\varepsilon_2}{2}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2 \\
&+ \frac{\delta_t^4 C}{2\varepsilon_3}\|\vec{\theta}^{(3)}\|_{L^2(0,T;\vec{L}_1^2)}^2 + \frac{\delta_t\varepsilon_3}{2}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2.
\end{aligned}$$

Employing again Lemma 5.3 in combination with (2.12b) we have

$$|\hat{I}_5| \le \frac{2\hat{C}_a^2\delta_t h^{2k}}{\varepsilon_4}\|\vec{\theta}\|^2_{L^\infty(0,T;H_1^{k+1})} + \frac{\delta_t\varepsilon_4}{2}\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}_1^1}.$$

In order to derive a bound for $\hat{I}_6$ we proceed as for the bound on I_7 in the proof of Theorem 5.2; namely adding and subtracting suitable terms in the definition of $\hat{I}_6$, defining $\tilde{I}_6$ in this case by

$$\hat{I}_6 = \tilde{I}_6 + 4\delta_t c_2^h\big(\boldsymbol{u}_h^{n+1}, \xi_{\vec{\theta}}^{n+1}, \xi_{\vec{\theta}}^{n+1}\big),$$

and applying (4.9), (4.6b) and Lemma 5.3 to the result, we get

$$\begin{aligned}|\tilde{I}_6| \le 4\delta_t\Bigg(&\frac{\tilde{C}^2C^*}{2\varepsilon_5}\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{\mathcal{T}_h^1}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)} + \frac{1}{2\varepsilon_5}\|\xi_{\vec{\theta}}\|^2_{\vec{H}_1^1}\\ &+ \frac{h^{2k}C\tilde{C}^2}{2\varepsilon_6}\|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{H}_1^1)}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^{k+1})} + \frac{\varepsilon_6}{2}\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}_1^1}\\ &+ \frac{Ch^{2k}\tilde{C}^2}{2\varepsilon_7}\|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)} + \frac{\varepsilon_7}{2}\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}_1^1}\Bigg).\end{aligned}$$

Next we add and subtract suitable terms in $\hat{I}_7$ to obtain

$$\begin{aligned}\hat{I}_7 = -4\delta_t\big(&d_2\big(\vec{s}_h^{n+1}, \mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) - d_2\big(\mathcal{L}_h\,\vec{s}(t_{n+1}), \mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big)\\ &+ d_2\big(\mathcal{L}_h\,\vec{s}(t_{n+1}), \mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) - d_2\big(\mathcal{L}_h\,\vec{s}(t_{n+1}), \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big)\\ &+ d_2\big(\mathcal{L}_h\,\vec{s}(t_{n+1}), \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big) - d_2\big(\vec{s}(t_{n+1}), \vec{\theta}(t_{n+1}), \xi_{\vec{\theta}}^{n+1}\big)\\ &+ d_2\big(\vec{s}_h^{n+1}, \xi_{\vec{\theta}}^{n+1}, \xi_{\vec{\theta}}^{n+1}\big)\big).\end{aligned}$$

After passing the last expression to the left-hand side and using (2.5), we can combine (2.6) and (2.7), to infer that the remaining terms in $\hat{I}_7$ (which we now denote as $\hat{I}_7^*$) are bounded as follows:

$$\begin{aligned}|\hat{I}_7^*| \le\ &\frac{8|f|^2_{\mathrm{Lip}}\delta_t}{\varepsilon_8}\|\xi_{\vec{s}}^{n+1}\|^2_{\vec{L}_1^2}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)} + \frac{\varepsilon_8\delta_t}{2}\|\xi_{\vec{\theta}}\|^2_{\vec{H}_1^1}\\ &+ \frac{8f_2^2\delta_t h^{2k}}{\varepsilon_9}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^{k+1})} + \frac{\delta_t\varepsilon_9}{2}\|\xi_{\vec{\theta}}\|^2_{\vec{H}_1^1}\\ &+ \frac{8|f|^2_{\mathrm{Lip}}\delta_t h^{2k}}{\varepsilon_{10}}\|\vec{s}\|^2_{L^\infty(0,T;\vec{H}_1^k)}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)} + \frac{\delta_t\varepsilon_{10}}{2}\|\xi_{\vec{\theta}}\|^2_{\vec{H}_1^1}.\end{aligned}$$

In this manner, and after choosing $\varepsilon_i = 3\hat{\alpha}_a/7$ for $i \in \{1,2,3,4,8,9,10\}$ and $\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \hat{\alpha}_a/4$, we can collect the above estimates and sum over $1 \le n \le m$, for all $m+1 \le N$, to get

$$\begin{aligned}&\|\xi_{\vec{\theta}}^{m+1}\|^2_{\vec{L}_1^2} + \|2\xi_{\vec{\theta}}^{m+1} - \xi_{\vec{\theta}}^m\|^2_{\vec{L}_1^2} + \sum_{n=1}^m\|\Lambda\xi_{\vec{\theta}}^n\|^2_{\vec{L}_1^2} + \sum_{n=1}^m\delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}_1^1} - 3\|\xi_{\vec{\theta}}^1\|^2_{\vec{L}_1^2}\\ &\le C(\delta_t^4 + h^{2k}) + \frac{56|f|^2_{\mathrm{Lip}}\delta_t}{3\hat{\alpha}_a}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)}\sum_{n=1}^m\|\xi_{\vec{s}}^{n+1}\|^2_{\vec{L}_1^2}\\ &\quad + \frac{8\delta_t\tilde{C}^2C^*}{\hat{\alpha}_a}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}_1^1)}\sum_{n=1}^m\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{\mathcal{T}_h^1}.\end{aligned}$$

Identifying the constants

$$\gamma_s = \frac{56|f|^2_{\mathrm{Lip}}}{3\hat{\alpha}_a}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}^1_1)}, \quad \gamma_u = \frac{8\tilde{C}^2 C^*}{\hat{\alpha}_a}\|\vec{\theta}\|^2_{L^\infty(0,T;\vec{H}^1_1)}$$

we may conclude the proof. □

Theorem 5.4 *Let $(\boldsymbol{u}, p, \vec{\theta}, \vec{s})$ be the solution of (2.2), (2.3) under the assumptions of Section 3, and $(\boldsymbol{u}_h, p_h, \vec{\theta}_h, \vec{s}_h)$ be the solution of (4.3). If*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^{k+1}_1) \cap L^\infty(0,T;\boldsymbol{V}^1_{1,\diamond}), \quad \vec{\theta} \in L^\infty(0,T;\vec{H}^{k+1}_{1,\diamond}),$$
$$\vec{s} \in L^\infty(0,T;\vec{H}^k_1), \quad \vec{s}\,' \in L^\infty(0,T;\vec{H}^k_1), \quad \vec{s}^{\,(3)} \in L^2(0,T;\vec{L}^2_1),$$

then there exist constants $C, \gamma_2 > 0$ that are independent of h and δ_t such that for all $m+1 \le N$

$$\begin{aligned}&\|\xi^{m+1}_{\vec{s}}\|^2_{\vec{L}^2_1} + \|2\xi^{m+1}_{\vec{s}} - \xi^m_{\vec{s}}\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \|\Lambda\xi^n_{\vec{s}}\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \delta_t g_1 \|\xi^{n+1}_{\vec{s}}\|^2_{\vec{L}^2_1} \\ &\le C(\delta_t^4 + h^{2k}) + \gamma_2 \sum_{n=1}^m \delta_t \|\xi^{n+1}_{\vec{\theta}}\|^2_{\vec{L}^2_1}.\end{aligned}$$

Proof. We choose as test function $\vec{l}_h = \xi^{n+1}_{\vec{s}}$ in the third equation of (4.3) and add and substract suitable terms. Analogously to (5.6) and (5.7), we obtain

$$\begin{aligned}&\frac{1}{2\delta_t}\big(3\xi^{n+1}_{\vec{s}} - 4\xi^n_{\vec{s}} + \xi^{n-1}_{\vec{s}}, \xi^{n+1}_{\vec{s}}\big)_{1,\Omega_{\mathrm{a}}} \\ &+ \frac{1}{2\delta_t}\big(3E^{n+1}_{\vec{s}} - 4E^n_{\vec{s}} + E^{n-1}_{\vec{s}} + 3\vec{s}(t_{n+1}) - 4\vec{s}(t_n) + \vec{s}(t_{n-1}), \xi^{n+1}_{\vec{s}}\big)_{1,\Omega_{\mathrm{a}}} \\ &+ d_3\big(\vec{\theta}^{n+1}_h, \xi^{n+1}_{\vec{s}}, \xi^{n+1}_{\vec{s}}\big) + d_3\big(\vec{\theta}^{n+1}_h; \mathcal{L}_h\,\vec{s}(t_{n+1}), \xi^{n+1}_{\vec{s}}\big) - d_4\big(\vec{\theta}^{n+1}_h, \xi^{n+1}_{\vec{s}}\big) = 0.\end{aligned} \tag{5.9}$$

Now we consider (2.4d) at time $t = t_{n+1}$, using also $\vec{l} = \xi^{n+1}_{\vec{s}}$ as test function. Adding and subtracting a suitable term, we deduce the relation

$$\begin{aligned}&\frac{1}{2\delta_t}\big(3\vec{s}(t_{n+1}) - 4\vec{s}(t_n) + \vec{s}(t_{n-1}), \xi^{n+1}_{\vec{s}}\big)_{1,\Omega_{\mathrm{a}}} + d_3\big(\vec{\theta}(t_{n+1}); \vec{s}(t_{n+1}), \xi^{n+1}_{\vec{s}}\big) \\ &= d_4\big(\vec{\theta}(t_{n+1}), \xi^{n+1}_{\vec{s}}\big) \\ &\quad - \left(\vec{s}\,'(t_{n+1}) - \frac{1}{2\delta_t}\big(3\vec{s}(t_{n+1}) - 4\vec{s}(t_n) + \vec{s}(t_{n-1})\big), \xi^{n+1}_{\vec{s}}\right)_{1,\Omega_{\mathrm{a}}}.\end{aligned} \tag{5.10}$$

As in the two previous proofs, we subtract (5.10) from (5.9) and multiply both sides by $4\delta_t$ to obtain $\bar{I}_1 + \bar{I}_2 + \cdots + \bar{I}_6 = 0$, where

$$\begin{aligned}&\bar{I}_1 := 2\big(3\xi^{n+1}_{\vec{s}} - 4\xi^n_{\vec{s}} + \xi^{n-1}_{\vec{s}}, \xi^{n+1}_{\vec{s}}\big)_{1,\Omega_{\mathrm{a}}}, \quad \bar{I}_2 := d_3(\vec{\theta}^{n+1}_h, \xi^{n+1}_{\vec{s}}, \xi^{n+1}_{\vec{s}}), \\ &\bar{I}_3 := 4\delta_t\left(\vec{s}\,'(t_{n+1}) - \frac{1}{2\delta_t}\big(3\vec{s}(t_{n+1}) - 4\vec{s}(t_n) + \vec{s}(t_{n-1})\big), \xi^{n+1}_{\vec{s}}\right)_{1,\Omega_{\mathrm{a}}}, \\ &\bar{I}_4 := 2\big(3E^{n+1}_{\vec{s}} - 4E^n_{\vec{s}} + E^{n-1}_{\vec{s}}, \xi^{n+1}_{\vec{s}}\big)_{1,\Omega_{\mathrm{a}}}, \\ &\bar{I}_5 := -4\delta_t\big(d_3(\vec{\theta}^{n+1}_h; \mathcal{L}_h\,\vec{s}(t_{n+1}), \xi^{n+1}_{\vec{s}}) - d_3(\vec{\theta}(t_{n+1}); \vec{s}(t_{n+1}), \xi^{n+1}_{\vec{s}})\big),\end{aligned}$$

$$\bar{I}_6 := -4\delta_t d_4\big(\vec{\theta}_h^{n+1} - \vec{\theta}(t_{n+1}), \xi_{\vec{s}}^{n+1}\big).$$

For the first, second, and third terms, we proceed to use (4.11), the ellipticity (2.8), and Taylor expansion to get

$$\bar{I}_1 = \|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2 + \|2\xi_{\vec{s}}^{n+1} - \xi_{\vec{s}}^n\|_{\vec{L}_1^2}^2 + \|\Lambda\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2 - \|\xi_{\vec{s}}^n\|_{\vec{L}_1^2}^2 - \|2\xi_{\vec{s}}^n - \xi_{\vec{s}}^{n-1}\|_{\vec{L}_1^2}^2,$$
$$\bar{I}_2 \geq 4\delta_t g_1\|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2,$$
$$|\bar{I}_3| \leq \frac{\delta_t^4}{24\varepsilon_1}\|\vec{s}^{(3)}\|_{L^2(t_{n-1},t_{n+1};\vec{L}_1^2)}^2 + \frac{\delta_t\varepsilon_1}{2}\|\xi_{\vec{s}}^{n+1}\|_{\vec{H}_1^1}^2.$$

For the fourth term we include $\pm 4\delta_t E_{\vec{s}}'(t_{n+1})$ and use Taylor's formula and Lemma 5.3, which leads to

$$\begin{aligned}|\bar{I}_4| &\leq \frac{C}{2\varepsilon_2}h^{2k}\|\vec{s}\,'\|_{L^\infty(0,T;\vec{H}_1^k)}^2 + \frac{\delta_t\varepsilon_2}{2}\|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2\\ &\quad + \frac{\delta_t^4 C}{2\varepsilon_3}\|\vec{s}^{(3)}\|_{L^2(0,T;\vec{L}_1^2)}^2 + \frac{\delta_t\varepsilon_3}{2}\|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2.\end{aligned}$$

To handle $\bar{I}_5$, we add and subtract the terms

$$d_3\big(\vec{\theta}(t_{n+1}); \vec{s}(t_{n+1}), \xi_{\vec{s}}^{n+1}\big) \quad \text{and} \quad d_3\big(\mathcal{I}_h\,\vec{\theta}(t_{n+1})M; \vec{s}(t_{n+1}), \xi_{\vec{s}}^{n+1}\big).$$

Then, owing to (2.9), (2.11), Lemma 5.1, and Young's inequality, we end up with

$$\begin{aligned}|\bar{I}_5| \leq{}& \frac{Cg_2^2\delta_t h^{2k}}{\varepsilon_4}\|\vec{s}\|_{L^\infty(0,T;\vec{H}_1^k)}^2 + \frac{\varepsilon_4\delta_t}{2}\|\xi_{\vec{s}}\|_{\vec{L}_1^2}^2 + \frac{8|g|_{\mathrm{Lip}}^2\delta_t}{\varepsilon_5}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{H}_1^1}^2\|\vec{s}\|_{L^\infty(0,T;\vec{H}_1^1)}^2\\ &+ \frac{\varepsilon_5\delta_t}{2}\|\xi_{\vec{s}}\|_{\vec{L}_1^2}^2 + \frac{C|g|_{\mathrm{Lip}}^2 h^{2k}\delta_t}{\varepsilon_6}\|\vec{\theta}\|_{L^\infty(0,T;\vec{H}_1^{k+1})}^2\|\vec{s}\|_{L^\infty(0,T;\vec{H}_1^1)}^2\frac{\varepsilon_6\delta_t}{2}\|\xi_{\vec{s}}\|_{\vec{L}_1^2}^2.\end{aligned}$$

Finally we insert $\pm 4\delta_t d_4(\mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{s}}^{n+1})$ in $\bar{I}_6$ and use Lemma 5.3 in order to deduce the bound

$$\begin{aligned}|\bar{I}_6| &= \Big|4\delta_t\big(d_4 bigl(\vec{\theta}_h^{n+1} - \mathcal{I}_h\,\vec{\theta}(t_{n+1}), \xi_{\vec{s}}^{n+1}\big) + d_4\big(\mathcal{I}_h\,\vec{\theta}(t_{n+1}) - \vec{\theta}(t_{n+1}), \xi_{\vec{s}}^{n+1}\big)\big)\Big|\\ &\leq \frac{8|g|_{\mathrm{Lip}}^2\delta_t}{\varepsilon_7}\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2 + \frac{C|g|_{\mathrm{Lip}}^2 h^{2k}}{\varepsilon_8}\|\vec{\theta}\|_{L^\infty(0,T;\vec{H}_1^{k+1})}^2 + \frac{\varepsilon_7+\varepsilon_8}{2}\delta_t\|\xi_{\vec{s}}\|_{\vec{L}_1^2}^2.\end{aligned}$$

It then suffices to take $\varepsilon_i = 3g_1/4$ for all $i \in \{1,\dots,10\}$ and to sum over $1 \leq n \leq m$, for all $m+1 \leq N$ in the above estimates, which, in combination with Theorem 5.2 implies that

$$\begin{aligned}&\|\xi_{\vec{s}}^{m+1}\|_{\vec{L}_1^2}^2 + \|2\xi_{\vec{s}}^{m+1} - \xi_{\vec{s}}^m\|_{\vec{L}_1^2}^2 + \sum_{n=1}^m\|\Lambda\xi_{\vec{s}}^n\|_{\vec{L}_1^2}^2 + \sum_{n=1}^m\delta_t g_1\|\xi_{\vec{s}}^{n+1}\|_{\vec{L}_1^2}^2\\ &\leq C(\delta_t^4 + h^{2k}) + \frac{32|g|_{\mathrm{Lip}}^2}{3g_1}\Big(1 + \|\vec{s}\|_{L^\infty(0,T;\vec{H}_1^1)}^2\Big)\,\delta_t\sum_{n=1}^m\|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2,\end{aligned}$$

and the result follows by choosing

$$\gamma_2 = \frac{32|g|_{\mathrm{Lip}}^2}{3g_1}\big(1 + \|\vec{s}\|_{L^\infty(0,T;\vec{H}_1^1)}^2\big).$$

□

Theorem 5.5 *Under the same assumptions of Theorems 5.2 - 5.4, there exist positive constants $\hat{\gamma}_u$, $\hat{\gamma}_\theta$ and $\hat{\gamma}_s$ independent of δ_t and h, such that for a sufficiently small δ_t and all $m+1 \leq N$, the following inequalities hold:*

$$\begin{aligned}
&\left(\|\xi_{\boldsymbol{u}}^{m+1}\|^2_{\boldsymbol{L}^2_1} + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|^2_{\boldsymbol{L}^2_1} + \sum_{n=1}^m \left(\|\Lambda\xi_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2_1} + \delta_t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{\mathcal{T}_h^1}\right)\right)^{1/2} \\
&\leq \hat{\gamma}_u(\delta_t^2 + h^k), \\
&\left(\|\xi_{\vec{\theta}}^{m+1}\|^2_{\vec{L}^2_1} + \|2\xi_{\vec{\theta}}^{m+1} - \xi_{\vec{\theta}}^m\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \left(\|\Lambda\xi_{\vec{\theta}}^n\|^2_{\vec{L}^2_1} + \delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}^1_1}\right)\right)^{1/2} \\
&\leq \hat{\gamma}_\theta(\delta_t^2 + h^k), \\
&\left(\|\xi_{\vec{s}}^{m+1}\|^2_{\vec{L}^2_1} + \|2\xi_{\vec{s}}^{m+1} - \xi_{\vec{s}}^m\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \left(\|\Lambda\xi_{\vec{s}}^n\|^2_{\vec{L}^2_1} + \delta_t g_1\|\xi_{\vec{s}}^{n+1}\|^2_{\vec{L}^2_1}\right)\right)^{1/2} \\
&\leq \hat{\gamma}_s(\delta_t^2 + h^k).
\end{aligned}$$

Proof. From Theorem 5.2 and 5.4 we have the estimates

$$\begin{aligned}
\sum_{n=1}^m \gamma_u\delta_t\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{\mathcal{T}_h^1} &\leq C(\delta_t^4 + h^{2k}) + \frac{\gamma_1\gamma_u}{\tilde{\alpha}_a}\sum_{n=1}^m \delta_t\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{L}^2_1}, \\
\sum_{n=1}^m \gamma_s\delta_t\|\xi_{\vec{s}}^{n+1}\|^2_{\vec{L}^2_1} &\leq C(\delta_t^4 + h^{2k}) + \frac{\gamma_s\gamma_2}{g_1}\sum_{n=1}^m \delta_t\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{L}^2_1},
\end{aligned}$$

which, after substituting them back into Theorem5.3, yield

$$\begin{aligned}
&\|\xi_{\vec{\theta}}^{m+1}\|^2_{\vec{L}^2_1} + \|2\xi_{\vec{\theta}}^{m+1} - \xi_{\vec{\theta}}^m\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \|\Lambda\xi_{\vec{\theta}}^n\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}^1_1} \\
&\leq C(\delta_t^4 + h^{2k}) + \frac{\gamma_1\gamma_u g_1 + \gamma_s\gamma_2\tilde{\alpha}_a}{\tilde{\alpha}_a g_1}\sum_{n=1}^m \delta_t\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{L}^2_1}.
\end{aligned}$$

For the last term on the right-hand side of this last bound we have

$$\|\xi_{\vec{\theta}}^{m+1}\|^2_{\vec{L}^2_1} \leq 2(\|\Lambda\xi_{\vec{\theta}}^m\|^2_{\vec{L}^2_1} + \|2\xi_{\vec{\theta}}^m - \xi_{\vec{\theta}}^{m-1}\|^2_{\vec{L}^2_1}),$$

and considering δ_t sufficiently small and applying Gronwall's lemma, we readily infer the estimate

$$\begin{aligned}
&\|\xi_{\vec{\theta}}^{m+1}\|^2_{\vec{L}^2_1} + \|2\xi_{\vec{\theta}}^{m+1} - \xi_{\vec{\theta}}^m\|^2_{\vec{L}^2_1} + \sum_{n=1}^m \left(\|\Lambda\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{L}^2_1} + \delta_t\hat{\alpha}_a\|\xi_{\vec{\theta}}^{n+1}\|^2_{\vec{H}^1_1}\right) \\
&\leq C(\delta_t^4 + h^{2k}).
\end{aligned} \tag{5.11}$$

The first and third bounds follow by combining (5.11) and Theorems 5.2 and 5.4. □

Lemma 5.5 *Under the same assumptions of Theorem 5.5, we have*

$$\left(\sum_{n=1}^m \delta_t\|p(t_{n+1}) - p_h^{n+1}\|^2_{L^2_1}\right)^{1/2} \leq \hat{\gamma}_p(\delta_t^2 + h^k).$$

Proof. Owing to the inf-sup condition (4.1), there exists $\boldsymbol{w}_h \in \boldsymbol{X}_h^{\perp}$ such that

$$b(\boldsymbol{w}_h, p(t_{n+1}) - p_h^{n+1}) = \|p(t_{n+1}) - p_h^{n+1}\|_{L_1^2}^2, \tag{5.12}$$

$$\|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} \leq \frac{1}{\tilde{\beta}} \|p(t_{n+1}) - p_h^{n+1}\|_{L_1^2}. \tag{5.13}$$

From (4.3) and Lemma 5.4, proceeding as in the proof of Theorem 5.2, we obtain

$$\begin{aligned}
&\delta_t b(\boldsymbol{w}_h, p(t_{n+1}) - p_h^{n+1}) \\
&= -\delta_t \left(\boldsymbol{u}'(t_{n+1}) - \frac{3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}}{2\delta_t}, \boldsymbol{w}_h \right)_{1,\Omega_\mathrm{a}} + \delta_t a_1^h(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h) \\
&\quad + \delta_t \big(c_1^h(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h) - c_1^h(\boldsymbol{u}(t_{n+1}); \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h) \big) \\
&\quad + \delta_t d_1(\vec{\theta}(t_{n+1}) - \vec{\theta}_h^{n+1}, \boldsymbol{w}_h) \\
&\leq \frac{\delta_t^2}{2\sqrt{3}} \|\boldsymbol{u}^{(3)}\|_{L^2(t_{n-1},t_{n+1},\boldsymbol{L}_1^2)} \sqrt{\delta_t} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} + \tilde{C}_a C^* h^k \delta_t \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} \\
&\quad + \tilde{C}_a \delta_t \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} + C^* \tilde{C}_c \delta_t \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)} \|\xi_{\boldsymbol{u}}\|_{\mathcal{T}_h^1} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} \\
&\quad + 2\delta_t C \tilde{C}_c h^k \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^1)} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}_1^{k+1})} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} \\
&\quad + C_{\boldsymbol{F}} \delta_t h^k C^* \|\vec{\theta}\|_{L^\infty(0,T;\vec{H}_1^{k+1})} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1} + C_{\boldsymbol{F}} \delta_t \|\xi_{\vec{\theta}}\|_{\vec{L}_1^2} \|\boldsymbol{w}_h\|_{\mathcal{T}_h^1}.
\end{aligned}$$

Summing over $1 \leq n \leq m$ for all $m+1 \leq N$ and substituting back into equations (5.12) and (5.13), we obtain

$$\begin{aligned}
&\left(\sum_{n=1}^{m} \delta_t \|p(t_{n+1}) - p_h^{n+1}\|_{L_1^2}^2 \right)^{1/2} \\
&\leq \frac{C}{\tilde{\beta}} \left(\delta_t^2 + h^k + \left(\sum_{n=1}^{m} \delta_t \|\xi_{\vec{\theta}}^{n+1}\|_{\vec{L}_1^2}^2 \right)^{1/2} + \left(\sum_{n=1}^{m} \delta_t \|\xi_{\boldsymbol{u}}^{n+1}\|_{\mathcal{T}_h^1}^2 \right)^{1/2} \right),
\end{aligned}$$

and the desired result readily follows from Theorem 5.5. □

6 Numerical tests

6.1 Example 1: Accuracy tests

In our first computational test we examine the convergence of the Galerkin method (4.2), taking as computational domain the square $\Omega = (0,1)^2$. We take the parameter values $\nu = 0.1$, $k^+(\boldsymbol{x}) = 1$, $\boldsymbol{g} = (0,-1)^T$, $\boldsymbol{K}^{-1} = \boldsymbol{I}$, $\boldsymbol{D} = 10^{-3}\boldsymbol{I}$, $D_s = 1$, $\rho_\mathrm{f} = \phi = 1$, $\rho_\mathrm{b} = 0.1$, $a_0 = 500 \cdot 10^k$, where k is the polynomial degree. Following the approach of manufactured solutions, we prescribe boundary data and additional external forces and adequate source terms so that the closed-form solutions to (1.1), (2.1) are given by the smooth functions

$$\boldsymbol{u}(r,z,t) = \begin{pmatrix} 0 \\ -\cos(r\pi/2)\exp(-t) \end{pmatrix}, \quad \vec{\theta}(r,z,t) = \begin{pmatrix} z^2 r^2 (3-2r)(1-\exp(-t)) \\ z^2 r^2 (3-2r)(1-\exp(-t)) \end{pmatrix},$$

$$p(r,z,t) = (r^3 - 2z^4)\sin(t), \quad \vec{s}(r,z,t) = \begin{pmatrix} 1 - \exp(-z^2 r^2 (3-2r)(t+\exp(t))) \\ 1 - \exp(-z^2 r^2 (3-2r)(t+\exp(t))) \end{pmatrix}.$$

k	DoF	$\mathtt{e}_{\boldsymbol{u}}$	rate	$\mathtt{e}_p$	rate	$\mathtt{e}_{\vec{\theta}}$	rate	$\mathtt{e}_s$	rate
1	75	0.05435	–	0.57400	–	0.26530	–	0.11760	–
	259	0.02894	0.909	0.12480	2.201	0.13940	0.928	0.05934	0.986
	963	0.01466	0.981	0.05242	1.252	0.07039	0.986	0.02978	0.995
	3715	0.00736	0.995	0.02545	1.042	0.03537	0.993	0.01490	0.999
	14595	0.00368	0.998	0.01202	1.083	0.01792	0.981	0.00746	0.999
2	195	0.00537	–	0.77890	–	0.00071	–	0.05373	–
	715	0.00149	1.848	0.11910	2.710	0.00018	1.947	0.01480	1.860
	2739	0.00038	1.953	0.01749	2.767	4.619e-5	2.001	0.00378	1.970
	10723	9.074e-5	2.084	0.00249	2.813	1.154e-5	2.001	0.00095	1.992
	42435	2.328e-5	1.963	0.00052	2.256	2.909e-6	1.988	0.00024	1.998

Table 6.1 Example 1 (Spatial accuracy test): experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, $\vec{\theta}_h$ and s_h. Values are displayed for schemes with first- and second-order in space

δ_t	$\hat{\mathtt{e}}_{\boldsymbol{u}}$	rate	$\hat{\mathtt{e}}_p$	rate	$\hat{\mathtt{e}}_{\vec{\theta}}$	rate	$\hat{\mathtt{e}}_s$	rate
2.5	0.5496	–	0.5663	–	17.691	–	0.6738	–
1.25	0.1408	1.964	0.1177	2.266	3.2720	2.435	0.1673	2.009
0.625	0.0289	2.284	0.0258	2.188	0.6621	2.305	0.0409	2.032
0.3125	0.0066	2.119	0.0061	2.091	0.1519	2.124	0.0105	1.965
0.1562	0.0016	2.047	0.0015	1.976	0.0366	2.054	0.0027	1.934

Table 6.2 Example 1 (time accuracy test): experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, $\vec{\theta}_h$ and s_h, computed for each refinement level

As $\boldsymbol{u}$ is prescribed everywhere on $\partial\Omega_{\mathrm{a}}$, for sake of uniqueness we impose $p \in L^2_{0,1}(\Omega_{\mathrm{a}})$ through a real Lagrange multiplier approach. Also note that the exact solutions satisfy the boundary conditions (2.3a), (2.3b), (2.3c) on the inlet, wall, and symmetry axis, respectively, whereas instead of (2.3d) we set

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{out}}, \quad \mathbb{D}\nabla\vec{\theta} \cdot \boldsymbol{n} = \vec{0},$$

on the outlet $\Gamma^{\mathrm{out}}_{\mathrm{a}} \times (0, T]$. The accuracy of the spatial semi-discretisation is tested by considering a sequence of uniformly refined meshes $\{\mathcal{T}_{h,l}\}_l$ of mesh size $h_l = 2^{-l}\sqrt{2}$, and fixing $T = 0.005$ with $\delta_t = 0.001$. Relative errors in their natural norms, along with the corresponding convergence rates are computed as

$$\mathtt{e}_{\boldsymbol{u}} = \frac{\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{T}_h^1}}{\|\boldsymbol{u}\|_{\mathcal{T}_h^1}}, \quad \mathtt{e}_p = \frac{\|p - p_h\|_{L^2_1(\Omega_{\mathrm{a}})}}{\|p\|_{L^2_1(\Omega_{\mathrm{a}})}}, \quad \mathtt{e}_{\vec{\theta}} = \frac{\|\vec{\theta} - \vec{\theta}_h\|_{\vec{H}^1_1(\Omega_{\mathrm{a}})}}{\|\vec{\theta}\|_{\vec{H}^1_1(\Omega_{\mathrm{a}})}},$$
$$\mathtt{e}_{\vec{s}} = \frac{\|\vec{s} - \vec{s}_h\|_{\vec{H}^1_1(\Omega_{\mathrm{a}})}}{\|\vec{s}\|_{\vec{H}^1_1(\Omega_{\mathrm{a}})}}, \quad \mathtt{rate} = \log(e_{(\cdot)}/\tilde{e}_{(\cdot)})[\log(h/\tilde{h})]^{-1},$$

where $e, \tilde{e}$ denote errors generated on two consecutive meshes of sizes h and $\tilde{h}$, respectively. These quantities are listed in Table 6.1 for $k = 0$ and $k = 1$, and they indicate optimal error decay in the light of Theorem 5.5.

Regarding the convergence of the time advancing scheme, now we set $T = 5$ and consider a sequence of uniform refined time partitions $\tau_l, l \in \{1,2,3,4,5\}$ where the time step is $5/2^l$. Absolute errors are computed as

$$\hat{\mathsf{e}}_{\boldsymbol{u}} = \left(\sum_{n=1}^{m} \delta_t \|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_h^{n+1}\|^2_{\mathcal{T}_h^1}\right)^{1/2}, \quad \hat{\mathsf{e}}_p = \left(\sum_{n=1}^{m} \delta_t \|p(t_{n+1}) - p_h^{n+1}\|^2_{L_1^2}\right)^{1/2},$$
$$\hat{\mathsf{e}}_{\vec{\theta}} = \left(\sum_{n=1}^{m} \delta_t \|\vec{\theta}(t_{n+1}) - \vec{\theta}_h^{n+1}\|^2_{\vec{H}_1^1}\right)^{1/2}, \quad \hat{\mathsf{e}}_{\vec{s}} = \left(\sum_{n=1}^{m} \delta_t \|\vec{s}(t_{n+1}) - \vec{s}_h^{n+1}\|^2_{\vec{L}_1^2}\right)^{1/2},$$

and we readily observe from Table 6.2 that the method converges to the exact solution with the expected second-order rate.

6.2 Example 2: Validation against experimental data

Now we define a different adimensionalisation of (1.1a)-(2.1d) that follows the recent model (tailored specifically for soil-based water filters for arsenic removal) proposed in [31]. This problem considers only one type of contaminant and only one type of adsorption. Defining as $L, v_i, \theta_0, s_{\max}$ the representative length of the column, the linear inflow rate, initial solids concentration, and maximum adsorption, respectively; we define dimensionless variables as

$$\bar{r} = \frac{r}{L},\ \bar{z} = \frac{z}{L},\ \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{v_i},\ \bar{\theta} = \frac{\theta}{\theta_0},\ \bar{p} = \frac{L(p - p_{\mathrm{atm}})}{\mu v_i},\ \bar{s} = \frac{s}{s_{\max}},\ \bar{t} = k^+\theta_0 t,$$

and we also define the constants

$$\mathrm{Re} = \frac{\rho_{\mathrm{f}} v_i L}{\nu}, \quad \mathrm{Pe} = \frac{v_i L}{D}, \quad \mathrm{Da} = \frac{\kappa}{L^2}, \quad \alpha = \frac{\rho_{\mathrm{b}} s_{\max}}{\theta_0}, \quad \beta = \frac{k^+ L^2 \theta_0}{D}. \tag{6.1}$$

Making abuse of notation, the problem defined in $\Omega_{\mathrm{a}} \times (0,T]$ adopts the form

$$\begin{aligned}\frac{\beta\,\mathrm{Re}}{\mathrm{Pe}}\partial_t \boldsymbol{u} + \mathrm{Re}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\mathrm{Da}}\boldsymbol{u} - \frac{1}{\phi}\mathbf{div}_{\mathrm{a}}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p + \frac{1}{\phi}(\boldsymbol{u}_r/r^2)\boldsymbol{e}_1 &= \mathbf{0},\\ \mathrm{div}_{\mathrm{a}}\,\boldsymbol{u} &= 0,\\ \frac{\phi\beta}{\mathrm{Pe}}\partial_t\theta - \frac{1}{\mathrm{Pe}}\mathrm{div}_{\mathrm{a}}(\nabla\theta) + \boldsymbol{u}\cdot\nabla\theta &= -\frac{\alpha\beta}{\mathrm{Pe}}\partial_t s,\\ \partial_t s &= \theta(1-s).\end{aligned}$$

The setup consists of a lab-scale filter (a column of height 1 and radius $\bar{R} = 0.11$, already in dimensionless units) where one varies the feed flow rate, the arsenic concentration at the feed, and also the bed height. Gravitational effects are not considered, and the boundary and initial conditions are precisely as in (2.3a)-(2.3e). The configuration of the system implies that the non-dimensional constants from (6.1) assume the values

$$\mathrm{Re} = 68.1, \quad \mathrm{Pe} = 1.11\cdot 10^5, \quad \mathrm{Da} = 8000, \quad \alpha = 248, \quad \beta = 136,$$

and the remaining parameter values are $\phi = 0.48$, $\boldsymbol{u}^{\mathrm{in}}(r,z) = (0, \frac{1}{\bar{R}^2}(r-\bar{R})(r+\bar{R}))^{\mathtt{t}}$, $\theta^{\mathrm{in}} = 1$. We employ a structured mesh of 8000 triangular elements and define a constant time step of $\delta_t = 0.15$ (adimensional time $t = 0.15 \approx 1$ day).

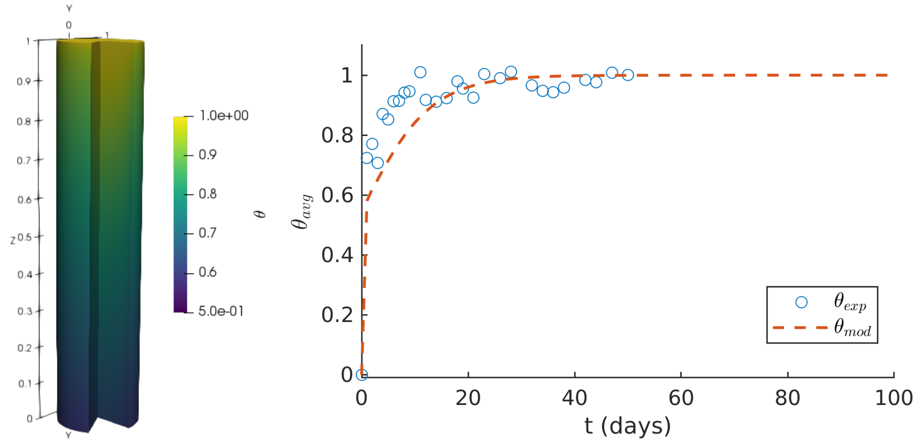

Fig. 2 Example 2 (validation against experimental data): contaminant concentration after one day (left). Value of $\theta|_{\mathrm{avg}}(t)$ (experimental observation from [31] and numerical simulation) using raw laterite as the adsorbent (right)

During the filtration process the soil-based bed reaches a point in time where it is no longer adequate for adsorption. This phenomenon can be observed in Figure 2 where we plot the evolution of the average concentration of the contaminant θ on the outlet, that is

$$\theta_{\mathrm{avg}}(t) = \frac{2}{\bar{R}^2} \int_{\Gamma_{\mathrm{a}}^{\mathrm{out}}} \theta\, r\, \mathrm{d}s.$$

We also compared the predictions of the model with experimental data, collected for a filter that uses raw laterite as an adsorbent medium, and to which an arsenic solution is injected in its upper part [31]. The qualitative results displayed on figure 2 seem to show an acceptable adjustment to the experimental data. This suggest that the model and the axisymmetric divergence-conforming scheme can be used effectively as a tool to study the behaviour of the filtration process under similar flow regimes.

6.3 Example 3: Two contaminants in a two-layer filter

We model a filter with two contaminants and two layers. The domain has a R/L ratio of 0.22. While the inlet is the top wall, the outlet is the region $\{(z,r)|z = 0 \text{ and } 0 \le r \le 0.25R\}$. For (2.2) we take (1.2) with $m = 2$ and and we consider $\mu = 8.94 \cdot 10^{-4}\,\mathrm{Pa\,s}$, $v_i = 6.0 \cdot 10^{-3}\,\mathrm{m/s}$, $\rho_{\mathrm{f}} = 10^3\,\mathrm{kg/m^3}$, $\theta_1^{\mathrm{in}} = 8.0 \cdot 10^{-5}\,\mathrm{kg/m^3}$, $\theta_2^{\mathrm{in}} = 2.0 \cdot 10^{-5}\,\mathrm{kg/m^3}$, $s_1^{\max} = 10^{-3}\,\mathrm{kg/kg}$, $s_2^{\max} = 10^{-2}\,\mathrm{kg/kg}$. In addition, the rheology of the grains is different in the top and bottom halves of the domain. More precisely, we have

$$D_{\mathrm{top}} = 3.8 \cdot 10^{-11}\,\mathrm{m^2/s}, \quad D_{\mathrm{bot}} = 7.6 \cdot 10^{-12}\,\mathrm{m^2/s}, \quad \phi_{\mathrm{top}} = 0.32, \quad \phi_{\mathrm{bot}} = 0.28,$$
$$\rho_{\mathrm{b,top}} = 1050\,\mathrm{kg/m^3}, \quad \rho_{\mathrm{b,bot}} = 1100\,\mathrm{kg/m^3}, \quad k_{1,\mathrm{top}}^{+} = 5.0 \cdot 10^{-3}\,\mathrm{m^3/(kg\,s)},$$
$$k_{2,\mathrm{top}}^{+} = 0\,\mathrm{m^3/(kg\,s)}, \quad k_{1,\mathrm{bot}}^{+} = 2.5 \cdot 10^{-3}\,\mathrm{m^3/(kg\,s)}, \quad k_{2,\mathrm{bot}}^{+} = 10^{-3}\,\mathrm{m^3/(kg\,s)},$$

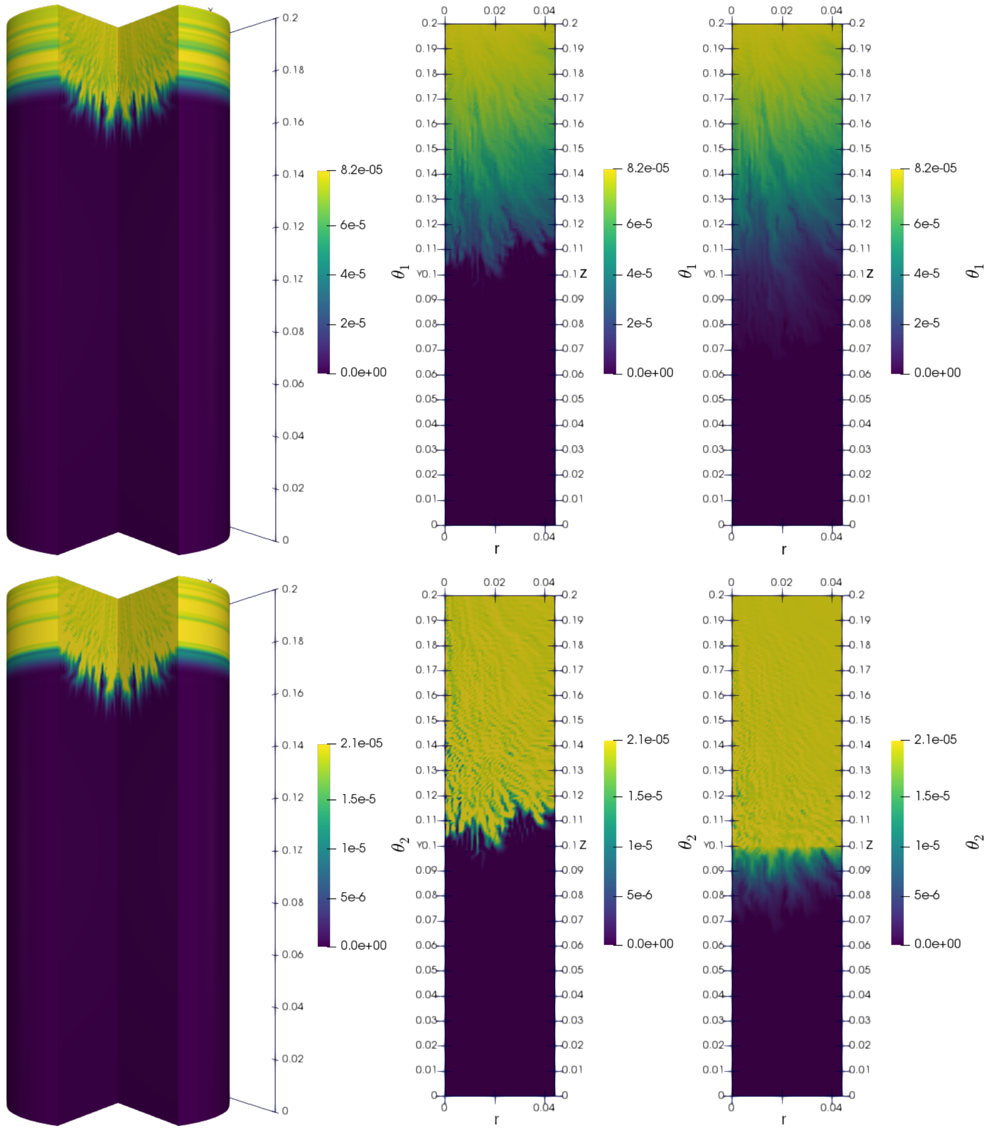

Fig. 3 Example 3 (two contaminants in two-layer filter): concentration of contaminants at times $t = 10, 100, 300$

and the permeability $\mathbb{K}(\boldsymbol{x}) = \kappa(\boldsymbol{x})\boldsymbol{I}$ has a log-uniform distribution in each layer that satisfies

$$1.57 \cdot 10^{-9}\,\mathrm{m}^2 \leq \kappa_{\mathrm{top}}(\boldsymbol{x}) \leq 3.04 \cdot 10^{-6}\,\mathrm{m}^2,$$
$$5.18 \cdot 10^{-10}\,\mathrm{m}^2 \leq \kappa_{\mathrm{bot}}(\boldsymbol{x}) \leq 10^{-6}\,\mathrm{m}^2.$$

Qualitative results for the concentration of the two contaminants at times $t = 10, 100$ and 300 are shown on Figure 3. As expected, most of the first contaminant is retained in the

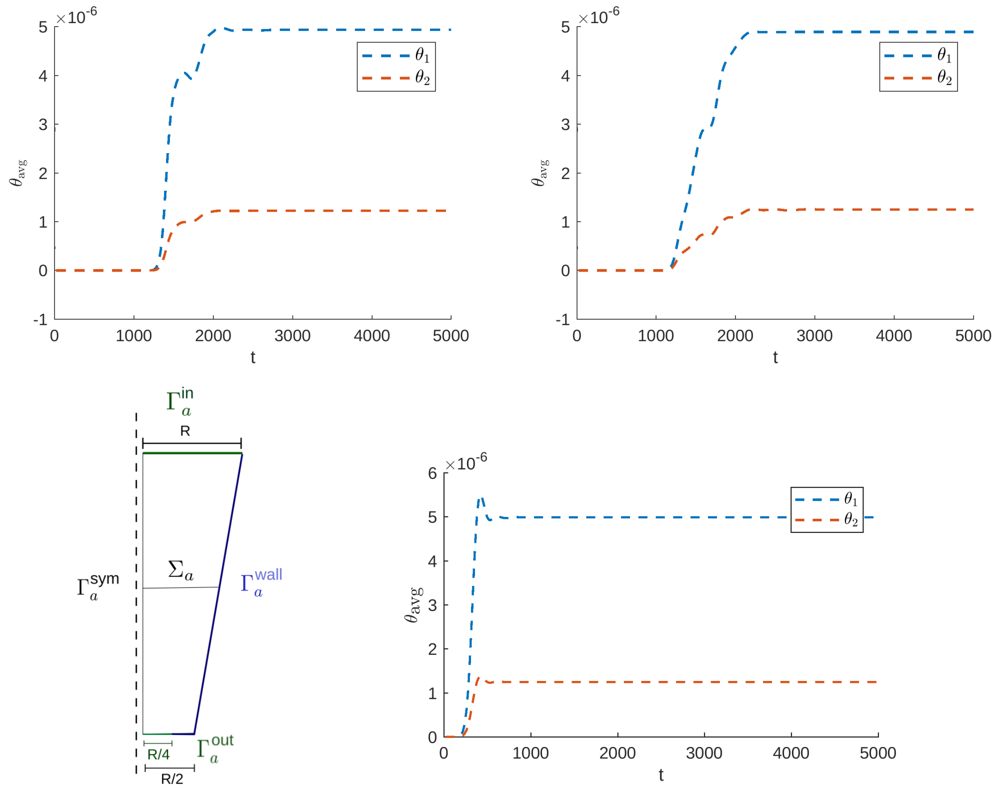

Fig. 4 Example 3 (two contaminants in two-layer filter): concentration of contaminants $\theta_{\mathrm{avg},i}(t)$ using a cylinder and changing order of layers (top); and similar computation using a truncated cone (bottom)

upper layer, whereas the second one passes the first layer to begin to be retained in the lower layer.

Now we change values to $s_1^{\max} = 10^{-7}\,\mathrm{kg/kg}$ and $s_2^{\max} = 10^{-6}\,\mathrm{kg/kg}$ and run the simulation for a longer time to assess how the swapping the order of layers and the geometry affect the contaminant removal, measured by $\theta_{\mathrm{avg}}(t)$. For the first two tests we use the same cylinder, altering only the order of the layers. As we can see from the top panels of Figure 4, reversing the order of the layers softens the transition towards saturation, but the most important behaviour is reached essentially at the same time in both cases. We also test with a truncated cone (see dimensions in the bottom left panel of Figure 4). The saturation is now achieved in a much shorter time, which could be explained by a combined effect of volume reduction (and therefore of adsorbent mass), and faster flow patterns that decrease the retention time and thus the adsorption of the system.

Acknowledgement. We are thankful to Ian Griffiths (Oxford) for the stimulating discussions about water filter models and for providing the experimental data employed in Section 6.2.

In addition, this work has been partially supported by projects CRHIAM, project CONICYT / FONDAP / 15130015; CONICYT / PIA / AFB170001; by CONICYT through the Becas-Chile program for foreign students; by the INRIA Associated Team "Efficient numerical schemes for non-local transport phenomena" (NOLOCO; 2018–2020); by SENESCYT Ecuador through the postgraduate scholarship program; by the Laboratory of Mathematical Modelling, Institute of Personalised Medicine, Sechenov

University; and by the Oxford Centre for Doctoral Training on Industrially Focused Mathematical Modelling.

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