An HDG method for Maxwell's equations in heterogeneous media

Liliana Camargo[a], Bibiana López-Rodríguez[a], Mauricio Osorio[a,*], Manuel Solano[b,c]

[a] *Escuela de Matemáticas, Universidad Nacional de Colombia, sede Medellín, Colombia*
[b] *Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas Universidad de Concepción, Concepción, Chile*
[c] *Centro de Investigación en Ingeniería Matemática CI²MA, Universidad de Concepción, Concepción, Chile*

Abstract

We analyze a hybridizable discontinuous Galerkin (HDG) method for the time harmonic Maxwell's equations arising from modeling photovoltaic solar cells. The problem is set in an inhomogeneous domain with a polyhedral connected boundary and the divergence-free condition is imposed using a Lagrange multiplier. We prove the HDG scheme is well-posed up to some frequencies and derive an stability estimate. Moreover, we prove that the method is optimal, that is, the L^2-norm of the error of the approximation in both, the electric and magnetic fields, are of order h^{k+1}, where h is the meshize and k the polynomial degree of the local approximation spaces. Numerical examples are shown to validate the theory.

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1. Introduction

Thin film photovoltaic solar cells are devices of thickness between few nanometers and tens micrometers that collect sunlight, converting it to electrical energy. In particular, it is of interest the study of devices comprising a periodically corrugated metallic backreflector because they are able to excite surface plasmonic polariton waves [1] and, as a consequence, enhance the intensity of the electromagnetic field in the cell. In this context, structures with periodic surface-relief gratings have been intensively studied during the last decade: amorphous silicon thin film tandem solar cell [2], rugate filters [1], periodic multilayered isotropic dielectric material on top of the metallic backreflector [3, 4], just to name a few.

The optimal design of these type of structures requires the maximization of quantities of interest such as light absorption, solar-spectrum-integrated power-flux density and spectrally averaged electron–hole pair density [4, 5], where the following Maxwell's equations [6] in the frequency domain must be solved for a wide range of geometrical and optical parameters:

$$\begin{aligned} \nabla \times \mathbf{E} &= i\,\omega\,\mu_0\,\mathbf{H} && \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H} &= \mathbf{J} - i\,\omega\,\varepsilon_0\,\varepsilon\,\mathbf{E} && \text{in } \mathbb{R}^3, \end{aligned} \tag{1}$$

*Corresponding author
Email addresses: `limcamargoma@unal.edu.co` (Liliana Camargo), `blopezr@unal.edu.co` (Bibiana López-Rodríguez), `maosorio@unal.edu.co` (Mauricio Osorio), `msolano@ing-mat.udec.cl` (Manuel Solano)

complemented with the effect of an incident field (sunlight). Here, $\mathbf{E}$ denotes the electric field, $\mathbf{H}$ the magnetic field, $\mathbf{J}$ the current density, ω the frequency, μ_0 the permeability of free space, ε_0 the permittivity of free space and ε the relative permittivity, which is a complex-valued function. Also, an $e^{-i\omega t}$ dependence on time $t > 0$ is implicit. It is convenient to define the free-space wavenumber $\kappa := \omega\sqrt{\varepsilon_0\mu_0}$, the free-space wavelength $\lambda_0 := 2\pi/\kappa$ and the intrinsic impedance of the free space $\eta_0 := \sqrt{\mu_0/\varepsilon_0}$.

The most commonly used numerical methods to approximate the solution of (1) are the exact modal method [7], the method of moments [8], the rigorous coupled-wave approach (RCWA) [9], the finite element method (FEM) [6], and the finite-difference time-domain (FDTD) method [10]. In the context of Galerkin methods, by taking the advantage of the periodicity of the surface relief, the equations are solved in a bounded domain $\Omega \subset \mathbb{R}^3$ occupied by one period of the periodic structure illuminated by the incident sunlight. On the vertical walls of Ω, quasi-periodic boundary conditions are imposed, whereas on top and bottom boundaries, suitable outgoing radiation conditions are considered [11, 12]. These radiation conditions can be handled by approximations of Dirichlet-to-Neumann operators that couple the solution in the free space with the solution in the bounded domain. An alternative to the latter approach, is to consider a perfectly matched layer technique in order to attenuate both outgoing and evanescent waves [13, 14].

The aim of this work is to contribute to the development of hybridizable discontinuous Galerkin (HDG) methods in simulations of photovoltaic devices. In particular, we restrict ourselves on analyzing an HDG scheme for (1) in a bounded domain with prescribed boundary conditions since, to the best of our knowledge, the theory is not fully developed when the electric permittiviy is complex-valued. Even though this is a simplification of the original problem, it involves several important issues that must be addressed first. In other words, our contribution constitutes a stepping stone towards developing an HDG method for the original problem arisen from photovoltaic cells modeling.

In recent years, most of the HDG methods for Maxwell's equations have been proposed and analysed assuming the electric permittivity ε is a positive real number, which is in general not true in light harvesting devices: Maxwell's operator [15, 16, 17], eddy current problems [18], Maxwell's equations in frequency-domain [19, 20], time-domain [21]. On the other hand, to the best of our knowledge, few contributions of HDG methods considering complex-valued permittivity can be found in the literature, for instance [22] and [23]. Motivated by the application to photovoltaic devices, the aim of our work is to introduce and analyze an HDG method considering an heterogeneous medium occupying the bounded domain Ω. In particular, Ω is divided in two disjoint domains: Ω_d and Ω_m, where Ω_d is occupied by an isotropic dielectric material and Ω_m is a metallic region. On the first region we assume a real and positive relative electric permittivity, while on the metallic one, the relative electric permittivity is assumed to be complex-valued with negative real part. This introduces additional difficulties that we address out using available techniques for the Helmholtz equation [24] and the Maxwell's equations with high wave number [19].

After defining a convenient tetrahedrization for the non homogeneous domain, we propose an HDG formulation for the Maxwell equation and we carry out a full error analysis. We highlight that in order to impose a divergence free condition to $\varepsilon\mathbf{E}$, a Lagrange multiplier p needs to be introduced as in [17], [18] and [25]. Then, by using a duality argument we deduce error estimates for $\mathbf{E}$ and $\nabla \times \mathbf{E}$.

The remaining of this paper is organized as follows. In Section 2 we define the truncated domain and introduce the boundary value problem. In Section 3, we set the approximation spaces, define the HDG formulation and prove it is well posed. Then, we employ a duality argument with the aim of getting a stability estimate. In Section 4, the error estimates for the HDG scheme are derived, and finally, in Section 5 we show some numerical results.

2. Problem statement

Let us begin by setting a preliminary notation that will be used through the manuscript. We use standard simplified terminology for Sobolev spaces and norms, where vector-valued functions are bold-faced. In particular, if $\mathscr{O}$ is a domain in $\mathbb{R}^3$, Σ is an open or closed Lipschitz curve, and $s \in \mathbb{R}$, we set $\mathbf{H}^s(\mathscr{O}) := [\mathrm{H}^s(\mathscr{O})]^3$ and $\mathbf{H}^s(\Sigma) := [\mathrm{H}^s(\Sigma)]^3$. However, when $s = 0$ we write $\mathbf{L}^2(\mathscr{O})$ and $\mathbf{L}^2(\Sigma)$ instead of $\mathbf{H}^0(\mathscr{O})$ and $\mathbf{H}^0(\Sigma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{s,\mathscr{O}}$ for $\mathrm{H}^s(\mathscr{O})$ and $\mathbf{H}^s(\mathscr{O})$; and $\|\cdot\|_{s,\Sigma}$ for $\mathrm{H}^s(\Sigma)$ and $\mathbf{H}^s(\Sigma)$, when $s = 0$ we will ignore the first subindex. For $s > 0$, we write $|\cdot|_{s,\mathscr{O}}$ for the $\mathbf{H}^s$- and H^s-seminorms. In addition, we introduce the following spaces

$$\mathrm{H}(\mathbf{div}_\varepsilon;\mathscr{O}) := \{\mathbf{w} \in \mathbf{L}^2(\mathscr{O}) : \nabla\cdot(\varepsilon\mathbf{w}) \in \mathbf{L}^2(\mathscr{O})\}, \qquad \mathrm{H}(\mathbf{div}_\varepsilon^0;\mathscr{O}) := \{\mathbf{w} \in \mathbf{L}^2(\mathscr{O}) : \nabla\cdot(\varepsilon\mathbf{w}) = 0\},$$
$$\mathrm{H}(\mathbf{curl};\mathscr{O}) := \{\mathbf{w} \in \mathbf{L}^2(\mathscr{O}) : \nabla\times\mathbf{w} \in \mathbf{L}^2(\mathscr{O})\}, \qquad \mathrm{H}_0(\mathbf{curl};\mathscr{O}) := \{\mathbf{w} \in \mathrm{H}(\mathbf{curl};\mathscr{O}) : \mathbf{w}\times\mathbf{n}|_{\partial\mathscr{O}} = 0\},$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\mathscr{O}$. For a vector-valued function $\mathbf{w}$ defined on a face F, we denote by $\mathbf{w}^t := (\mathbf{n}\times\mathbf{w})\times\mathbf{n}$ and $\mathbf{w}^n := (\mathbf{w}\cdot\mathbf{n})\mathbf{n}$ its tangential and normal components, respectively.

Finally, to avoid proliferation of unimportant constants, when there is no confusion will write $A \lesssim B$, whenever there exists $C > 0$, independent of the meshsize, such that $A \leq CB$.

As we mentioned in the introduction, we consider one period of a photovoltaic solar cell, denoted by $\Omega := (0,L)\times(0,L)\times(0,M)$, whose boundary $\Gamma := \partial\Omega$ is polyhedral and connected; and L, M are positive numbers. Ω is assumed to be simply connected and is divided into two parts by a piecewise plane interface as Fig. 1 shows: a metallic region Ω_m with relative electric permittivity $\varepsilon_m \in \mathbb{C}$ satisfying $\mathrm{Re}(\varepsilon_m) < 0$ and $\mathrm{Im}(\varepsilon_m) > 0$, and a dielectric region Ω_d : with relative permittivity $\varepsilon_d \in \mathbb{R}^+$. Specific values of the relative permittivities of these materials can be found in [3] .

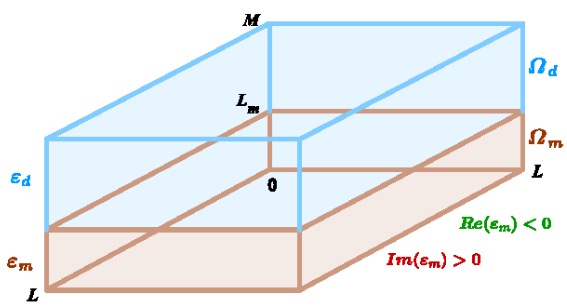

Figure 1: Example of domain having a flat interface between the metallic region Ω_m of thickness L_m and a dielectric region Ω_d.

Given $\hat{\mathbf{g}} \in \mathbf{L}^2(\Gamma)$ (or more precisely in $\gamma_t\left(\mathrm{H}(\mathbf{curl};\Omega)\cap\mathrm{H}(\mathbf{div}_\varepsilon^0;\Omega)\right)$, as will be explained in the appendix) and $\mathbf{J} \in \mathrm{H}(\mathbf{div}^0;\Omega)$, we look for $\mathbf{E}$ and $\mathbf{H}$ such that

$$\begin{aligned} \nabla\times\mathbf{E} &= i\,\omega\,\mu_0\,\mathbf{H} && \text{in} \quad \Omega, \\ \nabla\times\mathbf{H} &= \mathbf{J} - i\,\omega\,\varepsilon_0\,\varepsilon\,\mathbf{E} && \text{in} \quad \Omega, \\ \mathbf{E}\times\mathbf{n} &= \hat{\mathbf{g}} && \text{on} \quad \Gamma, \end{aligned} \tag{2}$$

where $\mathbf{n}$ denotes the outward unit normal of Γ. We recall that ε is the relative permittivity, $\varepsilon_0 = 8.854\times10^{-12}$ Fm^{-1} and $\mu_0 = 4\pi\times10^{-7}$ Hm^{-1}. It is convenient to write (2) in terms of relative quantities. To that end we introduce the change of variable $\mathbf{u} := \varepsilon_0^{1/2}\mathbf{E}$ and $\mathbf{v} := i\kappa\mu_0^{1/2}\mathbf{H}$. Moreover, the condition $\nabla\cdot(\varepsilon\mathbf{E}) = 0$ in Ω, deduced from the second equation, will be imposed by a Lagrange multiplier p. Then, from (2), the system to solve is the following: Find $\mathbf{u}$, $\mathbf{v}$ and p such that

$$\begin{aligned} \mathbf{v} - \nabla\times\mathbf{u} = 0 &\quad \text{in} \quad \Omega,\\ \nabla\times\mathbf{v} - \kappa^2\varepsilon\mathbf{u} + \overline{\varepsilon}\nabla p = \mathbf{f} &\quad \text{in} \quad \Omega,\\ \nabla\cdot\varepsilon\mathbf{u} = 0 &\quad \text{in} \quad \Omega,\\ \mathbf{u}\times\mathbf{n} = \mathbf{g} &\quad \text{on} \quad \Gamma,\\ p = 0 &\quad \text{on} \quad \Gamma, \end{aligned} \tag{3}$$

where $\mathbf{f} := i\kappa\mu_0^{1/2}\mathbf{J}$, $\mathbf{g} = \varepsilon_0^{1/2}\hat{\mathbf{g}}$ and $\overline{\varepsilon}$ is the complex conjugate of ε. By Lemma Appendix A.1, if $\kappa^2\varepsilon_d$ is not an eigenvalue of the operator $\nabla\times\nabla\times$, then (3) has a unique solution.

3. The HDG method

We consider a shape-regular simplicial tetrahedrization $\mathscr{T}_h$ of Ω, such that each $\overset{\circ}{K}\in\mathscr{T}_h$ is completely contained in Ω_m or Ω_d. Then, $\mathscr{T}_h^m$ and $\mathscr{T}_h^d$ will denote the sets of tetrahedra lying in Ω_m and Ω_d, resp. Furthermore, we define $\partial\mathscr{T}_h := \{\partial K : K\in\mathscr{T}_h\}$ and $\mathscr{E}_h := \mathscr{E}_I\cup\mathscr{E}_\Gamma$, where $\mathscr{E}_I$ and $\mathscr{E}_\Gamma$ denote the interior and boundary faces induced by $\mathscr{T}_h$, respectively. Given a region $\mathscr{O}\subset\mathbb{R}^3$, we denote by $(\cdot,\cdot)_{\mathscr{O}}$ and $\langle\cdot,\cdot\rangle_{\partial\mathscr{O}}$ the $L^2(\mathscr{O})$ and $L^2(\partial\mathscr{O})$ inner products, respectively. We set also $(\cdot,\cdot)_{\mathscr{T}_h} = \sum_{K\in\mathscr{T}_h}(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_{\partial\mathscr{T}_h} = \sum_{K\in\mathscr{T}_h}\langle\cdot,\cdot\rangle_{\partial K}$. In addition, $\mathbb{P}_k(A)$ denotes the space of complex-valued polynomials of degree less or equal to $k\geq 0$ defined over a region A and we set $\mathbf{P}_k(A) := [\mathbb{P}_k(A)]^3$.

Considering the above tetrahedrization of Ω, we define the following approximation spaces

$$\begin{aligned} \mathrm{V}_h &:= \left\{\mathbf{w}\in\mathbf{L}^2(\Omega) : \mathbf{w}\,|_K\in\mathbf{P}_k(K), \forall\, K\in\mathscr{T}_h\right\},\\ \mathrm{Q}_h &:= \left\{q\in\mathrm{L}^2(\Omega) : q\,|_K\in\mathbb{P}_k(K), \forall\, K\in\mathscr{T}_h\right\},\\ \mathrm{M}_h &:= \left\{\mu\in\mathrm{L}^2(\mathscr{E}_h) : \mu\,|_F\in\mathbb{P}_k(F), \forall\, F\in\mathscr{E}_h\right\},\\ \mathrm{M}_h^t &:= \left\{\beta\in\mathbf{L}^2(\mathscr{E}_h) : \beta\,|_F\in\mathbf{P}_k(F), (\beta\cdot\mathbf{n})\,|_F = 0, \forall\, F\in\mathscr{E}_h\right\}. \end{aligned}$$

The HDG scheme associated to (3) seeks the approximation $(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}_h^t,\widehat{p}_h)\in\mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$ of the exact solution $(\mathbf{v},\mathbf{u},p,\mathbf{u}^t|_{\mathscr{E}_h},p|_{\mathscr{E}_h})$, satisfying

$$(\mathbf{v}_h,\mathbf{w})_{\mathscr{T}_h} - (\mathbf{u}_h,\nabla\times\mathbf{w})_{\mathscr{T}_h} - \langle\widehat{\mathbf{u}}_h^t,\mathbf{w}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} = 0, \tag{4a}$$

$$(\mathbf{v}_h,\nabla\times\mathbf{z})_{\mathscr{T}_h} + \langle\widehat{\mathbf{v}}_h^t,\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \kappa^2(\varepsilon\mathbf{u}_h,\mathbf{z})_{\mathscr{T}_h} - (p_h,\nabla\cdot(\varepsilon\mathbf{z}))_{\mathscr{T}_h} + \langle\widehat{p}_h,\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} = \mathbf{F}(\mathbf{z}), \tag{4b}$$

$$-(\varepsilon\,\mathbf{u}_h,\nabla t)_{\mathscr{T}_h} + \langle\widehat{\varepsilon\,\mathbf{u}_h^n}\cdot\mathbf{n},t\rangle_{\partial\mathscr{T}_h} = 0, \tag{4c}$$

$$\langle\mathbf{n}\times\widehat{\mathbf{v}}_h^t,\eta\rangle_{\partial\mathscr{T}_h\setminus\Gamma} = 0, \tag{4d}$$

$$\langle\widehat{\varepsilon\,\mathbf{u}_h^n}\cdot\mathbf{n},\mu\rangle_{\partial\mathscr{T}_h\setminus\Gamma} = 0, \tag{4e}$$

$$\langle\widehat{\mathbf{u}}_h^t,\eta\rangle_\Gamma = \langle\mathbf{g},\eta\times\mathbf{n}\rangle_\Gamma, \tag{4f}$$

$$\langle\widehat{p}_h,\mu\rangle_\Gamma = 0, \tag{4g}$$

for all $(\mathbf{w},\mathbf{z},t,\eta,\mu)\in \mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$, where $\mathbf{F}:\mathbf{P}_k(\mathscr{T}_h)\to\mathbb{C}$ is such that $\mathbf{F}(\mathbf{z})=(\mathbf{f},\mathbf{z})_{\mathscr{T}_h}$ and the numerical fluxes defined on $\partial\mathscr{T}_h$ are given by

$$\mathbf{n}\times\widehat{\mathbf{v}}_h^t := \mathbf{n}\times\mathbf{v}_h^t+\tau(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t), \tag{4h}$$

$$\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n} := \varepsilon\mathbf{u}_h^n\cdot\mathbf{n}+\tau_n(p_h-\widehat{p}_h), \tag{4i}$$

being τ and τ_n positive stabilization parameters.

Remark 3.1. *Using similar arguments as in Section* 3 *of [18] and Section* 3 *of [20], it can be proved that the HDG formulation* (4) *is locally conservative and consistent.*

3.1. Well-posedness

Let us start by establishing a Gårding-type identity.

Lemma 3.1. *If* $(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}_h^t,\widehat{p}_h)\in \mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$ *satisfies (4), then*

$$\begin{aligned}\|\mathbf{v}_h\|^2_{\mathscr{T}_h}+\|\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right)\|^2_{\partial\mathscr{T}_h}+\|\tau_n^{1/2}\left(p_h-\widehat{p}_h\right)\|^2_{\partial\mathscr{T}_h}&\\ +\kappa^2\left(|Re(\varepsilon_m)|+i\,Im(\varepsilon_m)\right)\|\mathbf{u}_h\|^2_{\mathscr{T}_h^m}=\kappa^2\varepsilon_d\|\mathbf{u}_h\|^2_{\mathscr{T}_h^d}+\overline{\mathbf{F}}(\mathbf{u}_h)-\langle\mathbf{g},\widehat{\mathbf{v}}_h^t\rangle_\Gamma.&\end{aligned} \tag{5}$$

Proof. Let $\mathbf{w}:=\mathbf{v}_h$, $\mathbf{z}:=\mathbf{u}_h$ and $t:=p_h$ in (4). After applying the Green identity in $\mathrm{H}(\mathbf{curl};\mathscr{T}_h)$ to (4a) and the Green identity in $\mathrm{H}(\mathbf{div};\mathscr{T}_h)$ to (4c), we have

$$\begin{aligned}(\mathbf{v}_h,\mathbf{v}_h)_{\mathscr{T}_h}-(\nabla\times\mathbf{u}_h,\mathbf{v}_h)_{\mathscr{T}_h}+\langle\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t,\mathbf{v}_h\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}&=0,\\ (\nabla\cdot(\varepsilon\,\mathbf{u}_h),p_h)_{\mathscr{T}_h}-\langle\varepsilon\,\mathbf{u}_h\cdot\mathbf{n},p_h\rangle_{\partial\mathscr{T}_h}+\langle\widehat{\varepsilon\,\mathbf{u}_h^n}\cdot\mathbf{n},p_h\rangle_{\partial\mathscr{T}_h}&=0.\end{aligned}$$

Then, adding the two above expressions with the conjugate of equation (4b), we have

$$\begin{aligned}(\mathbf{v}_h,\mathbf{v}_h)_{\mathscr{T}_h}-\langle\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\mathbf{v}_h^t\rangle_{\partial\mathscr{T}_h}+\langle\mathbf{u}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_{\partial\mathscr{T}_h}-\kappa^2(\mathbf{u}_h,\varepsilon\mathbf{u}_h)_{\mathscr{T}_h}&\\ +\langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},p_h\rangle_{\partial\mathscr{T}_h}-\langle\varepsilon\mathbf{u}_h^n\cdot\mathbf{n},p_h-\widehat{p}_h\rangle_{\partial\mathscr{T}_h}=\overline{\mathbf{F}}(\mathbf{u}_h).&\end{aligned}$$

Furthermore, if here we add and subtract $\langle\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_{\partial\mathscr{T}_h}$ and $\langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},\widehat{p}_h\rangle_{\partial\mathscr{T}_h}$, it follows that

$$\begin{aligned}\|\mathbf{v}_h\|^2_{\mathscr{T}_h}+\langle\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t,\mathbf{n}\times(\widehat{\mathbf{v}}_h^t-\mathbf{v}_h^t)\rangle_{\partial\mathscr{T}_h}+\langle\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_{\partial\mathscr{T}_h}-\kappa^2(\mathbf{u}_h,\varepsilon\mathbf{u}_h)_{\mathscr{T}_h}&\\ +\langle(\widehat{\varepsilon\mathbf{u}_h^n}-\varepsilon\mathbf{u}_h^n)\cdot\mathbf{n},p_h-\widehat{p}_h\rangle_{\partial\mathscr{T}_h}+\langle\widehat{\varepsilon\,\mathbf{u}_h^n}\cdot\mathbf{n},\widehat{p}_h\rangle_{\partial\mathscr{T}_h}=\overline{\mathbf{F}}(\mathbf{u}_h),&\end{aligned}$$

from where, after using (4e), (4g), (4h) and (4i), we have

$$\begin{aligned}\|\mathbf{v}_h\|^2_{\mathscr{T}_h}+\langle\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t,\tau(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t)\rangle_{\partial\mathscr{T}_h}+\langle\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_{\partial\mathscr{T}_h}-\kappa^2(\mathbf{u}_h,\varepsilon\mathbf{u}_h)_{\mathscr{T}_h}&\\ +\langle\tau_n(p_h-\widehat{p}_h),p_h-\widehat{p}_h\rangle_{\partial\mathscr{T}_h}=\overline{\mathbf{F}}(\mathbf{u}_h).&\end{aligned} \tag{6}$$

Now, from (4d) and (4f), we have $\langle\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_{\partial\mathscr{T}_h}=\langle\widehat{\mathbf{u}}_h^t,\mathbf{n}\times\widehat{\mathbf{v}}_h^t\rangle_\Gamma=\langle\mathbf{g},\mathbf{n}\times(\widehat{\mathbf{v}}_h^t\times\mathbf{n})\rangle_\Gamma=\langle\mathbf{g},\widehat{\mathbf{v}}_h^t\rangle_\Gamma$. Thus, substituting the previous expression in (6), we obtain

$$\|\mathbf{v}_h\|^2_{\mathscr{T}_h}+\|\tau^{1/2}(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t)\|^2_{\partial\mathscr{T}_h}+\langle\mathbf{g},\widehat{\mathbf{v}}_h^t\rangle_\Gamma-\kappa^2\overline{\varepsilon_m}\|\mathbf{u}_h\|^2_{\mathscr{T}_h^m}-\kappa^2\varepsilon_d\|\mathbf{u}_h\|^2_{\mathscr{T}_h^d}+\|\tau_n^{1/2}(p_h-\widehat{p}_h)\|^2_{\partial\mathscr{T}_h}=\overline{\mathbf{F}}(\mathbf{u}_h)$$

and (5) follows. □

We will show that the HDG scheme (4) is well-posed assuming that $\kappa^2\varepsilon_d$ is not an eigenvalue of the following problem:
Find $\lambda \in \mathbb{R}$ and $(\varphi_h, \psi_h, \sigma_h, \widehat{\psi}_h^t, \widehat{\sigma}_h) \in \mathrm{V}_h \times \mathrm{V}_h \times \mathrm{Q}_h \times \mathrm{M}_h^t \times \mathrm{M}_h$ such that

$$\begin{aligned}
(\varphi_h, \mathbf{s})_{\mathscr{T}_h} - (\psi_h, \nabla\times\mathbf{s})_{\mathscr{T}_h} - \langle \widehat{\psi}_h^t, \gamma_t(\mathbf{s})\rangle_{\partial\mathscr{T}_h} &= 0,\\
(\varphi_h, \nabla\times\mathbf{q})_{\mathscr{T}_h} + \langle \widehat{\varphi}_h^t, \gamma_t(\mathbf{q})\rangle_{\partial\mathscr{T}_h} - \kappa^2(\varepsilon_m\psi_h, \mathbf{q})_{\mathscr{T}_h^m} - (\sigma_h, \nabla\cdot(\varepsilon\mathbf{q}))_{\mathscr{T}_h} + \langle \widehat{\sigma}_h, \varepsilon\mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} &= \lambda(\psi_h, \mathbf{q})_{\mathscr{T}_h^d},\\
-(\varepsilon\,\psi_h, \nabla r)_{\mathscr{T}_h} + \langle \widehat{\varepsilon\,\psi_h^n}\cdot\mathbf{n}, r\rangle_{\partial\mathscr{T}_h} &= 0,\\
\langle \mathbf{n}\times\widehat{\varphi}_h^t, \eta\rangle_{\partial\mathscr{T}_h\setminus\Gamma} &= 0,\\
\langle \widehat{\varepsilon\,\psi_h^n}\cdot\mathbf{n}, \zeta\rangle_{\partial\mathscr{T}_h\setminus\Gamma} &= 0,\\
\langle \widehat{\psi}_h^t, \eta\rangle_\Gamma &= 0,\\
\langle \widehat{\sigma}_h, \zeta\rangle_\Gamma &= 0,
\end{aligned} \tag{7}$$

for all $(\mathbf{s}, \mathbf{q}, r, \eta, \zeta) \in \mathrm{V}_h \times \mathrm{V}_h \times \mathrm{Q}_h \times \mathrm{M}_h^t \times \mathrm{M}_h$, where

$$\begin{aligned}
\mathbf{n}\times\widehat{\varphi}_h^t &= \mathbf{n}\times\varphi_h^t + \tau(\psi_h^t - \widehat{\psi}_h^t),\\
\widehat{\varepsilon\psi_h^n}\cdot\mathbf{n} &= \varepsilon\psi_h^n\cdot\mathbf{n} + \tau_n(\sigma_h - \widehat{\sigma}_h).
\end{aligned}$$

Theorem 3.1. *If $\kappa^2\varepsilon_d$ is not an eigenvalue of* (7)*, then (4) is well-posed.*

Proof. Here we use similar ideas to the ones used in proposition 2 of [20]. First notice that if we consider the homogeneous problem associate to (4), then the Gårding-type identity given by (5) has the form

$$\|\mathbf{v}_h\|_{\mathscr{T}_h}^2 + \|\tau^{1/2}\left(\mathbf{u}_h^t - \widehat{\mathbf{u}}_h^t\right)\|_{\partial\mathscr{T}_h}^2 + \kappa^2\left(|\mathrm{Re}\,(\varepsilon_m)| + i\,\mathrm{Im}\,(\varepsilon_m)\right)\|\mathbf{u}_h\|_{\mathscr{T}_h^m}^2 + \|\tau_n^{1/2}\left(p_h - \widehat{p}_h\right)\|_{\partial\mathscr{T}_h}^2 = \kappa^2\varepsilon_d\|\mathbf{u}_h\|_{\mathscr{T}_h^d}^2. \tag{8}$$

Now, after applying to (7) similar arguments to the ones used to proof Lemma 3.1, we get

$$\|\varphi_h\|_{\mathscr{T}_h}^2 + \|\tau^{1/2}\left(\psi_h^t - \widehat{\psi}_h^t\right)\|_{\partial\mathscr{T}_h}^2 + \kappa^2(|\mathrm{Re}(\varepsilon_m)| + i\,\mathrm{Im}(\varepsilon_m))\|\psi_h\|_{\mathscr{T}_h^m}^2 + \|\tau_n^{1/2}\left(\sigma_h - \widehat{\sigma}_h\right)\|_{\partial\mathscr{T}_h}^2 = \lambda\|\psi_h\|_{\mathscr{T}_h^d}^2. \tag{9}$$

Then, we conclude from (8) and (9) that $\mathbf{u}_h = 0$ in $\mathscr{T}_h^d$ since λ is not an eigenvalue of (7). Then, the real part of (8) is $\|\mathbf{v}_h\|_{\mathscr{T}_h}^2 + \|\tau^{1/2}\left(\mathbf{u}_h^t - \widehat{\mathbf{u}}_h^t\right)\|_{\partial\mathscr{T}_h}^2 + \kappa^2\,|\mathrm{Re}(\varepsilon_m)|\,\|\mathbf{u}_h\|_{\mathscr{T}_h^m}^2 + \|\tau_n^{1/2}\left(p_h - \widehat{p}_h\right)\|_{\partial\mathscr{T}_h}^2 = 0$, from which, $\mathbf{v}_h = 0$ in $\mathscr{T}_h$, $\mathbf{u}_h^t = \widehat{\mathbf{u}}_h^t$ on $\partial\mathscr{T}_h$, $\mathbf{u}_h = 0$ in $\mathscr{T}_h^m$ and $p_h = \widehat{p}_h$ on $\partial\mathscr{T}_h$. Therefore, $\mathbf{u}_h = 0$ in $\mathscr{T}_h$. Finally, by (4b) and (4h), we have $0 = -\left(p_h, \nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h} + \langle \widehat{p}_h, \varepsilon\,\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} = (\nabla p_h, \varepsilon\mathbf{z})_{\mathscr{T}_h}$, $\forall \mathbf{z} \in \mathrm{V}_h$, which together with (4g) and the fact that $\widehat{p}_h = p_h$, implies $p_h = 0$ on $\partial\Omega$. □

3.2. Stability estimate

We first state the main result of this section and postpone its proof to the end.

Theorem 3.2. *Suppose the regularity estimate* (11) *(stated below) holds and τ, τ_n are of order one. Then, there exists $h_0 > 0$, such that for all $h < h_0$, it holds*

$$\|\mathbf{v}_h\|_{\mathscr{T}_h}^2 + \|\tau^{1/2}\left(\mathbf{u}_h^t - \widehat{\mathbf{u}}_h^t\right)\|_{\partial\mathscr{T}_h}^2 + \|\tau_n^{1/2}\left(p_h - \widehat{p}_h\right)\|_{\partial\mathscr{T}_h}^2 + \kappa^2\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h}^2 \lesssim \|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})}^2 + h^{-1}\|\mathbf{g}\|_\Gamma^2.$$

Remark 3.2. *Theorem 3.2 implies* (4) *is well-posed without using problem* (7)*, relaxing this assumption in Theorem 3.1. In fact, for h small enough, it is enough to ask* $\kappa^2\varepsilon_d$ *not to be an eigenvalue of the operator* $\nabla\times\nabla\times$*. On the other hand, the price to pay is the additional regularity requirement* (11).

From (5), we observe that we need to bound $\kappa^2\varepsilon_d\|\mathbf{u}_h\|^2_{\mathscr{T}_h^d}$ in terms of the data, in order to obtain a stability estimate. To that end, we employ a duality argument and consider the following auxiliary problem: Given $\Theta\in\mathbf{L}^2(\Omega)$, we look for ϕ,ψ,ρ such that

$$\begin{aligned}
\phi-\nabla\times\psi&=0\quad\text{in }\Omega, &(10a)\\
\nabla\times\phi-\kappa^2\overline{\varepsilon}\psi+\overline{\varepsilon}\nabla\rho&=\Theta\quad\text{in }\Omega, &(10b)\\
\nabla\cdot(\varepsilon\psi)&=0\quad\text{in }\Omega, &(10c)\\
\psi\times\mathbf{n}&=0\quad\text{on }\Gamma, &(10d)\\
\rho&=0\quad\text{on }\Gamma. &(10e)
\end{aligned}$$

We assume the following regularity estimate holds true for all $s\in(0,1)$

$$\|\rho\|_{s+1,\Omega}+\|\phi\|_{s,\Omega_d}+\|\phi\|_{s,\Omega_m}+\|\varepsilon_d\psi\|_{s+1,\Omega_d}+\|\varepsilon_m\psi\|_{s+1,\Omega_m}\lesssim\|\Theta\|_{\Omega}. \tag{11}$$

In Appendix B we will comment on situations where this assumption is satisfied.

We now introduce the HDG projection operator: Given $(\mathbf{w},q)\in\mathbf{H}^1(\mathscr{T}_h)\times H^1(\mathscr{T}_h)$, $(\Pi_{\mathrm{V}}\mathbf{w},\Pi_{\mathrm{Q}}q)\in\mathrm{V}_h\times Q_h$ is the only solution of

$$\begin{aligned}
(\Pi_{\mathrm{V}}\mathbf{w},\mathbf{z})_K&=(\mathbf{w},\mathbf{z})_K, &&\forall\mathbf{z}\in\mathbf{P}_{k-1}(K), &(12a)\\
(\Pi_{\mathrm{Q}}q,t)_K&=(q,t)_K, &&\forall t\in\mathbb{P}_{k-1}(K), &(12b)\\
\langle\varepsilon\Pi_{\mathrm{V}}\mathbf{w}\cdot\mathbf{n}+\tau_n\Pi_{\mathrm{Q}}q,\eta\rangle_F&=\langle\varepsilon\mathbf{w}\cdot\mathbf{n}+\tau_nq,\eta\rangle_F, &&\forall\eta\in\mathbb{P}_k(F),\ \ \forall F\in\partial K, &(12c)
\end{aligned}$$

on each $K\in\mathscr{T}_h$. Moreover if $l_{\mathbf{w}}$, $l_q\in[0,k]$, $\tau_n^*=\min\tau_n$, $\mathbf{w}\in\mathrm{H}^{l_{\mathbf{w}}+1}(\mathscr{T}_h)$ and $q\in\mathrm{H}^{l_q+1}(\mathscr{T}_h)$, then

$$\begin{aligned}
\|\Pi_{\mathrm{V}}\mathbf{w}-\mathbf{w}\|_{\mathscr{T}_h}&\lesssim h^{l_{\mathbf{w}}+1}|\mathbf{z}|_{l_{\mathbf{z}}+1,\mathscr{T}_h}+\frac{h^{l_q+1}}{\tau_n^*}|q|_{l_q+1,\mathscr{T}_h}, &(12d)\\
\|\Pi_{\mathrm{Q}}q-q\|_{\mathscr{T}_h}&\lesssim h^{l_q+1}|q|_{l_q+1,\mathscr{T}_h}. &(12e)
\end{aligned}$$

We refer to [26] for further details. In addition, we denote by P_{V} the standard L^2- orthogonal projector from $\mathbf{L^2}(\mathscr{T}_\mathbf{h})$ into V_h. It is well-known (c.f. [27]) that if $l_{\mathbf{w}}\in[0,k]$ and $\mathbf{w}\in\mathrm{H}^{l_{\mathbf{w}}+1}(\mathscr{T}_h)$, then

$$\|P_{\mathrm{V}}\mathbf{w}-\mathbf{w}\|_{\mathscr{T}_h}\lesssim h^{l_{\mathbf{w}}+1}|\mathbf{w}|_{l_{\mathbf{w}}+1,\mathscr{T}_h}. \tag{13}$$

Likewise, we define $P_{\mathrm{M}_h^t}:\mathbf{L}^2(\mathscr{E}_h)\to\mathrm{M}_h^t$ and $P_{\mathrm{M}_h}:\mathrm{L}^2(\mathscr{E}_h)\to\mathrm{M}_h$ as standard L^2- orthogonal projectors on M_h^t and M_h, respectively.

Let us also define $\mathrm{X}_0:=\mathrm{H}_0(\mathbf{curl};\Omega)\cap\mathrm{H}(\mathbf{div}^0_\varepsilon;\Omega)$, endowed with the norm $\|\mathbf{w}\|_{\mathrm{X}_0}:=\left(\|\nabla\times\mathbf{w}\|^2_\Omega+\|\mathbf{w}\|^2_\Omega\right)^{1/2}$. In the following lemma we prove an identity that allow us to bound $\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h}$.

Lemma 3.2. *Given* $\Theta\in\mathbf{L}^2(\Omega)$*, let* $(\phi,\psi,\rho)\in\mathrm{H}(\mathbf{curl};\Omega)\times\mathrm{X}_0\times H_0^1(\Omega)$ *and* $\left(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}_h^t,\widehat{p}\right)\in\mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$ *be the solutions of (10) and (4), respectively. It holds*

$$\begin{aligned}
(\mathbf{u}_h,\Theta)_{\mathscr{T}_h}=\;&\mathbf{F}(\Pi_V\psi)+\langle\mathbf{g},P_{\mathrm{M}_h^t}(\phi^t)\rangle_\Gamma+\langle\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{1/2}(\Pi_V\psi-\psi)\rangle_{\partial\mathscr{T}_h}\\
&+\langle p_h-\widehat{p}_h,\varepsilon(\Pi_V\psi-\psi)\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}+\langle\tau_n(p_h-\widehat{p}_h),\Pi_Q\rho-\rho\rangle_{\partial\mathscr{T}_h}\\
&+\langle\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}(P_V\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}+\kappa^2(\varepsilon\mathbf{u}_h,\Pi_V\psi-\psi)_{\mathscr{T}_h}.
\end{aligned}\tag{14}$$

Proof. In order to prove this lemma we follow the guidelines given in the proof of Lemma 3.3 of [24]. To begin, let us test (10b) with $\mathbf{u}_h$

$$(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = (\mathbf{u}_h,\nabla\times\phi)_{\mathscr{T}_h} - \left(\mathbf{u}_h,\kappa^2\overline{\varepsilon}\psi\right)_{\mathscr{T}_h} + (\mathbf{u}_h,\overline{\varepsilon}\nabla\rho)_{\mathscr{T}_h}. \tag{15}$$

Now, using the Green identities, (12b) and the orthogonality property of P_{V}, we have

$$\begin{aligned}(\mathbf{u}_h,\nabla\times\phi)_{\mathscr{T}_h} &= (\mathbf{u}_h,\nabla\times P_{\mathrm{V}}\phi)_{\mathscr{T}_h} + \langle\mathbf{u}_h^t,(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h},\\ (\mathbf{u}_h,\overline{\varepsilon}\nabla\rho)_{\mathscr{T}_h} &= (\varepsilon\mathbf{u}_h,\nabla\Pi_{\mathrm{Q}}\rho)_{\mathscr{T}_h} + \langle\varepsilon\mathbf{u}_h^n\cdot\mathbf{n},\rho-\Pi_{\mathrm{Q}}\rho\rangle_{\partial\mathscr{T}_h}.\end{aligned}$$

Then, substituting these expressions in (15) and using (4a) with $\mathbf{w}=P_{\mathrm{V}}\phi$ in $\mathscr{T}_h$, we obtain

$$\begin{aligned}(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = {}& (\mathbf{v}_h,P_{\mathrm{V}}\phi)_{\mathscr{T}_h} - \langle\widehat{\mathbf{u}}_h^t,P_{\mathrm{V}}\phi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \langle\mathbf{u}_h^t,(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \kappa^2(\varepsilon\mathbf{u}_h,\psi)_{\mathscr{T}_h}\\ &+ (\varepsilon\mathbf{u}_h,\nabla\Pi_{\mathrm{Q}}\rho)_{\mathscr{T}_h} + \langle\varepsilon\mathbf{u}_h^n\cdot\mathbf{n},\rho-\Pi_{\mathrm{Q}}\rho\rangle_{\partial\mathscr{T}_h}.\end{aligned}$$

By (4c) with $t=\Pi_{\mathrm{Q}}\rho$, together with the fact that $(\mathbf{v}_h,P_{\mathrm{V}}\phi)_{\mathscr{T}_h} = (\mathbf{v}_h,\phi)_{\mathscr{T}_h} = (\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h}$ by (10a), we have

$$\begin{aligned}(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = {}& (\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} - \langle\widehat{\mathbf{u}}_h^t,P_{\mathrm{V}}\phi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \langle\mathbf{u}_h^t,(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \kappa^2(\varepsilon\mathbf{u}_h,\psi)_{\mathscr{T}_h}\\ &+ \langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},\Pi_{\mathrm{Q}}\rho\rangle_{\partial\mathscr{T}_h} + \langle\varepsilon\mathbf{u}_h^n\cdot\mathbf{n},\rho-\Pi_{\mathrm{Q}}\rho\rangle_{\partial\mathscr{T}_h}.\end{aligned}$$

Adding and subtracting $\langle\widehat{\mathbf{u}}_h^t,\phi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} = \langle\mathbf{n}\times\widehat{\mathbf{u}}_h^t,\phi^t\rangle_{\partial\mathscr{T}_h}$ and $\langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},\rho\rangle_{\partial\mathscr{T}_h}$, we obtain after rearranging terms that

$$\begin{aligned}(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = {}& (\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} + \langle\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \langle\mathbf{n}\times\widehat{\mathbf{u}}_h^t,\phi^t\rangle_{\partial\mathscr{T}_h}\\ &- \kappa^2(\varepsilon\mathbf{u}_h,\psi)_{\mathscr{T}_h} + \langle(\widehat{\varepsilon\mathbf{u}_h^n}-\varepsilon\mathbf{u}_h^n)\cdot\mathbf{n},\Pi_{\mathrm{Q}}\rho-\rho\rangle_{\partial\mathscr{T}_h} + \langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},\rho\rangle_{\partial\mathscr{T}_h}.\end{aligned}$$

By the ortogonality of P_{M_h}, (4e) and (10e), we obtain $\langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},\rho\rangle_{\partial\mathscr{T}_h} = \langle\widehat{\varepsilon\mathbf{u}_h^n}\cdot\mathbf{n},P_{\mathrm{M}_h}(\rho)\rangle_{\partial\mathscr{T}_h} = 0$ and then

$$\begin{aligned}(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = {}& (\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} + \langle\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \langle\mathbf{n}\times\widehat{\mathbf{u}}_h^t,\phi^t\rangle_{\partial\mathscr{T}_h}\\ &- \kappa^2(\varepsilon\mathbf{u}_h,\psi)_{\mathscr{T}_h} + \langle\tau_n(p_h-\widehat{p}_h),\Pi_{\mathrm{Q}}\rho-\rho\rangle_{\partial\mathscr{T}_h},\end{aligned}$$

where we have also used (4i). Moreover, since $\langle\mathbf{n}\times\widehat{\mathbf{u}}_h^t,\phi^t\rangle_{\partial\mathscr{T}_h} = \langle\mathbf{n}\times\widehat{\mathbf{u}}_h^t,\phi^t\rangle_\Gamma = \langle\widehat{\mathbf{u}}_h^t,P_{\mathrm{M}_h^t}(\phi^t)\times\mathbf{n}\rangle_\Gamma$, by (4f) with $\rho = P_{\mathrm{M}_h^t}(\phi^t)\times\mathbf{n}$,

$$\begin{aligned}(\mathbf{u}_h,\Theta)_{\mathscr{T}_h} = {}& (\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} + \langle\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}(P_{\mathrm{V}}\phi-\phi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \langle\mathbf{g},P_{\mathrm{M}_h^t}\left(\phi^t\right)\rangle_\Gamma\\ &- \kappa^2(\varepsilon\mathbf{u}_h,\psi)_{\mathscr{T}_h} + \langle\tau_n(p_h-\widehat{p}_h),\Pi_{\mathrm{Q}}\rho-\rho\rangle_{\partial\mathscr{T}_h}.\end{aligned} \tag{16}$$

Now, let us rewrite the term $(\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h}$. By the Green identities, (12a) with $\mathbf{z}=\nabla\times\mathbf{v}_h$ and (4b) with $\mathbf{z}=\Pi_{\mathrm{V}}\psi$, we have

$$\begin{aligned}(\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} = {}& \langle\mathbf{v}_h^t,(\Pi_{\mathrm{V}}\psi-\psi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \langle\widehat{\mathbf{v}}_h^t,\Pi_{\mathrm{V}}\psi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \kappa^2(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi)_{\mathscr{T}_h} + (p_h,\nabla\cdot(\varepsilon\,\Pi_{\mathrm{V}}\psi))_{\mathscr{T}_h}\\ &- \langle\widehat{p}_h,\varepsilon\,\Pi_{\mathrm{V}}\psi\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \mathbf{F}(\Pi_{\mathrm{V}}\psi).\end{aligned}$$

Hence, adding and subtracting $\langle\widehat{\mathbf{v}}_h^t,\psi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}$ and integrating by parts the fourth term, it holds

$$\begin{aligned}(\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} = {}& \langle\mathbf{v}_h^t-\widehat{\mathbf{v}}_h^t,(\Pi_{\mathrm{V}}\psi-\psi)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \langle\widehat{\mathbf{v}}_h^t,\psi\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \kappa^2(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi)_{\mathscr{T}_h} - (\nabla p_h,\varepsilon\,\Pi_{\mathrm{V}}\psi)_{\mathscr{T}_h}\\ &+ \langle p_h-\widehat{p}_h,\varepsilon\,\Pi_{\mathrm{V}}\psi\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \mathbf{F}(\Pi_{\mathrm{V}}\psi).\end{aligned}$$

By (4h), (10d), the orthogonality of $P_{\mathrm{M}_h^t}$, (12a), integrating by parts again the fourth term, adding and subtracting $\langle \widehat{p}_h, \varepsilon\psi\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}$, we obtain

$$\begin{aligned}(\mathbf{v}_h,\nabla\times\psi)_{\mathscr{T}_h} &= \langle \tau\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\Pi_{\mathrm{V}}\psi-\psi\rangle_{\partial\mathscr{T}_h} - \langle \mathbf{n}\times\widehat{\mathbf{v}}_h^t, P_{\mathrm{M}_h^t}\left(\psi^t\right)\rangle_{\partial\mathscr{T}_h\setminus\Gamma} + \kappa^2\left(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi\right)_{\mathscr{T}_h}\\ &\quad + \left(p_h,\nabla\cdot(\varepsilon\psi)\right)_{\mathscr{T}_h} + \mathbf{F}\left(\Pi_{\mathrm{V}}\psi\right) - \langle \widehat{p}_h,\varepsilon\psi\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \langle p_h-\widehat{p}_h,\varepsilon\left(\Pi_{\mathrm{V}}\psi-\psi\right)\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}\\ &= \langle \tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{1/2}\left(\Pi_{\mathrm{V}}\psi-\psi\right)\rangle_{\partial\mathscr{T}_h} + \kappa^2\left(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi\right)_{\mathscr{T}_h}\\ &\quad + \langle p_h-\widehat{p}_h,\varepsilon\left(\Pi_{\mathrm{V}}\psi-\psi\right)\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} + \mathbf{F}\left(\Pi_{\mathrm{V}}\psi\right),\end{aligned}$$

where for the last inequality we have used (4d), the facts that $\varepsilon\psi\in\mathrm{H}(\mathbf{div};\Omega)$ and $\nabla\cdot(\varepsilon\psi)=0$, together with (4g). Finally, replacing the above expression in (16), we obtain (14). $\square$

Lemma 3.3. *Let* $(\phi,\psi,\rho)\in\mathrm{H}(\mathbf{curl};\Omega)\times\mathrm{X}_0\times H_0^1(\Omega)$ *be the solution of (10) satisfying* (11) *and* $\left(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}_h^t,\widehat{p}\right)\in \mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$ *the solution of (4). If* τ *and* τ_n *are of order one and* $k\geq 1$, *then there exists* $h_0>0$ *such that, for all* $h<h_0$, *it holds*

$$\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h}\lesssim\|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})}+h^{-1/2}\|\mathbf{g}\|_{\Gamma}.$$

Proof. From Lemma 3.2 we have

$$\begin{aligned}\left|(\mathbf{u}_h,\Theta)_{\mathscr{T}_h}\right| &\leq \left|\mathbf{F}\left(\Pi_{\mathrm{V}}\psi\right)\right| + \left|\langle \mathbf{g},P_{\mathrm{M}_h^t}\left(\phi^t\right)\rangle_\Gamma\right| + \left|\langle \tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{1/2}\left(\Pi_{\mathrm{V}}\psi-\psi\right)\rangle_{\partial\mathscr{T}_h}\right|\\ &\quad + \left|\langle p_h-\widehat{p}_h,\varepsilon\left(\Pi_{\mathrm{V}}\psi-\psi\right)\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}\right| + \left|\langle \tau_n\left(p_h-\widehat{p}_h\right),\Pi_{\mathrm{Q}}\rho-\rho\rangle_{\partial\mathscr{T}_h}\right|\\ &\quad + \left|\langle \tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}\left(P_{\mathrm{V}}\phi-\phi\right)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}\right| + \kappa^2\left|\left(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi-\psi\right)_{\mathscr{T}_h}\right|.\end{aligned} \tag{17}$$

Using the Cauchy-Schwarz inequality, inverse inequality, (11) and (12d), we have

$$\left|\langle \tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{1/2}\left(\Pi_{\mathrm{V}}\psi-\psi\right)\rangle_{\partial\mathscr{T}_h}\right| \lesssim (\tau^*)^{1/2}h^{s+1/2}\|\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right)\|_{\partial\mathscr{T}_h}\|\Theta\|_\Omega,$$

where $\tau^*=\max_{F\in\mathscr{E}_h}\{\tau_F\}$. Similarly, taking $\widehat{\tau}^*=\max_{F\in\mathscr{E}_h}\{\tau_F^{-1}\}$, $\tau_*=\max_{F\in\mathscr{E}_h}\{\tau_n^F\}$ and $\widehat{\tau}_*=\max_{F\in\mathscr{E}_h}\left\{\left(\tau_n^F\right)^{-1}\right\}$, we have the following bounds

$$\begin{aligned}\left|\mathbf{F}\left(\Pi_{\mathrm{V}}\psi\right)\right| &\lesssim \|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})}\|\Theta\|_\Omega,\\ \langle \mathbf{g},P_{\mathrm{M}_h^t}\left(\phi^t\right)\rangle_\Gamma &\lesssim h^{-1/2}\|\mathbf{g}\|_\Gamma\|\Theta\|_\Omega,\\ \left|\kappa^2\left(\varepsilon\mathbf{u}_h,\Pi_{\mathrm{V}}\psi-\psi\right)_{\mathscr{T}_h}\right| &\lesssim h^{s+1}\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h}\|\Theta\|_\Omega,\\ \left|\langle \tau_n\left(p_h-\widehat{p}_h\right),\Pi_{\mathrm{Q}}\rho-\rho\rangle_{\partial\mathscr{T}_h}\right| &\lesssim h^{s+1/2}(\tau_*)^{1/2}\|\tau_n^{1/2}\left(p_h-\widehat{p}_h\right)\|_{\partial\mathscr{T}_h}\|\Theta\|_\Omega,\\ \left|\langle p_h-\widehat{p}_h,\varepsilon\left(\Pi_{\mathrm{V}}\psi-\psi\right)\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}\right| &\lesssim (\widehat{\tau}_*)^{1/2}h^{s+1/2}\|\tau_n^{1/2}\left(p_h-\widehat{p}_h\right)\|_{\partial\mathscr{T}_h}\|\Theta\|_\Omega,\\ \left|\langle \tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right),\tau^{-1/2}\left(P_{\mathrm{V}}\phi-\phi\right)\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}\right| &\lesssim (\widehat{\tau}^*)^{1/2}h^{s-1/2}\|\tau^{1/2}\left(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t\right)\|_{\partial\mathscr{T}_h}\|\Theta\|_\Omega.\end{aligned}$$

Hence, taking $\Theta=|\varepsilon|^2\mathbf{u}_h$, (17) becomes

$$\begin{aligned}(1-h^{s+1})\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h} &\lesssim ((\tau^*)^{1/2}h^{s+1/2}+(\widehat{\tau}^*)^{1/2}h^{s-1/2})\|\tau^{1/2}(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t)\|_{\partial\mathscr{T}_h}\\ &\quad + \|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})} + h^{s+1/2}((\widehat{\tau}_*)^{1/2}+(\tau_*)^{1/2})\|\tau_n^{1/2}\left(p_h-\widehat{p}_h\right)\|_{\partial\mathscr{T}_h} + h^{-1/2}\|\mathbf{g}\|_\Gamma\\ &\lesssim h^{s-1/2}\|\tau^{1/2}(\mathbf{u}_h^t-\widehat{\mathbf{u}}_h^t)\|_{\partial\mathscr{T}_h} + h^{s+1/2}\|\tau_n^{1/2}\left(p_h-\widehat{p}_h\right)\|_{\partial\mathscr{T}_h} + \|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})} + h^{-1/2}\|\mathbf{g}\|_\Gamma.\end{aligned} \tag{18}$$

On the other hand, from (5) we have

$$\|\mathbf{v}_h\|^2_{\mathscr{T}_h} + \|\tau^{1/2}\left(\mathbf{u}^t_h - \widehat{\mathbf{u}}^t_h\right)\|^2_{\partial\mathscr{T}_h} + \|\tau_n^{1/2}\left(p_h - \widehat{p}_h\right)\|^2_{\partial\mathscr{T}_h} + \kappa^2\left|\mathrm{Re}\left(\varepsilon_m\right)\right|\|\mathbf{u}_h\|^2_{\mathscr{T}_h^m} \\ \leq \kappa^2\varepsilon_d\|\mathbf{u}_h\|^2_{\mathscr{T}_h^d} + \left|\overline{\mathbf{F}}\left(\mathbf{u}_h\right)\right| + \left|\langle\mathbf{g},\widehat{\mathbf{v}}^t_h\rangle_\Gamma\right|.$$

Bounding the last two terms on the right hand side by using Young's inequality and the definition in (4h), it can be deduced that

$$\frac{1}{2}\|\mathbf{v}_h\|^2_{\mathscr{T}_h} + \frac{1}{2}\|\tau^{1/2}\left(\mathbf{u}^t_h - \widehat{\mathbf{u}}^t_h\right)\|^2_{\partial\mathscr{T}_h} + \|\tau_n^{1/2}\left(p_h - \widehat{p}_h\right)\|^2_{\partial\mathscr{T}_h} + \kappa^2\left|\mathrm{Re}\left(\varepsilon_m\right)\right|\|\mathbf{u}_h\|^2_{\mathscr{T}_h^m} \\ \leq \kappa^2\varepsilon_d\|\mathbf{u}_h\|^2_{\mathscr{T}_h^d} + \frac{1}{2}\|\mathbf{F}\|^2_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})} + \frac{1}{2}\|\varepsilon\mathbf{u}_h\|^2_{\mathscr{T}_h} + \left(h^{-1} + \tau^*\right)\|\mathbf{g}\|^2_\Gamma.$$

Thus, by inserting last expression in (18), considering $s > 1/2$ in (11), we obtain for h small enough

$$\|\varepsilon\mathbf{u}_h\|_{\mathscr{T}_h} \lesssim (h^{s-1/2} + 1)\|\mathbf{F}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})} + h^{-1/2}\|\mathbf{g}\|_\Gamma$$

and the result follows. □

Proof of Theorem 3.2. It follows directly from Lemmas 3.1 and 3.3.

4. Error analysis

In this section we deduce error estimates based on the stability results proved in the former section.

4.1. Error estimates

In order to develop the arguments in this section we start by defining the following projection of errors:

$$\begin{gathered} \mathbf{e}^{\mathbf{v}}_h := P_{\mathrm{V}}\mathbf{v} - \mathbf{v}_h, \quad \mathbf{e}^{\mathbf{u}}_h := \Pi_{\mathrm{V}}\mathbf{u} - \mathbf{u}_h, \quad \mathbf{e}^p_h := \Pi_{\mathrm{Q}}p - p_h, \\ \mathbf{e}^{\widehat{\mathbf{u}}^t_h}_h := P_{\mathrm{M}^t_h}\mathbf{u}^t - \widehat{\mathbf{u}}^t_h, \quad \mathbf{e}^{\widehat{p}}_h := P_{\mathrm{M}_h}p - \widehat{p}_h, \quad \mathbf{e}^{\widehat{\mathbf{v}}^t_h}_h := P_{\mathrm{M}^t_h}\mathbf{v}^t - \widehat{\mathbf{v}}^t_h, \quad \mathbf{e}^{\widehat{\varepsilon\mathbf{u}}^n_h}_h := P_{\mathrm{M}_h}\left(\varepsilon\mathbf{u}\cdot\mathbf{n}\right) - \widehat{\varepsilon\mathbf{u}^n_h}\cdot\mathbf{n}. \end{gathered} \tag{19}$$

Lemma 4.1. *Let* $(\mathbf{v},\mathbf{u},p)$ *and* $\left(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}^t_h,\widehat{p}_h\right)$ *be the solutions of (3) and (4), respectively. The projection of the errors* $(\mathbf{e}^{\mathbf{v}}_h,\mathbf{e}^{\mathbf{u}}_h,\mathbf{e}^p_h,\mathbf{e}^{\widehat{\mathbf{u}}^t_h}_h,\mathbf{e}^{\widehat{p}}_h,\mathbf{e}^{\widehat{\mathbf{v}}^t_h}_h,\mathbf{e}^{\widehat{\mathbf{u}}^n_h}_h)$ *satisfy the following system of equations*

$$\begin{aligned} \left(\mathbf{e}^{\mathbf{v}}_h,\mathbf{w}\right)_{\mathscr{T}_h} - \left(\mathbf{e}^{\mathbf{u}}_h,\nabla\times\mathbf{w}\right)_{\mathscr{T}_h} + \langle\mathbf{e}^{\widehat{\mathbf{u}}^t_h}_h\times\mathbf{n},\mathbf{w}\rangle_{\partial\mathscr{T}_h} &= 0, \\ \left(\mathbf{e}^{\mathbf{v}}_h,\nabla\times\mathbf{z}\right)_{\mathscr{T}_h} + \langle\mathbf{e}^{\widehat{\mathbf{v}}^t_h}_h,\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h} - \kappa^2\left(\varepsilon\mathbf{e}^{\mathbf{u}}_h,\mathbf{z}\right)_{\mathscr{T}_h} - \left(\mathbf{e}^p_h,\nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h} + \langle\mathbf{e}^{\widehat{p}}_h,\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} &= -\kappa^2\left(\Pi_V\mathbf{u} - \mathbf{u},\overline{\varepsilon}\mathbf{z}\right)_{\mathscr{T}_h}, \\ -\left(\varepsilon\mathbf{e}^{\mathbf{u}}_h,\nabla t\right)_{\mathscr{T}_h} + \langle\mathbf{e}^{\widehat{\varepsilon\mathbf{u}}^n_h}_h,t\rangle_{\partial\mathscr{T}_h} &= 0, \\ \langle\mathbf{n}\times\mathbf{e}^{\widehat{\mathbf{v}}^t_h}_h,\eta\rangle_{\partial\mathscr{T}_h\setminus\Gamma} &= 0, \\ \langle\mathbf{e}^{\widehat{\varepsilon\mathbf{u}}^n_h}_h,\mu\rangle_{\partial\mathscr{T}_h\setminus\Gamma} &= 0, \\ \langle\mathbf{e}^{\widehat{\mathbf{u}}^t_h}_h,\eta\rangle_\Gamma &= 0, \\ \langle\mathbf{e}^{\widehat{p}}_h,\mu\rangle_\Gamma &= 0, \end{aligned} \tag{20}$$

for all $(\mathbf{w},\mathbf{z},t,\eta,\mu)\in \mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times\mathrm{M}_h$. *Here,*

$$\mathbf{n}\times\mathbf{e}_h^{\widehat{\mathbf{v}}_h^t}=\mathbf{n}\times(\mathbf{e}_h^{\mathbf{v}})^t+\tau((\mathbf{e}_h^{\mathbf{u}})^t-\mathbf{e}_h^{\widehat{\mathbf{u}}_h^t}),$$
$$\mathbf{e}_h^{\widehat{\varepsilon\mathbf{u}}_h^n}=\varepsilon(\mathbf{e}_h^{\mathbf{u}})^n\cdot\mathbf{n}+\tau_n(\mathbf{e}_h^p-\mathbf{e}_h^{\widehat{p}}).$$

Proof. Let us focus on the second and ninth equations since the rest is proven in a similar way. First notice that using the definitions given in (19) and the orthogonality of the projectors, from (4b) we obtain

$$\begin{aligned}\mathbf{F}(\mathbf{z})&=(P_{\mathrm{V}}\mathbf{v}-\mathbf{e}_h^{\mathbf{v}},\nabla\times\mathbf{z})_{\mathscr{T}_h}+\langle P_{\mathrm{M}_h^t}\mathbf{v}^t-\mathbf{e}_h^{\widehat{\mathbf{v}}_h^t},\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}-\kappa^2\left(\varepsilon\left(\Pi_{\mathrm{V}}\mathbf{u}-\mathbf{e}_h^{\mathbf{u}}\right),\mathbf{z}\right)_{\mathscr{T}_h}\\&\quad-\left(\Pi_{\mathrm{Q}}p-\mathbf{e}_h^p,\nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h}+\langle P_{\mathrm{M}_h}p-\mathbf{e}_h^{\widehat{p}},\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}\\&=(\mathbf{v},\nabla\times\mathbf{z})_{\mathscr{T}_h}-(\mathbf{e}_h^{\mathbf{v}},\nabla\times\mathbf{z})_{\mathscr{T}_h}+\langle\mathbf{v}^t,\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}-\langle\mathbf{e}_h^{\widehat{\mathbf{v}}_h^t},\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}-\kappa^2\left(\varepsilon\,\Pi_{\mathrm{V}}\mathbf{u},\mathbf{z}\right)_{\mathscr{T}_h}\\&\quad+\kappa^2\left(\varepsilon\,\mathbf{e}_h^{\mathbf{u}},\mathbf{z}\right)_{\mathscr{T}_h}-\left(p,\nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h}+\left(\mathbf{e}_h^p,\nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h}+\langle p,\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}-\langle\mathbf{e}_h^{\widehat{p}},\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}.\end{aligned}\tag{21}$$

On the other hand, we know that the continuous solution $(\mathbf{v},\mathbf{u})$ satisfies

$$(\mathbf{v},\nabla\times\mathbf{z})_{\mathscr{T}_h}+\langle\mathbf{v}^t,\mathbf{z}\times\mathbf{n}\rangle_{\partial\mathscr{T}_h}-\kappa^2\left(\varepsilon\mathbf{u},\mathbf{z}\right)_{\mathscr{T}_h}-\left(p,\nabla\cdot(\varepsilon\mathbf{z})\right)_{\mathscr{T}_h}+\langle p,\varepsilon\mathbf{z}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h}=\mathbf{F}(\mathbf{z}).\tag{22}$$

Thus, the second error equation follows from combining (21) and (22).
Now, let $\mu\in\mathrm{M}_h$. By using (19) in (4i), we have

$$\langle P_{M_h}(\varepsilon\mathbf{u}^{\mathbf{n}}\cdot\mathbf{n})-\mathbf{e}_h^{\widehat{\varepsilon\mathbf{u}}_h^n},\mu\rangle_{\partial\mathscr{T}_h}=\langle\varepsilon\left(\Pi_{\mathrm{V}}\mathbf{u}-\mathbf{e}_h^{\mathbf{u}}\right)^n\cdot\mathbf{n}+\tau_n(\Pi_{\mathrm{Q}}p-\mathbf{e}_h^p-P_{\mathrm{M}_h}p+\mathbf{e}_h^{\widehat{p}}),\mu\rangle_{\partial\mathscr{T}_h},$$

from where, $\langle P_{M_h}(\varepsilon\mathbf{u}^{\mathbf{n}}\cdot\mathbf{n})-\varepsilon\Pi_{\mathrm{V}}\mathbf{u}\cdot\mathbf{n}-\tau_n(\Pi_{\mathrm{Q}}p-P_{\mathrm{M}_h}p),\mu\rangle_{\partial\mathscr{T}_h}=\langle\mathbf{e}_h^{\widehat{\varepsilon\mathbf{u}}_h^n}-\varepsilon\left(\mathbf{e}_h^{\mathbf{u}}\right)^n\cdot\mathbf{n}+\tau_n(\mathbf{e}_h^{\widehat{p}}-\mathbf{e}_h^p),\mu\rangle_{\partial\mathscr{T}_h}$. Then, by choosing $\mu=\mathbf{e}_h^{\widehat{\varepsilon\mathbf{u}}_h^n}-\varepsilon\left(\mathbf{e}_h^{\mathbf{u}}\right)^n\cdot\mathbf{n}+\tau_n(\mathbf{e}_h^{\widehat{p}}-\mathbf{e}_h^p)$ and using the orthogonality of the projectors together with (12c), we obtain $\mathbf{e}_h^{\widehat{\varepsilon\mathbf{u}}_h^n}=\varepsilon(\mathbf{e}_h^{\mathbf{u}})^n\cdot\mathbf{n}+\tau_n(\mathbf{e}_h^p-\mathbf{e}_h^{\widehat{p}})$. □

Theorem 4.1. *Let* $(\mathbf{v},\mathbf{u},p)\in\mathbf{H}^{l_{\mathbf{u}}+1}(\mathscr{T}_h)\times\mathbf{H}^{l_{\mathbf{v}}+1}(\mathscr{T}_h)\times H^{l_p+1}(\mathscr{T}_h)$ *and* $\left(\mathbf{v}_h,\mathbf{u}_h,p_h,\widehat{\mathbf{u}}_h^t,\widehat{p}_h\right)\in\mathrm{V}_h\times\mathrm{V}_h\times\mathrm{Q}_h\times\mathrm{M}_h^t\times M_h$ *be the solutions of (3) and (4), respectively, for* $l_{\mathbf{v}},l_{\mathbf{u}},l_p\in[0,k]$ *. There exists* $h_0>0$ *such that, for all* $h<h_0$*, it holds*

$$\|\mathbf{e}_h^{\mathbf{v}}\|_{\mathscr{T}_h}^2+\|\tau^{1/2}((\mathbf{e}_h^{\mathbf{u}})^t-\mathbf{e}_h^{\widehat{\mathbf{u}}_h^t})\|_{\partial\mathscr{T}_h}^2+\|\tau_n^{1/2}(\mathbf{e}_h^p-\mathbf{e}_h^{\widehat{p}})\|_{\partial\mathscr{T}_h}^2+\kappa\|\varepsilon\mathbf{e}_h^{\mathbf{u}}\|_{\mathscr{T}_h}^2\lesssim h^{2(l_{\mathbf{u}}+1)}|\mathbf{u}|_{l_{\mathbf{u}}+1,\mathscr{T}_h}^2+h^{2(l_p+1)}|p|_{l_p+1,\mathscr{T}_h}^2.$$

Proof. We observe that the formulation (20) is the same as (4), where $\widetilde{\mathbf{F}}:\mathbf{P}_k(\mathscr{T}_h)\to\mathbb{C}$, such that $\widetilde{\mathbf{F}}(\mathbf{z}):=-\kappa^2\left(\Pi_{\mathrm{V}}\mathbf{u}-\mathbf{u},\overline{\varepsilon}\mathbf{z}\right)_{\mathscr{T}_h}$, plays the role of $\mathbf{F}$, and $\mathbf{0}$ plays the role of $\mathbf{g}$. Then, by Theorem 3.2 there exists $h_0>0$, such that

$$\|\mathbf{e}_h^{\mathbf{v}}\|_{\mathscr{T}_h}^2+\|\tau^{1/2}((\mathbf{e}_h^{\mathbf{u}})^t-\mathbf{e}_h^{\widehat{\mathbf{u}}_h^t})\|_{\partial\mathscr{T}_h}^2+\|\tau_n^{1/2}(\mathbf{e}_h^p-\mathbf{e}_h^{\widehat{p}})\|_{\partial\mathscr{T}_h}^2+\kappa^2\|\varepsilon\mathbf{e}_h^{\mathbf{u}}\|_{\mathscr{T}_h}^2\lesssim\|\widetilde{\mathbf{F}}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})}^2,$$

for all $h<h_0$. It remains to bound the right hand side of this inequality.

Given $\mathbf{z}\in\mathbf{P}_(\mathscr{T}_h)$, by the Cauchy-Schwarz inequality and the properties in (12d), we obtain

$$|\widetilde{\mathbf{F}}(\mathbf{z})|\lesssim\kappa^2\left\|\Pi_{\mathrm{V}}\mathbf{u}-\mathbf{u}\right\|_{\mathscr{T}_h}\|\overline{\varepsilon}\mathbf{z}\|_{\mathscr{T}_h}\lesssim(h^{l_{\mathbf{u}}+1}|\mathbf{u}|_{l_{\mathbf{u}}+1,\mathscr{T}_h}+h^{l_p+1}|p|_{l_p+1,\mathscr{T}_h})\|\mathbf{z}\|_{\mathscr{T}_h}.$$

Then, $\|\widetilde{\mathbf{F}}\|_{\mathscr{L}(\mathbf{L}^2(\mathscr{T}_h),\mathbb{C})}\lesssim h^{l_{\mathbf{u}}+1}|\mathbf{u}|_{l_{\mathbf{u}}+1,\mathscr{T}_h}+h^{l_p+1}|p|_{l_p+1,\mathscr{T}_h}$ and the proof is completed. □

Corollary 4.1. *Under the same assumptions as in Theorem 4.1, if* $l:=\min\{l_{\mathbf{u}},l_{\mathbf{p}}\}$*, we have*

$$\|\mathbf{v}-\mathbf{v}_h\|_{\mathscr{T}_h}+\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{\mathscr{T}_h}\lesssim h^{l+1}(|\mathbf{u}|_{l\mathbf{u}+1,\mathscr{T}_h}+|p|_{l_p+1,\mathscr{T}_h}).$$

5. Numerical results

The implementation of the HDG method is based on the code developed by [28] in the context of diffusion and convection-diffusion problems. The numerical experiments are carried out on a unit cube $\Omega := [0,1]^3$ divided in two regions Ω_d and Ω_m and triangulated by a sequence of shape regular tetrahedral meshes. The interior of a tetrahedron belongs to either Ω_d or Ω_m. We set the wavelength $\lambda_0 = 4.5$ (450 nm), consider the isotropic dielectric region Ω_d is made of silicon oxynitride whose relative electric permittivity is $\varepsilon_d = 2.7124$, and the metal region is taken to be evaporated silver with $\varepsilon_m = -5.8828 + i0.6650$. These values of the relative electric permittivities depend on λ_0 and were taken from [3]. In all the examples we set $\tau = \tau_n = 1$.

The experimental order of convergence is computed as $r(w) := \dfrac{\log\left(\|w-w_h\|_{\mathscr{T}_h} / \|w-w_h\|_{\mathscr{T}_{\tilde{h}}}\right)}{\log(h/\tilde{h})}$, where h and $\tilde{h}$, are the sizes of two consecutive meshes and $w \in \{\mathbf{u}, \mathbf{v}, p\}$.

Example 5.1: The domain Ω is divided in $\Omega_d = [0,1]\times[0,1]\times[0,1/2]$ and $\Omega_m = [0,1]\times[0,1]\times[1/2,1]$. We test our HDG formulation, with the case $p(x,y,z) = 0$ and consider the exact solution $\mathbf{u}(x,y,z) := (0, u_2(x,y,z), 0)^T$, with

$$u_2(x,y,z) = \begin{cases} \exp\left(-i\kappa\sqrt{\varepsilon_d}\,(z-0.5)\right) + \exp\left(i\kappa\sqrt{\varepsilon_d}\,(z-0.5)\right), & \text{if } z \geq 0.5, \\ \exp\left(-i\kappa\sqrt{\varepsilon_m}\,(z-0.5)\right) + \exp\left(i\kappa\sqrt{\varepsilon_m}\,(z-0.5)\right), & \text{if } z < 0.5, \end{cases}$$

where we recall that $\kappa := 2\pi/\lambda_0$ and hence $\kappa = 1.3963$. The values of $\mathbf{f}$ and $\mathbf{g}$ are calculated according with the exact solution.

Table 1 shows the experimental convergence rates and absolute errors. We observe that, as Theorem 4.1 predicts, optimal rates of convergence are obtained, i.e., order $k+1$ for $\mathbf{u}$ and $\mathbf{v}$.

Table 1: Rate of convergence and errors of Example 5.1

k	$r(\mathbf{v})$	$r(\mathbf{u})$	$r(p)$	**Nelts**	$\|\mathbf{v}-\mathbf{v}_h\|_{\mathscr{T}_h}$	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{T}_h}$	$\|p-p_h\|_{\mathscr{T}_h}$
	-	-	-	48	1.4117e-01	2.5022e-01	1.9122e-01
	1.96	2.41	1.24	384	3.6382e-02	4.7241e-02	8.0744e-02
1	1.82	2.03	2.51	3072	1.0280e-02	1.1567e-02	1.4166e-02
	1.87	1.99	2.49	24576	2.8218e-03	2.9133e-03	2.5182e-03
	-	-	-	48	2.1954e-02	2.7236e-02	2.4103e-01
	2.68	2.31	2.31	384	3.4161e-03	5.4944e-03	4.8506e-02
2	2.85	2.90	2.89	3072	4.7249e-04	7.3397e-04	6.5483e-03
	2.93	3.00	2.97	24576	6.1861e-05	9.1955e-05	8.3516e-04
	-	-	-	48	3.5477e-03	1.2115e-02	1.3306e-01
	3.68	3.64	3.63	384	2.7665e-04	9.7457e-04	1.0771e-02
3	3.95	3.83	3.88	3072	1.7914e-05	6.8709e-05	7.2993e-04
	4.05	3.94	3.97	24576	1.0826e-06	4.4708e-06	4.6515e-05

Example 5.2: In this example we consider the same setting as in Example 5.1, but considering $p(x,y,z) = \sin(2\pi x)\sin(2\pi y)\sin(2\pi z)$ instead. Once again, the values of $\mathbf{f}$ and $\mathbf{g}$ are calculated according with the exact solution. Table 2 shows that the predicted optimal rates of convergence are achieved.

Table 2: Rate of convergence and errors of Example 5.2

k	$r(\mathbf{v})$	$r(\mathbf{u})$	$r(p)$	**Nelts**	$\|\mathbf{v}-\mathbf{v}_h\|_{\mathscr{T}_h}$	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{T}_h}$	$\|p-p_h\|_{\mathscr{T}_h}$
	-	-	-	48	9.7455e-02	1.4749e-01	1.8858e-02
	1.54	1.89	1.60	384	3.3616e-02	3.9763e-02	6.2176e-03
1	1.73	1.93	1.91	3072	1.0150e-02	1.0404e-02	1.6601e-03
	1.85	1.96	1.94	24576	2.8184e-03	2.6685e-03	4.3362e-04
	-	-	-	48	2.1492e-02	2.3376e-02	3.4517e-03
	2.71	2.85	2.95	384	3.2891e-03	3.2332e-03	4.4790e-04
2	2.86	2.94	2.84	3072	4.5457e-04	4.2035e-04	6.2567e-05
	2.94	3.00	2.91	24576	5.9395e-05	5.2433e-05	8.3503e-06
	-	-	-	48	1.5614e-03	2.0326e-03	3.2012e-04
	3.53	3.68	3.62	384	1.3546e-04	1.5808e-04	2.6109e-05
3	3.81	3.89	3.79	3072	9.6377e-06	1.0693e-05	1.8916e-06
	3.92	3.96	3.90	24576	6.3701e-07	6.8644e-07	1.2705e-07

Example 5.3: We now consider a metallic corrugation on the interface between Ω_m and Ω_d, that is, $\Omega_m := [0,1]\times[0,1]\times[0,0.5]\cup[0.25,0.75]\times[0.25,0.75]\times[0.5,0.75]$ and $\Omega_d := \Omega\setminus\Omega_m$. Motivated by the application to photovoltaic devices, we consider $\mathbf{f}\equiv\mathbf{0}$ and the top of the domain, the plane $z=1$, is illuminated by an incident field given by $\mathbf{u}(x,y,1) := (0,u_2(x,y,1),0)^T$, with $u_2(x,y,z) = \exp\left(-i\left[\kappa_x x+\kappa_y y-\kappa_z(z-1)\right]\right)$, where $\kappa_x = \kappa\sin\theta\cos\phi$, $\kappa_y = \kappa\sin\theta\sin\phi$, $\kappa_z = (\kappa^2-\kappa_x^2-\kappa_y^2)^{1/2}$, $\theta = \pi/4$ and $\phi = 0$. The boundary data $\mathbf{g}$ is computed according to $\mathbf{u}$. We emphasize the realistic problem involves quasi-periodic boundary conditions on the vertical walls and outgoing radiation conditions above and below the structure. This example constitutes a stepping stone towards that goal. Figure 2 shows the approximation of the intensity $|\mathbf{u}_h|$.

6. Concluding Remarks

We have been able to develop an *a priori* error analysis of an HDG method for a Maxwell equation in heterogeneous media arising from the application to photovoltaic solar cells that involve periodic surface-relief gratings. We proved that under enough regularity of the exact solution, the error of the method achieves optimal rate of convergence of the magnetic and electric fields. In despite of the simplification of the original problem, we consider this work constitutes a step forward to the ultimate goal of developing an error analysis of HDG methods for simulations of photovoltaic devices. The next step will consist on extending these results to the more realistic case where quasi-periodic boundary conditions are imposed on the vertical boundaries and radiation conditions are considered above and below the domain. This is the topic of our current research.

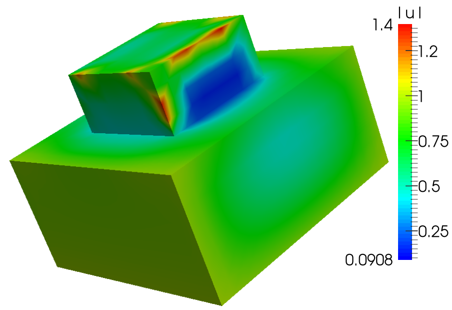

Figure 2: $|\mathbf{u}_h|$ for $k=1$ and 24576 elements.

Appendix A. Well-posedness of (3)

The aim of this part of the appendix is to prove that (3) has a unique solution. We begin by replacing $\mathbf{v} = \nabla\times\mathbf{u}$ into the second equation of (3), from which we get the following second order system

$$\begin{aligned} \nabla\times\nabla\times\mathbf{u} - \kappa^2\varepsilon\mathbf{u} + \overline{\varepsilon}\nabla p = \mathbf{f} &\quad \text{in} \quad \Omega,\\ \nabla\cdot\varepsilon\mathbf{u} = 0 &\quad \text{in} \quad \Omega,\\ \mathbf{u}\times\mathbf{n} = \mathbf{g} &\quad \text{on} \quad \Gamma,\\ p = 0 &\quad \text{on} \quad \Gamma. \end{aligned} \tag{A.1}$$

First let us suppose $\mathbf{g} = \mathbf{0}$ and define X_0 as in Section 3. We consider the following variational formulation equivalent to (A.1) (see Section 5 in [25]): Find $(\mathbf{u}, p) \in \mathrm{X}_0 \times \mathrm{H}^1_0(\Omega)$ such that

$$(\nabla\times\mathbf{u}, \nabla\times\mathbf{w})_\Omega - \kappa^2(\varepsilon\mathbf{u}, \mathbf{w})_\Omega = (\mathbf{f}, \mathbf{w})_\Omega, \tag{A.2a}$$

$$(\overline{\varepsilon}\nabla p, \nabla q)_\Omega = (\mathbf{f}, \nabla q)_\Omega, \tag{A.2b}$$

for all $(\mathbf{w}, q) \in \mathrm{X}_0 \times \mathrm{H}^1_0(\Omega)$.

Lemma Appendix A.1. *Suppose $\kappa^2\varepsilon_d$ is not an eigenvalue of $\nabla\times\nabla\times$. Then* (A.2a) *has a unique solution. Moreover, $\|\mathbf{u}\|_{\mathrm{X}_0} \lesssim \|\mathbf{f}\|_\Omega$.*

Proof. We will employ Fredholm alternative together with a compact perturbation theory of a bijective operator. To that end, we rewrite (A.2a) as: Find $\mathbf{u}\in\mathrm{X}_0$ such that $\widetilde{a}(\mathbf{u},\mathbf{v}) := a(\mathbf{u},\mathbf{w}) - k(\mathbf{u},\mathbf{w}) = (\mathbf{f},\mathbf{w})$, $\forall\mathbf{w}\in\mathrm{X}_0$, where $a(\mathbf{u},\mathbf{w}) := (\nabla\times\mathbf{u}, \nabla\times\mathbf{w})_\Omega - \kappa^2(\varepsilon\mathbf{u},\mathbf{w})_{\Omega_m} + \kappa^2(\varepsilon\mathbf{u},\mathbf{w})_{\Omega_d}$ and $k(\mathbf{u},\mathbf{w}) := 2\kappa^2(\varepsilon\mathbf{u},\mathbf{w})_{\Omega_d}$. It is easy to see that a and k are bounded sesquilinear forms. Moreover, a is elliptic in $(\mathrm{X}_0, \|\cdot\|_{\mathrm{X}_0})$. In fact, let $\mathbf{w}\in\mathrm{X}_0$. It follows that $|a(\mathbf{w},\mathbf{w})| \geq \mathrm{Re}(a(\mathbf{w},\mathbf{w})) \geq \min\{1, -\kappa^2\mathrm{Re}(\varepsilon_m), \kappa^2\varepsilon_d\}\|\mathbf{w}\|^2_{\mathrm{X}_0}$, where we recall that $\mathrm{Re}(\varepsilon_m) < 0$. In other words, $\widetilde{a}$ satisfies the Gårding inequality $|\widetilde{a}(\mathbf{w},\mathbf{w})| \geq \min\{1, -\kappa^2\mathrm{Re}(\varepsilon_m), \kappa^2\varepsilon_d\}\|\mathbf{w}\|^2_{\mathrm{X}_0} - k(\mathbf{w},\mathbf{w})$

for all $\mathbf{w} \in \mathrm{X}_0$. Then, since the inclusion X_0 in $\mathbf{L}^2(\Omega)$ is compact (c.f. Theorem 4.7 in [6]), by Fredholm alternative the solution exists if and only if it is unique (c.f. [29]).

Finally, to show uniqueness, let us take $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathrm{X}_0$ such that $(\mathrm{A}-\mathrm{K})\mathbf{u}_1 = (\mathrm{A}-\mathrm{K})\mathbf{u}_2$ and define $\phi := \mathbf{u}_1 - \mathbf{u}_2$. By the linearity of the problem and taking $\mathbf{w} = \phi$, (A.2a) implies that

$$0 = (\nabla\times\phi, \nabla\times\phi) - \kappa^2(\varepsilon\phi,\phi) = \|\nabla\times\phi\|_\Omega^2 + \kappa^2|\mathrm{Re}(\varepsilon_m)|\,\|\phi\|_{\Omega_m}^2 - i\kappa^2\,\mathrm{Im}(\varepsilon_m)\,\|\phi\|_{\Omega_m}^2 - \kappa^2\,\varepsilon_d\,\|\phi\|_{\Omega_d}^2.$$

Therefore, taking imaginary part we obtain that $-\kappa^2\mathrm{Im}(\varepsilon_m)\,\|\phi\|_{\Omega_m}^2 = 0$, from where, $\phi = 0$ in Ω_m. Thus, $0 = \|\nabla\times\phi\|_{\Omega_d}^2 - \kappa^2\varepsilon_d\,\|\phi\|_{\Omega_d}^2$, which implies that $\phi = 0$ in Ω_d if and only if $\kappa^2\varepsilon_d$ is not an eigenvalue of $\nabla\times\nabla\times\phi = \lambda\phi$. □

Lemma Appendix A.2. (A.2b) *has a unique solution.*

Proof. We consider the following sesquilinear forms: $b(p,q) := (\overline{\varepsilon}\nabla p, \nabla q) - i(\nabla p, \nabla q)$ and $c(p,q) := -i(\nabla p, \nabla q)$ in (A.2b). We notice that $|b(q,q)| \geq |\mathrm{Im}(b(q,q))| = \mathrm{Im}(\varepsilon_m)\|\nabla q\|_{\Omega_m}^2 + \|\nabla q\|_\Omega^2$, for all $q \in H_0^1(\Omega)$, then b is elliptic in $H_0^1(\Omega)$ by the Poincaré inequality. Let $\mathrm{S} : H_0^1(\Omega) \to H_0^1(\Omega)'$ the operator induced by the bounded sesquilinear form $c(\cdot,\cdot)$. We observe that S can be expressed as the composition of continuous functions, i, $\mathscr{R}^{-1}$, $\widetilde{i}$ and a compact injection $\tilde{\mathrm{S}}$, i.e. $\mathrm{S} := \widetilde{i} \circ \mathscr{R}^{-1} \circ \tilde{\mathrm{S}} \circ i$, where $i : \mathrm{H}_0^1(\Omega) \to \mathrm{H}^1(\Omega)$, $\tilde{\mathrm{S}} : \mathrm{H}^1(\Omega) \to L^2(\Omega)$, $\mathscr{R} : \left[L^2(\Omega)\right]' \to L^2(\Omega)$, $\tilde{i} : \left[L^2(\Omega)\right]' \to \left[\mathrm{H}_0^1(\Omega)\right]'$, then S is compact (see Theorem 9.11 in [29]). The result follows by similar arguments to the ones used in the proof of previous lemma. □

Remark Appendix A.1. *Using the arguments presented in [25], we can prove that (A.2) is equivalent to (A.1) and thus (A.1) has also a unique solution. On the other hand, for the case* $\mathbf{g} \neq \mathbf{0}$ *we just need to consider* $\mathbf{g} \in \gamma_t\left(\mathrm{H}(\mathbf{curl};\Omega)\cap\mathrm{H}(\mathbf{div}_\varepsilon^0;\Omega)\right) \subset \mathbf{H}^{-1/2}(\Gamma)$ *and take into account that there exists a unique* $\varphi \in \mathrm{H}(\mathbf{curl};\Omega)\cap\mathrm{H}(\mathbf{div}_\varepsilon^0;\Omega)$ *such that* $\gamma_t(\varphi) = \mathbf{g}$ *(see section* 3.5 *of [6]). So, if we set* $\chi := \mathbf{u} - \varphi$*,* χ *satisfies the homogeneous problem and the unicity of* $\mathbf{u}$*, solution of* (3)*, follows.*

Appendix B. On the regularity estimate (11)

By the same arguments in previous section, (10a) and (10b) are well-posed. Now, assumption (11) holds, for instance, when Ω is simply connected and the interface between Ω_m and Ω_d is a plane. In fact, by Theorem 7.1 of [30], $\psi|_{\Omega_j} \in \mathbf{H}^{s+1}(\Omega_j)$, for all $s \in (0,1)$, where $j \in \{m,d\}$. Then $\phi \in \mathbf{H}^s(\Omega_j)$, since $\phi := \nabla\times\psi$. Moreover, by elliptic regularity $\rho \in \mathrm{H}_{\mathrm{loc}}^2(\Omega)$.

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