An adaptive stabilized finite element method for the Stokes–Darcy coupled problem

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Abstract

For the Stokes–Darcy coupled problem, which models a fluid that flows in a free medium into a porous medium, we introduce and analyze an adaptive stabilized finite element method using Lagrange equal order element to approximate the velocity and pressure of the fluid. The interface conditions between both domains are given by mass conservation, the balance of normal forces, and the Beavers–Joseph-Saffman conditions. We prove the well-posedness of the discrete problem and present a convergence analysis with optimal error estimates in natural norms. Next, we introduce and analyze a residual-based a posteriori error estimator for the stabilized scheme. Finally, we present some numerical examples to show the quality of our scheme.

Keywords: Coupled Stokes–Darcy equation, stabilized finite element method, a priori error analysis, a posteriori error analysis.

1. Introduction

The coupling between a fluid that flows in a free medium into a porous medium is of particular interest by practitioners, scientists, engineering, and hydrology, for example, to predict how contamination spreads into streams, lakes, and rivers, affecting the water supply (see [35], and references therein). This coupling is also important in the filtration of blood through the arterial wall (for details of medical applications, see for example [6]). In the industrial sector, it is studied in the design of filters and in the hydrocarbon extraction process (for details, see [2, 25]). In this work, we consider the coupling of a fluid modeled by the Stokes equations with the fluid in a porous medium governed by the Darcy equations. At the interface, we have mass conservation, the balance of normal forces, and Beavers-Joseph-Saffman conditions [9, 32].

In the last decades, various ways of obtaining approximations of solutions by finite element methods for both problems separately can be found in the literature (for instance, for the Stokes equation [3, 8, 13, 22, 24, 33, 40], and for Darcy equation [4, 12, 14, 29]). Although it is known that both problems are mathematically very different, a unified study of both problems in a single domain can be found in [7, 11, 15, 17, 39]. In this shortlist of references, stabilized finite element methods are presented, assuming that the physical parameters of the problem allow generating the Darcy or Stokes equations in a limit case.

Regarding the coupled problem, there is a long list of discrete schemes to obtain approximated solutions. For example, in [23], a domain decomposition methodology introduces an iterative method to solve the problem. In [31], a complete theoretical analysis of a primal mixed scheme is presented using a Lagrange multiplier to obtain velocities with continuous normal trace. In [38], stable finite element space for the Stokes equations and a Galerkin least-squares formulation is used for the Darcy equations. In [27], the authors presents a conforming mixed finite element for the corresponding coupled problem using Bernardi–Raugel

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and Raviart–Thomas elements for the velocities and piecewise constants for the pressures. In [41], the mini–element for the Stokes equations and Brezzi–Douglas–Marini (BDM) with linear Lagrange approximations are used to solve the problem. For more schemes, see [18, 42, 19] and the references therein. Also, it is worth to mention the interior penalty method presented in [17], based in a Nitsche-type weak formulation, allowing to obtain approximations of piecewise linear continuous velocities and piecewise constant pressure; the scheme proposed in [37] which add jumps terms over the element edges of the velocities and used nonconforming Crouzeix–Raviart piecewise linear element for the velocities and piecewise constant for the pressures, and, finally, in [5], the authors used the classical mini–element discretization of the Darcy and Stokes problems.

In this work, we introduced and analyzed a new stabilized finite element method for the Stokes–Darcy coupled problem, which allows us to use equal order approximation spaces $\mathbb{P}_k^d \times \mathbb{P}_k^d$, for velocities and pressures, respectively, in both domains. Our discrete scheme uses the weak formulation presented in [38], and residual stabilized terms, for the Stokes equations, inspired in [26]. Using a weak inf–sup condition, the new method stability can be ensured; also by using a standard argument of stabilized finite element methods, we prove the convergence of the method with natural norms. Therefore, there are two main contributions in this work: the introduction and numerical analysis of this new scheme and the introduction of a reliable and efficient residual-based a posteriori error estimator (for other references in a posteriori error estimator for this problem, see [10, 18, 28]).

This paper is organized as follows: In Section 2, the model problem, with different boundary conditions on the interface between the free and porosity domains, is presented with its weak formulation. The new stabilized finite element method is introduced in Section 3, including its well-posedness. In Section 4, the a priori error analysis for equal order finite element spaces $\mathbb{P}_k^d \times \mathbb{P}_k^d$ is developed. The a posteriori error is presented and analyzed in Section 5. Numerical results, showing the theoretical convergence rate and the performance of the residual-based a posteriori error estimator, can be found in Section 6. Finally, a technical result is proved in Appendix A.

2. Model problem and preliminary results

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded domain with Lipschitz continuous boundary $\partial\Omega$. This domain is divided into two disjoint open subdomains Ω_S and Ω_D, both with Lipschitz continuous boundaries, such that $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_D$. Here Ω_S and Ω_D represent the domains in the free and porous media, respectively. The interface between both media is given by $\Gamma := \overline{\Omega}_S \cap \overline{\Omega}_D$ as shown in Figure 1. The remaining parts of the boundaries are $\Gamma_S := \partial\Omega_S \backslash \Gamma$ and $\Gamma_D := \partial\Omega_D \backslash \Gamma$, with Γ_D divided in Γ_D^{Dir} and Γ_D^{Neu}, with $\Gamma_D^{\mathrm{Dir}} \cap \Gamma_D^{\mathrm{Neu}} = \emptyset$ and $\Gamma_D^{\mathrm{Dir}} \neq \emptyset$.

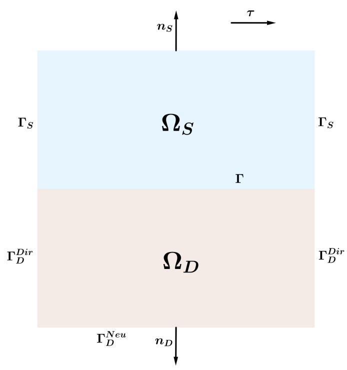

Figure 1: Representation of a possible computational domain Ω.

The Stokes–Darcy coupled problem consists on finding the velocities $\boldsymbol{u} := (\boldsymbol{u}_S, \boldsymbol{u}_D)$ and the pressures

$p := (p_S, p_D)$ such that they satisfy the system of equations

$$(P) \qquad \left\{ \begin{array}{rll} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S) & = \boldsymbol{f}_S & \text{in } \Omega_S, \\ \nabla \cdot \boldsymbol{u}_S & = 0 & \text{in } \Omega_S, \\ \boldsymbol{u}_S & = \boldsymbol{0} & \text{on } \Gamma_S, \\ \nu \, \boldsymbol{u}_D + \kappa \nabla p_D & = \boldsymbol{0} & \text{in } \Omega_D, \\ \nabla \cdot \boldsymbol{u}_D & = g_D & \text{in } \Omega_D, \\ p_D & = 0 & \text{on } \Gamma_D^{\text{Dir}}, \\ \boldsymbol{u}_D \cdot \boldsymbol{n}_D & = \boldsymbol{0} & \text{on } \Gamma_D^{\text{Neu}}, \end{array} \right. \tag{2.1}$$

where $\boldsymbol{\sigma}(\boldsymbol{u}_S, p_S) := 2\nu \, \boldsymbol{\varepsilon}(\boldsymbol{u}_S) - p_S \, \mathbf{I}$ is the stress rate tensor, $\mathbf{I}$ the identity matrix, $\boldsymbol{\varepsilon}(\boldsymbol{u}_S) := \frac{1}{2} (\nabla \boldsymbol{u}_S + \nabla \boldsymbol{u}_S^t)$ the deformation rate tensor, $\nu > 0$ the viscosity of the fluid, $\kappa > 0$ the permeability of the porous media and $\boldsymbol{n}_D$ the outward unit normal vector on Γ_D^{Neu}. Here $\boldsymbol{f}_S \in L^2(\Omega_S)^d$ and $g_D \in L^2(\Omega_D)$. This problem is completed with mass conservation, equilibrium of normal forces and the Beavers–Joseph–Saffman condition on Γ (for details see [9]):

$$\left\{ \begin{array}{l} \boldsymbol{u}_D \cdot \boldsymbol{n}_D + \boldsymbol{u}_S \cdot \boldsymbol{n}_S = 0, \\ -\boldsymbol{n}_S \cdot \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S) \, \boldsymbol{n}_S = p_D, \\ -\boldsymbol{n}_S \cdot \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S) \, \boldsymbol{\tau}_i = \dfrac{\alpha_i}{\kappa^{1/2}} \, \boldsymbol{u}_S \cdot \boldsymbol{\tau}_i, \qquad i = 1, \ldots, d-1, \end{array} \right. \tag{2.2}$$

where α_i are positives non-dimensional constants, and $\boldsymbol{\tau}_i$ are the tangent vectors on Γ.

In the sequel we will use the following Hilbert spaces,

$$\begin{aligned} \boldsymbol{H}^S &:= \left\{ \boldsymbol{v} \in H^1(\Omega_S)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_S \right\} \quad \text{and} \quad \boldsymbol{H}^D := L^2(\Omega_D)^d, \\ Q^S &:= L^2(\Omega_S) \quad \text{and} \quad Q^D := \left\{ q \in H^1(\Omega_D) : q = 0 \text{ on } \Gamma_D^{\text{Dir}} \right\}, \\ \boldsymbol{H} &:= \boldsymbol{H}^S \times \boldsymbol{H}^D \quad \text{and} \quad Q := Q^S \times Q^D. \end{aligned}$$

A variational formulation of problem (2.1)–(2.2) can be written as: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ *such that*

$$B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = F(\boldsymbol{v}, q), \tag{2.3}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$, where $B : (\boldsymbol{H} \times Q) \times (\boldsymbol{H} \times Q) \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B((\boldsymbol{u}, p); (\boldsymbol{v}, q)) := {} & 2\nu\kappa \, (\varepsilon(\boldsymbol{u}_S), \varepsilon(\boldsymbol{v}_S))_{\Omega_S} - \kappa \, (p_S, \nabla \cdot \boldsymbol{v}_S)_{\Omega_S} + \kappa \, (q_S, \nabla \cdot \boldsymbol{u}_S)_{\Omega_S} \\ & + \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i (\boldsymbol{u}_S \cdot \boldsymbol{\tau}_i, \boldsymbol{v}_S \cdot \boldsymbol{\tau}_i)_\Gamma + \kappa \, (p_D, \boldsymbol{v}_S \cdot \boldsymbol{n}_S)_\Gamma \\ & + \nu \, (\boldsymbol{u}_D, \boldsymbol{v}_D)_{\Omega_D} + \kappa \, (\nabla p_D, \boldsymbol{v}_D)_{\Omega_D} - \kappa \, (\boldsymbol{u}_D, \nabla q_D)_{\Omega_D} - \kappa \, (\boldsymbol{u}_S \cdot \boldsymbol{n}_S, q_D)_\Gamma \\ & + \frac{1}{2\nu} \, (\nu \, \boldsymbol{u}_D + \kappa \nabla p_D, -\nu \, \boldsymbol{v}_D + \kappa \nabla q_D)_{\Omega_D}, \end{aligned} \tag{2.4}$$

for all $(\boldsymbol{u}, p), (\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$, and $F : \boldsymbol{H} \times Q \longrightarrow \mathbb{R}$ is the linear functional defined by

$$F(\boldsymbol{v}, q) := \kappa(\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S} + \kappa(g_D, q_D)_{\Omega_D}, \tag{2.5}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$. Now, it is well-known [38, Theorem 1] that (2.3) has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$. In addition, we will use the following norm on the product space $\boldsymbol{H} \times Q$:

$$\|(\boldsymbol{w}, r)\| := \left\{ \nu\kappa \|\boldsymbol{w}_S\|_{1,\Omega_S}^2 + \kappa \|r_S\|_{0,\Omega_S}^2 + \nu \|\boldsymbol{w}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu} |r_D|_{1,\Omega_D}^2 \right\}^{1/2},$$

for all $(\boldsymbol{w}, r) \in \boldsymbol{H} \times Q$.

Throughout this paper C and C_i, $i > 0$ will denote positive constants independent of the mesh size h, but who may depend on the physical parameters of the equation.

The following result will be needed throughout the paper.

Lemma 1. *There exist positive constants C_{Korn}, C_t, such that*

$$C_{\mathrm{Korn}} \|\boldsymbol{v}_S\|_{1,\Omega_S} \le \|\varepsilon(\boldsymbol{v}_S)\|_{0,\Omega_S} \le C_t \|\boldsymbol{v}_S\|_{1,\Omega_S},$$

for all $\boldsymbol{v}_S \in \boldsymbol{H}^S$.

Proof. See [20, Theorem 1.2-2]. □

3. The stabilized finite element method

From now on, we denote by $\{\mathcal{T}_h^S\}_{h>0}$ and $\{\mathcal{T}_h^D\}_{h>0}$ two regular families of triangulations of $\bar\Omega_S$ and $\bar\Omega_D$, respectively, composed by simplexes that match at the interface Γ. For a $\mathcal{T}_h^S$ or $\mathcal{T}_h^D$, we will denote by K the elements of the triangulation, and by $\mathcal{E}_h^S$ the set of all edges (faces) of $\mathcal{T}_h^S$, with the splitting $\mathcal{E}_h^S := \mathcal{E}_{\Omega_S} \cup \mathcal{E}_S^{\mathrm{Dir}} \cup \mathcal{E}_{\Gamma_h}$, where $\mathcal{E}_{\Omega_S}$ stands for the edges (faces) lying in the interior of Ω_S, $\mathcal{E}_S^{\mathrm{Dir}}$ stands for the edges (faces) on the boundary Γ_S, and $\mathcal{E}_{\Gamma_h}$ stands for the edges (faces) on the boundaries Γ. In the same way, we denote by $\mathcal{E}_h^D$ the set of all edges (faces) of $\mathcal{T}_h^D$, with the splitting $\mathcal{E}_h^D := \mathcal{E}_{\Omega_D} \cup \mathcal{E}_D^{\mathrm{Neu}} \cup \mathcal{E}_D^{\mathrm{Dir}} \cup \mathcal{E}_{\Gamma_h}$, where $\mathcal{E}_{\Omega_D}$ stands for the edges (faces) lying in the interior of Ω_D, and $\mathcal{E}_D^{\mathrm{Neu}}$ and $\mathcal{E}_D^{\mathrm{Dir}}$ stand for the edges (faces) on the boundaries Γ_D^{Neu} and Γ_D^{Dir}, respectively. As usual h_T means the diameter of T, $h := \max\limits_{T\in\mathcal{T}_h^S\cup\mathcal{T}_h^D} h_T$, and $h_F := |F|$ for $F \in \mathcal{E}_h^S \cup \mathcal{E}_h^D$. Finally, for each $K \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ and $F \in \mathcal{E}_h^S \cup \mathcal{E}_h^D$, we denote by $\mathcal{N}(K)$ the set of nodes of K, $\mathcal{N}(F)$ the set of nodes of F, $\mathcal{E}(K)$ the set of edges of K, and define

$$\omega_F^S := \bigcup_{\substack{F\in\mathcal{E}(K')\\ K'\in\mathcal{T}_h^S}} K', \qquad \tilde\omega_K^S := \bigcup_{\substack{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset\\ K'\in\mathcal{T}_h^S}} K', \qquad \tilde\omega_F^S := \bigcup_{\substack{\mathcal{N}(F)\cap\mathcal{N}(K')\neq\emptyset\\ K'\in\mathcal{T}_h^S}} K',$$

$$\omega_F^D := \bigcup_{\substack{F\in\mathcal{E}(K')\\ K'\in\mathcal{T}_h^D}} K', \qquad \tilde\omega_K^D := \bigcup_{\substack{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset\\ K'\in\mathcal{T}_h^D}} K', \qquad \tilde\omega_F^D := \bigcup_{\substack{\mathcal{N}(F)\cap\mathcal{N}(K')\neq\emptyset\\ K'\in\mathcal{T}_h^D}} K'.$$

We introduce the following finite element subspaces of $\boldsymbol{H}^S$, $\boldsymbol{H}^D$, Q^S and Q^D, respectively:

$$\begin{aligned}
\boldsymbol{H}_{h,k}^S &:= \left\{\boldsymbol{v} \in C(\overline\Omega_S)^d \;:\; \boldsymbol{v}|_K \in \mathbb{P}_k(K)^d, \quad \forall K \in \mathcal{T}_h^S\right\} \cap \boldsymbol{H}^S,\\
\boldsymbol{H}_{h,k}^D &:= \left\{\boldsymbol{v} \in C(\overline\Omega_D)^d \;:\; \boldsymbol{v}|_K \in \mathbb{P}_k(K)^d, \quad \forall K \in \mathcal{T}_h^D\right\},\\
Q_{h,k}^S &:= \left\{q \in C(\overline\Omega_S) \;:\; q|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h^S\right\},\\
Q_{h,k}^D &:= \left\{q \in C(\overline\Omega_D) \;:\; q|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h^D\right\} \cap Q^D,
\end{aligned}$$

with $k \ge 1$, where $\mathbb{P}_k$ stands for the space of polynomials of total degree less or equal to k.

Let $\boldsymbol{H}_{h,k} := \boldsymbol{H}_{h,k}^S \times \boldsymbol{H}_{h,k}^D$ and $Q_{h,k} := Q_{h,k}^S \times Q_{h,k}^D$, and let $\boldsymbol{u}_h := (\boldsymbol{u}_{h,S}, \boldsymbol{u}_{h,D}) \in \boldsymbol{H}_{h,k}$ and $p_h := (p_{h,S}, p_{h,D}) \in Q_{h,k}$. We consider the following discrete stabilized scheme: *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_{h,k} \times Q_{h,k}$ such that

$$B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); (\boldsymbol{v}_h, q_h)) = F_{\mathrm{stab}}(\boldsymbol{v}_h, q_h), \tag{3.6}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_{h,k} \times Q_{h,k}$, where

$$\begin{aligned}
&B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); (\boldsymbol{v}_h, q_h)) := B((\boldsymbol{u}_h, p_h); (\boldsymbol{v}_h, q_h))\\
&+ \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \left(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}) + \nabla p_{h,S}, 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S} \right)_K\\
&+ \theta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \left(\nabla\cdot\boldsymbol{u}_{h,S}, \nabla\cdot\boldsymbol{v}_{h,S}\right)_K + \nu\,\lambda \sum_{K\in\mathcal{T}_h^D} h_K^2 \left(\nabla\cdot\boldsymbol{u}_{h,D}, \nabla\cdot\boldsymbol{v}_{h,D}\right)_K,
\end{aligned} \tag{3.7}$$

and

$$F_{\rm stab}(\boldsymbol{v}_h,q_h) := F(\boldsymbol{v}_h,q_h) + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu}\left(\boldsymbol{f}_S, 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S}\right)_K + \nu\,\lambda \sum_{K\in\mathcal{T}_h^D} h_K^2\left(g_D, \nabla\cdot\boldsymbol{v}_{h,D}\right)_K.$$

Here θ and λ are no negative constants, and $0<\beta<\dfrac{C_I}{2}$, where C_I is the constant appearing in the following inverse inequality

$$C_I \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S})\|_{0,K}^2 \le \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S})\|_{0,\Omega_S}^2 \qquad \forall \boldsymbol{v}_{h,S}\in \boldsymbol{H}_{h,k}^S. \tag{3.8}$$

Remark 1. *The stabilized finite element method* (3.6) *is inspired by the scheme proposed in [38], where a standard formulation for the Stokes equation and a Galerkin least–squares formulation for the Darcy equation was introduced. In this new method, we introduce an additional stabilization to the Stokes equation, allowing equal order interpolation spaces for the fluid flow as for the flow in the porous media.*

In the rest of this work, over $\boldsymbol{H}_{h,k}\times Q_{h,k}$ we will define the following mesh-dependent norm

$$|||(\boldsymbol{v}_h,q_h)|||_h := \left\{\nu\kappa\,\|\boldsymbol{v}_{h,S}\|_{1,\Omega_S}^2 + \kappa\,\|q_{h,S}\|_{0,\Omega_S}^2 + \kappa\,\beta\sum_{K\in\mathcal{T}_h^S}\frac{h_K^2}{\nu}\,|q_{h,S}|_{1,K}^2 + \nu\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu}\,|q_{h,D}|_{1,\Omega_D}^2\right\}^{1/2},$$

for all $(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_{h,k}\times Q_{h,k}$.

As it is well known, the subspaces $\boldsymbol{H}_{h,k}^S$ and $Q_{h,k}^S$ do not satisfy a discrete inf–sup condition, but they satisfy the following weak inf–sup condition

Lemma 2. *There exist positive constants C_1 and C_2, independent of ν and h, such that*

$$\sup_{\boldsymbol{v}_{h,S}\in\boldsymbol{H}_{h,k}^S\cap H_0^1(\Omega_S)^d}\frac{(q_{h,S},\nabla\cdot\boldsymbol{v}_{h,S})_{\Omega_S}}{\|\boldsymbol{v}_{h,S}\|_{1,\Omega_S}} \ge C_1\,\|q_{h,S}\|_{0,\Omega_S} - C_2\left\{\beta\sum_{K\in\mathcal{T}_h^S} h_K^2\,|q_{h,S}|_{1,K}^2\right\}^{1/2}, \tag{3.9}$$

for all $q_{h,S}\in Q_{h,k}^S$.

Proof. See [26, Lemma 3.3]. □

The following result will be necessary to prove the well-posedness of the stabilized finite element problem (3.6). The proof is based on similar arguments to those used in [4, Lemma 3].

Lemma 3. *For all $(\boldsymbol{u}_h,p_h)\in\boldsymbol{H}_{h,k}\times Q_{h,k}$, there exists a positive constant C, independent of h, ν, κ and α_i, such that*

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_{h,k}\times Q_{h,k}}\frac{B_{\rm stab}((\boldsymbol{u}_h,p_h);(\boldsymbol{v}_h,q_h))}{|||(\boldsymbol{v}_h,q_h)|||_h} \ge C\,|||(\boldsymbol{u}_h,p_h)|||_h.$$

Proof. Using Cauchy–Schwarz inequality, (3.8) and Lemma 1, we get

$$
\begin{aligned}
& B_{\text{stab}}((\boldsymbol{u}_h, p_h); (\boldsymbol{u}_h, p_h)) \\
\geq & \, 2\nu\,\kappa \|\varepsilon(\boldsymbol{u}_{h,S})\|_{0,\Omega_S}^2 + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \Bigg(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}) + \nabla p_{h,S}, 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}) + \nabla p_{h,S} \Bigg)_K \\
& + \nu\,\|\boldsymbol{u}_{h,D}\|_{0,\Omega_D}^2 + \frac{1}{2\nu} \left(\nu\,\boldsymbol{u}_{h,D} + \kappa\,\nabla p_{h,D}, -\nu\,\boldsymbol{u}_{h,D} + \kappa\,\nabla p_{h,D}\right)_{\Omega_D} \\
\geq & \, 2\nu\,\kappa \|\varepsilon(\boldsymbol{u}_{h,S})\|_{0,\Omega_S}^2 - 4\nu\,\kappa\,\beta \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S})\|_{0,K}^2 + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu}\,|p_{h,S}|_{1,K}^2 + \frac{\nu}{2}\,\|\boldsymbol{u}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|p_{h,D}|_{1,\Omega_D}^2 \\
\geq & \, 2\nu\,\kappa \left(1 - \frac{2\beta}{C_I}\right) \|\varepsilon(\boldsymbol{u}_{h,S})\|_{0,\Omega_S}^2 + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu}\,|p_{h,S}|_{1,K}^2 + \frac{\nu}{2}\,\|\boldsymbol{u}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|p_{h,D}|_{1,\Omega_D}^2 \\
\geq & \, C_3\,\nu\,\kappa\,\|\boldsymbol{u}_{h,S}\|_{1,\Omega_S}^2 + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu}\,|p_{h,S}|_{1,K}^2 + \frac{\nu}{2}\,\|\boldsymbol{u}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|p_{h,D}|_{1,\Omega_D}^2 . \qquad (3.10)
\end{aligned}
$$

Let $\boldsymbol{w}_S \in \boldsymbol{H}_{h,k}^S \cap H_0^1(\Omega_S)^d$ be a function for which the supremum of Lemma 2 is attained. Then

$$
\begin{aligned}
& B_{\text{stab}}((\boldsymbol{u}_h, p_h); ((-\boldsymbol{w}_S, \boldsymbol{0}), (0,0))) = B_{\text{stab}}(((\boldsymbol{u}_{h,S}, \boldsymbol{u}_{h,D}), (p_{h,S}, p_{h,D})); ((-\boldsymbol{w}_S, \boldsymbol{0}), (0,0))) \\
= & -2\nu\kappa\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}), \boldsymbol{\varepsilon}(\boldsymbol{w}_S))_{\Omega_S} - \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i (\boldsymbol{u}_{h,S}\cdot\boldsymbol{\tau}_i, \boldsymbol{w}_S\cdot\boldsymbol{\tau}_i)_\Gamma + \kappa\,(p_{h,S}, \nabla\cdot\boldsymbol{w}_S)_{\Omega_S} \\
& + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \Bigg(2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}), 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_S) \Bigg)_K - \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \Bigg(\nabla p_{h,S}, 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_S) \Bigg)_K \\
& - \theta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} (\nabla\cdot\boldsymbol{u}_{h,S}, \nabla\cdot\boldsymbol{w}_S)_K .
\end{aligned}
$$

Now assume that $\|\boldsymbol{w}_S\|_{1,\Omega_S} = \|p_{h,S}\|_{0,\Omega_S}$, using Cauchy–Schwarz inequality, the inverse inequality (3.8) with Lemma 1, we have

$$
\begin{aligned}
& B_{\text{stab}}((\boldsymbol{u}_h, p_h); ((-\boldsymbol{w}_S, \boldsymbol{0}), (0,0))) \\
\geq & -2\nu\kappa\,\|\boldsymbol{u}_{h,S}\|_{1,\Omega_S} \|\boldsymbol{w}_S\|_{1,\Omega_S} + \kappa\,(p_{h,S}, \nabla\cdot\boldsymbol{w}_S)_{\Omega_S} \\
& - \kappa\,\beta 4\nu \left\{ \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S})\|_{0,K}^2 \right\}^{1/2} \left\{ \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_S)\|_{0,K}^2 \right\}^{1/2} \\
& - 2\kappa \left\{ \beta \sum_{K\in\mathcal{T}_h^S} h_K^2\,|p_{h,S}|_{1,K}^2 \right\}^{1/2} \left\{ \beta \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_S)\|_{0,K}^2 \right\}^{1/2} - d\theta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \|\boldsymbol{u}_{h,S}\|_{1,K} \|\boldsymbol{w}_S\|_{1,K} \\
\geq & -C_4\nu\kappa\,\|\boldsymbol{u}_{h,S}\|_{1,\Omega_S} \|\boldsymbol{w}_S\|_{1,\Omega_S} + \kappa\,(p_{h,S}, \nabla\cdot\boldsymbol{w}_S)_{\Omega_S} - C_5\kappa \left\{ \beta \sum_{K\in\mathcal{T}_h^S} h_K^2\,|p_{h,S}|_{1,K}^2 \right\}^{1/2} \|\boldsymbol{w}_S\|_{1,\Omega_S} \\
& - \frac{C_6\theta}{\nu} \|\boldsymbol{u}_{h,S}\|_{1,\Omega_S} \|\boldsymbol{w}_S\|_{1,\Omega_S} \\
\geq & -\left(C_4 + \frac{C_6\theta}{\kappa\nu^2} \right) \nu\kappa\,\|\boldsymbol{u}_{h,S}\|_{1,\Omega_S} \|\boldsymbol{w}_S\|_{1,\Omega_S} + C_1\,\kappa\,\|p_{h,S}\|_{0,\Omega_S}^2 - (C_2 + C_5)\,\kappa \left\{ \beta \sum_{K\in\mathcal{T}_h^S} h_K^2\,|p_{h,S}|_{1,K}^2 \right\}^{1/2} \|p_{h,S}\|_{0,\Omega_S} .
\end{aligned}
$$

Moreover, using Young inequality with positive constants γ_1 and γ_2 chosen small enough, we can conclude that

$$
\begin{aligned}
&B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); ((-\boldsymbol{w}_S, \mathbf{0}), (0,0)))\\
&\geq -\frac{1}{2\gamma_1}\left(C_4 + \frac{C_6\theta}{\kappa\nu^2}\right)\nu\kappa\,\|\boldsymbol{u}_{h,S}\|^2_{1,\Omega_S} + \kappa\left(C_1 - \frac{\gamma_1}{2}\left(C_4 + \frac{C_6\theta}{\kappa\nu^2}\right)\nu - \frac{(C_2+C_5)\nu\gamma_2}{2}\right)\|p_{h,S}\|^2_{0,\Omega_S}\\
&\quad - \frac{C_2+C_5}{2\gamma_2}\,\kappa\,\beta\sum_{K\in\mathcal{T}_h^S}\frac{h_K^2}{\nu}\,|p_{h,S}|^2_{1,K}\\
&\geq -C_7\,\nu\kappa\,\|\boldsymbol{u}_{h,S}\|^2_{1,\Omega_S} + C_8\,\kappa\,\|p_{h,S}\|^2_{0,\Omega_S} - C_9\,\kappa\beta\sum_{K\in\mathcal{T}_h^S}\frac{h_K^2}{\nu}\,|p_{h,S}|^2_{1,K}. \qquad (3.11)
\end{aligned}
$$

Defining $(\boldsymbol{v}_h, q_h) := (\boldsymbol{u}_h, p_h) + \delta\,((-\boldsymbol{w}_S, \mathbf{0}), (0,0))$, and combining, (3.10) and (3.11), we get

$$
\begin{aligned}
&B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); (\boldsymbol{v}_h, q_h)) = B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); (\boldsymbol{u}_h, q_h)) + \delta\,B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h); ((-\boldsymbol{w}_S, \mathbf{0}), (0,0)))\\
&\geq \nu\kappa\,(C_3 - \delta C_7)\,\|\boldsymbol{u}_{h,S}\|^2_{1,\Omega_S} + \kappa\,(1-\delta C_9)\,\beta\sum_{K\in\mathcal{T}_h^S}\frac{h_K^2}{\nu}\,|p_{h,S}|^2_{1,K} + \frac{\nu}{2}\,\|\boldsymbol{u}_{h,D}\|^2_{0,\Omega_D} + \frac{\kappa^2}{2\nu}\,|p_{h,D}|^2_{1,\Omega_D}\\
&\quad + C_8\delta\,\kappa\,\|p_{h,S}\|^2_{0,\Omega_S}\\
&\geq C\left\{\nu\kappa\,\|\boldsymbol{u}_{h,S}\|^2_{1,\Omega_S} + \kappa\,\|p_{h,S}\|^2_{0,\Omega_S} + \kappa\,\beta\sum_{K\in\mathcal{T}_h^S}\frac{h_K^2}{\nu}\,|p_{h,S}|^2_{1,K} + \nu\,\|\boldsymbol{u}_{h,D}\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu}\,|p_{h,D}|^2_{1,\Omega_D}\right\}\\
&= C\,|||(\boldsymbol{u}_h, p_h)|||^2_h, \qquad (3.12)
\end{aligned}
$$

choosing $0 < \delta < \min\left\{\frac{C_3}{C_7}, \frac{1}{C_9}, \frac{1}{\nu^{1/2}}\right\}$.

On the other hand, we have

$$
|||(\boldsymbol{v}_h, q_h)|||_h \leq |||(\boldsymbol{u}_h, p_h)|||_h + \delta\nu^{1/2}\kappa^{1/2}\,\|p_{h,S}\|_{0,\Omega_S} \leq C\,|||(\boldsymbol{u}_h, p_h)|||_h,
$$

which combined with (3.12) proves the stability estimate. □

4. A priori error analysis

We begin this section by recalling a local trace theorem and the definition and properties of the interpolation operators that we will use during the proof of the convergence of the stabilized finite element scheme (3.6).

Lemma 4. *There exists a positive constant C, independent of h_K, such that*

$$
\|\psi\|^2_{0,\partial K} \leq C\left\{h_K^{-1}\|\psi\|^2_{0,K} + h_K|\psi|^2_{1,K}\right\},
$$

for all $K \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ and all $\psi \in H^1(K)$.

Proof. See [1, Theorem 3.10] or [16, (10.3.8)]. □

We consider the Lagrange interpolant operator $\mathbf{\Pi}_h^S : H^{k+1}(\Omega_S)^d \cap \boldsymbol{H}^S \longrightarrow \boldsymbol{H}^S_{h,k}$, and the Clément interpolation operator $\mathcal{C}_h^S : H^k(\Omega) \longrightarrow Q^S_{h,k}$, such that (see [24] for details) for all $K \in \mathcal{T}_h^S$, we have

$$
|\boldsymbol{u}_S - \mathbf{\Pi}_h^S\boldsymbol{u}_S|_{l,K} \leq Ch_K^{s-l}\,|\boldsymbol{u}_S|_{s,K}, \qquad (4.13)
$$

$$
\|p_S - \mathcal{C}_h^S p_S\|_{0,K} \leq Ch_K^{s}\,\|p_S\|_{s,\tilde{\omega}_K^S}, \qquad (4.14)
$$

for all $\boldsymbol{u}_S \in H^s(K)^d$ and all $p_S \in H^s(\tilde{\omega}_K^S)$ with $0 \leq l \leq 1$, $1 \leq s \leq k+1$. Here C are positive constants independent of h. Also, we consider the Clément interpolation operator $\boldsymbol{\mathcal{C}}_h^D : H^k(\Omega_D)^d \longrightarrow \boldsymbol{H}_{h,k}^D$, and the Lagrange interpolant operator $\Pi_h^D : H^{k+1}(\Omega_D) \cap Q^D \longrightarrow Q_{h,k}^D$, such that for all $K \in \mathcal{T}_h^D$, we have

$$\|\boldsymbol{u}_D - \boldsymbol{\mathcal{C}}_h^D\|_{0,K} \leq C h_K^s \, \|\boldsymbol{u}_D\|_{s,\tilde{\omega}_K^D}, \tag{4.15}$$

$$|p_D - \Pi_h^D p_D|_{l,K} \leq C h_K^{s-l} \, |p_D|_{s,K}, \tag{4.16}$$

for all $\boldsymbol{u}_D \in H^s(\tilde{\omega}_K^D)^d$ and all $p_D \in H^s(K)$ with $0 \leq l \leq 1$, $1 \leq s \leq k+1$, where C are positive constants independent of h.

Lemma 5. *Let $(\boldsymbol{u}, p) = ((\boldsymbol{u}_S, \boldsymbol{u}_D), (p_S, p_D))$ and $(\boldsymbol{u}_h, p_h) = ((\boldsymbol{u}_{h,S}, \boldsymbol{u}_{h,D}), (p_{h,S}, p_{h,D}))$ be the solutions of* (2.3) *and* (3.6)*, respectively. If $\boldsymbol{u}_S \in H^2(\Omega_S)^d$, $\boldsymbol{u}_D \in H^1(\Omega_D)^d$ and $p_S \in H^1(\Omega_S)$, then it holds*

$$B_{\text{stab}}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h); (\boldsymbol{v}_h, q_h)) = 0,$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_{h,k} \times Q_{h,k}$.

Proof. Using the fact that $\boldsymbol{u}_S \in H^2(\Omega_S)^d$ and $p_S \in H^1(\Omega_S)$, we get that $\boldsymbol{f}_S - \nabla \cdot (2\nu\varepsilon(\boldsymbol{u}_S)) + \nabla p_S = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{u}_S = 0$. Using the fact that $\boldsymbol{u}_D \in H^1(\Omega_D)^d$ we get $\nabla \cdot \boldsymbol{u}_D = 0$, thus the stabilized terms in the definition of B_{stab} vanish. □

Theorem 6. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of* (2.3) *and* (3.6)*, respectively. Assume that $(\boldsymbol{u}, p) = ((\boldsymbol{u}_S, \boldsymbol{u}_D), (p_S, p_D)) \in H^{k+1}(\Omega_S)^d \cap \boldsymbol{H}^S \times H^k(\Omega_D)^d \times H^k(\Omega_S) \times H^{k+1}(\Omega_D) \cap Q^D$, then*

$$\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\| \leq C \, h^k \left\{ \|\boldsymbol{u}_S\|_{k+1,\Omega_S} + \|p_S\|_{k,\Omega_S} + \|\boldsymbol{u}_D\|_{k,\Omega_D} + \|p_D\|_{k+1,\Omega_D} \right\},$$

with $C > 0$ independent of h.

Proof. We consider the following notations:

$$\begin{aligned}
\eta^{\boldsymbol{u}_S} &:= \boldsymbol{u}_S - \boldsymbol{\Pi}_h^S \boldsymbol{u}_S, \quad \eta^{p_S} := p_S - \mathcal{C}_h^S p_S, \\
\eta^{\boldsymbol{u}_D} &:= \boldsymbol{u}_D - \boldsymbol{\mathcal{C}}_h^D \boldsymbol{u}_D, \quad \eta^{p_D} := p_D - \Pi_h^D p_D, \\
\eta^{\boldsymbol{u}} &:= (\eta^{\boldsymbol{u}_S}, \eta^{\boldsymbol{u}_D}), \quad \eta^p := (\eta^{p_S}, \eta^{p_D}), \\
e_h^{\boldsymbol{u}} &:= (\boldsymbol{u}_{h,S} - \boldsymbol{\Pi}_h^S \boldsymbol{u}_S, \boldsymbol{u}_{h,D} - \boldsymbol{\mathcal{C}}_h^D \boldsymbol{u}_D), \quad e_h^p := (p_{h,S} - \mathcal{C}_h^S p_S, p_{h,D} - \Pi_h^D p_D).
\end{aligned}$$

Using the definition of $B_{\rm stab}$ given in (3.7), Lemma 5 and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
& B_{\rm stab}((e_h^{\boldsymbol{u}}, e_h^p); (\boldsymbol{v}_h, q_h)) = B_{\rm stab}((\eta^{\boldsymbol{u}}, \eta^p); (\boldsymbol{v}_h, q_h)) - B_{\rm stab}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); (\boldsymbol{v}_h, q_h)) \\
&\le 2\,\nu\,\kappa \|\eta^{\boldsymbol{u}_S}\|_{1,\Omega_S} \|\boldsymbol{v}_{h,S}\|_{1,\Omega_S} + \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i \sum_{F\in\mathcal{E}_{\Gamma_h}} \|\eta^{\boldsymbol{u}_S}\cdot\boldsymbol{\tau}_i\|_{0,F} \|\boldsymbol{v}_{h,S}\cdot\boldsymbol{\tau}_i\|_{0,F} + \kappa \sum_{F\in\mathcal{E}_{\Gamma_h}} \|\eta^{p_D}\|_{0,F} \|\boldsymbol{v}_{h,S}\cdot\boldsymbol{n}_S\|_{0,F} \\
&\quad + \kappa\,\|\eta^{p_S}\|_{0,\Omega_S} \|\nabla\cdot\boldsymbol{v}_{h,S}\|_{0,\Omega_S} + \kappa\,\|q_{h,S}\|_{0,\Omega_S} \|\nabla\cdot\eta^{\boldsymbol{u}_S}\|_{0,\Omega_S} \\
&\quad + \kappa\,\beta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu}\, \|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_S}) + \nabla\eta^{p_S}\|_{0,K}\, \|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S}\|_{0,K} \\
&\quad + \nu\,\|\eta^{\boldsymbol{u}_D}\|_{0,\Omega_D}\, \|\boldsymbol{v}_{h,D}\|_{0,\Omega_D} + \kappa\,|\eta^{p_D}|_{1,\Omega_D} \|\boldsymbol{v}_{h,D}\|_{0,\Omega_D} + \kappa\,\|\eta^{\boldsymbol{u}_D}\|_{0,\Omega_D} |q_{h,D}|_{1,\Omega_D} + \kappa \sum_{F\in\mathcal{E}_{\Gamma_h}} \|\eta^{\boldsymbol{u}_S}\cdot\boldsymbol{n}_S\|_{0,F} \|q_{h,D}\|_{0,F} \\
&\quad + \frac{1}{\nu}\,\|\nu\,\eta^{\boldsymbol{u}_D} + \kappa\,\nabla\eta^{p_D}\|_{0,\Omega_D}\, \|-\nu\,\boldsymbol{v}_{h,D} + \kappa\,\nabla q_{h,D}\|_{0,\Omega_D} \\
&\quad + \theta \sum_{K\in\mathcal{T}_h^S} \frac{h_K^2}{\nu} \|\nabla\cdot\eta^{\boldsymbol{u}_S}\|_{0,K} \|\nabla\cdot\boldsymbol{v}_{h,S}\|_{0,K} + \nu\lambda \sum_{K\in\mathcal{T}_h^D} h_K^2 \|\nabla\cdot\eta^{\boldsymbol{u}_D}\|_{0,K} \|\nabla\cdot\boldsymbol{v}_{h,D}\|_{0,K} \\
&\le C \Bigg\{ \|\eta^{\boldsymbol{u}_S}\|_{1,\Omega_S}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1} \|\eta^{\boldsymbol{u}_S}\|_{0,F}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1} \|\eta^{p_D}\|_{0,F}^2 + \|\eta^{p_S}\|_{0,\Omega_S}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h^S} h_K^2\, \|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_S}) + \nabla\eta^{p_S}\|_{0,K}^2 + \|\eta^{\boldsymbol{u}_D}\|_{0,\Omega_D}^2 + |\eta^{p_D}|_{1,\Omega_D}^2 + \|\nu\,\eta^{\boldsymbol{u}_D} + \kappa\,\nabla\eta^{p_D}\|_{0,\Omega_D}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h^S} h_K^2 |\eta^{\boldsymbol{u}_S}|_{1,K}^2 + \sum_{K\in\mathcal{T}_h^D} h_K^2 |\eta^{\boldsymbol{u}_D}|_{1,K}^2 \Bigg\}^{1/2} \\
&\quad \Bigg\{ \|\boldsymbol{v}_{h,S}\|_{1,\Omega_S}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F \|\boldsymbol{v}_{h,S}\|_{0,F}^2 + \|q_{h,S}\|_{0,\Omega_S}^2 + \sum_{K\in\mathcal{T}_h^S} h_K^2\, \|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S}\|_{0,K}^2 \\
&\quad + \|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + |q_{h,D}|_{1,\Omega_D}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F \|q_{h,D}\|_{0,F}^2 + \|-\nu\,\boldsymbol{v}_{h,D} + \kappa\,\nabla q_{h,D}\|_{0,\Omega_D}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h^S} h_K^2 \|\nabla\cdot\boldsymbol{v}_{h,S}\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h^D} h_K^2 \|\nabla\cdot\boldsymbol{v}_{h,D}\|_{0,K}^2 \Bigg\}^{1/2}. \qquad (4.17)
\end{aligned}
$$

Now, using mesh regularity, (4.13), and Lemma 4, we get

$$
\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1} \|\eta^{\boldsymbol{u}_S}\|_{0,F}^2 \le C \sum_{K\in\mathcal{T}_h^S} h_K^{-1} \left\{ h_K^{-1}\, \|\eta^{\boldsymbol{u}_S}\|_{0,K}^2 + h_K\, |\eta^{\boldsymbol{u}_S}|_{1,K}^2 \right\} \le C\, h^{2k} \|\boldsymbol{u}_S\|_{k+1,\Omega_S}. \qquad (4.18)
$$

Similar arguments allow us to obtain the following estimates:

$$
\begin{aligned}
\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1} \|\eta^{p_D}\|_{0,F}^2 &\le C\, h^{2k} \|p_D\|_{k+1,\Omega_D}, \\
\sum_{K\in\mathcal{T}_h^S} h_K^2\, \|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_S}) + \nabla\eta^{p_S}\|_{0,K}^2 &\le C\, h^{2k} \Big[\|\boldsymbol{u}_S\|_{k+1,\Omega_S}^2 + \|p_S\|_{k,\Omega_S}^2 \Big].
\end{aligned}
$$

Thus, it follows that

$$\begin{aligned}
&\|\eta^{\boldsymbol{u}_S}\|^2_{1,\Omega_S} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{\boldsymbol{u}_S}\|^2_{0,F} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{p_D}\|^2_{0,F} + \|\eta^{p_S}\|^2_{0,\Omega_S}\\
&+ \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_S}) + \nabla\eta^{p_S}\|^2_{0,K} + \|\eta^{\boldsymbol{u}_D}\|^2_{0,\Omega_D} + |\eta^{p_D}|^2_{1,\Omega_D} + \|\nu\,\eta^{\boldsymbol{u}_D} + \kappa\,\nabla\eta^{p_D}\|^2_{0,\Omega_D}\\
&+ \sum_{K\in\mathcal{T}_h^S} h_K^2\,|\eta^{\boldsymbol{u}_S}|^2_{1,K} + \sum_{K\in\mathcal{T}_h^D} h_K^2\,|\eta^{\boldsymbol{u}_D}|^2_{1,K} \le C\;h^{2k}\left\{\|\boldsymbol{u}_S\|^2_{k+1,\Omega_S} + \|p_S\|^2_{k,\Omega_S} + \|\boldsymbol{u}_D\|^2_{k,\Omega_D} + \|p_D\|^2_{k+1,\Omega_D}\right\}.
\end{aligned} \tag{4.19}$$

On the other hand, using inverse and Poincaré's inequalities, mesh regularity, and Lemma 4, we obtain

$$\begin{aligned}
&\|\boldsymbol{v}_{h,S}\|^2_{1,\Omega_S} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|\boldsymbol{v}_{h,S}\|^2_{0,F} + \|q_{h,S}\|^2_{0,\Omega_S} + \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S}\|^2_{0,K}\\
&+\ \|\boldsymbol{v}_{h,D}\|^2_{0,\Omega_D} + |q_{h,D}|^2_{1,\Omega_D} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|q_{h,D}\|^2_{0,F} + \|\,-\nu\,\boldsymbol{v}_{h,D} + \kappa\,\nabla q_{h,D}\|^2_{0,\Omega_D}\\
&+ \sum_{K\in\mathcal{T}_h^S} h_K^2\|\nabla\cdot\boldsymbol{v}_{h,S}\|^2_{0,K} + \sum_{K\in\mathcal{T}_h^D} h_K^2\|\nabla\cdot\boldsymbol{v}_{h,D}\|^2_{0,K}\\
&\le C\left\{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|^2_h + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|\boldsymbol{v}_{h,S}\|^2_{0,F} + \sum_{K\in\mathcal{T}_h^S} h_K^2\,\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,S}) + \nabla q_{h,S}\|^2_{0,K} + \frac{\kappa^2}{\nu}\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|q_{h,D}\|^2_{0,F}\right\}\\
&\le C\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|^2_h.
\end{aligned} \tag{4.20}$$

Now, from (4.17), (4.19), (4.20) and Lemma 3, we have

$$\|(e_h^{\boldsymbol{u}},e_h^p)\|^2 \le |\!|\!|(e_h^{\boldsymbol{u}},e_h^p)|\!|\!|^2_h \le C\,h^{2k}\left\{\|\boldsymbol{u}_S\|^2_{k+1,\Omega_S} + \|p_S\|^2_{k,\Omega_S} + \|\boldsymbol{u}_D\|^2_{k,\Omega_D} + \|p_D\|^2_{k+1,\Omega_D}\right\}. \tag{4.21}$$

Using interpolation properties (4.13)-(4.16), we obtain the estimate

$$\|(\eta^{\boldsymbol{u}},\eta^p)\|^2 \le C\,h^{2k}\left\{\|\boldsymbol{u}_S\|^2_{k+1,\Omega_S} + \|p_S\|^2_{k,\Omega_S} + \|\boldsymbol{u}_D\|^2_{k,\Omega_D} + \|p_D\|^2_{k+1,\Omega_D}\right\}. \tag{4.22}$$

The result follows using triangle inequality, (4.21) and (4.22). □

5. A posterior error analysis

This section introduces a residual a posteriori error estimator for the stabilized finite element method (3.6). Throughout this section, we will assume, for simplicity, that $\boldsymbol{f}_S$ and g_D are piecewise polynomials functions in Ω_S and Ω_D, respectively.

5.1. Preliminaries results

For the reliability of the error estimator, we need to define standard bubble functions and some of the results associated with them. We will define these functions considering the case in three dimensions ($d = 3$), but the results are still valid in two dimensions ($d = 2$). For all $K \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$, we define the element bubble function b_K by

$$b_K := (d+1)^{d+1} \prod_{x\in\mathcal{N}(K)} \lambda_x,$$

where λ_x corresponds to the barycentric coordinates associated to node x. Let $\hat{K}$ be the standard reference element with vertices $\hat{n}_1 := (1,0,0)$, $\hat{n}_2 := (0,1,0)$, $\hat{n}_3 := (0,0,1)$ and $\hat{n}_4 := (0,0,0)$, we define the edge bubble function by

$$b_{\hat{F}} := d^d \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_4,$$

where $\hat{F} := \big\{(\hat{x}, \hat{y}, 0) \in \mathbb{R}^d : 0 \leq \hat{x} + \hat{y} \leq 1,\ \hat{x} \in [0,1]\big\}$. For $F \in \mathcal{E}_h^S \cup \mathcal{E}_h^D$ assume that $\omega_F^S = K_1 \cup K_2$ with $K_1, K_2 \in \mathcal{T}_h^S$ or $\omega_F^D = K_1 \cup K_2$ with $K_1, K_2 \in \mathcal{T}_h^D$. Let $G_{F,i}$ be the (orientation preserving) affine transformation such that $G_{F,i}(\hat{K}) = K_i$ and $G_{F,i}(\hat{F}) = F$, with $i = 1, 2$. We define the bubble function associated with $F \in \mathcal{E}_h^S$ by

$$b_F^S := \begin{cases} b_{\hat{F}} \circ G_{F,i}^{-1}, & \text{on } K_i, \quad i = 1,2, \\ 0 & \text{on } \ \Omega_S \setminus \omega_F^S, \end{cases} \tag{5.23}$$

and with $F \in \mathcal{E}_h^D$ by

$$b_F^D := \begin{cases} b_{\hat{F}} \circ G_{F,i}^{-1}, & \text{on } K_i, \quad i = 1,2, \\ 0 & \text{on } \ \Omega_D \setminus \omega_F^D. \end{cases} \tag{5.24}$$

Let $\hat{\Pi} := \{(x,y,0) \,:\, (x,y) \in \mathbb{R}^2\}$ and let $\hat{Q} : \mathbb{R}^d \to \hat{\Pi}$ be the orthogonal projection from $\mathbb{R}^d$ to $\hat{\Pi}$. We introduce the lifting operator $\hat{P}_{\hat{F}} : \mathbb{P}_k(\hat{F}) \to \mathbb{P}_k(\hat{K})$ given by

$$\hat{s} \longmapsto \hat{P}_{\hat{F}}(\hat{s}) = \hat{s} \circ \hat{Q}.$$

Let $K_i \subseteq \omega_F^S \cup \omega_F^D$. We define the lifting operator $P_{F,K_i} : \mathbb{P}_k(F) \to \mathbb{P}_k(K_i)$ by

$$P_{F,K_i}(s) = \hat{P}_{\hat{F}}(s \circ G_{F,i}) \circ G_{F,i}^{-1}.$$

Using these notations, we can define a lifting operator $P_F^S : \mathbb{P}_k(F) \to \mathbb{P}_k(\omega_F^S)$ by

$$s \in \mathbb{P}_k(F) \longmapsto P_F^S(s) := \begin{cases} P_{F,K_1}(s) & \text{in } K_1, \\ P_{F,K_2}(s) & \text{in } K_2, \end{cases}$$

for $\boldsymbol{s} = (s_1, s_2, s_3) \in \mathbb{P}_k(F)^d$, we define $\mathcal{P}_F^S(\boldsymbol{s})$ by

$$\mathcal{P}_F^S(\boldsymbol{s}) := (P_F^S(s_1), P_F^S(s_2), P_F^S(s_3)).$$

Similary, we define

$$\mathcal{P}_F^D(\boldsymbol{s}) := (P_F^D(s_1), P_F^D(s_2), P_F^D(s_3)),$$

where $P_F^D : \mathbb{P}_k(F) \to \mathbb{P}_k(\omega_F^D)$ by

$$s \in \mathbb{P}_k(F) \longmapsto P_F^D(s) := \begin{cases} P_{F,K_1}(s) & \text{in } K_1, \\ P_{F,K_2}(s) & \text{in } K_2. \end{cases}$$

The next result can be prove using scaling arguments.

Theorem 7. *Let b_K be the bubble function corresponding to $K \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$. Then for all $\boldsymbol{v}_h \in \mathbb{P}_n$, $n \geq 0$ there exists a positive constant C, depending on the regularity of the mesh, such that*

$$\|\boldsymbol{v}_h\|_{0,K}^2 \leq C\,(b_K \boldsymbol{v}_h, \boldsymbol{v}_h)_K, \tag{5.25}$$

Moreover, let b_F^S and b_F^D be the bubbles functions defined in (5.23) *and* (5.24)*, respectively. Then there exist positive constants C, independent of h_K, such that*

$$\|\boldsymbol{v}_h\|_{0,F}^2 \leq C\,(b_F^S \boldsymbol{v}_h, \boldsymbol{v}_h)_F \qquad \forall F \in \mathcal{E}_h^S, \tag{5.26}$$

$$\|\boldsymbol{v}_h\|_{0,F}^2 \leq C\,(b_F^D \boldsymbol{v}_h, \boldsymbol{v}_h)_F \qquad \forall F \in \mathcal{E}_h^D, \tag{5.27}$$

$$\|b_F^S \mathcal{P}_F^S \boldsymbol{s}_h\|_{0,K} \leq C\,h_K |b_F^S \mathcal{P}_F^S \boldsymbol{s}_h|_{1,K} \qquad \forall F \in \mathcal{E}_h^S, K \in \mathcal{T}_h^S, \tag{5.28}$$

$$\|b_F^D \mathcal{P}_F^D \boldsymbol{s}_h\|_{0,K} \leq C\,h_K |b_F^D \mathcal{P}_F^D \boldsymbol{s}_h|_{1,K} \qquad \forall F \in \mathcal{E}_h^D, K \in \mathcal{T}_h^D, \tag{5.29}$$

$$|b_F^S \mathcal{P}_F^S \boldsymbol{s}_h|_{1,\omega_F^S} \leq C\,h_F^{-1/2} \|\boldsymbol{s}_h\|_{0,F} \qquad \forall F \in \mathcal{E}_h^S, \tag{5.30}$$

$$|b_F^D \mathcal{P}_F^D \boldsymbol{s}_h|_{1,\omega_F^D} \leq C\,h_F^{-1/2} \|\boldsymbol{s}_h\|_{0,F} \qquad \forall F \in \mathcal{E}_h^D, \tag{5.31}$$

for every polynomial $\boldsymbol{v}_h$, $\boldsymbol{s}_h$ of degree n defined in K and F, respectively.

Proof. See [3, Theorem 19]. □

We denote by $\boldsymbol{\mathcal{C}}_h^S : \boldsymbol{H}^S \to \boldsymbol{H}_{h,1}^S$, $\boldsymbol{\mathcal{C}}_h^D : \boldsymbol{H}^D \to \boldsymbol{H}_{h,1}^D$ and $\mathcal{C}_h^D : Q^D \to Q_{h,1}^D$, the Clément interpolation operators. Let $K \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ and $F \in \mathcal{E}_h^S$. These operators satisfy the following estimates (see [21] or [24]): There exist positive constants C, independent of h_K and the physical parameters, such that

$$\|\boldsymbol{v} - \boldsymbol{\mathcal{C}}_h^S \boldsymbol{v}\|_{0,K} \leq C\, h_K |\boldsymbol{v}|_{1,\tilde{\omega}_K^S}, \tag{5.32}$$

$$\|\boldsymbol{v} - \boldsymbol{\mathcal{C}}_h^S \boldsymbol{v}\|_{0,F} \leq C\, h_F^{1/2} |\boldsymbol{v}|_{1,\tilde{\omega}_F^S}, \tag{5.33}$$

$$\|\boldsymbol{w} - \boldsymbol{\mathcal{C}}_h^D \boldsymbol{w}\|_{0,K} \leq C\, \|\boldsymbol{w}\|_{0,\tilde{\omega}_K^D}, \tag{5.34}$$

$$\|q - \mathcal{C}_h^D q\|_{0,K} \leq C\, h_K |q|_{1,\tilde{\omega}_K^D}, \tag{5.35}$$

$$\|q - \mathcal{C}_h^D q\|_{0,F} \leq C\, h_F^{1/2} |q|_{1,\tilde{\omega}_F^D}, \tag{5.36}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^S$, $\boldsymbol{w} \in \boldsymbol{H}^D$ and $q \in Q^D$.

5.2. A residual error estimator

For each $K \in \mathcal{T}_h^S$ and each $F \in \mathcal{E}_{\Omega_S}$, we define the residuals

$$\mathcal{R}_S^K := \Big(\boldsymbol{f}_S + \nabla \cdot \boldsymbol{\sigma}_h \Big)\Big|_K, \tag{5.37}$$

$$\mathcal{R}_{\Omega_S}^F := [\![\boldsymbol{\sigma}_h \boldsymbol{n}_S]\!]_F, \tag{5.38}$$

where $\boldsymbol{\sigma}_h := 2\nu\, \boldsymbol{\varepsilon}(\boldsymbol{u}_{h,S}) - p_{h,S}\, \mathbf{I}$. For each $F \in \mathcal{E}_{\Gamma_h}$, we define

$$\mathcal{R}_{\text{ENF}}^F := \Big(-\boldsymbol{n}_S \cdot \boldsymbol{\sigma}_h \boldsymbol{n}_S - p_{h,D} \Big)\Big|_F, \tag{5.39}$$

$$\mathcal{R}_{\text{BJS,i}}^F := \Big(-\kappa \boldsymbol{n}_S \cdot \boldsymbol{\sigma}_h \boldsymbol{\tau}_i - \kappa^{1/2} \alpha_i\, \boldsymbol{u}_{h,S} \cdot \boldsymbol{\tau}_i \Big)\Big|_F, \quad 1 \leq i \leq d-1, \tag{5.40}$$

$$\mathcal{R}_{S,\Gamma}^F := \mathcal{R}_{\text{ENF}}^F \boldsymbol{n}_S + \sum_{i=1}^{d-1} \mathcal{R}_{\text{BJS,i}}^F \tau_i, \tag{5.41}$$

$$\mathcal{R}_{\text{MC}}^F := \Big(\boldsymbol{u}_{h,S} \cdot \boldsymbol{n}_S + \boldsymbol{u}_{h,D} \cdot \boldsymbol{n}_D \Big)\Big|_F. \tag{5.42}$$

Here, $[\![\boldsymbol{v}]\!]_F$ represents the jump of $\boldsymbol{v}$ across F. Similarly, for each $K \in \mathcal{T}_h^D$ and $F \in \mathcal{E}_h^D$, we define

$$\mathcal{R}_{1,D}^K := \Big(\nu \boldsymbol{u}_{h,D} + \kappa \nabla p_{h,D} \Big)\Big|_K, \tag{5.43}$$

$$\mathcal{R}_{2,D}^K := \Big(g_D - \nabla \cdot \boldsymbol{u}_{h,D} \Big)\Big|_K, \tag{5.44}$$

$$\mathcal{R}_D^F := \begin{cases} [\![\boldsymbol{u}_{h,D} \cdot \boldsymbol{n}_D]\!]_F, & F \in \mathcal{E}_{\Omega_D}, \\ \boldsymbol{u}_{h,D} \cdot \boldsymbol{n}_D \Big|_F, & F \in \mathcal{E}_D^{\text{Neu}}, \\ 0, & F \in \mathcal{E}_D^{\text{Dir}} \cup \mathcal{E}_{\Gamma_h}. \end{cases} \tag{5.45}$$

Our residual–based error estimator is given by

$$\eta := \left\{ \sum_{K \in \mathcal{T}_h^S} \eta_{S,K}^2 + \sum_{K \in \mathcal{T}_h^D} \eta_{D,K}^2 \right\}^{1/2}, \tag{5.46}$$

where, for each $K \in \mathcal{T}_h^S$, we define

$$\eta_{S,K} := \left\{ \frac{\kappa}{\nu}\, h_K^2 \left\| \mathcal{R}_S^K \right\|_{0,K}^2 + \kappa \left\| \nabla\cdot \boldsymbol{u}_{h,S} \right\|_{0,K}^2 + \frac{1}{2} \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Omega_S}} \frac{\kappa}{\nu} h_F \left\| \mathcal{R}_{\Omega_S}^F \right\|_{0,F}^2 + \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Gamma_h}} \frac{\kappa}{\nu} h_F \left\| \mathcal{R}_{S,\Gamma}^F \right\|_{0,F}^2 \right\}^{1/2}, \tag{5.47}$$

and for each $K \in \mathcal{T}_h^D$,

$$\begin{aligned} \eta_{D,K} := \Bigg\{ & \frac{1}{\nu} \left\| \mathcal{R}_{1,D}^K \right\|_{0,K}^2 + \nu h_K^2 \left\| \mathcal{R}_{2,D}^K \right\|_{0,K}^2 + \frac{1}{2} \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Omega_D}} \nu h_F \|\mathcal{R}_D^F\|_{0,F}^2 \\ & + \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_D^{\mathrm{Neu}}} \nu h_F \|\mathcal{R}_D^F\|_{0,F}^2 + \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Gamma_h}} \nu h_F \left\| \mathcal{R}_{\mathrm{MC}}^F \right\|_{0,F}^2 \Bigg\}^{1/2}. \end{aligned} \tag{5.48}$$

Remark 2. *Using integration by parts element–wise, we get for all* $(\boldsymbol{v}_S, q_S) \in \boldsymbol{H}^S \times Q^S$ *that*

$$\begin{aligned} & B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); ((\boldsymbol{v}_S, \boldsymbol{0}), (q_S, 0))) \\ &= \sum_{K\in\mathcal{T}_h^S} \kappa \left(\mathcal{R}_S^K, \boldsymbol{v}_S\right)_K + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa \left(\mathcal{R}_{S,\Gamma}^F, \boldsymbol{v}_S\right)_F + \sum_{F\in\mathcal{E}_{\Omega_S}} \kappa \left(\mathcal{R}_{\Omega_S}^F, \boldsymbol{v}_S\right)_F - \kappa \left(q_S, \nabla\cdot \boldsymbol{u}_{h,S}\right)_{\Omega_S}. \end{aligned} \tag{5.49}$$

Similarly, we have that for all $(\boldsymbol{v}_D, q_D) \in \boldsymbol{H}^D \times Q^D$

$$\begin{aligned} B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); ((\boldsymbol{0}, \boldsymbol{v}_D), (0, q_D))) = & \sum_{K\in\mathcal{T}_h^D} \kappa \left(\mathcal{R}_{2,D}^K, q_D\right)_K + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa \left(\mathcal{R}_{\mathrm{MC}}^F, q_D\right)_F + \sum_{F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}} \kappa \left(\mathcal{R}_D^F, q_D\right)_F \\ & - \frac{1}{2} \sum_{K\in\mathcal{T}_h^D} \left(\mathcal{R}_{1,D}^K, \boldsymbol{v}_D + \frac{\kappa}{\nu}\nabla q_D\right)_K. \end{aligned} \tag{5.50}$$

5.3. Reliability of the a posteriori error estimator

Theorem 8. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ *and* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_{h,k} \times Q_{h,k}$ *solutions of* (2.3) *and* (3.6)*, respectively. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\| \le C \left\{ \eta + \left[\sum_{K\in\mathcal{T}_h^S} \frac{h_K^4}{\nu^3\kappa} \|\nabla\cdot \boldsymbol{u}_{h,S}\|_{0,K}^2 \right]^{1/2} \right\}.$$

Proof. Let $(\boldsymbol{v}, q) := ((\boldsymbol{v}_S, \boldsymbol{v}_D), (q_S, q_D)) \in \boldsymbol{H} \times Q$ an arbitrary element, and $\boldsymbol{v}_{1,S} := \boldsymbol{\mathcal{C}}_h^S \boldsymbol{v}_S$. Applying (5.49) to $(\boldsymbol{v}_S - \boldsymbol{v}_{1,S}, q_S)$, using interpolation properties, and Cauchy-Schwarz inequalities, we get

$$
\begin{aligned}
& B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h);((\boldsymbol{v}_S-\boldsymbol{v}_{1,S},\boldsymbol{0}),(q_S,0))) \\
&\le \sum_{K\in\mathcal{T}_h^S} \kappa \left\|\mathcal{R}_S^K\right\|_{0,K} \|\boldsymbol{v}_S-\boldsymbol{v}_{1,S}\|_{0,K} + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa \left\|\mathcal{R}_{S,\Gamma}^F\right\|_{0,F} \|\boldsymbol{v}_S-\boldsymbol{v}_{1,S}\|_{0,F} \\
&\quad + \sum_{F\in\mathcal{E}_{\Omega_S}} \kappa \left\|\mathcal{R}_{\Omega_S}^F\right\|_{0,F} \|\boldsymbol{v}_S-\boldsymbol{v}_{1,S}\|_{0,F} + \sum_{K\in\mathcal{T}_h^S} \kappa \|q_S\|_{0,K}\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K} \\
&\le C\Bigg\{ \sum_{K\in\mathcal{T}_h^S} \kappa h_K \left\|\mathcal{R}_S^K\right\|_{0,K} |\boldsymbol{v}_S|_{1,\tilde{\omega}_K} + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa h_F^{1/2} \left\|\mathcal{R}_{S,\Gamma}^F\right\|_{0,F} |\boldsymbol{v}_S|_{1,\tilde{\omega}_F^S} \\
&\quad + \sum_{F\in\mathcal{E}_{\Omega_S}} \kappa h_F^{1/2} \left\|\mathcal{R}_{\Omega_S}^F\right\|_{0,F} |\boldsymbol{v}_S|_{1,\tilde{\omega}_F^S} + \sum_{K\in\mathcal{T}_h^S} \kappa \|q_S\|_{0,K}\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K} \Bigg\} \\
&\le C\Bigg\{ \sum_{K\in\mathcal{T}_h^S} \frac{\kappa}{\nu} h_K^2 \left\|\mathcal{R}_S^K\right\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} \frac{\kappa}{\nu} h_F \left\|\mathcal{R}_{S,\Gamma}^F\right\|_{0,F}^2 + \sum_{F\in\mathcal{E}_{\Omega_S}} \frac{\kappa}{\nu} h_F \left\|\mathcal{R}_{\Omega_S}^F\right\|_{0,F}^2 + \sum_{K\in\mathcal{T}_h^S} \kappa \|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}^2 \Bigg\}^{1/2} \\
&\quad \Bigg\{ \sum_{K\in\mathcal{T}_h^S} \kappa\nu \, |\boldsymbol{v}_S|_{1,\tilde{\omega}_K}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa\nu |\boldsymbol{v}_S|_{1,\tilde{\omega}_F^S}^2 + \sum_{F\in\mathcal{E}_{\Omega_S}} \kappa\nu |\boldsymbol{v}_S|_{1,\tilde{\omega}_F^S}^2 + \sum_{K\in\mathcal{T}_h^S} \kappa \|q_S\|_{0,K}^2 \Bigg\}^{1/2} \\
&\le C \Bigg\{ \sum_{K\in\mathcal{T}_h^S} \eta_{S,K}^2 \Bigg\}^{1/2} \|(\boldsymbol{v},q)\|. \qquad (5.51)
\end{aligned}
$$

On the other hand, let $\boldsymbol{v}_{1,D} := \boldsymbol{\mathcal{C}}_h^D \boldsymbol{v}_D$ and $q_{1,D} := \mathcal{C}_h^D q_D$. Using a similar procedure, we have

$$
\begin{aligned}
& B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h);((\boldsymbol{0},\boldsymbol{v}_D-\boldsymbol{v}_{1,D}),(0,q_D-q_{1,D}))) \\
&= \sum_{K\in\mathcal{T}_h^D} \kappa \left(\mathcal{R}_{2,D}^K, q_D-q_{1,D}\right)_K + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa \left(\mathcal{R}_{\mathrm{MC}}^F, q_D-q_{1,D}\right)_F + \sum_{F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}} \kappa \left(\mathcal{R}_D^F, q_D-q_{1,D}\right)_F \\
&\quad - \frac{1}{2} \sum_{K\in\mathcal{T}_h^D} \Big(\mathcal{R}_{1,D}^K, (\boldsymbol{v}_D-\boldsymbol{v}_{1,D}) + \frac{\kappa}{\nu}\nabla(q_D-q_{1,D})\Big)_K \\
&\le \sum_{K\in\mathcal{T}_h^D} \kappa \left\|\mathcal{R}_{2,D}^K\right\|_{0,K} \|q_D-q_{1,D}\|_{0,K} + \sum_{F\in\mathcal{E}_{\Gamma_h}} \kappa \left\|\mathcal{R}_{\mathrm{MC}}^F\right\|_{0,F} \|q_D-q_{1,D}\|_{0,F} + \sum_{F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}} \kappa \|\mathcal{R}_D^F\|_{0,F}\|q_D-q_{1,D}\|_{0,F} \\
&\quad + \frac{1}{2} \sum_{K\in\mathcal{T}_h^D} \left\|\mathcal{R}_{1,D}^K\right\|_{0,K} \left[\|\boldsymbol{v}_D-\boldsymbol{v}_{1,D}\|_{0,K} + \frac{\kappa}{\nu}|q_D-q_{1,D}|_{1,K}\right] \\
&\le C\Bigg\{ \sum_{K\in\mathcal{T}_h^D} \nu h_K^2 \left\|\mathcal{R}_{2,D}^K\right\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{\Gamma_h}} \nu h_F \left\|\mathcal{R}_{\mathrm{MC}}^F\right\|_{0,F}^2 + \sum_{F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}} \nu h_F \|\mathcal{R}_D^F\|_{0,F}^2 + \sum_{K\in\mathcal{T}_h^D} \frac{1}{\nu} \left\|\mathcal{R}_{1,D}^K\right\|_{0,K}^2 \Bigg\}^{1/2} \\
&\quad \Bigg\{ \nu \|\boldsymbol{v}_D\|_{0,\Omega_D} + \frac{\kappa^2}{\nu} |q_D|_{1,\Omega_D} \Bigg\}^{1/2} \\
&\le C \Bigg\{ \sum_{K\in\mathcal{T}_h^D} \eta_{D,K}^2 \Bigg\}^{1/2} \|(\boldsymbol{v},q)\|. \qquad (5.52)
\end{aligned}
$$

Let $\tilde{\boldsymbol{v}}_h := (\boldsymbol{v}_{1,S}, \boldsymbol{v}_{1,D})$ and $\tilde{q}_h := (0, q_{1,D})$. Combining (5.51) and (5.52), we obtain

$$
B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h);(\boldsymbol{v}-\tilde{\boldsymbol{v}}_h, q-\tilde{q}_h)) \le C\,\eta\,\|(\boldsymbol{v},q)\|. \qquad (5.53)
$$

Similar arguments proves that

$$B((\boldsymbol{u}-\boldsymbol{u}_h,p-p_h);(\tilde{\boldsymbol{v}}_h,\tilde{q}_h)) \leq C\left\{\eta+\left[\sum_{K\in\mathcal{T}_h^S}\frac{h_K^4}{\nu^3\kappa}\,\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}^2\right]^{1/2}\right\}\|(\boldsymbol{v},q)\|. \tag{5.54}$$

Therefore, from (5.53), (5.54) and Lemma 14, we arrive at

$$\begin{aligned}\|(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)\| &\leq C\sup_{(\boldsymbol{v},q)\in\boldsymbol{H}\times Q}\frac{B((\boldsymbol{u}-\boldsymbol{u}_h,p-p_h);(\boldsymbol{v},q))}{\|(\boldsymbol{v},q)\|}\\ &\leq C\left\{\eta+\left[\sum_{K\in\mathcal{T}_h^S}\frac{h_K^4}{\nu^3\kappa}\,\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}^2\right]^{1/2}\right\},\end{aligned}$$

which conclude the proof of the result. □

Remark 3. *The term*

$$T:=\left\{\sum_{K\in\mathcal{T}_h^S}\frac{h_K^4}{\nu^3\kappa}\,\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}^2\right\}^{1/2}, \tag{5.55}$$

appearing in Theorem 8, is only present if $\theta>0$ in the definition of B_{stab} in (3.7). In that case T is, at least, $\mathcal{O}(h^2)$ asymptotically, i.e. is a high order term compared to η.

5.4. Efficiency of the a posteriori error estimator

In order to prove the efficiency of error estimate η, we need some preliminary results.

Lemma 9. *For all $K\in\mathcal{T}_h^S$, there exist two positive constants C_1 and C_2, independent of ν, κ and h, such that*

$$\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}\leq C_1\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K},$$

and

$$\sqrt{\frac{\kappa}{\nu}}h_K\|\mathcal{R}_S^K\|_{0,K}\leq C_2\max\left\{1,\frac{1}{\sqrt{\nu}}\right\}\left\{\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K}+\sqrt{\kappa}\,\|p_S-p_{h,S}\|_{0,K}\right\}.$$

Proof. Let b_K be the bubble function associated to $K\in\mathcal{T}_h^S$. Since $\nabla\cdot\boldsymbol{u}_S\in L^2(\Omega_S)$ and using (2.3), we have that $\nabla\cdot\boldsymbol{u}_S=0$ a.e in Ω_S. Then it is clear that

$$\|\nabla\cdot\boldsymbol{u}_{h,S}\|_{0,K}\leq C_1\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K},$$

for all $K\in\mathcal{T}_h^S$. Now, let us define $\boldsymbol{b}_K:=b_K\mathcal{R}_S^K$. From (5.49), (2.4) and an inverse inequality, we obtain

$$\begin{aligned}\kappa\,(\mathcal{R}_S^K,\boldsymbol{b}_K)_K &= \sum_{K'\in\mathcal{T}_h^S}\kappa\,(\mathcal{R}_S^{K'},\boldsymbol{b}_K)_{K'} = B((\boldsymbol{u}-\boldsymbol{u}_h,p-p_h);((\boldsymbol{b}_K,\boldsymbol{0}),(0,0)))\\ &=2\nu\kappa\,(\varepsilon(\boldsymbol{u}_S-\boldsymbol{u}_{h,S}),\varepsilon(\boldsymbol{b}_K))_K-\kappa\,(p_S-p_{h,S},\nabla\cdot\boldsymbol{b}_K)_K\\ &\leq C\left\{\nu\kappa\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K}+\kappa\,\|p_S-p_{h,S}\|_{0,K}\right\}|\boldsymbol{b}_K|_{1,K}\\ &\leq C\left\{\nu\kappa\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K}+\kappa\,\|p_S-p_{h,S}\|_{0,K}\right\}h_K^{-1}\|\boldsymbol{b}_K\|_{0,K}\\ &\leq C\left\{\nu\kappa\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K}+\kappa\,\|p_S-p_{h,S}\|_{0,K}\right\}h_K^{-1}\|\mathcal{R}_S^K\|_{0,K}.\end{aligned}$$

Next, using Theorem 7, it follows, for all $K\in\mathcal{T}_h^S$, that

$$\kappa\,h_K\|\mathcal{R}_S^K\|_{0,K}\leq C\left\{\nu\kappa\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,K}+\kappa\,\|p_S-p_{h,S}\|_{0,K}\right\}, \tag{5.56}$$

hence, multiplying by $\sqrt{\frac{1}{\kappa\nu}}$, we arrive to the desired result. □

For the next result, we will consider the bubble function b_F^S defined for all $F \in \mathcal{E}_{\Omega_S} \cup \mathcal{E}_{\Gamma_h}$.

Lemma 10. *Let $F \in \mathcal{E}_{\Gamma_h}$. Then, there exist a positive constant C_3, independent of ν, κ, and h, such that:*

$$\sqrt{\frac{\kappa}{\nu}} h_F^{1/2} \, \|\mathcal{R}_{S,\Gamma}^F\|_{0,F} \leq C_3 \, \max\left\{1, \frac{1}{\sqrt{\nu}}, \frac{1}{\nu}, \frac{h_F}{\nu\sqrt{\kappa}}, \frac{h_F}{\sqrt{\kappa}}\right\} \left\{\sqrt{\nu\kappa}\, \|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\kappa}\, \|p_S - p_{h,S}\|_{0,\omega_F^S} + \frac{\kappa}{\sqrt{\nu}} \|p_D - p_{h,D}\|_{1,\omega_F^D}\right\}. \tag{5.57}$$

Furthermore, for $F \in \mathcal{E}_{\Omega_S}$ there holds

$$\sqrt{\frac{\kappa}{\nu}} h_F^{1/2} \|\mathcal{R}_{\Omega_S}^F\|_{0,F} \leq C_4 \, \max\left\{1, \frac{1}{\sqrt{\nu}}\right\} \left\{\sqrt{\nu\kappa}\, \|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\kappa}\, \|p_S - p_{h,S}\|_{0,\omega_F^S}\right\}, \tag{5.58}$$

where C_4 is a positive constant independent of h.

Proof. Applying Theorem 7, (5.49), the definition of B and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\kappa\|\mathcal{R}_{S,\Gamma}^F\|_{0,F}^2 \leq C\kappa(\mathcal{R}_{S,\Gamma}^F, b_F^S \mathcal{R}_{S,\Gamma}^F)_F \\
&\leq C\left\{B((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h); ((b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F, \mathbf{0}), (0,0))) - \sum_{K \in \omega_F^S} \kappa(\mathcal{R}_S^K, b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F)_K\right\} \\
&= C\left\{\nu\kappa(\varepsilon(\boldsymbol{u}_S - \boldsymbol{u}_{h,S}), \varepsilon(b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F))_{\omega_F^S} - \kappa(p_S - p_{h,S}, \nabla \cdot (b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F))_{\omega_F^S}\right. \\
&\quad \left. + \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i((\boldsymbol{u}_S - \boldsymbol{u}_{h,S}) \cdot \tau_i, b_F^S \mathcal{R}_{S,\Gamma}^F \cdot \tau_i)_F + \kappa(p_D - p_{h,D}, b_F^S \mathcal{R}_{S,\Gamma}^F \cdot \boldsymbol{n}_S)_F - \sum_{K \in \omega_F^S} \kappa(\mathcal{R}_S^K, b_F^S \mathcal{P}_F^S \mathcal{R}_S^F)_K\right\} \\
&\leq C\left\{\nu\kappa\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} |b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F|_{1,\omega_F^S} + \kappa\|p_S - p_{h,S}\|_{0,\omega_F^S} |b_F^S \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F|_{1,\omega_F^S} + \kappa^{1/2}\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{0,F} \|b_F^S \mathcal{R}_{S,\Gamma}^F\|_{0,F}\right. \\
&\quad \left. + \kappa\|p_D - p_{h,D}\|_{0,F} \|b_F^S \mathcal{R}_{S,\Gamma}^F\|_{0,F} + \sum_{K \in \omega_F^S} \kappa\|\mathcal{R}_S^K\|_{0,K} \|b_F \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F\|_{0,K}\right\}.
\end{aligned}$$

On the other hand, from Lemma 4, (5.28), Theorem 7, mesh regularity, and (5.56), we have

$$\begin{aligned}
\kappa\|\mathcal{R}_{S,\Gamma}^F\|_{0,F}^2 &\leq C\left\{\nu\kappa\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} h_F^{-1/2} \|\mathcal{R}_{S,\Gamma}^F\|_{0,F} + \kappa\|p_S - p_{h,S}\|_{0,\omega_F^S} h_F^{-1/2} \|\mathcal{R}_{S,\Gamma}^F\|_{0,F}\right. \\
&\quad \left. + \kappa^{1/2}\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{0,F} \|\mathcal{R}_{S,\Gamma}^F\|_{0,F} + \kappa\|p_D - p_{h,D}\|_{0,F} \|\mathcal{R}_{S,\Gamma}^F\|_{0,F} + \sum_{K \in \omega_F^S} \kappa h_K \|\mathcal{R}_S^K\|_{0,K} |b_F \mathcal{P}_F^S \mathcal{R}_{S,\Gamma}^F|_{1,\omega_F^S}\right\} \\
&\leq C\left\{\nu\kappa\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \kappa\|p_S - p_{h,S}\|_{0,\omega_F^S} + \kappa^{1/2} \|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{0,\omega_F^S}\right. \\
&\quad \left. + \kappa^{1/2} h_F \, |\boldsymbol{u}_S - \boldsymbol{u}_{h,S}|_{1,\omega_F^S} + \kappa \, \|p_D - p_{h,D}\|_{0,\omega_F^D} + \kappa h_F \, |p_D - p_{h,D}|_{1,\omega_F^D}\right\} h_F^{-1/2} \, \|\mathcal{R}_{S,\Gamma}^F\|_{0,F},
\end{aligned}$$

thus

$$\sqrt{\frac{\kappa}{\nu}} h_F^{1/2}\ \|\mathcal{R}_{S,\Gamma}^F\|_{0,F} \leq C\ \max\left\{1, \frac{1}{\sqrt{\nu}}, \frac{1}{\nu}, \frac{h_F}{\nu\sqrt{\kappa}}, \frac{h_F}{\sqrt{\kappa}}\right\} \left\{\sqrt{\nu\kappa}\, \|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} \right.$$
$$\left. + \sqrt{\kappa}\, \|p_S - p_{h,S}\|_{0,\omega_F^S} + \frac{\kappa}{\sqrt{\nu}}\, \|p_D - p_{h,D}\|_{1,\omega_F^D}\right\}.$$

Similar arguments, applied to $F \in \mathcal{E}_{\Omega_S}$, show that

$$\begin{aligned}
\kappa\|\mathcal{R}_{\Omega_S}^F\|_{0,F}^2 &\leq C\kappa(\mathcal{R}_{\Omega_S}^F, b_F^S\mathcal{R}_{\Omega_S}^F)_F \\
&\leq C\left[B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); ((b_F^S\mathcal{P}_F^S\mathcal{R}_{\Omega_S}^F, \mathbf{0}), (0,0))) - \sum_{K\in\omega_F^S} \kappa(\mathcal{R}_S^K, b_F^S\mathcal{P}_F^S\mathcal{R}_{\Omega_S}^F)_K\right] \\
&\leq C\left[\nu\kappa\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \kappa\|p_S - p_{h,S}\|_{0,\omega_F^S}\right] |b_F^S\mathcal{P}_F^S\mathcal{R}_{\Omega_S}^F|_{1,\omega_F^S} \\
&\leq C\left[\nu\kappa\|\boldsymbol{u}_S - \boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \kappa\|p_S - p_{h,S}\|_{0,\omega_F^S}\right] h_F^{-1/2}\, \|\mathcal{R}_{\Omega_S}^F\|_{0,F},
\end{aligned}$$

and the result follows. □

To prove the efficiency of η, we will need the next result

Lemma 11. *For all $K \in \mathcal{T}_h^D$, there exist two positive constants C_5 and C_6, independent of ν, κ, and h, such that*

$$\frac{1}{\sqrt{\nu}}\, \|\mathcal{R}_{1,D}^K\|_{0,K} \leq C_5 \left\{\sqrt{\nu}\, \|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa}{\sqrt{\nu}}\, |p_D - p_{h,D}|_{1,K}\right\}, \tag{5.59}$$

and

$$\sqrt{\nu}\, h_K \|\mathcal{R}_{2,D}^K\|_{0,K} \leq C_6 \left\{\sqrt{\nu}\, \|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa}{\sqrt{\nu}}\, |p_D - p_{h,D}|_{1,K}\right\}. \tag{5.60}$$

Proof. From Theorem 7, (5.50) and Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\|\mathcal{R}_{1,D}^K\|_{0,K}^2 &\leq C\, (\mathcal{R}_{1,D}^K, b_K\mathcal{R}_{1,D}^K)_K \leq C \sum_{K'\in\mathcal{T}_h^D} (R_{1,D}^{K'}, b_K\mathcal{R}_{1,D}^K)_{K'} \\
&\leq C\, B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); ((\mathbf{0}, -b_K\mathcal{R}_{1,D}^K), (0,0))) \\
&\leq C\left\{\nu\, (\boldsymbol{u}_D - \boldsymbol{u}_{h,D}, -b_K\mathcal{R}_{1,D}^K)_K + \kappa\, (\nabla(p_D - p_{h,D}), -b_K\mathcal{R}_{1,D}^K)_K + \right. \\
&\qquad \left. (\nu\, (\boldsymbol{u}_D - \boldsymbol{u}_{h,D}) + \kappa\, \nabla(p_D - p_{h,D}), b_K\mathcal{R}_{1,D}^K)_K\right\} \\
&\leq C\left\{\nu\, \|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,K} + \kappa\, |p_D - p_{h,D}|_{1,K}\right\} \|\mathcal{R}_{1,D}^K\|_{0,K},
\end{aligned}$$

which proves (5.59). We can now proceed analogously, using Theorem 7, (5.50), (5.59) and Cauchy–Schwarz inequality, to conclude that

$$
\begin{aligned}
&\kappa\,\|\mathcal{R}_{2,D}^K\|_{0,K}^2 \le C\,\kappa\,(\mathcal{R}_{2,D}^K, b_K\mathcal{R}_{2,D}^K)_K \le C \sum_{K'\in\mathcal{T}_h^D} \kappa\,(R_{2,D}^{K'}, b_K\mathcal{R}_{2,D}^K)_{K'} \\
&\le C\left\{ B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h); ((\mathbf{0},\mathbf{0}),(0,b_K\mathcal{R}_{2,D}^K))) + \left(\mathcal{R}_{1,D}^K, \frac{\kappa}{\nu}\nabla\left(b_K\,\mathcal{R}_{2,D}^K\right)\right)_K\right\} \\
&\le C\left\{ -\kappa\,\left(\boldsymbol{u}_D-\boldsymbol{u}_{h,D}, \nabla\left(b_K\mathcal{R}_{2,D}^K\right)\right)_K + \left(\nu\,(\boldsymbol{u}_D-\boldsymbol{u}_{h,D}) + \kappa\,\nabla(p_D-p_{h,D}), \frac{\kappa}{\nu}\nabla\left(b_K\,\mathcal{R}_{2,D}^K\right)\right)_K + \right. \\
&\left. \frac{\kappa}{\nu}\,\|\mathcal{R}_{1,D}^K\|_{0,K}\,\left|b_K\mathcal{R}_{2,D}^K\right|_{1,K}\right\} \\
&\le C\left\{\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \frac{\kappa}{\nu}\,\|\mathcal{R}_{1,D}^K\|_{0,K}\right\}|b_K\mathcal{R}_{2,D}^K|_{1,K} \\
&\le C\left\{\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \frac{\kappa}{\nu}\,\|\mathcal{R}_{1,D}^K\|_{0,K}\right\}h_K^{-1}\,\|\mathcal{R}_{2,D}^K\|_{0,K},
\end{aligned}
$$

thus, using (5.59) we prove (5.60). □

Lemma 12. *For all $F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}$, there exists two positive constants C_7 and C_8, independent of ν, κ, and h, such that*

$$
\sqrt{\nu}h_F^{1/2}\|\mathcal{R}_D^F\|_{0,F} \le C_7\left\{\sqrt{\nu}\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\omega_F^D} + \frac{\kappa}{\sqrt{\nu}}\,|p_D-p_{h,D}|_{1,\omega_F^D}\right\}. \tag{5.61}
$$

Furthermore, for all $F\in\mathcal{E}_{\Gamma_h}$ the following estimate holds:

$$
\sqrt{\nu}\,h_F^{1/2}\|\mathcal{R}_{\mathrm{MC}}^F\|_{0,F} \le C_8\,\max\left\{1,\frac{1}{\sqrt{\kappa}},\frac{h_F}{\sqrt{\kappa}}\right\}\left\{\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,\omega_F^S}+\sqrt{\nu}\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\omega_F^D}+\frac{\kappa}{\sqrt{\nu}}\,|p_D-p_{h,D}|_{1,\omega_F^D}\right\}. \tag{5.62}
$$

Proof. We use the same ideas that in previous results, but now for $F\in\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\mathrm{Neu}}$. In fact, using Theorem 7, (5.50), and Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\kappa\|\mathcal{R}_D^F\|_{0,F}^2 \le C\,\kappa\,\left(\mathcal{R}_D^F, b_F^D\mathcal{R}_D^F\right)_F \\
&\le C\left\{B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h);((\mathbf{0},\mathbf{0}),(0,b_F^D\,\mathcal{P}_F^D\mathcal{R}_D^F))) - \sum_{K\in\omega_F^D}\kappa\left(\mathcal{R}_{2,D}^K, b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)_K + \frac{1}{2}\sum_{K\in\omega_F^D}\left(\mathcal{R}_{1,D}^K,\frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)\right)_K\right\} \\
&\le C\left\{-\kappa\,\left(\boldsymbol{u}_D-\boldsymbol{u}_{h,D},\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)\right)_{\omega_F^D} + \left(\nu\,(\boldsymbol{u}_D-\boldsymbol{u}_{h,D})+\kappa\,\nabla(p_D-p_{h,D}),\frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)\right)_{\omega_F^D}\right. \\
&\left.-\sum_{K\in\omega_F^D}\kappa\,\left(\mathcal{R}_{2,D}^K, b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)_K + \frac{1}{2}\sum_{K\in\omega_F^D}\left(\mathcal{R}_{1,D}^K,\frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right)\right)_K\right\} \\
&\le C\sum_{K\in\omega_F^D}\left[\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \kappa\,h_K\,\left\|\mathcal{R}_{2,D}^K\right\|_{0,K} + \frac{\kappa}{\nu}\,\left\|\mathcal{R}_{1,D}^K\right\|_{0,K}\right]\,\left|b_F^D\mathcal{P}_F^D\mathcal{R}_D^F\right|_{1,\omega_F^D} \\
&\le C\sum_{K\in\omega_F^D}\left[\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \kappa\,h_K\,\left\|\mathcal{R}_{2,D}^K\right\|_{0,K} + \frac{\kappa}{\nu}\,\left\|\mathcal{R}_{1,D}^K\right\|_{0,K}\right]\,h_F^{-1/2}\,\left\|\mathcal{R}_D^F\right\|_{0,F},
\end{aligned}
$$

which, combined with Lemma 11, leads to (5.61). Again, using Theorem 7, and Cauchy–Schwarz inequality, we get that

$$
\begin{aligned}
&\kappa\|\mathcal{R}_{\text{MC}}^F\|_{0,F}^2 \leq C\,\kappa\,\left(\mathcal{R}_{\text{MC}}^F, b_F^D\mathcal{R}_{\text{MC}}^F\right)_F \\
\leq& C\left\{B((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h);((\mathbf{0},\mathbf{0}),(0,b_F^D\,\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F))) - \sum_{K\in\omega_F^D}\kappa\left(\mathcal{R}_{2,D}^K, b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)_K + \frac{1}{2}\sum_{K\in\omega_F^D}\left(\mathcal{R}_{1,D}^K, \frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)\right)_K\right\} \\
\leq& C\left\{-\kappa\left(\boldsymbol{u}_D-\boldsymbol{u}_{h,D}, \nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)\right)_{\omega_F^D} - \kappa\left((\boldsymbol{u}_S-\boldsymbol{u}_{h,S})\cdot\boldsymbol{n}_S, b_F^D\mathcal{R}_{\text{MC}}^F\right)_F\right. \\
&+\left(\nu\left(\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\right)+\kappa\nabla(p_D-p_{h,D}), \frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)\right)_{\omega_F^D} \\
&\left.-\sum_{K\in\omega_F^D}\kappa\left(\mathcal{R}_{2,D}^K, b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)_K + \sum_{K\in\omega_F^D}\left(\mathcal{R}_{1,D}^K, \frac{\kappa}{\nu}\nabla\left(b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F\right)\right)_K\right\} \\
\leq& C\left\{\sum_{K\in\omega_F^D}\left[\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \kappa\,h_K\,\left\|\mathcal{R}_{2,D}^K\right\|_{0,K} + \frac{\kappa}{\nu}\,\left\|\mathcal{R}_{1,D}^K\right\|_{0,K}\right]\,|b_F^D\mathcal{P}_F^D\mathcal{R}_{\text{MC}}^F|_{1,\omega_F^D}\right. \\
&\left.+\,\kappa\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{0,F}\,\left\|b_F^D\mathcal{R}_{\text{MC}}^F\right\|_{0,F}\right\} \\
\leq& C\left\{\sum_{K\in\omega_F^D}\left[\kappa\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa^2}{\nu}\,|p_D-p_{h,D}|_{1,K} + \kappa\,h_K\,\left\|\mathcal{R}_{2,D}^K\right\|_{0,K} + \frac{\kappa}{\nu}\,\left\|\mathcal{R}_{1,D}^K\right\|_{0,K}\right]\right. \\
&\left.+\,\kappa\,h_F^{1/2}\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{0,F}\right\}h_F^{-1/2}\,\left\|\mathcal{R}_{\text{MC}}^F\right\|_{0,F},
\end{aligned}
$$

thus (5.62) is obtained by lemmas 4 and 11. □

Theorem 13. *Let $(\boldsymbol{u},p)\in\boldsymbol{H}\times Q$ and $(\boldsymbol{u}_h,p_h)\in\boldsymbol{H}_{h,k}\times Q_{h,k}$ solutions of (2.3) and (3.6), respectively. Then, there is two positive constants C_9 and C_{10}, independent of h, such that*

$$
\begin{aligned}
\eta_{S,K} \leq& C_9\max\left\{1, \frac{1}{\sqrt{\nu}}, \frac{1}{\nu}, \frac{h_F}{\nu\sqrt{\kappa}}, \frac{h_F}{\sqrt{\kappa}}\right\}\left\{\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\kappa}\,\|p_S-p_{h,S}\|_{0,\omega_F^S}\right. \\
&+\frac{\kappa}{\sqrt{\nu}}\,\|p_D-p_{h,D}\|_{1,\omega_F^D} + \sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Omega_S}}\left[\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\kappa}\,\|p_S-p_{h,S}\|_{0,\omega_F^S}\right] \\
&\left.+\sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Gamma_h}}\left[\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\kappa}\,\|p_S-p_{h,S}\|_{0,\omega_F^S} + \frac{\kappa}{\sqrt{\nu}}\,\|p_D-p_{h,D}\|_{1,\omega_F^D}\right]\right\},
\end{aligned}
$$

for all $K\in\mathcal{T}_h^S$, and

$$
\begin{aligned}
\eta_{D,K} \leq& C_{10}\max\left\{1, \frac{1}{\sqrt{\kappa}}, \frac{h_K}{\sqrt{\kappa}}\right\}\left\{\sqrt{\nu}\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,K} + \frac{\kappa}{\sqrt{\nu}}\,|p_D-p_{h,D}|_{1,K}\right. \\
&+\sum_{F\in\mathcal{E}(K)\cap(\mathcal{E}_{\Omega_D}\cup\mathcal{E}_D^{\text{Neu}})}\left[\sqrt{\nu}\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\omega_F^D} + \frac{\kappa}{\sqrt{\nu}}\,|p_D-p_{h,D}|_{1,\omega_F^D}\right] \\
&\left.+\sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_{\Gamma_h}}\left[\sqrt{\nu\kappa}\,\|\boldsymbol{u}_S-\boldsymbol{u}_{h,S}\|_{1,\omega_F^S} + \sqrt{\nu}\,\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\omega_F^D} + \frac{\kappa}{\sqrt{\nu}}\,|p_D-p_{h,D}|_{1,\omega_F^D}\right]\right\},
\end{aligned}
$$

for all $K\in\mathcal{T}_h^D$.

Proof. The proof follows from a direct application of lemmas 9–12. □

6. Numerical experiments

In this section we present three numerical examples to show the quality of our stabilized adaptive scheme. All the examples were performed using the python libraries `Multiphenics` and `FEniCS` [34].

Recall that we use the notation $\boldsymbol{H}_{h,k} \times Q_{h,k}$ to mean that the velocity and the pressure, of Stokes and Darcy, are approximated using piecewise continuous polynomials of total degree at most k.

We define the effectivity index E as follows

$$E := \frac{\eta}{\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|}.$$

Thus our adaptive algorithm for the Stokes–Darcy coupled problem is given by

Algorithm 1 Adaptivity procedure

Require: $\vartheta \in (0,1)$ and a coarse mesh $\mathcal{T}_h = \mathcal{T}_h^S \cup \mathcal{T}_h^D$.

1: Solve the stabilized discrete scheme (3.6) on the current mesh.

2: For each $K \in \mathcal{T}_h$, compute the local error indicators $\eta_{S,K}$ or $\eta_{D,K}$ given by (5.47) and (5.48), respectively.

3: Given $K \in \mathcal{T}_h$ such that $\eta_K \geq \vartheta \max_{K' \in \mathcal{T}_h} \eta_{K'}$ mark K and generate a new mesh $\mathcal{T}_h$ refining the marked elements.

4: If the stop criterion is not satisfied, go to step 1.

6.1. *A smooth solution in two dimensions*

In this example (see [5]), $\Omega_S :=]0, 1/2[\times]0, 1[$ and $\Omega_D :=]1/2, 1[\times]0, 1[$, the interface between Ω_S and Ω_D is given by $\Gamma := \{(1/2, y) \in \mathbb{R}^2 : 0 < y < 1\}$ and, on the Beavers–Joseph–Saffman interface condition, $\alpha_1 = 1$. The boundary Γ_D is divided on $\Gamma_D^{\text{Dir}} := \{(1, y) \in \mathbb{R}^2 : 0 < y < 1\} \cup \{(x, 1) \in \mathbb{R}^2 : 1/2 < x < 1\}$ and $\Gamma_D^{\text{Neu}} := \{(x, 0) \in \mathbb{R}^2 : 1/2 < x < 1\}$ as shown in the Figure 6.1.

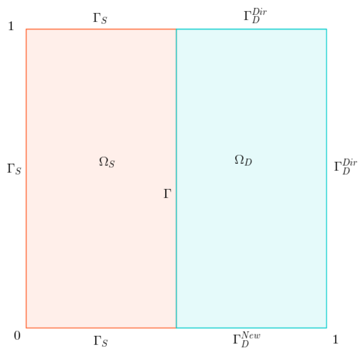

Figure 2: Configuration of the computational domain Ω.

The data $\boldsymbol{f}_S$ and g_D are such that the exact solution of our problem is given by

$$\begin{aligned}
\boldsymbol{u}_S(x,y) &:= (xy(1-y), x^2(1-y)\sin y), \quad p_S(x,y) := 12x^2 e^y, \\
\boldsymbol{u}_D(x,y) &:= (2xy(1-y)(1-x), xy^2(1-y)), \quad p_D(x,y) := 16xy^3 - e - 2.
\end{aligned}$$

Finally, we use $\nu = 1, 10^{-2}$ and $\kappa = 1$. The approximation subspaces used for this example are $\boldsymbol{H}_{h,k} \times Q_{h,k}$ with $k = 1$ and $k = 2$.

Figures 3 – 6, show the orders of convergence for our proposed scheme in a quasi-uniform refinement. We can observe that the error norms $\|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,\Omega_D}$ and $\|p_S - p_{h,S}\|_{0,\Omega_S}$ present a better behavior than predicted by Theorem 6; we assume that this is because the solutions of this example are smooth. In Tables 1–2, we present the behavior of the residual error estimator η defined through (5.46). Note that the error estimator has a quite good quality reflected on the fact that effectivity indexes are close to one.

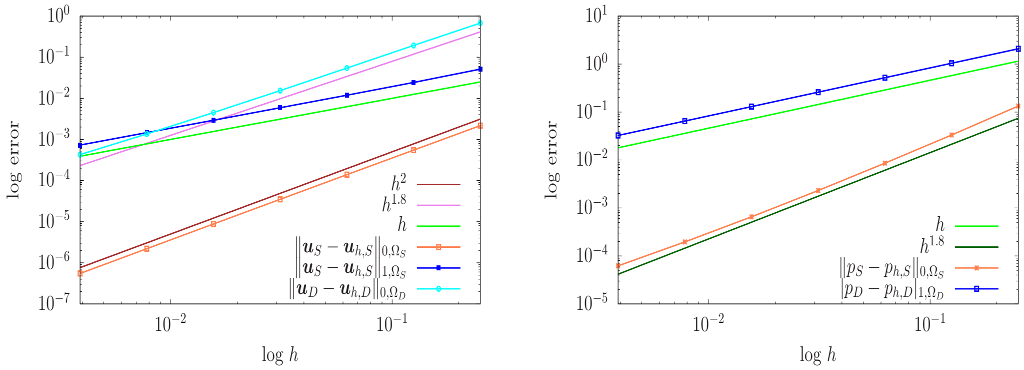

Figure 3: Convergence of the velocities (left) and pressures (right). Here $\nu = \kappa = 1$ and the interpolation order is $k = 1$.

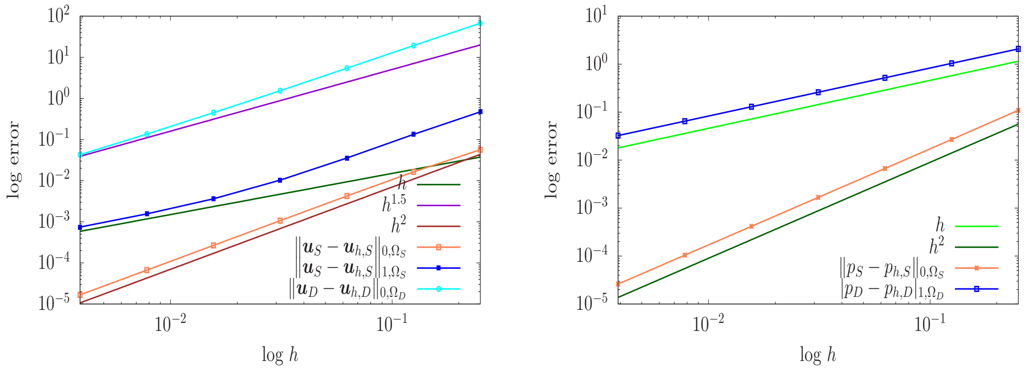

Figure 4: Convergence of the velocities (left) and pressures (right). Here $\nu = 10^{-2}$, $\kappa = 1$ and the interpolation order is $k = 1$.

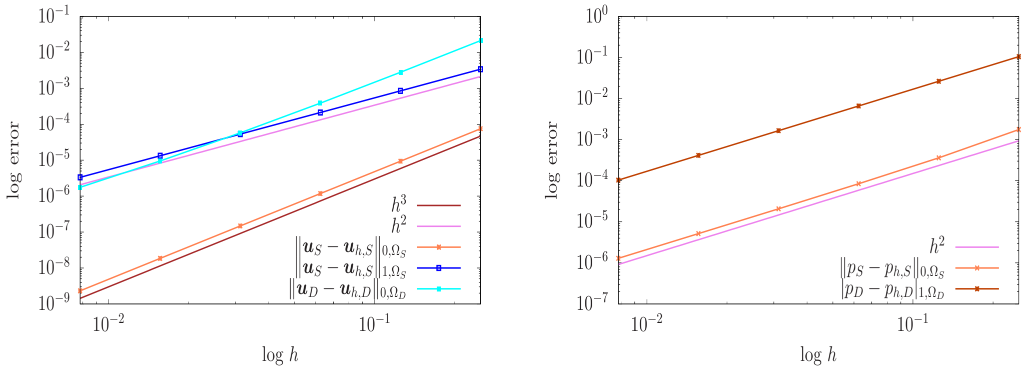

Figure 5: Convergence of the velocities (left) and pressures (right). Here $\nu = \kappa = 1$ and the interpolation order is $k = 2$.

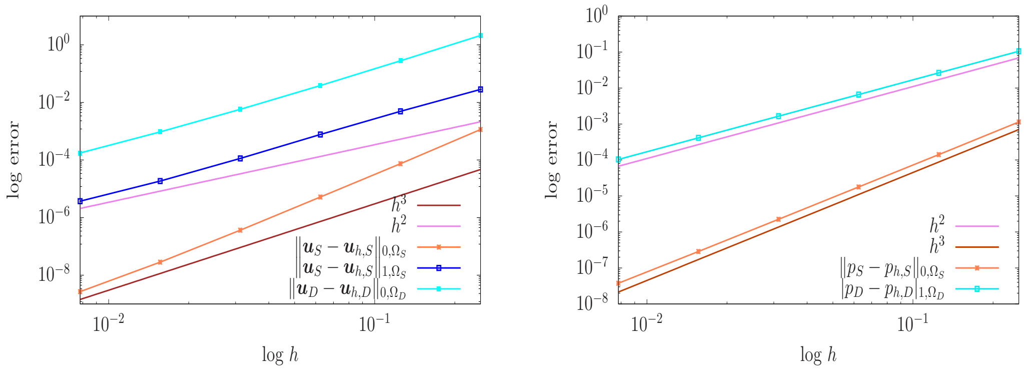

Figure 6: Convergence of the velocities (left) and pressures (right). Here $\nu = 10^{-2}, \kappa = 1$ and the interpolation order is $k = 2$.

k	h	$\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|$	order	η	order	E
1	0.0625	0.5223353	–	0.5199705	–	0.9954727
	0.03125	0.2601458	1.0056559	0.2605913	0.9966406	1.0017128
	0.015625	0.1299038	1.0018765	0.1304250	0.9985686	1.0040122
	0.0078125	0.0649220	1.0006647	0.0652396	0.9994017	1.0048916
	0.0039062	0.0324552	1.0002562	0.0326256	0.9997438	1.0052486
2	0.0625	0.0066269	–	0.0077845	–	1.1746854
	0.03125	0.0016570	1.9997191	0.0019468	1.9994783	1.1748814
	0.015625	0.0004144	1.9995227	0.0004869	1.9994128	1.1749709
	0.0078125	0.0001036	1.9996771	0.0001218	1.9996247	1.1750136

Table 1: Orders of convergence and effectivity index. Here $\nu = \kappa = 1$.

k	h	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	order	η	order	E
	0.0625	5.2212942	–	5.1711817	–	0.9904023
	0.03125	2.6006875	1.0055144	2.5912903	0.9968233	0.9963867
1	0.015625	1.2986902	1.0018357	1.2968388	0.9986714	0.9985745
	0.0078125	0.6490518	1.0006519	0.6486641	0.9994556	0.9994028
	0.0039062	0.3244693	1.0002517	0.3243835	0.9997714	0.9997356
	0.0625	0.0662295	–	0.0660592	–	0.9974282
2	0.03125	0.0165606	1.9997172	0.0165099	2.0004243	0.9969395
	0.015625	0.0041415	1.9995226	0.0041278	1.9999050	0.9966753
	0.0078125	0.0010356	1.9996773	0.0010320	1.9998723	0.9965406

Table 2: Orders of convergence and effectivity index. Here $\nu = 10^{-2}$, $\kappa = 1$.

6.2. *Simulation of a micro-filtration processes*

Inspired by [30], we build a test case based on the simulation of a micro-filtration process that satisfies our interface conditions. Let $\Omega_S :=]-1/2, 1[\times]0, 1/2[$, $\Omega_D :=]0, 1/2[\times]-1/4, 0[$, and the different boundaries conditions are represented in Figure 7. We consider $\nu = \kappa = 10^{-2}$, $\alpha_1 = 10^{-2}$ on the Beavers–Joseph–Saffman condition, and the data $\boldsymbol{f}_S = (0, 0)$ and $g_D = 0$.

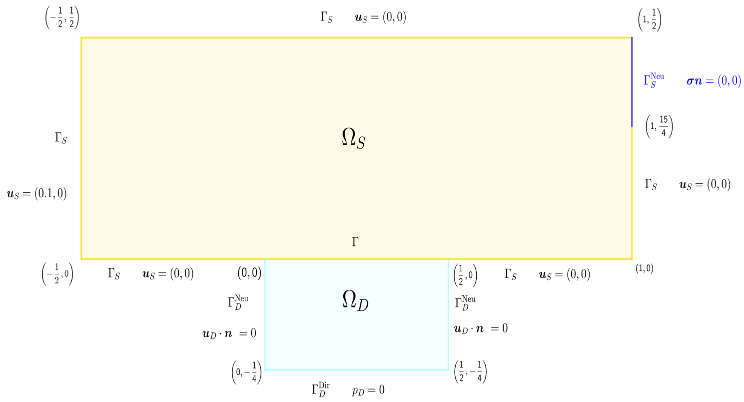

Figure 7: Boundaries conditions for simulation of a micro-filtration processes.

In this test case we do not have an analytical solution, thus we compare our adapted solution with a numerical solution obtained in a highly uniform refined mesh of 560.000 elements. In Figure 8 we show that our adaptive scheme refine the mesh in the corners of the domain, as well as, at the out flow boundary.

In Figures 9 and 10, we depict the pressure and velocity magnitude isolines, both in the case of our adapted mesh and in the reference solution. Note that we get a good agreement in both solutions.

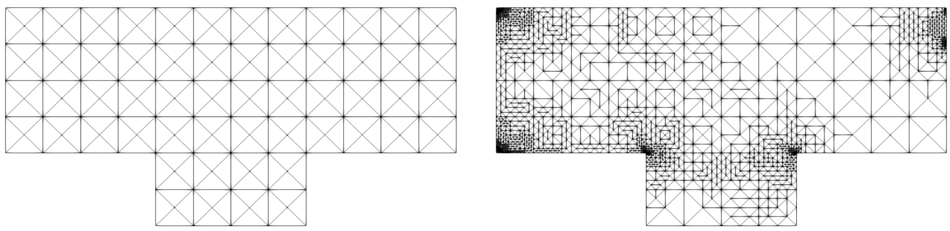

Figure 8: Initial mesh (left), with 224 elements, and final adapted mesh (right), with 8.674 elements. Here $k = 1$.

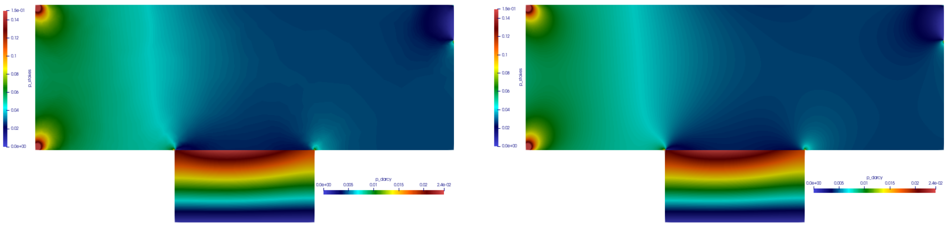

Figure 9: Isolines of the pressure on the adapted mesh (left) and on the highly uniform refined mesh. Here $k = 1$.

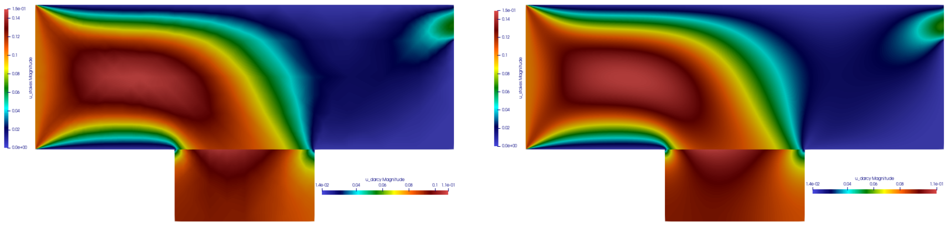

Figure 10: Isolines of the velocity magnitude on the adapted mesh (left) and on the highly uniform refined mesh. Here $k = 1$.

6.3. A analytical solution in three dimensions

The final test illustrates the performance of our adaptive stabilized finite element to simulate a 3D problem. In this example $\Omega := (0,1)\times(-1/2,1/2)\times(0,1)$, with $\Omega_S := (0,1)\times(-1/2,0)\times(0,1)$, $\Omega_D := (0,1)\times(0,1/2)\times(0,1)$, the interface $\Gamma := \overline{\Omega}_S \cap \overline{\Omega}_D$, $\nu = \kappa = 1$, and $\alpha_1 = \alpha_2 = 1$. The boundary Γ_D is split as follows

$$\Gamma_D^{\mathrm{Neu}} := \left\{(x,y,0) \in \mathbb{R}^3 : 0 < x < 1,\ 0 < y < 1/2\right\}, \qquad \Gamma_D^{\mathrm{Dir}} := \Gamma_D \backslash \overline{\Gamma_D^{\mathrm{Neu}}},$$

as it is shown in Figure 11. The data $\boldsymbol{f}_S$ and g_D are such that the exact solution is given by

$$\begin{aligned}
\boldsymbol{u}_S(x,y) &:= (e^z \sin y, -e^z \sin y, e^z \cos y - e^z \cos x),\\
p_S(x,y) &:= -\frac{1}{2}e^{2z} + \frac{1}{4}(e^2 - 1),\\
\boldsymbol{u}_D(x,y) &:= (\cos(\pi z)\sin(\pi x)\sin(\pi y), \sin(\pi z)\cos(\pi x)\sin(\pi y), -2\sin(\pi z)\sin(\pi x)\cos(\pi y)),\\
p_D(x,y) &= \sin(\pi z)\sin(\pi x)\cos(\pi y).
\end{aligned}$$

All calculations were performed using the approximation spaces $\boldsymbol{H}_{h,1} \times Q_{h,1}$.

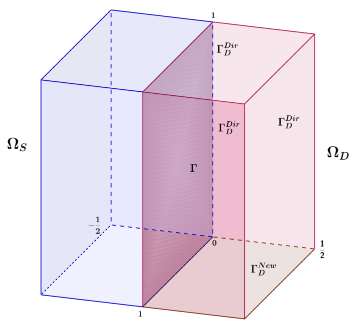

Figure 11: Configuration of the computational domain Ω.

In Figure 12 we present the convergence orders obtained using our stabilized scheme (3.6).

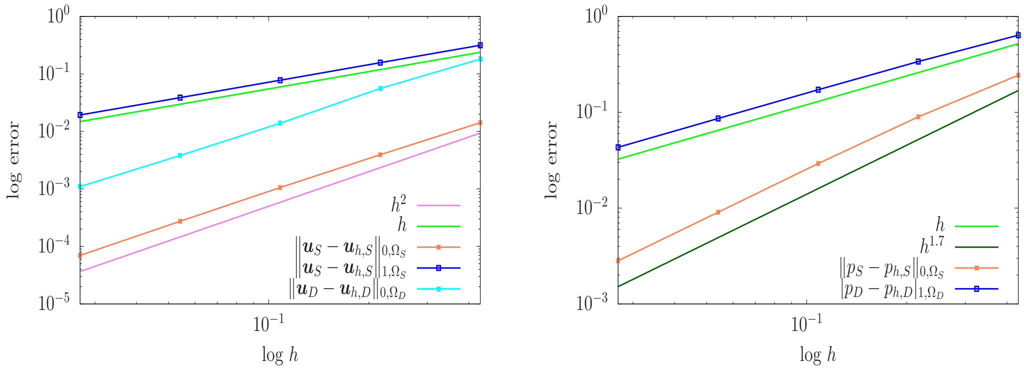

Figure 12: Convergence of the velocities (left) and pressures (right). Here the interpolation order is $k = 1$.

In Table 3 we depict the approximation error as well as the estimator η when h goes to 0. Note that the effectivity index E remains bounded close to a constant.

h	$\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|$	order	η	order	E
0.4330127	0.7736884	———	0.6859547	———	0.8866034
0.2165064	0.3876403	0.9970342	0.3053558	1.1676221	0.7877298
0.1082532	0.1912973	1.0189023	0.1472243	1.0524756	0.7696100
0.0541266	0.0949678	1.0103059	0.0728948	1.0141285	0.7675735
0.0270633	0.0473340	1.0045612	0.0362839	1.0064853	0.7665505

Table 3: Effectivity index in a quasi-uniform refinement for $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_{h,1} \times Q_{h,1}$.

Finally, in Figures 13 and 14 we compare the isovalues of the pressures and the velocities magnitude obtained by our adapted scheme with those of the exact solutions. Note the good agreement in both solutions.

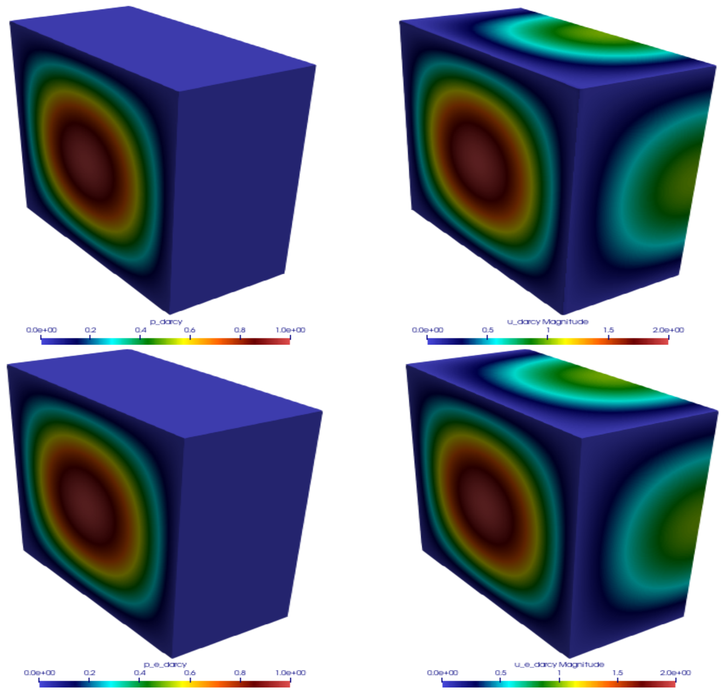

Figure 13: Isovalues of the pressure $p_{h,D}$ (top left) versus the exact pressure p_D (bottom left) and isovalues of the velocity magnitude $|\boldsymbol{u}_{h,D}|$ (top right) versus exact velocity magnitude $|\boldsymbol{u}_D|$ (bottom right). Here $k = 1$.

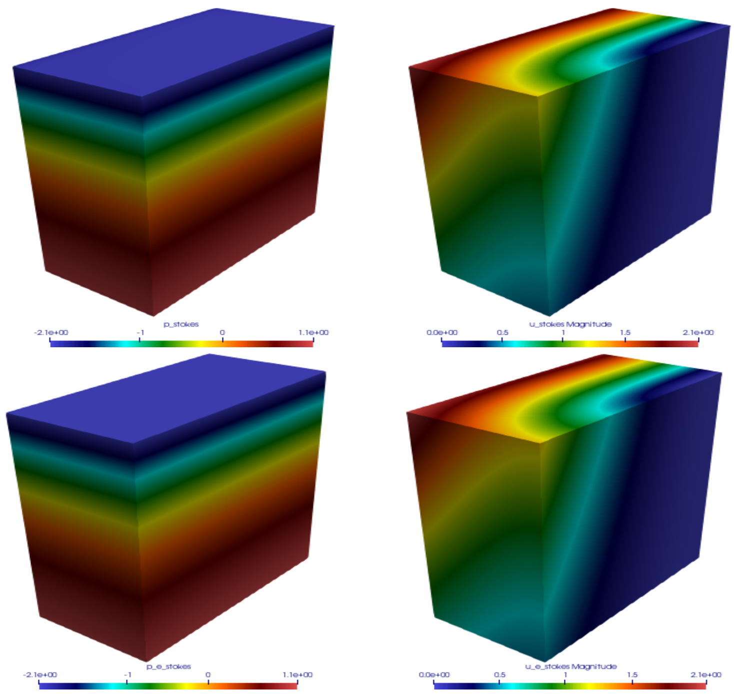

Figure 14: Isovalues of the pressure $p_{h,S}$ (top left) versus the exact pressure p_S (bottom left) and isovalues of the velocity magnitude $|\boldsymbol{u}_{h,S}|$ (top right) versus exact velocity magnitude $|\boldsymbol{u}_S|$ (bottom right). Here $k = 1$.

Conclusions

In this work, we have introduced a new stabilized finite element method for the coupled Stokes–Darcy problem. Stability and optimal convergence have been proven even when the polynomial degree used to approximate the pressures is the same as the one used to approximate the velocities. In addition, we have provided a residual-based error estimator composed by two estimators localized on each subdomain. For this error estimator, we proved its efficiency and reliability up to a high order term. These estimates have been verified by numerical experiments in 2D and 3D. The numerical examples presented testify that this approach is effective, for academic illustrations and realistic applications. Firstly, we can look that optimal rates are achieved, thus verifying the theoretical a priori results. Furthermore, it was verified the good performance of the estimator with respect to the viscosity ν, adapting on corner singularities and on interfaces. Finally, It was shown that the effectivity indexes are bounded as proven theoretically.

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Appendix A. Inf–sup condition for $\boldsymbol{B}$

The next result, used in Section 5, is an adaptation of the arguments presented in [4, Lemma 11].

Lemma 14. *There exists a positive constant C, depending on Ω, such that*

$$\sup_{(\boldsymbol{v},q)\in\boldsymbol{H}\times Q}\frac{B((\boldsymbol{u},p);(\boldsymbol{v},q))}{\|(\boldsymbol{v},q)\|}\geq C\|(\boldsymbol{u},p)\|,$$

for all $(\boldsymbol{u},p)\in\boldsymbol{H}\times Q$.

Proof. Using to the definition of B given in (2.4) and Lemma 1, we have

$$B((\boldsymbol{u},p);(\boldsymbol{u},p))\geq C_1\nu\kappa\|\boldsymbol{u}_S\|^2_{1,\Omega_S}+C_2\,\nu\|\boldsymbol{u}_D\|^2_{0,\Omega_D}+C_3\frac{\kappa^2}{\nu}|p_D|^2_{1,\Omega_D}. \tag{A.1}$$

On the other hand, by [36, Proposition 5.3.2] we know that for a given $p_S\in Q^S$ there exists $\boldsymbol{w}_S\in\boldsymbol{H}^S$, $\boldsymbol{w}_S\neq\boldsymbol{0}$ and a positive constant ζ, such that

$$-(p_S,\nabla\cdot\boldsymbol{w}_S)\geq\zeta\,\|\boldsymbol{w}_S\|_{1,\Omega_S}\|p_S\|_{0,\Omega_S}.$$

Now, if we define $(\boldsymbol{w},0):=((\boldsymbol{w}_S,\boldsymbol{0}),(0,0))$, assuming that $\|\boldsymbol{w}_S\|_{1,\Omega_S}=\|p_S\|_{0,\Omega_S}$, by the Young's inequality, a trace inequality and Lemma 1, we obtain

$$\begin{aligned}
&B((\boldsymbol{u},p);(\boldsymbol{w},0))\\
\geq&\,-C_4\,\nu\kappa\|\boldsymbol{u}_S\|_{1,\Omega_S}\|\boldsymbol{w}_S\|_{1,\Omega_S}-C_5\kappa^{1/2}\|\boldsymbol{u}_S\|_{1,\Omega_S}\|\boldsymbol{w}_S\|_{1,\Omega_S}-C_6\kappa|p_D|_{1,\Omega_D}\|\boldsymbol{w}_S\|_{1,\Omega_S}+\zeta\kappa\|p_S\|^2_{0,\Omega_S}\\
\geq&\,-\frac{\nu\kappa}{2}\left(\frac{C_4}{\gamma_1}+\frac{C_5}{\gamma_2}\right)\|\boldsymbol{u}_S\|^2_{1,\Omega_S}+\kappa\left(\zeta-\frac{C_4\gamma_1\nu}{2}-\frac{C_5\gamma_2}{2\nu\kappa}-\frac{C_6\gamma_3\nu}{2\kappa}\right)\|p_S\|^2_{0,\Omega_S}-\frac{C_6}{2\gamma_3}\frac{\kappa^2}{\nu}|p_D|^2_{1,\Omega_D}\\
\geq&\,-C_7\nu\kappa\,\|\boldsymbol{u}_S\|^2_{1,\Omega_S}+C_8\kappa\,\|p_S\|^2_{0,\Omega_S}-C_9\frac{\kappa^2}{\nu}|p_D|^2_{1,\Omega_D},
\end{aligned}\tag{A.2}$$

where we chose $\gamma_1, \gamma_2, \gamma_3$ small enough.

Let $(\boldsymbol{v}, q) := (\boldsymbol{u}, p) + \delta\,(\boldsymbol{w}, 0)$. Combining (A.1) with (A.2), we deduce that

$$
\begin{aligned}
&B((\boldsymbol{u}, p); (\boldsymbol{v}, q))\\
&\geq C_1 \nu\kappa \|\boldsymbol{u}_S\|_{1,\Omega_S}^2 + C_2\,\nu \|\boldsymbol{u}_D\|_{0,\Omega_D}^2 + C_3 \frac{\kappa^2}{\nu} |p_D|_{1,\Omega_D}^2 - C_7 \delta\nu\kappa\,\|\boldsymbol{u}_S\|_{1,\Omega_S}^2 + C_8 \delta\kappa\,\|p_S\|_{0,\Omega_S}^2 - C_9 \delta \frac{\kappa^2}{\nu} |p_D|_{1,\Omega_D}^2\\
&\geq \nu\kappa (C_1 - \delta C_7) \|\boldsymbol{u}_S\|_{1,\Omega_S}^2 + C_2\,\nu \|\boldsymbol{u}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu} (C_3 - \delta C_9) |p_D|_{1,\Omega_D}^2 + C_8 \delta\kappa\,\|p_S\|_{0,\Omega_S}^2\\
&\geq C_{10}\,\|(\boldsymbol{u}, p)\|^2, \qquad \text{(A.3)}
\end{aligned}
$$

choosing $0 < \delta < \min\left\{ \dfrac{C_1}{C_7}, \dfrac{C_3}{C_9}, \dfrac{1}{C_8}, \dfrac{1}{\nu^{1/2}} \right\}$.

Finally, using triangle inequality yields

$$
\|(\boldsymbol{v}, q)\| \leq \|(\boldsymbol{u}, p)\| + \delta \|(\boldsymbol{w}, 0)\| = \|(\boldsymbol{u}, p)\| + \delta \nu^{1/2} \kappa^{1/2} \|p_S\|_{0,\Omega_S} \leq C_{11}\,\|(\boldsymbol{u}, p)\|,
$$

thus, using (A.3), we get to the desired result. □

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