A Nitsche-type finite element method for the linear elasticity equation with discontinuity jumps

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Abstract

This article establishes a family of Nitsche-type finite element schemes to numerically approximate the solution of a linear elasticity problem with a jump condition on an interface. We detail the analysis of the existence and uniqueness of solution of the discrete problem, and the a priori error estimation for a mesh-dependent norm. An a posteriori error estimator is also introduced, which proves to be efficient and reliable. We show some numerical tests that confirm our findings and illustrate the application of adaptive refinement techniques to improve the numerical solution for modeling subduction earthquakes.

Keywords: Linear elasticity equation, Nitsche method

1. Introduction

The Nitsche method [28] has become popular as a technique for weakly imposing essential boundary conditions [9], primarily Dirichlet conditions [26], on numerical solutions of the finite element method. Although its use involves incorporating some additional terms into a discrete variational formulation, its main advantages lie in the fact that the formulations obtained are based on primal schemes, avoiding mixed formulations with Lagrange multipliers, and in several applications to unfitted mesh problems. While this method uses as an advantage the addition of penalty terms to discrete variational formulations seeking to preserve some properties of bilinear forms such as symmetry, some recent work [11, 7] shows that it is possible to obtain better results by adding terms that break the symmetry of the bilinear form.

A direct extension of this method is its application to various interface problems, where the Nitsche method framework can be applied to impose continuity or jump conditions on an interface [6, 17, 25], with applications to fluid-structure couplings [12], porous media [15] and non-linear coupling conditions [18]. The articles [14, 25] explore different applications of Nitsche method for imposing interface conditions on linear elliptic equations, showing this method as an alternative to using a Lagrange multiplier, allowing two subdomains to be mortared through a weak imposition of interface conditions. This variant better handles cases of non-matching [31] or unfitted meshes [24, 25] by using primal formulations that preserve the ellipticity of the original variational formulations in a new mesh-dependent norm, representing an algorithmic advantage in the numerical solution of these approximations.

One of the main motivations for using interface conditions in the elasticity equation is the analysis of subduction earthquakes. In this geological process, one lithospheric plate sinks, being recycled into the Earth's mantle at the convergence boundary of another plate. The denser plate subducts beneath the other and sinks into the mantle. A simple model of this phenomenon is derived from the linear elasticity equation applied to a continuous medium Ω composed by two continuous media Ω_1 and Ω_2 sharing an interface Γ_F. A momentum balance gives the coupling conditions between both equations, given by the continuity of normal stresses, the continuity of normal displacements, and a discontinuity jump of the tangential displacement on Γ_F (see Figure 1). Some previous way to obtain numerical approximations for this problem were given by [3, 27, 30], using a mortaring method with a Lagrange multiplier, a split-node technique and a mixed variational formulation with with weakly imposed symmetry, respectively.

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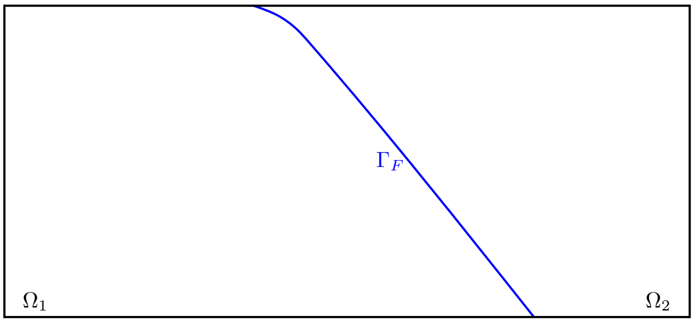

Figure 1: Domain Ω, divided in two subdomains Ω_1 and Ω_2 with an interface Γ_F.

The main novelty of this article lies in obtaining a family of primal discrete formulations, dependent on adjustable parameters, which allows to impose the jump condition on the interface by discretizing the displacements $\boldsymbol{u}$ in each subdomain with H^1 conformal finite elements for Dirichlet and mixed boundary conditions without adding any Lagrange multiplier when we use matching meshes, ensuring both computational efficiency and numerical precision.

The article is ordered as follows. Section 2 presents our model problem and some important notation. Section 3 introduces a new discrete variational formulation for our problem adding a penalty term to impose the discontinuity jump condition on an interface, based on continuous Lagrangian finite elements, and an a priori error estimate is established. Section 4 introduces a residual a posteriori error estimator for the solutions of this variational formulation, with detailed proof of its reliability and efficiency. Section 5 presents some numerical experiments that validate the results of sections 3 and 4, including adaptive refinement strategies. Finally, some conclusions are reported in Section 6.

2. Model problem

Consider $n \in \{2, 3\}$ and a non-empty bounded domain $\Omega \subseteq \mathbb{R}^n$. The Lebesgue measure of Ω is denoted by $|\Omega|$, which extends to lesser dimension spaces. The norm and seminorms for Sobolev spaces $W^{m,p}(\Omega)$ are denoted by $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$, respectively. For $p = 2$, the norm, seminorms, inner product and duality pairing of the space $W^{m,2}(\Omega) = H^m(\Omega)$ are denoted by $\|\cdot\|_{m,\Omega}$, $|\cdot|_{m,\Omega}$, $(\cdot,\cdot)_{m,\Omega}$ and $\langle\cdot,\cdot\rangle_{m,\Omega}$, respectively. Also, $\mathcal{C}^m(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ denote the space of functions with m continuous derivatives and all continuous derivatives, respectively. For Ω_1 and Ω_2 two open subsets of $\mathbb{R}^n$, we denote $\Omega_1 \Subset \Omega_2$ when there exists a compact set K such that $\Omega_1 \subseteq K \subseteq \Omega_2$.

The spaces $\mathbb{H}^m(\Omega)$, $\mathbb{W}^{m,p}(\Omega)$, $\boldsymbol{H}^m(\Omega)$, $\boldsymbol{W}^{m,p}(\Omega)$, $\boldsymbol{C}^m(\Omega)$ and $\boldsymbol{C}^\infty(\Omega)$ are defined by $\mathbb{H}^m(\Omega) = [H^m(\Omega)]^{n\times n}$, $\mathbb{W}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^{n\times n}$, $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^n$, $\boldsymbol{w}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^n$, $\boldsymbol{C}^m(\Omega) = [\mathcal{C}^m(\Omega)]^n$ and $\boldsymbol{C}^\infty(\Omega) = [\mathcal{C}^\infty(\Omega)]^n$. The notation for norms, seminorms and inner products will be extended from $W^{m,p}(\Omega)$ or $H^m(\Omega)$. Given $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$, a_{ij} denotes the entry in the $i-$th row and $j-$th column of matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$ denotes the transpose matrix of $\boldsymbol{A}$, $\operatorname{tr}(\boldsymbol{A})$ denotes the trace of $\boldsymbol{A}$ and $\boldsymbol{A} : \boldsymbol{B}$ denotes the inner product of $\mathbb{R}^{n\times n}$ given by

$$\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ij},$$

$\operatorname{sym} \boldsymbol{A} = \dfrac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$ and $\operatorname{skew} \boldsymbol{A} = \dfrac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$. We denote $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ as the identity matrix. Analogously, given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, a_j denotes the $j-$th component of vector $\boldsymbol{a}$ and $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the inner product of $\mathbb{R}^n$.

Consider Ω is a polyhedral domain with a boundary $\partial\Omega$, such that $\partial\Omega = \Gamma_D \cup \Gamma_N$, $|\Gamma_D| \neq 0$ and $|\Gamma_D \cap \Gamma_N| = 0$. Let two disjoint open sets $\Omega_1, \Omega_2 \subseteq \Omega$ such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$, $|\Gamma_D \cap \partial\Omega_j| \neq 0$ for $j \in \{1, 2\}$, and let $\Gamma_F = \partial\Omega_1 \cap \partial\Omega_2$ such that $\overline{\Gamma_F} \cap \overline{\Gamma_D} = \emptyset$. We denote by $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$, and by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ the outer normal vectors to $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Let $\mu_j, \lambda_j \in \mathbb{R}^+$ the Lamé coefficients of the material composing Ω_j, $j \in \{1, 2\}$. Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{u}_0 \in \boldsymbol{H}^{1/2}(\Gamma_D)$, and $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\Gamma_N)$. We consider the following model problem: *Find* $\boldsymbol{u}_j : \Omega_j \to \mathbb{R}^n$, $j \in \{1, 2\}$*, such that*

$$-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_j)) = \boldsymbol{f} \qquad \text{in } \Omega_j \tag{1}$$

$$\boldsymbol{u}_j = \boldsymbol{u}_D \qquad \text{on } \partial\Omega_j \cap \Gamma_D \tag{2}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}_j)\boldsymbol{n}_j = \boldsymbol{g} \qquad \text{on } \partial\Omega_j \cap \Gamma_N, \tag{3}$$

where $\boldsymbol{\sigma}(\boldsymbol{u}_j) = 2\mu_j \boldsymbol{\varepsilon}(\boldsymbol{u}_j) + \lambda_j \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}_j))\boldsymbol{I}$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}_j) = \operatorname{sym}(\nabla \boldsymbol{u}_j)$. We also have some interface conditions on Γ_F given by

$$\boldsymbol{\sigma}(\boldsymbol{u}_1)\,\boldsymbol{n}_1 + \boldsymbol{\sigma}(\boldsymbol{u}_2)\,\boldsymbol{n}_2 = \boldsymbol{0} \tag{4}$$

$$\boldsymbol{u}_1 - \boldsymbol{u}_2 = \boldsymbol{s} \tag{5}$$

where $\boldsymbol{s} \in \boldsymbol{H}^{1/2}(\Gamma_F)$. We define the spaces

$$V_j := \{\boldsymbol{v}_j \in \boldsymbol{H}^1(\Omega_j) \mid \boldsymbol{v}_j = \boldsymbol{0} \text{ on } \Gamma_D \cap \partial\Omega_j\} \quad \text{for } j \in \{1,2\}$$
$$V := V_1 \times V_2$$

$\langle \cdot, \cdot \rangle_{1/2,\Gamma_F}$ denotes the duality product between $\boldsymbol{H}^{-1/2}(\Gamma_F)$ and $\boldsymbol{H}^{1/2}(\Gamma_F)$ with respect to the inner product of $\boldsymbol{L}^2(\Gamma_F)$. For $\boldsymbol{u} \in V$, we denote $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ with $\boldsymbol{u}_1 \in V_1$ and $\boldsymbol{u}_2 \in V_2$.

Definition 1. *Let $\alpha \in [0,1]$ and $\boldsymbol{u} \in V$. On Γ_F, we denote*

$$\begin{aligned}
[\![\boldsymbol{u}]\!] &= \boldsymbol{u}_1 - \boldsymbol{u}_2\\
[\![\boldsymbol{\sigma}(\boldsymbol{u})]\!] &= \boldsymbol{\sigma}(\boldsymbol{u}_1)\boldsymbol{n}_1 + \boldsymbol{\sigma}(\boldsymbol{u}_2)\boldsymbol{n}_2\\
\{\boldsymbol{\sigma}(\boldsymbol{u})\}_\alpha &= \alpha\boldsymbol{\sigma}(\boldsymbol{u}_1)\boldsymbol{n}_1 - (1-\alpha)\boldsymbol{\sigma}(\boldsymbol{u}_2)\boldsymbol{n}_2\\
\{\boldsymbol{\sigma}(\boldsymbol{u})\} &= \{\boldsymbol{\sigma}(\boldsymbol{u})\}_{1/2}
\end{aligned}$$

The interface conditions (4) and (5) can be rewritten as $[\![\boldsymbol{\sigma}(\boldsymbol{u})]\!] = \boldsymbol{0}$ and $[\![\boldsymbol{u}]\!] = \boldsymbol{s}$, respectively.

3. A Nitsche method formulation

Consider $\Omega \subseteq \mathbb{R}^n$ as a bounded polygonal domain such that Γ_F is also polygonal. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of triangulations of $\overline{\Omega}$ composed of triangles (if $n=2$) or tetrahedron (if $n=3$), with $h := \max\{h_T \mid T \in \mathcal{T}_h\}$ and $h_T := \operatorname{diam}(T)$ be the diameter of $T \in \mathcal{T}_h$. We define $\mathcal{E}_h$ as the set of all edges (faces) of a given triangulation $\mathcal{T}_h$, thus for $j \in \{1,2\}$ we define

$$\begin{aligned}
\mathcal{T}_{h,j} &:= \{T \in \mathcal{T}_h \mid T \subseteq \Omega_j\},\\
\mathcal{E}_{h,F} &:= \{E \in \mathcal{E}_h \mid E \subseteq \Gamma_F\}.
\end{aligned}$$

For $T \in \mathcal{T}_h$, let $\mathcal{E}(T)$ be the set of edges of T. Furthermore, we define

$$\begin{aligned}
\mathcal{E}_{h,D} &:= \{E \in \mathcal{E}_h \mid E \subseteq \Gamma_D\},\\
\mathcal{E}_{h,N} &:= \{E \in \mathcal{E}_h \mid E \subseteq \Gamma_N\},\\
\mathcal{E}_{h,j} &:= \{E \in \mathcal{E}_h \mid E \subseteq \Omega_j\} \text{ for } j \in \{1,2\},\\
\mathcal{E}_{h,F} &:= \{E \in \mathcal{E}_h \mid E \subseteq \Gamma_F\}.
\end{aligned}$$

It is clear that $\mathcal{E}_h = \mathcal{E}_{h,1} \cup \mathcal{E}_{h,2} \cup \mathcal{E}_{h,D} \cup \mathcal{E}_{h,N} \cup \mathcal{E}_{h,F}$. For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ and $j \in \{1,2\}$ we define

$$\begin{aligned}
\omega_T &:= \bigcup\{T' \in \mathcal{T}_h \mid \overline{T} \cap \overline{T'} \neq \emptyset\},\\
\omega_E &:= \bigcup\{T' \in \mathcal{T}_h \mid E \cap \bar{T}' \neq \emptyset\},
\end{aligned}$$

$\omega_{T,j} := \omega_T \cap \overline{\Omega_j}$ and $\omega_{E,j} := \omega_E \cap \overline{\Omega_j}$.

Definition 2. *For $k \in \mathbb{N}$ and $j \in \{1,2\}$, we denote V_j^h the continuous Lagrange finite element vector space with degree k on $\overline{\Omega_j}$, that is,*

$$V_j^h := \{\boldsymbol{v}^h \in \boldsymbol{C}(\overline{\Omega_j}) \mid (\forall T \in \mathcal{T}_{h,j}) \quad \boldsymbol{v}^h|_T \in \mathbb{P}_k(T)^n\} \cap V_j$$

where $\mathbb{P}_k(T)$ is the space of polynomials of total degree at most k defined on T. Then, we define $V^h := V_1^h \times V_2^h$. For $\boldsymbol{u}^h \in V^h$, we denote by $\boldsymbol{u}^h = (\boldsymbol{u}_1^h, \boldsymbol{u}_2^h)$ with $\boldsymbol{u}_1^h \in V_1^h$ and $\boldsymbol{u}_2^h \in V_2^h$.

In that follows, we consider the case $\boldsymbol{u}_D = \boldsymbol{0}$. For some constants $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$, we define the following variational formulation

Find $\boldsymbol{u}^h \in V^h$ such that

$$(\forall \boldsymbol{v}^h \in V^h) \quad a_h(\boldsymbol{u}^h, \boldsymbol{v}^h) = L_h(\boldsymbol{v}^h) \tag{6}$$

where for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\boldsymbol{u}^h, \boldsymbol{v}^h \in V^h$

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &:= (\boldsymbol{\sigma}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{v}_1))_{0,\Omega_1} + (\boldsymbol{\sigma}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_2))_{0,\Omega_2} - \langle \{\boldsymbol{\sigma}(\boldsymbol{u})\}_\alpha, [\![\boldsymbol{v}]\!] \rangle_{1/2,\Gamma_F} - \theta \langle \{\boldsymbol{\sigma}(\boldsymbol{v})\}_\alpha, [\![\boldsymbol{u}]\!] \rangle_{1/2,\Gamma_F}\\
a_h(\boldsymbol{u}^h, \boldsymbol{v}^h) &:= a(\boldsymbol{u}^h, \boldsymbol{v}^h) + \gamma \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} ([\![\boldsymbol{u}^h]\!], [\![\boldsymbol{v}^h]\!])_{0,\Gamma_F}\\
L(\boldsymbol{v}) &:= (\boldsymbol{f}, \boldsymbol{v}_1)_{0,\Omega_1} + (\boldsymbol{f}, \boldsymbol{v}_2)_{0,\Omega_2} + \langle \boldsymbol{g}, \boldsymbol{v}_1 \rangle_{1/2,\Gamma_N \cap \partial\Omega_1} + \langle \boldsymbol{g}, \boldsymbol{v}_2 \rangle_{1/2,\Gamma_N \cap \partial\Omega_2} - \theta \langle \{\boldsymbol{\sigma}(\boldsymbol{v})\}_\alpha, \boldsymbol{s} \rangle_{1/2,\Gamma_F}\\
L_h(\boldsymbol{v}^h) &:= L(\boldsymbol{v}^h) + \gamma \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} (\boldsymbol{s}, [\![\boldsymbol{v}^h]\!])_{0,\Gamma_F}
\end{aligned}$$

The choice of the parameter θ allows to recover three remarkable schemes for this problem, inspired by the weak imposition of Dirichlet boundary conditions by adding Nitsche-type terms to the discrete variational formulations [16].

- The case $\theta = 1$ corresponds to a formulation inspired by the original formulation stated by Nitsche [28], based on a symmetric bilinear form. It can also be deduced from augmented Lagrangian formulations as in [13, Section 5.2.2].
- In the case $\theta = 0$, the simplest formulation is obtained [22, Section 37.1], although it is incomplete due to the absence of the term added to the bilinear form to induce symmetry in the case $\theta = 1$.
- The case $\theta = -1$ corresponds to a formulation with an anti-symmetric bilinear form [23], where coercivity is ensured for any positive value of the parameter γ. The case $\gamma = 0$ has been explored [11], also eliminating the Nitsche-type penalty in the formulation, obtaining an optimal error estimate in a mesh-dependent norm and a suboptimal error estimate in L^2 norm.

The parameter α can also be used to improve the stability of the scheme. While choosing the value $\alpha = 1/2$ we recover the usual definition of the average operator on the interface Γ_F, choosing $\alpha = 0$ or $\alpha = 1$ is a common practice in the useful master-slave formulations when $\mu_1 \gg \mu_2$ or $\mu_2 \gg \mu_1$ [24, Section 3.2]. In practice, often it is used $\alpha = \dfrac{\mu_2}{\mu_1 + \mu_2}$ as in [24, Section 3.3], which justifies the use of $\alpha = 1/2$ when $\mu_1 \approx \mu_2$ are close.

One important property of the formulation (6) is its consistency.

Lemma 3. *The exact solution of* (1)–(5) *satisfies the variational equations*

$$
\begin{aligned}
(\forall \boldsymbol{v}^h \in V^h) \quad & a(\boldsymbol{u}, \boldsymbol{v}^h) = L(\boldsymbol{v}^h) \\
(\forall \boldsymbol{v}^h \in V^h) \quad & a_h(\boldsymbol{u}, \boldsymbol{v}^h) = L_h(\boldsymbol{v}^h)
\end{aligned}
$$

Proof. It is a straightforward consequence of integration by parts and the interface conditions (4) and (5). We omit the details. □

In that follows, we consider the following mesh-dependent norm for V^h.

Definition 4. *We define the mesh-dependent norm* $\|\cdot\|_h$ *for* V^h *such that*

$$
(\forall \boldsymbol{v}^h \in V^h) \quad \|\boldsymbol{v}\|_h^2 := |\boldsymbol{v}_1^h|_{1,\Omega_1}^2 + |\boldsymbol{v}_2^h|_{1,\Omega_2}^2 + \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,\Gamma_F}^2 \tag{7}
$$

This auxiliary results are necessary to obtain good properties of this mesh-dependent norm.

Lemma 5. *There exists a constant* $C > 0$*, independent of* h*, such that for all* $T \in \mathcal{T}_h$ *and* $E \in \mathcal{E}_h$ *such that* $E \subseteq \partial T$ *and for all* $v \in H^1(T)$

$$
\|v\|_{0,E}^2 \leq C(h_E^{-1} \|v\|_{0,T}^2 + h_E |v|_{1,T}^2). \tag{8}
$$

Proof. See [1, Theorem 3.10]. □

Lemma 6. *Let* $m, l \in \mathbb{N} \cup \{0\}$ *such that* $0 \leq m \leq l$*. There exists a positive constant* C *independent on* h *such that for all* $T_j \in \mathcal{T}_j$*, for* $j \in \{1, 2\}$*, for all* $E \in \mathcal{E}(T_j)$ *and all* $\boldsymbol{v} \in V^h$

$$
|\boldsymbol{v}_j^h|_{l,T_j} \leq C h_{T_j}^{m-l} |\boldsymbol{v}_j^h|_{m,T_j} \tag{9}
$$

$$
\|\boldsymbol{v}_j^h\|_{0,E} \leq C h_{T_j}^{-1/2} \|\boldsymbol{v}_j^h\|_{0,T_j} \tag{10}
$$

Proof. See [21, Lemmas 12.1 and 12.8]. □

Lemma 7. *There exists a positive constant* C_I *independent on* α *and* h *such that for all* $\boldsymbol{v}^h \in V^h$

$$
\sum_{E \in \mathcal{E}_{h,F}} h_E \|\{\boldsymbol{\sigma}(\boldsymbol{v}^h)\}_\alpha\|_{0,E}^2 \leq C_I \sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{v}_j^h|_{1,\Omega_j}^2 \tag{11}
$$

Proof. Let $E \in \mathcal{E}_{h,F}$. There exist $T_1 \in \mathcal{T}_{h,1}$ and $T_2 \in \mathcal{T}_{h,2}$ such that $\omega_E = T_1 \cup T_2$. Then, applying (10),

$$
\begin{aligned}
\|\{\boldsymbol{\sigma}(\boldsymbol{v}^h)\}\|_{0,E} &\leq \|\boldsymbol{\sigma}(\boldsymbol{v}_1^h)\|_{0,E} + \|\boldsymbol{\sigma}(\boldsymbol{v}_2^h)\|_{0,E} \\
&\leq C_1 h_E^{-1/2} (\|\boldsymbol{\sigma}(\boldsymbol{v}_1^h)\|_{0,T_1} + \|\boldsymbol{\sigma}(\boldsymbol{v}_2^h)\|_{0,T_2}) \\
&\leq C_1 h_E^{-1/2} ((2\mu_1 + \lambda_1 n)|\boldsymbol{v}_1^h|_{0,T_1} + (2\mu_2 + \lambda_2 n)|\boldsymbol{v}_2^h|_{0,T_2})
\end{aligned}
$$

for a positive constant C_1 independent on α and h. The final result is obtained by some algebraic manipulations. □

The following results allow us to apply the Lax-Milgram lemma to determine the existence and uniqueness of solution of (6)

Lemma 8. *L_h is linear and continuous in V^h, and a_h is bilinear and continuous in V_h.*

Proof. First, the bilinearity of a_h and linearity of L_h are trivial.

Second, applying Lemma 7, Cauchy-Schwarz, Friedrichs-Poincaré, Trace and triangle inequalities, we have for all $\boldsymbol{v}^h \in V^h$

$$
\begin{aligned}
|L_h(\boldsymbol{v}^h)| \leq & C_1 \sum_{j=1}^{2} (\|\boldsymbol{f}\|_{0,\Omega_j} + \|\boldsymbol{g}\|_{-1/2,\Gamma_N\cap\partial\Omega_j}) |\boldsymbol{v}_j^h|_{1,\Omega_j} + |\theta| \left(\sum_{E\in\mathcal{E}_{h,F}} h_E \|[\![\boldsymbol{\sigma}(\boldsymbol{v}^h)]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|\boldsymbol{s}\|_{0,E}^2 \right)^{1/2} \\
& + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|\boldsymbol{s}\|_{0,E}^2 \right)^{1/2} \\
\leq & C_1 \sum_{j=1}^{2} (\|\boldsymbol{f}\|_{0,\Omega_j} + \|\boldsymbol{g}\|_{-1/2,\Gamma_D\cap\partial\Omega_j}) |\boldsymbol{v}_j^h|_{1,\Omega_j} + |\theta| C_2 \left(\sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{v}_j^h|_{1,\Omega_j}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|\boldsymbol{s}\|_{0,E}^2 \right)^{1/2} \\
& + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|\boldsymbol{s}\|_{0,E}^2 \right)^{1/2}
\end{aligned}
$$

The continuity of L_h is obtained by few algebraic manipulations. Analogously, we have for all $\boldsymbol{u}^h, \boldsymbol{v}^h \in V^h$

$$
\begin{aligned}
|a_h(\boldsymbol{u}^h, \boldsymbol{v}^h)| \leq & C_3 \sum_{j=1}^{2} (2\mu_j + n\lambda_j) |\boldsymbol{u}_j^h|_{1,\Omega_j} |\boldsymbol{v}_j^h|_{1,\Omega_j} + \left(\sum_{E\in\mathcal{E}_{h,F}} h_E \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \\
& + |\theta| \left(\sum_{E\in\mathcal{E}_{h,F}} h_E \|[\![\boldsymbol{\sigma}(\boldsymbol{v}^h)]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{u}^h]\!]\|_{0,E}^2 \right)^{1/2} \\
& + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{u}^h]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}]\!]\|_{0,E}^2 \right)^{1/2} \\
\leq & C_3 \sum_{j=1}^{2} (2\mu_j + n\lambda_j) |\boldsymbol{u}_j^h|_{1,\Omega_j} |\boldsymbol{v}_j^h|_{1,\Omega_j} + C_4 \left(\sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{u}_j^h|_{1,\Omega_j}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \\
& + |\theta| C_4 \left(\sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{v}_j^h|_{1,\Omega_j}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{u}^h]\!]\|_{0,E}^2 \right)^{1/2} \\
& + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|\boldsymbol{s}\|_{0,E}^2 \right)^{1/2}
\end{aligned}
$$

The continuity of a_h is obtained by few algebraic manipulations. □

Lemma 9. *For all $\theta \neq -1$, there exists a constant $\gamma_0 > 0$ such that a_h is coercive in V^h for all $\gamma > \gamma_0$. If $\theta = -1$, a_h is coercive in V^h for all $\gamma > 0$.*

Proof. First, applying Korn inequality, we have for all $\boldsymbol{v}^h \in V^h$

$$
\begin{aligned}
|a_h(\boldsymbol{v}^h, \boldsymbol{v}^h)| \geq & C_1 \sum_{j=1}^{2} (2\mu_j + n\lambda_j) |\boldsymbol{v}_j^h|_{1,\Omega_j}^2 \\
& - (1+\theta) \langle \{\boldsymbol{\sigma}(\boldsymbol{v}^h)\}_\alpha, [\![\boldsymbol{v}^h]\!] \rangle_{1/2,\Gamma_F} + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2}
\end{aligned}
$$

In the case $\theta = -1$, we have

$$
\begin{aligned}
|a_h(\boldsymbol{v}^h, \boldsymbol{v}^h)| & \geq C_1 \sum_{j=1}^{2} (2\mu_j + n\lambda_j) |\boldsymbol{v}_j^h|_{1,\Omega_j}^2 + \gamma \left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \\
& \geq \min\{C_1(2\mu_1 + n\lambda_1), C_1(2\mu_2 + n\lambda_2), \gamma\} \|\boldsymbol{v}^h\|_h^2
\end{aligned}
$$

proving the coercivity of a_h for all $\gamma > 0$ when $\theta = -1$.

Otherwise, for $\varepsilon > 0$ and $\gamma > \gamma_0 = \dfrac{C_I\,|1+\theta|^2}{C_1}\max\{2\mu_1 + n\lambda_1, 2\mu_2 + n\lambda_2\}$, where C_I is the constant from Lemma 7 and , applying Lemma 7,

$$\begin{aligned}
|a_h(\boldsymbol{v}^h, \boldsymbol{v}^h)| \geq& C_1 \sum_{j=1}^{2}(2\mu_j + n\lambda_j)|\boldsymbol{v}_j^h|^2_{1,\Omega_j} - (1+\theta)\left(\langle\{\boldsymbol{\sigma}(\boldsymbol{v}^h)\}_\alpha, [\![\boldsymbol{v}^h]\!]\rangle_{1/2,\Gamma_F}\right) + \gamma\left(\sum_{E\in\mathcal{E}_{h,F}}\frac{1}{h_E}\|[\![\boldsymbol{v}^h]\!]\|^2_{0,E}\right)^{1/2}\\
\geq& C_1 \sum_{j=1}^{2}(2\mu_j + n\lambda_j)|\boldsymbol{v}_j^h|^2_{1,\Omega_j} - \frac{|1+\theta|}{\varepsilon}\left(\sum_{E\in\mathcal{E}_{h,F}} h_E\|\{\boldsymbol{\sigma}(\boldsymbol{v}^h)\}_\alpha\|^2_{0,E}\right) + (\gamma - |1+\theta|\varepsilon)\left(\sum_{E\in\mathcal{E}_{h,F}}\frac{1}{h_E}\|[\![\boldsymbol{v}^h]\!]\|^2_{0,E}\right)\\
\geq& \sum_{j=1}^{2}\left(C_1(2\mu_j + n\lambda_j) - \frac{C_I|1+\theta|}{\varepsilon}(2\mu_j+n\lambda_j)^2\right)|\boldsymbol{v}_j^h|^2_{1,\Omega_j} + (\gamma - |1+\theta|\varepsilon)\left(\sum_{E\in\mathcal{E}_{h,F}}\frac{1}{h_E}\|[\![\boldsymbol{v}^h]\!]\|^2_{0,E}\right)
\end{aligned}$$

Choosing $\varepsilon = \dfrac{\gamma+\gamma_0}{2|1+\theta|} > 0$, we have $\gamma > |1+\theta|\varepsilon > \gamma_0$. Then, for all $j \in \{1,2\}$, we have

$$C_1(2\mu_j + \lambda_j n) - C_I\frac{|1+\theta|}{\varepsilon}(2\mu_j+\lambda_j n)^2 \geq C_1(2\mu_j+\lambda_j n)\left(1 - \frac{\gamma_0}{|1+\theta|\varepsilon}\right) = C_1(2\mu_j+\lambda_j n)\left(\frac{\gamma-\gamma_0}{\gamma+\gamma_0}\right)$$

$$\gamma - |1+\theta|\varepsilon = \frac{\gamma-\gamma_0}{2}$$

Taking

$$C_2 = (\gamma-\gamma_0)\min\left\{\frac{C_1(2\mu_1+\lambda_1 n)}{\gamma+\gamma_0}, \frac{C_1(2\mu_2+\lambda_2 n)}{\gamma+\gamma_0}, \frac{1}{2}\right\}$$

we obtain

$$|a_h(v,v)| \geq C_2\|v\|_h^2$$

□

Theorem 10. *If a_h is coercive, then* (6) *has an unique solution.*

Proof. It is a direct consequence of Lemmas 8 and 9, and Lax-Milgram lemma [20, Lemma 2.2]. □

The existence and uniqueness of solution of (6) is conditioned to $\gamma > \gamma_0 = \dfrac{C_I\,|1+\theta|^2}{C_1}\max\{2\mu_1+n\lambda_1, 2\mu_2+n\lambda_2\}$ for $\theta \neq -1$. For the case $\theta = -1$, the same conclusion is valid for all $\gamma > 0$.

In order to determine an a priori error estimate for this family of schemes, we cite some best approximation results that will be adapted for the mesh-dependent norm.

Definition 11. *We denote by $\mathcal{P}_j^h : \boldsymbol{H}^1(\Omega_j) \to V_j^h$ the orthogonal projection with respect to the $\boldsymbol{L}^2(\Omega_j)$-inner product. We also denote $\mathcal{P}^h(\boldsymbol{v}) = (\mathcal{P}_1^h(\boldsymbol{v}_1), \mathcal{P}_2^h(\boldsymbol{v}_2))$.*

Definition 12. *We denote by $\mathcal{I}_j^h : \boldsymbol{H}^{r+1}(\Omega_j) \to V_j^h$ the Lagrange interpolation operator. We also denote $\mathcal{I}^h(\boldsymbol{v}) = (\mathcal{I}_1^h(\boldsymbol{v}_1), \mathcal{I}_2^h(\boldsymbol{v}_2))$.*

The following Lemmas present some useful approximation results.

Lemma 13. *Let $k \in \mathbb{N}$, $r \in [1, k+1]$, $j \in \{1,2\}$ and $\boldsymbol{u}_j \in \boldsymbol{H}^{r+1}(\Omega_j)$. Then, there exist constants $C_j > 0$, independent of h, such that*

$$h|\boldsymbol{u}_j - \mathcal{P}_j^h(\boldsymbol{u}_j)|_{1,\Omega_j} + \|\boldsymbol{u}_j - \mathcal{P}_j^h(\boldsymbol{u}_j)\|_{0,\Omega_j} \leq C_j h^{r+1}|\boldsymbol{u}_j|_{r+1,\Omega_j}$$
$$h|\boldsymbol{u}_j - \mathcal{I}_j^h(\boldsymbol{u}_j)|_{1,\Omega_j} + \|\boldsymbol{u}_j - \mathcal{I}_j^h(\boldsymbol{u}_j)\|_{0,\Omega_j} \leq C_j h^{r+1}|\boldsymbol{u}_j|_{r+1,\Omega_j}$$

Proof. See [20, Corollary 109 and Proposition 1.134]. □

Lemma 14. *If $\boldsymbol{v} \in \boldsymbol{H}^r(\Omega_1) \times \boldsymbol{H}^r(\Omega_2)$ for $r \in [1, k+1]$, then $\|\boldsymbol{v} - \mathcal{I}(\boldsymbol{v})\|_h \leq Ch^r \sum_{j=1}^{2}|\boldsymbol{v}_j|_{r+1,\Omega_j}$*

Proof. From 13, we have $|\boldsymbol{v}_j - \mathcal{I}_1(\boldsymbol{v}_j)|_{1,\Omega_j} \leq C_j h^r|\boldsymbol{v}_j|_{r+1,\Omega_j}$. Now, for each $E \in \mathcal{E}_{h,F}$ we have

$$\|[\![v - \mathcal{I}(v)]\!]\|_{0,E} \leq \|\boldsymbol{v}_1 - \mathcal{I}_1(\boldsymbol{v}_1)\|_{0,E} + \|\boldsymbol{v}_2 - \mathcal{I}_2(\boldsymbol{v}_2)\|_{0,E}$$

Let $T_{1,E} \in \mathcal{T}_{h,1}$ and $T_{2,E} \in \mathcal{T}_{h,2}$ such that $\omega_E = T_{1,E} \cup T_{2,E}$. Then, applying (8) and local interpolation properties [20, Theorem 1.113], we have

$$\|\boldsymbol{v}_1 - \mathcal{I}_1(\boldsymbol{v}_1)\|_{0,E} \leq C_3(h_E^{-1/2}\|\boldsymbol{v}_1 - \mathcal{I}_1(\boldsymbol{v}_1)\|_{0,T} + h_E^{1/2}|\boldsymbol{v}_1 - \mathcal{I}_1(\boldsymbol{v}_1)|_{1,T})$$

$$\leq C_4 h_E^{r+1/2} |\boldsymbol{v}_1|_{r+1,T_{1,E}}$$

Later,

$$\sum_{E\in\mathcal{E}_{h,F}} \|\boldsymbol{v}_1 - \mathcal{I}_1(\boldsymbol{v}_1)\|_{0,E} \leq C_4 \sum_{E\in\mathcal{E}_{h,F}} h_E^{r+1/2} |\boldsymbol{v}_1|_{r+1,T_{1,E}}$$

Analogously, we have

$$\sum_{E\in\mathcal{E}_{h,F}} \|\boldsymbol{v}_2 - \mathcal{I}_2(\boldsymbol{v}_2)\|_{0,E} \leq C_5 \sum_{E\in\mathcal{E}_{h,F}} h_E^{r+1/2} |\boldsymbol{v}_2|_{r+1,T_{2,E}}$$

In conclusion,

$$\left(\sum_{E\in\mathcal{E}_{h,F}} \frac{1}{h_E}\|[\![v-\mathcal{I}(v)]\!]\|_{0,E}^2\right)^{1/2} \leq C_6 h^r \sum_{E\in\mathcal{E}_{h,F}} |\boldsymbol{v}_1|_{r+1,T_{1,E}} + |\boldsymbol{v}_2|_{r+1,T_{2,E}}$$

The final result is obtained after few algebraic manipulations. □

Finally, we present our a priori error estimate.

Theorem 15. *Given $k\in\mathbb{N}$ and $r>0$, if $\boldsymbol{u}_j \in \boldsymbol{H}^{r+1}(\Omega_j)$ for $j\in\{1,2\}$, then* $\|\boldsymbol{u}-\boldsymbol{u}^h\|_h \leq Ch^{\min\{k,r\}}\sum_{j=1}^{2}|v_j|_{r+1,\Omega_j}$

Proof. Let $\boldsymbol{e}^h = \mathcal{I}(\boldsymbol{u}) - \boldsymbol{u}^h$ and $\boldsymbol{\xi}^h = \boldsymbol{u} - \mathcal{I}(\boldsymbol{u})$. From Lemma 3, we have

$$a_h(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{e}^h) = a_h(\boldsymbol{\xi}^h + \boldsymbol{e}^h, \boldsymbol{e}^h) = 0.$$

From Lemmas 8 and 9, there exist positive constants α, C_1 independent on h such that

$$\alpha\|\boldsymbol{e}^h\|_h^2 \leq a_h(\boldsymbol{e}^h,\boldsymbol{e}^h) = -a_h(\boldsymbol{\xi}^h,\boldsymbol{e}^h) \leq C_1\|\boldsymbol{\xi}^h\|_h\|\boldsymbol{e}^h\|_h$$

Then,

$$\|\boldsymbol{e}^h\|_h \leq \frac{C_1}{\alpha}\|\boldsymbol{\xi}^h\|_h \leq C_2 h^r \sum_{j=1}^{2}|v_j|_{r+1,\Omega_j}$$

Finally,

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_h \leq \|\boldsymbol{\xi}^h\|_h + \|\boldsymbol{e}^h\|_h \leq Ch^r\sum_{j=1}^{2}|v_j|_{r+1,\Omega_j}$$

□

Remark 16. *For the antisymmetric scheme, i.e. $\theta=-1$, the case $\gamma=0$ requires a special discussion. The expression $\|\boldsymbol{v}\|_V^2 := |\boldsymbol{v}_1|_{1,\Omega_1}^2 + |\boldsymbol{v}_2|_{1,\Omega_2}^2$ is a norm in V, since $|\Gamma_D \cap (\partial\Omega_j \setminus \Gamma_F)| \neq 0$ for all $j\in\{1,2\}$, thus the results of Lemmas 8, 9, and 15 are derived directly by replacing the mesh-dependent norm $\|\cdot\|_h$ with the new norm.*

4. A posteriori error estimate

Let $E\in\mathcal{E}_{h,\Omega}$ and $T_1,T_2\in\mathcal{T}_h$ such that $E=\partial T_1\cap\partial T_2$. For each $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)$, such that $\boldsymbol{v}_j = \boldsymbol{v}|_{T_j} \in \boldsymbol{C}(T_j)$ for $j\in\{1,2\}$, we define the jump of $\boldsymbol{v}$ across E as

$$[\![\boldsymbol{v}]\!]_E := \boldsymbol{v}_1|_E - \boldsymbol{v}_2|_E.$$

Analogously, for each $\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)$ such that $\boldsymbol{\sigma}_j = \boldsymbol{\sigma}|_{T_j} \in [C(T_j)]^{d\times d}$ for $j\in\{1,2\}$, if $\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbb{R}^d$ are the outward normal vectors to T_1 and T_2, respectively, we define the normal jump of $\boldsymbol{\sigma}$ across E as

$$[\![\boldsymbol{\sigma}]\!]_E := \boldsymbol{\sigma}_1\boldsymbol{n}_1|_E + \boldsymbol{\sigma}_2\boldsymbol{n}_2|_E.$$

For $T\in\mathcal{T}_h$ and $E\in\mathcal{E}_h$, let ψ_T and ψ_E be the element and edge bubble functions and $\mathcal{P}: C(E)\to C(T)$ be the continuation operator (see [32, Section 3.1]). The element and edge bubble functions verify some important estimates.

Lemma 17. *Given $k\in\mathbb{N}$, there exists a constant $C>0$, depending only on k and the shape-regularity of $\mathcal{T}_h$, such that for all $T\in\mathcal{T}_h$, all E edge of T, all $q\in\mathbb{P}_k(T)$, and all $r\in\mathbb{P}_k(E)$ we have*

$$\|q\|_{0,T}^2 \leq C\|\psi_T^{1/2}q\|_{0,T}^2$$
$$\|r\|_{0,E}^2 \leq C\|\psi_E^{1/2}r\|_{0,E}^2$$
$$\|\psi_E^{1/2}\mathcal{P}(r)\|_{0,T}^2 \leq Ch_E\|r\|_{0,E}^2.$$

Proof. See [32, Lemma 3.3]. □

For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ and $k \in \mathbb{N}$, we define the local orthogonal projectors $P_T : \boldsymbol{L}^2(T) \to \mathbb{P}_k(T)$ and $P_E : \boldsymbol{L}^2(E) \to \mathbb{P}_k(E)$ with respect to the inner products in $\boldsymbol{L}^2(T)$ and $\boldsymbol{L}^2(E)$, respectively. In that follows, for $k \in \mathbb{N}$, we consider $l := \max\{k-2, 0\}$ and $m := \max\{k-1, 0\}$. For $j \in \{1,2\}$ and $T \in \mathcal{T}_j$, we define the local error estimates $\eta_{j,T}$ as

$$\eta_{j,T}^2 := \sum_{k=1}^{5} \eta_{j,T,k}^2$$

where

$$\begin{aligned}
\eta_{j,T,1}^2 &:= h_T^2 \|\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_j^h)\|_{0,T}^2, \\
\eta_{j,T,2}^2 &:= \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,j}} h_E \|[\![\boldsymbol{\sigma}(\boldsymbol{u}_j^h)]\!]_E\|_{0,E}^2, \\
\eta_{j,T,3}^2 &:= \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} h_E \|\boldsymbol{\sigma}(\boldsymbol{u}_j^h)\boldsymbol{n} - \boldsymbol{g}\|_{0,E}^2, \\
\eta_{j,T,4}^2 &:= \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,F}} \delta_j^2 h_E \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E}^2, \\
\eta_{j,T,5}^2 &:= \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{u}^h]\!] - \boldsymbol{s}\|_{0,E}^2,
\end{aligned}$$

with $\delta_1 = 1 - \alpha$ and $\delta_2 = \alpha$. For $T \in \mathcal{T}_h$, we define the local high order terms Θ_T given by

$$\Theta_T := h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} h_E \|\boldsymbol{g} - P_{m,E}(\boldsymbol{g})\|_{0,E}^2$$

Finally, we define the a posteriori error estimates η_1 and η_2, and the global high order term Θ by the expressions

$$\eta^2 := \sum_{T \in \mathcal{T}_{h,1}} \eta_{1,T}^2 + \sum_{T \in \mathcal{T}_{h,2}} \eta_{2,T}^2 \qquad \Theta^2 := \sum_{T \in \mathcal{T}_h} \Theta_T^2.$$

For $j \in \{1,2\}$, we denote by $\mathcal{J}_j^h : \boldsymbol{L}^2(\Omega_j) \to V_j^h$ the Clément interpolation operator defined in [19]. We also denote $\mathcal{J}^h(\boldsymbol{v}) = (\mathcal{J}_1^h(\boldsymbol{v}_1), \mathcal{J}_2^h(\boldsymbol{v}_2))$ for each $\boldsymbol{v} \in V$. This operator satisfies the following result.

Lemma 18. *Let $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$. There exists a constant $C > 0$, independent of h, such that for all $\boldsymbol{v} \in \boldsymbol{H}^1(\omega_T \cap \Omega_j)$*

$$\begin{aligned}
\|\boldsymbol{v} - \mathcal{J}_j^h(\boldsymbol{v})\|_{0,T} &\leq C h_T \|\boldsymbol{v}\|_{1,\omega_{T,j}}, \\
\|\boldsymbol{v} - \mathcal{J}_j^h(\boldsymbol{v})\|_{0,E} &\leq C h_E^{1/2} \|\boldsymbol{v}\|_{1,\omega_{E,j}}, \\
\|\mathcal{J}_j^h(\boldsymbol{v})\|_{1,T} &\leq C \|\boldsymbol{v}\|_{1,\omega_{T,j}}.
\end{aligned}$$

Proof. See [19, Theorem 1]. □

4.1. Reliability of the a posteriori estimator

In this subsection, we present an upper bound for the total error. First, we present some auxiliary lemmas.

Lemma 19. *Let $\boldsymbol{v}^h \in V^h$. Then,*

$$a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) \leq \gamma \left(\sum_{j=1}^{2} \sum_{T \in \mathcal{T}_j} \eta_{j,T,5}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2}$$

Proof. Since our scheme is consistent, we obtain by Holder inequality

$$\begin{aligned}
a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) &= L(\boldsymbol{v}^h) - L_h(\boldsymbol{u}^h, \boldsymbol{v}^h) + \gamma \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} ([\![\boldsymbol{u}^h]\!], [\![\boldsymbol{v}^h]\!])_{0,E}) \\
&= \gamma \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} ([\![\boldsymbol{u}^h]\!] - \boldsymbol{s}, [\![\boldsymbol{v}^h]\!])_{0,E}) \\
&\leq \gamma \left(\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{u}^h]\!] - \boldsymbol{s}\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2}
\end{aligned}$$

□

Lemma 20. *There exists a constant $C > 0$ such that for all $\boldsymbol{v} \in V$.*

$$a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) \leq C\eta \|\boldsymbol{v}\|_h$$

Proof. We have

$$a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) = \sum_{j=1}^{2} [(\boldsymbol{f}, \boldsymbol{v}_j)_{0,\Omega_j} + \langle \boldsymbol{g}, \boldsymbol{v}_j \rangle_{1/2,\Gamma_N \cap \partial\Omega_j} - (\boldsymbol{\sigma}(\boldsymbol{u}_j^h), \boldsymbol{\varepsilon}(\boldsymbol{v}_j))_{0,\Omega_j}] + \langle \{\boldsymbol{\sigma}(\boldsymbol{u}^h)\}_\alpha, [\![\boldsymbol{v}]\!] \rangle_{1/2,\Gamma_F}$$

Integrating by parts on each $T \in \mathcal{T}_h$, we have

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) \leq & \sum_{j=1}^{2} \left[\sum_{T \in \mathcal{T}_{h,j}} (\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_j^h), \boldsymbol{v}_j)_{0,\Omega_j} + \langle \boldsymbol{g} - \boldsymbol{\sigma}(\boldsymbol{u}_j^h)\boldsymbol{n}_j, \boldsymbol{v}_j \rangle_{1/2,\Gamma_N \cap \partial\Omega_j} - \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,j}} \langle [\![\boldsymbol{\sigma}(\boldsymbol{u}_j^h)]\!]_E, \boldsymbol{v}_j \rangle_{1/2,E} \right] \\ & + \sum_{E \in \mathcal{E}_{h,F}} \langle \{\boldsymbol{\sigma}(\boldsymbol{u}^h)\}_\alpha - \boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n}_1, \boldsymbol{v}_1 \rangle_{1/2,\Gamma_F} - \sum_{E \in \mathcal{E}_{h,F}} \langle \{\boldsymbol{\sigma}(\boldsymbol{u}^h)\}_\alpha + \boldsymbol{\sigma}(\boldsymbol{u}_2^h)\boldsymbol{n}_2, \boldsymbol{v}_2 \rangle_{1/2,\Gamma_F} \end{aligned}$$

Then, by Holder inequality,

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) \leq & \sum_{j=1}^{2} \left[\sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T,1} \left(\frac{1}{h_T} \|\boldsymbol{v}_j\|_{0,T} \right) + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,j}} \eta_{j,T,2} \left(\frac{1}{h_E^{1/2}} \|\boldsymbol{v}_j\|_{0,E} \right) + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} \eta_{j,T,3} \left(\frac{1}{h_E^{1/2}} \|\boldsymbol{v}_j\|_{0,E} \right) \right] \\ & + \sum_{E \in \mathcal{E}_{h,F}} (1-\alpha) \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E} \|\boldsymbol{v}_1\|_{0,E} + \alpha \sum_{E \in \mathcal{E}_{h,F}} \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E} \|\boldsymbol{v}_2\|_{0,E} \end{aligned}$$

Taking $\boldsymbol{v}^h = \mathcal{J}^h(\boldsymbol{v})$ and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^h$, we use Lemma 18, and Friedrichs-Poincaré and Holder inequalities. Then,

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{w}) \leq & C_1 \sum_{j=1}^{2} \left[\sum_{T \in \mathcal{T}_j} \eta_{j,T,1} \|\boldsymbol{v}_j\|_{1,\omega_T} + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,j}} \eta_{j,T,2} \|\boldsymbol{v}_j\|_{1,\omega_E} + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} \eta_{j,T,3} \|\boldsymbol{v}_j\|_{1,\omega_E} \right] \\ & + C_1 \sum_{E \in \mathcal{E}_{h,F}} (1-\alpha) h_E^{1/2} \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E} \|\boldsymbol{v}_1\|_{1,\omega_E \cap \overline{\Omega_1}} + C_1 \sum_{E \in \mathcal{E}_{h,F}} \alpha h_E^{1/2} \|[\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!]\|_{0,E} \|\boldsymbol{v}_2\|_{1,\omega_E \cap \overline{\Omega_2}} \\ \leq & C_2 \left(\sum_{j=1}^{2} |\boldsymbol{v}_j|_{1,\Omega_j}^2 \right)^{1/2} \left(\sum_{j=1}^{2} \sum_{T \in \mathcal{T}_j} \eta_{j,T,1}^2 + \eta_{j,T,2}^2 + \eta_{j,T,3}^2 + \eta_{j,T,4}^2 \right)^{1/2} \end{aligned}$$

Analogously, applying Lemmas 18, 19, and Friedrichs-Poincaré inequality, we have

$$a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) \leq C_3 \gamma \left(\sum_{j=1}^{2} \sum_{T \in \mathcal{T}_j} \eta_{j,T,5}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{v}^h]\!]\|_{0,E}^2 \right)^{1/2} \leq C_4 \gamma \left(\sum_{j=1}^{2} \sum_{T \in \mathcal{T}_j} \eta_{j,T,5}^2 \right)^{1/2} \left(\sum_{j=1}^{2} |\boldsymbol{v}_j|_{1,\Omega_j}^2 \right)^{1/2}$$

Finally,

$$a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) = a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{w}) + a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) \leq \max\{C_2, C_4\gamma\} \eta \|\boldsymbol{v}\|_h$$

□

We add the following saturation assumption

$$\sum_{E \in \mathcal{E}_{h,F}} h_E \|\{\boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{u}^h)\}_\alpha\|_{0,E}^2 \leq C_I \sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,\Omega_j}^2 \tag{12}$$

where $C_I > 0$ is the same constant that appears in Lemma 7. Similar assumptions are also used in [10, 33]. Using this inequality, we can prove the reliability of our a posteriori error estimate.

Theorem 21. *There exists a positive constant C independent on $h > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_h \leq C\eta$$

Proof. Taking $\boldsymbol{e}^h = \boldsymbol{u} - \boldsymbol{u}^h$, applying Lemmas 19, 20 and the assumption (12), we have

$$\begin{aligned} \alpha \|\boldsymbol{e}^h\|_h^2 & \leq a_h(\boldsymbol{e}^h, \boldsymbol{e}^h) = a(\boldsymbol{e}^h, \boldsymbol{e}^h) - (\theta + 1) \langle \{\boldsymbol{\sigma}(\boldsymbol{e}^h)\}_\alpha, [\![\boldsymbol{e}^h]\!] \rangle_{1/2,\Gamma_F} \\ & \leq C_1 \eta \|\boldsymbol{e}^h\|_h + |\theta + 1| \left(\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \|[\![\boldsymbol{e}^h]\!]\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{E}_{h,F}} h_E \|\{\boldsymbol{\sigma}(\boldsymbol{e}^h)\}_\alpha\|_{0,E}^2 \right)^{1/2} \\ & \leq C_1 \eta \|\boldsymbol{e}^h\|_h + C_I \, |\theta + 1| \left(\sum_{j=1}^{2} \sum_{T \in \mathcal{T}_j} \eta_{j,T,5}^2 \right)^{1/2} \left(\sum_{j=1}^{2} (2\mu_j + n\lambda_j)^2 |\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,\Omega_j}^2 \right)^{1/2} \\ & \leq (C_1 + C_2 \, |\theta + 1|) \, \eta \|\boldsymbol{e}^h\|_h \end{aligned}$$

proving the theorem. □

4.2. Efficiency of the a posteriori estimator

In this subsection, we present a lower bound for each term of our local a posteriori error estimate η following the same proof structure as in [32]. The consistency of our formulation will be conveniently exploited throughout the following results. Since the terms in $\eta_{1,T}$ and $\eta_{2,T}$ are similar, we only present detailed proofs for the terms in $\eta_{1,T}$.

Lemma 22. *Let $k \in \mathbb{N}$, $l := \max\{k-2, 0\}$. There exists a constant $C > 0$, independent on h, such that for all $j \in \{1,2\}$ and $T \in \mathcal{T}_{h,j}$*

$$\eta_{j,T,1}^2 \leq C \left(|\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right)$$

Proof. We only present the proof for $j = 1$ because the proof for $j = 2$ is analogous. Since $\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h) \in \mathbb{P}_l(T)$, we have

$$\eta_{1,T,1}^2 = h_T^2 \|\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)\|_{0,T}^2 \leq h_T^2 \left(\|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 + \|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T}^2 \right),$$

where, by Lemma 17,

$$\begin{aligned}
\|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T}^2 &\leq C_1 \int_T \psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)) \cdot P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)) dx \\
&\leq C_1 \int_T \psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)) \cdot (\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) dx.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
\int_T \psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)) \cdot (\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) dx &= \int_T (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) \cdot \nabla \left(\psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h)) \right) dx \\
&\leq \|\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)\|_{0,T} \, |\psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))|_{1,T},
\end{aligned}$$

where $\|\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)\|_{0,T} \leq C_2 |\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T}$ and, applying an inverse inequality (see [20, Lemma 1.138])

$$\begin{aligned}
|\psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))|_{1,T} &\leq C_3 h_T^{-1} \|\psi_T P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T} \\
&\leq C_3 h_T^{-1} \|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T}.
\end{aligned}$$

Then, we have

$$\|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T}^2 \leq C_4 h_T^{-1} \|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T} |\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T},$$

thus

$$\|P_{l,T}(\boldsymbol{f} + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h))\|_{0,T} \leq C_4 h_T^{-1} |\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T},$$

concluding our estimate. □

Lemma 23. *Let $k \in \mathbb{N}$ and $l := \max\{k-2, 0\}$. There exists a constant $C > 0$, independent on h, such that for all $E_1 \in \mathcal{E}_{h,1}$ and $E_2 \in \mathcal{E}_{h,2}$*

$$\begin{aligned}
h_{E_1} \|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]_{E_1}\|_{0,E_1}^2 &\leq C \left(\sum_{T \subseteq \omega_{E_1}} |\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right), \\
h_{E_2} \|[\![\boldsymbol{\sigma}(\boldsymbol{u}_2^h)]\!]_{E_2}\|_{0,E_2}^2 &\leq C \left(\sum_{T \subseteq \omega_{E_2}} |\boldsymbol{u}_2 - \boldsymbol{u}_2^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right).
\end{aligned}$$

Proof. We only show the proof of the first estimate; the deduction of the second inequality follows the same steps. Applying Lemma 17, since $[\![\boldsymbol{\sigma}(\boldsymbol{u}_1)]\!]_{E_1} = 0$, we have

$$\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]_{E_1}\|_{0,E_1}^2 \leq C_1 \int_{E_1} \psi_{E_1} [\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]_{E_1} \cdot [\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]_{E_1} dx = C_1 \int_{E_1} \psi_{E_1} \mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]_{E_1}) \cdot [\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)]\!]_{E_1} dx.$$

Applying the divergence theorem, and the fact that $\psi_{E_1} = 0$ on $\partial\omega_{E_1}$, we obtain

$$\begin{aligned}
\int_{E_1} \psi_{E_1} \mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]) \cdot [\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)]\!] dx &= \sum_{T \subseteq \omega_{E_1}} \int_T \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) \cdot \psi_{E_1} \mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]) dx \\
&\quad + \sum_{T \subseteq \omega_{E_1}} \int_{E_1} \nabla(\psi_{E_1} \mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])) \colon (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) dx,
\end{aligned}$$

where $\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)) = \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)) + \boldsymbol{f}$. Then, using Lemma 17, we arrive at

$$\begin{aligned}\sum_{T\subseteq\omega_{E_1}}\int_T \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\cdot\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])dx &\le \sum_{T\subseteq\omega_{E_1}}\|\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)) + \boldsymbol{f}\|_{0,T}\|\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])\|_{0,T}\\ &\le \sum_{T\subseteq\omega_{E_1}}\|\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)) + \boldsymbol{f}\|_{0,T}\|\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])\|_{0,T}\\ &\le C_2 h_{E_1}^{1/2}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1}\sum_{T\subseteq\omega_{E_1}}\|\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)) + \boldsymbol{f}\|_{0,T},\end{aligned}$$

and

$$\begin{aligned}\sum_{T\subseteq\omega_{E_1}}\int_{E_1}\nabla(\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])) : (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))dx &\le \sum_{T\subseteq\omega_{E_1}}\|\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)\|_{0,T}|\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])|_{1,T}\\ &\le C_3\sum_{T\subseteq\omega_{E_1}}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}|\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])|_{1,T}\\ &\le C_4\sum_{T\subseteq\omega_{E_1}}\frac{1}{h_T}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}\|\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])\|_{0,T}\\ &\le C_4\sum_{T\subseteq\omega_{E_1}}\frac{1}{h_T}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}\|\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])\|_{0,T}\\ &\le C_5 h_{E_1}^{1/2}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1}\sum_{T\subseteq\omega_{E_1}}\frac{1}{h_T}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}.\end{aligned}$$

Thus,

$$\begin{aligned}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1}^2 \le C_1\int_{E_1}\psi_{E_1}[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\cdot[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]dx &= C_1\int_{E_1}\psi_{E_1}\mathcal{P}([\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!])\cdot[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1)]\!]dx\\ \le C_6 h_{E_1}^{1/2}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1}&\left(\sum_{T\subseteq\omega_{E_1}}\|\operatorname{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{u}_1^h\right)\right) + \boldsymbol{f}\|_{0,T} + \frac{1}{h_T}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}\right),\end{aligned}$$

concluding that

$$\begin{aligned}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1} &\le C_6 h_{E_1}^{1/2}\left(\sum_{T\subseteq\omega_{E_1}}\|\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)) + \boldsymbol{f}\|_{0,T} + \frac{1}{h_T}|\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}\right)\\ h_{E_1}\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_1^h)]\!]\|_{0,E_1}^2 &\le C\left(\sum_{T\subseteq\omega_{E_1}}\eta_{1,T,1}^2 + |\boldsymbol{u}_1^h - \boldsymbol{u}_1|_{1,T}^2\right).\end{aligned}$$

The final estimate is obtained using Lemma 22. ☐

Lemma 24. *Let $k \in \mathbb{N}$, $l := \max\{k-2, 0\}$, and $m := \max\{k-1, 0\}$. There exists a constant $C > 0$, independent on h, such that for all $E_1 \in \mathcal{E}_{h,N}$ such that $E_1 \subseteq \partial T_1$, for $T_1 \in \mathcal{T}_{h,1}$, and for all $E_2 \in \mathcal{E}_{h,N}$ such that $E_2 \subseteq \partial T_2$, for $T_2 \in \mathcal{T}_{h,2}$, we have*

$$\begin{aligned}h_{E_1}\|\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n}_1 - \boldsymbol{g}\|_{0,E_1}^2 &\le C(|\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T}^2 + h_T^2\|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 + h_{E_1}\|\boldsymbol{g} - P_{m,E_1}(\boldsymbol{g})\|_{0,E_1}^2),\\ h_{E_2}\|\boldsymbol{\sigma}(\boldsymbol{u}_2^h)\boldsymbol{n}_2 - \boldsymbol{g}\|_{0,E_2}^2 &\le C(|\boldsymbol{u}_2 - \boldsymbol{u}_2^h|_{1,T}^2 + h_T^2\|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 + h_{E_2}\|\boldsymbol{g} - P_{m,E_2}(\boldsymbol{g})\|_{0,E_2}^2).\end{aligned}$$

Proof. First, we have

$$h_E\|\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g}\|_{0,E}^2 \le h_E(\|\boldsymbol{g} - P_{m,E_1}(\boldsymbol{g})\|_{0,E}^2 + \|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E}^2)$$

Since $\boldsymbol{\sigma}(\boldsymbol{u}_1)\boldsymbol{n} = \boldsymbol{g}$ in E, we have

$$\begin{aligned}\|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E}^2 &\le C_1\int_E \psi_E P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\cdot P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\,dS\\ &\le C_1\int_{\partial T}\psi_E P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\cdot(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\boldsymbol{n}\,dS\end{aligned}$$

since $\psi_E = 0$ in $\partial T \setminus E$. Then, applying divergence theorem, since $\boldsymbol{f} = -\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_1)$ in T, we have

$$\int_{\partial T}\psi_E P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\cdot(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\boldsymbol{n}\,dS = \int_T \psi_E\mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - P_{m,E_1}(\boldsymbol{g}))\cdot\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\,dS$$

$$
\begin{aligned}
&+ \int_T \nabla(\psi_E \mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - P_{m,E_1}(\boldsymbol{g}))) \cdot (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\, dS \\
&= \int_T \psi_E \mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - P_{m,E_1}(\boldsymbol{g})) \cdot (\boldsymbol{f} + \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h))\, dS \\
&+ \int_T \nabla(\psi_E \mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - P_{m,E_1}(\boldsymbol{g}))) \cdot (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\, dS
\end{aligned}
$$

where, repeating the same arguments as in the proof of Lemma 23, we have

$$
\begin{aligned}
\int_T \psi_E \mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - P_{m,E_1}(\boldsymbol{g})) \cdot (\boldsymbol{f} + \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h))\, dS &\le C_2 h_T^{1/2} \| \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h) + \boldsymbol{f}\|_{0,T} \|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E} \\
\int_T \nabla(\psi_E \mathcal{P}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)n - Pg)) \cdot (\boldsymbol{\sigma}(\boldsymbol{u}_1^h) - \boldsymbol{\sigma}(\boldsymbol{u}_1))\, dS &\le C_3 h_T^{-1/2} |u_1 - u_1^h|_{1,T} \|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E}^2 &\le C_4 (h_T \| \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_1^h) + \boldsymbol{f}\|_{0,T}^2 + h_T^{-1} |u_1 - u_1^h|_{1,T}^2) \\
h_E \|P_{m,E_1}(\boldsymbol{\sigma}(\boldsymbol{u}_1^h)\boldsymbol{n} - \boldsymbol{g})\|_{0,E}^2 &\le C_4 (\eta_{1,T,1}^2 + \boldsymbol{f}\|_{0,T}^2 + |u_1 - u_1^h|_{1,T}^2)
\end{aligned}
$$

The theorem is concluding by using Lemma 22 and some algebraic manipulations. □

Lemma 25. *Let $k \in \mathbb{N}$ and $l := \max\{k-2, 0\}$. There exists a constant $C > 0$, independent on h, such that for all $E \in \mathcal{E}_{h,F}$ and $T_1 \in \mathcal{T}_{h,1}$, $T_2 \in \mathcal{T}_{h,2}$ such that $E = \partial T_1 \cap \partial T_2$*

$$
h_E \| [\![\boldsymbol{\sigma}(\boldsymbol{u}^h)]\!] \|_{0,E}^2 \le C \left(|\boldsymbol{u}_1 - \boldsymbol{u}_1^h|_{1,T}^2 + |\boldsymbol{u}_2 - \boldsymbol{u}_2^h|_{1,T_2}^2 + h_{T_1}^2 \|\boldsymbol{f} - P_{l,T_1}(\boldsymbol{f})\|_{0,T}^2 + h_{T_2}^2 \|\boldsymbol{f} - P_{l,T_2}(\boldsymbol{f})\|_{0,T}^2 \right).
$$

Proof. The proof is similar to Lemma 24, thus we omit the details. □

Finally, we deduce the efficiency of our a posteriori error estimator by a direct application of the previous lemmas.

Theorem 26. *There exists a constant $C > 0$, independent on h, such that*

$$
\eta \le C \left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_h + \Theta \right).
$$

Proof. Since $[\![\boldsymbol{u}]\!] = \boldsymbol{s}$, we have

$$
\sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \| [\![\boldsymbol{u}^h]\!] - \boldsymbol{s}\|_{0,E}^2 = \sum_{E \in \mathcal{E}_{h,F}} \frac{1}{h_E} \| [\![\boldsymbol{u} - \boldsymbol{u}^h]\!] \|_{0,E}^2 \le \|\boldsymbol{u} - \boldsymbol{u}^h\|_h^2
$$

From Lemmas 22, 23, 24, and 25, we obtain for all $j \in \{1, 2\}$.

$$
\begin{aligned}
\sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T,1}^2 &\le C_1 \sum_{T \in \mathcal{T}_{h,j}} \left(|\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right) \\
\sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T,2}^2 &\le C_2 \sum_{T \in \mathcal{T}_{h,j}} \left(|\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right) \\
\sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T,3}^2 &\le C_3 \sum_{\substack{T \in \mathcal{T}_{h,j} \\ |\partial T \cap \Gamma_N| \neq 0}} \left(|\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} h_E \|\boldsymbol{g} - P_{m,T}(\boldsymbol{g})\|_{0,T}^2 \right) \\
\sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T,4}^2 &\le C_4 \sum_{\substack{T \in \mathcal{T}_{h,j} \\ |\partial T \cap \Gamma_F| \neq 0}} \left(|\boldsymbol{u}_j - \boldsymbol{u}_j^h|_{1,T}^2 + h_T^2 \|\boldsymbol{f} - P_{l,T}(\boldsymbol{f})\|_{0,T}^2 \right)
\end{aligned}
$$

The final result is deduced after some algebraic manipulations. We omit the details. □

Remark 27. *For the antisymmetric scheme, i.e. $\theta = -1$, our a posteriori error estimator for the $\gamma = 0$ case can be rewritten ignoring the $\eta_{j,T,5}$ terms. Furthermore, the reliability and efficiency results from Theorems 21 and 26 can be deduce without the Assumption (12). In this case, the new definitions for the a posteriori error estimators are*

$$
\eta_{j,T}^2 := \sum_{k=1}^{4} \eta_{j,T,k}^2 \qquad \eta_j^2 := \sum_{T \in \mathcal{T}_{h,j}} \eta_{j,T}^2 \qquad \eta^2 := \eta_1^2 + \eta_2^2
$$

Then, the proofs of Lemmas 19 and 20 can be reformulated, obtaining those new results

$$
\begin{aligned}
(\forall \boldsymbol{v}^h \in V^h) \quad & a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) = 0 \\
(\forall \boldsymbol{v} \in V) \quad & a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) \le C\eta \|\boldsymbol{v}\|_V.
\end{aligned}
$$

In this context, the results from Theorems 21 and 26 are the same as before, but using the $\|\cdot\|_V$ norm instead of $\|\cdot\|_h$. Since the proofs are similar to our previous results, we omit the details for the sake of brevity.

5. Numerical experiments

In this section, we report some numerical experiments in 2D to corroborate our a priori and a posteriori error estimates. Our examples were implemented in `FEniCS` [4] using the `Multiphenics` library [8], which provides tools for formulating multiphysics problems in conformal meshes, helping to define variables restricted to subdomains or boundaries through block structures. The linear system obtained by (6) was solved using `MUMPS` [5], a direct solver suitable for large linear systems with sparse matrices. The numerical results presented below were obtained by an Intel Core i7-10750H @ 2.60 GHz computer running Ubuntu 24.04.5 LTS inside a Windows Subsystem for Linux (WSL2) with 24 GB of RAM.

Given $\boldsymbol{u} \in V$ and $\boldsymbol{u}^h \in V^h$, the solutions to system (1)–(5) and (6), respectively, we denote by $e_{0,h}^2 := \|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_{0,\Omega_1}^2 + \|\boldsymbol{u}_2 - \boldsymbol{u}_2^h\|_{0,\Omega_2}^2$ and $e_h^2 := \|\boldsymbol{u} - \boldsymbol{u}^h\|_h$. Given an unknown φ and two approximations φ_h and $\varphi_{\tilde{h}}$ for two consecutive meshes of sizes h and $\tilde{h}$, respectively, the experimental order of convergence (e.o.c.) of the error of φ in a specific $\|\cdot\|$ norm is defined by eoc $= \dfrac{\log\|\varphi - \varphi_h\| - \log\|\varphi - \varphi_{\tilde{h}}\|}{\log h - \log \tilde{h}}$. Also, we denote by `dof` the number of degrees of freedom and define the effectivity index by

$$\rho := \frac{\eta}{e_h}.$$

For our last two experiments, we use a classical adaptive refinement procedure of solving (6) on a sequence of adapted meshes obtained using our a posteriori error estimation within a prescribed tolerance. To this end, the adaptive refinement process starts with a uniform mesh, and, in each stage, we create a new mesh that is better adapted to the solution of the problem. The adaptive refinement is done by computing the a posteriori local error estimations η_T for all $T \in \mathcal{T}_h$, refining the elements T such that

$$\eta_T \geq \delta \max\{\eta_{\tilde{T}} \mid \tilde{T} \in \mathcal{T}_h\},$$

where $\delta \in (0,1)$ is a prescribed parameter. We will give more details of the parameters, domains, meshes, polynomial degree k, and tolerances used in the description of each example.

5.1. Tests with analytic solution in 2D

In this example, we consider the domain $\Omega := (-1,1)^2$ divided in the subdomains $\Omega_1 := (-1,0) \times (-1,1)$ and $\Omega_2 := (0,1) \times (-1,1)$ with an interface given by $\Gamma_F = \{0\} \times (-1,1)$. We set $\Gamma_D = \{-1,1\} \times (-1,1)$ and $\Gamma_N = (-1,1) \times \{-1,1\}$ (see Figure 2).

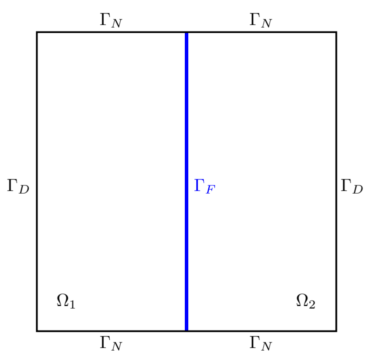

Figure 2: Domain Ω, divided into two subdomains Ω_1 and Ω_2, and boundary conditions.

We choose the Lamé parameters $\mu := 4$, $\lambda := 1$, and the functions $\boldsymbol{s}$, $\boldsymbol{f}$, $\boldsymbol{u}_D$ and $\boldsymbol{g}$ such that the solution of (1)–(5) is given by $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, where

$$\boldsymbol{u}_1(x,y) = \begin{pmatrix} 2\sin(\pi x)\cos(3\pi y) \\ 3\cos(\pi x)\sin(3\pi y) \end{pmatrix}, \qquad \boldsymbol{u}_2(x,y) = \begin{pmatrix} \sin(\pi x)\cos(3\pi y) \\ 6\cos(\pi x)\sin(3\pi y) \end{pmatrix}$$

In this test, we solved (6) for $k \in \{1,2,3\}$, $\alpha = 1/2$, $\gamma = 20$, and $\theta \in \{-1,0,1\}$. We summarize the a priori and a posteriori error estimations for the uniform refinement associated with this example in Table 1 and Figure 3 only on the case $\theta = 1$ because the error estimates $e_{0,h}$ and e_h for the chosen values of θ are very similar. Since the regularity of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we have, using Theorem 15, that the convergence order of e_h is $\mathcal{O}(h^k)$. Also it is expected to recover a $\mathcal{O}(h^{k+1})$ convergence order for $e_{0,h}$ by the Aubin–Nitsche trick (see [20, Section 2.3.4]) and some regularity results. According to the theory, our experimental error orders follow the expected rates for both $e_{0,h}$ and e_h.

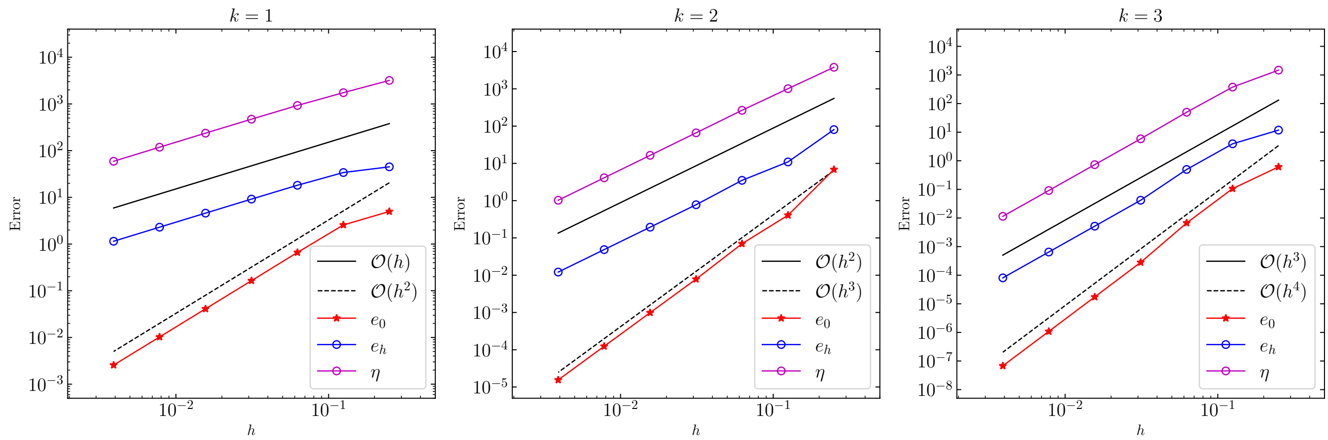

Figure 3: History of convergence.

k	h	dof	$e_{0,h}$	e.o.c	e_h	e.o.c	η	e.o.c	ρ
1	1/4	60	$4.972E+00$	–	$4.498E+01$	–	$3.186E+03$	–	70.831
	1/8	180	$2.570E+00$	0.952	$3.401E+01$	0.403	$1.744E+03$	0.869	51.292
	1/16	612	$6.592E-01$	1.963	$1.815E+01$	0.906	$9.288E+02$	0.909	51.164
	1/32	2244	$1.641E-01$	2.006	$9.174E+00$	0.985	$4.722E+02$	0.976	51.476
	1/64	8580	$4.099E-02$	2.001	$4.595E+00$	0.997	$2.369E+02$	0.995	51.564
	1/128	33540	$1.025E-02$	2.000	$2.298E+00$	1.000	$1.185E+02$	0.999	51.586
	1/256	132612	$2.562E-03$	2.000	$1.149E+00$	1.000	$5.926E+01$	1.000	51.592
2	1/4	180	$3.270E+00$	–	$4.004E+01$	–	$3.556E+03$	–	88.818
	1/8	612	$4.051E-01$	3.013	$1.087E+01$	1.882	$1.008E+03$	1.819	92.763
	1/16	2244	$5.827E-02$	2.797	$2.978E+00$	1.867	$2.632E+02$	1.937	88.369
	1/32	8580	$7.747E-03$	2.911	$7.674E-01$	1.956	$6.592E+01$	1.997	85.893
	1/64	33540	$9.881E-04$	2.971	$1.935E-01$	1.988	$1.644E+01$	2.003	84.965
	1/128	132612	$1.243E-04$	2.991	$4.848E-02$	1.997	$4.103E+00$	2.002	84.642
	1/256	527364	$1.557E-05$	2.997	$1.213E-02$	1.999	$1.025E+00$	2.001	84.523
3	1/4	364	$4.770E-01$	–	$9.471E+00$	–	$1.473E+03$	–	155.496
	1/8	1300	$7.031E-02$	2.762	$2.526E+00$	1.907	$3.460E+02$	2.090	136.963
	1/16	4900	$4.506E-03$	3.964	$3.302E-01$	2.936	$4.622E+01$	2.904	139.977
	1/32	19012	$2.796E-04$	4.010	$4.141E-02$	2.995	$5.847E+00$	2.983	141.198
	1/64	74884	$1.739E-05$	4.007	$5.170E-03$	3.002	$7.316E-01$	2.998	141.522
	1/128	297220	$1.084E-06$	4.003	$6.456E-04$	3.001	$9.140E-02$	3.001	141.590
	1/256	1184260	$6.784E-08$	3.998	$8.065E-05$	3.001	$1.142E-02$	3.001	141.600

Table 1: A priori and a posteriori error estimates with effectivity indexes, example with an analytic solution in 2D.

5.2. Adaptive refinement for a solution with singularities

In this example, we compute a numerical solution, using piecewise linear elements (i.e., $k = 1$), of the elasticity equation subject to the interface conditions (4)–(5) that models a mode II crack deformation (see [29]). This deformation model involves self-similar sliding of the sides of the cracked rock, which is analogous to the movement of tectonic plates during a subduction earthquake. The theoretical solution presents two singularity points located at the ends of a fracture zone of length $2a$ given by a subset of Γ_F. The domain for this example is shown in Figure 2, and the fracture zone is the segment with endpoints $(0, -a)$ and $(0, a)$. Taking the parameters $\nu = \dfrac{\lambda}{2(\mu+\lambda)}$ and $\sigma \in \mathbb{R}^+$, the jump $\boldsymbol{s}$ can be written as $\boldsymbol{t} = s\boldsymbol{t}$, where $\boldsymbol{t} = (0,1)^T$ and

$$(\forall y \in [-1,1]) \quad s(y) := \begin{cases} 0 & \text{if } |y| > a, \\ 2\sigma \left(\dfrac{1-\nu}{\mu}\right) \sqrt{a^2 - y^2} & \text{if } |y| \le a, \end{cases}$$

Assuming $\boldsymbol{f} = \boldsymbol{0}$, if (r, θ), (r_1, θ_1) and $r_2, \theta_2)$ are the polar coordinates of (x, y), $(x, y+a)$ and $(x, y-a)$ with respect to the polar axis $\{y < 0\}$, and defining $R := \sqrt{r_1 r_2}$ and $\omega = \dfrac{1}{2}(\theta_1 + \theta_2)$, the analytical solution of (1)–(5) is given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \dfrac{\sigma}{2\mu}\left((1-2\nu)(R\cos\omega - r\cos\theta) + r\sin\theta\left(\dfrac{r}{R}\cos(\theta-\omega) - 1\right)\right) \\ -\dfrac{\sigma}{2\mu}\left(2(1-\nu)(R\sin\omega - r\cos\theta) + \dfrac{r^2}{R}\sin\theta\sin(\theta-\omega)\right) \end{pmatrix}, \tag{13}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = -\sigma \begin{pmatrix} \dfrac{a^2 r}{R^3}\sin\theta\cos(3\omega) & \dfrac{r}{R}\cos(\theta-\omega) - 1 - \dfrac{a^2 r}{R^3}\sin\theta\sin(3\omega) \\ \dfrac{r}{R}\cos(\theta-\omega) - 1 - \dfrac{a^2 r}{R^3}\sin\theta\sin(3\omega) & \dfrac{2r}{R}\sin(\theta-\omega) - \dfrac{a^2 r}{R^3}\sin\theta\cos(3\omega) \end{pmatrix}. \tag{14}$$

Although $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, it is evident that $\boldsymbol{\sigma}(\boldsymbol{u})$ has two non-removable singularities located in $(0,-a)$ and $(0,a)$ because $R = 0$ in those points, which means that $\boldsymbol{\sigma}(\boldsymbol{u}) \notin \mathbb{L}^2(\Omega)$. Then, the assumptions of Theorem 15 are not satisfied, and we cannot expect the numerical solution to follow any convergence order. Note that, in Cartesian coordinates, the expression for $\boldsymbol{u}$ on Γ_F is given by

$$\boldsymbol{u}_1(0,y) = \begin{cases} \dfrac{\sigma}{2\mu}\begin{pmatrix} -(1-2\nu)y \\ 2(1-\nu)\sqrt{a^2-y^2} \end{pmatrix} & \text{if } |y| \le a \\ \dfrac{\sigma}{2\mu}\begin{pmatrix} (1-2\nu)\left(\operatorname{sgn}(y)\sqrt{y^2-a^2} - y\right) \\ 0 \end{pmatrix} & \text{if } |y| > a \end{cases}$$

$$\boldsymbol{u}_2(0,y) = \begin{cases} \dfrac{\sigma}{2\mu}\begin{pmatrix} -(1-2\nu)y \\ -2(1-\nu)\sqrt{a^2-y^2} \end{pmatrix} & \text{if } |y| \le a \\ \dfrac{\sigma}{2\mu}\begin{pmatrix} (1-2\nu)\left(\operatorname{sgn}(y)\sqrt{y^2-a^2} - y\right) \\ 0 \end{pmatrix} & \text{if } |y| > a \end{cases}$$

In this test, we set the parameters of (6) as $\theta = -1$, $\alpha = 1/2$, and $\gamma = 1$. We consider Lamé parameters $\mu = \lambda = 1$ (with $\nu = 1/4$), $\sigma = 1$, $a = 0.25$, and the boundary conditions $\boldsymbol{u}_D$ and $\boldsymbol{g}$ such that the theoretical solution is given by (13). Figures 4 and 5 show the initial mesh and some adapted meshes for $\delta = 0.75$ and a posteriori error history, respectively. We note that η follows a convergence order similar to the theoretical order for $e_h(\boldsymbol{u})$, but $e_h(\boldsymbol{u})$ is not computable due to $\boldsymbol{u}_j \notin \boldsymbol{H}^1(\Omega_j)$ for $j \in \{1,2\}$.

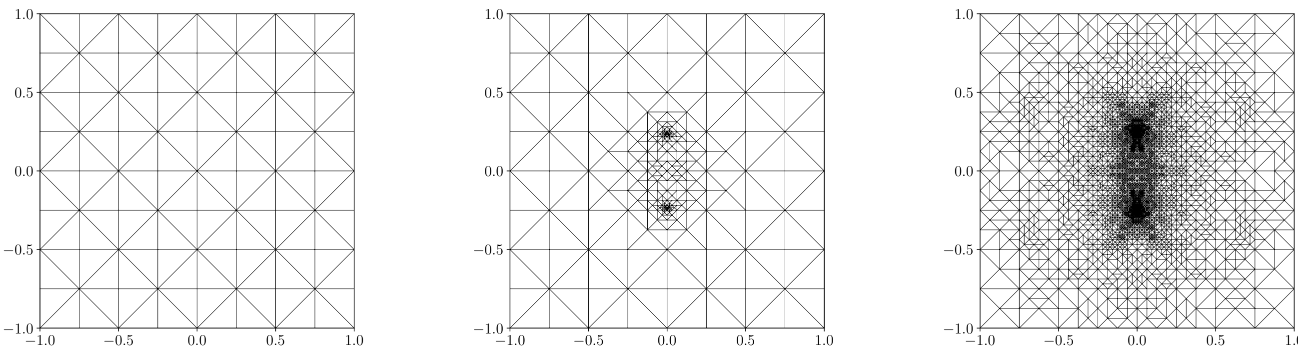

Figure 4: Original (left) and adapted meshes for stages 5 (center) and 15 (right).

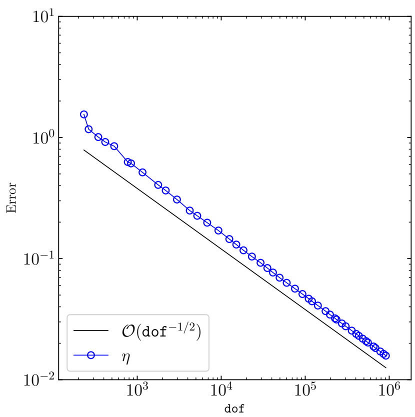

Figure 5: History of convergence, adaptive refinement with singularities.

Figures 6 and 7 show plots of the absolute error on tangential and normal components of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ on Γ_F, respectively, where a decrease in the error near the singularities located in $(0,-a)$ and $(0,a)$ is evident thanks to the adaptive refinement. Figure 8 shows isovalues of $|\boldsymbol{u}|$ and the mean normal stress $\sigma_n = \dfrac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}(\boldsymbol{u}))$ with similar results to Figure 5 in [30] and Figure 8.7.B in [29], respectively.

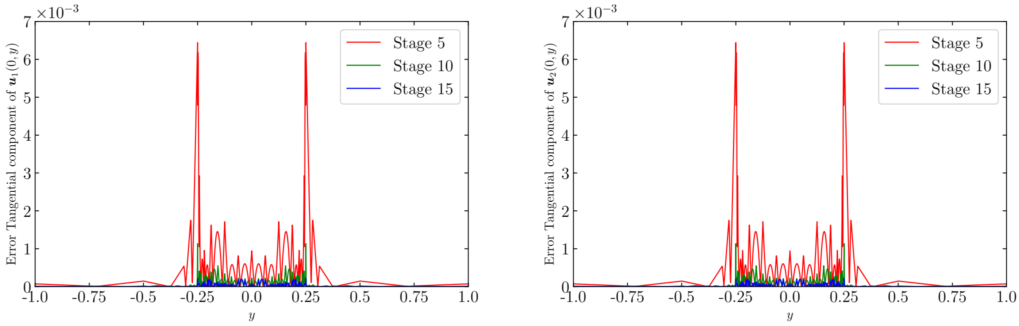

Figure 6: Absolute error on tangential component of $\boldsymbol{u}_1$ (left) and $\boldsymbol{u}_2$ (right) on Γ_F.

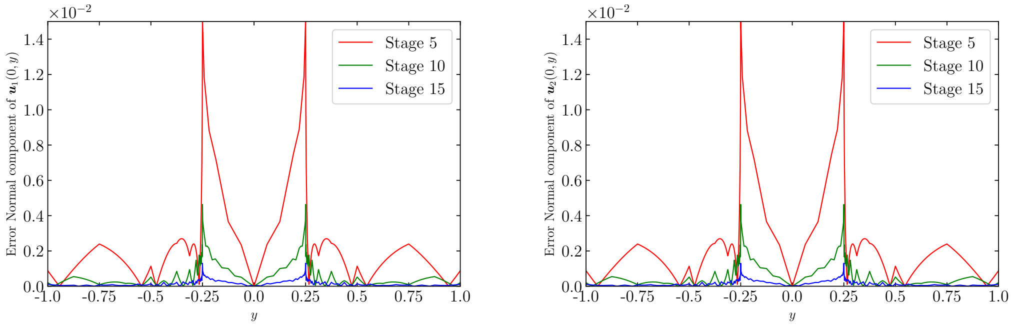

Figure 7: Absolute error on normal component of $\boldsymbol{u}_1$ (left) and $\boldsymbol{u}_2$ (right) on Γ_F.

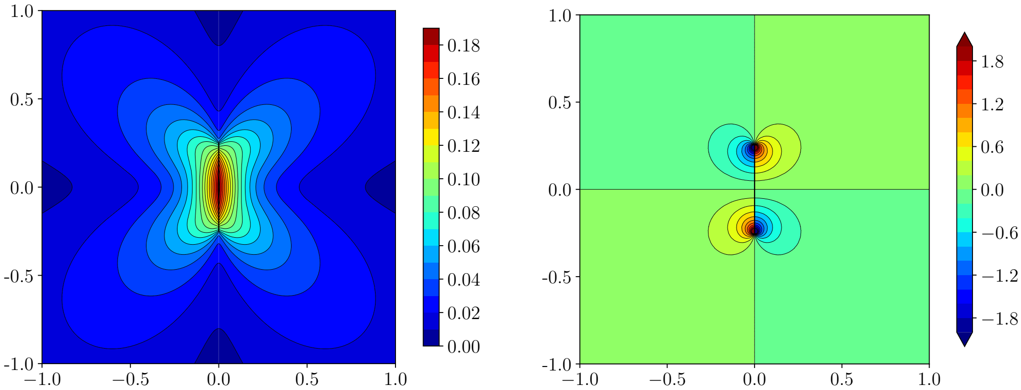

Figure 8: Isovalues of $|\boldsymbol{u}|$ (left) and σ_n (right) on Ω for stage 39 ($\texttt{dof} = 360,542$).

5.3. *Adaptive refinement for a solution with an unknown analytic solution and curved interface simulating a subduction earthquake*

This final example simulates the effects of a subduction earthquake prescribing a coseismic slip $\boldsymbol{s}$ between the tectonic plates, based on the experiment in Section 4 of [30].

The domain is given by $\Omega = [-400, 700] \times [-500, 0]$, representing a domain of size 1100km × 500km. The tectonic fault is the interface Γ_F that divides Ω into the subdomains Ω_1 and Ω_2, on the left-hand and right-hand side respectively, representing two tectonic plates. The geometry of Γ_F is given by an circular arc, centered at $(a, b) = (-40, -157.5)$ and connecting the points $(0, 0)$ and $(90, -60)$, and a segment tangent to the arc connecting the points $(90, -60)$ and $(420, -500)$. We set $\Gamma_D = \{-400, 700\} \times (-500, 0)$ and $\Gamma_N = (-400, 700) \times \{-500, 0\}$ (see Figure 9).

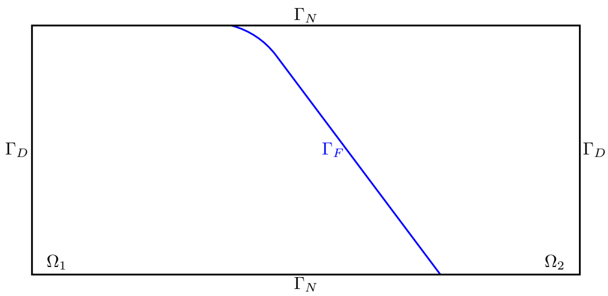

Figure 9: Domain Ω with a tectonic fault and boundary conditions.

We choose the Lamé parameters $\mu := 33,65$GPa, $\lambda := 30,11$GPa, $\boldsymbol{f} = \boldsymbol{0}$, applying homogenous Dirichlet boundary condition on Γ_D and a free surface Neumann condition on Γ_N, given by $\boldsymbol{g} = \boldsymbol{0}$. The prescribed coseismic slip is given by $\boldsymbol{s} = s\boldsymbol{t}$[m], where s is a scalar function and $\boldsymbol{t}$ is the unit tangent vector on Γ_F, both given by

$$(\forall(x,y) \in \Gamma_F) \quad s(x,y) = \begin{cases} A \exp\left(-\left(\dfrac{y+50}{15}\right)^2\right) + B, & \text{if } y \in [-95,-5] \\ 0, & \text{otherwise} \end{cases}$$

$$(\forall(x,y) \in \Gamma_F) \quad \boldsymbol{t}(x,y) = \begin{cases} \left(\dfrac{y-b}{r}, -\dfrac{x-a}{r}\right), & \text{if } y \in [-60,0] \\ \left(\dfrac{3}{5}, -\dfrac{4}{5}\right), & \text{otherwise} \end{cases}$$

where r is the radius of the circular arc and $A, B \in \mathbb{R}$ are positive constants such that s is continuous and the maximum of s is equal to 1. We compute a numerical solution using piecewise linear elements using adaptive refinement with $\delta = 0.5$ until we obtain a mesh with more than 500000 triangles, verifying this criterion in stage 18. Figures 10, 11 , and 12 show the initial mesh and a zoom of two adapted meshes, respectively. The refinement is concentrated on Γ_F, with the densest refinement near the point $(90, -60)$, corresponding to the intersection between the segment and the circle arc. Figure 13 shows the a posteriori error history, where we recover again our convergence order.

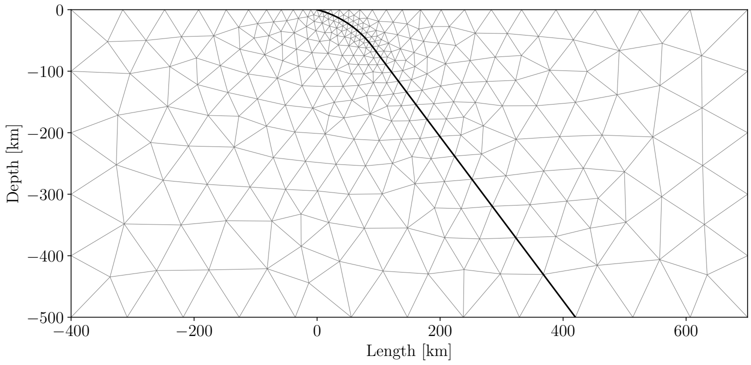

Figure 10: Initial mesh. 593 triangles, `dof` = 702.

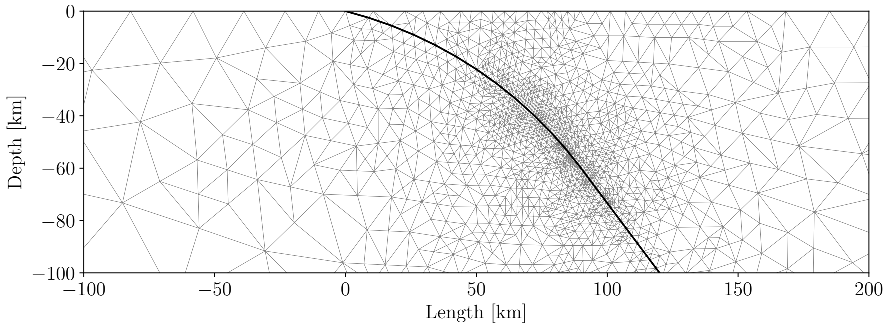

Figure 11: Zoom to an adapted mesh, stage 5. 3749 triangles, `dof` = 4044.

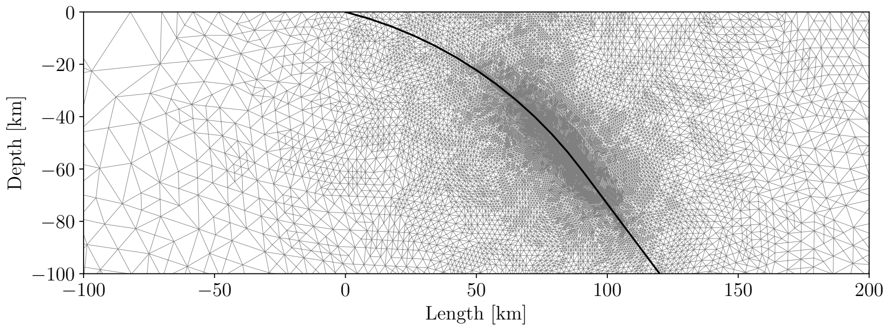

Figure 12: Zoom to an adapted mesh, stage 10. 35502 triangles, `dof` = 36372.

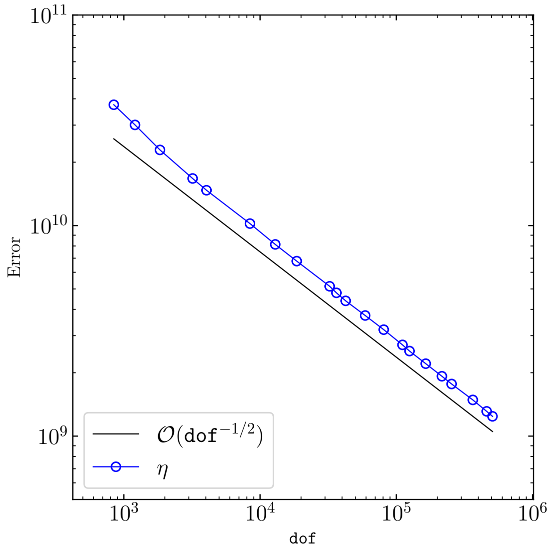

Figure 13: History of convergence, adaptive refinement for a subduction earthquake simulation.

Figure 14 illustrates a comparison between the numerical solution obtained after 21 stages of adaptive refinement (505020 triangles, 508458 degrees of freedom) of the initial mesh and the one obtained on a highly refined mesh (2867938 degrees of freedom, 2861152 triangles), which better approximates the geometry of Γ_F. Figure 14 shows that adaptive refinement can generate meshes that allow obtaining highly accurate numerical solutions for $\boldsymbol{u}$ with fewer degrees of freedom.

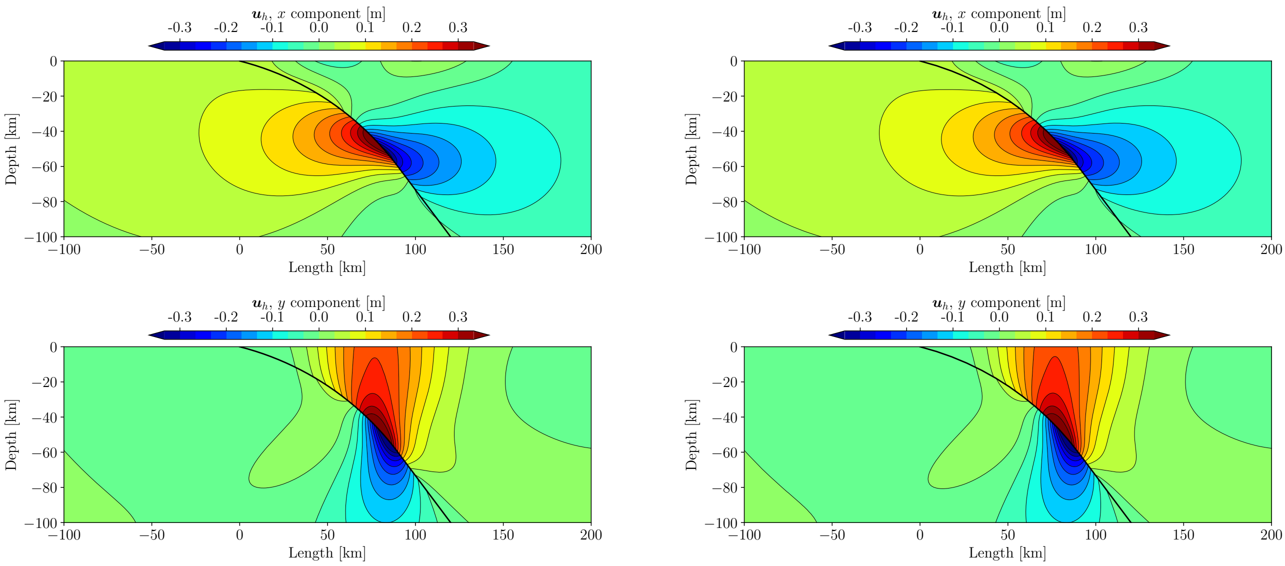

Figure 14: Zoom of isovalues of $\boldsymbol{u}_h$ for the final adapted mesh (left) and the highly refined mesh (right).

6. Conclusions

This article presented a family of conformal schemes for numerically solving the linear elasticity equation with a displacement jump condition on an interface using a Nitsche-type penalty term. An optimal prior error estimate was obtained for a mesh-dependent norm, along with a residual a posteriori error estimator that is efficient and reliable with respect to the error norm. Our numerical experiments recovered the theoretical convergence orders and showed that the posterior error estimator can be used to develop adapted meshes for realistic applied problems.

A direct extension of our result is the derivation of a scheme that uses independent meshes for each subdomain, such as non-matching or unfitted meshes, where the meshes may not coincide at the interface. In such a case, the incorporation of a Lagrange multiplier can be avoided by following a scheme similar to [31] based on an $(n-1)$–dimensional mesh for the interface. Another future application is the analysis of optimal control problems for coseismic slip analysis [30]. Since this family of schemes requires less storage memory compared to other mixed methods [3], it is possible to reduce the optimality conditions to a coupled linear system as in [2].

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