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Ramiro Acevedo, Salim Meddahi

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An $\boldsymbol{E}$-based mixed-FEM and BEM coupling for a time-dependent eddy current problem

Ramiro Acevedo*
Departamento de Matemática, Universidad del Cauca, Popayán, Colombia
Calle 5 No 4-70, Popayán, Colombia.

Salim Meddahi†
Departamento de Matemáticas, Universidad de Oviedo
Calvo Sotelo s/n, 33007, Oviedo, España.

Abstract

In this paper, we analyze a mixed-FEM and BEM coupling for a time-dependent eddy current problem posed in the whole space and formulated in terms of the electric field $\boldsymbol{E}$. The coupled problem is obtained by first proposing a mixed formulation of the interior problem in order to handle efficiently the divergence free constraint satisfied by $\boldsymbol{E}$ in the dielectric material. Next, we incorporate the far field effect to the latter formulation through boundary integral equations defined on the coupling interface. We show that the resulting degenerate parabolic problem (with saddle point structure) is well-posed and use Nédélec edge elements and standard nodal finite elements to define a semi-discrete Galerkin scheme. Furthermore, we introduce the corresponding backward-Euler fully-discrete formulation and analyze the asymptotic behavior of the error in terms of the discretization parameters for both schemes.

Keywords: eddy current problem, saddle point problems, mixed finite elements, Nédélec finite elements, boundary elements.

1 Introduction

The eddy current problem is naturally formulated in the whole space with decay conditions on the fields at infinity; see, for instance, [5]. Consequently, to apply conventional numerical methods, such as the finite element method (FEM), it is necessary to reduce the problem to a bounded domain. The most common approach consists in restricting the equations to a sufficiently large computational domain containing the region of interest and imposing an artificial homogeneous boundary condition on its border (which must be "sufficiently" far away from the conductor). This strategy yields the difficulty of fixing a convenient cut-off distance a priori. Moreover, in case of conductors with a "special" shape or a very large computational domain, a finite element mesh can lead to a very large number of elements. On the other hand, methods based on boundary integral equations, like the boundary element method (BEM), in general can not be directly applied because the equations are not homogeneous and have variable coefficients.

Since the equations of the eddy current problem are complex only in a bounded region, techniques combining BEM and FEM look convenient. The first FEM-BEM couplings for the eddy current model have been proposed by engineers: [9], [10] (using the magnetic field $\boldsymbol{H}$ in the conductor and the Steklov-Poincaré operator) and [17] (using the electric field $\boldsymbol{E}$ in the conductor and certain harmonic basis functions near its boundary Σ). From a mathematical point of view, more recent results based on the well-known symmetric method by [14] are due to [16] (using $\boldsymbol{E}$ in the conductor and $\boldsymbol{H} \times \boldsymbol{n}$ on Σ) and [19] (using $\boldsymbol{H}$ in the conductor and the normal trace

*Email: rmacevedo@unicauca.edu.co (This author was partially supported by MECESUP UCO0406 and a CONICYT Ph.D. fellowship at Universidad de Concepción, Chile)

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of the magnetic induction on Σ) for the time-harmonic problem. Another FEM-BEM approach for the same problem in terms of vector and scalar potentials has been also recently analyzed by [4].

When the conductor is multiply-connected, the approach mentioned above requires the construction of cumbersome (and expensive) cutting surfaces in order to deal correctly with the discrete problem, see also [7]. Recently, [2] showed that the time-harmonic $\mathbf{H}$-based formulation of the eddy current problem (posed in a bounded domain) admits a saddle point structure that is free from the above restriction. (See also [20] for a similar strategy applied to the case of a time-dependent eddy current problem posed in the whole space.) Such a formulation is obtained by solving the problem in a box Ω completely containing the conductor Ω_c and by introducing a Lagrange multiplier associated to the **curl**-free constraint satisfied by the magnetic field in the insulating region $\Omega_d := \Omega \setminus \overline{\Omega}_c$ surrounding the conductor. We adopt here the same point of view for the problem under consideration.

Actually, our goal is to introduce a new method to solve the time-dependent eddy current problem, based on a mixed-FEM and BEM coupling. We use as main variable a time primitive of $\boldsymbol{E}$ in Ω (see also [8]). The divergence free condition in the insulating material is handled through a Lagrange multiplier, which gives rise to a saddle point formulation in the interior domain. Besides, the integral representation of the electric field in the complementary unbounded domain provides non-local boundary conditions for the interior mixed formulation. This approach extends our previous work [1], where the eddy current problem is assumed to be posed in a bounded domain.

A feature of our formulation is that the compact support of the current density is not necessarily assumed to be completely contained in the conductor or in its exterior. Furthermore, we choose Ω simply connected with a connected boundary in order to be able to introduce a certain scalar potential as a boundary variable and use standard nodal finite elements to approximate it. On the other hand, in contrast with the formulation given in [20], our approach fits well into the theory of monotone operators, because the reluctivity (the inverse of the magnetic permeability) appears as a diffusion coefficient in the degenerate parabolic problem at hand. Consequently, this approach seems convenient when the relation between the magnetic field and the magnetic induction (given by the reluctivity) depends on the magnetic induction intensity, which is typical for the ferromagnetic materials.

We perform a space discretization of our weak formulation by using Nédélec edge elements for the main unknown and standard finite elements for the Lagrange multiplier and the boundary variable. We show that our semi-discrete Galerkin scheme is uniquely solvable and provide error estimates in terms of the space discretization parameter h. We also propose a fully-discrete Galerkin scheme based on a backward-Euler time-stepping. Here again we provide error estimates that prove optimal convergence. Moreover, we obtain error estimates for the eddy currents and the magnetic induction field.

The paper is organized as follows. In Section 2, we summarize some results from [12, 11, 13] concerning tangential differential operators and traces in $\mathbf{H}(\mathbf{curl}; \Omega)$. In Section 3, we introduce the model problem. We derive a symmetric mixed-FEM and BEM coupling of our problem in Section 4 and prove that it is uniquely solvable in Section 5. The construction of a semi-discretization in space and the analysis of its convergence are reported in Section 6. Finally, a backward Euler method is employed to obtain a time discretization of the problem. The results presented in Section 7 prove that the resulting fully discrete scheme is convergent with optimal order.

2 Preliminaries

We use boldface letters to denote vectors as well as vector–valued functions and the symbol $|\cdot|$ represents the standard Euclidean norm for vectors. In this section Ω is a generic Lipschitz bounded domain of $\mathbb{R}^3$. We denote by Γ its boundary and by $\boldsymbol{n}$ the unit outward normal to Ω. Let

$$(f, g)_{0,\Omega} := \int_\Omega fg$$

be the inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|_{0,\Omega}$ the corresponding norm. As usual, for all $s > 0$, $\|\cdot\|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$ and $|\cdot|_{s,\Omega}$ for the corresponding seminorm. The space $\mathrm{H}^{1/2}(\Gamma)$ is defined by localization on the Lipschitz surface Γ. We denote by $\|\cdot\|_{1/2,\Gamma}$ the norm in $\mathrm{H}^{1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ stands for the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual $\mathrm{H}^{-1/2}(\Gamma)$. From now on we denote by

$\gamma : \mathrm{H}^1(\Omega) \to \mathrm{H}^{1/2}(\Gamma)$ and $\boldsymbol{\gamma} : \mathrm{H}^1(\Omega)^3 \to \mathrm{H}^{1/2}(\Gamma)^3$ the standard trace operator acting on scalar and vector fields respectively.

2.1 Tangential differential operators and traces

We consider the space

$$\mathbf{L}^2_\tau(\Gamma) := \left\{\boldsymbol{\lambda} \in \mathrm{L}^2(\Gamma)^3 : \quad \boldsymbol{\lambda}\cdot\boldsymbol{n} = 0\right\},$$

endowed with the standard norm in $\mathrm{L}^2(\Gamma)^3$. We define the tangential trace $\boldsymbol{\gamma}_\tau : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathbf{L}^2_\tau(\Gamma)$ and the tangential component trace $\boldsymbol{\pi}_\tau : C^\infty\left(\overline{\Omega}\right)^3 \to \mathbf{L}^2_\tau(\Gamma)$ as $\boldsymbol{\gamma}_\tau\boldsymbol{v} := \boldsymbol{\gamma}\boldsymbol{v}\times\boldsymbol{n}$ and $\boldsymbol{\pi}_\tau\boldsymbol{v} := \boldsymbol{n}\times(\boldsymbol{\gamma}\boldsymbol{v}\times\boldsymbol{n})$ respectively. The previous traces can be extended by completeness to $\mathrm{H}^1(\Omega)^3$. The spaces $\mathbf{H}^{1/2}_\perp(\Gamma) := \boldsymbol{\gamma}_\tau(\mathrm{H}^1(\Omega)^3)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma) := \boldsymbol{\pi}_\tau\left(\mathrm{H}^1(\Omega)^3\right)$, are respectively endowed with the Hilbert norms

$$\begin{aligned}
\|\boldsymbol{\eta}\|_{\mathbf{H}^{1/2}_\perp(\Gamma)} &:= \inf_{\boldsymbol{w}\in\mathrm{H}^1(\Omega)^3}\left\{\|\boldsymbol{w}\|_{1,\Omega} : \quad \boldsymbol{\gamma}_\tau\boldsymbol{w} = \boldsymbol{\eta}\right\},\\
\|\boldsymbol{\eta}\|_{\mathbf{H}^{1/2}_\parallel(\Gamma)} &:= \inf_{\boldsymbol{w}\in\mathrm{H}^1(\Omega)^3}\left\{\|\boldsymbol{w}\|_{1,\Omega} : \quad \boldsymbol{\pi}_\tau\boldsymbol{w} = \boldsymbol{\eta}\right\}.
\end{aligned}$$

Let us notice that the density of $\mathrm{H}^{1/2}(\Gamma)^3$ in $\mathrm{L}^2(\Gamma)^3$ ensures that $\mathbf{H}^{1/2}_\perp(\Gamma)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma)$ are dense subspaces of $\mathbf{L}^2_\tau(\Gamma)$. We denote by $\mathbf{H}^{-1/2}_\perp(\Gamma)$ and $\mathbf{H}^{-1/2}_\parallel(\Gamma)$ the dual spaces of $\mathbf{H}^{1/2}_\perp(\Gamma)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma)$ with $\mathbf{L}^2_\tau(\Gamma)$ as pivot space, with duality pairing $\langle\cdot,\cdot\rangle_{\perp,\Gamma}$ and $\langle\cdot,\cdot\rangle_{\parallel,\Gamma}$ respectively.

We introduce the tangential differential operators

$$\mathbf{grad}_\Gamma\,\varphi := \boldsymbol{\pi}_\tau(\mathbf{grad}\,\varphi) \quad \text{and} \quad \mathbf{curl}_\Gamma\,\varphi := \boldsymbol{\gamma}_\tau(\mathbf{grad}\,\varphi) \qquad \forall\varphi\in\mathrm{H}^2(\Omega).$$

Let $\mathrm{H}^{3/2}(\Gamma) := \gamma(\mathrm{H}^2(\Omega))$. It is well known that the previous operators depend only on the trace $\gamma(\varphi)$ on Γ, which implies that

$$\mathbf{grad}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{H}^{1/2}_\parallel(\Gamma) \quad \text{and} \quad \mathbf{curl}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{H}^{1/2}_\perp(\Gamma) \tag{2.1}$$

are linear and continuous, cf. [13, Proposition 3.4]. Let $\mathrm{H}^{-3/2}(\Gamma)$ be the dual space of $\mathrm{H}^{3/2}(\Gamma)$ with $\mathrm{L}^2(\Gamma)$ as pivot space. We define

$$\mathrm{div}_\Gamma : \mathbf{H}^{-1/2}_\parallel(\Gamma) \to \mathrm{H}^{-3/2}(\Gamma) \quad \text{and} \quad \mathrm{curl}_\Gamma : \mathbf{H}^{-1/2}_\perp(\Gamma) \to \mathrm{H}^{-3/2}(\Gamma), \tag{2.2}$$

by the dualities

$$\begin{aligned}
\langle\mathrm{div}_\Gamma\boldsymbol{\eta},\phi\rangle_{3/2,\Gamma} &= -\left\langle\boldsymbol{\eta},\mathbf{grad}_\Gamma\,\phi\right\rangle_{\parallel,\Gamma} && \forall\phi\in\mathrm{H}^{3/2}(\Gamma) \quad \forall\boldsymbol{\eta}\in\mathbf{H}^{-1/2}_\parallel(\Gamma),\\
\langle\mathrm{curl}_\Gamma\boldsymbol{\xi},\phi\rangle_{3/2,\Gamma} &= \left\langle\boldsymbol{\xi},\mathbf{curl}_\Gamma\,\phi\right\rangle_{\perp,\Gamma} && \forall\phi\in\mathrm{H}^{3/2}(\Gamma) \quad \forall\boldsymbol{\xi}\in\mathbf{H}^{-1/2}_\perp(\Gamma).
\end{aligned} \tag{2.3}$$

The following proposition is proved in [13, Proposition 3.6].

Proposition 1 *The operators* $\mathbf{grad}_\Gamma$ *and* $\mathbf{curl}_\Gamma$ *given in* (2.1) *can be extended to* $\mathrm{H}^{1/2}(\Gamma)$. *Moreover,* $\mathbf{grad}_\Gamma : \mathrm{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_\perp(\Gamma)$ *and* $\mathbf{curl}_\Gamma : \mathrm{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_\parallel(\Gamma)$ *are linear and continuous. Analogously, the transpose operators introduced in* (2.2) *are also continuous for the following choice of spaces:* $\mathrm{div}_\Gamma : \mathbf{H}^{1/2}_\perp(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma : \mathbf{H}^{1/2}_\parallel(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$. *Furthermore, analogous identities to* (2.3) *still hold for any* $\phi\in\mathrm{H}^{1/2}(\Gamma)$, $\boldsymbol{\eta}\in\mathbf{H}^{1/2}_\perp(\Gamma)$ *and* $\boldsymbol{\xi}\in\mathbf{H}^{1/2}_\parallel(\Gamma)$. *More precisely, we have*

$$\begin{aligned}
\langle\mathrm{div}_\Gamma\boldsymbol{\eta},\phi\rangle_{1/2,\Gamma} &= -\left\langle\mathbf{grad}_\Gamma\,\phi,\boldsymbol{\eta}\right\rangle_{\perp,\Gamma} && \forall\phi\in\mathrm{H}^{1/2}(\Gamma) \quad \forall\boldsymbol{\eta}\in\mathbf{H}^{1/2}_\perp(\Gamma),\\
\langle\mathrm{curl}_\Gamma\boldsymbol{\xi},\phi\rangle_{1/2,\Gamma} &= \left\langle\mathbf{curl}_\Gamma\,\phi,\boldsymbol{\xi}\right\rangle_{\parallel,\Gamma} && \forall\phi\in\mathrm{H}^{1/2}(\Gamma) \quad \forall\boldsymbol{\xi}\in\mathbf{H}^{1/2}_\parallel(\Gamma).
\end{aligned}$$

Let

$$\mathbf{H}(\mathbf{curl};\Omega) := \left\{\boldsymbol{v}\in\mathrm{L}^2(\Omega)^3 : \quad \mathbf{curl}\,\boldsymbol{v}\in\mathrm{L}^2(\Omega)^3\right\},$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} := \left(\|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega}\right)^{1/2}. \tag{2.4}$$

Using the Green formula (see, for instance, [11] for the case of Lipschitz polyhedra and [13] for arbitrary Lipschitz domains)

$$(\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v})_{0,\Omega} = \langle \boldsymbol{\gamma}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v} \rangle_{\|,\Gamma} = - \langle \boldsymbol{\pi}_\tau \boldsymbol{v}, \boldsymbol{\gamma}_\tau \boldsymbol{u} \rangle_{\perp,\Gamma} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^3,$$

and the density of $\mathcal{C}^\infty(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl};\Omega)$ (see, for instance, [21, Theorem 3.26]) and in $\mathrm{H}^1(\Omega)$, it follows that

$$\boldsymbol{\gamma}_\tau : \ \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_{\|}^{-1/2}(\Gamma), \qquad \boldsymbol{\pi}_\tau : \ \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_{\perp}^{-1/2}(\Gamma)$$

are continuous. The space $\mathbf{H}_0(\mathbf{curl};\Omega)$ stands for the kernel of $\boldsymbol{\gamma}_\tau$ in $\mathbf{H}(\mathbf{curl};\Omega)$. The ranges of $\boldsymbol{\gamma}_\tau$ and $\boldsymbol{\pi}_\tau$ are characterized in the following result.

Theorem 2 *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) := \left\{ \boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\Gamma) : \ \mathrm{div}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma) \right\}$$

and

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma) := \left\{ \boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma) : \ \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma) \right\}.$$

Then

$$\boldsymbol{\gamma}_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma), \quad \boldsymbol{\pi}_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$$

are surjective and possess a continuous right inverse.

The spaces $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$ are dual to each other, when $\mathbf{L}^2_\tau(\Gamma)$ is used as pivot space, i.e. the usual $\mathbf{L}^2_\tau(\Gamma)$-inner product can be extended to a duality pairing $\langle \cdot,\cdot \rangle_{\tau,\Gamma}$ between $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$. Moreover, the following integration by parts formula holds true

$$(\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v})_{0,\Omega} = \langle \boldsymbol{\gamma}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v} \rangle_{\tau,\Gamma} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega). \tag{2.5}$$

Proof. See Theorem 4.1 and Lemma 5.6 of [13]. □

Let Ω be a Lipschitz polyhedron. The following Theorem gives a characterization of the space

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma) := \left\{ \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) : \quad \mathrm{div}_\Gamma \boldsymbol{\eta} = 0 \right\}.$$

Theorem 3 *Let $\mathcal{O}$ be a regular bounded open connected and simply connected subset of $\mathbb{R}^3$, such that $\overline{\Omega} \subset \mathcal{O}$. We set $\Omega_{\mathrm{ext}} := \mathcal{O} \setminus \overline{\Omega}$. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ the spaces of the so-called harmonic Neumann fields associated to Ω and Ω_{ext} respectively, i.e.*

$$\mathbb{H}_1 := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathrm{H}(\mathrm{div};\Omega) : \quad \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \quad \mathrm{div}\, \boldsymbol{v} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0 \},$$
$$\mathbb{H}_2 := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{ext}}) \cap \mathrm{H}(\mathrm{div};\Omega_{\mathrm{ext}}) : \quad \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \quad \mathrm{div}\, \boldsymbol{v} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_{\mathrm{ext}}} = 0 \}.$$

Let $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$. Then, $\mathrm{div}_\Gamma \boldsymbol{\eta} = 0$ if and only if there exists $\lambda \in \mathrm{H}^{1/2}(\Gamma)$, $\boldsymbol{v}_1 \in \mathbb{H}_1$ and $\boldsymbol{v}_2 \in \mathbb{H}_2$ such that

$$\boldsymbol{\eta} = \mathbf{curl}_\Gamma\, \lambda + \boldsymbol{\pi}_\tau \boldsymbol{v}_1 + \boldsymbol{\pi}_\tau \boldsymbol{v}_2|_\Gamma.$$

Proof. See [12, Section 3]. □

If Ω is simply-connected, it is well know that $\mathbb{H}_1 = \mathbb{H}_2 = \{\mathbf{0}\}$; see, for instance, [6, Subsection 3.3]. Therefore, the previous theorem implies

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma) = \mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma)).$$

Furthermore, if Γ is connected then $\ker(\mathbf{curl}_\Gamma) \cap \mathrm{H}^{1/2}(\Gamma) = \mathbb{R}$; cf. [13, Corollary 3.7]. Consequently, the next result follows immediately from Proposition 1.

Corollary 4 *Let*

$$\mathrm{H}_0^{1/2}(\Gamma) := \left\{ \eta \in \mathrm{H}^{1/2}(\Gamma) : \quad \int_\Gamma \eta = 0 \right\}.$$

If Ω is simply connected and Γ is connected, then the operator

$$\mathbf{curl}_\Gamma : \ \mathrm{H}_0^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma)$$

is an isomorphism.

We will also use the normal trace $\gamma_{\boldsymbol{n}} : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)$ given by $\boldsymbol{q} \mapsto \boldsymbol{\gamma q} \cdot \boldsymbol{n}$. It is well known that this operator can be extended to a continuous and surjective mapping (see, for instance, [21, Theorem 3.24])

$$\gamma_{\boldsymbol{n}} : \ \mathbf{H}(\mathrm{div};\Omega) \to \mathrm{H}^{-1/2}(\Gamma),$$

where

$$\mathbf{H}(\mathrm{div};\Omega) := \left\{\boldsymbol{q} \in \mathrm{L}^2(\Omega)^3 : \quad \mathrm{div}\, \boldsymbol{q} \in \mathrm{L}^2(\Omega)\right\}$$

is endowed with the norm $\|\boldsymbol{v}\|_{\mathrm{H}(\mathrm{div};\Omega)} := \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\, \boldsymbol{v}\|_{0,\Omega}^2\right)^{1/2}$. We denote by $\mathbf{H}_0(\mathrm{div},\Omega)$ the kernel of $\gamma_{\boldsymbol{n}}$ in $\mathbf{H}(\mathrm{div};\Omega)$.

2.2 Basic spaces for time dependent problems

Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0,T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0,T];V)$ for the Banach space consisting of all continuous functions $f : \ [0,T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0,T];V)$ denotes the subspace of $\mathcal{C}^0([0,T];V)$ of all functions f with (strong) derivatives of order at most k in $\mathcal{C}^0([0,T];V)$, *i.e.*

$$\mathcal{C}^k([0,T];V) := \left\{f \in \mathcal{C}^0([0,T];V) : \quad \frac{d^j f}{dt^j} \in \mathcal{C}^0([0,T];V), \quad 1 \le j \le k\right\}.$$

We also consider the space $\mathrm{L}^2(0,T;V)$ of classes of functions $f : \ (0,T) \to V$ that are Böchner-measurable and such that

$$\|f\|_{\mathrm{L}^2(0,T;V)}^2 := \int_0^T \|f(t)\|_V^2 \, dt < +\infty.$$

Furthermore, we will use the space

$$\mathrm{H}^1(0,T;V) := \left\{f \in \mathrm{L}^2(0,T;V) : \quad \frac{d}{dt} f \in \mathrm{L}^2(0,T;V)\right\},$$

where $\frac{d}{dt} f$ is the (generalized) time derivative of f; see, for instance [23, Section 23.5]. In what follows, we will use indistinctly the notations

$$\frac{d}{dt} f = \partial_t f$$

to express the time derivative of f. Analogously, we define $\mathrm{H}^k(0,T;V)$ for all $k \in \mathbb{N}$.

3 The model problem

We assume that the conductor is represented by a connected and bounded polyhedron $\Omega_{\mathrm{c}} \subset \mathbb{R}^3$ with a Lipschitz boundary Σ. We denote by Σ_i, $i = 0, \ldots, I$ the connected components of Σ and assume that Σ_0 is the boundary of the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}_c$. The unit normal vector $\boldsymbol{n}$ on Σ is pointed outwards.

Given a time-dependent compactly supported current density $\boldsymbol{J}$, our aim is to find an electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and a magnetic field $\boldsymbol{H}(\boldsymbol{x},t)$ satisfying the following equations:

$$\begin{aligned}
\partial_t(\mu \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0} && \text{in } \mathbb{R}^3 \times (0,T), && (3.1)\\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} + \sigma \boldsymbol{E} && \text{in } \mathbb{R}^3 \times [0,T), && (3.2)\\
\mathrm{div}(\varepsilon \boldsymbol{E}) &= 0 && \text{in } \left(\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}\right) \times [0,T), && (3.3)\\
\int_{\Sigma_i} \varepsilon \boldsymbol{E} \cdot \boldsymbol{n} &= 0 && \text{in } [0,T), \quad i = 0, \cdots, I, && (3.4)\\
\boldsymbol{H}(\boldsymbol{x},0) &= \boldsymbol{H}_0(\boldsymbol{x}) && \text{in } \mathbb{R}^3, && (3.5)
\end{aligned}$$

$$\boldsymbol{H}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \qquad (3.6)$$

where the asymptotic behavior (3.6) holds uniformly in $[0,T]$. The electric permittivity ε, the electric conductivity σ, and the magnetic permeability μ are piecewise smooth real valued functions satisfying:

$$\begin{aligned}
\varepsilon_1 \geq \varepsilon(\boldsymbol{x}) \geq \varepsilon_0 > 0 \quad &\text{a.e. in } \Omega_{\mathrm{c}} \quad \text{and} \quad \varepsilon(\boldsymbol{x}) = \varepsilon_0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_{\mathrm{c}},\\
\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad &\text{a.e. in } \Omega_{\mathrm{c}} \quad \text{and} \quad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_{\mathrm{c}},\\
\mu_1 \geq \mu(\boldsymbol{x}) \geq \mu_0 > 0 \quad &\text{a.e. in } \Omega_{\mathrm{c}} \quad \text{and} \quad \mu(\boldsymbol{x}) = \mu_0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_{\mathrm{c}}.
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^3$ be a connected and simply connected polyhedron with a connected boundary $\Gamma := \partial\Omega$ and such that $\overline{\Omega}_{\mathrm{c}} \cup \operatorname{supp} \boldsymbol{J} \subset \Omega$. We introduce $\Omega_{\mathrm{d}} := \Omega \setminus \overline{\Omega}_{\mathrm{c}}$ and $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$. We also denote by $\boldsymbol{n}$ the outward normal unit vector on Γ. It is important to notice that, since $\sigma = 0$ in Ω_{d}, (3.2) implies that $\boldsymbol{J}$ must satisfy the compatibility conditions

$$\operatorname{div} \boldsymbol{J} = 0 \text{ in } \Omega_{\mathrm{d}} \qquad \text{and} \qquad \langle \gamma_{\boldsymbol{n}}(\boldsymbol{J}|_{\Omega_{\mathrm{d}}}), 1\rangle_{1/2,\Sigma_i} = 0, \quad i = 0, \ldots, I, \tag{3.7}$$

for all $t \in (0,T)$.

For reasons that will be clear later, we need to consider a modified electric field. To this end, let us denote by Ω_{d}^i, $i = 0, \ldots, I$, the connected components of Ω_{d} with $\partial\Omega_{\mathrm{d}}^i = \Sigma_i$, $i = 1, \ldots, I$, and $\partial\Omega_{\mathrm{d}}^0 = \Gamma \cup \Sigma_0$. We introduce the function

$$F := \begin{cases} \boldsymbol{0} & \text{in } \Omega_{\mathrm{c}} \cup \Omega_{\mathrm{d}}^1 \cup \cdots \cup \Omega_{\mathrm{d}}^I,\\ \psi & \text{in } \Omega_{\mathrm{d}}^0,\\ \psi_{\mathrm{ext}} & \text{in } \Omega'. \end{cases}$$

where $\psi \in \mathrm{H}^1(\Omega_{\mathrm{d}}^0)$ is the unique harmonic function satisfying $\gamma_{\boldsymbol{n}}(\mathbf{grad}\,\psi) = \gamma_{\boldsymbol{n}}\boldsymbol{E}$ on Γ and $\gamma(\psi) = 0$ on Σ_0 and ψ_{ext} is the unique harmonic function from

$$W^1(\Omega') := \left\{ \varphi \in \mathcal{D}'(\Omega'); \quad \frac{\varphi}{\sqrt{1+|\boldsymbol{x}|}} \in \mathrm{L}^2(\Omega'), \quad \mathbf{grad}\,\varphi \in \mathrm{L}^2(\Omega')^3 \right\}$$

satisfying the boundary condition $\gamma\psi_{\mathrm{ext}} = \gamma\psi$ on Γ. It turns out that the shifted electric field $\boldsymbol{E}^* := \boldsymbol{E} - \varepsilon_0\,\mathbf{grad}\,F$ and the magnetic field $\boldsymbol{H}$ solve the equations:

$$\begin{aligned}
\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E}^* &= \boldsymbol{0} && \text{in } \Omega \times (0,T),\\
\mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{J} + \sigma\boldsymbol{E}^* && \text{in } \Omega \times [0,T),\\
\operatorname{div}(\varepsilon_0\boldsymbol{E}^*) &= 0 && \text{in } \Omega_{\mathrm{d}} \times [0,T),\\
\textstyle\int_{\Sigma_i} \varepsilon_0\boldsymbol{E}^* \cdot \boldsymbol{n} &= 0 && \text{in } [0,T),\ i = 0, \cdots, I,\\
\gamma_{\boldsymbol{n}}^-(\boldsymbol{E}^*) &= 0 && \text{on } \Gamma \times [0,T),\\
\boldsymbol{\gamma}_\tau^-(\boldsymbol{E}^*) &= \boldsymbol{\gamma}_\tau^+(\boldsymbol{E}^*) && \text{on } \Gamma \times [0,T),\\
\boldsymbol{\gamma}_\tau^-(\boldsymbol{H}) &= \boldsymbol{\gamma}_\tau^+(\boldsymbol{H}) && \text{on } \Gamma \times [0,T),\\
\partial_t(\mu_0\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E}^* &= \boldsymbol{0} && \text{in } \Omega' \times (0,T),\\
\mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{0} && \text{in } \Omega' \times [0,T),\\
\operatorname{div}(\varepsilon_0\boldsymbol{E}^*) &= 0 && \text{in } \Omega' \times [0,T),\\
\boldsymbol{H}(\boldsymbol{x},0) &= \boldsymbol{H}_0(\boldsymbol{x}) && \text{in } \mathbb{R}^3,\\
\boldsymbol{H}(\boldsymbol{x},t) = O(1/|\boldsymbol{x}|) \quad &\text{and} \quad \boldsymbol{E}^*(\boldsymbol{x},t) = O(1/|\boldsymbol{x}|) && \text{as } |\boldsymbol{x}| \to \infty.
\end{aligned} \tag{3.8}$$

It is important to notice that the change of variable leaves the electric field unchanged in the conductor since $\boldsymbol{E}^* = \boldsymbol{E}$ in Ω_c. In the equations above, $\boldsymbol{\gamma}_\tau^+$ refers to the tangential trace on Γ taken from Ω' and $\boldsymbol{\gamma}_\tau^-$ to the tangential trace taken from Ω. We adopt the same convention for any other kind of trace operator.

In order to obtain a suitable variational formulation for the previous problem, we proceed as in [1, Section 3] and introduce the variable $\boldsymbol{u}(\boldsymbol{x},t) := \int_0^t \boldsymbol{E}^*(\boldsymbol{x},s)\,ds$. Next, we integrate the first equation of (3.8) with respect to t to obtain the expression $\boldsymbol{H} = -\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} + \boldsymbol{H}_0$ of the magnetic field in terms of $\boldsymbol{u}$. This leads us to the following formulation of the problem:

$$
\begin{aligned}
&\text{Find } \boldsymbol{u} : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3 \text{ such that:}\\
\sigma\partial_t \boldsymbol{u} + \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} &= \boldsymbol{f} && \text{in } \Omega\times(0,T),\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega_{\mathrm{d}}\times[0,T),\\
\textstyle\int_{\Sigma_i}\varepsilon_0\boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{in } [0,T),\ \ i=0,\cdots,I,\\
\boldsymbol{u}(\boldsymbol{x},0) &= \mathbf{0} && \text{in } \mathbb{R}^3\\
\gamma_{\boldsymbol{n}}^-(\boldsymbol{u}) &= 0 && \text{on } \Gamma\times[0,T),\\
\boldsymbol{\pi}_\tau^+\boldsymbol{u} &= \boldsymbol{\pi}_\tau^-\boldsymbol{u} && \text{on } \Gamma\times[0,T),\\
\boldsymbol{\gamma}_\tau^-\left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}\right) &= \boldsymbol{\gamma}_\tau^+\left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}\right) && \text{on } \Gamma\times[0,T),\\
\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} &= \mathbf{0} && \text{in } \Omega'\times[0,T),\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega'\times[0,T),\\
\boldsymbol{u}(\boldsymbol{x},t) &= O\left(1/|\boldsymbol{x}|\right) && \text{as } |\boldsymbol{x}|\to\infty,\\
\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x},t) &= O\left(1/|\boldsymbol{x}|\right) && \text{as } |\boldsymbol{x}|\to\infty,
\end{aligned}
\tag{3.9}
$$

where

$$\boldsymbol{f} := \mathbf{curl}\,\boldsymbol{H}_0 - \boldsymbol{J}. \tag{3.10}$$

We assume that both $\boldsymbol{J}$ and $\mathbf{curl}\,\boldsymbol{H}_0$ belong to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Hence, the right handside $\boldsymbol{f}$ also belongs to the same space. Moreover, we deduce from (3.7) and (3.10) that $\boldsymbol{f}$ inherits from $\boldsymbol{J}$ the same compatibility conditions, i.e.,

$$\operatorname{div}\boldsymbol{f} = 0 \text{ in } \Omega_{\mathrm{d}} \qquad \text{and} \qquad \langle\gamma_{\boldsymbol{n}}(\boldsymbol{f}|_{\Omega_{\mathrm{d}}}),1\rangle_{1/2,\Sigma_i} = 0, \quad i=0,\dots,I, \tag{3.11}$$

for all $t\in(0,T)$. Let us also remark that equation (3.2) provides at the initial time $t=0$ the relation

$$\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}(\boldsymbol{x},0) + \sigma(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},0) \quad \text{in } \mathbb{R}^3. \tag{3.12}$$

It then follows from our hypotheses on $\boldsymbol{J}$ and σ that the support of $\boldsymbol{f}$ is compact and contained in Ω.

4 The variational formulation

4.1 A mixed formulation in Ω

We introduce the space

$$M(\Omega_{\mathrm{d}}) := \left\{ q\in \mathrm{H}^1(\Omega_{\mathrm{d}}) : \quad \int_{\Omega_{\mathrm{d}}^i} q = 0, \quad \text{and} \quad \gamma q|_{\Sigma_i} = C_i, \quad i=0,\dots,I \right\}.$$

It is well known that $|\cdot|_{1,\Omega_{\mathrm{d}}}$ is a norm in $M(\Omega_{\mathrm{d}})$ equivalent to the $\mathrm{H}^1(\Omega_{\mathrm{d}})$-norm. Let us consider now the kernel

$$V(\Omega) := \{\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega) : \ b(\boldsymbol{v},q) = 0 \quad \forall q\in M(\Omega_{\mathrm{d}})\} \tag{4.1}$$

of the bilinear form

$$b(\boldsymbol{v},q) := (\varepsilon\boldsymbol{v},\mathbf{grad}\,q)_{0,\Omega_{\mathrm{d}}}\,.$$

Taking into account that ε is constant in $\mathbb{R}^3\setminus\overline{\Omega}_{\mathrm{c}}$, it straightforward to obtain the following characterization of $V(\Omega)$.

Lemma 5 *There holds*

$$V(\Omega) = \left\{ \boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega) : \ \operatorname{div}\boldsymbol{v} = 0 \ \textit{in } \Omega_{\mathrm{d}}; \quad \gamma_{\boldsymbol{n}}\boldsymbol{v} = 0 \ \textit{on } \Gamma; \quad \langle\gamma_{\boldsymbol{n}}\boldsymbol{v},1\rangle_{1/2,\Sigma_i} = 0,\ i=0,\dots,I \right\}.$$

Let $\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'$ be the dual space of $\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})$ with respect to the pivot space

$$\mathrm{L}^2(\Omega_{\mathrm{c}},\sigma)^3 := \left\{ \boldsymbol{v} : \Omega_{\mathrm{c}} \to \mathbb{R}^3 \text{ Lebesgue-measurable} : \int_{\Omega_{\mathrm{c}}} \sigma\,|\boldsymbol{v}|^2 < \infty \right\}.$$

We define

$$\mathcal{W}_0 := \left\{ \boldsymbol{v} \in \mathrm{L}^2(0,T;V(\Omega)) : \quad \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})) \right\},$$

with

$$W^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})) : \quad \partial_t \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})') \right\}.$$

We also introduce

$$\mathcal{W} := \left\{ \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})) \right\}.$$

Notice that $\mathcal{W}$, endowed with the graph norm

$$\|\boldsymbol{v}\|^2_{\mathcal{W}} := \int_0^T \|\boldsymbol{v}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt + \int_0^T \|\partial_t \boldsymbol{v}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}})'}\,dt,$$

is a Hilbert space and that $\mathcal{W}_0$ is a closed subspace of $\mathcal{W}$.

We test the first equation of (3.9) with $\boldsymbol{v} \in V(\Omega)$ and use the Green formula (2.5) to obtain the following variational formulation:

Find $\boldsymbol{u} \in \mathcal{W}_0$ such that

$$\frac{d}{dt}\left(\sigma \boldsymbol{u}(t), \boldsymbol{v}\right)_{0,\Omega_{\mathrm{c}}} + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t), \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} - \left\langle \boldsymbol{\gamma}_\tau\left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\right), \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} = \left(\boldsymbol{f}(t), \boldsymbol{v}\right)_{0,\Omega}$$

for all $\boldsymbol{v} \in V(\Omega)$. Next, we introduce a Lagrange multiplier $p(t)$ to relax the divergence-free restriction (implicit in the definition of $V(\Omega)$) and end up with the mixed variational formulation:

Find $\boldsymbol{u} \in \mathcal{W}$ and $p \in \mathrm{L}^2(0,T;M(\Omega_{\mathrm{d}}))$ such that

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},p(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} - \left\langle \boldsymbol{\gamma}_\tau^-\left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\right), \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} &= \left(\boldsymbol{f}(t),\boldsymbol{v}\right)_{0,\Omega} \\ b(\boldsymbol{u}(t),q) &= 0 \\ \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) &= \mathbf{0}, \end{aligned} \tag{4.2}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$ and for all $q \in M(\Omega_{\mathrm{d}})$. Finally, testing $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} = \mathbf{0}$ with $\mathbf{grad}\,r$, $r \in \mathrm{H}^1(\Omega')$, and applying again (2.5) we deduce that

$$\mathrm{div}_\Gamma\left[\boldsymbol{\gamma}_\tau^+\left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}\right)\right] = 0.$$

Consequently, Corollary 4 shows that there exists a unique $\lambda(t) \in \mathrm{H}_0^{1/2}(\Gamma)$ such that

$$\boldsymbol{\gamma}_\tau^-\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\right) = \mathbf{curl}_\Gamma\,\lambda(t) \qquad \text{on } \Gamma \quad \text{for a.e. } t \in (0,T). \tag{4.3}$$

With the last identity at hand, we can rewrite (4.2) as follows:

Find $\boldsymbol{u} \in \mathcal{W}$ and $p \in \mathrm{L}^2(0,T;M(\Omega_{\mathrm{d}}))$ such that

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},p(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} - \left\langle \mathbf{curl}_\Gamma\,\lambda, \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} &= \left(\boldsymbol{f}(t),\boldsymbol{v}\right)_{0,\Omega} \\ b(\boldsymbol{u}(t),q) &= 0 \\ \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) &= \mathbf{0}, \end{aligned} \tag{4.4}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$ and for all $q \in M(\Omega_{\mathrm{d}})$.

4.2 Non local boundary conditions on Γ

We deduce from the last four equations of (3.9) that $\boldsymbol{u}$ admits the following integral representation; see for instance [16, Section 5]:

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau^+ \boldsymbol{u}\, dS_{\boldsymbol{y}} - \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{\gamma}_\tau^+ (\mathbf{curl}\, \boldsymbol{u})\; dS_{\boldsymbol{y}} - \mathbf{grad}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \gamma_{\boldsymbol{n}}^+ \boldsymbol{u}\, dS_{\boldsymbol{y}} \tag{4.5}$$

for any $\boldsymbol{x} \in \Omega'$. Here, E is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, i.e.,

$$E(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}, \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3,\ \boldsymbol{x} \neq \boldsymbol{y}.$$

We will make repeated use of the integral operators formally defined below, for smooth densities $\phi:\ \Gamma \to \mathbb{R}$ and $\boldsymbol{\eta}:\ \Gamma \to \mathbb{R}^3$, by:

$$\begin{aligned}
S\phi(\boldsymbol{x}) &:= \gamma\left(\boldsymbol{x} \mapsto \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\phi(y)\, dS_{\boldsymbol{y}}\right),\\
\boldsymbol{V\eta}(\boldsymbol{x}) &:= \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\eta}(y)\, dS_{\boldsymbol{y}}\right),\\
\boldsymbol{K\eta}(\boldsymbol{x}) &:= \boldsymbol{\gamma}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\eta}(y)\, dS_{\boldsymbol{y}}\right),\\
\boldsymbol{K}^*\boldsymbol{\eta}(\boldsymbol{x}) &:= \boldsymbol{\pi}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\eta}(\boldsymbol{y})\, dS_{\boldsymbol{y}}\right) - \boldsymbol{\eta}(\boldsymbol{x}),\\
\boldsymbol{W\eta}(\boldsymbol{x}) &:= \boldsymbol{\gamma}_\tau^+ \left[\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \left(\mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\eta}(\boldsymbol{y})\, dS_{\boldsymbol{y}}\right)\right].
\end{aligned}$$

In the following theorem we summarize some fundamental tools concerning the properties of these integral operators when mapping between Sobolev spaces.

Theorem 6 *The linear mappings*

$$S : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma), \quad \boldsymbol{V} : \mathbf{H}_\parallel^{-1/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma), \quad \boldsymbol{K} : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma),$$
$$\boldsymbol{K}^* : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma), \quad \boldsymbol{W} : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$$

are bounded and satisfy the following properties:

- *There exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that:*

$$\langle \phi, S\phi\rangle_{1/2,\Gamma} \geq \alpha_1 \|\phi\|^2_{-1/2,\Gamma} \qquad \forall \phi \in \mathrm{H}^{-1/2}(\Gamma) \tag{4.6}$$

and

$$\langle \boldsymbol{\eta}, \boldsymbol{V\eta}\rangle_{\tau,\Gamma} \geq \alpha_2 \|\boldsymbol{\eta}\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma). \tag{4.7}$$

- *The operator $\boldsymbol{W}$ is related to S through the following identity:*

$$\langle \boldsymbol{W\lambda}, \boldsymbol{\eta}\rangle_{\tau,\Gamma} = -\langle \mathrm{curl}_\Gamma \boldsymbol{\eta}, S(\mathrm{curl}_\Gamma \boldsymbol{\lambda})\rangle_{1/2,\Gamma} \qquad \forall \boldsymbol{\lambda},\ \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma). \tag{4.8}$$

- *The operator $\boldsymbol{K}^*$ is the transpose of $\boldsymbol{K}$, i.e.,*

$$\langle \boldsymbol{K\eta}, \boldsymbol{\xi}\rangle_{\tau,\Gamma} = \langle \boldsymbol{\eta}, \boldsymbol{K}^*\boldsymbol{\xi}\rangle_{\tau,\Gamma} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma),\ \forall \boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma). \tag{4.9}$$

Proof. See Theorems 6.1, 6.2 and 6.3 of [16]. □

Finally, we will need the following result proved in Lemma 2.3 of [18].

Lemma 7 *For* $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ *we have that*

$$\mathrm{div}\left(\boldsymbol{x} \mapsto \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\eta}(\boldsymbol{y})\,dS_{\boldsymbol{y}}\right) = \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\mathrm{div}_\Gamma\boldsymbol{\eta}(\boldsymbol{y})\,dS_{\boldsymbol{y}} \qquad \textit{in } L^2(\mathbb{R}^3).$$

A coupled FEM-BEM formulation of (3.9) is obtained by relating the mixed formulation (4.4) of the interior problem with (4.5) through the transmission conditions on Γ. We begin by applying $\boldsymbol{\gamma}_\tau^+ \circ \mu_0^{-1}\,\mathbf{curl}$ to (4.5) and using (4.3) to obtain

$$\mathbf{curl}_\Gamma\,\lambda = \mu_0^{-1}\boldsymbol{W}\boldsymbol{\pi}_\tau^+\boldsymbol{u} - \boldsymbol{K}\left(\mathbf{curl}_\Gamma\,\lambda\right). \tag{4.10}$$

Next, we take the tangential trace $\boldsymbol{\pi}_\tau^+$ of both sides of (4.5) to derive

$$\boldsymbol{\pi}_\tau^+\boldsymbol{u} = \boldsymbol{\pi}_\tau^+\left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}}\int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{n}\times\boldsymbol{\pi}_\tau^+\boldsymbol{u}\,dS_{\boldsymbol{y}}\right) - \boldsymbol{V}\boldsymbol{\gamma}_\tau^+(\mathbf{curl}\,\boldsymbol{u}) - \mathbf{grad}_\Gamma\,S\gamma_{\boldsymbol{n}}^+\boldsymbol{u},$$

or equivalently

$$\boldsymbol{K}^*\left(\mu_0^{-1}\boldsymbol{\pi}_\tau^+\boldsymbol{u}\right) - \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda) - \mu_0^{-1}\,\mathbf{grad}_\Gamma\,S\gamma_{\boldsymbol{n}}^+\boldsymbol{u} = \mathbf{0}.$$

Testing the previous equation with $\mathrm{curl}_\Gamma\boldsymbol{\eta}$, $\boldsymbol{\eta} \in \mathrm{H}_0^{1/2}(\Gamma)$, yields

$$-\left\langle \mathbf{curl}_\Gamma\,\eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\,\eta), \boldsymbol{\pi}_\tau\boldsymbol{u}\right\rangle_{\tau,\Gamma} = 0 \qquad \forall\eta \in \mathrm{H}_0^{1/2}(\Gamma).$$

Combining the last identity with (4.4) and (4.10), we obtain a symmetric mixed-FEM and BEM coupling for our problem:

Find $\boldsymbol{u} \in \mathcal{W}$, $p \in \mathrm{L}^2(0,T;M(\Omega_\mathrm{d}))$ and $\lambda \in \mathrm{L}^2(0,T;\mathrm{H}_0^{1/2}(\Gamma))$ such that

$$\begin{aligned}
&\frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},p(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} \\
&\qquad +\mu_0^{-1}\left\langle S(\mathrm{curl}_\Gamma\boldsymbol{\pi}_\tau\boldsymbol{u}),\mathrm{curl}_\Gamma\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{1/2,\Gamma} + \left\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\,\lambda(t),\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{\tau,\Gamma} = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega}\,, \\
&\qquad -\left\langle \mathbf{curl}_\Gamma\,\eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\,\eta),\boldsymbol{\pi}_\tau\boldsymbol{u}\right\rangle_{\tau,\Gamma} = 0, \\
&\qquad b(\boldsymbol{u}(t),q) = 0, \\
&\qquad \boldsymbol{u}|_{\Omega_\mathrm{c}}(0) = \mathbf{0},
\end{aligned} \tag{4.11}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$, $\eta \in \mathrm{H}_0^{1/2}(\Gamma)$ and $q \in M(\Omega_\mathrm{d})$.

In the sequel, for the theoretical analysis, it will be convenient to eliminate the boundary variable λ from the previous formulation. To this end, we introduce the operator $R : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}_0^{1/2}(\Gamma)$ characterized by

$$\left\langle \mathbf{curl}_\Gamma\,\chi, \boldsymbol{V}(\mathbf{curl}_\Gamma\,R\xi)\right\rangle_{\tau,\Gamma} = \left\langle \xi,\chi\right\rangle_{1/2,\Gamma} \qquad \forall\chi \in \mathrm{H}_0^{1/2}(\Gamma) \quad \forall\xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{4.12}$$

It is straightforward to deduce from Corollary 4, Theorem 6 and the Lax-Milgramm lemma that R is well-defined and bounded. Furthermore, the second equation of (4.11) may be equivalently written $\lambda = \mu_0^{-1}R(\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{u})$. Consequently, (4.11) admits the following equivalent reduced form:

Find $\boldsymbol{u} \in \mathcal{W}$, $p \in \mathrm{L}^2(0,T;M(\Omega_\mathrm{d}))$ such that:

$$\begin{aligned}
\frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},p(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} + c(\boldsymbol{u},\boldsymbol{v}) &= (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega} \quad \forall\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega), \\
b(\boldsymbol{u}(t),q) &= 0 \qquad\qquad\quad \forall q \in M(\Omega_\mathrm{d}), \\
\boldsymbol{u}|_{\Omega_\mathrm{c}}(0) &= \mathbf{0},
\end{aligned} \tag{4.13}$$

where $c(\cdot,\cdot) : \mathbf{H}(\mathbf{curl};\Omega) \times \mathbf{H}(\mathbf{curl};\Omega) \to \mathbb{R}$ is the bounded, symmetric and nonnegative bilinear form given by

$$c(\boldsymbol{u},\boldsymbol{v}) := \mu_0^{-1}\left\langle \left(\mathbf{curl}_\Gamma\,S\mathrm{curl}_\Gamma + \boldsymbol{K}\,\mathbf{curl}_\Gamma\,R\mathrm{curl}_\Gamma\boldsymbol{K}^*\right)\boldsymbol{\pi}_\tau\boldsymbol{u},\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{\tau,\Gamma} \qquad \forall\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega).$$

5 Existence and uniqueness.

From now on, we assume that $\Omega_{\rm d}$ satisfies the following topological assumption, which is necessary to prove Corollary 10 below: there exists a set $\{\omega_j,\, j=1,\dots,J\}$ of admissible cuts of $\Omega_{\rm d}$ such that $\cup_{j=1}^J \partial\omega_j \subset \Sigma$ and any connected component of

$$\Omega_{\rm d}^0 := \Omega_{\rm d} \setminus \left(\cup_{j=1}^J \omega_j\right)$$

is simply connected. This assumption is satisfied for any geometry in practice.

We introduce the space

$$V(\Omega_{\rm d}) := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm d}) : \quad \boldsymbol{\gamma}_\tau \boldsymbol{v} = 0 \text{ on } \Sigma; \quad b(\boldsymbol{v}, q) = 0 \quad \forall q \in M(\Omega_{\rm d})\}$$

Notice that, as $\varepsilon(\boldsymbol{x}) = \varepsilon_0$ for all $\boldsymbol{x} \in \Omega_{\rm d}$,

$$V(\Omega_{\rm d}) = \Big\{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm d}) : \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega_{\rm d}, \quad \boldsymbol{\gamma}_\tau \boldsymbol{v} = 0 \text{ on } \Sigma, \quad \gamma_{\boldsymbol{n}} \boldsymbol{v} = 0 \text{ on } \Gamma,$$
$$\langle \gamma_{\boldsymbol{n}} \boldsymbol{v}, 1\rangle_{1/2,\Sigma_i} = 0, \quad i = 0,\dots,I\Big\}.$$

Let us clear up here that the shifted electric field $\boldsymbol{E}^*$ has been introduced in order to obtain a variable $\boldsymbol{u}$ with a vanishing normal component on Γ. This boundary condition will play a central role in the proof of the following Lemma.

Lemma 8 *The embedding of* $\mathrm{L}^2(\Omega_{\rm d})^3$ *into* $V(\Omega_{\rm d})$ *is compact.*

Proof. It is well know that the spaces $\mathbf{H}_0(\mathbf{curl};\Omega_{\rm d}) \cap \mathrm{H}(\operatorname{div};\Omega_{\rm d})$ and $\mathbf{H}(\mathbf{curl};\Omega_{\rm d}) \cap \mathrm{H}_0(\operatorname{div};\Omega_{\rm d})$ are continuously embedded in $\mathrm{H}^s(\Omega_{\rm d})^3$, for some $s > 1/2$; see, [6, Proposition 3.7]. Let $\psi \in C_0^\infty(\Omega)$ be such that $0 \le \psi \le 1$ and $\psi \equiv 1$ in $\overline{\Omega}_{\rm c}$. Notice that $\boldsymbol{v} = \psi\boldsymbol{v} + (1-\psi)\boldsymbol{v}$ for any $\boldsymbol{v} \in V(\Omega_{\rm d})$ with

$$\psi\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\rm d}) \cap \mathrm{H}(\operatorname{div};\Omega_{\rm d}) \quad \text{and} \quad (1-\psi)\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm d}) \cap \mathrm{H}_0(\operatorname{div};\Omega_{\rm d}).$$

Hence, $\boldsymbol{v} \in \mathrm{H}^s(\Omega_{\rm d})^3$ and there exists $C > 0$ (depending only on $\Omega_{\rm d}$ and ψ) such that

$$\begin{aligned}
\|\boldsymbol{v}\|_{s,\Omega_{\rm d}} &\le \|\psi\boldsymbol{v}\|_{s,\Omega_{\rm d}} + \|(1-\psi)\boldsymbol{v}\|_{s,\Omega_{\rm d}} \\
&\le C_0\left(\|\psi\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\rm d})} + \|(1-\psi)\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\rm d})}\right) \le C\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\rm d})}.
\end{aligned}$$

The result follows now from the compactness of the embedding $\mathrm{L}^2(\Omega_{\rm d})^3 \hookrightarrow \mathrm{H}^s(\Omega_{\rm d})^3$. □

We now recall the well-known Petree-Tartar Lemma; see, for instance [15, Chapter I, Theorem 2.1].

Lemma 9 *Let X, Y and Z be three Banach spaces. Let $A : X \to Y$ and $T : X \to Z$ be linear and bounded operators, with A injective and T compact. If there exists $\kappa > 0$ such that $\kappa\|x\|_X \le \|Ax\|_Y + \|Tx\|_Z$ for any $x \in X$, then there exists $\alpha > 0$ such that $\alpha\|x\|_X \le \|Ax\|_Y$ for any $x \in X$.*

Corollary 10 *On the space $V(\Omega_{\rm d})$, the seminorm $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm d}}$ is equivalent to the $\mathbf{H}(\mathbf{curl};\Omega_{\rm d})$-norm.*

Proof. We apply Lemma 9 with $X := V(\Omega_{\rm d})$ (endowed with the $\mathbf{H}(\mathbf{curl};\Omega_{\rm d})$-norm) and $Y = Z = \mathrm{L}^2(\Omega_{\rm d})^3$ and with bounded operators $A : X \to Y$ and $T : X \to Z$ given by

$$A\boldsymbol{v} := \mathbf{curl}\,\boldsymbol{v}, \quad T\boldsymbol{v} := \boldsymbol{v} \qquad \forall \boldsymbol{v} \in V(\Omega_{\rm d}).$$

As T is compact (cf. Lemma 8), we only need to prove that A is injective.

Thanks to our topological hypothesis on Ω_d (see the beginning of this section) we know from [6, Subsection 3.5] that for any $\boldsymbol{v} \in V(\Omega_{\rm d})$ there exists a unique vector potential $\psi \in \mathbf{H}(\mathbf{curl};\Omega_{\rm d}) \cap \mathrm{H}(\operatorname{div};\Omega_{\rm d})$ such that:

$$\begin{gathered}
\boldsymbol{v} = \mathbf{curl}\,\psi \text{ in } \Omega_{\rm d}, \quad \operatorname{div}\psi = 0 \text{ in } \Omega_{\rm d} \quad \boldsymbol{\gamma}_\tau\psi = \mathbf{0} \text{ on } \Gamma, \quad \gamma_{\boldsymbol{n}}\psi = 0 \text{ on } \Sigma, \\
\langle \gamma_{\boldsymbol{n}}\psi, 1\rangle_{1/2,\Gamma} = \langle \gamma_{\boldsymbol{n}}\psi, 1\rangle_{1/2,\omega_j} = 0, \qquad j = 1,\dots,J.
\end{gathered}$$

Hence, if $\mathbf{curl}\,\boldsymbol{v} = \mathbf{0}$,

$$\int_{\Omega_{\rm d}} \boldsymbol{v}\cdot\boldsymbol{v} = \int_{\Omega_{\rm d}} \boldsymbol{v}\cdot\mathbf{curl}\,\psi = \langle \boldsymbol{\gamma}_\tau\boldsymbol{v}, \boldsymbol{\pi}_\tau\psi\rangle_{\tau,\Gamma} + \langle \boldsymbol{\gamma}_\tau\boldsymbol{v}, \boldsymbol{\pi}_\tau\psi\rangle_{\tau,\Sigma} = 0,$$

which implies that $\boldsymbol{v} = \mathbf{0}$ and the result follows. □

With the aid of the last result, the proofs of the next two lemmas are similar to the corresponding ones from Section 4 of [1].

Lemma 11 *The linear mapping* $\mathcal{E}: \mathbf{H}(\mathbf{curl};\Omega_{\rm c}) \to V(\Omega)$ *characterized, for any* $\boldsymbol{v}_{\rm c} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm c})$, *by* $(\mathcal{E}\boldsymbol{v}_{\rm c})|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$\mu_0^{-1}\left(\mathbf{curl}\,\mathcal{E}\boldsymbol{v}_{\rm c}, \mathbf{curl}\,\boldsymbol{w}\right)_{0,\Omega_{\rm d}} + c(\mathcal{E}\boldsymbol{v}_{\rm c}, \boldsymbol{w}) = 0 \qquad \forall \boldsymbol{w} \in V(\Omega_{\rm d}) \tag{5.1}$$

is well defined and bounded.

Lemma 12 *The inner product in* $V(\Omega)$

$$(\boldsymbol{u},\boldsymbol{v})_{V(\Omega)} := (\boldsymbol{u},\boldsymbol{v})_\sigma + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} + c(\boldsymbol{u},\boldsymbol{v}) \tag{5.2}$$

induces a norm $\|\cdot\|_{V(\Omega)}$ *that is equivalent to the* $\mathbf{H}(\mathbf{curl};\Omega)$ *norm in* $V(\Omega)$. *Moreover, the following decomposition is orthogonal with respect to the inner product* $(\cdot,\cdot)_{V(\Omega)}$:

$$V(\Omega) = \widetilde{V(\Omega_{\rm d})} \oplus \mathcal{E}(\mathbf{H}(\mathbf{curl};\Omega_{\rm c})), \tag{5.3}$$

where $\widetilde{V(\Omega_{\rm d})}$ *is the subspace of* $V(\Omega)$ *obtained by extending by zero the functions of* $V(\Omega_{\rm d})$ *to the whole domain* Ω.

Theorem 13 *Problem* (4.13) *has a unique solution* $(\boldsymbol{u}, p)$ *and*

$$\max_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{0,\Omega_{\rm c}} + \int_0^T \|\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt \le C\int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt, \tag{5.4}$$

for some constant $C>0$. *Moreover, if we define* $\lambda = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u})$ *then* $(\boldsymbol{u}, \lambda, p)$ *is the unique solution of Problem* (4.11).

Proof. The second equation of (4.13) means that $\boldsymbol{u} \in \mathcal{W}_0$. Hence, we can apply the orthogonal decomposition (5.3) to write that $\boldsymbol{u} = \boldsymbol{u}_{\rm d} + \mathcal{E}\boldsymbol{u}_{\rm c}$, with $\boldsymbol{u}_{\rm d} \in \mathrm{L}^2(0,T;\widetilde{V(\Omega_{\rm d})})$ and $\mathcal{E}\boldsymbol{u}_{\rm c} \in \mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl};\Omega_{\rm c})))$. It is easy to show that the first component $\boldsymbol{u}_{\rm d}(t)$ of this decomposition solves the elliptic problem

$$\mu_0^{-1}\left(\mathbf{curl}\,\boldsymbol{u}_{\rm d}(t), \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\rm d}} + c(\boldsymbol{u}_{\rm d}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega_{\rm d}} \qquad \forall \boldsymbol{v} \in V(\Omega_{\rm d}), \tag{5.5}$$

for a.e. t. On the other hand, $\boldsymbol{u}_{\rm c}$ satisfies the parabolic equation

$$\frac{d}{dt}(\boldsymbol{u}_{\rm c}(t), \boldsymbol{v})_\sigma + \left(\mu^{-1}\,\mathbf{curl}\,\mathcal{E}\boldsymbol{u}_{\rm c}(t), \mathbf{curl}\,\mathcal{E}\boldsymbol{v}\right)_{0,\Omega} + c(\mathcal{E}\boldsymbol{u}_{\rm c}(t), \mathcal{E}\boldsymbol{v}) = (\boldsymbol{f}(t), \mathcal{E}\boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\rm c}),$$

with the initial condition $\boldsymbol{u}_{\rm c}(0) = \mathbf{0}$. Now, using that $c(\cdot,\cdot)$ is nonnegative, we can proceed exactly as in [1, Theorem 4.4] to prove the existence and uniqueness of $\boldsymbol{u}_{\rm c}$ and $\boldsymbol{u}_{\rm d}$.

Notice that, for any $q \in M(\Omega_{\rm d})$, the extension by zero of $\mathbf{grad}\,q$ to the whole Ω belongs to $\mathbf{H}(\mathbf{curl};\Omega)$. Hence, we deduce that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\sup_{\boldsymbol{z}\in\mathbf{H}(\mathbf{curl};\Omega)} \frac{b(\boldsymbol{z}, q)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)}} \ge \frac{b(\mathbf{grad}\,q, q)}{\|\,\mathbf{grad}\,q\|_{\mathbf{H}(\mathbf{curl};\Omega)}} = \varepsilon_0 |q|_{1,\Omega_{\rm d}} \qquad \forall q \in M(\Omega_{\rm d}) \tag{5.6}$$

and a similar reasoning to the one presented in [1, Theorem 4.4] proves that there exists a unique $p(t) \in M(\Omega_d)$ satisfying

$$b(\boldsymbol{v}, p(t)) = \langle \mathcal{G}(t), \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega) \tag{5.7}$$

for all $t \in [0,T]$, where $\mathcal{G} \in \mathcal{C}^0([0,T], \mathbf{H}(\mathbf{curl};\Omega)')$ is given by

$$\langle \mathcal{G}(t), \boldsymbol{v}\rangle := -(\boldsymbol{u}(t), \boldsymbol{v})_\sigma - \int_0^t \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(s), \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} ds - \int_0^t c(\boldsymbol{u}(s), \boldsymbol{v})\,ds + \int_0^t (\boldsymbol{f}(s), \boldsymbol{v})_{0,\Omega}\,ds.$$

We conclude that $(\boldsymbol{u}, p)$ solves (4.13) by differentiating the last identity with respect to t in the sense of distributions.

The last assertion of the theorem follows directly from the definition of R. $\square$

Lemma 14 *The Lagrange multiplier p of Problem (4.11) vanishes identically.*

Proof. Testing the first equation of (4.11) with $\mathbf{grad}\, q$ yields

$$\frac{d}{dt} b(\mathbf{grad}\, q, p(t)) + \langle \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda(t), \mathbf{grad}_\Gamma\, q\rangle_{\tau,\Gamma} = (\boldsymbol{f}(t), \mathbf{grad}\, q)_{0,\Omega_{\mathrm{d}}} = 0,$$

where the last equality follows from the compatibility conditions (3.7). Moreover, as $\mathrm{div}_\Gamma \boldsymbol{\gamma}_\tau \boldsymbol{q} = \mathbf{curl}\, \boldsymbol{q} \cdot \boldsymbol{n}$ in $H^{-1/2}(\Gamma)$ for all $\boldsymbol{q} \in \mathbf{H}(\mathbf{curl}, \Omega')$, we have that

$$\mathrm{div}_\Gamma(\boldsymbol{K}\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}) := \mathrm{div}_\Gamma \boldsymbol{\gamma}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}(y)\, dS_{\boldsymbol{y}} \right)$$

$$= \mathbf{curl} \left(\mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}(y)\, dS_{\boldsymbol{y}} \right) \cdot \boldsymbol{n}.$$

Using the property $\mathbf{curl}\,\mathbf{curl} = -\boldsymbol{\Delta} + \mathbf{grad}\, \mathrm{div}$ together with Lemma 7 and the fact that $\boldsymbol{x} \mapsto E(\boldsymbol{x}, \boldsymbol{y})$ solves the Laplace equation in Ω' lead us to the identity

$$\mathbf{curl} \left(\mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}(y)\, dS_{\boldsymbol{y}} \right) = \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \mathrm{div}_\Gamma\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}(y)\, dS_{\boldsymbol{y}} = \mathbf{0} \quad \text{in } \Omega',$$

or equivalently,

$$\mathrm{div}_\Gamma(\boldsymbol{K}\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda}) = 0. \tag{5.8}$$

This means that $\frac{d}{dt} b(\mathbf{grad}\, q, p(t)) = 0$ for all $q \in M(\Omega_{\mathrm{d}})$. Next, taking $t = 0$ in (5.7) and using the fact that $\mathcal{G}(0) = \mathbf{0}$ we deduce that $t \mapsto b(\mathbf{grad}\, q, p(t))$ vanishes identically in $[0, T]$ for all $q \in M(\Omega_{\mathrm{d}})$. In particular $\varepsilon_0 |p(t)|^2_{1,\Omega_{\mathrm{d}}} = b(\mathbf{grad}\, p(t), p(t)) = 0$ for all $t \in [0, T]$, and the result follows. □

Remark 15 *As a consequence of* (3.12), $\boldsymbol{f}(\boldsymbol{x}, 0) := \mathbf{curl}\, \boldsymbol{H}_0 - \boldsymbol{J}(\boldsymbol{x}, 0) = \mathbf{0}$. *Hence, solving* (5.5) *at* $t = 0$ *shows that* $\boldsymbol{u}_{\mathrm{d}}(\boldsymbol{x}, 0) = 0$ *in* Ω_{d} *and then, the global initial condition*

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \mathbf{0} \quad \text{in } \Omega$$

holds true.

Theorem 16 *If* $(\boldsymbol{u}, \lambda, p)$ *is the solution of Problem* (4.11), *then*

$$\gamma_\tau \left(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u} \right) = \mathbf{curl}_\Gamma\, \lambda \qquad \text{in } \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma; \Gamma). \tag{5.9}$$

Proof. Testing the first equation of (4.11) with $\boldsymbol{v} \in C_0^\infty(\Omega_{\mathrm{d}})$ and using the previous lemma, we obtain

$$\mathbf{curl}(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})|_{\Omega_{\mathrm{d}}} = \boldsymbol{f}|_{\Omega_{\mathrm{d}}}.$$

Testing again the first equation of (4.11) with a function $\boldsymbol{v}$ that belongs to the space

$$\mathbf{H}_\Sigma(\mathbf{curl}; \Omega_{\mathrm{d}}) := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega_{\mathrm{d}}); \quad \boldsymbol{\gamma}_\tau \boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma \}$$

we obtain

$$\gamma_\tau(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u}) = \mu_0^{-1} \boldsymbol{W} \boldsymbol{\pi}_\tau \boldsymbol{u} - \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda \qquad \text{in } \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma; \Gamma). \tag{5.10}$$

Owing to (5.8) and (4.8) we deduce that

$$\mathrm{div}_\Gamma(\boldsymbol{\gamma}_\tau(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u})) = 0. \tag{5.11}$$

The second equation of (4.11) implies that $\boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda) - \mu_0^{-1} \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma; \Gamma) \cap \ker(\mathrm{curl}_\Gamma)$. Then, there exists $\varphi \in \mathrm{H}^{1/2}(\Gamma)$ such that (cf. Theorem 5.1 of [13])

$$\boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda) - \mu_0^{-1} \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u} = \mathbf{grad}_\Gamma\, \varphi.$$

According to the definition of $\boldsymbol{K}^*$, this equation may be written

$$\boldsymbol{\pi}_\tau \boldsymbol{u} = \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, dS_{\boldsymbol{y}} \right) - \mu_0 \boldsymbol{V}(\mathbf{curl}_\Gamma \lambda) + \mu_0\, \mathbf{grad}_\Gamma \varphi. \tag{5.12}$$

Let us now consider the unique harmonic function $\psi \in W^1(\Omega')$ satisfying the boundary condition $\psi = \varphi$ on Γ, and let $\boldsymbol{z} : \Omega' \to \mathbb{R}^3$ be given by

$$\boldsymbol{z}(\boldsymbol{x}) := \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, dS_{\boldsymbol{y}} - \mu_0 \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \mathbf{curl}_\Gamma \lambda(\boldsymbol{y})\, dS_{\boldsymbol{y}} + \mu_0\, \mathbf{grad}\, \psi. \tag{5.13}$$

We deduce from (5.12) and (5.10) that

$$\boldsymbol{\pi}_\tau \boldsymbol{z} = \boldsymbol{\pi}_\tau \boldsymbol{u} \quad \text{and} \quad \mu_0^{-1} \boldsymbol{\gamma}_\tau\, \mathbf{curl}\, \boldsymbol{z} = \mu_0^{-1} \boldsymbol{\gamma}_\tau\, \mathbf{curl}\, \boldsymbol{u}. \tag{5.14}$$

Moreover, (5.8) together with Lemma 7 show that $\operatorname{div} \boldsymbol{z} = 0$ in Ω' and $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{z} = (-\boldsymbol{\Delta} + \mathbf{grad}\operatorname{div})\boldsymbol{z} = \mathbf{0}$ in Ω'. Consequently, taking into account that $\boldsymbol{z}$ satisfies adequate asymptotic conditions at infinity, this function is also given by the following integral representation

$$\boldsymbol{z}(\boldsymbol{x}) = \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, dS_{\boldsymbol{y}} - \int_\Gamma E(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u}(\boldsymbol{y}))\, dS_{\boldsymbol{y}} + \mathbf{grad}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y}) \gamma_{\boldsymbol{n}} \boldsymbol{z}\, dS_{\boldsymbol{y}}.$$

Applying $\boldsymbol{\pi}_\tau$ to both sides of the previous equation yields

$$\boldsymbol{\pi}_\tau \boldsymbol{z} = \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, dS_{\boldsymbol{y}} \right) - \boldsymbol{V} \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u}) + \mathbf{grad}_\Gamma\, S(\gamma_{\boldsymbol{n}} \boldsymbol{z}).$$

Next, subtracting the last identity from (5.12) and using (5.14) provide

$$\boldsymbol{V}(\mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u})) = \mathbf{grad}_\Gamma (\mu_0 \varphi - S(\gamma_{\boldsymbol{n}} \boldsymbol{z})).$$

Finally, taking the duality product of this equation with $\mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u}) \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma 0; \Gamma)$ (cf. (5.11)) and using (4.6), give

$$\begin{aligned}\alpha_2 \|\mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u})\|^2_{\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma;\Gamma)} &\leq \langle \mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u}), \boldsymbol{V}(\mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u})) \rangle_{\tau,\Gamma} \\ &= \langle \mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau (\mathbf{curl}\, \boldsymbol{u}), \mathbf{grad}_\Gamma (\mu_0 \varphi - S \gamma_{\boldsymbol{n}} \boldsymbol{z}) \rangle_{\tau,\Gamma} = 0\end{aligned}$$

and the result follows. □

6 Analysis of the semi-discrete scheme

6.1 Well-posedness

Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of Ω such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_c$ or in $\overline{\Omega}_d$. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h$. Furthermore, we denote by $\{\mathcal{T}_h(\Sigma)\}_h$ and $\{\mathcal{T}_h(\Gamma)\}_h$ the families of triangulations induced by $\{\mathcal{T}_h\}_h$ on Σ and Γ respectively. We assume that $\{\mathcal{T}_h(\Sigma)\}_h$ is quasi-uniform. From now on C denotes a positive constant independent of h and that may take different values at different occurrences.

We define a semi-discrete version of (4.11) by means of Nédélec finite elements. The local representation of the mth-order element of this family on a tetrahedron K is given by $\mathcal{N}_m(K) := \mathbb{P}^3_{m-1} \oplus S_m$, where $\mathbb{P}_m$ is the set of polynomials of degree not greater than m and $S_m := \left\{ p \in \widetilde{\mathbb{P}}^3_m : \ \boldsymbol{x} \cdot p(\boldsymbol{x}) = 0 \right\}$, with $\widetilde{\mathbb{P}}_m$ being the set of homogeneous polynomials of degree m. The corresponding global space $X_h(\Omega)$ to approximate $\mathbf{H}(\mathbf{curl}; \Omega)$ is the space of functions that are locally in $\mathcal{N}_m(K)$ and have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$X_h(\Omega) := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \ \boldsymbol{v}|_K \in \mathcal{N}_m(K)\ \forall K \in \mathcal{T} \}.$$

On the other hand, we use standard mth-order Lagrange finite elements to approximate $M(\Omega_{\rm d})$ and $\mathrm{H}_0^{1/2}(\Gamma)$:

$$M_h(\Omega_{\rm d}) := \left\{ q \in \mathrm{H}^1(\Omega_{\rm d}) : \quad q|_K \in \mathbb{P}_m \quad \forall K \in \mathcal{T}_h, \quad \int_{\Omega_{\rm d}^i} q = 0, \quad q|_{\Sigma_i} = C_i,\ i = 0, \dots, I \right\}$$

and

$$\Lambda_h(\Gamma) := \left\{ \vartheta \in \mathrm{H}_0^{1/2}(\Gamma) : \quad \vartheta|_F \in \mathbb{P}_m \quad \forall F \in \mathcal{T}_h(\Gamma) \right\}.$$

We are now ready to introduce a semi-discretization of problem (4.11):

Find $\boldsymbol{u}_h(t) : [0,T] \to X_h(\Omega)$, $\lambda_h(t) : [0,T] \to \Lambda_h(\Gamma)$ and $p_h(t) : [0,T] \to M_h(\Omega_{\rm d})$ such that

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} & \\ +\mu_0^{-1}\left\langle S(\mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{u}_h), \mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{1/2,\Gamma} + \left\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\, \lambda_h(t), \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} &= (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega}, \\ -\left\langle \mathbf{curl}_\Gamma\, \eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda_h)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\, \eta), \boldsymbol{\pi}_\tau \boldsymbol{u}_h\right\rangle_{\tau,\Gamma} &= 0 \\ b(\boldsymbol{u}_h(t), q) &= 0, \\ \boldsymbol{u}_h|_{\Omega_{\rm c}}(0) &= \mathbf{0}. \end{aligned} \tag{6.1}$$

for all $\boldsymbol{v} \in X_h(\Omega)$, $\eta \in \Lambda_h(\Gamma)$ and $q \in M_h(\Omega_{\rm d})$.

Remark 17 *For piecewise smooth functions, the boundary integral operators in* (6.1) *are structurally equal to those for second order elliptic problems. The terms involving the operator S and $\boldsymbol{V}$ are immediately written in terms of integrals. The same happens with the terms involving $\boldsymbol{K}$. In fact, for any $\eta \in \Lambda_h(\Gamma)$ and $\boldsymbol{v} \in X_h(\Omega)$, we have ([16])*

$$\begin{aligned} \left\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\, \eta, \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} = &\int_\Gamma \int_\Gamma \mathbf{curl}_\Gamma\, \eta(\boldsymbol{y}) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x}) \frac{\partial E(\boldsymbol{x}, \boldsymbol{y})}{\partial \boldsymbol{n}(\boldsymbol{x})}\, dS_{\boldsymbol{y}}\, dS_{\boldsymbol{x}} \\ &+ \int_\Gamma \int_\Gamma \mathbf{grad}_{\boldsymbol{x}}\, E(\boldsymbol{x}, \boldsymbol{y})(\mathbf{curl}_\Gamma\, \eta(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{x})) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x})\, dS_{\boldsymbol{y}}\, dS_{\boldsymbol{x}} \\ &- \frac{1}{2}\int_\Gamma \mathbf{curl}_\Gamma\, \eta(\boldsymbol{x}) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x})\, dS_{\boldsymbol{x}}. \end{aligned}$$

We proceed as in the continuous case to prove existence and uniqueness for (6.1). Indeed, let $R_h : \mathrm{H}^{-1/2}(\Gamma) \to \Lambda_h(\Gamma)$ be the operator characterized by

$$\left\langle \mathbf{curl}_\Gamma\, \chi, \boldsymbol{V}(\mathbf{curl}_\Gamma\, R_h \xi)\right\rangle_{\tau,\Gamma} = \langle \xi, \chi\rangle_{1/2,\Gamma} \qquad \forall \chi \in \Lambda_h(\Gamma) \quad \forall \xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{6.2}$$

Notice that (6.2) is a Galerkin discretization of the elliptic problem (4.12). Consequently, using Corollary 4, we have the following Céa estimate

$$\|R\xi - R_h\xi\|_{1/2,\Gamma} \le C \inf_{\eta \in \Lambda_h(\Gamma)} \|R\xi - \eta\|_{1/2,\Gamma} \qquad \forall \xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{6.3}$$

Here again, using that $\lambda_h = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h)$ we deduce the following equivalent formulation of (6.1):

Find $\boldsymbol{u}_h : [0,T] \to X_h(\Omega)$ and $p_h : [0,T] \to M_h(\Omega_{\rm d})$ such that,

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} + c_h(\boldsymbol{u}_h, \boldsymbol{v}) &= (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in X_h(\Omega) \\ b(\boldsymbol{u}_h(t), q) &= 0 \qquad\qquad\quad \forall q \in M_h(\Omega_{\rm d}) \\ \boldsymbol{u}_h|_{\Omega_{\rm c}}(0) &= \mathbf{0}, \end{aligned} \tag{6.4}$$

where $c_h(\cdot,\cdot) : X_h(\Omega) \times X_h(\Omega) \to \mathbb{R}$ is the uniformly bounded and nonnegative bilinear form given by:

$$c_h(\boldsymbol{u},\boldsymbol{v}) := \mu_0^{-1} \left\langle (\mathbf{curl}_\Gamma\, S \mathrm{curl}_\Gamma + \boldsymbol{K}\,\mathbf{curl}_\Gamma\, R_h \mathrm{curl}_\Gamma \boldsymbol{K}^*)\, \boldsymbol{\pi}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v} \right\rangle_{\tau,\Gamma} \qquad \forall \boldsymbol{u},\boldsymbol{v} \in X_h(\Omega).$$

Notice that the discrete kernel

$$V_h(\Omega) := \{\boldsymbol{v} \in X_h(\Omega) :\ b(\boldsymbol{v},q) = 0\ \forall q \in M_h(\Omega_{\mathrm{d}})\}\,.$$

of the bilinear form b is not a subspace of $V(\Omega)$. Let us also introduce the space

$$V_h(\Omega_{\mathrm{d}}) := \{\boldsymbol{v}|_{\Omega_{\mathrm{d}}} :\ \boldsymbol{v} \in V_h(\Omega)\} \cap \mathbf{H}_\Sigma(\mathbf{curl};\Omega_{\mathrm{d}}).$$

The following result is a variation of Proposition 4.6 from [6].

Proposition 18 *On the space $V_h(\Omega_{\mathrm{d}})$, the seminorm $\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{0,\Omega_{\mathrm{d}}}$ is equivalent to the usual norm in $\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{d}})$.*

Proof. Let $\boldsymbol{\varphi}_h$ be an arbitrary function from $V_h(\Omega_{\mathrm{d}})$. We consider the unique solution $p \in M(\Omega_{\mathrm{d}})$ of

$$\int_{\Omega_{\mathrm{d}}} \mathbf{grad}\, p \cdot \mathbf{grad}\, q = \int_{\Omega_{\mathrm{d}}} \boldsymbol{\varphi}_h \cdot \mathbf{grad}\, q \qquad \forall q \in M(\Omega_{\mathrm{d}}).$$

Reasoning as in the proof of Lemma 8 we deduce that there exists $\delta > 0$ such that $\boldsymbol{v} := \boldsymbol{\varphi}_h - \mathbf{grad}\, p \in V(\Omega_{\mathrm{d}}) \hookrightarrow \mathrm{H}^{1/2+\delta}(\Omega_{\mathrm{d}})^3$. In particular, there exits $C_1 > 0$ independent of $\boldsymbol{v}$ such that

$$\|\boldsymbol{v}\|_{1/2+\delta,\Omega_{\mathrm{d}}} \le C_1 \|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{d}})}. \tag{6.5}$$

Moreover, as $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\varphi}_h$ in Ω_{d}, the Nédélec interpolant $\mathcal{I}_h \boldsymbol{v}$ of $\boldsymbol{v}$ is well-defined, cf. [6, Lemma 4.7]. Actually, there exists $C_2 > 0$ independent of $\boldsymbol{v}$ and h such that (cf. [6, Proposition 4.6])

$$\|\mathcal{I}_h \boldsymbol{v}\|_{0,\Omega_{\mathrm{d}}} \le C_2 \left(h \|\mathbf{curl}\,\boldsymbol{\varphi}_h\|_{0,\Omega_{\mathrm{d}}} + \|\boldsymbol{v}\|_{1/2+\delta,\Omega_{\mathrm{d}}}\right). \tag{6.6}$$

Now, following the strategy given in [15, Chapter III, Proposition 5.10], we are able to build a $p_h \in M_h(\Omega_{\mathrm{d}})$ such that $\mathcal{I}_h(\mathbf{grad}\, p) = \mathbf{grad}\, p_h$. Thus, $\boldsymbol{\varphi}_h = \mathbf{grad}\, p_h + \mathcal{I}_h \boldsymbol{v}$ and

$$\int_{\Omega_{\mathrm{d}}} |\boldsymbol{\varphi}_h|^2 = \int_{\Omega_{\mathrm{d}}} \boldsymbol{\varphi}_h \cdot (\mathbf{grad}\, p_h + \mathcal{I}_h \boldsymbol{v}) = \int_{\Omega_{\mathrm{d}}} \boldsymbol{\varphi}_h \cdot \mathcal{I}_h \boldsymbol{v}.$$

Then, the Cauchy-Schwarz inequality, (6.6) and (6.5) yield

$$\|\boldsymbol{\varphi}_h\|_{0,\Omega_{\mathrm{d}}} \le \|\mathcal{I}_h \boldsymbol{v}\|_{0,\Omega_{\mathrm{d}}} \le C_2 \left(h \|\mathbf{curl}\,\boldsymbol{\varphi}_h\|_{0,\Omega_{\mathrm{d}}} + C_1 \|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{d}})}\right). \tag{6.7}$$

Finally, Corollary 10 the fact that $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\varphi}_h$ show that there exists $C > 0$ independent of h such that

$$\|\boldsymbol{\varphi}_h\|_{0,\Omega_{\mathrm{d}}} \le C \|\mathbf{curl}\,\boldsymbol{\varphi}_h\|_{0,\Omega_{\mathrm{d}}}$$

and the result follows. □

From now on, the proof of the well-posedness of (6.1) runs parallel to the one given in the continuous case. First of all, using Proposition (18) and the fact that $\{\mathcal{T}_h(\Sigma)\}_h$ is quasi-uniform, one can obtain the following technical tool (cf. Lemma 5.3 and Lemma 5.4 of [1] for more details).

Lemma 19 *The linear mapping $\mathcal{E}_h : X_h(\Omega_{\mathrm{c}}) \to V_h(\Omega)$ characterized by $(\mathcal{E}_h \boldsymbol{v}_{\mathrm{c}})|_{\Omega_{\mathrm{c}}} = \boldsymbol{v}_{\mathrm{c}}$ and*

$$\mu_0^{-1} \left(\mathbf{curl}\,\mathcal{E}_h \boldsymbol{v}_{\mathrm{c}}, \mathbf{curl}\,\boldsymbol{w}\right)_{0,\Omega_{\mathrm{d}}} + c_h(\mathcal{E}_h \boldsymbol{v}_{\mathrm{c}}, \boldsymbol{w}) = 0 \qquad \forall \boldsymbol{w} \in V_h(\Omega_{\mathrm{d}}) \quad \forall \boldsymbol{v}_{\mathrm{c}} \in X_h(\Omega_{\mathrm{c}}) \tag{6.8}$$

is well defined and bounded uniformly in h. Furthermore, the inner product

$$(\boldsymbol{u},\boldsymbol{v})_{V_h(\Omega)} := (\boldsymbol{u},\boldsymbol{v})_\sigma + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\mathrm{d}}} + c_h(\boldsymbol{u},\boldsymbol{v}) \tag{6.9}$$

induces in $V_h(\Omega)$ a norm $\|\cdot\|_{V_h(\Omega)}$ that is equivalent to the $\mathbf{H}(\mathbf{curl};\Omega)$-norm in $V_h(\Omega)$. Moreover, the decomposition $V(\Omega) = \widetilde{V_h(\Omega_{\mathrm{d}})} \oplus \mathcal{E}_h(\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{c}}))$ is orthogonal with respect to the inner product $(\cdot,\cdot)_{V_h(\Omega)}$, where $\widetilde{V_h(\Omega_{\mathrm{d}})}$ is the subspace of $V_h(\Omega)$ obtained by extending by zero the functions of $V_h(\Omega_{\mathrm{d}})$ to the whole domain Ω.

Theorem 20 *Problem* (6.4) *has a unique solution* $(\boldsymbol{u}_h, p_h)$. *Moreover, if* $\lambda_h := \mu_0^{-1} R_h(\operatorname{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h)$, *then* $(\boldsymbol{u}_h, \lambda_h, p_h)$ *is the unique solution of Problem* (6.1).

Proof. The orthogonal decomposition provided by the last Lemma permits to split the principle variable $\boldsymbol{u}_h$ into two components. Each component is easily shown to be the unique solution of the problem obtained by restricting (6.4) to the corresponding subspace of $V_h(\Omega)$, see the proof of Theorem 5.5 of [1] for more details.

The existence and uniqueness of the Lagrange multiplier p_h is also obtained as in the aforementioned paper. It is a direct consequence of the discrete inf-sup condition

$$\sup_{\boldsymbol{z}\in X_{h,\Sigma}(\Omega_{\mathrm{d}})} \frac{b(\boldsymbol{z}, q)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{d}})}} \geq \varepsilon_0 \frac{(\mathbf{grad}\, q, \mathbf{grad}\, q)_{0,\Omega_{\mathrm{d}}}}{\|\,\mathbf{grad}\, q\|_{\mathbf{H}(\mathbf{curl};\Omega_{\mathrm{d}})}} = \varepsilon_0 |q|_{1,\Omega_{\mathrm{d}}} \qquad \forall q \in M_h(\Omega_{\mathrm{d}}). \tag{6.10}$$

that follows immediately from the fact that $\mathbf{grad}(M_h(\Omega_{\mathrm{d}})) \subset X_{h,\Sigma}(\Omega_{\mathrm{d}})$. □

6.2 Error estimates.

Consider the linear projection operator $\Pi_h : \mathbf{H}(\mathbf{curl};\Omega) \to V_h(\Omega)$ defined by

$$\Pi_h \boldsymbol{v} \in V_h(\Omega): \qquad (\Pi_h \boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} = (\boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{z} \in V_h(\Omega). \tag{6.11}$$

We deduce easily from (6.10) the following Céa estimate, cf. [15, Chapter II, Theorem 1.1],

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \inf_{\boldsymbol{z}\in X_h(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{v} \in V(\Omega). \tag{6.12}$$

We introduce the notations

$$a(\boldsymbol{v}, \boldsymbol{w}) := \left(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{v}, \mathbf{curl}\, \boldsymbol{w}\right)_{0,\Omega}, \quad \boldsymbol{\rho}_h(t) := \boldsymbol{u}(t) - \Pi_h \boldsymbol{u}(t), \quad \boldsymbol{\delta}_h(t) := \Pi_h \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$$

and

$$\beta_h(\boldsymbol{w}) := \|(R - R_h)\operatorname{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{w}\|_{1/2,\Gamma}. \tag{6.13}$$

Notice that, as a consequence of Proposition 18 and Lemma 19, we have that

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} = \|\boldsymbol{v} - \mathcal{E}_h(\boldsymbol{v}|_{\Omega_{\mathrm{c}}}) + \mathcal{E}_h(\boldsymbol{v}|_{\Omega_{\mathrm{c}}})\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \left(\|\boldsymbol{v}\|_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega}\right) \tag{6.14}$$

for all $\boldsymbol{v} \in V_h(\Omega)$. In particular,

$$\|\boldsymbol{\delta}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \left(\|\boldsymbol{\delta}_h(t)\|_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\, \boldsymbol{\delta}_h(t)\|_{0,\Omega}\right) \qquad \forall t \in [0, T]. \tag{6.15}$$

Lemma 21 *Assume that the solution* $\boldsymbol{u}$ *of* (4.11) *belongs to* $\mathrm{H}^1(0, T; \mathbf{H}(\mathbf{curl};\Omega))$, *then there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \|\boldsymbol{\delta}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t \boldsymbol{\delta}_h(s)\|^2_\sigma ds \\ &\quad \leq C \left[\int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} dt + \sup_{t\in[0,T]} \|\mathbf{curl}\, \boldsymbol{\rho}_h(t)\|^2_{0,\Omega} + \sup_{t\in[0,T]} \beta_h(\boldsymbol{u}(t))^2 + \int_0^T \beta_h(\partial_t \boldsymbol{u}(t))^2 dt\right]. \end{aligned} \tag{6.16}$$

Proof. A straightforward computation yields

$$\begin{aligned} (\partial_t \boldsymbol{\delta}_h(t), \boldsymbol{v})_\sigma + a(\boldsymbol{\delta}_h(t), \boldsymbol{v}) + c_h(\boldsymbol{\delta}_h(t), \boldsymbol{v}) = -(\partial_t \boldsymbol{\rho}_h(t), \boldsymbol{v})_\sigma& \\ - a(\boldsymbol{\rho}_h(t), \boldsymbol{v}) - c_h(\boldsymbol{\rho}_h(t), \boldsymbol{v}) + \left[c_h(\boldsymbol{u}(t), \boldsymbol{v}) - c(\boldsymbol{u}(t), \boldsymbol{v})\right],& \end{aligned} \tag{6.17}$$

for all $\boldsymbol{v} \in V_h(\Omega)$. Then, it follows from (6.14) that

$$\begin{aligned} &(\partial_t \boldsymbol{\delta}_h(t), \boldsymbol{v})_\sigma + a(\boldsymbol{\delta}_h(t), \boldsymbol{v}) + c_h(\boldsymbol{\delta}_h(t), \boldsymbol{v}) \\ &\qquad \leq \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma \|\boldsymbol{v}\|_\sigma + C_1 \left(\|\boldsymbol{v}\|_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega}\right) \left[\|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))\right] \\ &\qquad \leq \frac{1}{2}\|\boldsymbol{v}\|^2_\sigma + \frac{1}{2\mu_1}\|\mathbf{curl}\, \boldsymbol{v}\|^2_{0,\Omega} + C_2 \left[\|\partial_t \boldsymbol{\rho}_h(t)\|^2_\sigma + \|\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))^2\right]. \end{aligned}$$

Taking $\boldsymbol{v} = \boldsymbol{\delta}_h(t)$ in the last inequality and recalling that $c_h(\cdot,\cdot)$ is nonnegative give

$$\frac{d}{dt}\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega} \leq \|\boldsymbol{\delta}_h(t)\|_\sigma^2 + C_3\left[\|\partial_t\boldsymbol{\rho}_h(t)\|_\sigma^2 + \|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \beta_h(\boldsymbol{u}(t))^2\right].$$

We now integrate over $[0,t]$ (we recall that $\boldsymbol{\delta}_h(0) = \mathbf{0}$) and use Gronwall's inequality to obtain

$$\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \mu_1^{-1}\int_0^t \|\mathbf{curl}\,\boldsymbol{\delta}_h(s)\|_{0,\Omega}^2 ds \leq C_4\int_0^T \left[\|\partial_s\boldsymbol{\rho}_h(s)\|_\sigma^2 + \|\boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \beta_h(\boldsymbol{u}(s))^2\right] ds. \tag{6.18}$$

Analogously, taking $\boldsymbol{v} = \partial_t\boldsymbol{\delta}_h(t)$ in (6.17) gives

$$\begin{aligned}
&\|\partial_t\boldsymbol{\delta}_h(t)\|_\sigma^2 + \frac{1}{2}\frac{d}{dt}\left[a(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t)) + c_h(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))\right] \\
&\quad = -(\partial_t\boldsymbol{\rho}_h(t),\partial_t\boldsymbol{\delta}_h(t))_\sigma - \frac{d}{dt}\left[a(\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t)) + c_h(\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))\right] + a(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t)) \\
&\qquad + c_h(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t)) + \frac{d}{dt}\left[c_h(\boldsymbol{u}(t),\boldsymbol{\delta}_h(t)) - c(\boldsymbol{u}(t),\boldsymbol{\delta}_h(t))\right] - \left[c_h(\partial_t\boldsymbol{u}(t),\boldsymbol{\delta}_h(t)) - c(\partial_t\boldsymbol{u}(t),\boldsymbol{\delta}_h(t))\right].
\end{aligned}$$

Next, integrating over $[0,t]$ and using the Cauchy-Schwartz inequality and (6.15) provide

$$\begin{aligned}
&\int_0^t \|\partial_s\boldsymbol{\delta}_h(s)\|_\sigma^2 ds + \|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\
&\quad \leq C_5\Bigg[\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \int_0^t \|\boldsymbol{\delta}_h(s)\|_\sigma^2 ds + \int_0^t \|\mathbf{curl}\,\boldsymbol{\delta}_h(s)\|_{0,\Omega} ds + \int_0^T \|\partial_s\boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 ds \\
&\qquad + \sup_{s\in[0,T]}\|\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|_{0,\Omega}^2 + \sup_{s\in[0,T]}\beta_h(\boldsymbol{u}(s))^2 + \int_0^T \beta_h(\partial_s\boldsymbol{u}(s))^2 ds\Bigg].
\end{aligned}$$

Finally, using (6.18) we conclude that

$$\begin{aligned}
&\int_0^t \|\partial_s\boldsymbol{\delta}_h(s)\|_\sigma^2 ds + \|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\
&\leq C_6\left[\int_0^T \|\partial_s\boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 ds + \sup_{s\in[0,T]}\|\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|_{0,\Omega}^2 + \sup_{s\in[0,T]}\beta_h(\boldsymbol{u}(s))^2 + \int_0^T \beta_h(\partial_s\boldsymbol{u}(s))^2 ds\right].
\end{aligned}$$

The result is now a direct consequence of the last inequality, (6.18) and (6.15). $\square$

Theorem 22 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of Problems* (4.11) *and* (6.1) *respectively. Assume that $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega))$ and let $\boldsymbol{e}_h(t) := \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. There exists $C > 0$ such that*

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \int_0^T \|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt + \int_0^T \|\partial_t\boldsymbol{e}_h(t)\|_\sigma^2\,dt \\
&\quad \leq C\Bigg\{\int_0^T \left[\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\partial_t\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \inf_{\chi\in\Lambda_h(\Gamma)}\|\partial_t\lambda(t) - \chi\|_{1/2,\Gamma}^2\right] dt \\
&\qquad + \sup_{[0,T]}\inf_{\chi\in\Lambda_h(\Gamma)}\|\lambda(t) - \chi\|_{1/2,\Gamma}^2 + \sup_{t\in[0,T]}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\Bigg\}
\end{aligned} \tag{6.19}$$

Proof. Recall that $\lambda(t) = \mu_0^{-1}R\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{u}(t)$. Hence, the regularity assumption on $\boldsymbol{u}$ implies that

$$\lambda \in \mathrm{H}^1(0,T;\mathrm{H}_0^{1/2}(\Gamma))$$

and $\partial_t\lambda(t) = \mu_0^{-1}R\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\partial_t\boldsymbol{u}(t)$. It follows from (6.3) that

$$\beta_h(\boldsymbol{u}(t)) \leq C\inf_{\chi\in\Lambda_h(\Gamma)}\|\lambda(t) - \chi\|_{1/2,\Gamma}, \qquad \beta_h(\partial_t\boldsymbol{u}(t)) \leq C\inf_{\chi\in\Lambda_h(\Gamma)}\|\partial_t\lambda(t) - \chi\|_{1/2,\Gamma}. \tag{6.20}$$

Furthermore, since $\partial_t \Pi_h \boldsymbol{u}(t) = \Pi_h(\partial_t \boldsymbol{u}(t))$, the result follows by writing $\boldsymbol{e}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using Lemma 21 and (6.12). □

For any $r \geq 0$, we consider the Sobolev space

$$\mathbf{H}^r(\mathbf{curl}; Q) := \left\{ \boldsymbol{v} \in \mathrm{H}^r(Q)^3 : \ \mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}^r(Q)^3 \right\},$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl};Q)} := \|\boldsymbol{v}\|^2_{r,Q} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{r,Q}$, where Q is either Ω_{c} or Ω_{d}. It is well known that the Nédélec interpolant $\mathcal{I}_h \boldsymbol{v} \in X_h(Q)$ is well defined for any $\boldsymbol{v} \in \mathbf{H}^r(\mathbf{curl}, Q)$ with $r > 1/2$; see for instance [3, Lemma 5.1] or [6, Lemma 4.7]. We fix now an index $r > 1/2$ and introduce the space

$$\mathbf{X} := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \quad \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in \mathbf{H}^r(\mathbf{curl}; \Omega_{\mathrm{c}}) \ \text{ and } \boldsymbol{v}|_{\Omega_{\mathrm{d}}} \in \mathbf{H}^r(\mathbf{curl}; \Omega_{\mathrm{d}})\} \tag{6.21}$$

endowed with the broken norm

$$\|\boldsymbol{v}\|_{\mathbf{X}} := (\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl};\Omega_{\mathrm{c}})} + \|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl};\Omega_{\mathrm{d}})})^{1/2}.$$

Then, the Nédélec interpolation operator $\mathcal{I}_h : \mathbf{X} \to X_h(\Omega)$ is uniformly bounded and the following interpolation error estimate holds true (see [7, Lemma 5.1] or [3, Proposition 5.6]):

$$\|\boldsymbol{v} - \mathcal{I}_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C h^{\min\{r,m\}} \|\boldsymbol{v}\|_{\mathbf{X}} \qquad \forall \boldsymbol{v} \in \mathbf{X}. \tag{6.22}$$

Lemma 23 *Let $(\boldsymbol{u}, p, \lambda)$ be the solution of (4.11). If we assume that*

$$\boldsymbol{u} \in \mathrm{H}^1(0, T; \mathbf{X}) \quad \textit{and} \quad \mu^{-1}\, \mathbf{curl}\, \boldsymbol{u} \in \mathrm{H}^1(0, T; \mathbf{X}),$$

then

$$\inf_{\chi \in \Lambda_h(\Gamma)} \|\lambda(t) - \chi\|_{1/2,\Gamma} \leq C h^{\min\{r,m\}} \|\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t)\|_{\mathbf{X}} \tag{6.23}$$

and

$$\inf_{\chi \in \Lambda_h(\Gamma)} \|\partial_t \lambda(t) - \chi\|_{1/2,\Gamma} \leq C h^{\min\{r,m\}} \|\partial_t(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t))\|_{\mathbf{X}}. \tag{6.24}$$

Proof. Let $\mathcal{I}_h^\Gamma$ be the $2D$ Nédélec interpolant on $\mathcal{T}_h(\Gamma)$. Using commuting diagram property

$$\boldsymbol{\pi}_\tau \circ \mathcal{I}_h = \mathcal{I}_h^\Gamma \circ \boldsymbol{\pi}_\tau$$

and recalling that $\mathbf{curl}_\Gamma\, \lambda = \boldsymbol{\gamma}_\tau(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})$ we obtain

$$\begin{aligned} \boldsymbol{\pi}_\tau(\mathcal{I}_h(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})) = \mathcal{I}_h^\Gamma(\boldsymbol{\pi}_\tau(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})) &= \mathcal{I}_h^\Gamma(\boldsymbol{n} \times \boldsymbol{\gamma}_\tau(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})) \\ &= \mathcal{I}_h^\Gamma(\boldsymbol{n} \times \mathbf{curl}_\Gamma\, \lambda) = \mathcal{I}_h^\Gamma(\mathbf{grad}_\Gamma\, \lambda). \end{aligned}$$

Then, we can find $\chi(t) \in \Lambda_h(\Gamma)$ such that (see the proof of Proposition 18 for a similar argument)

$$\boldsymbol{\gamma}_\tau(\mathcal{I}_h(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t))) = \mathbf{curl}_\Gamma\, \chi(t).$$

Now, by virtue of Corollary 4,

$$\begin{aligned} \inf_{\chi \in \Lambda_h(\Gamma)} \|\lambda(t) - \chi\|_{1/2,\Gamma} &\leq C_1 \inf_{\chi \in \Lambda_h(\Gamma)} \|\mathbf{curl}_\Gamma\, \lambda(t) - \mathbf{curl}_\Gamma\, \chi\|_{-1/2,\Gamma} \\ &\leq C_1 \|\mathbf{curl}_\Gamma\, \lambda(t) - \boldsymbol{\gamma}_\tau \mathcal{I}_h(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t))\|_{-1/2,\Gamma} \\ &= C_1 \|\boldsymbol{\gamma}_\tau(\mathbf{I}_{\mathrm{d}} - \mathcal{I}_h)(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t))\|_{-1/2,\Gamma} \\ &\leq C_2 \|(\mathbf{I}_{\mathrm{d}} - \mathcal{I}_h)(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(t))\|_{\mathbf{H}(\mathbf{curl};\Omega)} \end{aligned}$$

and (6.23) follows by using the interpolation error estimate (6.22).

Finally, the regularity assumption on $\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}$ allow us to write $\boldsymbol{\pi}_\tau(\mathcal{I}_h(\partial_t(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}))) = \mathcal{I}_h^\Gamma(\mathbf{grad}_\Gamma\, \partial_t \lambda)$ and (6.24) follows by using the same arguments as above. □

The following convergence result is a direct consequence of Theorem 22, Lemma 23 and the interpolation error estimate (6.22).

Corollary 24 *Let $l := \min\{r, m\}$. Under the assumptions of Lemma 23, there holds*

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \int_0^T \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt + \int_0^T \|\partial_t \boldsymbol{e}_h(t)\|^2_\sigma\,dt$$
$$\leq Ch^{2l}\left\{\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{\mathbf{X}} + \sup_{t\in[0,T]} \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\|^2_{\mathbf{X}} + \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\mathbf{X}}\,dt + \int_0^T \|\partial_t(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t))\|^2_{\mathbf{X}}\,dt\right\}.$$

Remark 25 *Let us recall that*

$$\lambda(t) = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}(t)) \quad \textit{and} \quad \lambda_h(t) = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h(t)).$$

Therefore, using (6.20) *and the uniform boundedness of R_h, we obtain*

$$\begin{aligned}\mu_0\|\lambda(t) - \lambda_h(t)\|_{1/2,\Gamma} &\leq \beta_h(\boldsymbol{u}(t)) + \|R_h \mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau(\boldsymbol{u} - \boldsymbol{u}_h)(t)\|_{1/2,\Gamma} \\ &\leq C\left\{\inf_{\chi\in\Lambda_h(\Gamma)} \|\lambda(t) - \chi\|_{1/2,\Gamma} + \|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}\right\}.\end{aligned}$$

Consequently, using Lemma 23 and Corollary 24 we have

$$\int_0^T \|\lambda(t) - \lambda_h(t)\|^2_{1/2,\Gamma} dt \leq Ch^{2l},$$

with $l := \min\{r, m\}$.

7 Analysis of a fully-discrete scheme.

7.1 Well-posedness

We consider a uniform partition $\{t_n := n\Delta t :\ n = 0, \ldots, N\}$ of $[0, T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n :\ n = 0, \cdots, N\}$ we denote

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1, 2, \ldots, N.$$

A fully-discrete version of problem (4.11) reads as follows:

For $n = 1, \cdots, N$, find $(\boldsymbol{u}_h^n, p_h^n, \lambda_h^n) \in X_h(\Omega) \times M_h(\Omega_\mathrm{d}) \times \Lambda_h(\Gamma)$ such that

$$\begin{aligned}(\bar{\partial}\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \bar{\partial}p_h^n) + a(\boldsymbol{u}_h^n, \boldsymbol{v}) + \mu_0^{-1}\left\langle S(\mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n), \mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{1/2,\Gamma}& \\ + \left\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\, \lambda_h^n(t), \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma} &= (\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in X_h(\Omega), \\ -\left\langle \mathbf{curl}_\Gamma\, \eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda_h^n)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\, \eta), \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n\right\rangle_{\tau,\Gamma} &= 0 \quad \forall \eta \in \mathrm{H}_0^{1/2}(\Gamma), \\ b(\boldsymbol{u}_h^n, q) &= 0 \quad \forall q \in M_h(\Omega_\mathrm{d}), \\ \boldsymbol{u}_h^0|_{\Omega_\mathrm{c}} &= \mathbf{0}, \\ p_h^0 &= 0, \\ \lambda_h^0 &= 0.\end{aligned} \tag{7.1}$$

Writing the second equation of (7.1) $\lambda_h^n = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n)$ we can reformulate the problem as follows:

For $n = 1, \cdots, N$, find $(\boldsymbol{u}_h^n, p_h^n) \in X_h(\Omega) \times M_h(\Omega_\mathrm{d})$ such that

$$\begin{aligned}(\bar{\partial}\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \bar{\partial}p_h^n) + a(\boldsymbol{u}_h^n, \boldsymbol{v}) + c_h(\boldsymbol{u}_h^n, \boldsymbol{v}) &= (\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h^n, q) &= 0 \quad \forall q \in M_h(\Omega_\mathrm{d}), \\ \boldsymbol{u}_h^0|_{\Omega_\mathrm{c}} &= \mathbf{0}, \\ p_h^0 &= 0 \quad .\end{aligned} \tag{7.2}$$

Hence, at each iteration step we have to find $(\boldsymbol{u}_h^n, p_h^n) \in X_h(\Omega) \times M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned}(\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + \Delta t\,[a(\boldsymbol{u}_h^n, \boldsymbol{v}) + c_h(\boldsymbol{u}_h^n, \boldsymbol{v})] + b(\boldsymbol{v}, p_h^n) &= F_n(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in X_h(\Omega),\\ b(\boldsymbol{u}_h^n, q) &= 0 \qquad \forall q \in M_h(\Omega_{\mathrm{d}}),\end{aligned}$$

where

$$F_n(\boldsymbol{v}) := \Delta t(\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h^{n-1}).$$

The existence and uniqueness of $(\boldsymbol{u}_h^n, \lambda_h^n)$ is a direct consequence of the Babuška-Brezzi theory. Indeed, the bilinear form b satisfies the discrete inf-sup condition (6.10) and the bilinear form

$$(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\boldsymbol{v}, \boldsymbol{w})_\sigma + \Delta t\,[a(\boldsymbol{v}, \boldsymbol{w}) + c_h(\boldsymbol{v}, \boldsymbol{w})]$$

is elliptic on its kernel $V_h(\Omega)$ (cf. Lemma 19).

7.2 Error estimates.

Lemma 26 *Let $\boldsymbol{\rho}^n := \boldsymbol{u}(t_n) - \Pi_h \boldsymbol{u}(t_n)$, $\boldsymbol{\delta}^n := \Pi_h \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$, $\boldsymbol{\tau}^n := \bar{\partial}\boldsymbol{u}(t_n) - \partial_t \boldsymbol{u}(t_n)$ and let β_h be defined as in* (6.13)*. There exists $C > 0$ independent of h and Δt such that*

$$\begin{aligned}\max_{1\le k\le n} \|\boldsymbol{\delta}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^n \|\bar{\partial}\boldsymbol{\delta}^k\|^2_\sigma \le\ & C\{\Delta t \sum_{k=1}^n \left[\|\bar{\partial}\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\tau}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\partial_t \boldsymbol{u}(t_k))^2\right]\\ & + \max_{1\le k\le n} \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \max_{1\le k\le n} \beta_h(\boldsymbol{u}(t_k))^2\}. \qquad (7.3)\end{aligned}$$

Proof. It is straightforward to show that

$$\begin{aligned}&(\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{v})_\sigma + a(\boldsymbol{\delta}^k, \boldsymbol{v}) + c_h(\boldsymbol{\delta}^k, \boldsymbol{v})\\ &\qquad = -(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{v})_\sigma - a(\boldsymbol{\rho}^k, \boldsymbol{v}) + (\boldsymbol{\tau}^k, \boldsymbol{v})_\sigma - c_h(\boldsymbol{\rho}^k, \boldsymbol{v}) + c_h(\boldsymbol{u}(t_k), \boldsymbol{v}) - c(\boldsymbol{u}(t_k), \boldsymbol{v})\end{aligned} \qquad (7.4)$$

for any $\boldsymbol{v} \in V_h(\Omega)$. Choosing $\boldsymbol{v} = \boldsymbol{\delta}^k$ in the last identity, recalling that $c_h(\cdot,\cdot)$ is nonnegative and uniformly bounded, and using the estimates

$$a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) \ge \mu_1^{-1} \|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega} \quad \text{and} \quad (\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{\delta}^k)_\sigma \ge \frac{1}{2\Delta t}\left(\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma\right),$$

together with (cf. (6.15))

$$\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\left[\|\boldsymbol{\delta}^k\|_\sigma + \|\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega}\right] \qquad k = 1,\dots,n \qquad (7.5)$$

and the Cauchy-Schwartz inequality lead us to the following inequality:

$$\begin{aligned}&\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma + \Delta t\,\mu_1^{-1} \|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega}\\ &\qquad \le \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|^2_\sigma + C\Delta t\left[\|\bar{\partial}\boldsymbol{\rho}^k\|^2_\sigma + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\tau}^k\|^2_\sigma + \beta_h(\boldsymbol{u}(t_k))^2\right].\end{aligned} \qquad (7.6)$$

Next, summing over k in

$$\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma \le \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|^2_\sigma + C\Delta t\left[\|\bar{\partial}\boldsymbol{\rho}^k\|^2_\sigma + \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\tau}^k\|^2_\sigma + \beta_h(\boldsymbol{u}(t_k))^2\right].$$

and using the discrete Gronwall's Lemma (see, for instance, Lemma 1.4.2 from [22]) and the fact that $\boldsymbol{\delta}^0 = \mathbf{0}$ yield

$$\|\boldsymbol{\delta}^n\|^2_\sigma \le C\Delta t \sum_{k=1}^n \left(\|\bar{\partial}\boldsymbol{\rho}^k\|^2_\sigma + \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\tau}^k\|^2_\sigma + \beta_h(\boldsymbol{u}(t_k))^2\right), \qquad (7.7)$$

for $n = 1, \ldots, N$. Inserting the last inequality in (7.6) and summing over k we have the estimate

$$\|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t \sum_{k=1}^n \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^k\|_{0,\Omega}^2 \leq C\Delta t \left(\sum_{k=1}^n \|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \sum_{k=1}^n \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \sum_{k=1}^n \|\boldsymbol{\tau}^k\|_\sigma^2 + \sum_{k=1}^n \beta_h(\boldsymbol{u}(t_k))^2 \right). \tag{7.8}$$

Taking now $\boldsymbol{v} = \bar{\partial}\boldsymbol{\delta}^k$ in (7.4) produces the identity

$$\begin{aligned} &\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) + c_h(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \\ &\quad = -(\bar{\partial}\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + (\boldsymbol{\tau}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) + c_h(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) + c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) \\ &\qquad - c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) + c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - \frac{1}{\Delta t}(\gamma_k - \gamma_{k-1}) \end{aligned} \tag{7.9}$$

with $\gamma_k := a(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) + c_h(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) + c(\boldsymbol{u}(t_k), \boldsymbol{\delta}^k) - c_h(\boldsymbol{u}(t_k), \boldsymbol{\delta}^k)$. On the other hand, as $a(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ are nonnegative, it is easy to check that

$$a(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \geq \frac{1}{2\Delta t}\left[a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right], \quad c_h(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \geq \frac{1}{2\Delta t}\left[c_h(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - c_h(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right].$$

Using these inequalities together with the Cauchy-Schwartz inequality in (7.9) lead to

$$\begin{aligned} &\frac{1}{2}\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + \frac{1}{2\Delta t}\left[a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right] + \frac{1}{2\Delta t}\left[c_h(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - c_h(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right] \\ &\quad \leq C\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\boldsymbol{\tau}^k\|_\sigma^2\right) + a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) + c_h(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) + c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) \\ &\qquad + c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - \frac{1}{\Delta t}(\gamma_k - \gamma_{k-1}). \end{aligned}$$

Then, summing over k and recalling that $c_h(\cdot,\cdot)$ is nonnegative, we deduce that

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^n \|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + \frac{1}{2\mu_1 \Delta t}\|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^n\|_{0,\Omega}^2 \\ &\qquad \leq C_1 \sum_{k=1}^n \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\boldsymbol{\tau}^k\|_\sigma^2\right) + \sum_{k=1}^n (\theta_{1,k} + \theta_{2,k} + \theta_{3,k}) + \frac{1}{\Delta t}|\gamma_n|, \end{aligned} \tag{7.10}$$

with $\theta_{1,k} := \left|a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right|$, $\theta_{2,k} := \left|c_h(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right|$, $\theta_{3,k} := \left|c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1})\right|$ and

$$\theta_{4,k} := \left|c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1})\right|.$$

It is easy to obtain from the Cauchy-Schwartz inequality and (7.5) the bounds

$$\begin{aligned} &\sum_{k=1}^n \theta_{1,k} \leq \sum_{k=1}^n \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C_2 \sum_{k=1}^n \|\operatorname{\mathbf{curl}} \bar{\partial}\boldsymbol{\rho}^k\|_{0,\Omega}^2, \\ &\sum_{k=1}^n \theta_{2,k} \leq \sum_{k=1}^n \left[\|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C_3\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right], \\ &\sum_{k=1}^n \theta_{3,k} \leq \sum_{k=1}^n \left[\|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C_4\|\boldsymbol{\tau}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right], \\ &\sum_{k=1}^n \theta_{4,k} \leq \sum_{k=1}^n \left[\|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C_5 \beta_h(\partial_t \boldsymbol{u}(t_k))^2\right], \\ &|\gamma_n| \leq \|\boldsymbol{\delta}^n\|_\sigma^2 + \frac{1}{4\mu_1}\|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^n\|_{0,\Omega}^2 + C_6\left[\|\operatorname{\mathbf{curl}} \boldsymbol{\rho}^n\|_{0,\Omega}^2 + \beta_h(\boldsymbol{u}(t_n))^2\right]. \end{aligned}$$

Substituting the last inequalities in (7.10) and using (7.8), we obtain

$$\Delta t \sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\delta}^k\|_{\sigma}^2 + \|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2 \leq C_7 \left\{ \Delta t \sum_{k=1}^{n} \left[\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{\tau}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \right.\right.$$
$$\left.\left. + \beta_h(\boldsymbol{u}(t_k))^2 + \beta_h(\partial_t \boldsymbol{u}(t_k))^2 \right] + \|\mathbf{curl}\,\boldsymbol{\rho}^n\|_{0,\Omega}^2 + \beta_h(\boldsymbol{u}(t_n))^2 \right\}.$$

The estimate (7.3) follows directly from a combination of the last inequality with (7.8) and (7.5). □

Theorem 27 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h^n$ be the solutions of Problems* (4.11) *and* (7.1) *respectively. Assume that $\boldsymbol{u} \in \mathrm{H}^2(0,T;\mathbf{X})$ and let $\boldsymbol{e}^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$. Then, there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\leq n\leq N} \|\boldsymbol{e}^n\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{e}^k\|_{\sigma}^2$$
$$\leq C \left\{ \max_{1\leq n\leq N} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t_n) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \max_{1\leq n\leq N} \inf_{\xi\in\Lambda_h(\Gamma)} \|\lambda(t_n) - \xi\|_{1/2,\Gamma}^2 \right.$$
$$+ \Delta t \sum_{n=1}^{N} \inf_{\xi\in\Lambda_h(\Gamma)} \|\partial_t\lambda(t_n) - \eta\|_{1/2,\Gamma}^2 + \int_0^T \left(\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t \boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \right) dt$$
$$\left. + (\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \, dt \right\}.$$

Proof. The result is obtained by using (6.20) and Lemma 21 and proceeding as in Theorem 6.2 of [1]. □

Finally, with the aid of Lemma 23, Theorem 27 and the interpolation error estimate (6.22), we deduce the following asymptotic error estimate for our fully-discrete scheme.

Corollary 28 *Under the assumptions of Lemma 23 and Theorem 27, there holds*

$$\max_{1\leq n\leq N} \|\boldsymbol{e}^n\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{e}^k\|_{\sigma}^2 \leq C h^{2l} \left\{ \max_{1\leq n\leq N} \|\boldsymbol{u}(t_n)\|_{\mathbf{X}}^2 + \max_{1\leq n\leq N} \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t_n)\|_{\mathbf{X}}^2 \right.$$
$$\left. + \max_{1\leq n\leq N} \|\partial_t(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t_n))\|_{\mathbf{X}}^2 + \int_0^T \|\partial_t \boldsymbol{u}(t)\|_{\mathbf{X}}^2 \, dt \right\} + C(\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_{\sigma}^2 \, dt,$$

with $l := \min\{m, r\}$.

Remark 29 *As $\lambda_h^n = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n))$, we can proceed as in Remark 25 to obtain*

$$\Delta t \sum_{k=1}^{n} \|\lambda(t_n) - \lambda_h^n\|_{1/2,\Gamma}^2 \leq C[h^{2l} + (\Delta t)^2],$$

with $l := \min\{r, m\}$.

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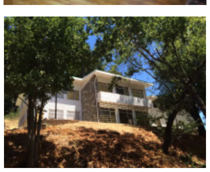

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