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Relaxing the hypotheses of Bielak–MacCamy's BEM–FEM coupling

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Abstract

In this paper we show that the quasi–symmetric coupling of Finite and Boundary Elements of Bielak and MacCamy can be freed of two very restricting hypotheses that appeared in the original paper: the coupling boundary can be taken polygonal/polyhedral and coupling can be done using the normal stress instead of the pseudostress. We will do this by first considering a model problem associated to the Yukawa equation, where we prove how compactness arguments can be avoided to show stability of Galerkin discretizations of a coupled system in the style of Bielak–MacCamy's. We also show how discretization properties are robust in the continuation parameter that appears in the formulation. This analysis is carried out using a new and very simplified proof of the ellipticity of the Johnson–Nédélec BEM–FEM coupling operator. Finally, we show how to apply the techniques that we have fully developed in the model problem to the linear elasticity system.

1 Introduction

The idea of coupling finite and boundary elements goes back to the late years of the decade of the seventies, just some years after the first mathematically rigorous proofs of ellipticity of some boundary integral equations of the first kind had appeared [21, 12], a fact that considerably widened the area of boundary element methods, taking it one step closer to the world of numerical partial differential equations and one step further away from the realm of numerical integral equations. The simplest and in principle most naïf way of coupling BEM and FEM can be found in [14, 2, 3] or [25] for example. It uses Green's Third Identity to create a non–local boundary condition that is coupled to an interior Neumann solver. The interior Neumann solver uses finite elements, whereas the integral identity is discretized using a finite element space on the artificial boundary that has been introduced to cut–off the computational domain.

This kind of methods has been used ever since (they are often referred to as non–symmetric or one–equation coupling methods) but they appeared to have a serious drawback. From the way their analysis was carried out, using Fredholm theory, it seemed not to be possible to prove that

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the Galerkin schemes under study were stable when the coupling boundary was non–smooth, which is not optimal from the point of view of the finite elements discretization. Moreover, the methods looked not to be applicable at all for the elasticity system. Both problems reflect the lack of compactness of a certain boundary integral operator. It has to be admitted that the drawback was mainly theoretical and that no failure of the non–symmetric coupled schemes was reported by users of the method in situations where the smoothness hypotheses were not met.

The paper [1] solved partially the second problem, by using as underlying transmission problem one that employed the displacement field as well as the normal pseudostress (the physical and natural problem would require coupling in the normal stress, but nothing seemed to be analyzable with that system). However, the coupling boundary was still required to be smooth enough.

A way around both problems was found in [7] and [10], by using two integral identities on the coupling boundary instead of only one. These papers gave birth to the rich family of symmetric couplings of finite and boundary element methods, a family that has grown since then, including coupling methods with mixed [18], [5] or non–conforming FEM [4] among other interior solvers, such as Discontinuous Galerkin methods. A very general frame that includes most of these methods has been presented in [6], mixing ideas of hybridizable methods with the theoretical tools of boundary integral equations. Early surveys on different ways of coupling BEM and FEM appear in [11] and in the introductory chapter of [9].

The theoretical panorama has recently changed with [22]. In this paper it was proved that all Galerkin methods for Johnson–Nédélec's coupling are stable without any recourse to Fredholm theory, thus allowing the interface to be polygonal/polyhedral. The same technique gives theoretical support to the application of the method to the linear elasticity equations. In a way, this result restores the importance of the non–symmetric BEM–FEM coupling as competitor of the symmetric coupling. The technique employed in [22] is based on rewritting the discrete equations as a non–standard transmission problem in free space, an idea that had already been used in [16] to deal with BEM and BEM–FEM discretizations in the resolvent set of the Laplace operator. It was also the origin of the method in [19], which gives a non–symmetric coupling of BEM with mixed FEM on any Lipschitz interface.

In this paper we advance in the program of redoing the analysis of the original non–symmetric BEM–FEM coupling schemes by eliminating the very restricting hypothesis of the smoothness of coupling interface. We now undertake the task of revising Bielak and MacCamy's coupling [1], renamed as quasi–symmetric coupling in the monograph [9]. Because of the way we are going to approach the proof, the result will also hold true for the elasticity system, with coupling on the normal stress instead of on the pseudostress.

As a novelty, *the present paper includes a new and extremely simplified proof of the main result in* [22], by showing directly ellipticity of the operator equation (Theorem 3.1 below). This is done by exploiting the same idea that was underlying in the analysis in [22]: eliminate variables on the interface and move them to free space by the use of potentials. This is, after all, the gist of the original analysis of ellipticity of the single layer integral operator in [21]. Note that an alternative proof of this ellipticity result has been recently given by Olaf Steinbach in [23], using an operator expression of the Steklov–Poincaré operator (based on a Schur complement of a perturbation of the Calderón projector) and a theorem on contractivity of some integral operator of the second kind proved in [24].

As in [22], we will work with the Yukawa operator $u \mapsto -\Delta u + u$ in free space to avoid unimportant annoyances about boundary conditions, energy–free solutions and behavior at infinity.

The modifications needed to deal with the Laplace operator are detailed in the final section of [22]: they involve the use of weighted Sobolev spaces and a very minor additional hypothesis on the discrete spaces (they are required to include constant functions), which is needed to deal with energy–free solutions and to tackle constant behavior at infinity in the two dimensional case. In this paper we will also show how to handle boundary conditions on an interior obstacle. The Bielak–MacCamy BEM–FEM coupling depends on a parameter $0 < \delta < 1$. We will show that the convergence properties are uniform in this parameter. The limiting values are precisely the Johnson–Nédélec BEM–FEM coupling and its transpose, both including a posprocessing of the solution to approximate another variable. An application to an exterior problem in three dimensional elasticity will be given in a final section to show how the technique applies with small modifications to very general situations.

The paper is organized as follows. In Section 2 we introduce the Bielak–MacCamy coupling applied to the Yukawa transmission problem as a convex combination of two postprocessed non–symmetric coupling methods. We also state the main result of this paper (Theorem 2.2). In Section 3, we prove this result by first giving a simple proof of the ellipticity of the bilinear form associated to the Johnson–Nédélec non–symmetric coupling (Theorem 3.1) in a new and very simple way. In Section 4 we show how to prove equivalent results when the problem is exterior to a bounded domain, that is when there are given boundary conditions on an interior obstacle. Finally, in Section 5 we show a simple extension to an exterior problem in linear elasticity.

We end this section mentioning some background material and notation used throughout this article. For basic results on Sobolev spaces on Lipschitz domains as well as on the elementary properties of layer potentials, a convenient reference is [17]. The norm of the Sobolev space $H^1(\mathcal{O})$ ($\mathcal{O}$ being an open set in $\mathbb{R}^d$) will be denoted $\|\cdot\|_{1,\mathcal{O}}$. The norms of the fractional Sobolev spaces $H^{\pm 1/2}(\Gamma)$ (Γ being a closed Lipschitz surface) will be denoted $\|\cdot\|_{\pm 1/2,\Gamma}$. Note that the spaces $H^{\pm 1/2}(\Gamma)$ are reciprocally dual. We will employ angled brackets $\langle\cdot,\cdot\rangle$ for the $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ duality product. The identification of $H^{1/2}(\Gamma)$ as the dual space of $H^{-1/2}(\Gamma)$ will be used without further warning: for instance we will identify a bounded linear functional $\ell : H^{-1/2}(\Gamma) \to \mathbb{R}$ with an element of $\beta \in H^{1/2}(\Gamma)$, so that $\ell(\mu) = \langle \mu, \beta\rangle$ for all $\mu \in H^{-1/2}(\Gamma)$.

2 BEM–FEM coupling

The problem we are going to consider in a first instance is a transmission problem in free space for the Yukawa equation. The geometrical setting is the following. We assume that Ω_- is a bounded Lipschitz domain (or the finite union of Lipschitz domains), with boundary Γ and such that the exterior domain $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ ($d = 2$ or 3) is connected.

The transmission problem is: find $u \in H^1(\mathbb{R}^d \setminus \Gamma)$ such that

$$
\begin{aligned}
-\Delta u + u &= f && \text{in } \Omega_-, && \text{(2.1a)}\\
[\gamma u] &= \beta_0 && \text{on } \Gamma, && \text{(2.1b)}\\
[\partial_\nu u] &= \beta_1 && \text{on } \Gamma, && \text{(2.1c)}\\
-\Delta u + u &= 0 && \text{in } \Omega_+. && \text{(2.1d)}
\end{aligned}
$$

The interface conditions involve the following jump operators

$$[\gamma u] = \gamma^- u - \gamma^+ u, \qquad [\partial_\nu u] = \partial_\nu^- u - \partial_\nu^+ u,$$

where $\gamma^\pm : H^1(\Omega_\pm) \to H^{1/2}(\Gamma)$ is the trace operator and $\partial_\nu^\pm$ is the interior/exterior normal derivative, with the normal vector pointing from Ω_- to Ω_+. The unsuperscripted symbol γ will be used whenever there is no doubt on which trace we are taking, be it because the function is only defined on one side of Γ or because the jump across the interface is zero. The basic assumptions on the data functions are: $f \in L^2(\Omega_-)$, $\beta_0 \in H^{1/2}(\Gamma)$ and $\beta_1 \in H^{-1/2}(\Gamma)$.

There are two choices for representing the solution of $-\Delta u + u = 0$ in Ω_+: the use of potentials (indirect method) or the use of Green's formula (direct method). As explained in [9], when applied to the transmission problem at hand, they lead to mutually transposed coupled systems and Bielak–MacCamy's method can be understood as a convex combination of post–processed versions of both possibilities.

2.1 Potentials and integral operators

We first introduce the single and double layer potentials

$$\mathrm{S}\psi := \int_\Gamma E(\,\cdot\,,\mathbf{y})\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.2a}$$

$$\mathrm{D}\varphi := \int_\Gamma \partial_{\nu(\mathbf{y})} E(\,\cdot\,,\mathbf{y})\varphi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.2b}$$

where

$$E(\mathbf{x},\mathbf{y}) := \begin{cases} K_0(|\mathbf{x}-\mathbf{y}|)/(2\pi), & \text{when } d=2, \\ \exp(-|\mathbf{x}-\mathbf{y}|)/(4\pi|\mathbf{x}-\mathbf{y}|), & \text{when } d=3. \end{cases}$$

The function K_0 is the Macdonald function (or modified Bessel function of the second kind) of order zero. Both potentials can be interpreted as solutions of transmission problems. For single layer potentials we have the equivalence

$$\left.\begin{array}{rr} & u \in H^1(\mathbb{R}^d \setminus \Gamma) \\ -\Delta u + u = 0 & \text{in } \mathbb{R}^d \setminus \Gamma \\ [\gamma u] = 0, & [\partial_\nu u] = \psi \end{array}\right] \quad \Longleftrightarrow \quad \left[\begin{array}{l} \psi \in H^{-1/2}(\Gamma), \\ u = \mathrm{S}\psi, \end{array}\right. \tag{2.3}$$

which for double layer potentials becomes

$$\left.\begin{array}{rr} & u \in H^1(\mathbb{R}^d \setminus \Gamma) \\ -\Delta u + u = 0 & \text{in } \mathbb{R}^d \setminus \Gamma \\ [\gamma u] = -\varphi, & [\partial_\nu u] = 0 \end{array}\right] \quad \Longleftrightarrow \quad \left[\begin{array}{l} \varphi \in H^{1/2}(\Gamma) \\ u = \mathrm{D}\varphi. \end{array}\right. \tag{2.4}$$

To describe the traces of the layer potentials we will use two integral operators

$$\mathrm{V}\psi := \int_\Gamma E(\,\cdot\,,\mathbf{y})\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.5a}$$

$$\mathrm{K}\varphi := \int_\Gamma \partial_{\nu(\mathbf{y})} E(\,\cdot\,,\mathbf{y})\varphi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{2.5b}$$

that define functions in $H^{1/2}(\Gamma)$ for arbitrary densities $\psi \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$. The traces of the layer potentials are

$$\gamma^\pm \mathrm{S}\psi = \mathrm{V}\psi, \qquad \gamma^\pm \mathrm{D}\varphi = \pm\tfrac{1}{2}\varphi + \mathrm{K}\varphi. \tag{2.6}$$

Finally, we will also use the transpose $\mathrm{K}^t : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, that allows us to describe the normal derivatives of the single layer potential:

$$\partial_\nu^\pm \mathrm{S}\psi = \mp\tfrac{1}{2}\psi + \mathrm{K}^t\psi. \tag{2.7}$$

2.2 Two coupling schemes and their postprocessings

The only operator left to define the coupling procedures is $\mathrm{A} : H^1(\Omega_-) \to H^1(\Omega_-)'$, given by

$$(\mathrm{A}u)(v) := a(u,v) := \int_{\Omega_-} \Big(\nabla u \cdot \nabla v + u\, v\Big), \qquad u, v \in H^1(\Omega_-).$$

(The superscript $'$ is henceforth used to denote the dual space.) For the right–hand side we will use $\ell_1 : H^1(\Omega_-) \to \mathbb{R}$ and $\ell_2 : H^{-1/2}(\Gamma) \to \mathbb{R}$ defined by

$$\ell_1(v) := \int_{\Omega_-} f\, v - \langle \beta_1, \gamma v\rangle, \qquad \ell_2(\mu) := \langle \mu, (\tfrac{1}{2}I - \mathrm{K})\beta_0\rangle.$$

Johnson–Nédélec's coupling [14] can be described as the operator equation

$$\begin{bmatrix} \mathrm{A} & -\gamma^t \\ (\frac{1}{2}\mathrm{I} - \mathrm{K})\gamma & \mathrm{V} \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}. \tag{2.8}$$

The solution of this system is related to that of (2.1) by

$$\lambda = \partial_\nu^+ u, \qquad u = \mathrm{D}(\gamma u - \beta_0) - \mathrm{S}\lambda \quad \text{in } \Omega_+.$$

The transpose operator leads to this other formulation

$$\begin{bmatrix} \mathrm{A} & \gamma^t(\frac{1}{2}\mathrm{I} - \mathrm{K})^t \\ -\gamma & \mathrm{V} \end{bmatrix} \begin{bmatrix} u \\ \psi \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_3 \end{bmatrix}, \tag{2.9}$$

where $\ell_3 : H^{-1/2}(\Gamma) \to \mathbb{R}$ is defined by

$$\ell_3(\varphi) := -\langle \varphi, \beta_0\rangle.$$

In this case the exterior solution is given by a potential expression: $u = \mathrm{S}\psi$ in Ω_+.

In the case of (2.9) (a coupling with an indirect Boundary Integral Equation) we might be interested in obtaining the normal derivative $\lambda := \partial_\nu^+ u$ as a quantity of physical interest. This can be done by solving the operator equation

$$\mathrm{V}\lambda = \ell_2 - (\tfrac{1}{2}\mathrm{I} - \mathrm{K})\gamma u = (\tfrac{1}{2}\mathrm{I} - \mathrm{K})(\beta_0 - \gamma u).$$

In the original Johnson–Nédélec method (2.8), we might be interested in obtaining a potential representation of the exterior solution as a single layer potential $\mathrm{S}\psi$, which can be attained by solving another elliptic problem

$$\mathrm{V}\psi = \ell_3 + \gamma u = \gamma u - \beta_0.$$

Note that each of these postprocessed methods use the missing equation of the other one for the postprocessing step. We are thus led to the systems

$$\begin{bmatrix} \mathrm{A} & -\gamma^t & 0 \\ (\frac{1}{2}\mathrm{I}-\mathrm{K})\gamma & \mathrm{V} & 0 \\ -\gamma & 0 & \mathrm{V} \end{bmatrix} \begin{bmatrix} u \\ \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \tag{2.10}$$

and

$$\begin{bmatrix} \mathrm{A} & 0 & \gamma^t(\frac{1}{2}\mathrm{I}-\mathrm{K})^t \\ (\frac{1}{2}\mathrm{I}-\mathrm{K})\gamma & \mathrm{V} & 0 \\ -\gamma & 0 & \mathrm{V} \end{bmatrix} \begin{bmatrix} u \\ \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}, \tag{2.11}$$

which share right–hand side, meaning of the three unknowns and the last two equations. The systems are solved in different ways: in (2.10) we first solve the first two equations and then the third one, whereas in (2.11) we group the first and third equations to begin with and use the second equation as postprocessing step.

2.3 The Bielak–MacCamy coupling

A simple convex combination of (2.10) and (2.11) with parameter $\delta \in (0,1)$ leads to the coupling procedure of Bielak and MacCamy:

$$\begin{bmatrix} \mathrm{A} & -\delta\gamma^t & (1-\delta)\gamma^t(\frac{1}{2}\mathrm{I}-\mathrm{K})^t \\ (\frac{1}{2}\mathrm{I}-\mathrm{K})\gamma & \mathrm{V} & 0 \\ -\gamma & 0 & \mathrm{V} \end{bmatrix} \begin{bmatrix} u \\ \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}. \tag{2.12}$$

As a byproduct of our analysis we will obtain a direct proof that this system is well–posed.

Proposition 2.1 *Any solution of* (2.12) *solves* (2.1).

Proof. Assume that we have a solution of (2.12) and let $\varphi = \gamma u - \beta_0$. Then the third equation proves that $\mathrm{S}\psi$ is a solution of the exterior Dirichlet problem with data φ. The function $\mathrm{D}\varphi - \mathrm{S}\lambda$ solves the same problem by the second equation. Therefore the exterior normal derivatives of both functions coincide and $\lambda = -(\frac{1}{2}\mathrm{I}-\mathrm{K})^t\psi$. Finally, the first equation shows that $\lambda + \beta_1 = \partial_\nu^- u$ and we have the solution of the transmission problem. □

Before going any further let us write the main result of this paper. We need three closed spaces

$$V_h \subset H^1(\Omega_-), \qquad X_h \subset H^{-1/2}(\Gamma), \qquad Y_h \subset H^{-1/2}(\Gamma). \tag{2.13}$$

From the point of view of numerics, these spaces are members of sequences of finite dimensional subspaces. From the theoretical point of view we will only need them to be closed and we even cover the case when $V_h = H^1(\Omega_-)$, $X_h = Y_h = H^{-1/2}(\Gamma)$. Next we define a Galerkin scheme for (2.12). We take a general right–hand side $\ell_{1,h} \in V_h'$, $\ell_{2,h} \in X_h'$, $\ell_{3,h} \in Y_h'$ and consider the variational problem (we write it with bilinear forms) that looks for $(u_h, \lambda_h, \psi_h) \in V_h \times X_h \times Y_h$

such that

$$
\begin{aligned}
a(u_h, v_h) - \delta\langle\lambda_h, \gamma v_h\rangle + (1-\delta)\langle(\tfrac{1}{2}\mathrm{I} - \mathrm{K})^t\psi_h, \gamma v_h\rangle &= \ell_{1,h}(v_h), \quad \forall v_h \in V_h, && \text{(2.14a)}\\
\langle\mu_h, (\tfrac{1}{2}\mathrm{I} - \mathrm{K})\gamma u_h\rangle + \langle\mu_h, \mathrm{V}\lambda_h\rangle &= \ell_{2,h}(\mu_h), \quad \forall \mu_h \in X_h, && \text{(2.14b)}\\
-\langle\phi_h, \gamma u_h\rangle \qquad\qquad + \langle\phi_h, \mathrm{V}\psi_h\rangle &= \ell_{3,h}(\phi_h), \quad \forall \phi_h \in Y_h. && \text{(2.14c)}
\end{aligned}
$$

Explicit dependence on δ as well as the limiting cases $\delta \in \{0, 1\}$ are going to be examined as part of the analysis: we will see how the system (2.12) is elliptic for any value of $\delta \in (0, 1)$, that the ellipticity degenerates in the limiting values $\delta = 0$ and $\delta = 1$ but, nevertheless, the Galerkin schemes become stable for a different reason and the stability is robust in the parameter δ.

Theorem 2.2 *The system* (2.14) *admits a unique solution for any* $\delta \in [0, 1]$. *There exists a constant* $C > 0$ *independent of the choice of the spaces* (2.13) *and of* δ *such that*

$$\|u_h\|_{1,\Omega_-} + \|\lambda_h\|_{-1/2,\Gamma} + \|\psi_h\|_{-1/2,\Gamma} \le C\Big(\|\ell_{1,h}\|_{V_h'} + \|\ell_{2,h}\|_{X_h'} + \|\ell_{3,h}\|_{Y_h'}\Big). \tag{2.15}$$

When applied to discrete (finite dimensional) spaces, the stability estimate (2.15) is equivalent to a Céa estimate for the approximation of (2.12) with the Galerkin scheme (2.14). Hence in the natural $H^1(\Omega_-) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$–norm the discretization method is quasi–optimal.

Before proceeding to prove Theorem 2.2, it seems adequate to make some observations concerning different aspects of the continuous and discrete problems above.

1. The solution of (2.12) gives therefore a double representation of the exterior solution. This fact can be considered as an advantage of this formulation, that yields at the same time a 'physical' quantity (λ, as opposed to ψ which is just a density) and a simple potential representation of the exterior solution.

2. Assume now that K is compact (this happens when Γ has Liapunov regularity). After multiplication of the second row of the operator matrix in (2.12) by 2δ and of the third row by $(1-\delta)/2$, we obtain that the principal part of the operator is

$$\begin{bmatrix} \mathrm{A} & -\delta\gamma^t & \frac{1}{2}(1-\delta)\gamma^t \\ \delta\gamma & 2\delta\mathrm{V} & 0 \\ -\frac{1}{2}(1-\delta)\gamma & 0 & \frac{1}{2}(1-\delta)\mathrm{V} \end{bmatrix}. \tag{2.16}$$

   All operators in the diagonal are elliptic: ellipticity of V for the case of the three dimensional Laplacian is a well–known result that can be traced back to [21] and extended with the same techniques to our simpler operator (see [17] for a general proof). Off–diagonal elements of (2.16) form a skew–symmetric matrix. Therefore the operator in (2.12) is the sum of an invertible and a compact operator. Moreover, because of the ellipticity of the principal part (up to scaling of its rows), Galerkin methods for this equation (2.14) are asymptotically stable. (General results on discrete Fredholm theory can be found in [15, Chapter 13] for instance.) This argument, resembling the one in [14] uses compactness of K in a crucial step and the technique does not seem extendable to non–smooth interfaces.

3. In the particular case $\delta = 1/2$, switching the second and third equations and multiplying the first one by 2, we obtain the symmetric operator

$$\begin{bmatrix} 2\mathrm{A} & -\gamma^t & \gamma^t(\frac{1}{2}\mathrm{I}-\mathrm{K})^t \\ -\gamma & 0 & \mathrm{V} \\ (\frac{1}{2}\mathrm{I}-\mathrm{K})\gamma & \mathrm{V} & 0 \end{bmatrix}. \tag{2.17}$$

The Galerkin method (2.14) uses the ellipticity of V to be stable (see (2.16)). Therefore, when the equations are reordered in this way, but we keep the same discretization method, what we obtain is not a Galerkin scheme for (2.17), but a Petrov–Galerkin scheme: the second unknown is taken in X_h but the second equation is tested in Y_h, for example. However, the very reasonable choice $X_h = Y_h$ restores symmetry to the system. This is the reason why this coupling scheme is called quasi–symmetric in [9] and symmetric in the original reference [1].

4. Apart from the advantage of the double representation of the exterior solution (good for near and far field knowledge of the solution), symmetry seems to be a desirable property for a linear system that makes this coupling procedure attractive. Note again that symmetry is obtained when $X_h = Y_h$ and $\delta = 1/2$. In this case the block matrices for the Galerkin equations (2.14) are the same ones as for a Galerkin scheme applied to (2.8) or (2.9), so from the point of view of matrix storage (in case we are going to use iterative methods for the linear system), this method is not more expensive than Johnson–Nédélec's. The same is true for any $\delta \in (0,1)$ if $X_h = Y_h$.

5. Note that when $X_h = Y_h$ there are four matrices to be stored: the large sparse matrix for the interior finite element block corresponding to A, the smaller full matrix for the boundary element discretization of V, a sparse rectangular matrix corresponding to γ and a full–by–rows sparse–by–columns matrix corresponding to $\mathrm{K}\gamma$. If we consider the boundary spaces X_h and γV_h, the coupling blocks can be considered as reorderings of Petrov–Galerkin boundary element discretizations of the identity operator and of K.

3 Analysis

We start by giving a surprisingly simple proof of the ellipticity of the operator associated to the Johnson–Nédélec coupling (or to its transpose). We remark that a variant of this result appears in [23] for the Laplace operator, using the whole machinery of integral representations for Steklov–Poincaré operators and a contractivity result shown in [24] some years ago. The current proof relies uniquely on a very elementary variational argument.

Consider the bilinear form $b : \big(H^1(\Omega_-) \times H^{-1/2}(\Gamma)\big) \times \big(H^1(\Omega_-) \times H^{-1/2}(\Gamma)\big) \to \mathbb{R}$ given by

$$b\big((u,\psi),(v,\phi)\big) := a(u,v) + \langle(\tfrac{1}{2}\mathrm{I}-\mathrm{K})^t\psi, \gamma v\rangle - \langle\phi, \gamma u\rangle + \langle\phi, \mathrm{V}\psi\rangle. \tag{3.1}$$

Note that this is the bilinear form associated to the operator in (2.9), while its transpose is associated to (2.8). Note that the ellipticity of b (Theorem 3.1) is the same property for b and for its transpose.

Theorem 3.1 *There exists $C > 0$ such that*

$$b\big((u,\psi),(u,\psi)\big) \geq C\Big(\|u\|^2_{1,\Omega_-} + \|\psi\|^2_{-1/2,\Gamma}\Big), \qquad \forall (u,\psi) \in H^1(\Omega_-) \times H^{-1/2}(\Gamma).$$

Proof. Let $u^* := \mathrm{S}\psi$. Note that $\partial_\nu^- u^* = (\frac{1}{2}\mathrm{I} + \mathrm{K})^t \psi$, that $\langle \partial_\nu^- u^*, \gamma v\rangle = a(u^*, v)$ for all $v \in H^1(\Omega_-)$ and that by the variational formulation of the transmission problem in (2.3)

$$\langle \psi, \mathrm{V}\psi\rangle = \int_{\mathbb{R}^d} \big(|\nabla u^*|^2 + |u^*|^2\big) = \|u^*\|^2_{1,\mathbb{R}^d}.$$

The ellipticity property follows now from the following argument:

$$\begin{aligned}
b\big((u,\psi),(u,\psi)\big) &= a(u,u) - \langle (\tfrac{1}{2}\mathrm{I} + \mathrm{K})^t\psi, \gamma u\rangle + \langle \psi, \mathrm{V}\psi\rangle \\
&= \|u\|^2_{1,\Omega_-} - \langle \partial_\nu^- u^*, \gamma u\rangle + \|u^*\|^2_{1,\mathbb{R}^d} \\
&= \|u\|^2_{1,\Omega_-} - a(u^*, u) + \|u^*\|^2_{1,\mathbb{R}^d} \\
&\geq \|u\|^2_{1,\Omega_-} - \|u\|_{1,\Omega_-}\|u^*\|_{1,\Omega_-} + \|u^*\|^2_{1,\mathbb{R}^d} \geq \tfrac{1}{2}\|u\|^2_{1,\Omega_-} + \tfrac{1}{2}\|u^*\|^2_{1,\mathbb{R}^d} \\
&= \tfrac{1}{2}\|u\|^2_{1,\Omega_-} + \tfrac{1}{2}\langle \psi, \mathrm{V}\psi\rangle.
\end{aligned}$$

The result is then a consequence of the ellipticity of V in $H^{-1/2}(\Gamma)$. □

Note that if use

$$\Big(\|u\|^2_{1,\Omega_-} + \langle \psi, \mathrm{V}\psi\rangle\Big)^{1/2}$$

as equivalent norm in $H^1(\Omega_-) \times H^{-1/2}(\Gamma)$, the ellipticity constant is just $1/2$.

Corollary 3.2 *The operator equations* (2.10) *and* (2.11) *are well–posed. Moreover, any Galerkin discretization of these equations (i.e.,* (2.14) *for $\delta = 0$ or $\delta = 1$) is stable, with stability constant independent of the choice of discrete spaces.*

Proof. In both cases, the system can be written as a triangular system with a 2×2 operator equation in $H^1(\Omega_-) \times H^{-1/2}(\Gamma)$, that is elliptic because of Theorem 3.1, followed by an elliptic operator equation based on inverting the operator V. It is clear that stability constants of Galerkin discretization depend only on the inverses of the ellipticity constants of the two 'diagonal' operators as well as on the norms of the operators that are out of these diagonal positions. □

The bilinear form associated to problem (2.12) is defined on $H^1(\Omega_-) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ by

$$\begin{aligned}
d_\delta\big((u,\lambda,\psi),(v,\mu,\phi)\big) \;:=\;& a(u,v) - \delta\langle \lambda, \gamma v\rangle + (1-\delta)\langle (\tfrac{1}{2}\mathrm{I} - \mathrm{K})^t\psi, \gamma v\rangle \\
&+ \langle \mu, (\tfrac{1}{2}\mathrm{I} - \mathrm{K})\gamma u\rangle + \langle \mu, \mathrm{V}\lambda\rangle - \langle \phi, \gamma u\rangle + \langle \phi, \mathrm{V}\psi\rangle.
\end{aligned}$$

As the next result shows, a simple scaling of rows (which is different to the one that gives ellipticity to the principal part of the operator) makes d_δ elliptic.

Proposition 3.3 *There exists $C > 0$ independent of $\delta \in (0,1)$ such that*

$$d_\delta\big((u,\lambda,\psi),(u,\delta\lambda,(1-\delta)\psi)\big) \geq C\Big(\|u\|^2_{1,\Omega_-} + \delta\|\lambda\|^2_{-1/2,\Gamma} + (1-\delta)\|\psi\|^2_{-1/2,\Gamma}\Big)$$

for all $(u,\lambda,\psi) \in H^1(\Omega_-) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

Proof. The key to this result is the following identity

$$d_\delta\big((u,\lambda,\psi),(v,\delta\mu,(1-\delta)\phi)\big) = \delta\, b\big((v,\mu),(u,\lambda)\big) + (1-\delta)\, b\big((u,\psi),(v,\phi)\big),$$

used together with Theorem 3.1. Note that in particular we have the very precise bound

$$d_\delta\big((u,\lambda,\psi),(u,\delta\lambda,(1-\delta)\psi)\big) \geq \tfrac{1}{2}\Big(\|u\|^2_{1,\Omega_-} + \delta\langle\lambda, \mathrm{V}\lambda\rangle + (1-\delta)\langle\psi, \mathrm{V}\psi\rangle\Big).$$

□

Corollary 3.4 *The discrete equations* (2.14) *are uniquely solvable for any choice of discrete spaces and there exists $C > 0$ independent of h and δ such that*

$$\begin{aligned}\|u_h\|_{1,\Omega_-} + \delta^{1/2}\|\lambda_h\|_{-1/2,\Gamma} + (1-\delta)^{1/2}\|\psi_h\|_{-1/2,\Gamma} \\ \leq C\left(\|\ell_{1,h}\|_{V_h'} + \delta^{1/2}\|\ell_{2,h}\|_{X_h'} + (1-\delta)^{1/2}\|\ell_{3,h}\|_{Y_h'}\right),\end{aligned}$$

Proof. The unique solvability of the discrete equations is a simple consequence of Proposition 3.3, since scaling the equation leads to an elliptic system. Using the weighted norm

$$\|(u,\lambda,\psi)\|_\delta := \Big(\|u\|^2_{1,\Omega_-} + \delta\|\lambda\|^2_{-1/2,\Gamma} + (1-\delta)\|\psi\|^2_{-1/2,\Gamma}\Big)^{1/2}$$

and Proposition 3.3 we obtain

$$\begin{aligned} C\|(u_h,\lambda_h,\psi_h)\|^2_\delta &\leq d_\delta\big((u_h,\lambda_h,\psi_h),(u_h,\delta\lambda_h,(1-\delta)\psi_h)\big) \\ &= \ell_{1,h}(u_h) + \delta\ell_{2,h}(\lambda_h) + (1-\delta)\ell_{3,h}(\psi_h) \\ &\leq \Big(\|\ell_{1,h}\|_{V_h'} + \delta^{1/2}\|\ell_{2,h}\|_{X_h'} + (1-\delta)^{1/2}\|\ell_{3,h}\|_{Y_h'}\Big)\|(u_h,\lambda_h,\psi_h)\|_\delta. \end{aligned}$$

□

In principle, given a choice of spaces (a discretization level h) the Galerkin estimates provided by Corollary 3.4 degenerate as $\delta \to \{0,1\}$. Note, however, that the equations have been scaled to apply an ellipticity argument. We finish this section by finishing the proof of Theorem 2.2, which requires showing $\delta-$independence of all constants involved.

Proof of Theorem 2.2. Using Corollary 3.4 we can obtain a stability estimate

$$\|u_h\|_{1,\Omega_-} + \|\lambda_h\|_{-1/2,\Gamma} + \|\psi_h\|_{-1/2,\Gamma} \leq \frac{C}{\sqrt{\delta_0(1-\delta_0)}}\Big(\|\ell_{1,h}\|_{V_h'} + \|\ell_{2,h}\|_{X_h'} + \|\ell_{3,h}\|_{Y_h'}\Big) \qquad (3.2)$$

for all $\delta \in [\delta_0, 1-\delta_0]$ and for any $\delta_0 \in (0,1)$.

This shows that we only have to take care of neighborhoods of $\delta = 0$ and $\delta = 1$. The operator in (2.14) is an affine function of δ, so it will be enough to prove bounds on the limits and extend them to a neighborhood by perturbation (continuity) arguments.

Equations (2.14) define an operator from $V_h \times X_h \times Y_h$ to its dual, that can be decomposed as $\mathbb{A}_h + \delta\mathbb{B}_h$. Actually $\mathbb{A}_h$ corresponds to the Galerkin discretization of (2.11) and hence Corollary 3.2 guarantees that $\mathbb{A}_h^{-1}$ is bounded independently of the choice of the spaces. Also $\mathbb{B}_h$ is uniformly bounded and we can take a constant such that $M \geq \|\mathbb{A}_h^{-1}\mathbb{B}_h\|$. Using then Neumann series (see [15, Theorem 2.9] or any text on basic functional analysis), we next prove that

$$\|(\mathbb{A}_h + \delta\mathbb{B}_h)^{-1}\| = \|(\mathbb{I}_h + \delta\mathbb{A}_h^{-1}\mathbb{B}_h)^{-1}\mathbb{A}_h^{-1}\| \leq 2\|\mathbb{A}_h^{-1}\|, \qquad \forall\delta \in [0, 1/(2M)].$$

The argument for $\delta = 1$ is exactly the same and we will not repeat the details. □

Theorem 2.2 shows that there is a continuous (actually it is analytic) transition between the Johnson–Nédélec coupling (based on a direct boundary integral equation) and its transpose (based on an indirect integral representation) and that this smooth transition is preserved by any Galerkin discretization. Independently on its usefulness or lack thereof, this result provides a natural, and somewhat surprising, connection between a numerical method and its transpose. An interesting phenomenon that can be observed in the discretization of this family of problems is that while interior situations ($0 < \delta < 1$) are elliptic after scaling by rows, the limiting equations can be considered as block triangular systems with elliptic diagonal operators.

4 Boundary conditions on an interior boundary

In this section we examine some extensions of the results for the model problem (2.1).

4.1 Dirichlet boundary conditions

Assume now that there is a new Lipschitz domain Ω_{obs}, interior to Ω_-, that their common interface is denoted Σ and that $\mathbb{R}^d \setminus \Omega_{\text{obs}}$ is connected. We denote $\Omega_{\text{int}} := \Omega_- \setminus \overline{\Omega_{\text{obs}}}$. A pictorial representation of this situation is shown in Figure 1.

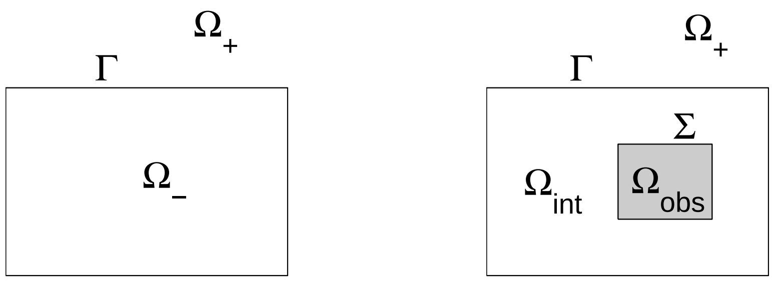

Figure 1: On the left the original geometric configuration. On the right, the domain Ω_- has been divided into an interior obstacle and the surrounding interior domain.

We substitute (2.1a) by

$$\gamma_\Sigma u = 0, \qquad -\Delta u + u = f \quad \text{in } \Omega_{\text{int}}, \tag{4.1}$$

where γ_Σ is the trace operator on the boundary Σ. We keep (2.1b), (2.1c) and (2.1d) unchanged. The solution is looked for in $H^1(\Omega_{\text{int}} \cup \Omega_+)$. Because the proof of Theorem 2.2 is based on ellipticity principles, it never uses the fact that the discrete spaces are finite dimensional or that they are good approximations of the full spaces. Everything is true as long as we can use Lax–Milgram's lemma, so we only need the subspaces to be closed. In particular we can take V_h equal to

$$H^1_{\mathrm{obs}}(\Omega_-) := \{u \in H^1(\Omega_-) \,:\, u \equiv 0 \quad \text{in } \Omega_{\mathrm{obs}}\} \cong \{u \in H^1(\Omega_{\mathrm{int}}) \,:\, \gamma_\Sigma u = 0\}$$

or any finite dimensional subspace of it and let X_h and Y_h as before. Working with these spaces is equivalent to dealing with the problem in Ω_{int} with homogeneous Dirichlet conditions on Σ. Note that the effective interior bilinear is just

$$\int_{\Omega_{\mathrm{int}}} \Big(\nabla u \cdot \nabla v + u\, v\Big),$$

and in practice we are not required to do the extension by zero to the interior of the obstacle. Taking a non–homogeneous condition in (4.1) does not make any change from the analytical point of view, since Dirichlet conditions are essential in the primal formulation of the interior problem and the analysis has to be carried out for the homogeneous problem.

4.2 Neumann boundary conditions

For reasons that will become apparent as we proceed through the proofs (and for which we will attempt to give some kind of explanation), imposition of Neumann conditions on the boundary of an interior obstacle poses a slightly more difficult problem from the theoretical point of view.

The geometric frame for this section is the same as the one in Section 4.1 (see Figure 1). We are going to solve slightly modified problem

$$\begin{aligned}
\partial_\nu u &= 0 && \text{on } \Sigma, && (4.2a)\\
-\Delta u + c^2\, u &= f && \text{in } \Omega_{\mathrm{int}}, && (4.2b)\\
[\gamma u] &= \beta_0 && \text{on } \Gamma, && (4.2c)\\
[\partial_\nu u] &= \beta_1 && \text{on } \Gamma, && (4.2d)\\
-\Delta u + c^2\, u &= 0 && \text{in } \Omega_+, && (4.2e)
\end{aligned}$$

for some $c > 0$. Non–homogeneous Neumann conditions only affect the right–hand side of the system, so they will be covered with our stability analysis. The normal vector on Σ is taken to point towards Ω_{obs}.

We weight the Sobolev norms with the reaction coefficient c:

$$\|u\|^2_{1,\Omega,c} := \|\nabla u\|^2_{0,\Omega} + c^2 \|u\|^2_{0,\Omega}.$$

The fundamental solution is changed to

$$E(\mathbf{x}, \mathbf{y}) := \begin{cases} K_0(c\,|\mathbf{x} - \mathbf{y}|)/(2\pi), & \text{when } d = 2, \\ \exp(-c\,|\mathbf{x} - \mathbf{y}|)/(4\pi|\mathbf{x} - \mathbf{y}|), & \text{when } d = 3, \end{cases}$$

and with it all potentials and integral operators are redefined. Dependence on c will be made explicit by use of the subindex c in the operators.

We substitute the bilinear form a in (2.14) by

$$a_c(u,v) := \int_{\Omega_{\rm int}} \Big(\nabla u \cdot \nabla v + c^2\, u\, v\Big) \tag{4.3}$$

and now $V_h \subset H^1(\Omega_{\rm int})$. We accordingly redefine

$$b_c\big((u,\psi),(v,\phi)\big) := a_c(u,v) + \langle(\tfrac{1}{2}\mathrm{I}-\mathrm{K}_c)^t\psi, \gamma v\rangle - \langle \phi, \gamma u\rangle + \langle \phi, \mathrm{V}_c\psi\rangle.$$

It has to be understood that now $\gamma : H^1(\Omega_{\rm int}) \to H^{1/2}(\Gamma)$.

We redefine the operator $\mathrm{A}_c : H^1(\Omega_{\rm int}) \to H^1(\Omega_{\rm int})'$ by $(\mathrm{A}_c u)(v) := a_c(u,v)$. The Bielak–MacCamy coupling corresponds to the Galerkin discretization of the equations associated to the following operator

$$\mathbb{D}_{c,\delta} := \left[\begin{array}{ccc} \mathrm{A}_c & -\delta\gamma^t & (1-\delta)\gamma^t(\tfrac{1}{2}\mathrm{I}-\mathrm{K}_c)^t \\ (\tfrac{1}{2}\mathrm{I}-\mathrm{K}_c)\gamma & \mathrm{V}_c & 0 \\ -\gamma & 0 & \mathrm{V}_c \end{array}\right],$$

which is the analogue of (2.12). The main result of this section is just a copy of the result for the situation in free space.

Theorem 4.1 *For the given geometric setting there exists $c > 0$ (large enough) such that all the conclusions of Theorem 2.2 hold in this situation. The stability constant depends on c but not on the subspaces or on δ.*

Proof. It is simple to see that once we prove that b_c is elliptic for c large enough, the remainder of the proof of Theorem 2.2 can be applied verbatim. Take now a measurable function $w : \Omega_{\rm int} \to \mathbb{R}$ in the following conditions: $0 \le w \le 1$ in $\Omega_{\rm int}$, $\nabla w \in (L^\infty(\Omega_{\rm int}))^d$, $w \equiv 1$ in a neighborhood of Γ and $\omega \equiv 0$ in a neighborhood of Σ. If

$$\tfrac{2}{3}\|\nabla w\|_{\infty,\Omega_{\rm int}} < c, \tag{4.4}$$

where $\|\cdot\|_{\infty,\Omega_{\rm int}}$ denotes the $(L^\infty(\Omega_{\rm int}))^d$ norm, then [19, Lemma 19] proves that there exists $0 < C < 2$ (depending on $\|\nabla w\|_{\infty,\Omega_{\rm int}}/c$) such that

$$\|w\, u\|_{1,\Omega_{\rm int},c} \le C\|u\|_{1,\Omega_{\rm int},c}, \qquad \forall u \in H^1(\Omega_{\rm int}). \tag{4.5}$$

Noticing that $\gamma u = \gamma(w\, u)$ and that $\gamma_\Sigma(w\, u) = 0$, we can easily follow the steps of the proof of Theorem 3.1 by introducing w conveniently. Namely, we still write $u^* := \mathrm{S}_c\psi$ and show that

$$\begin{aligned} b_c\big((u,\psi),(u,\psi)\big) &= a_c(u,u) - \langle(\tfrac{1}{2}\mathrm{I}+\mathrm{K}_c)^t\psi, \gamma u\rangle + \langle\psi, \mathrm{V}_c\psi\rangle \\ &= \|u\|^2_{1,\Omega_{\rm int},c} - \langle \partial_\nu^- u^*, \gamma(w\, u)\rangle - \langle \partial_\nu u^*, \gamma(w\, u)\rangle_\Sigma + \|u^*\|^2_{1,\mathbb{R}^d,c} \\ &= \|u\|^2_{1,\Omega_{\rm int},c} - a_c(u^*, w\, u) + \|u^*\|^2_{1,\mathbb{R}^d,c} \\ &\ge \|u\|^2_{1,\Omega_{\rm int},c} - C\|u^*\|_{1,\Omega_{\rm int},c}\,\|u\|_{1,\Omega_{\rm int},c} + \|u^*\|^2_{1,\mathbb{R}^d,c} \\ &\ge (1-\tfrac{C}{2})\Big(\|u\|^2_{1,\Omega_{\rm int},c} + \langle\psi, \mathrm{V}_c\psi\rangle\Big), \end{aligned}$$

where in the last but one inequality we have used (4.5). The angled bracket subindexed with Σ denotes the duality product of the pair $H^{\pm 1/2}(\Sigma)$.

Note that $1 - C/2 > 0$ and the bilinear form b_c is elliptic. If we want to use the natural norm of $H^1(\Omega_{\rm int}) \times H^{-1/2}(\Gamma)$, we introduce further dependence on c in the ellipticity constant for b_c. As already mentioned, from this ellipticity result everything else in the statement can be easily proved using exactly the same arguments as in Section 3. $\square$

If we fix $c = 1$ we can approach the problem from the point of view of w: we want to find w in the conditions above with $\|\nabla w\|_{\infty,\Omega_{\rm int}} < 3/2$. This can be understood as a hypothesis asking for a sufficiently wide interior annulus. It is not clear whether this hypothesis is purely technical or if there is really some loss of ellipticity in the formulation (a minor one as we will see next) due to the fact that the layer potentials define a solution in Ω_- and we need 'enough room' to reduce its influence on the interior boundary Σ.

4.3 More on Neumann conditions

In the general case, when c is smaller and (4.4) is not satisfied, we can use a compactness argument to obtain asymptotic stability provided that we have a certain approximation property. In this section we place ourselves in the usual frame of numerical PDEs: we have three families of spaces (2.13) directed (partially ordered) in the parameter h that is allowed to converge to zero. We will further assume the following **approximation property:** for all $(u, \lambda, \psi) \in H^1(\Omega_{\rm int}) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ we have

$$\lim_{h\to 0} \Big(\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega_{\rm int}} + \inf_{\mu_h \in X_h} \|\lambda - \mu_h\|_{-1/2,\Gamma} + \inf_{\phi_h \in Y_h} \|\psi - \phi_h\|_{-1/2,\Gamma} \Big) = 0. \tag{4.6}$$

We will prove asymptotic convergence for the Galerkin method associated to operator equations with operator $\mathbb{D}_{c,\delta}$ for any $c > 0$ (including then $c = 1$, which is our original model problem) and with all constants independent of δ, although there is going to be a certain amount of dependence on the approximation properties of the sequence of spaces. We start with well posedness of the problem and showing an important compactness result.

Proposition 4.2 *Let $c, d > 0$ and $\delta \in [0, 1]$. Then $\mathbb{D}_{c,\delta} - \mathbb{D}_{d,\delta}$ is compact. Moreover, $\mathbb{D}_{c,\delta}$ is invertible for all $c > 0$ and there exists C independent of δ such that*

$$\|\mathbb{D}_{c,\delta}\| + \|\mathbb{D}_{c,\delta}^{-1}\| \le C, \qquad \forall \delta \in [0, 1].$$

Proof. We start by proving that $\mathbb{D}_{c,\delta} - \mathbb{D}_{d,\delta}$ is compact. This is clear for the term $\mathrm{A}_c - \mathrm{A}_d$, where the discrepancy is in a lower order term. Using the definitions of the layer potentials on Lipschitz domains of [8] (see [17]), we can also prove that $\mathrm{V}_c - \mathrm{V}_d$ is compact as an operator from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and so is $\mathrm{K}_c - \mathrm{K}_d$ as an operator from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

Given the domain $\Omega_{\rm int}$ we can construct the cut–off funcion w as in Section 4.2 and choose $d > 0$ such that $d > 2\|\nabla w\|_{\infty,\Omega_{\rm int}}/3$. Theorem 4.1 applied to the continuous operator proves that $\mathbb{D}_{d,\delta}$ is invertible. Then, by the Fredholm alternative, injectivity of $\mathbb{D}_{c,\delta}$ is equivalent to its invertibility.

We will however give a direct proof of invertibility by factorizing the operator

$$\mathbb{D}_{c,\delta} = \begin{bmatrix} \mathrm{I} & -\delta\gamma^t \mathrm{V}_c^{-1} & (1-\delta)\gamma^t(\frac{1}{2}\mathrm{I} - \mathrm{K}_c)^t \mathrm{V}_c^{-1} \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{Q}_c & 0 & 0 \\ (\frac{1}{2}\mathrm{I} - \mathrm{K}_d)\gamma & \mathrm{V}_c & 0 \\ -\gamma & 0 & \mathrm{V}_c \end{bmatrix}, \tag{4.7}$$

where

$$\mathrm{Q}_c := \mathrm{A}_c + \gamma^t \left(\delta \mathrm{V}_c^{-1}(\tfrac{1}{2}\mathrm{I} - \mathrm{K}_c) + (1-\delta)(\tfrac{1}{2}\mathrm{I} - \mathrm{K}_c)^t \mathrm{V}_c^{-1}\right)\gamma.$$

Note that

$$\mathrm{T}_c := -\mathrm{V}_c^{-1}(\tfrac{1}{2}\mathrm{I} - \mathrm{K}_c) = -(\tfrac{1}{2}\mathrm{I} - \mathrm{K}_c)^t \mathrm{V}_c^{-1}$$

is the exterior Steklov–Poincaré (Dirichlet–to–Neumann) operator for $-\Delta + c^2$. This is easily inferred from (2.3), (2.4), (2.6) and (2.7). Therefore

$$\mathrm{Q}_c = \mathrm{A}_c - \gamma^t \mathrm{T}_c \gamma$$

is selfadjoint and elliptic, hence invertible. Note that Q_c does not depend on δ even if δ was present in its definition. The decomposition (4.7) can be used to find (we can give an explicit expression of the inverse of the upper triangular matrix with identities in the diagonal):

$$\mathbb{D}_{c,\delta}^{-1} = \begin{bmatrix} \mathrm{Q}_c & 0 & 0 \\ (\tfrac{1}{2}\mathrm{I} - \mathrm{K}_d)\gamma & \mathrm{V}_c & 0 \\ -\gamma & 0 & \mathrm{V}_c \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{I} & \delta\gamma^t \mathrm{V}_c^{-1} & -(1-\delta)\gamma^t(\tfrac{1}{2}\mathrm{I} - \mathrm{K}_c)^t \mathrm{V}_c^{-1} \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix}.$$

The first of this two matrices is lower triangular with elliptic operators in the diagonal and independent of δ. Note that the existence of a constant independent of δ that bounds the norms of $\mathbb{D}_{c,\delta}^{-1}$ follows from this same factorization. □

The corresponding Galerkin equations look for $(u_h, \lambda_h, \psi_h) \in V_h \times X_h \times Y_h$ such that

$$\Big(\mathbb{D}_{c,\delta}(u_h, \lambda_h, \psi_h)\Big)(v_h, \mu_h, \phi_h) = \ell_h(v_h, \mu_h, \phi_h), \qquad \forall (v_h, \mu_h, \phi_h) \in V_h \times X_h \times Y_h, \tag{4.8}$$

where $\ell_h \in (V_h \times X_h \times Y_h)'$ is the restriction of the right–hand side of the continuous operator equation to the discrete test spaces.

Theorem 4.3 *Take $c > 0$ and $\delta \in [0,1]$. Assume that the approximation property* (4.6) *holds. Then the Galerkin equations* (4.8) *are uniquely solvable for $h \le h_0$ and there exists a constant C, independent of h, such that*

$$\|u_h\|_{1,\Omega_-} + \|\lambda_h\|_{-1/2,\Gamma} + \|\psi_h\|_{-1/2,\Gamma} \le C\|\ell_h\|. \tag{4.9}$$

The stability threshold h_0 is allowed to depend on c and the sequence of spaces but can be taken independent of δ. The constant C can be taken depending only on c (and on the geometry of the problem).

Proof. Throughout the proof, c is taken as a fixed value. We take $d > 0$ large enough so that Theorem 4.1 holds. Theorem 4.1 proves that Galerkin discretizations of the operator $\mathbb{D}_{d,\delta}$ are automatically stable and that the stability constant depends neither on δ nor on the particular sequence of discrete spaces. Stability and the approximation property (4.6) prove convergence of the Galerkin method for $\mathbb{D}_{d,\delta}$.

Since $\mathbb{D}_{c,\delta} = \mathbb{D}_{d,\delta} + (\mathbb{D}_{c,\delta} - \mathbb{D}_{d,\delta})$ is invertible and $\mathbb{D}_{c,\delta} - \mathbb{D}_{d,\delta}$ is compact (Proposition 4.2), the discrete version of the Fredholm theory (see [15, Chapter 13] for instance) proves that Galerkin methods with property (4.6) for $\mathbb{D}_{c,\delta}$ are also convergent and hence stable (4.9).

This means that there exists h_0 such that for $h \le h_0$ the equations (4.8) are invertible and that there exists $C > 0$ such that (4.9) holds. The stability constant C depends on the stability constant for the Galerkin methods applied to $\mathbb{D}_{d,\delta}$ (being therefore independent of δ and of the sequence of subspaces) and on upper bounds of $\|\mathbb{D}_{c,\delta}^{-1}\|$, which shows that C can be taken independent of δ.

The stability threshold h_0 depends on the sequence of discrete spaces as well as on the operator $\mathbb{D}_{c,\delta}$ itself. Let us finish the proof by showing how, given the sequence of discrete spaces satisfying (4.6), we can take h_0 independent of $\delta \in [0,1]$.

If we fix δ_0, we have $h_0 = h_0(\delta_0)$ that makes the system (4.8) invertible and the stability estimate (4.9) holds true for $h \le h_0$, with C independent of h. By a perturbation argument similar to the one used at the end of the proof of Theorem 2.2, the same h_0 and a fixed proportional constant, say $C/2$, can be used for a neighborhood of δ_0, say for all $\delta \in (\delta_0 - \varepsilon(\delta_0), \delta_0 + \varepsilon(\delta_0))$. Since $\{(\delta_0 - \varepsilon(\delta_0), \delta_0 + \varepsilon(\delta_0)) \,:\, \delta_0 \in [0,1]\}$ is an open covering of the compact interval $[0,1]$ we can also choose

$$[0,1] \subset \bigcup_{j=1}^{n} (\delta_j - \varepsilon(\delta_j), \delta_j + \varepsilon(\delta_j))$$

and take $h_{\mathrm{st}} := \min\{h_0(\delta_j) \,:\, j = 1, \ldots, n\}$. For $h \le h_{\mathrm{st}}$ and all δ, the discrete systems are invertible and the stability constant (which has been slightly modified) can be again taken independent of δ. □

5 An application to linear elasticity

Let us consider the same geometric configuration as the one in Section 4. We are going to deal with a transmission problem with a linear homogeneous isotropic medium occupying Ω_+ (that is Hooke's law is the constitutive material equation) and a linear but possibly heterogeneous anisotropic material in Ω_{int}. We will write

$$\boldsymbol{\sigma}_{\mathrm{H}}(\mathbf{u}) := 2\mu\, \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{div} \mathbf{u})\mathrm{I}, \qquad \boldsymbol{\varepsilon}(\mathbf{u}) := \tfrac{1}{2}\nabla \mathbf{u} + \tfrac{1}{2}(\nabla \mathbf{u})^\top$$

for the Hookean material definition of the stress tensor (I is the 3×3 identity matrix and λ and μ are the Lamé constants). On the other hand we will write $\boldsymbol{\sigma}_{\mathrm{C}}(\mathbf{u}) := \mathrm{C}(\boldsymbol{\varepsilon}(\mathbf{u}))$ for the interior definition of the stress tensor. The operator C is required to be linear and $\Omega_{\mathrm{int}}-$uniformly positive definite on the space of symmetric matrices, so that the bilinear form

$$\int_{\Omega_{\mathrm{int}}} \mathrm{C}(\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) = \int_{\Omega_{\mathrm{int}}} \boldsymbol{\sigma}_{\mathrm{C}}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})$$

is symmetric positive semidefinite and defines a seminorm in $H^1(\Omega_{\mathrm{int}})$ that is equivalent to the seminorm

$$\Big(\int_{\Omega_{\mathrm{int}}} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})\Big)^{1/2}.$$

Normal stresses on Γ will be subscripted depending on the constitutive law:

$$\mathbf{t}_{\mathrm{H}}^{\pm}(\mathbf{u}) := \boldsymbol{\sigma}_{\mathrm{H}}^{\pm}(\mathbf{u})\boldsymbol{\nu}, \qquad \mathbf{t}_{\mathrm{C}}(\mathbf{u}) := \boldsymbol{\sigma}_{\mathrm{C}}(\mathbf{u})\boldsymbol{\nu}.$$

Note that we will need both lateral limits for the Hookean material from inside and outside of Γ. The symbol $\boldsymbol{\nu}$ denotes the outwards normal vector field on Γ and we will use $\boldsymbol{\gamma}\pm$ for the traces on Γ of vector fields and $\boldsymbol{\gamma}_\Sigma$ for those on Σ. With these ingredients we can define the transmission problem, whose data is a force density $\mathbf{f} \in (L^2(\Omega_{\rm int}))^3$:

$$\boldsymbol{\gamma}_\Sigma \mathbf{u} = \mathbf{0} \qquad \text{on } \Sigma, \tag{5.1a}$$

$$\mathbf{div}\, \boldsymbol{\sigma}_{\rm C}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad \text{in } \Omega_{\rm int}, \tag{5.1b}$$

$$\boldsymbol{\gamma}^- \mathbf{u} - \boldsymbol{\gamma}^+ \mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma, \tag{5.1c}$$

$$\mathbf{t}_{\rm C}(\mathbf{u}) - \mathbf{t}^+_{\rm H}(\mathbf{u}) = \mathbf{0} \qquad \text{on } \Gamma, \tag{5.1d}$$

$$\mathbf{div}\, \boldsymbol{\sigma}_{\rm H}(\mathbf{u}) = \mathbf{0} \qquad \text{in } \Omega_+, \tag{5.1e}$$

Behavior at infinity will be that of a decaying vector field, so that we demand that $\mathbf{u}(\mathbf{x}) \le C/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$ uniformly in all directions. It is possible to impose that at infinity the solution behaves like a prescribed infinitesimal rigid motion $\mathbf{m}(\mathbf{x}) := \mathbf{a} + \mathbf{b} \times \mathbf{x}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are given vectors. From the point of view of our forthcoming formulation, this can be subtracted from the exterior solution, thus creating a jump of the displacement field across the interface.

To describe in variational terms the behavior at infinity of decaying solutions of the Lamé system (the one that is satisfied in Ω_+) we consider the spaces ($\mathcal{O}$ is any open set)

$$\mathbf{W}^1(\mathcal{O}) := \Big\{ \mathbf{u} : \mathcal{O} \to \mathbb{R}^3 \ : \ \rho\, \mathbf{u} \in (L^2(\mathcal{O}))^3, \quad \nabla \mathbf{u} \in (L^2(\mathcal{O}))^{3\times 3} \Big\}$$

where $\rho(\mathbf{x}) := (1 + |\mathbf{x}|^2)^{-1/2}$. Note that these weighted Sobolev spaces correspond to those denoted $(W^{1,-1}(\mathcal{O}))^3$ in [20, Chapter 2]. They are Hilbert spaces when endowed with their natural norms.

We will then demand that

$$\mathbf{u}|_{\Omega_{\rm int}} \in \mathbf{H}^1_\Sigma(\Omega_{\rm int}) := \{ \mathbf{u} \in (H^1(\Omega_{\rm int}))^3 \ : \ \boldsymbol{\gamma}_\Sigma \mathbf{u} = \mathbf{0} \}, \qquad \mathbf{u}|_{\Omega_+} \in \mathbf{W}^1(\Omega_+),$$

to include the finite energy condition, the correct behavior at infinity as well as the Dirichlet boundary condition on Σ (this one can also be taken non–homogeneous with little modifications of the forthcoming arguments). We will use the following notation for the energy norms

$$\|\mathbf{u}\|_{{\rm C},\Omega_{\rm int}} := \Big(\int_{\Omega_{\rm int}} \boldsymbol{\sigma}_{\rm C}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) \Big)^{1/2}, \qquad \|\mathbf{u}\|_{{\rm H},\mathcal{O}} := \Big(\int_{\mathcal{O}} \boldsymbol{\sigma}_{\rm H}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) \Big)^{1/2}.$$

In the second case case we consider both $\mathcal{O} = \Omega_{\rm int}$ and $\mathcal{O} = \mathbb{R}^3$.

Note that because of the demands given to the operator defining the constitutive law in $\Omega_{\rm int}$ and Korn's inequality, it follows that $\|\cdot\|_{{\rm C},\Omega_{\rm int}}$ and $\|\cdot\|_{{\rm H},\Omega_{\rm int}}$ are both norms in $\mathbf{H}^1_\Sigma(\Omega_{\rm int})$ and that they are equivalent. Using the fact that the space of $\mathcal{C}^\infty$ compactly supported vector fields in $\mathbb{R}^3$ is dense in $\mathbf{W}^1(\mathbb{R}^3)$ (this can be easily proved from arguments and results given in [20, Chapter 2]), it follows that $\|\cdot\|_{{\rm H},\mathbb{R}^3}$ is a norm in $\mathbf{W}^1(\mathbb{R}^3)$, endowed with its natural norm. Note that $\mathbf{W}^1(\mathbb{R}^3)$ does not contain any non–trivial infinitesimal rigid motion.

For the exterior formulation we need the fundamental solution to the three dimensional Lamé equation (see [13, Chapter 2])

$$\mathbf{E}(\mathbf{x},\mathbf{y}) := \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} + \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} (\mathbf{x}-\mathbf{y})\,(\mathbf{x}-\mathbf{y})^\top$$

($\mathbf{x}$ and $\mathbf{y}$ are understood as column vectors in the final expression). We can then define the single and double layer potentials

$$\mathbf{S}\boldsymbol{\lambda} := \int_\Gamma \mathbf{E}(\,\cdot\,,\mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) {\rm d}\Gamma(\mathbf{y}), \qquad \mathbf{D}\boldsymbol{\varphi} := \int_\Gamma \mathbf{t}_{{\rm H},\mathbf{y}} \mathbf{E}(\,\cdot\,,\mathbf{y})\, \boldsymbol{\varphi}(\mathbf{y})\, {\rm d}\Gamma(\mathbf{y})$$

where $\mathbf{t}_{\mathrm{H},\mathbf{y}}$ has been used to denote the application of the normal stress operator using Hooke's law when $\mathbf{y}$ is considered as the space variable. Then if $\mathbf{u} \in \mathbf{W}^1(\Omega_+)$ satisfies $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{H}}(\mathbf{u}) = \mathbf{0}$ in Ω_+, it can be written using the Betti–Somogliana identity

$$\mathbf{u} = \mathbf{D}(\boldsymbol{\gamma}^+\mathbf{u}) - \mathbf{S}(\mathbf{t}_{\mathrm{H}}^+(\mathbf{u}))$$

as well as a single layer potential $\mathbf{u} = \mathbf{S}\boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \mathbf{H}^{-1/2}(\Gamma) := (H^{-1/2}(\Gamma))^3$. The corresponding boundary integral operators $\mathbf{V}$, $\mathbf{K}$ and $\mathbf{K}^t$ are defined as in the case of the Yukawa equation. The following key identity holds:

$$\langle \boldsymbol{\psi}, \mathbf{V}\boldsymbol{\psi}\rangle = \|\mathbf{S}\boldsymbol{\psi}\|^2_{\mathrm{H},\mathbb{R}^3}, \qquad \forall \boldsymbol{\psi} \in \mathbf{H}^{-1/2}(\Gamma). \tag{5.2}$$

Here we have used the angled bracket for the duality product between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma) := (H^{1/2}(\Gamma))^3$. This formula allows to prove ellipticity of $\mathbf{V}$. Recall that its scalar counterpart was at the core of the ellipticity analysis of the Johnson–Nédélec coupling (Theorem 3.1).

In this case, we can again begin with the postprocessed Johnson–Nédélec coupling and its postprocessed transpose or think directly in terms of the Bielak–MacCamy type coupling (which unlike in [1] we are doing in the normal stress on Γ instead of in the normal pseudostress): we look for $(\mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\psi}) \in \mathbf{H}^1_\Sigma(\Omega_{\mathrm{int}}) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\Gamma)$ such that

$$a_{\mathrm{C}}(\mathbf{u}, \mathbf{v}) - \delta\langle \boldsymbol{\lambda}, \boldsymbol{\gamma}\mathbf{v}\rangle + (1-\delta)\langle(\tfrac{1}{2}\mathbf{I} - \mathbf{K})^t\boldsymbol{\psi}, \boldsymbol{\gamma}\mathbf{v}\rangle = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^1_\Sigma(\Omega_{\mathrm{int}}), \tag{5.3a}$$

$$\langle \boldsymbol{\mu}, (\tfrac{1}{2}\mathbf{I} - \mathbf{K})\boldsymbol{\gamma}\mathbf{u}\rangle + \langle \boldsymbol{\mu}, \mathbf{V}\boldsymbol{\lambda}\rangle = 0, \quad \forall \boldsymbol{\mu} \in \mathbf{H}^{-1/2}(\Gamma), \tag{5.3b}$$

$$-\langle \boldsymbol{\phi}, \boldsymbol{\gamma}\mathbf{u}\rangle + \langle \boldsymbol{\phi}, \mathbf{V}\boldsymbol{\psi}\rangle = 0, \quad \forall \boldsymbol{\phi} \in \mathbf{H}^{-1/2}(\Gamma), \tag{5.3c}$$

where

$$a_{\mathrm{C}}(\mathbf{u}, \mathbf{v}) := \int_{\Omega_{\mathrm{int}}} \boldsymbol{\sigma}_{\mathrm{C}}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}), \qquad \ell(\mathbf{v}) := \int_{\Omega_{\mathrm{int}}} \mathbf{f}\cdot\mathbf{v}.$$

We will also deal with a general Galerkin approximation of this problem.

Theorem 5.1 *Assume that there exists $C_{\mathrm{mat}} < 2$ such that*

$$\|\mathbf{u}\|_{\mathrm{H},\Omega_{\mathrm{int}}} \le C_{\mathrm{mat}}\|\mathbf{u}\|_{\mathrm{C},\Omega_{\mathrm{int}}}, \qquad \forall \mathbf{u} \in \mathbf{H}^1_\Sigma(\Omega_{\mathrm{int}}). \tag{5.4}$$

In this case Galerkin methods for (5.3) *are stable with constants independent of the sequence of discrete spaces and of $\delta \in [0,1]$.*

Proof. As in all the preceding cases, all the proof hinges on the proof of ellipticity of the bilinear form

$$b\big((\mathbf{u}, \boldsymbol{\psi}), (\mathbf{v}, \boldsymbol{\phi})\big) := a_{\mathrm{C}}(\mathbf{u}, \mathbf{v}) + \langle(\tfrac{1}{2}\mathbf{I} - \mathbf{K})^t\boldsymbol{\psi}, \boldsymbol{\gamma}\mathbf{v}\rangle - \langle\boldsymbol{\phi}, \boldsymbol{\gamma}\mathbf{u}\rangle + \langle\boldsymbol{\phi}, \mathbf{V}\boldsymbol{\psi}\rangle$$

in $\mathbf{H}^1_\Sigma(\Omega_{\mathrm{int}}) \times \mathbf{H}^{-1/2}(\Gamma)$. However, if we define $\mathbf{u}^* := \mathbf{S}\boldsymbol{\psi}$, then

$$\begin{aligned} b\big((\mathbf{u}, \boldsymbol{\psi}), (\mathbf{u}, \boldsymbol{\psi})\big) &= \|\mathbf{u}\|^2_{\mathrm{C},\Omega_{\mathrm{int}}} - \langle \mathbf{t}^-_{\mathrm{H}}(\mathbf{u}^*), \boldsymbol{\gamma}\mathbf{u}\rangle + \|\mathbf{u}^*\|^2_{\mathrm{H},\mathbb{R}^3} \\ &= \|\mathbf{u}\|^2_{\mathrm{C},\Omega_{\mathrm{int}}} - a_{\mathrm{H}}(\mathbf{u}^*, \mathbf{u}) + \|\mathbf{u}^*\|^2_{\mathrm{H},\mathbb{R}^3} \\ &\ge \|\mathbf{u}\|^2_{\mathrm{C},\Omega_{\mathrm{int}}} - C_{\mathrm{mat}}\|\mathbf{u}^*\|_{\mathrm{H},\Omega_{\mathrm{int}}}\|\mathbf{u}\|_{\mathrm{C},\Omega_{\mathrm{int}}} + \|\mathbf{u}^*\|^2_{\mathrm{H},\mathbb{R}^3} \\ &\ge \Big(1 - \frac{C_{\mathrm{mat}}}{2}\Big)\Big(\|\mathbf{u}\|^2_{\mathrm{H},\Omega_{\mathrm{int}}} + \langle\boldsymbol{\psi}, \mathbf{V}\boldsymbol{\psi}\rangle\Big) \end{aligned}$$

where we have used (5.2), (5.4) and a_{H} has been used to denote the energy bilinear form for the Hookean material law in Ω_{int} . Moreover, (5.2) can be used to prove that $\langle \boldsymbol{\psi}, \mathbf{V}\boldsymbol{\psi}\rangle^{1/2}$ is an equivalent norm in $\mathbf{H}^{-1/2}(\Gamma)$. This finishes the proof of the ellipticity of the bilinear form b. The remainder of the proof follows step by step the proof of Theorem 2.2 (that is, Section 3). □

Corollary 5.2 *If the interior material law is the same as the exterior one, then the results of Theorem 5.1 hold true.*

Proof. In this case, (5.4) is satisfied with $C_{\mathrm{mat}} = 1$. □

If the energy inequality (5.4) is not satisfied, the proof cannot be carried out. It remains as an open question whether the system is still elliptic (which does not appear to be the case) or a compactness argument (in the spirit of the one used for the consideration of Neumann boundary conditions) can be used to show convergence of Galerkin methods with the corresponding approximation property for the sequence of discrete spaces.

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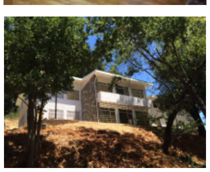

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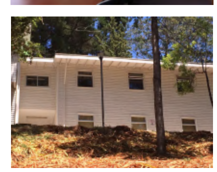

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