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Abstract

We analyze the convergence of a numerical scheme for a class of degenerate parabolic problems modelling reactions in porous media, and involving a nonlinear, possibly vanishing diffusion. The scheme involves the Kirchhoff transformation of the regularized nonlinearity, as well as an Euler implicit time stepping and triangle based finite volumes. We prove the convergence of the approach by giving error estimates in terms of the discretization and regularization parameter.

1. Introduction

Degenerate parabolic equations appear in the mathematical modeling of numerous real life processes. A well known example in this sense is the porous medium equation, describing the flow of an ideal gas in a homogeneous porous medium. More complex situations are encountered in petroleum reservoir and groundwater aquifer simulations. Compared to regular parabolic problems and in particular to the heat equation, the diffusive term in the degenerate case depends on the unknown solution and may vanish or blow up. Thus the parabolic character of the equation may change into an elliptic, or even hyperbolic one. The interfaces separating the domains of regularity - also called free boundaries - have finite speed of propagation. Generally these are not known in advance and have to be determined together with the solution.

Typically the solutions of such problems are lacking regularity. Eventual singularities do not smooth out in time; these may even develop in time, giving the problem a strongly nonlinear character. Consequently, the numerical approximation of such solutions require adequate algorithms that are able to deal with both the free boundary, as well as the singularities of the solution.

This paper is motivated by a combined mixed finite element (MFE) - finite volume (FV) scheme of a two phase flow model for the heap leaching of copper ores [5]. The convergence of such schemes has been investigated in [6] and [18], where the MFE method is employed for the flow component, whereas the saturation is discretized by using a FV scheme. The convergence results there are obtained under several simplifying assumptions that rule out the degeneracy of the model.

For the convergence of numerical schemes for degenerate parabolic problems we refer to [17] for the finite element discretization, [3], [22], [23], [26] for MFE schemes, [24] for a DG approach, and [15] for a multipoint flux approximation method. FV schemes for porous media models are analyzed rigorously in [1], [10], whereas a MFEM-FV approach is considered in [11]. There convergence is obtained by compactness, and no estimates for the approximation error are given.

Here we consider the FV discretization of the degenerate parabolic equation

$$\partial_t u - \Delta\beta(u) = r(u), \quad \text{in } Q_T \equiv (0,T)\times\Omega. \tag{1}$$

Initially we have $u(0) = u^0$ in Ω, whereas $u = 0$ on $\partial\Omega$. In the above $0 < T < \infty$ is fixed, Ω is a bounded polygonal domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary. The function $\beta : \mathbb{R} \to \mathbb{R}$ is non-decreasing and differentiable. Specifically, we assume the following:

(A1) β is Lipschitz and differentiable, $\beta(0) = 0$, $0 \le \beta'(u) \le L_\beta$.

(A2) $u^0 \in L^2(\Omega)$ and $\beta(u^0) \in H_0^1(\Omega)$.

(A3) $r : \mathbb{R} \to \mathbb{R}$ is continuous in u; furthermore,

$$|r(u) - r(v)|^2 \le C(u - v)(\beta(u) - \beta(v))$$

for any $u, v \in \mathbb{R}$, where $C > 0$ does not depend on x, t, u and v. Moreover, $r(0) = 0$.

By degeneracy we mean a vanishing diffusion, namely $\beta'(u) = 0$ for some u. An important example that can be written in the above form is the porous medium equation, where $\beta(u) = u^m$ for some $m > 1$ whenever $u \ge 0$, while $r = 0$. Another example is a simple model for melting and solidification, where β is increasing, piecewise linear, and vanishes on a certain interval, say $[0, 1]$. More complex is the Richards equation, which models unsaturated flow in porous media and involves nonzero convection.

For the ease of presentation restrict here to homogeneous Dirichlet boundary conditions; considering more general ones is straightforward. We use standard notations for function spaces, norms and scalar products: $L^2(\Omega)$, $H_0^1(\Omega)$, or its dual $H^{-1}(\Omega)$. With X being one of the spaces above, $L^2(0,T;X)$ extends the square integrability to time dependent functions. We let $(\cdot,\cdot)$ stand for the inner product on $L^2(\Omega)$, or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|$ for the norm in $L^2(\Omega)$, whereas $\|\cdot\|_k$ denotes the norm in $H^k(\Omega)$. Moreover, we write u or $u(t)$ instead of $u(t,x)$ and use C to denote a positive constant independent of the discretization parameters or the function itself. Saying this, we seek for the weak solution of Problem P, solving:

Problem WP. Find $u \in H^1(0,T;H^{-1}(\Omega))$ such that $\beta(u) \in L^2(0,T;H_0^1(\Omega))$, and

$$(\partial_t u, \varphi) + (\nabla\beta(u), \nabla\varphi) = (r(u), \varphi), \tag{2}$$

for all $\varphi \in H_0^1(\Omega)$ and for all $t \in (0,T]$, whereas $u(0) = u_0$ in $H^{-1}(\Omega)$.

Existence and uniqueness for Problem WP is proven e. g. in [2] and [19]. Notice that $\beta(u)$ is more regular than u, this being a property that we exploit below. We also employ a regularization step in constructing the numerical scheme: with $\varepsilon > 0$ a given parameter, the nonlinear function β is approximated by β_ε satisfying $\beta' \ge \varepsilon$. For simplicity we consider the global perturbation

$$\beta_\varepsilon(u) \equiv \beta(u) + \varepsilon u. \tag{3}$$

Other perturbations can be considered and the analysis is similar. In general, β_ε should be invertible satisfying

$$\varepsilon \le \beta_\varepsilon'(u) \le C,\ 0 \le \beta_\varepsilon'(u) - \beta'(u) \le \varepsilon, \text{ and } C \le \left(\beta_\varepsilon^{-1}\right)'(\beta_\varepsilon(u)) \le 1/\varepsilon. \tag{4}$$

2. The time discretization

Due to the lacking regularity, we employ the Euler implicit scheme to discretize the regularized Problem WP in time. This idea is used for constructing effective numerical algorithms; see

e.g. [14], where compactness arguments are considered for showing the convergence in a general setting. Further, since $\beta(u)$ is more regular than u, we first approximate $\theta \approx \beta(u)$, and then $u = \beta_\varepsilon^{-1}(\theta)$. With $n \in \mathbb{N}$ and $\tau = T/n$ denoting the (fixed) time step, we let $t_k = k\tau$. For $k \in \{1, \dots, n\}$ we define the time discrete approximation θ^k of $\beta(u(t_k))$ as the solution of
Problem WT$_k$. Given $\theta^{k-1} \in H_0^1(\Omega)$, find $\theta^k \in H_0^1(\Omega)$ such that for all $\varphi \in H_0^1(\Omega)$

$$(\beta_\varepsilon^{-1}(\theta^k) - \beta_\varepsilon^{-1}(\theta^{k-1}), \varphi) + \tau(\nabla\theta^k, \nabla\varphi) = \tau(r(\beta_\varepsilon^{-1}(\theta^k)), \varphi). \tag{5}$$

The scheme is completed by the initial data. A straightforward choice is $\theta^0 = \beta_\varepsilon(u^0)$. Whenever $u^0 \in H_0^1(\Omega)$, this gives θ^0 in the same space. However, in (A2) we have only required $u^0 \in L^2(\Omega)$. Following the discussion in [20], Chapter 3, one can consider $\theta^0 = \beta(u^0) + \varepsilon\rho_\mu * u^0$, where ρ_μ is a mollifier having a compact support in $B(0,\mu)$. With $\mu = O(\varepsilon)$, θ^0 is bounded uniformly in H^1, whereas $\|u^0 - \beta_\varepsilon^{-1}(\theta^0)\|$ vanishes as $\varepsilon \searrow 0$. It is worth mentioning that the convolution can be replaced by the solution of the heat equation at a (small) time, where the initial data is u^0.

Remark 2.1. *Inverting β_ε may be tedious. Further, since function calls increase the computing time significantly, for implementing the scheme we construct a look-up table of values for β_ε. Requiring little computer memory, as well as linear interpolation for the values not included in the table, this leads to a reduction of the computing time. Moreover, the monotonicity of β_ε allows a fast searching in this table and thus the values of β_ε or its inverse can be obtained efficiently.*

Existence and uniqueness for the time discrete problems WP$_k$ is provided by standard results for nonlinear elliptic equations. Furthermore, assuming that the initial data is essentially bounded, the sequence of solutions θ^k remain essentially bounded uniformly in k. The following estimates are obtained in [20], Chapter 3 (see also [21]).

Lemma 2.1. *Assume (A1) - (A3). With θ^k solving the problems WT$_k$, for all $0 \le p \le n$ we have*

$$\|\theta^p\|^2 + \sum_{k=1}^{n} \left\{(\beta_\varepsilon^{-1}(\theta^k) - \beta_\varepsilon^{-1}(\theta^{k-1}), \theta^k - \theta^{k-1}) + \|\theta^k - \theta^{k-1}\|^2 + \tau\|\nabla\theta^k\|^2\right\} \le C. \tag{6}$$

Remark 2.2. *The estimate (6) immediately implies*

$$\sum_{k=1}^{n} \|\beta_\varepsilon^{-1}(\theta^k) - \beta_\varepsilon^{-1}(\theta^{k-1})\|_{-1}^2 \le C\tau. \tag{7}$$

To prove the above we use (5) and the Poincaré inequality to obtain

$$|(\beta_\varepsilon^{-1}(\theta^k) - \beta_\varepsilon^{-1}(\theta^{k-1}), \varphi)| \le \tau\big(\|\nabla\theta^k\| + C\|r(\beta_\varepsilon^{-1}(\theta^k))\|\big)$$

for all $\varphi \in H_0^1(\Omega)$ s.t. $\|\nabla\varphi\| = 1$. Now (2.2) follows by (A3) and the above estimates.

We use the following notations. Given a function $f : Q_T \to \mathbb{R}$ integrable in time, define

$$\bar{f}^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(s,\cdot)ds, \quad \text{if } k \ge 1.$$

Further, $\bar{f}^0 := f(0,\cdot)$. The errors are obtained in terms of e_u^k and e_θ^k defined as

$$e_u^k := \bar{u}^k - \beta_\varepsilon^{-1}(\theta^k), \quad e_\theta^k := \overline{\beta(u)}^k - \theta^k, \tag{8}$$

where $k \geq 0$. Given a sequence $\{f^k \in H_0^1(\Omega), k = \overline{1,n}\}$, the piecewise constant extension in time f_Δ is defined as $f_\Delta(t) = \theta^k$ for $t \in (t_{k-1}, t_k]$.

Further, $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ stands for the Green operator defined by

$$(\nabla G\psi, \nabla\varphi) = (\psi, \varphi), \tag{9}$$

for all $\varphi \in H_0^1(\Omega)$, where $\psi \in H^{-1}(\Omega)$. Therefore

$$\|\nabla G\psi\| = \|\psi\|_{-1}, \quad \|\psi\|_{-1} \leq C\|\psi\|, \tag{10}$$

where the last inequality applies only if $\psi \in L^2(\Omega)$. We have ([20], Chapter 3):

Theorem 2.1. *Assume (A1) -(A3). If u and θ^k ($k = \overline{1,n}$) solve the problems given above, then*

$$\sup_{k=\overline{1,n}} \|e_u^k\|_{-1}^2 + \int_0^T (\beta_\varepsilon(u(t)) - \theta_\Delta(t), u(t) - \beta_\varepsilon^{-1}(\theta_\Delta(t)))dt + \|\beta(u) - \theta_\Delta\|_{L^2(Q_T)}^2 \leq C\left\{\tau + \varepsilon\right\}.$$

These estimates hold in a more general setting, where convection terms can also be included. Using the results in [25], the estimates become optimal, $C\{\tau^2 + \varepsilon^2\}$. This holds in a simplified case, e.g. in the absence of convection and if β is maximal monotone having the range $\mathbb{R}$.

3. The finite volume discretization

Here we refer to the framework in [9] (see also [12] and [18]) and let $\mathcal{T} := \{T_i, i \in I \subset \mathbb{N}\}$ be a regular and acute decomposition of Ω into triangles. We assume that the diameter of any triangle $T \in \mathcal{T}$ does not exceed h. Further, $\mathcal{E}$ and $\mathcal{P}$ stand for the set of triangle edges, respectively the set of nodes. Since Ω is assumed polygonal, such a decomposition is possible without introducing additional errors occurring when discretizing curved boundaries. In this case we also have $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$, where $\mathcal{E}_{ext} = \mathcal{E} \cap \partial\Omega$ and $\mathcal{E}_{int} = \mathcal{E} \backslash \mathcal{E}_{ext}$. In what follows we use the following notation:

$|T|$ - the area of $T \in \mathcal{T}$, $|\ell|$ - the length of $\ell \in \mathcal{E}$,

N_i - the triangles adjacent to $T_i \in \mathcal{T}$, $\mathcal{E}_i$ - the edges of T_i,

$\mathbf{x}_i$ - the center of the circumcircle of T_i,

ℓ_{ij} - the edge between T_i and T_j (where $T_j \in N_i$),

$d(\mathbf{x}_i, \ell_{ij})$ - the distance from $\mathbf{x}_i$ to ℓ_{ij}, and $d_{ij} = d(\mathbf{x}_i, \ell_{ij}) + d(\mathbf{x}_j, \ell_{ij})$ if $\ell_{ij} \in \mathcal{E}_{int}$,

$\sigma_{ij} = |\ell_{ij}|/d_{ij}$ - the "transmissibility" through ℓ_{ij}

$\mathbf{n}_{ij}$ - the outward unit normal to ℓ_{ij} pointing into T_j ($T_j \in N_i$).

The assumptions on $\mathcal{T}$ ensure that $\mathbf{x}_i \in int(T_i)$ for all i. Furthermore, if $T_j \in N_i$, then the line through $\mathbf{x}_i$ and $\mathbf{x}_j$ is orthogonal to ℓ_{ij}. Given $\mathcal{T}$ we define the finite dimensional subspace of $L^2(\Omega)$

$$W_h := \{v \in L^2(\Omega) / \ v \text{ is constant on any } T \in \mathcal{T}\}, \tag{11}$$

which is spanned by the triangle indicator functions $\{\chi_T \ / \ T \in \mathcal{T}\}$. Furthermore we define

$$P_h : L^2(\Omega) \to W_h, \quad (P_h w - w, w_h) = 0 \tag{12}$$

for any $w_h \in W_h$. With $s \in \{0,1\}$, a constant $C > 0$ exists such that

$$\|P_h w - w\| \leq C h^s \|w\|_s \tag{13}$$

for all $w \in H^s(\Omega)$. Moreover, for any $w \in L^2(\Omega)$ and $T_i \in \mathcal{T}$ we have

$$\bar{w}_i := (P_h w)|_{T_i} = \frac{1}{|T_i|} \int_{T_i} w(\mathbf{x}) d\mathbf{x}. \tag{14}$$

As in the spatially continuous case, for any $u, v \in L^2(\Omega)$ we define the discrete inner products

$$(u,v)_h := \sum_{T_i \in \mathcal{T}} |T_i| \bar{u}_i \bar{v}_i, \quad (u,v)_{1,h} := \sum_{\ell_{ij} \in \mathcal{E}} \sigma_{ij} (\bar{u}_i - \bar{u}_j)(\bar{v}_i - \bar{v}_j), \tag{15}$$

where the value of $\bar{u}$ and $\bar{v}$ are extended by 0 outside Ω in view of the homogeneous Dirichlet boundary conditions. The associated discrete norms are denoted by $\|\cdot\|_h$, respectively $\|\cdot\|_{1,h}$. It is easy to see that for all $u, v \in L^2(\Omega)$,

$$(u,v)_h = (u, P_h v) = (P_h u, v). \tag{16}$$

Furthermore, in [9] the following discrete Poincaré inequality is given:

$$\|u\| \leq C \|u\|_{1,h}, \tag{17}$$

for all $u \in W_h$, where $C > 0$ does not depend on h or u. Using (16) and (13), one obtains

$$|(u,v) - (u,v)_h| = |(u - P_h u, v)| = |(u - P_h u, v - P_h v)| \leq C h^{s+p} \|u\|_s \|v\|_p, \tag{18}$$

for all $u \in H^s(\Omega)$ and $v \in H^p(\Omega)$, $s, p \in \{0,1\}$. Further we define the discrete H^{-1} norm

$$\|u\|_{-1,h} = \sup_{w_h \in W_h, \|w_h\|_{1,h} = 1} |(u, w_h)_h|. \tag{19}$$

Following [9] and [12] we write the finite volume scheme for the time discrete problem WT$_k$. With $\theta_{h,i} = \theta_h|_{T_i}$ and given $\theta_h^{k-1} \in W_h$, we seek $\theta^k \in W_h$ such that for all $T_i \in \mathcal{T}$ it holds

$$|T_i| (\beta_\varepsilon^{-1}(\theta_{h,i}^k) - \beta_\varepsilon^{-1}(\theta_{h,i}^{k-1})) + \tau \sum_{\ell_{ij} \in \mathcal{E}_i} \sigma_{ij} (\theta_{h,i}^k - \theta_{h,j}^k) = \tau |T_i| r(\beta_\varepsilon^{-1}(\theta_{h,i}^k)). \tag{20}$$

To give a weak form of the above scheme, for any $w_h \in W_h$ we multiply (20) by $w_i = w_h|_{T_i}$ and sum up the resulting for all $T_i \in \mathcal{T}$. Recalling the definitions in (15), after changing the summation order in the second term on the left we obtain the following

Problem WD$_k$. Given $\theta_h^{k-1} \in W_h$, find $\theta_h^k \in W_h$ such that for all $w_h \in W_h$

$$(\beta_\varepsilon^{-1}(\theta_h^k) - \beta_\varepsilon^{-1}(\theta_h^{k-1}), w_h)_h + \tau (\theta_h^k, w_h)_{1,h} = \tau (r(\beta_\varepsilon^{-1}(\theta_h^k)), w_h)_h. \tag{21}$$

To complete the scheme, we define the initial data $\theta_h^0 = \beta_\varepsilon(P_h(\beta_\varepsilon^{-1}(\theta^0))) \in W_h$.

The stability properties below are similar to the ones in the time discrete case.

Lemma 3.1. *Assume (A1) - (A3), and let θ_h^k solving (21). For any $1 \leq p \leq n$ we have*

$$\begin{aligned} \|\beta_\varepsilon^{-1}(\theta_h^p)\|_h^2 + \textstyle\sum_{k=1}^n \|\beta_\varepsilon^{-1}(\theta_h^k) - \beta_\varepsilon^{-1}(\theta_h^{k-1})\|_h^2 + \tau \sum_{k=1}^n \|\theta_h^k\|_{1,h}^2 &\leq C, \\ \textstyle\sum_{k=1}^n (\beta_\varepsilon^{-1}(\theta_h^k) - \beta_\varepsilon^{-1}(\theta_h^{k-1}), \theta_h^k - \theta_h^{k-1})_h + \sum_{k=1}^n \|\theta_h^k - \theta^{k-1}\|_h^2 &\leq C. \end{aligned} \tag{22}$$

Proof. We start by noticing that for any $w_h \in W_h$, by (A3) and (4) we get

$$C\|w_h\|_{1,h}^2 \le (\beta_\varepsilon^{-1}(w_h), w_h)_{1,h}, \text{ and } |(r(w_h), w_h)_h| \le C(w_h, \beta(w_h))_h^{\frac{1}{2}} \|w_h\|_h \le C\|w_h\|_h^2. \quad (23)$$

Next we take $w_h = \beta_\varepsilon^{-1}(\theta_h^k)$ into (21), sum the resulting up for $k = 1, \ldots, p$ and use the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and obtain

$$\begin{aligned} &\tfrac{1}{2}\big(\|\beta_\varepsilon^{-1}(\theta_h^p)\|_h^2 - \|\beta_\varepsilon^{-1}(\theta_h^0)\|_h^2 + \textstyle\sum_{k=1}^p \|\beta_\varepsilon^{-1}(\theta_h^k) - \beta_\varepsilon^{-1}(\theta_h^{k-1})\|_h^2\big) \\ &\quad + \tau \textstyle\sum_{k=1}^p (\theta_h^k, \beta_\varepsilon^{-1}(\theta_h^k))_{1,h} = \tau \textstyle\sum_{k=1}^p (r(\beta_\varepsilon^{-1}(\theta_h^k)), \beta_\varepsilon^{-1}(\theta_h^k))_h. \end{aligned}$$

Using (23), the first part of the estimates is a direct consequence of the Gronwall lemma.

By the assumptions on β and r, testing (21) with $w_h = \theta_h^k - \theta_h^{k-1}$ completes the proof. ■

Remark 3.1. *As in the spatially continuous case, the estimates above immediately imply*

$$\sum_{k=1}^n \|\beta_\varepsilon^{-1}(\theta_h^k) - \beta_\varepsilon^{-1}(\theta_h^{k-1})\|_{-1,h}^2 \le C\tau. \quad (24)$$

The fully discrete problems have unique solutions, as follows from:

Theorem 3.1. *Assume (A1) - (A3), and let $\theta_h^{k-1} \in W_h$ be given. Then the fully discrete problem WD_k has a unique solution θ_h^k, at least for moderately small time steps τ.*

Proof. We start with the uniqueness, which is a direct consequence of the monotonicity of β. To see this we consider two piecewise constant functions $\theta_h, \bar{\theta}_h \in W_h$ satisfying (21) for any $w_h \in W_h$. Subtracting the resulting two equalities we obtain

$$(\beta_\varepsilon^{-1}(\theta_h) - \beta_\varepsilon^{-1}(\bar{\theta}_h), w_h)_h + \tau\|w_h\|_{1,h}^2 = \tau(r(\beta_\varepsilon^{-1}(\theta_h)) - r(\beta_\varepsilon^{-1}(\bar{\theta}_h)), w_h)_h. \quad (25)$$

With $w_h = \theta_h - \bar{\theta}_h$, using (A3), the Cauchy inequality and the inequality of means, we obtain

$$\tau(r(\beta_\varepsilon^{-1}(\theta_h)) - r(\beta_\varepsilon^{-1}(\bar{\theta}_h)), \theta_h - \bar{\theta}_h)_h \le \frac{1}{2}(\beta_\varepsilon^{-1}(\theta_h) - \beta_\varepsilon^{-1}(\bar{\theta}_h), \theta_h - \bar{\theta}_h)_h + \frac{\tau^2}{2\bar{C}}\|\theta_h - \bar{\theta}_h\|_h^2.$$

By the discrete Poincaré inequality, uniqueness follows whenever $\tau \le C$, where $C > 0$ is a constant that does not depend on the parameters τ, h, or ε.

For the existence we use Lemma 1.4, p. 140 in [27], which is an abstract result for finite dimensional Hilbert spaces. In this sense we define the continuous mapping $\mathcal{P} : W_h \to W_h$

$$\mathcal{P}\theta = \Phi = \sum_{T_i \in \mathcal{T}} \alpha_i \chi_{T_i},$$

where χ_{T_i} is the indicator function of the triangle T_i, while $\alpha_i \in \mathbb{R}$ are given by

$$\alpha_i = |T_i|(\beta_\varepsilon^{-1}(\theta_i) - \beta_\varepsilon^{-1}(\theta_{h,i}^{k-1})) + \tau \sum_{\ell_{ij} \in \mathcal{E}_i} \sigma_{ij}(\theta_i - \theta_j) - \tau|T_i| r(\beta_\varepsilon^{-1}(\theta_i)). \quad (26)$$

For any $\theta \in W_h$, we use (21) to estimate the inner product $(\mathcal{P}\theta, \theta)_h$:

$$(\mathcal{P}\theta, \theta)_h = (\beta_\varepsilon^{-1}(\theta), \theta)_h + \tau\|\theta\|_{1,h}^2 - (\beta_\varepsilon^{-1}(\theta_h^{k-1}), \theta)_h - \tau(r(\beta_\varepsilon^{-1}(\theta)), \theta)_h.$$

The first term on the right is bounded by

$$(\beta_\varepsilon^{-1}(\theta),\theta)_h \geq \frac{1}{2}(\beta_\varepsilon^{-1}(\theta),\theta)_h + \frac{C}{2}\|\theta|_h^2,$$

whereas for the third term we use (4) and the Cauchy inequality to obtain

$$|(\beta_\varepsilon^{-1}(\theta_h^{k-1}),\theta)_h| \leq C\|\theta_h^{k-1}\|_h\|\theta\|_h \leq \frac{C'}{\delta}\|\theta_h^{k-1}\|_h^2 + \delta\|\theta|_h^2.$$

Proceeding as for the apriori estimates (22), the last term yields

$$\tau|(r(\beta_\varepsilon^{-1}(\theta)),\theta)_h| \leq \frac{1}{4}(\beta_\varepsilon^{-1}(\theta),\theta)_h + C\tau^2\|\theta|_h^2.$$

Choosing δ properly, for moderately small τ the above inequalities as well as (17) imply

$$(\mathcal{P}\theta,\theta)_h \geq \frac{1}{4}(\beta_\varepsilon^{-1}(\theta),\theta)_h + \frac{\tau}{2}\|\theta\|_{1,h}^2 - K,$$

with $K = \tilde{C}\|\theta_h^{k-1}\|_h^2$ for some fixed $\tilde{C}$. Now existence follows by the result mentioned above. ■

For obtaining the error estimates we proceed as in the time discrete case and use the discrete Green operator $G_h : L^2(\Omega) \to W_h$ defined by

$$(G_h\psi,\varphi)_{1,h} = (\psi,\varphi)_h, \tag{27}$$

for all $\varphi \in W_h$. As in the spatially continuous case, for any $\psi \in W_h$ one gets

$$\|G_h\psi\|_{1,h} = \|\psi\|_{-1,h}, \quad \|\psi\|_{-1,h} \leq C\|\psi\|_h. \tag{28}$$

G_h is well defined as the FV approximation of the Poisson equation with homogeneous Dirichlet boundary conditions and an L^2 right hand side (see [9] and [12]). As shown in [8], [9] and [12], for any $\psi \in L^2(\Omega)$ one has the estimates

$$\|(G - G_h)\Psi\|_{1,h} \leq Ch\|\Psi\|. \tag{29}$$

To estimate the error due to the spatial discretization we define for each time step $t = k\tau$

$$e_u^{k,h} := \beta_\varepsilon^{-1}(\theta^k) - \beta_\varepsilon^{-1}(\theta_h^k), \quad e_\theta^{k,h} := \theta^k - \theta_h^k, \tag{30}$$

see also (8). The errors defined above are estimated in the following lemma:

Lemma 3.2. *Assume (A1) - (A3), let θ^k and θ_h^k solving (5), respectively (21). We have*

$$\sup_{k=\overline{1,n}} \|e_u^{k,h}\|_{-1}^2 + C\tau\sum_{k=1}^n \|e_\theta^{k,h}\|^2 + C\tau\sum_{k=1}^n (e_u^{k,h},e_\theta^{k,h}) \leq C\left(\|e_u^{0,h}\|_{-1}^2 + h^2/\varepsilon\right),$$

provided τ is reasonably small.

Proof. We take $\varphi = Ge_u^{k,h} \in H_0^1$ in (5) and $\varphi = G_h e_u^{k,h} \in W_h$ in (21), subtract the resulting and use (16) to obtain

$$\begin{aligned}(e_u^{k,h} - e_u^{k-1,h}, Ge_u^{k,h}) &+ \tau\left[(\nabla\theta^k,\nabla Ge_u^{k,h}) - (\theta_h^k, G_h e_u^{k,h})_{1,h}\right] \\ &= -(\beta^{-1}(\theta_h^k) - \beta^{-1}(\theta_h^{k-1}), (G-G_h)e_u^{k,h})_h \\ &\quad +\tau(r(\beta^{-1}(\theta_h^k)), (G-G_h)e_u^{k,h})_h + \tau(r(\beta^{-1}(\theta^k)) - r(\beta^{-1}(\theta_h^k)), Ge_u^{k,h}).\end{aligned} \tag{31}$$

We sum up the above for $k = 1, \ldots, p$, denote the resulting terms by $S_1, \ldots, S_5$, and proceed by estimating them separately. By (9), for S_1 we have

$$2S_1 = 2\sum_{k=1}^{p}(\nabla G(e_u^{k,h} - e_u^{k-1,h}), \nabla G e_u^{k,h}) = \|e_u^{p,h}\|_{-1}^2 - \|e_u^{0,h}\|_{-1}^2 + \sum_{k=1}^{p}\|e_u^{k,h} - e_u^{k-1,h}\|_{-1}^2. \quad (32)$$

Using (4), (9), (16) and (27), S_2 becomes

$$\begin{aligned} S_2 &= \tau\textstyle\sum_{k=1}^{p}(\theta^k, e_u^{k,h}) - (\theta_h^k, e_u^{k,h})_h = \tau\sum_{k=1}^{p}(e_\theta^{k,h}, e_u^{k,h}) \\ &\geq \tfrac{\tau}{3}\textstyle\sum_{k=1}^{p}(e_\theta^{k,h}, e_u^{k,h}) + C\tau\sum_{k=1}^{p}\|e_\theta^{k,h}\|^2 + \frac{\tau\varepsilon}{3}\sum_{k=1}^{p}\|e_u^{k,h}\|^2. \end{aligned} \quad (33)$$

To estimate S_3 we use the estimates (24) and (29), as well as (19) and (28) to obtain

$$\begin{aligned} |S_3| &\leq \textstyle\sum_{k=1}^{p}\|\beta^{-1}(\theta_h^k) - \beta^{-1}(\theta_h^{k-1})\|_{-1,h}\|(G - G_h)e_u^{k,h}\|_{1,h} \\ &\leq \delta_1\textstyle\sum_{k=1}^{p}\|\beta^{-1}(\theta_h^k) - \beta^{-1}(\theta_h^{k-1})\|_{-1,h}^2 + \frac{Ch^2}{4\delta_1}\sum_{k=1}^{p}\|e_u^{k,h}\|^2 \\ &\leq\ \leq C\tfrac{h^2}{\varepsilon} + \tfrac{\tau\varepsilon}{12}\textstyle\sum_{k=1}^{p}\|e_u^{k,h}\|^2, \end{aligned} \quad (34)$$

where in the above we have taken $\delta_1 = O(h^2/(\tau\varepsilon))$. Alternatively, the L^2 estimates for $\beta^{-1}(\theta_h)$ and $\beta^{-1}(\theta_h^k)$ imply $\tau\sum_{k=1}^{p}\|e_u^{k,h}\|^2 \leq C$. For $\delta_1 = h/\tau$ this yields $|S_3| \leq Ch$.

For S_4 we use (A3) and (17), and proceed in a similar manner to get

$$\begin{aligned} |S_4| &\leq \tau\textstyle\sum_{k=1}^{p}\|r(\beta^{-1}(\theta_h^k))\|_{-1,h}\|(G - G_h)e_u^{k,h}\|_{1,h} \\ &\leq C\tau h\textstyle\sum_{k=1}^{p}\|r(\beta^{-1}(\theta_h^k))\|_h\|e_u^{k,h}\| \leq C\frac{h^2}{\varepsilon} + \frac{\tau\varepsilon}{12}\sum_{k=1}^{p}\|e_u^{k,h}\|^2. \end{aligned} \quad (35)$$

As above, the alternative estimate is $|S_4| \leq Ch$. By (A3), the last term gives

$$\begin{aligned} |S_5| &\leq \tau\textstyle\sum_{k=1}^{p}\|r(\beta^{-1}(\theta_h^k)) - r(\beta^{-1}(\theta_h^{k-1}))\|\|Ge_u^{k,h}\| \\ &\leq \tilde{C}\tau\delta_2\textstyle\sum_{k=1}^{p}(e_\theta^{k,h}, e_u^{k,h}) + \frac{\tau}{\delta_2}\sum_{k=1}^{p}\|e_u^{k,h}\|_{-1}^2. \end{aligned} \quad (36)$$

Taking $\delta_2 = 1/(6\tilde{C})$, and using (31) - (36), the discrete Gronwall lemma provides the result. ■

Remark 3.2. *As following from the proof, the ratio h^2/ε in the estimates can be replaced by h. Furthermore, the initial error can be made arbitrarily small. To see this, notice that $e_u^{0,h} = \beta_\varepsilon^{-1}(\theta^0) - P_h(\beta_\varepsilon^{-1}(\theta^0))$. As mentioned in the beginning of Section 2, a mollifying step is involved in constructing a θ^0 that is H^1, having an ε-uniformly bounded norm. By (13) this gives $\|e_u^{0,h}\| \leq Ch$.*

The FV scheme is convergent, as follows from Theorem 2.1 and of Lemma 3.1:

Theorem 3.2. *Assuming (A1) -(A3), the FV approximation converges to the solution of Problem WP as τ, h and $\varepsilon \searrow 0$. The following estimates hold*

$$\begin{aligned} \sup_{k=\overline{1,n}}\|\bar{u}^k - \beta_\varepsilon^{-1}(\theta_h^k)\|_{-1}^2 + \textstyle\int_0^T(\beta_\varepsilon(u(t)) - \theta_{\Delta,h}(t), u(t) - \beta_\varepsilon^{-1}(\theta_{\Delta,h}(t)))dt \\ +\|\beta(u) - \theta_{\Delta,h}\|_{L^2(Q_T)}^2 \leq C\left\{\tau + \varepsilon + \tfrac{h^2}{\varepsilon}\right\}, \end{aligned}$$

where (similarly to the time discrete case) $\theta_{\Delta,h}(t) = \theta_h^k$ whenever $t \in (t_{k-1}, t_k]$.

Remark 3.3. *The above estimates are sub-optimal when compared to the ones for the heat equation. As mentioned before, in a certain framework one can obtain optimal (first order) estimates for the time discretization. To extend such a result to the FV discretization one needs higher order estimates in (29), as suggested e.g. in [9], [4] and [7].*

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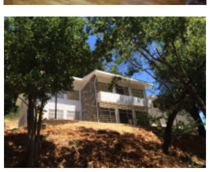
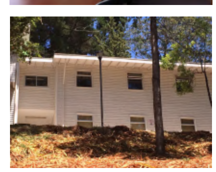

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