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PREPRINT 2009-22

SERIE DE PRE-PUBLICACIONES

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Abstract

We consider an electromagnetic scattering problem produced by a perfect conductor. We pose the problem in a bounded region surrounding the obstacle and impose on the exterior boundary of the computational domain an impedance boundary condition inspired from the asymptotic behavior of the scattered field at infinity. The operator associated to our problem belongs to a class of operators for which a suitable decomposition of the energy space plays an essential role in the analysis. This decomposition is performed here through a regularising projector that takes into account the boundary conditions. The discrete version of this projector is the key tool to prove that a Galerkin scheme based on Nédélec's edge elements is well-posed and convergent under general topological asumptions on the scatterer and without assuming special requirements on the triangulations.

Key words: Maxwell equations, edge finite elements

Mathematics subject classifications (1991): 65N30, 65N12, 65N15

1 Introduction

This paper deals with the finite element approximation of an electromagnetic scattering problem. More precisely, we consider a conductor occupying a bounded region Ω_c and assume that it is subject to a given time-harmonic incident wave. Our purpose is to provide a finite element approximation of the scattered electromagnetic wave. We avoid here the difficulty related with the fact that the problem is posed in an unbounded domain by introducing an artificial boundary Γ (located sufficiently far from the obstacle) and considering a computational domain represented by the region Ω delimited by Γ and $\Sigma := \partial\Omega_c$. We impose on the exterior closed surface Γ an absorbing boundary condition that mimics the Silver-Müller radiation condition. Moreover, since the scatterer is assumed to be a perfect conductor, the eventual penetration of the electric field inside the obstacle can be neglected, whence the tangential trace of the electric field vanishes on the interface Σ.

The importance of the time harmonic Maxwell system in real word applications is undeniable. In spite of this, convergence results concerning the Galerkin approximations of this problem with Nédeléc's finite elements only appeared at the beginning of the nineties. Monk [14] was the first to prove quasi-optimal error estimates for this model problem posed in a convex domain Ω. Then, Hiptmair [12] and Monk [15] extended these error estimates to the case of general Lipschitz polyhedrons. The essential tool underlying all the strategies used to deal efficiently with this problem consists in a suitable Helmholtz-type decomposition of the unknown. It reveals hidden compactness properties and

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allows one to handle the problem through a classical analysis. Actually, Buffa [3] succeeded in setting up this technique in a general abstract framework for a certain class of noncoercive operators and applied it to this model problem.

All the aformentioned articles consider a vanishing tangential trace as a boundary condition, i.e., they deal with a model problem posed in $\mathbf{H}_0(\mathbf{curl};\Omega)$. Here we are interested in a generalization of the technique to the case of an impedance boundary condition in a connected component of $\partial\Omega$. This task has already been considered in [16, Chapter 4]. The error analysis of this problem is achieved in Chapter 7 of this book by using a collective compactness property when the finite element families of triangulations are quasiuniform on Γ. Our aim here is to provide a new convergence proof that circumvents this restriction. Our technique is more in the spirit of [12, Section 5] and [3]. In fact, we show that the bilinear form we deal with fits exactly within the theory exposed in [3] and prove that this approach can be successfully adapted to deal with the boundary conditions considered for the present Maxwell system (formulated in terms of the electric field). The construction of a suitable projector is essential in our analysis since it yields an appropriate Helmholtz decomposition of the energy space arising in the formulation. A similar idea was employed in [9] for a decomposition of $H(\mathrm{div},\Omega)$. Moreover, the approximation property satisfied by the discrete version of this projector (see (5.7) below) provides stability and convergence for a Galerkin scheme based on Nédélec's edge finite element. Compared to Monk's strategy, our analysis is free from any special requirement on the finite element triangulations and it does not need the regularity result proved in [16, Lemma 7.15] under the hypothesis of a simply connected domain Ω.

The rest of the paper is organized as follows. In Section 2 we collect some known results on tangential trace operators in a generic space $\mathbf{H}(\mathbf{curl};\Omega)$. In Section 3 we describe the boundary value problem of interest. Then, in Section 4 we derive and analyze the continuous variational formulation. In particular, we use an adequate Helmholtz decomposition to prove its well-posedness. Finally, in Section 5 we introduce the corresponding Galerkin scheme and show that it is convergent.

We end this section with some notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. In addition, given any Hilbert space U, we let $[U]^3$ denote the space of vectors with entries in U. When no confusion arises we simply use U^3 instead of $[U]^3$. Finally, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C , with or without subscripts, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 Preliminaries

We denote by $\Omega \subset \mathbb{R}^3$ a generic bounded polyhedral domain and let $\boldsymbol{n}$ be the outward normal vector on its boundary Γ. We recall that

$$\mathbf{H}(\mathbf{curl};\Omega) := \left\{ \boldsymbol{w} \in [L^2(\Omega)]^3 : \quad \mathbf{curl}(\boldsymbol{w}) \in [L^2(\Omega)]^3 \right\}$$

endowed with the norm $\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} := \|\boldsymbol{w}\|^2_{[L^2(\Omega)]^3} + \|\mathbf{curl}(\boldsymbol{w})\|^2_{[L^2(\Omega)]^3}$ is a Hilbert space and that $[\mathcal{C}^\infty(\overline{\Omega})]^3$ is dense in $\mathbf{H}(\mathbf{curl};\Omega)$. As usual, $\mathbf{curl}(\boldsymbol{w})$ stands for the vector defined formally by $\nabla \times \boldsymbol{w}$. We also recall that

$$\mathbf{H}(\mathrm{div};\Omega) := \left\{\boldsymbol{\tau} \in [L^2(\Omega)]^3 : \quad \mathrm{div}(\boldsymbol{\tau}) \in L^2(\Omega)\right\} ,$$

endowed with the norm $\|\boldsymbol{\tau}\|^2_{\mathbf{H}(\mathrm{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^3} + \|\mathrm{div}(\boldsymbol{\tau})\|^2_{L^2(\Omega)}$ is a Hilbert space and that $[\mathcal{C}^\infty(\overline{\Omega})]^3$ is dense in $\mathbf{H}(\mathrm{div};\Omega)$. It is well known that the mapping

$$\begin{aligned} \gamma_{\boldsymbol{n}} :\ [\mathcal{C}^\infty(\overline{\Omega})]^3 &\longrightarrow L^2(\Gamma) \\ \boldsymbol{\tau} &\longrightarrow \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}) := \boldsymbol{\tau}|_\Gamma \cdot \boldsymbol{n} \end{aligned}$$

can be extended to define a normal trace operator

$$\begin{aligned} \gamma_{\boldsymbol{n}} :\ \mathbf{H}(\mathrm{div};\Omega) &\longrightarrow H^{-1/2}(\Gamma) \\ \boldsymbol{\tau} &\longrightarrow \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}) \end{aligned} \tag{2.1}$$

which is bounded, surjective, and possesses a right inverse.

Tangential traces of functions in $\mathbf{H}(\mathbf{curl};\Omega)$ are also well understood even in the case of polyhedral domains thanks to the recent results of [4, 5]. We give here a brief summary of these fundamental tools. To this end, we begin by defining the space

$$\mathbf{L}^2_{\mathbf{t}}(\Gamma) := \left\{ \boldsymbol{\mu} \in [L^2(\Gamma)]^3 : \quad \boldsymbol{\mu}\cdot\boldsymbol{n} = 0 \right\}$$

and the tangential trace mapping

$$\begin{aligned} \boldsymbol{\gamma}_{\mathbf{t}} :\ [\mathcal{C}^\infty(\overline{\Omega})]^3 &\rightarrow \mathbf{L}^2_{\mathbf{t}}(\Gamma) \\ \boldsymbol{v} &\mapsto \boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{v}) := \boldsymbol{v}|_\Gamma \times \boldsymbol{n} \end{aligned}$$

together with the tangential projection operator

$$\begin{aligned} \boldsymbol{\pi}_{\mathbf{t}} :\ [\mathcal{C}^\infty(\overline{\Omega})]^3 &\rightarrow \mathbf{L}^2_{\mathbf{t}}(\Gamma) \\ \boldsymbol{v} &\mapsto \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{v}) := \boldsymbol{n} \times (\boldsymbol{v}|_\Gamma \times \boldsymbol{n}). \end{aligned}$$

Notice that, because of the orthogonality condition defining $\mathbf{L}^2_{\mathbf{t}}(\Gamma)$, this subspace of $[L^2(\Gamma)]^3$ may be identified in what follows with a space of two dimensional tangent fields. At this point we also recall that

$$\mathbf{H}_0(\mathbf{curl};\Omega) := \left\{ \boldsymbol{w} \in \mathbf{H}(\mathbf{curl};\Omega) : \quad \boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{w}) = \mathbf{0} \quad \text{on} \quad \partial\Omega \right\}.$$

Let us now introduce the spaces

$$\mathbf{H}^{1/2}_\perp(\Gamma) := \boldsymbol{\gamma}_{\mathbf{t}}([H^1(\Omega)]^3) \qquad \text{and} \qquad \mathbf{H}^{1/2}_\parallel(\Gamma) := \boldsymbol{\pi}_{\mathbf{t}}([H^1(\Omega)]^3),$$

which are endowed with the natural Hilbert space structure that makes both $\boldsymbol{\gamma}_{\mathbf{t}} : [H^1(\Omega)]^3 \to \mathbf{H}^{1/2}_\perp(\Gamma)$ and $\boldsymbol{\pi}_{\mathbf{t}} : [H^1(\Omega)]^3 \to \mathbf{H}^{1/2}_\parallel(\Gamma)$ bounded and surjective. Similarly, for any $\delta \in (0,1)$, we define

$$\mathbf{H}^\delta_\parallel(\Gamma) := \boldsymbol{\pi}_{\mathbf{t}}([H^{\delta+1/2}(\Omega)]^3) \tag{2.2}$$

and provide it with an inner product that renders $\boldsymbol{\pi}_{\mathbf{t}} : [H^{\delta+1/2}(\Omega)]^3 \to \mathbf{H}^\delta_\parallel(\Gamma)$ continuous. We refer to [4] for and explicit definition of these spaces in the case of Lipschitz boundaries with piecewise smooth components. In the following, we will also write $\boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{\varphi})$ (or $\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{\varphi})$) for $\boldsymbol{\varphi} \in [H^{1/2}(\Gamma)]^3$, which should be understood as $\boldsymbol{\gamma}_{\mathbf{t}}(\gamma^{-1}(\boldsymbol{\varphi}))$ (or $\boldsymbol{\pi}_{\mathbf{t}}(\gamma^{-1}\boldsymbol{\varphi})$) where $\gamma^{-1} : [H^{1/2}(\Gamma)]^3 \to [H^1(\Omega)]^3$ is a given bounded right-inverse of the usual trace operator $\gamma : [H^1(\Omega)]^3 \to [H^{1/2}(\Gamma)]^3$.

Next, we introduce the dual $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ of $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ and the dual $\mathbf{H}_{\|}^{-1/2}(\Gamma)$ of $\mathbf{H}_{\|}^{1/2}(\Gamma)$ with respect to the pivot space $\mathbf{L}^2_{\mathbf{t}}(\Gamma)$. Then, it is easy to deduce from the Green formula

$$\int_\Omega \Big\{ \boldsymbol{u}\cdot\mathbf{curl}(\boldsymbol{v}) \;-\; \boldsymbol{v}\cdot\mathbf{curl}(\boldsymbol{u}) \Big\} \;=\; \int_\Gamma \boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{u})\cdot\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{v}) \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \,\in\, [\mathcal{C}^\infty(\overline{\Omega})]^3 \tag{2.3}$$

and the fact that $[\mathcal{C}^\infty(\overline{\Omega})]^3$ is dense in $\mathbf{H}(\mathbf{curl};\Omega)$, that $\boldsymbol{\gamma}_{\mathbf{t}}$ and $\boldsymbol{\pi}_{\mathbf{t}}$ can be extended to define bounded tangential mappings from $\mathbf{H}(\mathbf{curl};\Omega)$ onto $\mathbf{H}_{\|}^{-1/2}(\Gamma)$ and from $\mathbf{H}(\mathbf{curl};\Omega)$ onto $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$, respectively. A more precise result is given by the following theorem (see [6]) (we refer to [4, 6] for the definition of the differential operators div_Γ and curl_Γ on piecewise smooth Lipschitz boundaries).

Theorem 2.1 *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) \;:=\; \Big\{\boldsymbol{\mu}\in\mathbf{H}_{\|}^{-1/2}(\Gamma): \quad \mathrm{div}_\Gamma(\boldsymbol{\mu}) \,\in\, H^{-1/2}(\Gamma)\Big\}$$

and

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma) \;:=\; \Big\{\boldsymbol{\mu}\in\mathbf{H}_{\perp}^{-1/2}(\Gamma): \quad \mathrm{curl}_\Gamma(\boldsymbol{\mu}) \,\in\, H^{-1/2}(\Gamma)\Big\}\,.$$

Then

$$\boldsymbol{\gamma}_{\mathbf{t}}\,:\,\mathbf{H}(\mathbf{curl};\Omega)\to\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) \quad \textit{and} \quad \boldsymbol{\pi}_{\mathbf{t}}\,:\,\mathbf{H}(\mathbf{curl};\Omega)\to\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$$

are bounded, surjective and possess continuous right inverses. Moreover, the $[L^2(\Gamma)]^3$-inner product can be extended to define a duality product $\langle\,\cdot,\cdot\,\rangle_{\mathbf{t},\Gamma}$ between the spaces $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$.

As a consequence of this theorem, Green's formula (2.3) can be extended to functions $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathbf{H}(\mathbf{curl};\Omega)$ if the boundary integral of the right hand side is interpreted as $\langle\,\boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{u}),\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{v})\,\rangle_{\mathbf{t},\Gamma}$, that is

$$\int_\Omega \Big\{ \boldsymbol{u}\cdot\mathbf{curl}(\boldsymbol{v}) \;-\; \boldsymbol{v}\cdot\mathbf{curl}(\boldsymbol{u}) \Big\} \;=\; \langle\,\boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{u}),\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{v})\,\rangle_{\mathbf{t},\Gamma} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \,\in\, \mathbf{H}(\mathbf{curl};\Omega)\,. \tag{2.4}$$

3 The model problem

We consider a perfect conductor occupying a region represented by a bounded and connected polyhedra $\Omega_c\subset\mathbb{R}^3$ and immersed in an electromagnetic medium filling the whole space. We denote by Σ_j, $j=0,\cdots,J$ the connected components of $\Sigma:=\partial\Omega_c$, Σ_0 being the boundary of the unbounded component of $\mathbf{R}^3\setminus\overline{\Omega}_c$.

Let ϵ, μ, and σ be the electric permittivity, the magnetic permeability and the conductivity of the medium, respectively. These coefficients are piecewise regular real valued scalar functions satisfying in $\mathbb{R}^3\setminus\Omega_c$,

$$\mu_0\,\le\,\mu(\boldsymbol{x})\,\le\,\bar{\mu}, \qquad \epsilon_0\,\le\,\epsilon(\boldsymbol{x})\,\le\,\bar{\epsilon} \qquad \text{and} \qquad 0\,\le\,\sigma(\boldsymbol{x})\,\le\,\bar{\sigma}\,, \tag{3.1}$$

where the constants ϵ_0 and μ_0 denote the electric permittivity and magnetic permeability of free space, respectively, and $\bar{\mu}$, $\bar{\epsilon}$, and $\bar{\sigma}$ are given upper bounds. Moreover, we assume that we have vacuum conditions sufficiently far from the obstacle, i.e., there exists $R>0$ such that

$$\mu(\boldsymbol{x})\,=\,\mu_0, \qquad \epsilon(\boldsymbol{x})\,=\,\epsilon_0 \quad \text{and} \quad \sigma(\boldsymbol{x})\,=\,0 \qquad \forall\,\boldsymbol{x},\ |\boldsymbol{x}|\,\ge\,R\,. \tag{3.2}$$

The incident electric and magnetic fields $\mathcal{E}^i$ and $\mathcal{H}^i$ are supposed to exhibit a time-harmonic behavior with frequency ω and complex amplitudes $\boldsymbol{e}^i$ and $\boldsymbol{h}^i$, respectively. Hence, the total electric and magnetic fields have also a time harmonic behavior with frequency ω, namely,

$$\begin{aligned}\mathcal{E}(\boldsymbol{x},t) &= \mathrm{Re}\left\{exp\,(-\imath\,\omega\,t)\,\epsilon_0^{-1/2}\,\boldsymbol{e}(\boldsymbol{x})\right\},\\ \mathcal{H}(\boldsymbol{x},t) &= \mathrm{Re}\left\{exp\,(-\imath\,\omega\,t)\,\mu_0^{-1/2}\,\boldsymbol{h}(\boldsymbol{x})\right\},\end{aligned}$$

where the complex amplitudes $\boldsymbol{e}$ and $\boldsymbol{h}$ satisfy

$$\begin{aligned}\mathbf{curl}\,(\boldsymbol{e}) - \imath\,k\,b\,\boldsymbol{h} &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3\backslash\Omega_c\,,\\ \mathbf{curl}\,(\boldsymbol{h}) + \imath\,k\,a\,\boldsymbol{e} &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3\backslash\Omega_c\,,\end{aligned} \tag{3.3}$$

$k := \omega\,\sqrt{\epsilon_0\,\mu_0}$ is the wave number,

$$a(\boldsymbol{x}) := \frac{\epsilon(\boldsymbol{x})}{\epsilon_0} + \imath\frac{\sigma(\boldsymbol{x})}{\epsilon_0\,\omega} \quad \text{and} \quad b(\boldsymbol{x}) := \frac{\mu(\boldsymbol{x})}{\mu_0} \quad \forall\,\boldsymbol{x} \in \mathbb{R}^3\,. \tag{3.4}$$

It is clear from (3.2) that

$$a(\boldsymbol{x}) = b(\boldsymbol{x}) = 1 \qquad \forall\,\boldsymbol{x},\ |\boldsymbol{x}| \geq R\,. \tag{3.5}$$

We now let $\boldsymbol{n}$ denote the unit normal on Σ oriented towards the exterior of Ω_c. Then, according to our hypothesis,

$$\boldsymbol{e}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Sigma\,. \tag{3.6}$$

In addition, the scattered electromagnetic field exhibits the Silver-Müller asymptotic behaviour

$$(\boldsymbol{h}-\boldsymbol{h}^i)\times\frac{\boldsymbol{x}}{|\boldsymbol{x}|} - (\boldsymbol{e}-\boldsymbol{e}^i) = o(\frac{1}{|\boldsymbol{x}|})\,, \tag{3.7}$$

as $|\boldsymbol{x}| \to +\infty$, uniformly for all directions $\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$. We notice that this asymptotic behaviour implies that the outgoing waves are absorbed by the far field. Motivated by this fact, and aiming to obtain a suitable simplification of our model problem, we now introduce a sufficiently large sphere Γ centered at the origin, let Ω be the complement of $\overline{\Omega}_c$ in the ball whose boundary is Γ, and consider the impedance boundary condition:

$$(\boldsymbol{h}-\boldsymbol{h}^i)\times\boldsymbol{n} - (\boldsymbol{e}-\boldsymbol{e}^i) = \mathbf{0} \quad \text{on} \quad \Gamma\,, \tag{3.8}$$

where $\boldsymbol{n}$ denotes also the unit outward normal on Γ. Actually, in order to avoid introducing later a nonconforming Galerkin scheme, we may simply think of Γ as the polyhedral surface resulting from a sufficiently accurate approximation of the given sphere.

In this way, equations (3.3), (3.6), (3.8), the expression $\boldsymbol{h} = (\imath\,k\,b)^{-1}\,\mathbf{curl}\,(\boldsymbol{e})$ of the magnetic field in terms of $\boldsymbol{e}$, and the fact that $b \equiv 1$ on Γ (cf. (3.5)), lead us to the following formulation of the problem: Find $\boldsymbol{e}:\ \Omega \to \mathbb{C}^3$ such that

$$\begin{aligned}\mathbf{curl}\left(b^{-1}\,\mathbf{curl}\,(\boldsymbol{e})\right) - k^2\,a\,\boldsymbol{e} &= \mathbf{0} \quad \text{in} \quad \Omega\,,\\ \boldsymbol{e}\times\boldsymbol{n} &= \mathbf{0} \quad \text{on} \quad \Sigma\,,\\ \mathbf{curl}\,(\boldsymbol{e})\times\boldsymbol{n} - \imath\,k\,\boldsymbol{e} &= \mathbf{g} \quad \text{on} \quad \Gamma\,,\end{aligned} \tag{3.9}$$

where $\mathbf{g} := \imath k(\boldsymbol{h}^i\times\boldsymbol{n} - \boldsymbol{e}^i)$. Note here that the boundary conditions on Σ and Γ can be expressed in terms of the tangential trace mapping $\boldsymbol{\gamma}_{\mathbf{t}}$, respectively, as follows

$$\boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{e}) = \mathbf{0} \quad \text{on} \quad \Sigma\,, \tag{3.10}$$

and

$$\boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{curl}\,(\boldsymbol{e})) = \imath\,k\,\boldsymbol{e} + \mathbf{g} \quad \text{on} \quad \Gamma\,. \tag{3.11}$$

4 The continuous variational formulation

In this section we derive and analyze the full continuous variational formulation of (3.9). We begin by noticing, as we will see below, that the natural space for the electric field is given by

$$\mathbf{X} := \left\{ \boldsymbol{w} \in \mathbf{H}(\mathbf{curl};\Omega) : \quad \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \in \mathbf{L}^2_{\mathbf{t}}(\Gamma) \quad \text{and } \boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{w}) = \mathbf{0} \text{ on } \Sigma \right\},$$

which, equipped with the graph norm

$$\|\boldsymbol{w}\|^2_{\mathbf{X}} := \|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w})\|^2_{[L^2(\Gamma)]^3}, \tag{4.1}$$

is a Hilbert space.

Now, we test the first equation of (3.9) with a function $\boldsymbol{w} \in \mathbf{X}$, use Green's formula (2.4) and the fact that $b \equiv 1$ on Γ (cf. (3.5)) to obtain

$$\begin{aligned} &\int_\Omega \left\{ b^{-1}\mathbf{curl}(\boldsymbol{e}) \cdot \mathbf{curl}(\boldsymbol{w}) - k^2\, a\, \boldsymbol{e} \cdot \boldsymbol{w} \right\} \\ &\qquad + \langle \boldsymbol{\gamma}_{\mathbf{t}}(b^{-1}\mathbf{curl}(\boldsymbol{e})), \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \rangle_{\mathbf{t},\Sigma} - \langle \boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{curl}(\boldsymbol{e})), \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \rangle_{\mathbf{t},\Gamma} = 0. \end{aligned} \tag{4.2}$$

Then, noting from (2.4) that

$$\langle \boldsymbol{\gamma}_{\mathbf{t}}(b^{-1}\,\mathbf{curl}(\boldsymbol{e})), \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \rangle_{\mathbf{t},\Sigma} = -\langle \boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{w}), \boldsymbol{\pi}_{\mathbf{t}}(b^{-1}\,\mathbf{curl}(\boldsymbol{e})) \rangle_{\mathbf{t},\Sigma} = 0,$$

and incorporating the boundary condition (3.11) we find that (4.2) yields the following global variational formulation of problem (3.9): Find $\boldsymbol{e} \in \mathbf{X}$ such that

$$\mathbf{A}(\boldsymbol{e},\boldsymbol{w}) = \int_\Gamma \mathbf{g} \cdot \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in \mathbf{X}, \tag{4.3}$$

where $\mathbf{A} : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ is the bounded bilinear form defined by

$$\mathbf{A}(\boldsymbol{e},\boldsymbol{w}) := \int_\Omega \left\{ b^{-1}\,\mathbf{curl}(\boldsymbol{e}) \cdot \mathbf{curl}(\boldsymbol{w}) - k^2\, a\, \boldsymbol{e} \cdot \boldsymbol{w} \right\} - \imath\, k \int_\Gamma \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e}) \cdot \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \quad \forall\, \boldsymbol{e},\, \boldsymbol{w} \in \mathbf{X}. \tag{4.4}$$

In what follows we employ a suitable decomposition of $\mathbf{X}$ to prove that (4.3) becomes a compact perturbation of a well-posed problem.

4.1 A Helmholtz decomposition

Let us first introduce a sphere Γ_0 containing $\bar{\Omega} \cup \bar{\Omega}_c$ in its interior. We consider now the open annular domain Q delimited by the boundaries Γ and Γ_0 and denote $\widetilde{\Omega}$ the set $Q \cup \Gamma \cup \Omega$. Then, we define the spaces

$$\mathbf{V}(\widetilde{\Omega}) := \left\{ \boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega}); \quad \mathrm{div}\boldsymbol{w} = 0 \text{ in } Q, \quad \langle \boldsymbol{\gamma}_{\boldsymbol{n}}(\boldsymbol{w}), 1 \rangle_{\Gamma_0} = 0 \right\},$$

$$\mathbf{V}_0(Q) := \{ \boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl};Q); \quad \mathrm{div}\boldsymbol{w} = 0 \text{ in } Q, \quad \langle \boldsymbol{\gamma}_{\boldsymbol{n}}(\boldsymbol{w}), 1 \rangle_{\Gamma_0} = 0 \},$$

where $\langle \cdot, \cdot \rangle_{\Gamma_0}$ stands for the $H^{-1/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$ bracket. We recall the following useful result.

Lemma 4.1 *The seminorm $\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{[L^2(Q)]^3}$ is a norm on $\mathbf{V}_0(Q)$ equivalent to the usual norm in $\mathbf{H}(\mathbf{curl};Q)$.*

Proof. See for instance [2, Corollary 3.19]. □

Lemma 4.2 *The linear extension mapping*

$$\begin{array}{rcl}\mathcal{E}:\ \mathbf{X} & \to & \mathbf{V}(\widetilde{\Omega}) \\ \boldsymbol{w} & \mapsto & \mathcal{E}\boldsymbol{w}\end{array}$$

characterized by $\mathcal{E}\boldsymbol{w} \in \mathbf{V}(\widetilde{\Omega})$, $\mathcal{E}\boldsymbol{w} = \boldsymbol{w}$ *in* Ω, *and*

$$\int_Q \mathbf{curl}\,\mathcal{E}\boldsymbol{w}\cdot\mathbf{curl}\boldsymbol{q} = 0 \qquad \forall \boldsymbol{q}\in\mathbf{V}_0(Q)\,, \tag{4.5}$$

is bounded.

Proof. Let us denote by $\gamma_{\mathbf{t}}^-$ and $\gamma_{\mathbf{t}}^+$ the tangential traces on Γ taken from Ω and Q, respectively. We know from Theorem 2.1 that there exists a continuous right inverse $(\gamma_{\mathbf{t}}^+)^{-1}:\ \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\to \mathbf{H}_{\Gamma_0}(\mathbf{curl};Q)$ of $\gamma_{\mathbf{t}}^+$, where

$$\mathbf{H}_{\Gamma_0}(\mathbf{curl};Q) := \left\{\boldsymbol{w}\in\mathbf{H}(\mathbf{curl};Q);\quad \gamma_{\mathbf{t}}(\boldsymbol{w}) = \mathbf{0}\quad \text{on } \Gamma_0\right\}.$$

It follows that the linear operator

$$\begin{array}{rcl}\mathcal{L}:\ \mathbf{X} & \to & \mathbf{H}_{\Gamma_0}(\mathbf{curl};Q) \\ \boldsymbol{w} & \mapsto & \mathcal{L}\boldsymbol{w} := (\gamma_{\mathbf{t}}^+)^{-1}(\gamma_{\mathbf{t}}^-\boldsymbol{w})\end{array}$$

is bounded, that is there exists a constant $C_0 > 0$ such that

$$\|\mathcal{L}(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};Q)} \;\le\; C_0\,\|\boldsymbol{w}\|_{\mathbf{X}} \quad \forall\,\boldsymbol{w}\in\mathbf{X}. \tag{4.6}$$

Now, we let

$$M(Q) := \left\{\theta\in H^1(Q);\quad \theta|_\Gamma = 0,\quad \theta|_{\Gamma_0} = \text{constant}\right\},$$

and, given $\boldsymbol{w}\in\mathbf{X}$, we seek $\boldsymbol{z}_w\in\mathcal{L}(\boldsymbol{w})+\mathbf{H}_0(\mathbf{curl};Q)$ and $\chi\in M(Q)$ satisfying

$$\begin{array}{rcll}\displaystyle\int_Q \mathbf{curl}\boldsymbol{z}_w\cdot\mathbf{curl}\boldsymbol{q} + \int_Q \boldsymbol{q}\cdot\nabla\chi & = & 0 & \forall\boldsymbol{q}\in\mathbf{H}_0(\mathbf{curl};Q)\,, \\ \displaystyle\int_Q \boldsymbol{z}_w\cdot\nabla\theta & = & 0 & \forall\theta\in M(Q)\,.\end{array} \tag{4.7}$$

The well-posedness of this problem is guaranteed by the Babuška-Brezzi theory. Indeed, the fact that $\nabla(M(Q))\subset\mathbf{H}_0(\mathbf{curl};Q)$ and the Poincaré inequality yield the inf-sup condition

$$\sup_{\boldsymbol{q}\in\mathbf{H}_0(\mathbf{curl};Q)} \frac{\displaystyle\int_Q \boldsymbol{q}\cdot\nabla\theta}{\|\boldsymbol{q}\|_{\mathbf{H}(\mathbf{curl};Q)}} \ge \frac{\displaystyle\int_Q |\nabla\theta|^2}{\|\nabla\theta\|_{\mathbf{H}(\mathbf{curl};Q)}} = \|\nabla\theta\|_{[L^2(Q)]^3} \ge \beta\,\|\theta\|_{H^1(Q)},$$

for all $\theta\in M(Q)$, whereas Lemma 4.1 ensures the ellipticity on the kernel

$$\mathbf{V}_0(Q) = \left\{\boldsymbol{q}\in\mathbf{H}_0(\mathbf{curl};Q);\quad \int_Q \boldsymbol{q}\cdot\nabla\chi = 0 \quad \forall\chi\in M(Q)\right\},$$

which means that there exists $C_1 > 0$ such that

$$\|\mathbf{curl}\boldsymbol{q}\|_{[L^2(Q)]^3} \;\ge\; C_1\,\|\boldsymbol{q}\|^2_{\mathbf{H}(\mathbf{curl};Q)} \quad \forall\boldsymbol{q}\in\mathbf{V}_0(Q).$$

It is clear now that $\mathcal{E}\boldsymbol{w} := \begin{cases} \boldsymbol{w} & \text{in } \Omega \\ \boldsymbol{z}_w & \text{in } Q \end{cases}$ satisfies the required conditions. Moreover, thanks to the stability results for (4.7), there exists a constant $C_2 > 0$ such that

$$\|\mathcal{E}(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};Q)} \,\leq\, C_2\, \|\mathcal{L}(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};Q)}\,.$$

Finally, (4.6) yields the estimate

$$\|\mathcal{E}(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})} \,\leq\, \sqrt{1+(C_0C_2)^2}\; \|\boldsymbol{w}\|_{\mathbf{X}} \quad \forall\, \boldsymbol{w} \in \mathbf{X}\,. \tag{4.8}$$

□

Lemma 4.3 *Let* $\mathcal{R}:\, \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega}) \to \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})$ *be the linear and bounded operator defined by*

$$\mathcal{R}(\boldsymbol{w}) \,:=\, \boldsymbol{w} - \nabla\varphi_w \qquad \forall\, \boldsymbol{w} \,\in\, \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})\,,$$

where

$$\varphi_w \in M(\widetilde{\Omega}) := \Big\{\theta \in H^1(\widetilde{\Omega});\quad \theta|_{\Gamma_0} = \textit{constant},\quad \theta|_{\Sigma_j} = 0,\quad j = 0,\cdots,J\Big\}$$

is the unique solution of

$$\int_{\widetilde{\Omega}} \nabla\varphi_w \cdot \nabla\psi = \int_{\widetilde{\Omega}} \boldsymbol{w} \cdot \nabla\psi \quad \forall \psi \in M(\widetilde{\Omega})\,. \tag{4.9}$$

Then there hold:

- $\operatorname{div}(\mathcal{R}(\boldsymbol{w})) = 0$ *and* $\langle\, \boldsymbol{\gamma_n}\mathcal{R}(\boldsymbol{w}), 1\,\rangle_{\Gamma_0} = 0,$
- $\mathbf{curl}(\mathcal{R}(\boldsymbol{w})) = \mathbf{curl}(\boldsymbol{w}) \quad \forall\, \boldsymbol{w} \,\in\, \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})\,,$
- $\mathcal{R}\circ\mathcal{R} = \mathcal{R}$,
- *there exists a constant* $\widetilde{C} > 0$ *such that*

$$\|\mathcal{R}(\boldsymbol{w})\|_{\mathbf{X}} \,\leq\, \widetilde{C}\, \|\boldsymbol{w}\|_{\mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})} \quad \forall\, \boldsymbol{w} \,\in\, \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})\,. \tag{4.10}$$

Proof. The properties listed in the first two items follow immediately from the definition of $\mathcal{R}$. It is also clear that $\mathcal{R}$ is idempotent and bounded. Finally, it is known (see [2, Proposition 3.7]) that there exists $s \in (1/2, 1]$ such that

$$\mathbf{H}_0(\mathbf{curl};\widetilde{\Omega}) \cap \mathbf{H}(\operatorname{div};\widetilde{\Omega}) \hookrightarrow [H^s(\widetilde{\Omega})]^3. \tag{4.11}$$

Then, by virtue of (2.2) and (4.11),

$$\begin{aligned}\|\mathcal{R}(\boldsymbol{w})\|^2_{\mathbf{X}} &= \|\mathcal{R}(\boldsymbol{w})\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{\mathbf{t}}\mathcal{R}(\boldsymbol{w})\|^2_{[L^2(\Gamma)]^3} \leq C_1\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})} + \|\boldsymbol{\pi}_{\mathbf{t}}\mathcal{R}(\boldsymbol{w})\|^2_{\mathbf{H}_{\|}^{s-1/2}(\Gamma)} \\ &\leq C_1\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})} + C_2\|\mathcal{R}(\boldsymbol{w})\|^2_{[H^s(\widetilde{\Omega})]^3} \leq C_1\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})} + C_3\|\mathcal{R}(\boldsymbol{w})\|^2_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})} \leq \widetilde{C}\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\widetilde{\Omega})},\end{aligned}$$

for all $\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl};\widetilde{\Omega})$, which proves the result. □

With the aid of these tools, we are able to introduce the following projector.

Lemma 4.4 *Let $\mathcal{P} : \mathbf{X} \to \mathbf{X}$ be the linear and bounded operator defined by*

$$\mathcal{P}(\boldsymbol{w}) := (\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega \qquad \forall\, \boldsymbol{w} \in \mathbf{X}.$$

Then $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$ and

$$\mathbf{curl}\big(\mathcal{P}(\boldsymbol{w})\big) = \mathbf{curl}(\boldsymbol{w}) \quad \forall\, \boldsymbol{w} \in \mathbf{X}. \tag{4.12}$$

Proof. The boundedness of $\mathcal{P}$ is a consequence of (4.10) and (4.8). The property (4.12) follows immediately from Lemma 4.3. Indeed,

$$\mathbf{curl}\big(\mathcal{P}(\boldsymbol{w})\big) = \mathbf{curl}\left((\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega\right) = (\mathbf{curl}\mathcal{R}(\mathcal{E}\boldsymbol{w}))|_\Omega = (\mathbf{curl}(\mathcal{E}\boldsymbol{w}))|_\Omega = \mathbf{curl}(\mathcal{E}\boldsymbol{w}|_\Omega) = \mathbf{curl}(\boldsymbol{w})$$

for all $\boldsymbol{w} \in \mathbf{X}$.

To prove that $\mathcal{P}$ is a projector we first recall that $\mathcal{R} \circ \mathcal{R} = \mathcal{R}$. Now, notice that the field $\boldsymbol{z} := \mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega) - \mathcal{R}\mathcal{E}\boldsymbol{w}$ vanishes identically in Ω. Moreover, it is straightforward from the definitions of $\mathcal{E}$ and $\mathcal{R}$ that $\boldsymbol{z}|_Q \in \mathbf{V}_0(Q)$. Hence, by virtue of (4.5),

$$\begin{aligned} \int_Q \mathbf{curl}\boldsymbol{z} \cdot \mathbf{curl}\boldsymbol{z} &= \int_Q \mathbf{curl}\mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega) \cdot \mathbf{curl}\boldsymbol{z} - \int_Q \mathbf{curl}\mathcal{R}\mathcal{E}\boldsymbol{w} \cdot \mathbf{curl}\boldsymbol{z} \\ &= -\int_Q \mathbf{curl}\mathcal{E}\boldsymbol{w} \cdot \mathbf{curl}\boldsymbol{z} = 0, \end{aligned}$$

which proves that $\mathbf{curl}\boldsymbol{z} = \mathbf{0}$ in Q. Consequently, thanks again to Lemma 4.1, $\boldsymbol{z}$ also vanishes identically in Q. This means that

$$\mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega) = \mathcal{R}\mathcal{E}\boldsymbol{w} \quad \text{in } \Omega.$$

Using the last identity together with the fact that $\mathcal{R}$ is idempotent yield

$$\mathcal{P}(\mathcal{P}\boldsymbol{w}) = (\mathcal{R}\mathcal{E}(\mathcal{P}\boldsymbol{w}))|_\Omega = \Big(\mathcal{R}\mathcal{E}\Big(\mathcal{R}\mathcal{E}\boldsymbol{w}\Big)|_\Omega\Big)|_\Omega = \Big(\mathcal{R}\mathcal{R}\mathcal{E}\boldsymbol{w}\Big)|_\Omega = (\mathcal{R}\mathcal{E}\boldsymbol{w})|_\Omega = \mathcal{P}\boldsymbol{w}$$

and the result follows. □

We deduce from the last results that $\mathcal{P}$ provides the stable and direct Helmholtz-type decomposition

$$\mathbf{X} = \mathcal{P}(\mathbf{X}) \;\oplus\; (\mathcal{I} - \mathcal{P})(\mathbf{X}), \tag{4.13}$$

where $\mathcal{I}$ represents the identity operator. Hence, any element $\boldsymbol{w} \in \mathbf{X}$ admits the unique splitting

$$\boldsymbol{w} = \mathcal{P}(\boldsymbol{w}) + (\mathcal{I} - \mathcal{P})(\boldsymbol{w}) \tag{4.14}$$

and the norm $\boldsymbol{w} \to |||\boldsymbol{w}|||_{\mathbf{X}} := \Big(\|\mathcal{P}(\boldsymbol{w})\|^2_{\mathbf{X}} + \|(\mathcal{I} - \mathcal{P})(\boldsymbol{w})\|^2_{\mathbf{X}}\Big)^{1/2}$ is equivalent to $\boldsymbol{w} \to \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl};\,\Omega)}$ on $\mathbf{X}$. Namely, since $\|\mathcal{P}\| = \|\mathcal{I} - \mathcal{P}\|$,

$$\frac{1}{\sqrt{2}\|\mathcal{P}\|} |||\boldsymbol{w}|||_{\mathbf{X}} \leq \|\boldsymbol{w}\|_{\mathbf{X}} \leq \sqrt{2}\, |||\boldsymbol{w}|||_{\mathbf{X}}, \tag{4.15}$$

for all $\boldsymbol{w} \in \mathbf{X}$.

Lemma 4.5 *The mappings $\mathcal{P} : \mathbf{X} \to [L^2(\Omega)]^3$ and $\boldsymbol{\pi}_{\mathbf{t}} \circ \mathcal{P} : \mathbf{X} \to \mathbf{L}^2_{\mathbf{t}}(\Gamma)$ are compact.*

Proof. The first assertion is a consequence of (4.11) and the compactness of the canonical injection $H^s(\Omega) \hookrightarrow L^2(\Omega)$. On the other hand, we choose $0 < \epsilon < s - 1/2$ and notice that the embedding $[H^s(\Omega)]^3 \hookrightarrow [H^{s-\epsilon}(\Omega)]^3$ is compact and the tangential trace operator $\boldsymbol{\pi}_{\mathbf{t}}$ is bounded from $[H^{s-\epsilon}(\Omega)]^3$ to $\mathbf{L}^2_{\mathbf{t}}(\Gamma)$, see (2.2). □

4.2 A Fredholm alternative

In this section we apply the stable decomposition (4.13) to reformulate (4.3) as a compact perturbation of a well-posed problem. To this end, we first introduce the bounded bilinear form

$$\mathbf{A}^+(\boldsymbol{e},\boldsymbol{w}) := \int_\Omega \left\{ b^{-1}\,\mathbf{curl}(\boldsymbol{e})\cdot\mathbf{curl}(\boldsymbol{w}) + k^2\,a\,\boldsymbol{e}\cdot\boldsymbol{w}\right\} + \imath\,k\int_\Gamma \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e})\cdot\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \tag{4.16}$$

which arises from the form $\mathbf{A}$ (cf. (4.4)) after performing suitable changes of sign. More precisely, note that

$$\mathbf{A}^+(\boldsymbol{e},\boldsymbol{w}) = \mathbf{A}(\boldsymbol{e},\boldsymbol{w}) + 2\,k^2\int_\Omega a\,\boldsymbol{e}\cdot\boldsymbol{w} + 2\,\imath\,k\int_\Gamma \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e})\cdot\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \tag{4.17}$$

Then, employing (4.14) for each $\boldsymbol{h}$, $\boldsymbol{w}\in\mathbf{X}$ we deduce from (4.17) that the bilinear form $\mathbf{A}$ can be decomposed as

$$\mathbf{A}(\boldsymbol{e},\boldsymbol{w}) = \mathbf{A}_0(\boldsymbol{e},\boldsymbol{w}) + \mathbf{K}(\boldsymbol{e},\boldsymbol{w}), \tag{4.18}$$

where

$$\mathbf{A}_0(\boldsymbol{e},\boldsymbol{w}) = \mathbf{A}^+(\mathcal{P}(\boldsymbol{e}),\mathcal{P}(\boldsymbol{w})) - \mathbf{A}^+((\mathcal{I}-\mathcal{P})(\boldsymbol{e}),(\mathcal{I}-\mathcal{P})(\boldsymbol{w})) \tag{4.19}$$

and

$$\begin{aligned}\mathbf{K}(\boldsymbol{e},\boldsymbol{w}) = {} & \mathbf{A}(\mathcal{P}(\boldsymbol{e}),(\mathcal{I}-\mathcal{P})(\boldsymbol{w})) + \mathbf{A}((\mathcal{I}-\mathcal{P})(\boldsymbol{e}),\mathcal{P}(\boldsymbol{w}))\\ & -2\,k^2\int_\Omega a\mathcal{P}(\boldsymbol{e})\cdot\mathcal{P}(\boldsymbol{w}) - 2\,\imath\,k\int_\Gamma \boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{e}))\cdot\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w})).\end{aligned} \tag{4.20}$$

Next, we let $\mathcal{A}_0$, $\mathcal{K}:\mathbf{X}\to\mathbf{X}'$ be the linear and bounded operators induced by the corresponding bilinear forms $\mathbf{A}_0(\cdot,\cdot)$ and $\mathbf{K}(\cdot,\cdot)$ respectively. Then, the continuous variational formulation (4.3) can be rewritten as the following operator equation: Find $\boldsymbol{e}\in\mathbf{X}$ such that

$$(\mathcal{A}_0+\mathcal{K})\boldsymbol{e} = \mathcal{G} \tag{4.21}$$

where $\mathcal{G}\in\mathbf{X}'$ represents the linear form $\boldsymbol{w}\mapsto \int_\Gamma \mathbf{g}\cdot\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w})$.

In what follows we prove that the operators on the left-hand side of (4.21) are invertible and compact, respectively. We begin by showing that $\mathcal{A}_0$ is bijective. To this end, we first observe from (4.16), recalling the definition of the coefficient a (cf. (3.4)), that for each $\boldsymbol{w}\in\mathbf{X}$ there holds

$$\mathrm{Re}\Big\{(1-\imath)\mathbf{A}^+(\boldsymbol{w},\overline{\boldsymbol{w}})\Big\} = \int_\Omega\left\{\frac{\mu_0}{\mu}\,|\mathbf{curl}(\boldsymbol{w})|^2 + k^2\,(\frac{\omega\epsilon+\sigma}{\epsilon_0\omega})\,|\boldsymbol{w}|^2\right\} + k\,\|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w})\|^2_{[L^2(\Gamma)]^3}, \tag{4.22}$$

which, according to the assumptions (3.1), yields

$$\mathrm{Re}\Big\{(1-\imath)\mathbf{A}^+(\boldsymbol{w},\overline{\boldsymbol{w}})\Big\} \geq \alpha\,\|\boldsymbol{w}\|^2_{\mathbf{X}} \qquad \forall\,\boldsymbol{w}\in\mathbf{X}, \tag{4.23}$$

where

$$\alpha := \min\left\{\frac{\mu_0}{\bar{\mu}},\,k^2,\,k\right\}.$$

Lemma 4.6 *There exists $\alpha > 0$ such that*

$$\sup_{\boldsymbol{w}\in\mathbf{X}\setminus\{\mathbf{0}\}} \frac{|\,\mathbf{A}_0(\boldsymbol{e},\boldsymbol{w})\,|}{\|\boldsymbol{w}\|_{\mathbf{X}}} \geq \alpha\,\|\boldsymbol{e}\|_{\mathbf{X}} \qquad \forall\,\boldsymbol{e}\in\mathbf{X}. \tag{4.24}$$

In addition, there holds

$$\sup_{\boldsymbol{e}\in\mathbf{X}} |\,\mathbf{A}_0(\boldsymbol{e},\boldsymbol{w})\,| > 0 \qquad \forall\,\boldsymbol{w}\in\mathbf{X}\setminus\{\mathbf{0}\}. \tag{4.25}$$

Proof. Let $\Xi : \mathbf{X} \to \mathbf{X}$ be the linear operator defined by

$$\Xi(\boldsymbol{w}) \,:=\, (1+\imath)(2\,\mathcal{P} - \mathcal{I})(\boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \,\in\, \mathbf{X}\,. \tag{4.26}$$

It follows from the properties of $\mathcal{P}$ that Ξ is bounded and bijective. Thus, we have that

$$\sup_{\boldsymbol{w}\in\mathbf{X}\setminus\{\mathbf{0}\}} \frac{|\,\mathbf{A}_0(\boldsymbol{e},\boldsymbol{w})\,|}{\|\boldsymbol{w}\|_{\mathbf{X}}} \,\ge\, \frac{|\,\mathbf{A}_0(\boldsymbol{e},\overline{\Xi(\boldsymbol{e})})\,|}{\|\Xi(\boldsymbol{e})\|_{\mathbb{X}}} \,\ge\, \frac{\mathrm{Re}\Big\{\,\mathbf{A}_0(\boldsymbol{e},\overline{\Xi(\boldsymbol{e})})\,\Big\}}{\|\Xi(\boldsymbol{e})\|_{\mathbb{X}}} \qquad \forall\,\boldsymbol{e}\in\mathbf{X}\setminus\{\mathbf{0}\}\,. \tag{4.27}$$

Since $\mathcal{P}^2 \,=\, \mathcal{P}$, we observe that

$$\mathcal{P}\,(2\,\mathcal{P}-\mathcal{I}) \,=\, \mathcal{P} \quad \text{and} \quad (\mathcal{I}-\mathcal{P})\,(2\,\mathcal{P}-\mathcal{I}) \,=\, -\,(\mathcal{I}-\mathcal{P})\,,$$

whence we obtain from (4.19) that

$$\mathbf{A}_0(\boldsymbol{e},\overline{\Xi(\boldsymbol{e})}) \,=\, (1-\imath)\,\mathbf{A}^+(\mathcal{P}(\boldsymbol{e}),\mathcal{P}(\overline{\boldsymbol{e}})) \,+\, (1-\imath)\,\mathbf{A}^+((\mathcal{I}-\mathcal{P})(\boldsymbol{e}),(\mathcal{I}-\mathcal{P})(\overline{\boldsymbol{e}}))\,. \tag{4.28}$$

Applying (4.23) to both terms on the right-hand side of (4.28), we deduce that

$$\begin{aligned} &\mathrm{Re}\Big\{(1-\imath)\,\mathbf{A}^+(\mathcal{P}(\boldsymbol{e}),\mathcal{P}(\overline{\boldsymbol{e}})) \,+\, (1-\imath)\,\mathbf{A}^+((\mathcal{I}-\mathcal{P})(\boldsymbol{e}),(\mathcal{I}-\mathcal{P})(\overline{\boldsymbol{e}}))\Big\} \\ &\quad\ge\, \alpha\,\Big\{\,\|\mathcal{P}(\boldsymbol{e})\|_{\mathbf{X}}^2 \,+\, \|(\mathcal{I}-\mathcal{P})(\boldsymbol{e})\|_{\mathbf{X}}^2\,\Big\} \quad \forall\,\boldsymbol{e}\,\in\,\mathbf{X}\,. \end{aligned} \tag{4.29}$$

In this way, thanks to (4.28), (4.29) and (4.15), we deduce that

$$\mathrm{Re}\Big\{\mathbf{A}_0(\boldsymbol{e},\overline{\Xi(\boldsymbol{e})})\Big\} \,\ge\, \frac{\alpha}{2}\,\|\boldsymbol{e}\|_{\mathbf{X}}^2 \quad \forall\,\boldsymbol{e}\,\in\,\mathbf{X}\,, \tag{4.30}$$

which, using the boundedness of Ξ, yields

$$\frac{\mathrm{Re}\Big\{\mathbf{A}_0(\boldsymbol{e},\overline{\Xi(\boldsymbol{e})})\Big\}}{\|\Xi(\boldsymbol{e})\|_{\mathbf{X}}} \,\ge\, \beta\,\|\boldsymbol{e}\|_{\mathbf{X}} \qquad \forall\,\boldsymbol{e}\,\in\,\mathbf{X}\setminus\{\mathbf{0}\} \tag{4.31}$$

with $\beta = \alpha/(2\|\Xi\|)$. The above estimate and (4.27) proves the inf-sup condition (4.24). Finally, the symmetry of $\mathbf{A}_0$ and (4.24) provide the inf-sup condition (4.25). $\square$

Therefore, as a consequence of Lemma 4.6 and the well-known Nečas theorem (see [10, Theorem 3.2.3]), the operator $\mathcal{A}_0 :\, \mathbf{X} \to \mathbf{X}'$ is an isomorphism.

Lemma 4.7 *The operator* $\mathcal{K} : \mathbf{X} \to \mathbf{X}'$ *is compact.*

Proof. Using that

$$\mathbf{curl}\big((\mathcal{I}-\mathcal{P})(\boldsymbol{w})\big) \,=\, \mathbf{0} \qquad \forall\,\boldsymbol{w}\,\in\,\mathbf{X}\,,$$

we deduce from (4.20) that

$$\begin{aligned} \mathbf{K}(\boldsymbol{e},\boldsymbol{w}) \,=\, &-k^2\int_\Omega a\Big\{2\mathcal{P}(\boldsymbol{e})\cdot\mathcal{P}(\boldsymbol{w}) + \mathcal{P}(\boldsymbol{e})\cdot(\mathcal{I}-\mathcal{P})(\boldsymbol{w}) + (\mathcal{I}-\mathcal{P})(\boldsymbol{e})\cdot\mathcal{P}(\boldsymbol{w})\Big\} \\ &-\,\imath\,k\int_\Gamma 2\,\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{e}))\cdot\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w})) + \boldsymbol{\pi}_{\mathbf{t}}(\mathcal{I}-\mathcal{P})(\boldsymbol{e})\cdot\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w})) + \boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{e}))\cdot\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{I}-\mathcal{P})(\boldsymbol{w})\,. \end{aligned}$$

The compactness of $\mathcal{P} : \mathbf{X} \to [L^2(\Omega)]^3$ and $\boldsymbol{\pi}_{\mathbf{t}} \circ \mathcal{P} : \mathbf{X} \to \mathbf{L}_{\mathbf{t}}^2(\Gamma)$ (see Lemma 4.5) guarantees that the operators associated, in the last expression of $\mathbf{K}(\cdot,\cdot)$, to both the integrals on Ω and Γ are compact and the result follows. $\square$

We are now ready to establish the main result of this section.

Theorem 4.1 *Assume that the homogeneous problem associated to* (4.3) *has only the trivial solution. Then* $\mathcal{A} : \mathbf{X} \to \mathbf{X}'$ *is an isomorphism and, consequently, given an incident wave* $(\boldsymbol{h}^i, \boldsymbol{e}^i)$*, there exists a unique solution* $\boldsymbol{e} \in \mathbf{X}$ *to* (4.3)*. In addition, there exists* $C > 0$ *such that*

$$\|\boldsymbol{e}\|_{\mathbf{X}} \leq C \, \|\mathcal{G}\|_{\mathbf{X}'} \,. \tag{4.32}$$

Proof. It suffices to observe, in virtue of Lemmas 4.6, and 4.7, that $\mathcal{A} := \mathcal{A}_0 + \mathcal{K}$ is a Fredholm operator of index zero, and hence the well-posedness of (4.3) follows from uniqueness. □

We end this section with a uniqueness result for (4.3). At this point we need to make more restrictive assumptions on the coefficients. We denote by Ω^j the connected components of Ω represented by the cavities whose boundaries are Σ_j , $j = 1, \cdots, J$. Let $\mathcal{J} := \{1, \cdots, J\}$ and $\mathcal{J}_0 \subset \mathcal{J}$ be the subset of indices such that $\sigma(\boldsymbol{x}) = 0$ for a.e. $\boldsymbol{x} \in \cup_{j \in \mathcal{J}_0} \Omega^j$. We assume that σ is a stricty positive function in $\cup_{j \in \mathcal{J} \setminus \mathcal{J}_0} \Omega^j$. For each $j \in \mathcal{J}_0$, we consider the positive increasing sequence $\{k^j_\ell\}_\ell$ diverging to ∞ and solving the eigenvalue problem:

$$\begin{aligned}
&\text{find } (k^j_\ell, \boldsymbol{w}^j) \in \mathbb{R} \times \mathbf{H}_0(\mathbf{curl}; \Omega^j) \text{ such that} \\
&\int_{\Omega^j} b^{-1} \mathbf{curl} \boldsymbol{w}^j \cdot \mathbf{curl} \boldsymbol{v} = (k^j)^2 \int_{\Omega^j} \frac{\epsilon(\boldsymbol{x})}{\epsilon_0} \boldsymbol{w}^j \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega^j).
\end{aligned} \tag{4.33}$$

Finally, let us denote now by Ω^0 the connected component of Ω delimited by the boundaries Σ_0 and Γ. We assume that Ω^0 can be decomposed into L connected polyhedra Q^l such that $\overline{\Omega}^0 = \cup_{l=1}^L \overline{Q}^l$ with $Q^l \cap Q^m = \emptyset$ if $l \neq m$ and such that μ, ϵ and σ are constant functions in each Q^l.

Theorem 4.2 *Assume that* k *does not belong to the set* $\{0\} \cup (\cup_{j \in \mathcal{J}_0} \{k^j_\ell\}_\ell)$*. Then, there is at most one solution to* (4.3)*.*

Proof. Let $\boldsymbol{e}$ be a solution of the homogeneous system corresponding to (4.3), that is when $\mathbf{g} = \mathbf{0}$. Then, taking $\boldsymbol{w} = \overline{\boldsymbol{e}}$ in (4.3) gives

$$\int_\Omega \Big\{ b^{-1} |\mathbf{curl}(\boldsymbol{e})|^2 - k^2 \, a \, |\boldsymbol{e}|^2 \Big\} - \imath \, k \, \|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e})\|^2_{[L^2(\Gamma)]^3} = 0 \,. \tag{4.34}$$

This shows that the imaginary part of (4.34) reduces to

$$-k^2 \int_\Omega \mathrm{Im}\big(a\big) \, |\boldsymbol{e}|^2 - k \, \|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e})\|^2_{[L^2(\Gamma)]^3} = 0 \,.$$

Noting that $\mathrm{Im}\big(a\big) > 0$ in $\cup_{j \in \mathcal{J} \setminus \mathcal{J}_0} \Omega^j$ we deduce immediately that $\boldsymbol{e}$ vanishes identically in this domain. In the other cavities Ω^j, $j \in \mathcal{J}_0$, $k > 0$ is not a solution of (4.33) and then $\boldsymbol{e}$ also vanishes in $\cup_{j \in \mathcal{J}_0} \Omega^j$. Finally, since $\mathrm{Im}\big(a\big) \geq 0$ in Ω^0 we have that $\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e}) = \mathbf{0}$ on Γ. Thus, applying the unique continuation principle of [8, Theorem 9.3] (as done also in [16, Theorem 4.12]), we deduce that $\boldsymbol{e} = \mathbf{0}$ in Ω^0. □

5 The discrete problem

In order to introduce a Galerkin approximation of (4.3) we first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ by tetrahedrons K of diameter h_K. As usual, the parameter h denotes in each case the mesh size of the corresponding triangulation. Then, we denote by $\mathcal{T}_h(\Gamma)$ the triangulations induced by $\mathcal{T}_h(\Omega)$ on Γ.

For any $K \in \mathcal{T}_h(\Omega)$ we let $\mathbb{ND}_1(K) := [\mathbb{P}_0(K)]^3 \oplus [\mathbb{P}_0(K)]^3 \times \boldsymbol{x}$ be the local edge space of Nédélec, that is

$$\mathbb{ND}_1(K) := \left\{ \boldsymbol{v} : K \to \mathbb{C}^3 , \quad \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} \quad \forall \boldsymbol{x} \in K, \quad \boldsymbol{a} ,\ \boldsymbol{b} \in \mathbb{C}^3 \right\}.$$

Then, the finite element subspace for the unknown $\boldsymbol{e}$ is defined by $\mathbf{X}_h := \mathbf{X} \cap \mathcal{ND}_h(\Omega)$, where

$$\mathcal{ND}_h(\Omega) := \left\{ \boldsymbol{w} \in \mathbf{H}(\mathbf{curl};\Omega) : \quad \boldsymbol{w}|_K \in \mathbb{ND}_1(K) \quad \forall K \in \mathcal{T}_h(\Omega) \right\}.$$

We define the finite element scheme associated to (4.3) as follows: Find $\boldsymbol{e}_h \in \mathbf{X}_h$ such that

$$\mathbf{A}(\boldsymbol{e}_h, \boldsymbol{w}) = \int_\Gamma \mathbf{g} \cdot \boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \mathbf{X}_h , \tag{5.1}$$

In order to show that (5.1) is well-posed (see Section 5.2 below) we need some technical results.

5.1 Technical results

For any $\delta \geq 0$ we introduce the Sobolev space

$$\mathbf{H}^\delta(\mathbf{curl};\Omega) := \left\{ \boldsymbol{w} \in [H^\delta(\Omega)]^3 : \quad \mathbf{curl}(\boldsymbol{w}) \in [H^\delta(\Omega)]^3 \right\}$$

and endow it with its Hilbertian norm

$$\|\boldsymbol{w}\|^2_{\mathbf{H}^\delta(\mathbf{curl};\Omega)} := \|\boldsymbol{w}\|^2_{[H^\delta(\Omega)]^3} + \|\mathbf{curl}(\boldsymbol{w})\|^2_{[H^\delta(\Omega)]^3} .$$

Then for any edge E of $\mathcal{T}_h(\Omega)$, we denote by $\mathbf{t}_E$ a unit tangential vector along E. It follows from [2, Lemma 4.7] that if $\boldsymbol{w} \in \mathbf{H}^\delta(\mathbf{curl};\Omega)$ with $\delta > 1/2$, then the moments $\int_E \boldsymbol{w} \cdot \mathbf{t}_E$ are meaningful. This guarantees that the interpolation operator $\Pi_h : \mathbf{H}^\delta(\mathbf{curl};\Omega) \to \mathcal{ND}_h(\Omega)$ associated to the edge finite element, which is characterized by

$$\int_E \Pi_h(\boldsymbol{w}) \cdot \mathbf{t}_E = \int_E \boldsymbol{w} \cdot \mathbf{t}_E \qquad \text{for all edge } E \text{ of } \mathcal{T}_h(\Omega),$$

is well-defined and uniformly bounded. In addition, the following interpolation error estimate holds (see [1, Proposition 5.6]):

$$\|\boldsymbol{w} - \Pi_h(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C\, h^\delta \,\|\boldsymbol{w}\|_{\mathbf{H}^\delta(\mathbf{curl};\Omega)} \quad \forall \boldsymbol{w} \in \mathbf{H}^\delta(\mathbf{curl};\Omega), \quad \forall \delta > 1/2 . \tag{5.2}$$

Another useful property of Π_h is given by the following result.

Lemma 5.1 *For each $\delta \in (1/2, 1]$ define the space*

$$\mathbf{H}^\delta_h(\mathbf{curl};\Omega) := \left\{ \boldsymbol{w} \in [H^\delta(\Omega)]^3 : \qquad \mathbf{curl}(\boldsymbol{w}) \in \mathbf{curl}(\mathcal{ND}_h(\Omega)) \right\}. \tag{5.3}$$

Then, the operator Π_h is also well defined in $\mathbf{H}^\delta_h(\mathbf{curl};\Omega)$ and there exists a constant $C > 0$, independent of h, such that

$$\|\boldsymbol{w} - \Pi_h(\boldsymbol{w})\|_{[L^2(\Omega)]^3} \leq C\, h^\delta \,\|\boldsymbol{w}\|_{[H^\delta(\Omega)]^3} \qquad \forall \boldsymbol{w} \in \mathbf{H}^\delta_h(\mathbf{curl};\Omega). \tag{5.4}$$

Proof. See [12, Lemma 4.6]. □

Next, we need to introduce curl_Γ-conforming surface finite elements on the manifold Γ. Actually, div_Γ-conforming finite elements on manifolds are more frequently used in the literature since they arise naturally in the BEM-theory for Maxwell equations, (see, e.g. [7] and the references therein). We still can benefit here from the result announced in the last reference for the Raviart-Thomas finite elements since they may be translated to the bidimensional Nédélec finite element by a simple $\pi/2$-rotation in the space variable on each one of the faces compounding Γ. To be more specific, the lowest order bidimensional Nédélec finite element (also known as the rotated Raviart-Thomas finite element) approximation of the space

$$\mathbf{H}(\mathrm{curl}_\Gamma;\Gamma) := \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2_{\mathbf{t}}(\Gamma) : \quad \mathrm{curl}_\Gamma(\boldsymbol{\varphi}) \in L^2(\Gamma) \right\},$$

relatively to the mesh $\mathcal{T}_h(\Gamma)$, is given by

$$\boldsymbol{\mathcal{ND}}_h(\Gamma) := \boldsymbol{\pi}_{\mathbf{t}}(\mathcal{ND}_h(\Omega)).$$

The corresponding interpolation operator $\Pi_h^\Gamma : \mathbf{H}^\delta_{\|}(\Gamma) \cap \mathbf{H}(\mathrm{curl}_\Gamma;\Gamma) \to \boldsymbol{\mathcal{ND}}_h(\Gamma)$ ($\delta \in (0,1]$) satisfies the following error estimate.

Lemma 5.2 *For each $\delta \in (0,1]$ there exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{\varphi} - \Pi_h^\Gamma(\boldsymbol{\varphi})\|_{[L^2(\Gamma)]^3} \le C\, h^\delta \left\{ \|\boldsymbol{\varphi}\|_{\mathbf{H}^\delta_{\|}(\Gamma)} + \|\mathrm{curl}_\Gamma(\boldsymbol{\varphi})\|_{L^2(\Gamma)} \right\} \quad \forall\, \boldsymbol{\varphi} \in \mathbf{H}^\delta_{\|}(\Gamma) \cap \mathbf{H}(\mathrm{curl}_\Gamma;\Gamma).$$

Proof. See [7, Lemma 15]. □

For tangential vector fields with a discrete curl_Γ, there holds the following variant.

Lemma 5.3 *For each $\delta \in (0,1]$ there exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{\varphi} - \Pi_h^\Gamma(\boldsymbol{\varphi})\|_{[L^2(\Gamma)]^3} \le C\, h^\delta\, \|\boldsymbol{\varphi}\|_{\mathbf{H}^\delta_{\|}(\Gamma)}$$

for all $\boldsymbol{\varphi} \in \mathbf{H}^\delta_{\|}(\Gamma)$ satisfying $\mathrm{curl}_\Gamma(\boldsymbol{\varphi}) \in \mathrm{curl}_\Gamma(\boldsymbol{\mathcal{ND}}_h(\Gamma))$.

Proof. See [7, Lemma 16]. □

In this way, recalling the definition of the norm $\|\cdot\|_{\mathbf{X}}$ (see (4.1)), and using the commuting diagram property $\boldsymbol{\pi}_{\mathbf{t}}\, \Pi_h = \Pi_h^\Gamma\, \boldsymbol{\pi}_{\mathbf{t}}$ together with (5.2) and Lemma 5.2, we deduce that for each $\delta \in (1/2,1]$ there exists a constant $C > 0$, independent of h, such that for all $\boldsymbol{w} \in \mathbf{H}^\delta(\mathbf{curl};\Omega)$ satisfying $\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}) \in \mathbf{H}^\delta_{\|}(\Gamma) \cap \mathbf{H}(\mathrm{curl}_\Gamma;\Gamma)$, there holds

$$\begin{aligned} \|\boldsymbol{w} - \Pi_h(\boldsymbol{w})\|_{\mathbf{X}} &:= \left\{ \|\boldsymbol{w} - \Pi_h(\boldsymbol{w})\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{\mathbf{t}}\big(\boldsymbol{w} - \Pi_h(\boldsymbol{w})\big)\|^2_{[L^2(\Gamma)]^3} \right\}^{1/2} \\ &\le C\, h^\delta \left\{ \|\boldsymbol{w}\|_{\mathbf{H}^\delta(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w})\|_{\mathbf{H}^\delta_{\|}(\Gamma)} + \|\mathrm{curl}_\Gamma(\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{w}))\|_{L^2(\Gamma)} \right\}, \end{aligned} \tag{5.5}$$

which constitutes an approximation property of the space $\mathbf{X}_h$.

5.2 Well-posedness of the discrete problem

In this section we prove the well-posedness of the discrete problem (5.1). For this purpose, according to a classical result on projection methods for compact perturbations of invertible operators (see, e.g., Theorem 13.7 in [13]), it suffices to show that the Galerkin scheme associated to the isomorphism $\mathcal{A}_0$ is well posed. Hence, in what follows we prove that $\mathbf{A}_0$ satisfies an inf-sup condition on the finite element subspace $\mathbf{X}_h$, thus providing the discrete analogue of Lemma 4.6.

In the sequel, we define discrete version of the operator $\mathcal{P}$. Let us first notice that for each $\boldsymbol{w} \in \mathbf{X}_h$ there holds

$$\mathbf{curl}(\mathcal{P}(\boldsymbol{w})) \;=\; \mathbf{curl}(\boldsymbol{w}) \;\in\; \mathbf{curl}(\mathbf{X}_h)\,,$$

which, recalling that $\mathcal{P}(\boldsymbol{w}) \in [H^s(\Omega)]^3$, shows that $\mathcal{P}(\boldsymbol{w})$ belongs to $\mathbf{H}^s_h(\mathbf{curl};\Omega)$ (cf. (5.3) with $\delta = s$).

In this way, Lemma 5.1 implies that Π_h can be applied to $\mathcal{P}(\boldsymbol{w})$, and hence we define the discrete version of the operator $\mathcal{P}$ as follows

$$\begin{array}{rcl} \mathcal{P}_h :\, \mathbf{X}_h & \rightarrow & \mathbf{X}_h \\ \boldsymbol{w} & \mapsto & \mathcal{P}_h(\boldsymbol{w}) \,:=\, \Pi_h(\mathcal{P}(\boldsymbol{w}))\,. \end{array} \tag{5.6}$$

Lemma 5.4 *There exists a constant $C > 0$, independent of h, such that*

$$\|\mathcal{P}(\boldsymbol{w}) \,-\, \mathcal{P}_h(\boldsymbol{w})\|_{\mathbf{X}} \;\leq\; C\, h^{s-1/2}\, \|\boldsymbol{w}\|_{\mathbf{X}} \qquad \forall\, \boldsymbol{w} \,\in\, \mathbf{X}_h\,. \tag{5.7}$$

Proof. Let $\tilde{\Pi}_h$ be the lowest order Raviart-Thomas interpolation operator associated to the triangulation $\mathcal{T}_h(\Omega)$, cf. [16]. By virtue of the well-known commuting diagram property

$$\mathbf{curl}\,\Pi_h \;=\; \tilde{\Pi}_h\,\mathbf{curl}\,,$$

we have that

$$\mathbf{curl}(\Pi_h\mathcal{P}(\boldsymbol{w})) \;=\; \tilde{\Pi}_h\big\{\mathbf{curl}(\mathcal{P}(\boldsymbol{w}))\big\} \;=\; \tilde{\Pi}_h\big\{\mathbf{curl}(\boldsymbol{w})\big\} \;=\; \mathbf{curl}(\boldsymbol{w})\,.$$

Thus $\mathbf{curl}\big(\mathcal{P}(\boldsymbol{w})\big) \;=\; \mathbf{curl}\big(\mathcal{P}_h(\boldsymbol{w})\big)$, which yields

$$\|\mathcal{P}(\boldsymbol{w}) \,-\, \mathcal{P}_h(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \;=\; \|\mathcal{P}(\boldsymbol{w}) \,-\, \mathcal{P}_h(\boldsymbol{w})\|_{[L^2(\Omega)]^3}.$$

Hence, applying Lemma 5.1 (cf. (5.4)) we deduce that for each $\boldsymbol{w} \in \mathbf{X}_h$ there holds

$$\|\mathcal{P}(\boldsymbol{w}) - \mathcal{P}_h(\boldsymbol{w})\|_{\mathbf{H}(\mathbf{curl};\,\Omega)} \;\leq\; C_0\, h^s\, \|\mathcal{P}(\boldsymbol{w})\|_{[H^s(\Omega)]^3} \leq C_1\, h^s\, \|\boldsymbol{w}\|_{\mathbf{X}}, \tag{5.8}$$

where the last inequality follows from the boundedness of $\mathcal{P} :\, \mathbf{X} \rightarrow [H^s(\Omega)]^3$.

On the other hand, using the commuting diagram property $\boldsymbol{\pi}_{\mathbf{t}}\,\Pi_h \;=\; \Pi^\Gamma_h\,\boldsymbol{\pi}_{\mathbf{t}}$, we have that

$$\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}_h(\boldsymbol{w})) \;=\; \boldsymbol{\pi}_{\mathbf{t}}\big(\Pi_h(\mathcal{P}(\boldsymbol{w})\big) \;=\; \Pi^\Gamma_h\big(\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w}))\big)\,.$$

In addition, since $\mathrm{curl}_\Gamma\,\boldsymbol{\pi}_{\mathbf{t}} \;=\; \mathrm{div}_\Gamma\,\boldsymbol{\gamma}_{\mathbf{t}}$ and $\mathrm{div}_\Gamma(\boldsymbol{\gamma}_{\mathbf{t}}(\boldsymbol{w})) \;=\; \mathbf{curl}(\boldsymbol{w})\cdot\boldsymbol{n} \;\in\; H^{-1/2}(\Gamma)$ for each $\boldsymbol{w} \in \mathbf{H}(\mathbf{curl};\,\Omega)$ (see [6]), we deduce that

$$\mathrm{curl}_\Gamma\big(\boldsymbol{\pi}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w}))\big) = \mathrm{div}_\Gamma(\boldsymbol{\gamma}_{\mathbf{t}}(\mathcal{P}(\boldsymbol{w})) \;=\; \mathbf{curl}(\mathcal{P}(\boldsymbol{w}))\cdot\boldsymbol{n} = \mathbf{curl}(\boldsymbol{w})\cdot\boldsymbol{n} = \mathrm{curl}_\Gamma\big(\boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{w}\big) \in \mathrm{curl}_\Gamma(\boldsymbol{\mathcal{ND}}_h(\Gamma))$$

for all $\boldsymbol{w} \in \mathbf{X}_h$. Consequently, applying now the boundedness of $\boldsymbol{\pi}_{\mathbf{t}} : [H^s(\Omega)]^3 \rightarrow \mathbf{H}^{s-1/2}_{\|}(\Gamma)$ and the estimate provided by Lemma 5.3, we find that

$$\begin{aligned} \|\boldsymbol{\pi}_{\mathbf{t}}\big(\mathcal{P}(\boldsymbol{w}) - \mathcal{P}_h(\boldsymbol{w})\big)\|_{[L^2(\Gamma)]^3} \;&=\; \|\boldsymbol{\pi}_{\mathbf{t}}\big(\mathcal{P}(\boldsymbol{w})\big) \,-\, \Pi^\Gamma_h\big(\boldsymbol{\pi}_{\mathbf{t}}\big(\mathcal{P}(\boldsymbol{w})\big)\big)\|_{[L^2(\Gamma)]^3} \\ \leq\; C_2\, h^{s-1/2}\, \|\boldsymbol{\pi}_{\mathbf{t}}\big(\mathcal{P}(\boldsymbol{w})\big)\|_{\mathbf{H}^{s-1/2}_{\|}(\Gamma)} \;&\leq\; C_3\, h^{s-1/2}\, \|\mathcal{P}(\boldsymbol{w})\|_{[H^s(\Omega)]^3} \;\leq\; C_4\, h^{s-1/2}\, \|\boldsymbol{w}\|_{\mathbf{X}}\,, \end{aligned}$$

which, together with (5.8), yields the required estimate and completes the proof. $\square$

We are now ready to establish the discrete inf-sup condition for $\mathbf{A}_0$.

Lemma 5.5 *There exist constants β^*, $h_0 > 0$, independent of h, such that for each $h \le h_0$ there holds*

$$\sup_{\boldsymbol{w}\in\mathbf{X}_h\backslash\{\mathbf{0}\}} \frac{|\,\mathbf{A}_0(\boldsymbol{e},\boldsymbol{w})\,|}{\|\boldsymbol{w}\|_{\mathbf{X}}} \;\ge\; \beta^*\,\|\boldsymbol{e}\|_{\mathbf{X}} \qquad \forall\,\boldsymbol{e}\in\mathbf{X}_h\,. \tag{5.9}$$

Proof. Following the definition of the operator $\Xi:\mathbf{X}\to\mathbf{X}$ (see (4.26)), we now introduce its discrete version as follows

$$\begin{array}{rcl} \Xi_h:\mathbf{X}_h & \to & \mathbf{X}_h \\[1ex] \boldsymbol{w} & \mapsto & (1+\imath)(2\,\mathcal{P}_h - \mathcal{I})(\boldsymbol{w})\,. \end{array}$$

It follows straightforwardly from Lemma 5.4 that

$$\|\Xi(\boldsymbol{w}) \,-\, \Xi_h(\boldsymbol{w})\|_{\mathbf{X}} \;\le\; C_0\,h^{s-1/2}\,\|\boldsymbol{w}\|_{\mathbf{X}} \qquad \forall\,\boldsymbol{w}\,\in\,\mathbf{X}_h\,. \tag{5.10}$$

Hence, using (5.10), (4.30) and the boundedness of $\mathbf{A}_0$, we find that for each $\boldsymbol{w}\,\in\,\mathbf{X}_h$ there holds

$$\begin{aligned} \mathrm{Re}\Big\{\,\mathbf{A}_0(\boldsymbol{w},\overline{\Xi_h(\boldsymbol{w})})\,\Big\} &\ge \mathrm{Re}\Big\{\,\mathbf{A}_0(\boldsymbol{w},\overline{\Xi(\boldsymbol{w})})\,\Big\} \,-\, C_0\,\|\mathbf{A}_0\|\,h^{s-1/2}\,\|\boldsymbol{w}\|^2_{\mathbf{X}} \\ &\ge \frac{\alpha}{2}\,\|\boldsymbol{w}\|^2_{\mathbf{X}} \,-\, C_0\,\|\mathbf{A}_0\|\,h^{s-1/2}\,\|\boldsymbol{w}\|^2_{\mathbf{X}} \;\ge\; \frac{\alpha}{4}\,\|\boldsymbol{w}\|^2_{\mathbf{X}}\,, \end{aligned} \tag{5.11}$$

for all $h \,\le\, h_1 \,:=\, \big(\frac{\alpha}{4\,C_0\,\|\mathbf{A}_0\|}\big)^{2/(2s-1)}$.

On the other hand, the boundedness of Ξ and (5.10) imply the existence of C_1, C_2, $h_2 \,>\, 0$, independent of h, such that

$$C_1\,\|\boldsymbol{w}\|_{\mathbf{X}} \;\le\; \|\Xi_h(\boldsymbol{w})\|_{\mathbf{X}} \;\le\; C_2\,\|\boldsymbol{w}\|_{\mathbf{X}} \qquad \forall\,\boldsymbol{w}\,\in\,\mathbf{X}_h\,, \quad \forall\,h\le h_2\,. \tag{5.12}$$

Hence, (5.9) follows immediately from (5.11) and (5.12) defining $h_0 := \min\{h_1,h_2\}$. $\square$

The well-posedness and convergence of the discrete scheme (4.3) can finally be established.

Theorem 5.1 *Assume that there exists at most one solution to* (4.3). *Then, there exists $h_0 > 0$ such that for each $h \,\le\, h_0$, the Galerkin scheme* (4.3) *has a unique solution $\boldsymbol{e}_h \in \mathbf{X}_h$. In addition, there exist C_1, $C_2 \,>\, 0$, independent of h, such that*

$$\|\boldsymbol{e}_h\|_{\mathbf{X}} \;\le\; C_1\,\|\mathcal{G}\|_{\mathbb{X}'}\,, \tag{5.13}$$

and

$$\|\boldsymbol{e} \,-\, \boldsymbol{e}_h\|_{\mathbf{X}} \;\le\; C_2\,\inf_{\boldsymbol{w}_h\in\mathbf{X}_h}\,\|\boldsymbol{e} \,-\, \boldsymbol{w}_h\|_{\mathbf{X}}\,. \tag{5.14}$$

Furthermore, if there exists $\delta \,\in\, (1/2,1]$ such that $\boldsymbol{e} \,\in\, \mathbf{H}^\delta(\mathbf{curl},\Omega)$ and $\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e}) \,\in\, \mathbf{H}^\delta_{\|}(\Gamma)\cap\mathbf{H}(\mathrm{curl}_\Gamma;\Gamma)$ then there holds

$$\|\boldsymbol{e} \,-\, \boldsymbol{e}_h)\|_{\mathbf{X}} \;\le\; C_3\,h^\delta\,\Big\{\,\|\boldsymbol{e}\|_{\mathbf{H}^\delta(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e})\|_{\mathbf{H}^\delta_{\|}(\Gamma)} \,+\, \|\mathrm{curl}_\Gamma(\boldsymbol{\pi}_{\mathbf{t}}(\boldsymbol{e}))\|_{L^2(\Gamma)}\,\Big\}\,, \tag{5.15}$$

with a constant $C_3 > 0$, independent of h.

Proof. Thanks to Lemma 5.5, the first part of the proof is a direct application of Theorem 13.7 in [13], whereas the rate of convergence (5.15) follows from the Céa estimate (5.14) and the approximation properties of the finite element subspaces provided in Section 5.1 (cf. (5.5)). $\square$

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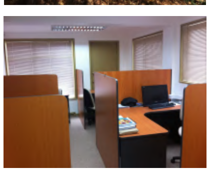

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