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Numerical solution of transient eddy current problems with input current intensities as boundary data

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The aim of this paper is to analyze a numerical method to solve transient eddy current problems with input current intensities as data, formulated in terms of the magnetic field in a bounded domain including conductors and dielectrics. To this end, we introduce a time-dependent weak formulation and prove its well-posedness. Under appropriate hypotheses on the input current intensities, we show that the weak solution has additional regularity and satisfies strong forms of the equations. We propose a finite element method for space discretization based on Nédélec edge elements on tetrahedral mesh, for which we prove well-posedness and error estimates. Furthermore, we introduce an implicit Euler scheme for time discretization and prove error estimates for the fully discrete problem. Moreover, a magnetic scalar potential is introduced to deal with the curl-free condition in the dielectric domain. This approach leads to an important saving in computational effort. Finally, the method is applied to solve two problems: a test with a known analytical solution and an application to electromagnetic forming.

Keywords: Eddy current problems, time-dependent electromagnetic problems, input current intensities, finite elements.

1. Introduction

The objective of this work is to analyze a time-dependent eddy current problem defined in a 3D bounded domain including conducting and dielectric materials when the current source is given in terms of current intensities. This model arises in applications where the problem is written in a bounded domain and it is necessary to link the electromagnetic fields with the sources provided by an external circuit, voltage drops and/or current intensities (see, for instance, Bossavit (2000)). In particular, we are inter-

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ested in imposing the current intensities entering some conducting regions by using non-local boundary conditions. In this framework, we refer the reader to Alonso Rodríguez & Valli (2008), where the authors give a systematic approach to eddy-current problems driven by voltage or current intensity in the harmonic regime. Numerical analysis of different finite element methods to solve this kind of models can be found in Bermúdez *et al.* (2005b); Alonso Rodríguez *et al.* (2009); in both cases, the proposed numerical method has been applied to simulate metallurgical furnaces by means of harmonic eddy current models subjected to boundary conditions proposed in Bossavit (2000). However, if the exciting source is non-sinusoidal or if the materials have a non-linear behavior, a genuine transient eddy current problem must be solved. This is why the present paper pretends to extend the analysis of the model studied in Bermúdez *et al.* (2005b) for the harmonic regime to the general transient situation.

In the literature, we can find several papers devoted to the numerical analysis of the 3D time-dependent eddy current model, both in bounded and unbounded domains by using FEM and BEM-FEM methods Acevedo & Meddahi (2010); Acevedo *et al.* (2009); Kang Kim (2009); Kang *et al.* (2006); Ma (2008); Meddahi & Selgas (2008); Zheng *et al.* (2006). However, in all these works, the current source is given as a volume current in a conducting region and the publications differ in the primary unknown of each formulation. Moreover, the models proposed in bounded domains only deal with homogeneous essential and/or natural boundary conditions. Thus, to the author's knowledge, the transient linear eddy current problem by imposing the current intensities has not been analyzed before, and this is the main objective of the present paper.

By following Bermúdez *et al.* (2005b), we propose a formulation based on the magnetic field in the conductor regions and a scalar magnetic potential in the dielectric ones. The scalar potential is defined from the curl-free condition of the magnetic field in the air and can be multivalued in order to consider general topologies. Notice that the introduction of this potential has two main advantages: it leads to an important saving from a computational point of view and allows us to impose directly the current intensities in terms of the jumps of the scalar potential. From a mathematical point of view, we will obtain a parabolic problem and prove its well posedness by using a suitable lifting from the boundary conditions. If the intensities are smooth enough, we prove additional regularity properties for the magnetic field; these properties are used to prove that the weak solution satisfies in some sense the strong eddy current model posed initially. We propose a finite element method combined with an implicit Euler time discretization to numerically solve the problem. Concerning the space discretization, the magnetic field is approximated by the lowest order Nédélec edge finite elements and the magnetic potential by standard piecewise linear continuous elements. The current intensities are imposed as jumps of the multivalued magnetic potential on some prescribed cut surfaces. We obtain convergence results for the main physical quantities, namely the magnetic field and the current density.

The outline of the paper is as follows: In Section 2 we introduce the transient eddy current model and state the geometrical framework for our analysis. In Section 3 we obtain a weak formulation of the problem. We prove that it is well-posed as well as a regularity result. In Section 4 we introduce a semi-discretization based on finite elements and prove error estimates. In Section 5 we propose an implicit Euler scheme for time discretization and obtain error estimates for the fully discretized problem. In Section 6, we report some numerical results; first, we present the results obtained for an example with known analytical solution, which confirms the order of convergence predicted by the theory and allows us to assess the performance of the method; secondly, we simulate an application of electromagnetic forming, where the transient simulation in the time domain is mandatory.

Throughout the paper, we use standard notation for function spaces, norms, and duality pairings.

2. Time-dependent eddy current problem with input current intensities as boundary data

Three dimensional eddy current problems describe low-frequency electromagnetic phenomena. In this case, displacement currents may be neglected (see (Bossavit, 2000, Chapter 8)) so Maxwell's equations become

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.1}$$

$$\partial_t \mu \boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.2}$$

$$\operatorname{div}(\mu \boldsymbol{H}) = 0 \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.3}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.4}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ is the magnetic field, $\boldsymbol{J}(t,\boldsymbol{x})$ the current density, μ the magnetic permeability and σ the electric conductivity. Here and thereafter, we use boldface letters to denote vector fields and variables as well as vector-valued operators.

We are interested in solving these equations for $t \in [0,T]$ in a simply connected three-dimensional bounded domain Ω, which consists of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. The electric conductivity σ vanishes in Ω_{D}. The mathematical framework we are going to analyze covers transient eddy current problems posed on different geometrical settings. In Figure 1 we sketch a particular case including several connected components of the conducting domain with different topological properties.

The domain Ω is assumed to have a Lipschitz-continuous connected boundary $\partial\Omega$, which splits into two parts: $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{D}}$, with $\Gamma_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ and $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ being the outer boundaries of the conducting and dielectric domains, respectively. We denote $\Gamma_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$, the interface between dielectrics and conductors. We also denote by $\boldsymbol{n}$ the outer unit normal vector to $\partial\Omega$.

As shown in Figure 1, the connected components of the conducting domain are of two types: "inductors" which go through the boundary of Ω, and "workpieces" which have their closure included in Ω. We denote $\Omega_{\mathrm{C}}^1, \ldots, \Omega_{\mathrm{C}}^N$ the former and $\Omega_{\mathrm{C}}^{N+1}, \ldots, \Omega_{\mathrm{C}}^M$ the latter.

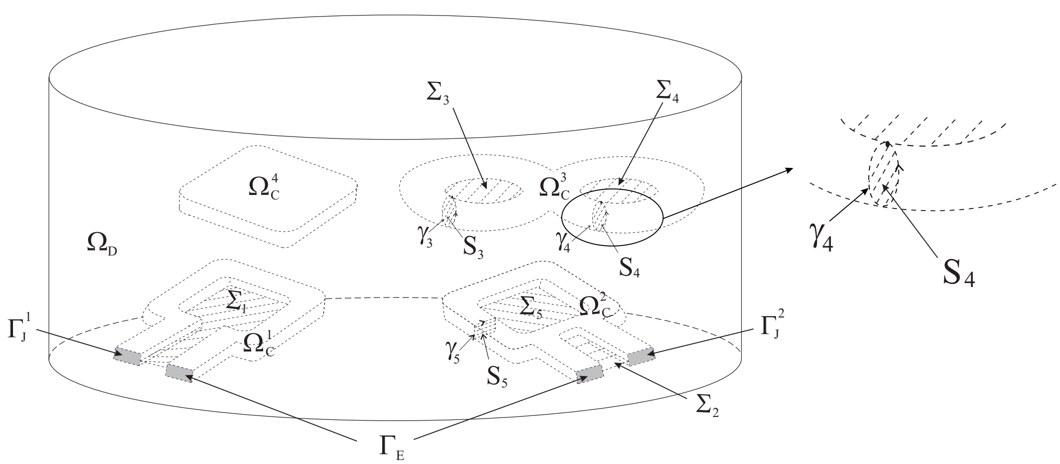

FIG. 1. Sketch of the domain and zoom around S_4.

We assume that the outer boundary of each inductor, $\partial\Omega_{\mathrm{C}}^n \cap \partial\Omega$ ($n = 1,\dots,N$), has two connected components, both with non-zero measure: the current entrance, Γ_{J}^n, where the inductor is connected to a transient electric current source, and the current exit, Γ_{E}^n. Finally, we denote $\Gamma_{\mathrm{J}} := \Gamma_{\mathrm{J}}^1 \cup \cdots \cup \Gamma_{\mathrm{J}}^N$ and $\Gamma_{\mathrm{E}} := \Gamma_{\mathrm{E}}^1 \cup \cdots \cup \Gamma_{\mathrm{E}}^N$. Further, we assume that $\Gamma_{\mathrm{J}} \cap \Gamma_{\mathrm{E}} = \emptyset$.

We consider that μ and σ are time-independent, and that there exist constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$
\begin{aligned}
&0 < \underline{\mu} \leqslant \mu(\boldsymbol{x}) \leqslant \overline{\mu}, \quad \text{a.e. } \boldsymbol{x} \in \Omega, \\
&0 < \underline{\sigma} \leqslant \sigma(\boldsymbol{x}) \leqslant \overline{\sigma}, \quad \text{a.e. } \boldsymbol{x} \in \Omega_{\mathrm{C}}, \qquad \sigma \equiv 0 \text{ in } \Omega_{\mathrm{D}}.
\end{aligned}
$$

We have to complete the model with an initial condition $\boldsymbol{H}(0) = \boldsymbol{H}_0$ and suitable boundary conditions. For the latter, we consider the following ones which were proposed in Bossavit (2000) and analyzed in Bermúdez *et al.* (2005b) in the harmonic regime:

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n} = I_n(t), \quad n = 1,\dots,N, \quad t \in [0,T], \tag{2.5}$$

$$\boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } [0,T]\times\Gamma_{\mathrm{E}}, \tag{2.6}$$

$$\boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } [0,T]\times\Gamma_{\mathrm{J}}, \tag{2.7}$$

$$\mu\boldsymbol{H}\cdot\boldsymbol{n} = 0 \quad \text{on } [0,T]\times\partial\Omega, \tag{2.8}$$

where the only data are the current intensities I_n through each surface Γ_{J}^n, which are assumed to satisfy $I_n \in \mathrm{H}^1(0,T)$, $n = 1,\dots,N$.

Conditions (2.5) account for the input current intensities through each Γ_{J}^n. Conditions (2.6), (2.7) and (2.8) have been proposed in Bossavit (2000) in a more general setting. They will appear as natural boundary conditions of our weak formulation of the problem. The former implies the assumption that the electric current is normal to the current entrance and exit surfaces, whereas the latter means that the magnetic field is tangential to the boundary. (See Bermúdez *et al.* (2005a) for further discussions on these boundary conditions and Bermúdez *et al.* (2005b) for its application on the modeling of an electric furnace.)

3. Variational formulation, existence and uniqueness

Our first goal is to give a variational formulation in terms of the magnetic field to solve the transient eddy current problem. To do this, we follow the arguments from Bermúdez *et al.* (2005b), which we include for the sake of completeness.

By testing (2.2) with a smooth function $\boldsymbol{G}$ such that

$$\mathbf{curl}\,\boldsymbol{G} = \boldsymbol{0} \quad \text{in } \Omega_{\mathrm{D}} \qquad \text{and} \qquad \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0, \quad n = 1,\dots,N, \tag{3.1}$$

we have

$$\int_\Omega \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{G} = 0. \tag{3.2}$$

Moreover, by formal calculations, boundary condition (2.8) implies that the tangential component of the electric field $\boldsymbol{E}$ is a gradient. Indeed, after integrating $\mu\partial_t\boldsymbol{H}\cdot\boldsymbol{n}$ on any surface S contained in $\partial\Omega$, by using (2.2) and Stokes Theorem, we obtain

$$0 = \int_S \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{n} = -\int_S \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{n} = -\int_{\partial S} \boldsymbol{E}\cdot\boldsymbol{t} = -\int_{\partial S} \boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n})\cdot\boldsymbol{t},$$

with $\boldsymbol{t}$ being a unit vector tangent to ∂S. Therefore, since $\partial\Omega$ is simply connected, we can assert that there exists a sufficiently smooth function V defined in Ω up to a constant, such that $V|_{\partial\Omega}$ is a surface potential of the tangential component of $\boldsymbol{E}$; namely, $\boldsymbol{E}\times\boldsymbol{n} = -\nabla V\times\boldsymbol{n}$ on $\partial\Omega$. On the other hand, (2.6) and (2.7) imply that V must be constant on each connected component of Γ_{J} and Γ_{E}. Furthermore, in our model case, we will assume that the potential is the same on the whole Γ_{E}. Hence V can be chosen to be null on Γ_{E}. Then, we can transform the second term of (3.2) by using Green's formulas as follows:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{G} = \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G} - \int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G}, \tag{3.3}$$

the latter equality because

$$-\int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_{\partial\Omega}\nabla V\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_\Omega \nabla V\cdot\mathbf{curl}\,\boldsymbol{G} = \int_{\partial\Omega} V\,\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0,$$

where, in the last equality, we have used that $V=0$ on Γ_{E}, V is constant on each Γ_{J}^n and (3.1).

Now, by substituting (3.3) in (3.2), we obtain

$$\int_\Omega \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G} = 0.$$

Moreover, because of the first equation in (3.1), the second integral above reduces to the conducting domain Ω_{C}, where (2.1) and (2.4) lead to $\boldsymbol{E} = \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}$. Thus, we obtain

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G} = 0.$$

Let

$$\mathscr{X} := \{\boldsymbol{G}\in \mathrm{H}(\mathbf{curl};\Omega) : \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\}.$$

For all $\boldsymbol{G}\in\mathscr{X}$, $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}\in \mathrm{H}^{-1/2}(\partial\Omega)$ and $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0$ on Γ_{D}. Then $\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{J}}^n}$ is well defined. Indeed, let ζ be any smooth function defined in $\partial\Omega$ such that $\zeta = 1$ on Γ_{J}^n and $\zeta = 0$ on Γ_{E} (such functions exist because $\Gamma_{\mathrm{J}}\cap\Gamma_{\mathrm{E}} = \emptyset$). Then $\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{J}}^n} := \langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},\zeta\rangle_{\partial\Omega}$ is well defined and its value does not depend on the particular extension ζ. Here and thereafter, $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing in $\mathrm{H}^{-1/2}(\partial\Omega)\times\mathrm{H}^{1/2}(\partial\Omega)$. Let

$$\mathscr{V} := \left\{\boldsymbol{G}\in\mathscr{X} : \langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{J}}^n} = 0,\ n = 1,\ldots,N\right\},$$

which is a closed subspace of $\mathscr{X}$.

We are led to the following problem: Find $\boldsymbol{H}$ such that

$$\int_{\Gamma_{\mathrm{J}}^n}\mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n} = I_n(t), \qquad n = 1,\ldots,N, \tag{3.4}$$

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G} = 0 \qquad \forall\boldsymbol{G}\in\mathscr{V}, \tag{3.5}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0. \tag{3.6}$$

3.1 *Introducing a magnetic potential*

In what follows we show how problem (3.4)–(3.6) can be rewritten by replacing the magnetic field in the dielectric domain Ω_{D} by a (scalar) magnetic potential.

We assume there exist L connected "cut" surfaces $\Sigma_n \subset \Omega_{\mathrm{D}}$, $n = 1, \cdots, L$, such that $\partial\Sigma_n \subset \partial\Omega_{\mathrm{D}}$ and $\widetilde{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_{n=1}^{L} \Sigma_n$ is pseudo-Lipschitz and simply connected (see, for instance, Amrouche *et al.* (1998)). We also assume that $\overline{\Sigma}_n \cap \overline{\Sigma}_m = \emptyset$ for $n \neq m$ (see Figure 1). For each inductor, Ω_{C}^n, $n = 1, \dots, N$, there exists one cut surface Σ_n such that, necessarily, $\partial\Sigma_n \cap \partial\Omega_{\mathrm{D}} \neq \emptyset$ (see Figure 1). The remaining cut surfaces, $\Sigma_{N+1}, \dots, \Sigma_L$, are assumed to be contained in the interior of Ω_{D} (see Figure 1, again).

For each cut surface Σ_n, we assume there exists a surface $S_n \subset \Omega_{\mathrm{C}}^n$, with $\partial S_n \subset \partial\Omega_{\mathrm{C}}^n$, and such that its boundary γ_n is a simple closed curve. We assume that γ_n intersects once and only once Σ_n and it does not intersect Σ_m, $m \neq n$. Moreover, for $n = 1, \dots, N$, we chose $\overline{S}_n = \Gamma_{\mathrm{J}}^n$.

We denote the two faces of each Σ_n by Σ_n^- and Σ_n^+ and fix a unit normal $\boldsymbol{n}_n$ on Σ_n as the "outer" normal to $\Omega_{\mathrm{D}} \setminus \Sigma_n$ along Σ_n^+. We choose an orientation for each γ_n by taking its initial and end points on Σ_n^- and Σ_n^+, respectively. We denote by $\boldsymbol{t}_n$ the unit vector tangent to γ_n according with this orientation.

Each function $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ has in general different traces on each face of Σ_n and we denote by

$$[\![\widetilde{\Psi}]\!]_{\Sigma_n} := \widetilde{\Psi}|_{\Sigma_n^-} - \widetilde{\Psi}|_{\Sigma_n^+}$$

the jump of $\widetilde{\Psi}$ through Σ_n along $\boldsymbol{n}_n$. The gradient of $\widetilde{\Psi}$ in $\mathscr{D}'(\widetilde{\Omega}_{\mathrm{D}})$ can be extended to $\mathrm{L}^2(\Omega_{\mathrm{D}})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

Let Θ be the linear subspace of $\mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ defined by

$$\Theta := \left\{ \widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = \text{constant}, \ n = 1, \dots, L \right\}.$$

Then, for $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we have that $\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{D}})$ if and only if $\widetilde{\Psi} \in \Theta$, in which case $\mathbf{curl}(\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) = \mathbf{0}$ (see (Amrouche *et al.*, 1998, Lemma 3.11)).

We use the following notation: given $\boldsymbol{G}_{\mathrm{C}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$ and $\boldsymbol{G}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$, $(\boldsymbol{G}_{\mathrm{C}}|\boldsymbol{G}_{\mathrm{D}})$ denotes the field $\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3$ defined by $\boldsymbol{G}|_{\Omega_{\mathrm{C}}} := \boldsymbol{G}_{\mathrm{C}}$ and $\boldsymbol{G}|_{\Omega_{\mathrm{D}}} := \boldsymbol{G}_{\mathrm{D}}$.

Let us denote by $\mathscr{Y}$ the linear space given by

$$\mathscr{Y} := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}}) \times (\Theta/\mathbb{R}) : \ (\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}; \Omega) \right\}.$$

Then $(\boldsymbol{G}, \widetilde{\Psi}) \in \mathscr{Y}$ if and only if $(\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) \in \mathscr{X}$.

When a magnetic potential $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ is used, boundary condition (3.4) can be imposed by fixing its jumps on the cut surfaces. Indeed, if $(\boldsymbol{G}, \widetilde{\Psi}) \in \mathscr{Y}$ is smooth enough for the following integrals to make sense, we have that

$$\langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\mathrm{J}}^n} = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n} = \int_{\gamma_n} \boldsymbol{G} \cdot \boldsymbol{t}_n = \int_{\gamma_n} \widetilde{\mathbf{grad}}\,\widetilde{\Psi} \cdot \boldsymbol{t}_n = [\![\widetilde{\Psi}]\!]_{\Sigma_n}, \tag{3.7}$$

where we have used Stokes Theorem and the fact that $\boldsymbol{G} \times \boldsymbol{n} = \widetilde{\mathbf{grad}}\,\widetilde{\Psi} \times \boldsymbol{n}$ on $\Gamma_{\mathrm{I}} \supset \gamma_n$.

Therefore, problem (3.4)–(3.6) reduces to find $(\boldsymbol{H}, \widetilde{\Phi}) : [0,T] \to \mathscr{Y}$ such that

$$[\![\widetilde{\Phi}(t)]\!]_{\Sigma_n} = I_n(t), \quad n = 1,\dots,N, \tag{3.8}$$

$$\int_{\Omega_C} \mu\, \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_D} \mu\, \partial_t \widetilde{\mathbf{grad}}\, \widetilde{\Phi}(t) \cdot \widetilde{\mathbf{grad}}\, \widetilde{\Psi} + \int_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \mathbf{curl}\, \boldsymbol{G} = 0 \quad \forall (\boldsymbol{G}, \widetilde{\Psi}) \in \mathscr{Y}^0, \tag{3.9}$$

$$(\boldsymbol{H}(0) | \widetilde{\mathbf{grad}}\, \widetilde{\Phi}(0)) = \boldsymbol{H}_0, \tag{3.10}$$

where

$$\mathscr{Y}^0 := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \mathscr{Y} \;:\; [\![\widetilde{\Psi}]\!]_{\Sigma_n} = 0,\; n = 1,\dots,N \right\}.$$

3.2 *Existence and uniqueness of the solution*

In this section we will prove the existence and uniqueness of the solution to the transient eddy current problem (3.4)–(3.6). With this aim, we introduce an adequate functional framework for functions defined on a bounded time interval $[0,T]$ and with values in a separable Hilbert space X. We use the notation $\mathscr{C}^0([0,T];X)$ for the Banach space consisting of all continuous functions $f : [0,T] \to X$. More generally, for any $k \in \mathbb{N}$, $\mathscr{C}^k([0,T];X)$ denotes the subspace of $\mathscr{C}^0([0,T];X)$ of all functions f with (strong) $\dfrac{d^j f}{dt^j}$ derivatives in $\mathscr{C}^0([0,T];X)$ for all $1 \leqslant j \leqslant k$. We will use indistinctly the notations $\dfrac{df}{dt} = \partial_t f$ to express the derivative with respect to the variable t. We also consider the spaces $\mathrm{L}^2(0,T;X)$ of classes of functions $f : [0,T] \to X$ that are Böchner-measurable and such that

$$\|f\|_{\mathrm{L}^2(0,T;X)} := \left(\int_0^T \|f(t)\|_X^2 \, dt \right)^{1/2} < \infty.$$

Furthermore, we will use the space

$$\mathrm{H}^1(0,T;X) := \{ f \in \mathrm{L}^2(0,T;X) : \partial_t f \in \mathrm{L}^2(0,T;X) \}.$$

Analogously, we define $\mathrm{H}^k(0,T;X)$ for all $k \in \mathbb{N}$.

On the other hand, we denote by $\mathscr{H}_{\mathscr{V}}$ the closure of $\mathscr{V}$ in $\mathrm{L}^2(\Omega)^3$ and by $\mathscr{V}'$ the dual space of $\mathscr{V}$ with respect to the pivot space $\mathscr{H}_{\mathscr{V}}$ with measure $\mu(\boldsymbol{x})\, d\boldsymbol{x}$ (which is topologically equivalent to $\mathrm{L}^2(\Omega)^3$ with the standard Lebesgue measure). Hence, for $\boldsymbol{F} \in \mathscr{H}_{\mathscr{V}}$ we have

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle_{\mathscr{V}' \times \mathscr{V}} = \int_\Omega \mu \boldsymbol{F} \cdot \boldsymbol{G} \qquad \forall \boldsymbol{G} \in \mathscr{V}.$$

Thus, problem (3.4)–(3.6) can be rewritten as follows:

Problem 3.1 Find $\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathscr{X}) \cap \mathrm{H}^1(0,T;\mathscr{V}')$ such that

$$\langle \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_J^n} = I_n(t), \quad n = 1,\dots,N, \tag{3.11}$$

$$\langle \partial_t \boldsymbol{H}(t), \boldsymbol{G} \rangle_{\mathscr{V}' \times \mathscr{V}} + a(\boldsymbol{H}(t), \boldsymbol{G}) = 0 \quad \forall \boldsymbol{G} \in \mathscr{V}, \tag{3.12}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0. \tag{3.13}$$

The bilinear form a is defined over $\mathscr{X} \times \mathscr{X}$ by

$$a(\boldsymbol{K}, \boldsymbol{G}) := \int_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{K} \cdot \mathbf{curl}\, \boldsymbol{G}.$$

It is continuous and satisfies the following Gårding's inequality: for each $\lambda > 0$ there exists $\alpha > 0$ such that

$$a(\boldsymbol{G},\boldsymbol{G}) + \lambda \|\boldsymbol{G}\|^2_{\mathrm{L}^2(\Omega)^3} \geqslant \alpha \|\boldsymbol{G}\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{G} \in \mathscr{X}. \tag{3.14}$$

For the initial data we assume that

$$\boldsymbol{H}_0 \in \mathscr{X} \quad \text{and} \quad \langle \mathbf{curl}\,\boldsymbol{H}_0 \cdot \boldsymbol{n}, 1\rangle_{\Gamma^n_{\mathrm{J}}} = I_n(0), \quad n = 1,\ldots,N. \tag{3.15}$$

The reason for this assumption will be made clear below.

The next step is to write an equivalent form of Problem 3.1 more amenable for the analysis. The first goal is to build $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathscr{X})$ satisfying (3.11). With this aim we will use the unique solutions $w_n \in \mathrm{H}^1(\Omega^n_{\mathrm{C}})$, $n = 1,\ldots,N$, of the following problems:

$$\begin{aligned} -\Delta w_n &= 0 \quad \text{in } \Omega^n_{\mathrm{C}}, \\ \frac{\partial w_n}{\partial n} &= \begin{cases} \dfrac{I_n(0)}{|\Gamma^n_{\mathrm{J}}|} & \text{on} \quad \Gamma^n_{\mathrm{J}}, \\ 0 & \text{on} \quad \partial\Omega^n_{\mathrm{C}} \cap \Gamma_{\mathrm{I}}, \end{cases} \\ w_n &= 0 \quad \text{on } \Gamma^n_{\mathrm{E}}. \end{aligned} \tag{3.16}$$

Straightforward computations allow us to show that $\|w_n\|_{\mathrm{H}^1(\Omega^n_{\mathrm{C}})} \leqslant C|I_n(0)|$.

Let $\boldsymbol{Q} \in \mathrm{L}^2(\Omega)^3$ be defined by

$$\boldsymbol{Q} := \begin{cases} \nabla w_n & \text{in } \Omega^n_{\mathrm{C}}, \quad n = 1,\ldots,N, \\ \mathbf{0} & \text{in } \Omega_{\mathrm{D}}, \\ \mathbf{0} & \text{in } \Omega^n_{\mathrm{C}}, \quad n = N+1,\ldots,M. \end{cases}$$

Since $\operatorname{div}(\nabla w_n) = 0$ in Ω^n_{C} and $\nabla w_n \cdot \boldsymbol{n} = 0$ on $\Gamma_{\mathrm{I}} \cap \partial\Omega^n_{\mathrm{C}}$, $\boldsymbol{Q} \in \mathrm{H}(\operatorname{div},\Omega)$ and $\operatorname{div}\boldsymbol{Q} = 0$ in Ω. Then, since $\partial\Omega$ is connected, there exists a vector potential $\widehat{\boldsymbol{H}}_0 \in \mathrm{H}^1(\Omega)^3$ such that

$$\mathbf{curl}\,\widehat{\boldsymbol{H}}_0 = \boldsymbol{Q} \tag{3.17}$$

and $\operatorname{div}\widehat{\boldsymbol{H}}_0 = 0$ (see (Girault & Raviart, 1986, Theorem I.3.4)). Moreover, as a consequence of the open mapping theorem, we obtain

$$\|\widehat{\boldsymbol{H}}_0\|_{\mathrm{H}^1(\Omega)^3} \leqslant C\|\boldsymbol{Q}\|_{\mathrm{L}^2(\Omega)^3} \leqslant C\left(\sum_{n=1}^{N} |I_n(0)|^2\right)^{1/2}; \tag{3.18}$$

here and thereafter C denotes a generic constant not necessarily the same at each occurrence.

Similarly, let $v_n : [0,T] \to \mathrm{H}^1(\Omega^n_{\mathrm{C}})$ be the unique solution of

$$\begin{aligned} -\Delta v_n(t) &= 0 \quad \text{in } \Omega^n_{\mathrm{C}}, \\ \frac{\partial v_n}{\partial n}(t) &= \begin{cases} \dfrac{I'_n(t)}{|\Gamma^n_{\mathrm{J}}|} & \text{on} \quad \Gamma^n_{\mathrm{J}}, \\ 0 & \text{on} \quad \partial\Omega^n_{\mathrm{C}} \cap \Gamma_{\mathrm{I}}, \end{cases} \\ v_n(t) &= 0 \quad \text{on } \Gamma^n_{\mathrm{E}}. \end{aligned} \tag{3.19}$$

Then, as above, $\|v_n(t)\|_{\mathrm{H}^1(\Omega_{\mathrm{C}}^n)} \leqslant C|I_n'(t)|$. (Recall that we have assumed $I_n \in \mathrm{H}^1(0,T)$.)

We repeat the procedure above and define $\boldsymbol{P}(t) \in \mathrm{L}^2(\Omega)^3$ by $\boldsymbol{P}(t) := \nabla v_n(t)$ in Ω_{C}^n, $n = 1,\dots,N$, extended by zero to the whole Ω. Once more $\boldsymbol{P}(t) \in \mathrm{H}(\mathrm{div},\Omega)$ with $\mathrm{div}\,\boldsymbol{P}(t) = 0$. Then, there exists a vector potential $\boldsymbol{F}(t) \in \mathrm{H}^1(\Omega)^3$ such that

$$\mathbf{curl}\,\boldsymbol{F}(t) = \boldsymbol{P}(t), \tag{3.20}$$

$\mathrm{div}\,\boldsymbol{F}(t) = 0$ and

$$\|\boldsymbol{F}(t)\|_{\mathrm{H}^1(\Omega)^3} \leqslant C\|\boldsymbol{P}(t)\|_{\mathrm{L}^2(\Omega)^3} \leqslant C\left(\sum_{n=1}^{N} |I_n'(t)|^2\right)^{1/2}. \tag{3.21}$$

Function $\boldsymbol{F}$ is Böchner-integrable (i.e., it is Böchner-measurable and the real-valued function $\|\boldsymbol{F}(t)\|_{\mathscr{X}} : [0,T] \to \mathbb{R}$ has a finite Lebesgue integral). Indeed, using the Cauchy-Schwartz inequality we obtain

$$\int_0^T \|\boldsymbol{F}(s)\|_{\mathscr{X}}\, ds \leqslant C\left(\sum_{n=1}^{N}\int_0^T |I_n'(s)|^2\, ds\right)^{1/2} < \infty,$$

since $I_n \in \mathrm{H}^1(0,T)$, $n = 1,\dots,N$. Therefore, if we define

$$\widehat{\boldsymbol{H}}(t) := \widehat{\boldsymbol{H}}_0 + \int_0^t \boldsymbol{F}(s)\, ds, \tag{3.22}$$

then it follows that (see (Ženíšek, 1990, Remark 131(b))) a.e. in $[0,T]$ and in the distributional sense

$$\partial_t \widehat{\boldsymbol{H}}(t) = \boldsymbol{F}(t).$$

On the other hand, from (3.21) we have

$$\int_0^T \|\partial_t \widehat{\boldsymbol{H}}(t)\|_{\mathscr{X}}^2\, dt = \int_0^T \|\boldsymbol{F}(t)\|_{\mathscr{X}}^2\, dt \leqslant C\sum_{n=1}^{N}\int_0^T |I_n'(t)|^2\, dt < \infty.$$

Straightforward computations yield $\int_0^T \|\widehat{\boldsymbol{H}}(t)\|_{\mathscr{X}}^2\, dt < \infty$ too, so that we conclude that $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathscr{X})$; furthermore, from (3.17), (3.20) and (Ženíšek, 1990, Theorem 111 & 127) we have that

$$\mathbf{curl}\,\widehat{\boldsymbol{H}}(t) = \boldsymbol{Q} + \int_0^t \boldsymbol{P}(s)\, ds.$$

Consequently, for $n = 1,\dots,N$,

$$\begin{aligned}\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\widehat{\boldsymbol{H}}(t)\cdot\boldsymbol{n} &= \int_{\Gamma_{\mathrm{J}}^n} \boldsymbol{Q}\cdot\boldsymbol{n} + \int_{\Gamma_{\mathrm{J}}^n}\int_0^t \boldsymbol{P}(s)\cdot\boldsymbol{n}\, ds = \int_{\Gamma_{\mathrm{J}}^n} \frac{\partial w_n}{\partial n} + \int_{\Gamma_{\mathrm{J}}^n}\int_0^t \frac{\partial v_n}{\partial n}(s)\, ds \\ &= I_n(0) + \int_0^t I_n'(s)\, ds = I_n(t).\end{aligned}$$

Now, if we write $\boldsymbol{H} = \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$, then Problem 3.1 is equivalent to finding $\widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0,T;\mathscr{V}) \cap \mathrm{H}^1(0,T;\mathscr{V}')$ such that

$$\langle \partial_t \widetilde{\boldsymbol{H}}(t), \boldsymbol{G}\rangle_{\mathscr{V}'\times\mathscr{V}} + a(\widetilde{\boldsymbol{H}}(t),\boldsymbol{G}) = \langle f(t),\boldsymbol{G}\rangle_{\mathscr{V}'\times\mathscr{V}} \quad \forall \boldsymbol{G} \in \mathscr{V}, \tag{3.23}$$

$$\widetilde{\boldsymbol{H}}(0) = \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}_0, \tag{3.24}$$

where $f:[0,T]\to\mathscr{V}'$ is defined by

$$\langle f(t),\boldsymbol{G}\rangle_{\mathscr{V}'\times\mathscr{V}} := -\int_{\Omega}\mu\,\partial_t\widehat{\boldsymbol{H}}(t)\cdot\boldsymbol{G}-\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\widehat{\boldsymbol{H}}(t)\cdot\mathbf{curl}\,\boldsymbol{G}\qquad\forall\boldsymbol{G}\in\mathscr{V}. \tag{3.25}$$

Notice that from (3.18) and (3.21)

$$\|f(t)\|^2_{\mathscr{V}'}\leqslant C\left\{\sum_{n=1}^{N}|I_n(0)|^2+\sum_{n=1}^{N}|I_n'(t)|^2+\sum_{n=1}^{N}\int_0^t|I_n'(s)|^2\,ds\right\} \tag{3.26}$$

and hence $f\in\mathrm{L}^2(0,T;\mathscr{V}')$.

Regarding the initial condition, since $\mathscr{H}_{\mathscr{V}}$ is the closure of $\mathscr{V}$ in $\mathrm{L}^2(\Omega)^3$, we have that $\mathrm{L}^2(0,T;\mathscr{V})\cap\mathrm{H}^1(0,T;\mathscr{V}')\hookrightarrow\mathscr{C}^0([0,T];\mathscr{H}_{\mathscr{V}})$ (see (Dautray & Lions, 1992, Chapter XVIII)) and consequently $\boldsymbol{H}_0-\widehat{\boldsymbol{H}}_0$ has to belong to $\mathscr{H}_{\mathscr{V}}$. This is fulfilled in our case since, because of assumption (3.15), $\boldsymbol{H}_0-\widehat{\boldsymbol{H}}_0\in\mathscr{V}\subset\mathscr{H}_{\mathscr{V}}$.

Now, we are in a position to prove the following result:

THEOREM 3.2 Given $I_n\in\mathrm{H}^1(0,T)$, $n=1,\dots,N$, and $\boldsymbol{H}_0$ satisfying (3.15), Problem 3.1 has a unique solution $\boldsymbol{H}$. Furthermore, there exists $C>0$ such that

$$\|\boldsymbol{H}\|^2_{\mathscr{C}^0(0,T;\mathrm{L}^2(\Omega)^3)}+\|\boldsymbol{H}\|^2_{\mathrm{L}^2(0,T;\mathscr{X})}\leqslant C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3}+\sum_{n=1}^{N}\|I_n\|^2_{\mathrm{H}^1(0,T)}\right\}.$$

Proof. Let $\widehat{\boldsymbol{H}}_0$ and $\widehat{\boldsymbol{H}}(t)$ be defined as above. Since $\widetilde{\boldsymbol{H}}(0)\in\mathscr{H}_{\mathscr{V}}$, $a(\cdot,\cdot)$ is a continuous bilinear form satisfying the Gårding inequality (3.14), and $f\in\mathrm{L}^2(0,T;\mathscr{V}')$ (cf. (3.26)), by applying Lions Theorem (see Dautray & Lions (1992)) problem (3.23)–(3.24) has a unique solution $\widetilde{\boldsymbol{H}}$ and there exists $C>0$ such that

$$\max_{t\in[0,T]}\|\widetilde{\boldsymbol{H}}(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\int_0^T\|\widetilde{\boldsymbol{H}}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt\leqslant C\left\{\|\boldsymbol{H}_0-\widehat{\boldsymbol{H}}_0\|^2_{\mathrm{L}^2(\Omega)^3}+\|f\|^2_{\mathrm{L}^2(0,T;\mathscr{V}')}\right\}.$$

Therefore, Problem 3.1 has a unique solution $\boldsymbol{H}=\widetilde{\boldsymbol{H}}+\widehat{\boldsymbol{H}}$ and

$$\max_{t\in[0,T]}\|\boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\int_0^T\|\boldsymbol{H}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt\leqslant C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3}+\sum_{n=1}^{N}|I_n(0)|^2+\sum_{n=1}^{N}\int_0^T|I_n'(s)|^2\,ds\right\}.$$

Thus we conclude the theorem. □

3.3 *Additional regularity*

Our next goal is to show that the solution of Problem 3.1 satisfies somehow equations (2.1)–(2.8). First, we will show an additional regularity result for this kind of evolution problems.

We consider two real, separable Hilbert spaces X and H. Moreover, we suppose that X is dense in H, so that, by identifying H with its dual H', we have $X\hookrightarrow H\hookrightarrow X'$, both inclusions being dense.

Given $u_0\in H$ and $g\in\mathrm{L}^2(0,T;X')$ we consider the following problem: Find $u\in\mathrm{L}^2(0,T;X)\cap\mathrm{H}^1(0,T;X')$ such that

$$\langle\partial_t u(t),v\rangle_{X'\times X}+c(u(t),v)=\langle g(t),v\rangle_{X'\times X}\quad\forall v\in X, \tag{3.27}$$

$$u(0)=u_0, \tag{3.28}$$

where $c : X \times X \to \mathbb{R}$ is bilinear, bounded and elliptic. We know that problem (3.27)–(3.28) has a unique solution (see Dautray & Lions (1992)). Moreover, we have the following additional regularity result.

LEMMA 3.1 If $u_0 \in X$, $g \in \mathrm{H}^1(0,T;X')$ and u is the solution of problem (3.27)–(3.28), then $u \in \mathrm{L}^\infty(0,T;X)$ and $\partial_t u \in \mathrm{L}^2(0,T;H)$.

Proof. The proof follows the line of that of Theorem 5 from (Evans, 1998, Chapter 7) by using a Galerkin approximation method (see Dautray & Lions (1992)). Let $\{X_m\}_{m\in\mathbb{N}}$ be a family of finite dimensional vector spaces satisfying

$$X_m \subseteq X, \qquad \dim X_m < \infty, \qquad X_m \to X \text{ as } m \to \infty;$$

the convergence above must be understood in the following sense: there exists a dense subspace $\mathscr{U}$ of X, such that, for all $v \in \mathscr{U}$, we can find a sequence $\{v_m\}_{m\in\mathbb{N}}$, $v_m \in X_m$ such that $v_m \to v$ in X as $m \to \infty$.

Therefore, for $u_0 \in X$ let $\{u_{0m}\}_{m\in\mathbb{N}}$, $u_{0m} \in X_m$ be such that $u_{0m} \to u_0$ in X as $m \to \infty$. Let $\{w_j\}_{j\in\mathbb{N}}$ be such that $\{w_j\}_{j=1}^m$ in a basis of X_m.

Consider the following problem: Find $u_m(t) := \sum_{j=1}^m \xi_j(t) w_j$ satisfying

$$(\partial_t u_m(t), w_j)_H + c(u_m(t), w_j) = \langle g(t), w_j\rangle_{X'\times X}, \quad 1 \leqslant j \leqslant m, \tag{3.29}$$

$$u_m(0) = u_{0m}. \tag{3.30}$$

We know that there exists a subsequence of $\{u_m\}_{m\in\mathbb{N}}$, that we also denote $\{u_m\}_{m\in\mathbb{N}}$, such that

$$u_m \to u \text{ weakly in } \mathrm{L}^2(0,T;X), \qquad \partial_t u_m \to \partial_t u \text{ weakly in } \mathrm{L}^2(0,T;X').$$

For fixed $m \geqslant 1$, we multiply (3.29) by $\xi_j'(t)$ and sum from $j = 1$ to m, to obtain

$$\|\partial_t u_m(t)\|_H^2 + \frac{1}{2}\frac{d}{dt} c(u_m(t), u_m(t)) = \frac{d}{dt}\langle g(t), u_m(t)\rangle_{X'\times X} - \langle \partial_t g(t), u_m(t)\rangle_{X'\times X}.$$

Integrating over t and using Cauchy-Schwartz inequality yield

$$\int_0^t \|\partial_t u_m(s)\|_H^2\, ds + \|u_m(t)\|_X^2 \leqslant C\left\{\|u_m(0)\|_X^2 + \sup_{0\leqslant t\leqslant T}\|g(t)\|_{X'}^2 + \int_0^T \|\partial_t g(t)\|_{X'}^2\, dt + \int_0^t \|u_m(s)\|_X^2\, ds\right\}.$$

Using Gronwall's inequality, we obtain

$$\|u_m(t)\|_X^2 \leqslant C\left\{\|u_{0m}\|_X^2 + \sup_{0\leqslant t\leqslant T}\|g(t)\|_{X'}^2 + \int_0^T \|\partial_t g(t)\|_{X'}^2\, dt\right\}.$$

Therefore

$$\int_0^T \|\partial_t u_m(t)\|_H^2\, dt + \sup_{0\leqslant t\leqslant T}\|u_m(t)\|_X^2 \leqslant C\left\{\|u_{0m}\|_X^2 + \sup_{0\leqslant t\leqslant T}\|g(t)\|_{X'}^2 + \|\partial_t g\|_{\mathrm{L}^2(0,T;X')}^2\right\}$$

and passing to the limit as $m \to \infty$, we deduce that $u \in \mathrm{L}^\infty(0,T;X)$ and $\partial_t u \in \mathrm{L}^2(0,T;H)$. □

The previous lemma is also valid for any bilinear form that satisfies a Gårding inequality like (3.14). In fact, if we write $u = we^{\lambda t}$, $\lambda > 0$, then w satisfies

$$\langle \partial_t w(t), v\rangle_{X'\times X} + \widetilde{c}(w(t), v) = \langle e^{-\lambda t} g(t), v\rangle_{X'\times X} \quad \forall v \in X,$$

$$w(0) = u_0,$$

where $\widetilde{c}(w(t),v) := c(w(t),v) + \lambda(w(t),v)_H$ is bilinear, bounded and elliptic.

In what follows we will also use the closure of $\mathscr{X}$ in $\mathrm{L}^2(\Omega)^3$, which we denote $\mathscr{H}_{\mathscr{X}}$. We have the following characterization of this space.

LEMMA 3.2

$$\mathscr{H}_{\mathscr{X}} = \left\{ \boldsymbol{G} \in \mathrm{L}^2(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \right\}.$$

Proof. Given that $\{\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\}$ is a closed subspace of $\mathrm{L}^2(\Omega)^3$, it is enough to prove that $\mathscr{X}$ is densely included in this subspace.

Let $\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3$ such that $\mathbf{curl}\,\boldsymbol{G} = \mathbf{0}$ in Ω_{D}. Let $\widetilde{\boldsymbol{G}} \in \mathrm{H}(\mathbf{curl};\Omega)$ such that $\widetilde{\boldsymbol{G}} = \boldsymbol{G}$ in Ω_{D}. Hence, $(\boldsymbol{G} - \widetilde{\boldsymbol{G}})|_{\Omega_{\mathrm{C}}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$. Then, there exists $\{\boldsymbol{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathscr{D}(\Omega_{\mathrm{C}})^3$ such that $\|\boldsymbol{\Phi}_k - (\boldsymbol{G} - \widetilde{\boldsymbol{G}})\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \to 0$ as $k \to \infty$. If we denote by $\widetilde{\boldsymbol{\Phi}}_k$ the extension by zero of $\boldsymbol{\Phi}_k$ to Ω, then $\widetilde{\boldsymbol{\Phi}}_k + \widetilde{\boldsymbol{G}} \in \mathscr{X}$ for all $k \in \mathbb{N}$ and $\|(\widetilde{\boldsymbol{\Phi}}_k + \widetilde{\boldsymbol{G}}) - \boldsymbol{G}\|_{\mathrm{L}^2(\Omega)^3} \to 0$ as $k \to \infty$. $\square$

In what follows, we will apply Lemma 3.1 to our problem. With this end, we will assume more regularity on the input currents intensities, namely, $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\ldots,N$. In such a case we can modify the definition (3.22) of $\widehat{\boldsymbol{H}}$ so that $\partial_t \boldsymbol{H} \in \mathrm{L}^2(0,T;\mathscr{H}_{\mathscr{X}})$. Indeed, let $u_n : [0,T] \to \mathrm{H}^1(\Omega_{\mathrm{C}}^n)$ be the unique solution of

$$\begin{aligned}
-\Delta u_n(t) &= 0 \quad \text{in } \Omega_{\mathrm{C}}^n,\\
\frac{\partial u_n(t)}{\partial n} &= \begin{cases} \dfrac{I_n''(t)}{|\Gamma_{\mathrm{J}}^n|} & \text{on} \quad \Gamma_{\mathrm{J}}^n,\\ 0 & \text{on} \quad \partial\Omega_{\mathrm{C}}^n \cap \Gamma_{\mathrm{I}}, \end{cases}\\
u_n(t) &= 0 \quad \text{on } \Gamma_{\mathrm{E}}^n.
\end{aligned}$$

Proceeding as was done for problem (3.16), we obtain that $\|u_n(t)\|_{\mathrm{H}^1(\Omega_{\mathrm{C}}^n)} \leqslant C|I_n''(t)|$.

Let $\boldsymbol{R}(t) \in \mathrm{L}^2(\Omega)^3$ be defined by $\boldsymbol{R}(t) := \nabla u_n(t)$ in Ω_{C}^n, $n = 1,\ldots,N$, extended by zero to the whole Ω. Hence $\boldsymbol{R}(t) \in \mathrm{H}(\mathrm{div},\Omega)$ with $\mathrm{div}\,\boldsymbol{R}(t) = 0$ and there exists a vector potential $\boldsymbol{K}(t) \in \mathrm{H}^1(\Omega)^3$ such that

$$\mathbf{curl}\,\boldsymbol{K}(t) = \boldsymbol{R}(t), \tag{3.31}$$

$\mathrm{div}\,\boldsymbol{K}(t) = 0$ and

$$\|\boldsymbol{K}(t)\|_{\mathrm{H}^1(\Omega)^3} \leqslant C\|\boldsymbol{R}(t)\|_{\mathrm{L}^2(\Omega)^3} \leqslant C\left(\sum_{n=1}^{N} |I_n''(t)|^2\right)^{1/2}. \tag{3.32}$$

Now, if instead of $\widehat{\boldsymbol{H}}$ as defined in (3.22) we use

$$\widehat{\boldsymbol{H}}(t) := \widehat{\boldsymbol{H}}_0 + t\,\boldsymbol{F}(0) + \int_0^t \left(\int_0^s \boldsymbol{K}(r)dr\right) ds, \tag{3.33}$$

then $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathscr{X})$ and, from (3.17), (3.20) and (3.31), we have that

$$\mathbf{curl}\,\widehat{\boldsymbol{H}}(t) = \boldsymbol{Q} + t\,\boldsymbol{P}(0) + \int_0^t \left(\int_0^s \boldsymbol{R}(r)\,dr\right) ds. \tag{3.34}$$

Hence, $\int_{\Gamma_J^n} \mathbf{curl}\,\widehat{\boldsymbol{H}}(t)\cdot\boldsymbol{n} = I_n(t)$, $n = 1,\dots,N$.

To apply Lemma 3.1 to our problem we take $f : [0,T] \to \mathscr{V}'$ defined as in (3.25) with $\widehat{\boldsymbol{H}}$ given by (3.33). For $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\dots,N$, we have that $f \in \mathrm{H}^1(0,T;\mathscr{V}')$. In fact, $\partial_t f : [0,T] \to \mathscr{V}'$ is given by

$$\langle \partial_t f(t), \boldsymbol{G}\rangle_{\mathscr{V}'\times\mathscr{V}} := -\int_\Omega \mu\,\partial_{tt}\widehat{\boldsymbol{H}}(t)\cdot\boldsymbol{G} - \int_{\Omega_C}\frac{1}{\sigma}\partial_t(\mathbf{curl}\,\widehat{\boldsymbol{H}}(t))\cdot\mathbf{curl}\,\boldsymbol{G}.$$

Since, $\partial_{tt}\widehat{\boldsymbol{H}}(t) = \boldsymbol{K}(t)$ (cf. (3.33)) and $\partial_t(\mathbf{curl}\,\widehat{\boldsymbol{H}}(t)) = \boldsymbol{P}(0) + \int_0^t \boldsymbol{K}(s)\,ds$ (cf. (3.34)), thanks to (3.21) and (3.32) we have

$$\|\partial_t f(t)\|^2_{\mathscr{V}'} \leqslant C\left\{\sum_{n=1}^N |I_n'(0)|^2 + \sum_{n=1}^N |I_n''(t)|^2 + \sum_{n=1}^N \int_0^t |I_n''(s)|^2\,ds\right\}$$

and, consequently, $f \in \mathrm{H}^1(0,T;\mathscr{V}')$, which allows us to use Lemma 3.1 to conclude the following result:

THEOREM 3.3 Given $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\dots,N$, and $\boldsymbol{H}_0$ satisfying (3.15), the unique solution $\boldsymbol{H}$ of Problem 3.1 satisfies $\boldsymbol{H} \in \mathrm{L}^\infty(0,T;\mathscr{H}_{\mathscr{X}})$ and $\partial_t\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathscr{H}_{\mathscr{X}})$.

Proof. Let $\boldsymbol{H}$ be the solution of Problem 3.1. Let $\widehat{\boldsymbol{H}}$ be defined by (3.33). Let $\widetilde{\boldsymbol{H}} := \boldsymbol{H} - \widehat{\boldsymbol{H}}$. Then $\widetilde{\boldsymbol{H}}$ satisfies (3.23)–(3.24) with f given by (3.25). From the assumptions on $\boldsymbol{H}_0$ and the definition of $\widehat{\boldsymbol{H}}_0$ we have that $\boldsymbol{H}_0 - \widehat{\boldsymbol{H}}_0 \in \mathscr{V}$. Moreover, as was shown above, $f \in \mathrm{H}^1(0,T;\mathscr{V}')$. Hence, we can apply Lemma 3.1 to conclude that $\widetilde{\boldsymbol{H}} \in \mathrm{L}^\infty(0,T;\mathscr{V})$ and $\partial_t\widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0,T;\mathscr{H}_{\mathscr{V}})$. Thus the theorem follows from the fact that $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathscr{X})$, which was also shown above. □

Let $\mathscr{X}'$ be the dual space of $\mathscr{X}$ with respect to the pivot space $\mathscr{H}_{\mathscr{X}}$ with measure $\mu(\boldsymbol{x})\,d\boldsymbol{x}$. In order to prove that the solution of Problem 3.1 satisfies equations (2.1)–(2.8) we introduce the following mixed formulation:

Find $(\boldsymbol{H},\vec{V}) \in \mathrm{L}^2(0,T;\mathscr{X}) \cap \mathrm{H}^1(0,T;\mathscr{X}') \times \mathrm{L}^2(0,T;\mathbb{R}^N)$ such that

$$\langle \partial_t\boldsymbol{H}(t), \boldsymbol{G}\rangle_{\mathscr{X}'\times\mathscr{X}} + a(\boldsymbol{H}(t),\boldsymbol{G}) + b(\boldsymbol{G},\vec{V}(t)) = 0 \quad \forall \boldsymbol{G} \in \mathscr{X}, \tag{3.35}$$

$$b(\boldsymbol{H}(t),\vec{W}) = \sum_{n=1}^N \boldsymbol{I}_n(t)\,W_n \quad \forall \vec{W} = (W_1,\dots,W_N) \in \mathbb{R}^N, \tag{3.36}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0, \tag{3.37}$$

where $b : \mathscr{X} \times \mathbb{R}^N \to \mathbb{R}$ is the bilinear form defined by

$$b(\boldsymbol{G},\vec{W}) := \sum_{n=1}^N W_n\langle \mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_J^n}.$$

Next we prove that (3.35)–(3.37) is actually equivalent to Problem 3.1.

LEMMA 3.3 Given $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\dots,N$, and $\boldsymbol{H}_0$ satisfying (3.15), let $\boldsymbol{H}$ be the solution of Problem 3.1. Then there exists $\vec{V} \in \mathrm{L}^2(0,T;\mathbb{R}^N)$ such that $(\boldsymbol{H},\vec{V})$ is the unique solution of (3.35)–(3.37).

Proof. Let $\boldsymbol{H}$ be the solution of Problem 3.1. For $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\dots,N$, because of Theorem 3.3 $\partial_t\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathscr{H}_{\mathscr{X}}) \subset \mathrm{L}^2(0,T;\mathscr{X}')$. Hence, we can define $h : [0,T] \to \mathscr{X}'$ by

$$\langle h(t), \boldsymbol{G}\rangle_{\mathscr{X}'\times\mathscr{X}} := -\langle \partial_t\boldsymbol{H}(t), \boldsymbol{G}\rangle_{\mathscr{X}'\times\mathscr{X}} - a(\boldsymbol{H}(t),\boldsymbol{G}), \quad \boldsymbol{G} \in \mathscr{X}$$

and we have that $h \in \mathrm{L}^2(0,T;\mathscr{X}')$. On the other hand it was proved in Bermúdez *et al.* (2005b) (see the proof of Theorem 7) that b satisfies the following *inf-sup* condition

$$\sup_{\boldsymbol{G}\in\mathscr{X}} \frac{b(\boldsymbol{G},\vec{W})}{\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geqslant \beta|\vec{W}| \qquad \forall \vec{W} \in \mathbb{R}^N.$$

Consequently, for each $t \in [0,T]$, there exists a unique $\vec{V}(t) \in \mathbb{R}^N$ such that $b(\boldsymbol{G},\vec{V}(t)) = \langle h(t),\boldsymbol{G}\rangle_{\mathscr{X}'\times\mathscr{X}}$ $\forall \boldsymbol{G} \in \mathscr{X}$ (see (Girault & Raviart, 1986, Lemma 4.1)). Furthermore, since $h \in \mathrm{L}^2(0,T;\mathscr{X}')$, $\vec{V} \in \mathrm{L}^2(0,T;\mathbb{R}^N)$, and (3.35) holds true. Moreover, (3.36) and (3.37) follows directly from (3.11) and (3.13). Hence, $(\boldsymbol{H},\vec{V})$ is a solution of (3.35)–(3.37). There only remains to prove that this problem has at most one solution. With this aim consider $(\check{\boldsymbol{H}},\check{\vec{V}})$ a solution of (3.35)–(3.37) with data $I_n = 0$, $n = 1,\ldots,N$, and $\boldsymbol{H}_0 = \mathbf{0}$. For each $t \in [0,T]$ we take $\boldsymbol{G} = \check{\boldsymbol{H}}(t)$ and $\vec{W} = \check{\vec{V}}(t)$ as test functions in (3.35) and (3.36), respectively. Subtracting the resulting equations we obtain

$$\langle \partial_t \check{\boldsymbol{H}}(t),\check{\boldsymbol{H}}(t)\rangle_{\mathscr{X}'\times\mathscr{X}} + a(\check{\boldsymbol{H}}(t),\check{\boldsymbol{H}}(t)) = 0.$$

Since $a(\check{\boldsymbol{H}}(t),\check{\boldsymbol{H}}(t)) \geqslant 0$, $\langle \partial_t \check{\boldsymbol{H}}(t),\check{\boldsymbol{H}}(t)\rangle_{\mathscr{X}'\times\mathscr{X}} = \frac{1}{2}\frac{d}{dt}\|\check{\boldsymbol{H}}(t)\|^2_{\mathscr{X}}$ and $\check{\boldsymbol{H}}(0) = \mathbf{0}$, it follows that $\check{\boldsymbol{H}}(t) = \mathbf{0}$. Finally we also have $\check{\vec{V}}(t) = \mathbf{0}$ because of the *inf-sup* condition for b and (3.35). □

Now we are in a position to prove that the solution of (3.35)–(3.37), and consequently of Problem 3.1, satisfies equations (2.1)–(2.8).

THEOREM 3.4 Let $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\ldots,N$, and $\boldsymbol{H}_0$ satisfying (3.15) and $\mu\boldsymbol{H}_0 \in \mathrm{H}_0(\mathrm{div}^0;\Omega)$ (i.e., $\mathrm{div}(\mu\boldsymbol{H}_0) = 0$ in Ω). Let $(\boldsymbol{H},\vec{V})$ be the solution of (3.35)–(3.37). Let $\boldsymbol{J}(t) := \mathbf{curl}\,\boldsymbol{H}(t)$ and $\boldsymbol{E}(t) := \left(\frac{1}{\sigma}\boldsymbol{J}(t)\right)\big|_{\Omega_{\mathrm{C}}}$. Then the following properties hold true:

$$\mathrm{div}(\mu\boldsymbol{H}(t)) = 0 \quad \text{in } \Omega, \tag{3.38}$$

$$\mu\partial_t\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{C}}, \tag{3.39}$$

$$\boldsymbol{J}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.40}$$

$$\langle \mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{J}}^n} = I_n(t), \quad n = 1,\ldots,N, \tag{3.41}$$

$$\mu\boldsymbol{H}(t)\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3.42}$$

$$\boldsymbol{E}(t)\times\boldsymbol{n} = -\nabla V_*(t)\times\boldsymbol{n} \quad \text{in } \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{C}})^3, \tag{3.43}$$

where, in the last equation, $V_*(t) \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ is such that $V_*(t)|_{\Gamma_{\mathrm{J}}^n} = V_n(t)$, $n = 1,\ldots,N$, and $V_*(t)|_{\Gamma_{\mathrm{E}}} = 0$. Hence, in particular,

$$\boldsymbol{E}(t)\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{E}} \qquad \text{and} \qquad \boldsymbol{E}(t)\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{J}}^n, \quad n = 1,\ldots,N.$$

Proof. The proof follows the lines of that of Theorem 7 from Bermúdez *et al.* (2005b). Given $v \in \mathscr{D}(\Omega)$ it follows that $\nabla v \in \mathscr{V}$. Then since according to Theorem 3.3 $\partial_t\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathscr{H}_{\mathscr{X}})$, (3.35) yields

$$\int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\nabla v = 0.$$

Hence, $\mathrm{div}(\mu\partial_t\boldsymbol{H}(t)) = 0$ and consequently $\partial_t(\mu\,\mathrm{div}\,\boldsymbol{H}(t)) = 0$ (see (Ženíšek, 1990, Theorems 111 & 113)). Therefore, (3.38) follows from the fact that $\mathrm{div}(\mu\boldsymbol{H}(0)) = 0$.

Now, let $\boldsymbol{G} \in \mathscr{D}(\Omega)^3$ be such that $\operatorname{supp} \boldsymbol{G} \subset \Omega_{\mathrm{C}}$. Then $\boldsymbol{G} \in \mathscr{V}$ too and (3.35) yields

$$\int_{\Omega_{\mathrm{C}}} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{H}(t) \cdot \operatorname{\mathbf{curl}} \boldsymbol{G} = 0.$$

Hence, $\boldsymbol{E}(t) := \left(\frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{H}(t)\right)\big|_{\Omega_{\mathrm{C}}}$ satisfies (3.39).

Equation (3.40) follows from the definition of $\boldsymbol{J}(t)$ and the fact that $\boldsymbol{H}(t) \in \mathscr{X}$, whereas equation (3.41) follows from (3.36).

To prove (3.42), notice that $\mu \partial_t \boldsymbol{H}(t) \in \mathrm{H}(\operatorname{div}, \Omega)$ because of (3.38). Then $\mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial \Omega)$. Moreover, given $v \in \mathrm{H}^1(\Omega)$, we have

$$\langle \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{n}, v \rangle_{\partial \Omega} = \int_{\Omega} \operatorname{div}(\mu \partial_t \boldsymbol{H}(t))\, v + \int_{\Omega} \mu \partial_t \boldsymbol{H}(t) \cdot \nabla v = 0,$$

the last equality because of (3.38) and (3.35), since $\nabla v \in \mathscr{V}$. Therefore $\partial_t(\mu \boldsymbol{H}(t)) = 0$ in $\mathrm{H}^{1/2}(\partial \Omega)$, which together with the fact that $\mu \boldsymbol{H}_0 \cdot \boldsymbol{n} = 0$ on $\partial \Omega$ leads to (3.42).

Finally, let $V_*(t) \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ be any function such that $V_*(t)|_{\Gamma_{\mathrm{J}}^n} = V_n(t)$, $n = 1, \ldots, N$, and $V_*(t)|_{\Gamma_{\mathrm{E}}} = 0$; functions of this type clearly exist since $\Gamma_{\mathrm{J}} \cap \Gamma_{\mathrm{E}} = \emptyset$. On the other hand, notice that $\boldsymbol{E}(t) \in \mathrm{H}(\operatorname{\mathbf{curl}}; \Omega_{\mathrm{C}})$ because of (3.39), and consequently $\boldsymbol{E}(t) \times \boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial \Omega_{\mathrm{C}})^3$. Hence, to prove (3.43), it is enough to show that $\langle \boldsymbol{E}(t) \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \Omega_{\mathrm{C}}} = -\langle \nabla V_*(t) \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \Omega_{\mathrm{C}}}$ $\forall \boldsymbol{v} \in \mathrm{H}^{1/2}(\partial \Omega_{\mathrm{C}})^3$ with $\operatorname{supp} \boldsymbol{v} \subset\subset \Gamma_{\mathrm{C}}$.

Given one such $\boldsymbol{v}$, notice that there exists $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$ vanishing in Ω_{D} and such that $\boldsymbol{G}|_{\partial \Omega_{\mathrm{C}}} = \boldsymbol{v}$. Then $\boldsymbol{G} \in \mathscr{X}$ and, from (3.35), (3.39), Green's formula, and the fact that $\boldsymbol{E}(t) = \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{H}(t)$ in Ω_{C}, we obtain

$$\begin{aligned}
0 &= \int_{\Omega} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \boldsymbol{E}(t) \cdot \operatorname{\mathbf{curl}} \boldsymbol{G} + b(\boldsymbol{G}, \vec{V}(t)) \\
&= \int_{\Omega_{\mathrm{C}}} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \operatorname{\mathbf{curl}} \boldsymbol{E}(t) \cdot \boldsymbol{G} + \left\langle \boldsymbol{E}(t) \times \boldsymbol{n}, \boldsymbol{G}|_{\partial \Omega_{\mathrm{C}}} \right\rangle_{\partial \Omega_{\mathrm{C}}} + b(\boldsymbol{G}, \vec{V}(t)) \\
&= \langle \boldsymbol{E}(t) \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \Omega_{\mathrm{C}}} + \langle \nabla V_*(t) \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \Omega_{\mathrm{C}}},
\end{aligned}$$

the last equality because of the fact that

$$b(\boldsymbol{G}, \vec{V}(t)) = \left\langle \operatorname{\mathbf{curl}} \boldsymbol{G} \cdot \boldsymbol{n}, V_*(t)|_{\partial \Omega_{\mathrm{C}}} \right\rangle_{\partial \Omega_{\mathrm{C}}} = \int_{\Omega_{\mathrm{C}}} \operatorname{\mathbf{curl}} \boldsymbol{G} \cdot \nabla V_*(t) = \langle \nabla V_*(t) \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \Omega_{\mathrm{C}}},$$

which in its turn follows from the definition of b and Green's formulas. Therefore, we conclude the proof. □

4. Space discretization

We assume that Ω, Ω_{C}, and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathscr{T}_h$ of Ω, such that each element $K \in \mathscr{T}_h$ is contained either in Ω_{C} or in Ω_{D} (h stands as usual for the corresponding mesh-size). We employ edge finite elements to approximate the magnetic field, more precisely, the lowest-order finite elements of the family introduced by Nédélec:

$$\mathscr{N}_h(\Omega) := \{\boldsymbol{G}_h \in \mathrm{H}(\operatorname{\mathbf{curl}}; \Omega) : \boldsymbol{G}_h|_K \in \mathscr{N}(K)\ \forall K \in \mathscr{T}_h\}.$$

The field is approximated in each tetrahedron K by a polynomial vector field in the space

$$\mathscr{N}(K) := \left\{ \boldsymbol{G}_h \in \mathbb{P}_1^3 : \boldsymbol{G}_h(\boldsymbol{x}) = \mathbf{a} \times \boldsymbol{x} + \mathbf{b}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \boldsymbol{x} \in K \right\}.$$

We introduce

$$\mathscr{X}_h := \{ \boldsymbol{G}_h \in \mathscr{N}_h(\Omega) : \mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \} \subset \mathscr{X},$$

$$\mathscr{V}_h := \left\{ \boldsymbol{G}_h \in \mathscr{X}_h(\Omega) : \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n} = 0, n = 1, \ldots, N \right\} \subset \mathscr{V}.$$

Then, the space-discretization of Problem 3.1 leads as follows:

Problem 4.1 Find $\boldsymbol{H}_h : [0,T] \to \mathscr{X}_h$ such that

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{H}_h(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \ldots, N, \tag{4.1}$$

$$\int_\Omega \mu \partial_t \boldsymbol{H}_h(t) \cdot \boldsymbol{G}_h + a(\boldsymbol{H}_h(t), \boldsymbol{G}_h) = 0 \quad \forall \boldsymbol{G}_h \in \mathscr{V}_h, \tag{4.2}$$

$$\boldsymbol{H}_h(0) = \boldsymbol{H}_{0h}, \tag{4.3}$$

where $\boldsymbol{H}_{0h} \in \mathscr{X}_h$ is an approximation of $\boldsymbol{H}_0$.

To prove that this problem is well posed, first we will use a function $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\mathscr{X}_h)$ such that $\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{n} = I_n(t)$, $n = 1, \ldots, N$. Let

$$\widehat{\boldsymbol{H}}_h(t, \boldsymbol{x}) := \sum_{e \in \mathscr{E}} c_e(t) \Psi_e(\boldsymbol{x}), \tag{4.4}$$

where $\{\Psi_e\}_{e \in \mathscr{E}}$ the nodal basis on $\mathscr{X}_h$ (with $\mathscr{E}$ being the set of edges associated to the mesh $\mathscr{T}_h$) and

$$c_e(t) := \frac{I_n(t)}{|e| N_n},$$

with N_n being the number of edges $e \in \mathscr{E}$ lying on γ_n. Hence,

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{n} = \int_{\gamma_n} \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{t}_n = \sum_{e \in \mathscr{E} : e \subseteq \gamma_n} c_e(t) \int_e \Psi_e \cdot \boldsymbol{t}_n = I_n(t),$$

where for the last equality we have used that $\int_e \Psi_e \cdot \boldsymbol{t}_n = |e|$. Since $I_n \in \mathrm{H}^1(0,T)$ $n = 1, \ldots, N$, we conclude that $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\mathscr{X}_h)$.

Now, if we write $\boldsymbol{H}_h = \widetilde{\boldsymbol{H}}_h + \widehat{\boldsymbol{H}}_h$, Problem 4.1 is equivalent to finding $\widetilde{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\mathscr{V}_h)$ such that

$$\int_\Omega \mu \partial_t \widetilde{\boldsymbol{H}}_h(t) \cdot \boldsymbol{G}_h + a(\widetilde{\boldsymbol{H}}_h(t), \boldsymbol{G}_h) = -\int_\Omega \mu \partial_t \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{G}_h - a(\widehat{\boldsymbol{H}}_h(t), \boldsymbol{G}_h) \quad \forall \boldsymbol{G}_h \in \mathscr{V}_h, \tag{4.5}$$

$$\widetilde{\boldsymbol{H}}_h(0) = \boldsymbol{H}_{0h} - \widehat{\boldsymbol{H}}_h(0). \tag{4.6}$$

Next, let be a basis of $\mathscr{V}_h$, $\{\boldsymbol{\Phi}_i\}_{i=1}^K$. We write

$$\widetilde{\boldsymbol{H}}_h(t, \boldsymbol{x}) = \sum_{i=1}^K \beta_i(t) \boldsymbol{\Phi}_i(\boldsymbol{x}).$$

Let $\boldsymbol{\beta}(t) := (\beta_i(t))_{1\leqslant i\leqslant K}$ and $\boldsymbol{F}_h(t) := (f_{h_i}(t))_{1\leqslant i\leqslant K}$, with

$$f_{h_i}(t) = -\int_\Omega \mu\,\partial_t \widehat{\boldsymbol{H}}_h(t)\cdot\boldsymbol{\Phi}_i - a(\widehat{\boldsymbol{H}}_h(t),\boldsymbol{\Phi}_i), \qquad 1\leqslant i\leqslant K,$$

and the matrices $\mathscr{K}\in\mathbb{R}^{K\times K}$ and $\mathscr{K}\in\mathbb{R}^{K\times K}$ given by

$$\mathscr{K}_{i,j} := a(\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j), \qquad \mathscr{M}_{i,j} := \int_\Omega \mu\boldsymbol{\Phi}_i\cdot\boldsymbol{\Phi}_j, \qquad 1\leqslant i,j\leqslant K.$$

Then, problem (4.5)–(4.6) leads as follows: Find $\beta(t)\in\mathbb{R}^K$ such that

$$\begin{aligned}\mathscr{M}\,\boldsymbol{\beta}'(t) &= -\mathscr{K}\,\boldsymbol{\beta}(t)+\boldsymbol{F}_h(t),\\ \boldsymbol{\beta}(0) &= \boldsymbol{\beta}_0\end{aligned}$$

since $\mathscr{M}$ is a positive definite symmetric matrix, this linear system of differential equations has a unique solution. Thus, we conclude that Problem 4.1 admits a unique solution, too.

Our next goal is to obtain error estimates for the semi-discrete scheme of Problem 4.1. For $r\in\left(\frac{1}{2},1\right]$, let

$$\mathscr{X}^r := \left\{\boldsymbol{G}\in\mathscr{X} \,:\, \boldsymbol{G}|_{\Omega_\mathrm{C}}\in \mathrm{H}^r(\mathbf{curl},\Omega_\mathrm{C}) \text{ and } \boldsymbol{G}|_{\Omega_\mathrm{D}}\in\mathrm{H}^r(\Omega_\mathrm{D})^3\right\}$$

where $\mathrm{H}^r(\mathbf{curl},\Omega_\mathrm{C}) := \left\{\boldsymbol{G}\in\mathrm{H}^r(\Omega_\mathrm{C})^3 : \mathbf{curl}\,\boldsymbol{G}\in\mathrm{H}^r(\Omega_\mathrm{C})^3\right\}$. If $\boldsymbol{G}\in\mathscr{X}^r$, then its Nédélec interpolant $\mathscr{I}_h\boldsymbol{G}\in\mathscr{N}_h(\Omega)$ is well defined (see (Bermúdez *et al.*, 2002, Lemma 5.1) and Amrouche *et al.* (1998)).

From now on, we assume that the solution of Problem 3.1 satisfies $\boldsymbol{H}\in\mathrm{H}^1(0,T;\mathscr{X}^r)$, which in particular implies that the initial condition $\boldsymbol{H}_0\in\mathscr{X}^r$. Therefore, the Nédélec interpolant $\mathscr{I}_h\boldsymbol{H}(t)$ is well defined and satisfies

$$\int_{\Gamma_\mathrm{J}^n}\mathbf{curl}\,\mathscr{I}_h\boldsymbol{H}(t)\cdot\boldsymbol{n} = \int_{\gamma_n}\mathscr{I}_h\boldsymbol{H}(t)\cdot\mathbf{t}_n = \int_{\gamma_n}\boldsymbol{H}(t)\cdot\mathbf{t}_n = \langle\mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_\mathrm{J}^n} = I_n(t).$$

Thus, we are allowed to use $\boldsymbol{H}_{0h} := \mathscr{I}_h\boldsymbol{H}_0$.

Let $\boldsymbol{\rho}_h(t) := \boldsymbol{H}(t)-\mathscr{I}_h\boldsymbol{H}(t)$ and $\boldsymbol{\delta}_h(t) := \mathscr{I}_h\boldsymbol{H}(t)-\boldsymbol{H}_h(t)$. Notice that from the last equality we have that $\boldsymbol{\delta}_h(t)\in\mathscr{V}_h$. A straightforward computation yields

$$\int_\Omega\mu\,\partial_t\boldsymbol{\delta}_h(t)\cdot\boldsymbol{G}_h + a(\boldsymbol{\delta}_h(t),\boldsymbol{G}_h) = -\int_\Omega\mu\,\partial_t\boldsymbol{\rho}_h(t)\cdot\boldsymbol{G}_h - a(\boldsymbol{\rho}_h(t),\boldsymbol{G}_h) \qquad \forall\boldsymbol{G}_h\in\mathscr{V}_h. \tag{4.7}$$

By taking $\boldsymbol{G}_h := \boldsymbol{\delta}_h(t)$, using (3.14) and the Cauchy-Schwartz inequality, we obtain

$$\partial_t\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \leqslant C\left\{\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}. \tag{4.8}$$

Using Gronwall's inequality and the fact that $\boldsymbol{\delta}_h(0)=\mathbf{0}$, we obtain

$$\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \leqslant C\left\{\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt + \int_0^T\|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt\right\}.$$

Integrating over t in (4.8), and using the last inequality, we obtain

$$\begin{aligned}&\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^t\|\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds\\ &\qquad\leqslant C\left\{\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt + \int_0^T\|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt\right\}.\end{aligned} \tag{4.9}$$

On the other hand, by taking $\boldsymbol{G}_h := \partial_t \boldsymbol{\delta}_h(t)$ in (4.7) and using the fact that

$$\frac{d}{dt} a(\boldsymbol{S}(t), \boldsymbol{T}(t)) = a(\partial_t \boldsymbol{S}(t), \boldsymbol{T}(t)) + a(\boldsymbol{S}(t), \partial_t \boldsymbol{T}(t)),$$

we obtain that

$$\begin{aligned}
&\underline{\mu} \, \|\partial_t \boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \frac{1}{2}\frac{d}{dt} a(\boldsymbol{\delta}_h(t), \boldsymbol{\delta}_h(t)) \\
&\qquad \leqslant -\int_\Omega \mu \partial_t \boldsymbol{\rho}_h(t) \cdot \partial_t \boldsymbol{\delta}_h(t) + a(\partial_t \boldsymbol{\rho}_h(t), \boldsymbol{\delta}_h(t)) - \frac{d}{dt} a(\boldsymbol{\rho}_h(t), \boldsymbol{\delta}_h(t)).
\end{aligned}$$

Integrating over t, since $\frac{1}{\overline{\sigma}} \| \mathbf{curl}\, \boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \leqslant a(\boldsymbol{\delta}_h(t), \boldsymbol{\delta}_h(t))$, Cauchy-Schwartz inequality yields

$$\begin{aligned}
&\int_0^t \|\partial_t \boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\, ds + \| \mathbf{curl}\, \boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \\
&\qquad \leqslant C \left\{ \int_0^t \| \mathbf{curl}\, \boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\, ds + \sup_{0 \leqslant t \leqslant T} \| \mathbf{curl}\, \boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt \right\}.
\end{aligned}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
&\int_0^t \|\partial_t \boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\, ds + \| \mathbf{curl}\, \boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \\
&\qquad \leqslant C \left\{ \sup_{0 \leqslant t \leqslant T} \| \mathbf{curl}\, \rho_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt \right\}.
\end{aligned} \tag{4.10}$$

Combining the equations (4.9) and (4.10), we obtain

$$\begin{aligned}
&\sup_{0 \leqslant t \leqslant T} \|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t \boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\, dt \\
&\qquad \leqslant C \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt + \sup_{0 \leqslant t \leqslant T} \| \mathbf{curl}\, \boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \right\}.
\end{aligned} \tag{4.11}$$

Now, we are in a position to prove the following error estimates.

THEOREM 4.2 Suppose that the solution of Problem 3.1 satisfies $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathscr{X}^r)$ with $r \in \left(\frac{1}{2}, 1\right]$. Then, there exists a constant $C > 0$ independent of h such that the solution of Problem 4.1 satisfies

$$\begin{aligned}
&\sup_{0 \leqslant t \leqslant T} \|\boldsymbol{H}(t) - \boldsymbol{H}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t (\boldsymbol{H}(t) - \boldsymbol{H}_h(t))\|^2_{\mathrm{L}^2(\Omega)^3}\, dt \\
&\qquad \leqslant C h^{2r} \left[\int_0^T \left\{ \|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right\} dt \right. \\
&\qquad\qquad \left. + \sup_{0 \leqslant t \leqslant T} \left\{ \|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right\} \right].
\end{aligned}$$

Proof. Notice that the regularity on $\boldsymbol{H}$ implies that $\partial_t(\mathscr{I}_h \boldsymbol{H}(t)) = \mathscr{I}_h(\partial_t \boldsymbol{H}(t))$ for a.e. $t \in [0,T]$ (see Ženíšek (1990)). Therefore (see Bermúdez *et al.* (2002)),

$$\begin{aligned}
\|\boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} &\leqslant C h^r \left\{ \|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right\}, \\
\|\partial_t \boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} &\leqslant C h^r \left\{ \|\partial_t \boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\partial_t \boldsymbol{H}(t)\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right\}.
\end{aligned}$$

Thus, the result follows by writing $\boldsymbol{H}(t)-\boldsymbol{H}_h(t)=\boldsymbol{\rho}_h(t)+\boldsymbol{\delta}_h(t)$ and using the estimates (4.11). $\square$

For the implementation of Problem 4.1, we resort to its formulation in terms of a magnetic potential. With this aim, we assume that the cut surfaces Σ_n are polyhedral and the meshes are compatible with them, in the sense that each Σ_n is a union of faces of tetrahedra $K \in \mathscr{T}_h$. Therefore, $\mathscr{T}_h^{\Omega_{\mathrm{D}}} := \{K \in \mathscr{T}_h :\ K \subset \Omega_{\mathrm{D}}\}$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{D}}$.

We introduce an approximation of the space Θ. Let

$$\mathscr{L}_h(\widetilde{\Omega}_{\mathrm{D}}) := \left\{\widetilde{\Psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) :\ \widetilde{\Psi}_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathscr{T}_h^{\Omega_{\mathrm{D}}}\right\}$$

and consider the finite-dimensional subspace of Θ given by

$$\Theta_h := \left\{\widetilde{\Psi}_h \in \mathscr{L}_h(\widetilde{\Omega}_{\mathrm{D}}) :\ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = \text{constant},\ n=1,\dots,L\right\}.$$

We introduce the following finite-dimensional subsets of $\mathscr{Y}$ and $\mathscr{Y}^0$, respectively,

$$\mathscr{Y}_h := \left\{(\boldsymbol{G}_h,\widetilde{\Psi}_h) \in \mathscr{N}_h(\Omega_{\mathrm{C}}) \times (\Theta_h/\mathbb{R}) :\ (\boldsymbol{G}_h|\widetilde{\mathbf{grad}}\,\widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl};\Omega)\right\},$$
$$\mathscr{Y}_h^0 := \left\{(\boldsymbol{G}_h,\widetilde{\Psi}_h) \in \mathscr{Y}_h :\ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0,\ n=1,\dots,N\right\}.$$

Proceeding as in Bermúdez *et al.* (2005b) it is immediate to show that Problem 4.1 is equivalent to finding $(\boldsymbol{H}_h,\widetilde{\Phi}_h) : [0,T] \to \mathscr{Y}_h$ such that

$$[\![\widetilde{\Phi}_h(t)]\!]_{\Sigma_n} = I_n(t), \quad n=1,\dots,N, \tag{4.12}$$

$$\int_{\Omega_{\mathrm{C}}} \mu\, \partial_t \boldsymbol{H}_h(t)\cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\, \mathbf{curl}\,\boldsymbol{H}_h(t)\cdot \mathbf{curl}\,\boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mu\, \partial_t \widetilde{\mathbf{grad}}\,\widetilde{\Phi}_h(t)\cdot \widetilde{\mathbf{grad}}\,\widetilde{\Psi}_h = 0 \quad \forall (\boldsymbol{G}_h,\widetilde{\Psi}_h) \in \mathscr{Y}_h^0, \tag{4.13}$$

$$(\boldsymbol{H}_h(0)|\widetilde{\Phi}_h(0)) = \mathscr{I}_h \boldsymbol{H}_0. \tag{4.14}$$

Let us remark that the first equation above is actually equivalent to (4.1) because $\boldsymbol{H}_h(t)$ and $\widetilde{\Phi}_h(t)$ are smooth enough for (3.7) to hold. The above problem can be seen as a discretization of the magnetic field - magnetic potential formulation (3.8)–(3.10).

5. Time discretization

We consider a uniform partition of $[0,T]$, $t_k := k\Delta t$, $k=0,\dots,M$, with time step $\Delta t := \frac{T}{M}$. A fully discrete approximation of Problem 3.1 is defined as follows:

Problem 5.1 Find $\boldsymbol{H}_h^m \in \mathscr{X}_h$, $m=1,\dots,M$, such that

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\boldsymbol{H}_h^m \cdot \boldsymbol{n} = I_n(t_m), \quad n=1,\dots,N,$$
$$\int_{\Omega} \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t}\cdot \boldsymbol{G}_h + a(\boldsymbol{H}_h^m,\boldsymbol{G}_h) = 0 \quad \forall \boldsymbol{G}_h \in \mathscr{V}_h$$
$$\boldsymbol{H}_h^0 = \mathscr{I}_h \boldsymbol{H}_0.$$

Hence, at each iteration step we have to find $\boldsymbol{H}_h^m \in \mathscr{X}_h$ such that

$$\int_{\Gamma_J^n} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \boldsymbol{n} = I_n(t_m), \quad n = 1, \ldots, N,$$

$$\int_\Omega \mu \boldsymbol{H}_h^m \cdot \boldsymbol{G}_h + \Delta t\, a(\boldsymbol{H}_h^m, \boldsymbol{G}_h) = \int_\Omega \mu \boldsymbol{H}_h^{m-1} \cdot \boldsymbol{G}_h \quad \forall \boldsymbol{G}_h \in \mathscr{V}_h.$$

The problem above has a unique solution. In fact, taking $t = t_m$ in (4.4) and writing $\boldsymbol{H}_h^m = \widetilde{\boldsymbol{H}}_h^m + \widehat{\boldsymbol{H}}_h^m$, we have to find $\widetilde{\boldsymbol{H}}_h^m \in \mathscr{V}_h$ such that

$$\int_\Omega \mu \widetilde{\boldsymbol{H}}_h^m \cdot \boldsymbol{G}_h + \Delta t\, a(\widetilde{\boldsymbol{H}}_h^m, \boldsymbol{G}_h) = \int_\Omega \mu \widetilde{\boldsymbol{H}}_h^{m-1} \cdot \boldsymbol{G}_h + \int_\Omega \mu \widehat{\boldsymbol{H}}_h^{m-1} \cdot \boldsymbol{G}_h - \int_\Omega \mu \widehat{\boldsymbol{H}}_h^m \cdot \boldsymbol{G}_h - \Delta t\, a(\widehat{\boldsymbol{H}}_h^m, \boldsymbol{G}_h)$$

for all $\boldsymbol{G}_h \in \mathscr{V}_h$, which is a linear system of equations with a positive definite symmetric matrix.

Our next goal is to obtain error estimates for this fully-discrete scheme. Let $\boldsymbol{\rho}^k := \boldsymbol{H}(t_k) - \mathscr{I}_h \boldsymbol{H}(t_k)$, $\boldsymbol{\delta}^k := \mathscr{I}_h \boldsymbol{H}(t_k) - \boldsymbol{H}_h^k$ and $\boldsymbol{\tau}^k := \frac{\boldsymbol{H}(t_k) - \boldsymbol{H}(t_{k-1})}{\Delta t} - \partial_t \boldsymbol{H}(t_k)$. A straightforward computation allows us to show that

$$\int_\Omega \mu \frac{\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + a(\boldsymbol{\delta}^k, \boldsymbol{G}_h) = \int_\Omega \mu \boldsymbol{\tau}^k \cdot \boldsymbol{G}_h - \int_\Omega \mu \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - a(\boldsymbol{\rho}^k, \boldsymbol{G}_h) \quad \boldsymbol{G}_h \in \mathscr{V}_h. \tag{5.1}$$

Choosing $\boldsymbol{G}_h := \boldsymbol{\delta}^k$ and using that

$$\int_\Omega \frac{\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1}}{\Delta t} \cdot \boldsymbol{\delta}^k \geqslant \frac{1}{2\Delta t} \left\{ \|\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} - \|\boldsymbol{\delta}^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} \right\}$$

and $a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) \geqslant \frac{1}{\sigma} \|\mathbf{curl}\, \boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3}$, together with the Cauchy-Schwartz inequality, yield

$$\begin{aligned} &\|\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} - \|\boldsymbol{\delta}^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \|\mathbf{curl}\, \boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} \\ &\qquad \leqslant \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} + C\Delta t \left\{ \left\| \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{\rho}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}. \end{aligned} \tag{5.2}$$

In particular

$$\begin{aligned} &\|\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} - \|\boldsymbol{\delta}^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} \\ &\qquad \leqslant \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} + C\Delta t \left\{ \left\| \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{\rho}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}. \end{aligned}$$

Using the discrete Gronwall's inequality in the last inequality, the fact that $\boldsymbol{\delta}^0 = \mathbf{0}$ and summing over k in (5.2), we obtain

$$\begin{aligned} &\|\boldsymbol{\delta}^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^m \|\mathbf{curl}\, \boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3} \\ &\qquad \leqslant C\, \Delta t \sum_{k=1}^m \left\{ \left\| \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{\rho}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}. \end{aligned} \tag{5.3}$$

On the other hand, by taking $\boldsymbol{G}_h := \frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}$ in (5.1) and using that

$$a\left(\boldsymbol{\delta}^k, \frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}\right) \geqslant \frac{1}{2\Delta t}\left\{a(\boldsymbol{\delta}^k,\boldsymbol{\delta}^k)-a(\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\right\}$$

and

$$a\left(\boldsymbol{\rho}^k, \frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}\right) = \frac{1}{\Delta t}\left\{a(\boldsymbol{\rho}^k,\boldsymbol{\delta}^k)-a(\boldsymbol{\rho}^{k-1},\boldsymbol{\delta}^{k-1})\right\} - a\left(\frac{\boldsymbol{\rho}^k-\boldsymbol{\rho}^{k-1}}{\Delta t},\boldsymbol{\delta}^{k-1}\right),$$

together with the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\Delta t\left\|\frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}+a(\boldsymbol{\delta}^k,\boldsymbol{\delta}^k)-a(\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\\ &\leqslant C\Delta t\left\{\|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3}+\left\|\frac{\boldsymbol{\rho}^k-\boldsymbol{\rho}^{k-1}}{\Delta t}\right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}+\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3}\right\}-2\left\{a(\boldsymbol{\rho}^k,\boldsymbol{\delta}^k)+a(\boldsymbol{\rho}^{k-1},\boldsymbol{\delta}^{k-1})\right\}.\end{aligned}$$

Summing over k leads to

$$\begin{aligned}&\Delta t\sum_{k=1}^m\left\|\frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}+\|\mathbf{curl}\,\boldsymbol{\delta}^m\|^2_{\mathrm{L}^2(\Omega)^3}\\ &\qquad\leqslant C\,\Delta t\sum_{k=1}^m\left\{\left\|\frac{\boldsymbol{\rho}^k-\boldsymbol{\rho}^{k-1}}{\Delta t}\right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}+\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3}+\|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3}\right\}.\end{aligned}$$

Adding this inequality to (5.3) and using again (5.3) to estimate $\Delta t\sum_{k=1}^m\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3}$, we obtain

$$\begin{aligned}&\|\boldsymbol{\delta}^m\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}+\Delta t\sum_{k=1}^m\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{\mathrm{L}^2(\Omega)^3}+\Delta t\sum_{k=1}^m\left\|\frac{\boldsymbol{\delta}^k-\boldsymbol{\delta}^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}\\ &\qquad\leqslant C\,\Delta t\sum_{k=1}^m\left\{\left\|\frac{\boldsymbol{\rho}^k-\boldsymbol{\rho}^{k-1}}{\Delta t}\right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}+\|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3}+\|\mathbf{curl}\,\rho^k\|^2_{\mathrm{L}^2(\Omega)^3}\right\}.\end{aligned}$$

Therefore, we are in position to write the main result of this paper which involves error estimates for the physical quantities of interest, the magnetic field $\boldsymbol{H}$ and the current density $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$.

THEOREM 5.2 Let $\boldsymbol{H}$ be the solution of Problem 3.1 and $\boldsymbol{H}_h^k$, $k = 1,\dots,M$, that of Problem 5.1. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathscr{X}^r)\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$, with $r \in \left(\frac{1}{2},1\right]$, then there exists a constant $C > 0$, independent of h and Δt, such that

$$\begin{aligned}&\max_{1\leqslant k\leqslant M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}+\Delta t\sum_{k=1}^M\left\|\partial_t\boldsymbol{H}(t_k)-\frac{\boldsymbol{H}_h^k-\boldsymbol{H}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}\\ &\quad\leqslant C\left\{(\Delta t)^2\|\boldsymbol{H}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)}+h^{2r}\sup_{0\leqslant t\leqslant T}\left[\|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})}+\|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right]\right.\\ &\qquad\left.+h^{2r}\int_0^T\left[\|\partial_t\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})}+\|\partial_t\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right]dt\right\}.\end{aligned}$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{M} \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} = \sum_{k=1}^{M} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_k - s) \partial_{tt} \boldsymbol{H}(s)\, ds \right\|^2_{\mathrm{L}^2(\Omega)^3} \leqslant \Delta t \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}\, dt.$$

Moreover,

$$\sum_{k=1}^{M} \left\| \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \leqslant \frac{1}{\Delta t} \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt.$$

Let $\boldsymbol{e}^k := \boldsymbol{H}(t_k) - \boldsymbol{H}_h^k = \boldsymbol{\rho}^k + \boldsymbol{\delta}^k$. Using the estimates for $\boldsymbol{\delta}^k$ and the fact that

$$\partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} = \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} + \frac{\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1}}{\Delta t} - \boldsymbol{\tau}^k$$

we obtain

$$\begin{aligned}
&\|\boldsymbol{e}^m\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \left\| \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \\
&\quad \leqslant C \left\{ \|\boldsymbol{\rho}^m\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\rho}^k - \boldsymbol{\rho}^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \|\mathbf{curl}\, \rho^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\} \\
&\quad \leqslant C \left\{ (\Delta t)^2 \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}\, dt + \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt + \max_{0\leqslant m\leqslant M} \|\boldsymbol{\rho}_h(t_m)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \right\}.
\end{aligned}$$

Thus, since $\boldsymbol{\rho}_h := \boldsymbol{H} - \mathscr{I}_h \boldsymbol{H}$, the result follows from the assumed regularity of $\boldsymbol{H}$ and standard error estimates for the Nédélec interpolant. □

For the actual computation of Problem 5.1 we proceed as in the semidiscrete problem and rewrite it in terms of a magnetic potential: Find $\left(\boldsymbol{H}_h^m, \widetilde{\Phi}_h^m\right) \in \mathscr{Y}_h$, $m = 1, \ldots, M$, such that

$$\begin{aligned}
&[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = I_n(t_m), \quad n = 1, \ldots, N, \\
&\int_{\Omega_{\mathrm{C}}} \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mu \frac{\widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m - \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^{m-1}}{\Delta t} \cdot \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h = 0 \\
&\hspace{25em} \forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \mathscr{Y}_h^0, \\
&\left(\boldsymbol{H}_h^0 | \widetilde{\Phi}_h^0\right) = \mathscr{I}_h \boldsymbol{H}_0.
\end{aligned}$$

Notice that the problem above can be seen as a backward Euler time discretization of (4.12)–(4.14). This is the discrete problem implemented in the computer, because a scalar variable $\left(\Phi_h^m\right)$ is used instead of a vector field $(\boldsymbol{H}_h^m)$ in the dielectric domain.

Notice that for all time m the following constraints must be imposed:

- $\left(\boldsymbol{H}_h^m | \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m\right) \in \mathrm{H}(\mathbf{curl};\Omega)$, which arises in the definition of $\mathscr{Y}_h$;
- $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} =$ constant, $n = 1, \ldots, L$, which arise in the definition of Θ_h.

To deal with these conditions, we employ the following procedure (see Bermúdez *et al.* (2002) for more details).

For the first one we use that, for $\left(\boldsymbol{H}_h^m | \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m\right) \in \mathrm{H}(\mathbf{curl};\Omega)$,

$$\int_\ell \boldsymbol{H}_h^m \cdot \boldsymbol{t}_\ell = \int_\ell \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m \cdot \boldsymbol{t}_\ell = \widetilde{\Phi}_h^m \left(P_\ell^+\right) - \widetilde{\Phi}_h^m \left(P_\ell^-\right) \quad \forall \ell \text{ edge of } \mathscr{T}_h : \ \ell \subset \Gamma_{\mathrm{I}},$$

where P_ℓ^- and P_ℓ^+ are the initial and end points of ℓ, respectively, and $\boldsymbol{t}_\ell$ the unit tangent vector pointing from P_ℓ^- to P_ℓ^+. Then the degrees of freedom of $\boldsymbol{H}_h^m$ associated with the edges $\ell \subset \Gamma_{\mathrm{I}}$ are eliminated by static condensation in terms of those of $\widetilde{\Phi}_h^m$ corresponding to the vertices of the mesh on Γ_{I}.

Regarding the second constraint, for each cut surface Σ_n, we in principle distinguish the degrees of freedom of $\widetilde{\Phi}_h^m$ on Σ_n^+ from those on Σ_n^-. Then the latter are eliminated by using

$$\widetilde{\Phi}_h^m|_{\Sigma_n^-} = \widetilde{\Phi}_h^m|_{\Sigma_n^+} + [\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n},$$

with $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0$ for the test functions and $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = I_n(t_m)$ for the trial functions where $I_n(t_m)$, $n = 1, \cdots, N$, are the input current intensities and $I_n(t_m)$, $n = N+1, \cdots, L$, will be additional degrees of freedom of the problem. We notice that these additional unknowns, $I_n(t_m)$ $n = N+1, \cdots, L$, are the intensities crossing through the conductors called workpieces, which are only due to induced currents because they are not connected with any power source.

6. Numerical experiments

In this section we report some numerical result obtained with a MATLAB code implementing the numerical method described above. First, we present a test with a known analytical solution to validate the computer code and to test the error estimates proved above. Finally, we will apply the method to a problem arising from an electromagnetic forming process.

6.1 *A test with known analytical solution*

The problem solved in this section has been already solved in Bermúdez *et al.* (2002) in harmonic regime. This is the reason why we only give here a brief description and refer the reader to the quoted paper for further details. Figure 2 shows a sketch of the domain where the conducting part Ω_{C} and the whole domain Ω are coaxial cylinders. An alternating current of intensity $I(t) = I_0 cos(\omega t)$ enters the conductor through Γ_{J}^1 and goes through Ω_{C} in the axial direction; I_0 denotes the amplitude of the intensity and ω the angular frequency. It is easy to obtain an analytical solution of the eddy current problem in Ω by writing all the fields in the form $\boldsymbol{F}(t,\boldsymbol{x}) = \mathrm{R}e(e^{\mathrm{i}\omega t}\mathscr{F}(\boldsymbol{x}))$. In particular, the solution leads to a magnetic field which has only an azimuthal component and is defined by a scalar multivalued potential in the dielectric domain. Notice that in this case we only need one cutting surface in the dielectric domain. To determine the order of convergence, the numerical method has been used on several successively refined meshes and the time-step has been conveniently reduced to analyze the convergence with respect to both, the mesh-size and the time-step. We have compared the obtained numerical solutions with the analytical one.

In order to analyze the linear convergence respect to the mesh-size and the time-step, we have computed the relative errors of the different fields corresponding to $\left(\frac{h}{n}, \frac{\Delta t}{n}\right)$, $n = 1, \ldots, 6$. Figure 3 shows log-log plots of the relative error for the magnetic field $\boldsymbol{H}$ in $\mathscr{C}^0([0,T];\mathrm{H}(\mathbf{curl};\Omega))$-norm (left)

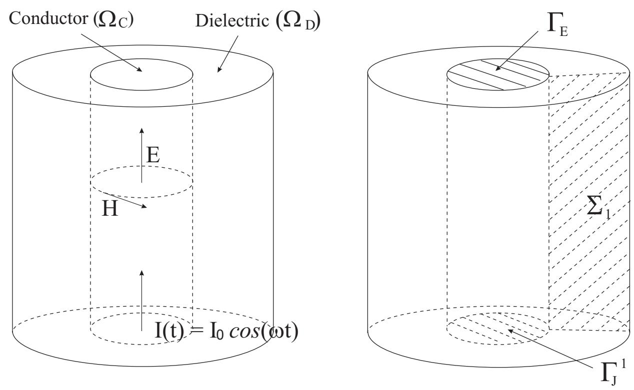

FIG. 2. Sketch of the domain in the analytical example.

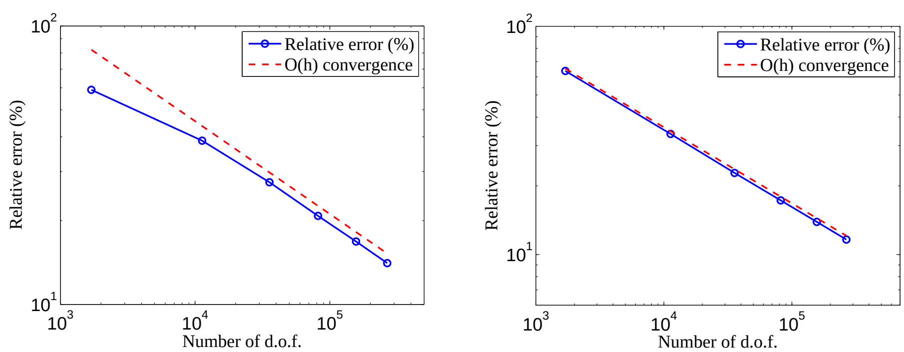

FIG. 3. $\dfrac{\max\limits_{1\leqslant k\leqslant M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)}}{\max\limits_{1\leqslant k\leqslant M}\|\boldsymbol{H}(t_k)\|_{\mathrm{H}(\mathbf{curl};\Omega)}}$ (left) and $\dfrac{\sqrt{\Delta t}\sum_{k=1}^{M}\left\|\partial_t\boldsymbol{H}(t_k)-\dfrac{\boldsymbol{H}_h^k-\boldsymbol{H}_h^{k-1}}{\Delta t}\right\|_{\mathrm{L}^2(\Omega)^3}}{\sqrt{\Delta t}\sum_{k=1}^{M}\|\partial_t\boldsymbol{H}(t_k)\|_{\mathrm{L}^2(\Omega)^3}}$ (right) versus number of d.o.f. (log-log scale).

and for its derivative $\partial_t\boldsymbol{H}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)$-norm (right) versus the number of degrees of freedom (d.o.f.).

The slopes of the curves clearly show an order of convergence $\mathscr{O}(h+\Delta t)$ for all the quantities, which agrees with the theoretical results, since the solution is smooth and hence the hypotheses of Theorem 5.2 are fulfilled for $r=1$.

6.2 *A problem arising from an electromagnetic forming process*

Electromagnetic Forming is a metal working process that relies on the use of electromagnetic forces to deform metallic workpieces at high speeds. A transient electric current is induced in a coil which produces a magnetic field that penetrates a nearby conductive workpiece where an eddy current is generated. The magnetic field, together with the eddy current, induce Lorentz forces that drive the deformation of the workpiece (see, for instance El-Azab *et al.* (2003)). In this section, we have simulated the electromagnetic behavior of a 3D workpiece under the action of a coil. It corresponds to a similar configuration to the one presented in Ulacia *et al.* (2009), but with simpler geometry and workpiece data. The coil and workpiece are presented in Figure 4, which also shows a typical mesh of the conducting domain. Domain Ω has been chosen as a box surrounding the conductor. Notice that we only need to build a cutting surface in the dielectric domain. The current intensity which enters the coil is shown in Figure 5; a typical curve in electromagnetic forming. Concerning the physical properties, the workpiece is a magnesium alloy and the coil is made with copper (see Table 2 of Ulacia *et al.* (2009)). Figure 6 shows

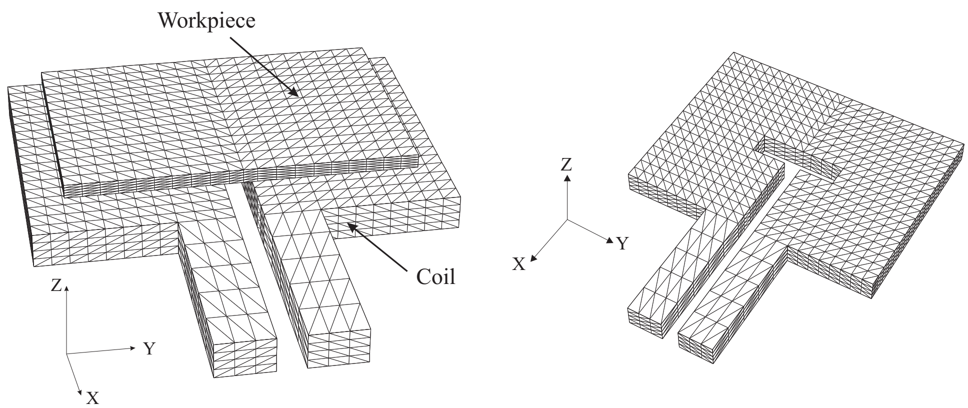

FIG. 4. Mesh of the conducting domain (left). Detail of the coil mesh (right).

the computed resultant of the Lorentz force versus time in the workpiece; the peak value corresponds to the time in which the input current intensity reaches its maximum (0.00018 s). All the other reported results correspond to this time. Figure 7 shows the modulus of the current density in the conducting domain. Figure 8 shows the current density vector field. Finally, Figure 9 shows the Lorentz force in the workpiece.

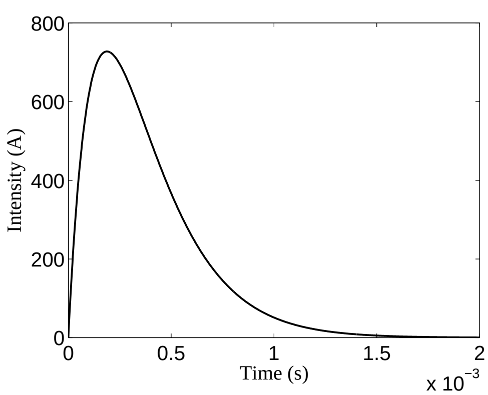

FIG. 5. Current intensity (A) vs. time (s).

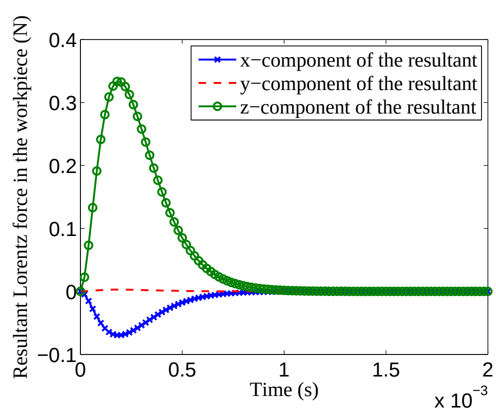

FIG. 6. Resultant of the Lorentz force (N) in the workpiece vs. time (s).

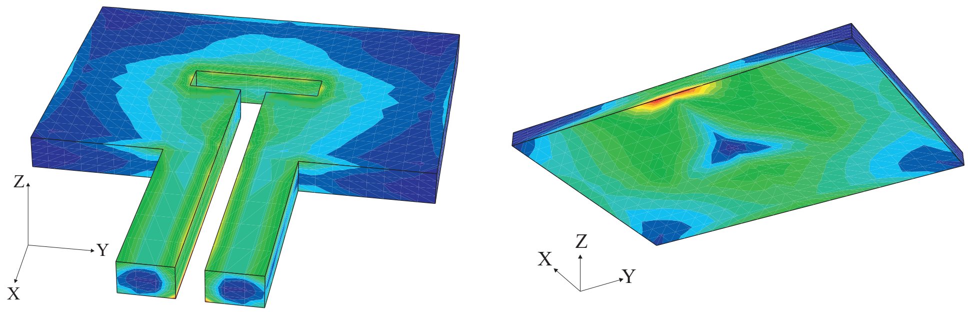

FIG. 7. Modulus of the current density in coil and workpiece at time 0.00018 s.

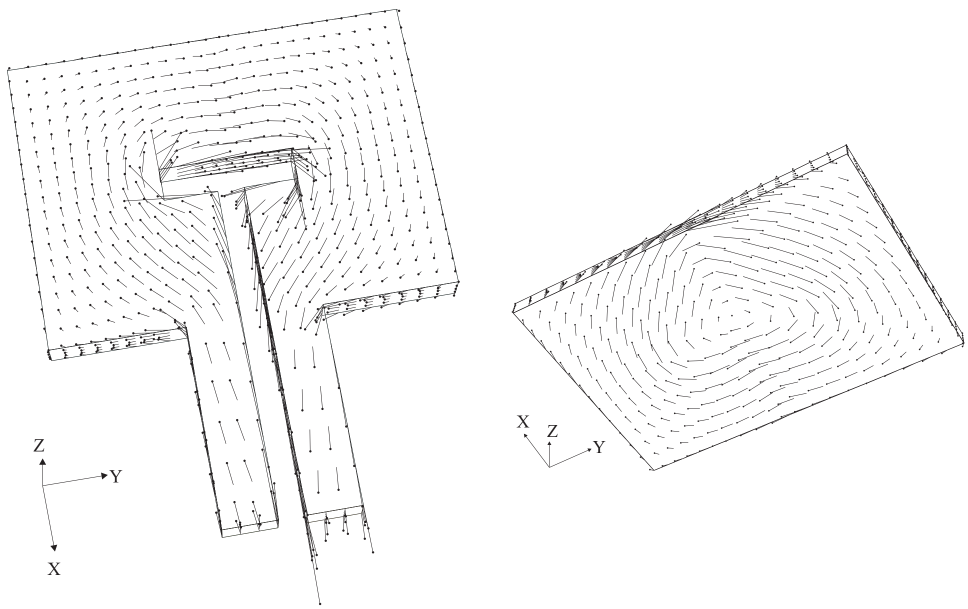

FIG. 8. Distribution of the current density (vector field) in coil and workpiece (underside) at time 0.00018 s.

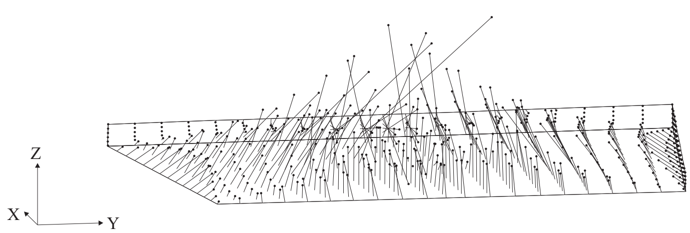

FIG. 9. Lorentz force in the workpiece at time 0.00018 s.

Since this approach is also able to deal with non simply connected conductors we have solved another example in which the workpiece is as shown in Figure 10. In this case, two cut surfaces are needed, one contained in the interior of Ω_D and the other touching $\partial\Omega_\mathrm{D}$. As we have explained above, in this example the induced current intensity in the workpiece is an additional unknown which must be computed at each time step. Figure 11 shows the modulus of the induced current density in the workpiece at $t = 0.00018$ s. Figure 12 shows the additional unknown (induced current intensity in the workpiece) versus time.

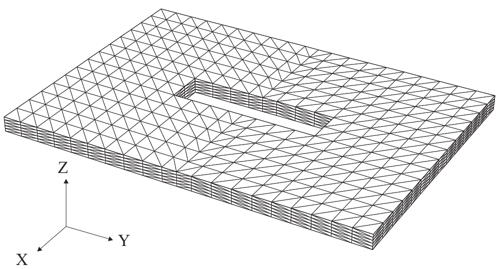

FIG. 10. Mesh of the workpiece.

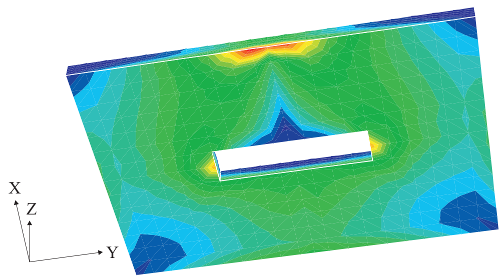

FIG. 11. Modulus of the current density in workpiece at time 0.00018 s.

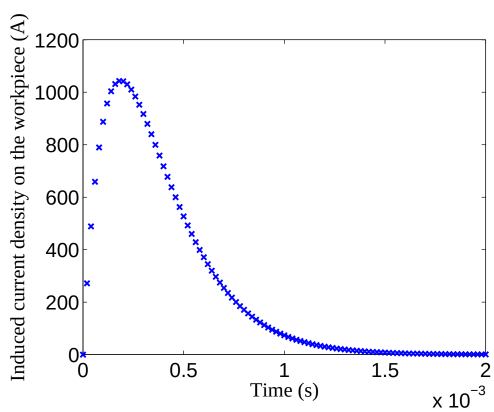

FIG. 12. Induced current intensity (A) vs. time (s).

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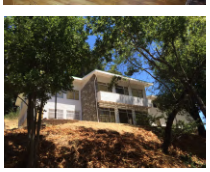

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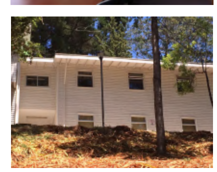

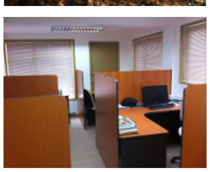

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