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# A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem

Gabriel N. Gatica* Ricardo Oyarzúa† Francisco-Javier Sayas‡

**Abstract**

In this paper we develop an a posteriori error analysis of a new fully mixed finite element method for the coupling of fluid flow with porous media flow in 2D. Flows are governed by the Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We consider dual-mixed formulations in both media, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, and the traces of the porous media pressure and the fluid velocity on the interface, as the resulting unknowns. The set of feasible finite element subspaces includes Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the traces. We derive a reliable and efficient residual-based a posteriori error estimator for the coupled problem. The proof of reliability makes use of the global inf-sup condition, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on triangle-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator. Finally, some numerical results confirming the theoretical properties of this estimator, and illustrating the capability of the corresponding adaptive algorithm to localize the singularities of the solution, are reported.

**Key words**: a posteriori error analysis, efficiency, reliability, Stokes, Darcy, fully-mixed

**Mathematics Subject Classifications (2000)**: 65N15, 65N30, 74F10, 74S05

## 1 Introduction

The derivation of new finite element methods for the Stokes-Darcy coupled problem, in which the respective interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, has become a very active research area lately (see, e.g. [5], [10], [13], [14], [20], [22], [23], [24], [29], [30], [35], [38], [40], [41], [42], [43], [47] and the references therein). The above list includes porous media with cracks, nonlinear problems, and the incorporation of the Brinkman equation in the model (see [10], [23], and [47]). In addition,

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most of the formulations employed are based on appropriate combinations of stable elements for the free fluid flow and for the porous medium flow, and the first theoretical results in this direction go back to [22] and [35]. Indeed, an iterative subdomain method employing the primal variational formulation and standard finite element subspaces in both domains is proposed in [22], whereas the primal method in the fluid and the dual-mixed method in the porous medium are applied in [35]. In this way, the approach from [35] yields the velocity and the pressure in both domains, together with the trace of the porous medium pressure on the interface, as the main unknowns of the coupled problem. This trace unknown is motivated by the fact that one of the transmission conditions becomes essential. Then, new mixed finite element discretizations of the variational formulation from [35] have been introduced and analyzed in [29] and [30]. The stability of a specific Galerkin method is the main result in [29], and the resulting mixed finite element method is the first one that is conforming for the primal/dual-mixed formulation proposed in [35]. The results from [29] are improved in [30] where it is shown that the use of any pair of stable Stokes and Darcy elements implies the stability of the corresponding Stokes-Darcy Galerkin scheme. The analysis in [30] hinges on the fact that the operator defining the continuous variational formulation is given by a compact perturbation of an invertible mapping. Further techniques utilized in the literature include mortar finite element methods, discontinuous Galerkin (DG) schemes, and stabilized formulations (see, e.g. [5], [13], [14], [20], [21], [24], [38], [40], [41], [42], [43]). In particular, the main motivation for employing stabilized formulations either in both domains or in one of them, is the possibility of approximating the Stokes and Darcy flows with the same finite element subpaces. Certainly, different finite element subspaces in each flow region may lead to different approximation properties for each subproblem. On the contrary, using the same spaces guarantees the same accurateness along the entire domain and leads to simpler and more efficient computational codes.

Now, in the recent paper [31] we have developed a new variational approach for the 2D Stokes-Darcy coupled problem, which allows, on one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. More precisely, in [31] we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, as the main unknowns. The pressure and the gradient of the velocity in the fluid can then be computed as a very simple postprocess of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous media pressure and the fluid velocity, which are also variables of importance from a physical point of view, as the corresponding Lagrange multipliers. Then, we apply the well known Fredholm and Babuška-Brezzi theories to prove the unique solvability of the resulting continuous formulation and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. Among the several different ways in which the equations and unknowns can be ordered, we choose the one yielding a doubly mixed structure for which the inf-sup conditions of the off-diagonal bilinear forms follow straightforwardly. In this way, the arguments of the continuous analysis can be easily adapted to the discrete case. In particular, a feasible choice of subspaces is given by Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the Lagrange multipliers.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\eta}$ that are expressed in terms of local indicators $\eta_T$ defined on each element $T$ of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\eta}$ is said to be efficient (resp. reliable) if there exists $C_{\mathtt{eff}} > 0$ (resp. $C_{\mathtt{rel}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\, \boldsymbol{\eta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\mathtt{rel}}\, \boldsymbol{\eta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. In particular, the a posteriori error analysis of variational formulations with saddle-point structure has already been widely investigated by many authors (see, e.g. [2], [3], [4], [11], [15], [17], [27], [33], [36], [37], [39], [44], and the references therein). These contributions refer mainly to reliable and efficient a posteriori error estimators based on local and global residuals, local problems, postprocessing, and functional-type error estimates. In addition, the applications include Stokes and Oseen equations, Poisson problem, linear elasticity, and general elliptic partial differential equations of second order. However, up to our knowledge, the first a posteriori error analysis for the Stokes-Darcy coupled problem has been provided recently in [8], where a reliable and efficient residual-based a posteriori error estimator for the variational formulation analyzed in [29] is derived. Partially following known approaches, the proof of reliability makes use of suitable auxiliary problems, diverse continuous inf-sup conditions satisfied by the bilinear forms involved, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. Similarly, Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions, are the main tools for proving the efficiency of the estimator.

Motivated by the discussion in the above paragraphs, our purpose now is to additionally contribute in the direction of [8] and provide the a posteriori error analysis of the fully-mixed variational approach introduced in [31]. According to this, the rest of this work is organized as follows. In Section 2 we recall from [31] the Stokes-Darcy coupled problem and its continuous and discrete fully-mixed variational formulations. The kernel of the present work is given by Section 3, where we develop the a posteriori error analysis. In Section 3.1 we employ the global continuous inf-sup condition, Helmholtz decompositions in both domains, and the local approximation properties of the Clément and Raviart-Thomas operators, to derive a reliable residual-based a posteriori error estimator. An interesting feature of our proof of reliability is the previous transformation of the global continuous inf-sup condition into an equivalent estimate involving global inf-sup conditions for each one of the components of the product space to which the vector of unknowns belongs. Then, in Section 3.2 we apply again Helmholtz decompositions, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions to prove the efficiency of the estimator. This proof benefits partially from the fact that some components of the a posteriori error estimator coincide with those obtained in [8] and the related work [15]. Finally, numerical results confirming the reliability and efficiency of the a posteriori error estimator and showing the good performance of the associated adaptive algorithm, are presented in Section 4.

We end this section with some notations to be used below. In particular, in what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, $\Gamma$ is a closed

Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^2.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see, e.g. [12] or [32]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. On the other hand, the symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\,\lambda \quad \forall\, \xi,\, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use $C$ and $c$, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

# 2 The Stokes-Darcy coupled problem

In this section we follow very closely the presentation from [31] to introduce the model problem and the corresponding continuous and discrete mixed variational formulations.

## 2.1 The model problem

The Stokes-Darcy coupled problem consists of an incompressible viscous fluid occupying a region $\Omega_{\mathrm{S}}$, which flows back and forth across the common interface into a porous medium living in another region $\Omega_{\mathrm{D}}$ and saturated with the same fluid. Physically, we consider a simplified 2D model where $\Omega_{\mathrm{D}}$ is surrounded by a bounded region $\Omega_{\mathrm{S}}$ (see Figure 2.1 below). Their common interface is supposed to be a Lipschitz curve $\Sigma$ and we assume that $\partial\Omega_{\mathrm{D}} = \Sigma$. The remaining part of the boundary of $\Omega_{\mathrm{S}}$ is also assumed to be a Lipschitz curve $\Gamma_{\mathrm{S}}$. For practical purposes, we can assume that both $\Gamma_{\mathrm{S}}$ and $\Sigma$ are polygons. The unit normal vector field on the boundaries $\mathbf{n}$ is chosen pointing outwards from $\Omega_{\mathrm{S}}$ (and therefore inwards to $\Omega_{\mathrm{D}}$ when seen on $\Sigma$). On $\Sigma$ we also consider a unit tangent vector field $\mathbf{t}$ in any fixed orientation of this closed curve.

The governing equations in $\Omega_{\mathrm{S}}$ are those of the Stokes problem, which are written in the following non-standard velocity-pressure-pseudostress formulation:

$$\begin{gathered} \boldsymbol{\sigma}_{\mathrm{S}} = -p_{\mathrm{S}}\,\mathrm{I} + \nu\,\nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}, \\ \mathrm{div}\,\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}, \end{gathered} \tag{2.1}$$

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_{\mathrm{S}}$ is the fluid velocity, $p_{\mathrm{S}}$ is the pressure, $\boldsymbol{\sigma}_{\mathrm{S}}$ is the pseudostress tensor, I is the $2\times 2$ identity matrix, and $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ are known source terms.

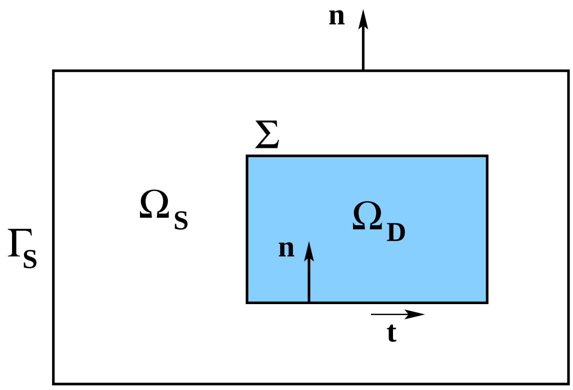


Figure 2.1: Geometry of the problem

Here, div is the usual divergence operator acting on vector fields, and **div** denotes the action of div along the rows of each tensor. On the other hand, the flow equations in $\Omega_D$ are those of the linearized Darcy model:

$$\mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in} \quad \Omega_D, \qquad \operatorname{div} \mathbf{u}_D = f_D \quad \text{in} \quad \Omega_D, \tag{2.2}$$

where the unknowns are the pressure $p_D$ and the flow $\mathbf{u}_D$, and the source term, given by $f_D \in L^2(\Omega_D)$, satisfies $\int_{\Omega_D} f_D = 0$. The matrix valued function $\mathbf{K}$, describing permeability of $\Omega_D$ divided by the viscosity $\nu$, is symmetric, has $L^\infty(\Omega_D)$ components and is uniformly elliptic. Finally, the transmission conditions on $\Sigma$ are given by

$$\begin{aligned} \mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad &\text{on} \quad \Sigma, \\ \boldsymbol{\sigma}_S\,\mathbf{n} + \nu\,\kappa^{-1}(\mathbf{u}_S \cdot \mathbf{t})\,\mathbf{t} &= -p_D\,\mathbf{n} \quad &\text{on} \quad \Sigma, \end{aligned} \tag{2.3}$$

where $\kappa := \dfrac{\sqrt{(\nu\,\mathbf{K}\,\mathbf{t})\cdot\mathbf{t}}}{\alpha}$ is the friction coefficient, and $\alpha$ is a positive parameter to be determined experimentally. The first equation in (2.3) corresponds to mass conservation on $\Sigma$, whereas the normal and tangential components of the second one constitute the balance of normal forces and the Beavers-Joseph-Saffman law, respectively. Throughout the rest of the paper we assume, without loss of generality, that $\kappa$ is a positive constant.

We complete the description of our model problem by observing that the equations in the Stokes domain (cf. (2.1)) can be rewritten equivalently as

$$\begin{gathered} \nu^{-1}\boldsymbol{\sigma}_S^d = \nabla\mathbf{u}_S \quad \text{in} \quad \Omega_S, \qquad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S, \\ p_S = -\tfrac{1}{2}\,\mathbf{tr}\,\boldsymbol{\sigma}_S \quad \text{in} \quad \Omega_S, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S, \end{gathered} \tag{2.4}$$

where **tr** stands for the usual trace of tensors, that is $\mathbf{tr}\,\boldsymbol{\tau} := \tau_{11} + \tau_{22}$, and

$$\boldsymbol{\tau}^d := \boldsymbol{\tau} - \tfrac{1}{2}(\mathbf{tr}\,\boldsymbol{\tau})\,\mathrm{I}$$

is the deviatoric part of the tensor $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$.

We end this section by remarking that, though the geometry described by Figure 2.1 was choosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not yield further complications for the a posteriori error analysis of the problem. We already discussed this issue in [31, Section 2.1], in connection with the respective a priori error analysis, and further details can be found in [24].

## 2.2 The fully-mixed variational formulation

We first define the global unknows $\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda)$ and $\underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}})$, where $\boldsymbol{\varphi}$ and $\lambda$ are the traces $\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}}|_{\Sigma}$ and $\lambda := p_{\mathrm{D}}|_{\Sigma}$. Then we recall from [31, Lemma 3.5] that the coupled problem given by (2.2), (2.3), and (2.4) has the one-dimensional kernel defined by

$$\{ ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}})) : \quad \boldsymbol{\sigma}_{\mathrm{S}} = -c\,\mathbf{I},\ \mathbf{u}_{\mathrm{D}} = \mathbf{0},\ \boldsymbol{\varphi} = \mathbf{0},\ \lambda = c,\ \mathbf{u}_{\mathrm{S}} = \mathbf{0},\ p_{\mathrm{D}} = c\,;\ c \in \mathbb{R} \}\,.$$

Hence, in order to solve this indetermination, we introduce

$$L_0^2(\Omega_{\mathrm{D}}) := \left\{ q \in L^2(\Omega_{\mathrm{D}}) : \quad \int_{\Omega_{\mathrm{D}}} q = 0 \right\},$$

and define the product spaces

$$\mathbb{X} := \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}) \times \mathbf{H}(\mathrm{div}\,; \Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)\,, \quad \mathbb{M} := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L_0^2(\Omega_{\mathrm{D}})\,,$$

endowed with the product norms

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}} + \|\boldsymbol{\psi}\|_{1/2,\Sigma} + \|\xi\|_{1/2,\Sigma} \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}\,,$$

and

$$\|\underline{\mathbf{v}}\|_{\mathbb{M}} := \|\mathbf{v}_{\mathrm{S}}\|_{0.\Omega_{\mathrm{S}}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{M}\,.$$

In this way, as explained in [31, Sections 2 and 3]), it suffices to consider from now on the following modified variational formulation of (2.2), (2.3), and (2.4): Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}\,, \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{M}\,, \end{aligned} \tag{2.5}$$

where

$$\mathcal{F}(\underline{\boldsymbol{\tau}}) := 0, \qquad \mathcal{G}(\underline{\mathbf{v}}) = \mathcal{G}((\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}})) := -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}\,, \tag{2.6}$$

and $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) := &\ \mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) + \mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\varphi}, \lambda)) \\ &+ \mathbf{b}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\psi}, \xi)) - \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi))\,, \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} \mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) &:= \nu^{-1} (\boldsymbol{\sigma}_{\mathrm{S}}^d, \boldsymbol{\tau}_{\mathrm{S}}^d)_{\mathrm{S}} + (\mathbf{K}^{-1} \mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}}\,, \\ \mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\psi}, \xi)) &:= \langle \boldsymbol{\tau}_{\mathrm{S}}\, \mathbf{n}, \boldsymbol{\psi} \rangle_{\Sigma} - \langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_{\Sigma}\,, \\ \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)) &:= \nu\, \kappa^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_{\Sigma} + \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_{\Sigma} - \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_{\Sigma}\,, \end{aligned}$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \,:=\, (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \,-\, (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}. \tag{2.8}$$

Hereafter we utilize, for each $\star \in \{\mathrm{S}, \mathrm{D}\}$, the following notations

$$(u,v)_\star \,:=\, \int_{\Omega_\star} u\,v, \qquad (\mathbf{u},\mathbf{v})_\star \,:=\, \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star \,:=\, \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}\,,$$

for all $u,\, v \,\in\, L^2(\Omega_\star)$, $\mathbf{u},\, \mathbf{v} \,\in\, \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma},\, \boldsymbol{\tau} \,\in\, \mathbb{L}^2(\Omega_\star)$, where $\boldsymbol{\sigma} : \boldsymbol{\tau} \,:=\, \mathbf{tr}(\boldsymbol{\sigma}^t\boldsymbol{\tau})$.

We find it important to remark that $\boldsymbol{\varphi}$ and $\lambda$ constitute the Lagrange multipliers associated with the transmission conditions (2.3). In addition, we notice that (2.5) is equivalent to the variational formulation defined in [31, Section 3.2, eq. (3.2)], in which $\boldsymbol{\sigma}_{\mathrm{S}}$ is decomposed into $\boldsymbol{\sigma}_{\mathrm{S}} \,=\, \boldsymbol{\sigma} \,+\, \mu$, with $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mu \in \mathbb{R}$, where

$$\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \,:\, \int_{\Omega_{\mathrm{S}}} \mathbf{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}.$$

The following result taken from [31] establishes, in particular, the well-posedness of (2.5).

Theorem 2.1 *For each pair* $(\mathcal{F},\mathcal{G}) \,\in\, \mathbb{X}' \times \mathbb{M}'$ *there exists a unique* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,\in\, \mathbb{X} \times \mathbb{M}$ *solution to* (2.5)*, and there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, C \left\{ \|\mathcal{F}\|_{\mathbb{X}'} \,+\, \|\mathcal{G}\|_{\mathbb{M}_0'} \right\}. \tag{2.9}$$

*Proof.* See [31, Theorem 3.9]. □

We end this section with the converse of the derivation of (2.5). More precisely, the following theorem establishes that the unique solution of (2.5), with $\mathcal{F}$ and $\mathcal{G}$ given by (2.6), solves the original transmission problem described in Section 2.1. This result will be used later on in Section 3.2 to prove the efficiency of our a posteriori error estimator. We remark that no extra regularity assumptions on the data, but only $\mathbf{f}_{\mathrm{S}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $f_{\mathrm{D}} \,\in\, L^2(\Omega_{\mathrm{D}})$, are required here.

Theorem 2.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,\in\, \mathbf{H} \times \mathbf{Q}$ *be the unique solution of the variational formulation* (2.5) *with* $\mathcal{F}$ *and* $\mathcal{G}$ *given by* (2.6)*. Then* $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \,=\, -\,\mathbf{f}_{\mathrm{S}}$ *in* $\Omega_{\mathrm{S}}$*,* $\nu^{-1}\,\boldsymbol{\sigma}^{d}_{\mathrm{S}} \,=\, \nabla\,\mathbf{u}_{\mathrm{S}}$ *in* $\Omega_{\mathrm{S}}$*,* $\mathbf{u}_{\mathrm{S}} \,\in\, \mathbf{H}^1(\Omega_{\mathrm{S}})$*,* $\mathrm{div}\,\mathbf{u}_{\mathrm{D}} \,=\, f_{\mathrm{D}}$ *in* $\Omega_{\mathrm{D}}$*,* $\mathbf{u}_{\mathrm{D}} \,=\, -\,\mathbf{K}\,\nabla\,p_{\mathrm{D}}$ *in* $\Omega_{\mathrm{D}}$*,* $p_{\mathrm{D}} \,\in\, H^1(\Omega_{\mathrm{D}})$*,* $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \,+\, \boldsymbol{\varphi}\cdot\mathbf{n} \,=\, 0$ *on* $\Sigma$*,* $\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} \,+\, \lambda\,\mathbf{n} \,-\, \frac{\nu}{\kappa}\,(\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t} \,=\, 0$ *on* $\Sigma$*,* $\lambda \,=\, p_{\mathrm{D}}$ *on* $\Sigma$*,* $\boldsymbol{\varphi} \,=\, -\,\mathbf{u}_{\mathrm{S}}$ *on* $\Sigma$*, and* $\mathbf{u}_{\mathrm{S}} \,=\, 0$ *on* $\Gamma_{\mathrm{S}}$*.*

*Proof.* It basically follows by applying integration by parts backwardly in (2.5) and using suitable test functions. We omit further details. □

## 2.3 The Galerkin formulation

Although the analysis in [31] provides general hypotheses for the well-posedness of a Galerkin scheme of (2.5), it suffices to consider in what follows the particular case described in [31, Section 5]. Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains $\Omega_{\mathrm{S}}$ and $\Omega_{\mathrm{D}}$ formed by shape-regular triangles $T$ of diameter $h_T$, and assume that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ match in $\Sigma$, so that their union is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathrm{RT}_0(T)$ be the local Raviart-Thomas space of order 0, that is

$$\mathrm{RT}_0(T) := \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\},$$

where $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbb{R}^2$, and for each $\star \in \{\mathrm{S}, \mathrm{D}\}$ we define the global spaces

$$\mathbf{H}_h(\Omega_\star) := \left\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,; \Omega_\star) : \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\}, \tag{2.10}$$

and

$$L_h(\Omega_\star) := \left\{ q_h : \Omega_\star \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\}.$$

Hereafter, given a non-negative integer $k$ and a subset $S$ of $\mathbb{R}^2$, $\mathbb{P}_k(S)$ stands for the space of polynomials defined on $S$ of degree $\leq k$. Next, we let $\Sigma_h$ be the partition of $\Sigma$ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$), and assume, without loss of generality, that the number of edges of $\Sigma_h$ is even. The case of an odd number of edges is easily reduced to the even case (see [31]). Then, we let $\Sigma_{2h}$ be the partition of $\Sigma$ arising by joining pairs of adjacent edges of $\Sigma_h$. Note that because $\Sigma_h$ is inherited from one of the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is $\Sigma_{2h}$.

Employing the above notations, we now introduce

$$\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \{ \boldsymbol{\tau} : \Omega_{\mathrm{S}} \to \mathbb{R}^{2\times 2} : \quad \mathbf{c}^t \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \}, \\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}}), \\
L_{h,0}(\Omega_{\mathrm{D}}) &:= L_h(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}}), \\
\Lambda_h(\Sigma) &:= \{ \xi_h \in C(\Sigma) : \quad \xi_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \text{ edge of } \Sigma_{2h} \}, \\
\boldsymbol{\Lambda}_h(\Sigma) &:= \Lambda_h(\Sigma) \times \Lambda_h(\Sigma),
\end{aligned}$$

and the product spaces

$$\mathbb{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma) \quad \text{and} \quad \mathbb{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}}).$$

In this way, the Galerkin scheme of (2.5) becomes: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}_h) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \quad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_h, \\
\mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \quad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{M}_h,
\end{aligned} \tag{2.11}$$

where $\underline{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h)$ and $\underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})$.

The following theorems, also taken from [31], provide the well-posedness of (2.11), the associated Cea estimate, and the corresponding theoretical rate of convergence.

THEOREM 2.3 *Assume that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ are quasiuniform in a neighborhood of $\Sigma$. Then the Galerkin scheme (2.11) has a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$. Moreover, there exist $C_1$, $C_2 > 0$, independent of $h$, such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \leq C_1 \left\{ \|\mathcal{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} + \|\mathcal{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'} \right\},$$

*and*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \leq C_2 \left\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_h} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} + \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \right\}.$$

*Proof.* See [31, Theorems 5.3 and 5.4]. $\square$

THEOREM 2.4 *Assume the same hypotheses of Theorem* 2.3, *and let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ *be the unique solutions of the continuous and discrete formulations* (2.5) *and* (2.11), *respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, *and* $\operatorname{div} \mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$. *Then,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, *and there exists* $C > 0$, *independent of* $h$ *and the continuous and discrete solutions, such that*

$$
\begin{aligned}
\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \\
+ \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\operatorname{div} \mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} & + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}.
\end{aligned} \tag{2.12}
$$

*Proof.* See [31, Theorem 5.5]. $\square$

# 3 A residual-based a posteriori error estimator

We first introduce some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathcal{E}(T)$ be the set of edges of $T$, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, that is

$$
\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),
$$

where $\mathcal{E}_h(\Gamma_{\mathrm{S}}) := \{ e \in \mathcal{E}_h : e \subseteq \Gamma_{\mathrm{S}} \}$, $\mathcal{E}_h(\Omega_\star) := \{ e \in \mathcal{E}_h : e \subseteq \Omega_\star \}$ for each $\star \in \{\mathrm{S}, \mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{ e \in \mathcal{E}_h : e \subseteq \Sigma \}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition $\Sigma_h$. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition $\Sigma_{2h}$. In what follows, $h_e$ stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{D}, \mathrm{S}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1, 2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of $q$ across $e$, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where $T'$ and $T''$ are the triangles of $\mathcal{T}_h^\star$ having $e$ as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge $e$, which points either inward $T'$ or inward $T''$, and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along $e$. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v} \cdot \mathbf{t}_e]$ and $[\boldsymbol{\tau}\, \mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across $e$, that is $[\mathbf{v} \cdot \mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\} \cdot \mathbf{t}_e$ and $[\boldsymbol{\tau}\, \mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\, \mathbf{t}_e$, respectively. From now on, when no confusion arises, we simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for suffiently smooth scalar, vector and tensors fields $q$, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$
\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \operatorname{curl} q := \left( \frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1} \right)^{\mathtt{t}},
$$

$$
\operatorname{rot} \mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} := \left( \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \right)^{\mathtt{t}}.
$$

Next, let $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ and $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h), (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})) \in \mathbb{X}_h \times \mathbb{M}_h$ be the unique solutions of (2.5) and (2.11), respectively. Then, we introduce the global a posteriori error estimator:

$$\Theta \;:=\; \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 \;+\; \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{3.1}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned}
\Theta_{\mathrm{S},T}^2 \;:=\; & \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \;+\; h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \;+\; h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\,\|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2 + h_e\,\left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \nabla\boldsymbol{\varphi}_h\,\mathbf{t}\right\|_{0,e}^2 \right\},
\end{aligned}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}
\Theta_{\mathrm{D},T}^2 \;:=\; & \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \;+\; h_T^2\,\|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \;+\; h_T^2\,\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e\,\left\|[\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\left\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \frac{d\lambda_h}{d\mathbf{t}}\right\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} \;+\; \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \;+\; h_e\,\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}.
\end{aligned}$$

### 3.1 Reliability of the a posteriori error estimator

The main result of this section is stated as follows.

THEOREM 3.1 *There exists $C_{\mathtt{rel}} > 0$, independent of $h$, such that*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \;+\; \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \;\leq\; C_{\mathtt{rel}}\,\Theta\,. \tag{3.2}$$

We begin the derivation of (3.2) by recalling that the continous dependence result given by (2.9) is equivalent to the global inf-sup condition for the continuous formulation (2.5). Then, applying this estimate to the error $(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) \in \mathbb{X}\times\mathbb{M}$, we obtain

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \;\leq\; C \sup_{\substack{(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\in\mathbb{X}\times\mathbb{M}\\ (\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\neq\mathbf{0}}} \frac{|R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})|}{\|(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\|_{\mathbb{X}\times\mathbb{M}}}\,, \tag{3.3}$$

where $R : \mathbb{X}\times\mathbb{M} \to \mathbb{R}$ is the residual operator defined by

$$R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) \;:=\; \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) \;+\; \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) \;+\; \mathcal{B}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}), \quad \forall\,(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) \in \mathbb{X}\times\mathbb{M}\,.$$

More precisely, according to (2.5) and the definitions of $\mathcal{A}$ and $\mathcal{B}$ (cf. (2.7), (2.8)), we find that for any $(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) := ((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi), (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}})) \in \mathbb{X}\times\mathbb{M}$ there holds

$$R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) \;=\; R_1(\boldsymbol{\tau}_{\mathrm{S}}) \;+\; R_2(\mathbf{v}_{\mathrm{D}}) \;+\; R_3(\boldsymbol{\psi}) \;+\; R_4(\xi) \;+\; R_5(\mathbf{v}_{\mathrm{S}}) \;+\; R_6(q_{\mathrm{D}})\,,$$

where

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) \,:=\, -\,\nu^{-1}\int_{\Omega_{\mathrm{S}}}\boldsymbol{\sigma}_{\mathrm{S},h}^{d}:\boldsymbol{\tau}_{\mathrm{S}}^{d} \,-\, \int_{\Omega_{\mathrm{S}}}\mathbf{u}_{\mathrm{S},h}\cdot\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}} \,-\, \langle\boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\varphi}_h\rangle_\Sigma\,,$$

$$R_2(\mathbf{v}_{\mathrm{D}}) \,:=\, -\int_{\Omega_{\mathrm{D}}}\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{v}_{\mathrm{D}} \,+\, \int_{\Omega_{\mathrm{D}}}p_{\mathrm{D},h}\,\mathrm{div}\,\mathbf{v}_{\mathrm{D}} \,+\, \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda_h\rangle_\Sigma\,,$$

$$R_3(\boldsymbol{\psi}) \,:=\, -\,\langle\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma \,-\, \langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda_h\rangle_\Sigma \,+\, \frac{\nu}{\kappa}\langle\boldsymbol{\psi}\cdot\mathbf{t},\boldsymbol{\varphi}_h\cdot\mathbf{t}\rangle_\Sigma\,,$$

$$R_4(\xi) \,:=\, \langle\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n},\xi\rangle_\Sigma \,+\, \langle\boldsymbol{\varphi}_h\cdot\mathbf{n},\xi\rangle_\Sigma\,,$$

$$R_5(\mathbf{v}_{\mathrm{S}}) \,:=\, -\int_{\Omega_{\mathrm{S}}}\mathbf{v}_{\mathrm{S}}\cdot(\mathbf{f}_{\mathrm{S}}+\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h})\,,$$

and

$$R_6(q_{\mathrm{D}}) \,:=\, -\int_{\Omega_{\mathrm{D}}}q_{\mathrm{D}}\,(f_{\mathrm{D}} - \,\mathrm{div}\,\mathbf{u}_{\mathrm{D},h})\,.$$

Hence, the supremum in (3.3) can be bounded in terms of $R_i\,,\ i\in\{1,...,6\}$, which yields

$$\begin{gathered}\|(\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\sigma}}_h,\underline{\mathbf{u}}-\underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C\,\Bigg\{\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S}}\neq\mathbf{0}}}\frac{|R_1(\boldsymbol{\tau}_{\mathrm{S}})|}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} + \sup_{\substack{\mathbf{v}_{\mathrm{D}}\in\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})\\ \mathbf{v}_{\mathrm{D}}\neq\mathbf{0}}}\frac{|R_2(\mathbf{v}_{\mathrm{D}})|}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}}\\ + \sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}^{1/2}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}}\frac{|R_3(\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_{1/2,\Sigma}} + \sup_{\substack{\xi\in H^{1/2}(\Sigma)\\ \xi\neq\mathbf{0}}}\frac{|R_4(\xi)|}{\|\xi\|_{1/2,\Sigma}} + \sup_{\substack{\mathbf{v}_{\mathrm{S}}\in\mathbf{L}^2(\Omega_{\mathrm{S}})\\ \mathbf{v}_{\mathrm{S}}\neq\mathbf{0}}}\frac{|R_5(\mathbf{v}_{\mathrm{S}})|}{\|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}} + \sup_{\substack{q_{\mathrm{D}}\in L_0^2(\Omega_{\mathrm{D}})\\ q_{\mathrm{D}}\neq\mathbf{0}}}\frac{|R_6(q_{\mathrm{D}})|}{\|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}}\Bigg\}\,.\end{gathered} \tag{3.4}$$

Throughout the rest of this section we provide suitable upper bounds for each one of the terms on the right hand side of (3.4). The following lemma, whose proof follows from straightforward applications of the Cauchy-Schwarz inequality, is stated first.

Lemma 3.1 *There hold*

$$\sup_{\substack{\mathbf{v}_{\mathrm{S}}\in\mathbf{L}^2(\Omega_{\mathrm{S}})\\ \mathbf{v}_{\mathrm{S}}\neq\mathbf{0}}}\frac{|R_5(\mathbf{v}_{\mathrm{S}})|}{\|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}} \,\le\, \|\mathbf{f}_{\mathrm{S}}+\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} \,=\, \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}}\|\mathbf{f}_{\mathrm{S}}+\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2\right\}^{1/2}\,, \tag{3.5}$$

*and*

$$\sup_{\substack{q_{\mathrm{D}}\in L_0^2(\Omega_{\mathrm{D}})\\ q_{\mathrm{D}}\neq\mathbf{0}}}\frac{|R_6(q_{\mathrm{D}})|}{\|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}} \,\le\, \|f_{\mathrm{D}} - \,\mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \,=\, \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{D}}}\|f_{\mathrm{D}} - \,\mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\right\}^{1/2}\,. \tag{3.6}$$

The next lemma estimates the suprema on the spaces defined in the interface $\Sigma$.

Lemma 3.2 *There exist $C_3$, $C_4 > 0$, independent of $h$, such that*

$$\sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}^{1/2}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}} \frac{|R_3(\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_{1/2,\Sigma}} \,\le\, C_3 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{0,e}^2 \right\}^{1/2}, \tag{3.7}$$

*and*

$$\sup_{\substack{\xi\in H^{1/2}(\Sigma)\\ \xi\neq\mathbf{0}}} \frac{|R_4(\xi)|}{\|\xi\|_{1/2,\Sigma}} \,\le\, C_4 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \right\}^{1/2}. \tag{3.8}$$

*Proof.* It is clear from the definition of $R_3$ that

$$R_3(\boldsymbol{\psi}) \,=\, -\,\langle \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma \qquad \forall\boldsymbol{\psi}\in\mathbf{H}^{1/2}(\Sigma)\,,$$

and hence

$$\sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}^{1/2}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}} \frac{|R_3(\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_{1/2,\Sigma}} \,=\, \left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{-1/2,\Sigma}. \tag{3.9}$$

In order to estimate $\left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{-1/2,\Sigma}$ in terms of local quantities we now apply a technical result from [16]. In fact, taking $\boldsymbol{\tau}_{\mathrm{S}} = \mathbf{0}$, $\mathbf{v}_{\mathrm{D}} = \mathbf{0}$ and $\xi = 0$ in the first equation of (2.11), we have

$$\langle \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma \,=\, 0 \qquad \forall\,\boldsymbol{\psi}\,\in\,\boldsymbol{\Lambda}_h(\Sigma)\,,$$

which says that $\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}$ is $\mathbf{L}^2(\Sigma)$-orthogonal to $\boldsymbol{\Lambda}_h(\Sigma)$. Hence, applying [16, Theorem 2], and recalling that $\Sigma_h$ and $\Sigma_{2h}$ are of bounded variation, we deduce that

$$\begin{aligned} &\left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{-1/2,\Sigma}^2 \\ &\quad\le\, C \sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{0,e}^2 \\ &\quad\le\, C \sum_{e\in\mathcal{E}_{h}(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \frac{\nu}{\kappa}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|_{0,e}^2, \end{aligned}$$

which, together with (3.9), yields (3.7).

The proof of (3.8) proceeds analogously. In fact, it is easy to see that

$$\sup_{\substack{\xi\in H^{1/2}(\Sigma)\\ \xi\neq\mathbf{0}}} \frac{|R_4(\xi)|}{\|\xi\|_{1/2,\Sigma}} \,=\, \|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{-1/2,\Sigma}\,,$$

and hence, noting also from the first equation of (2.11) that $\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}$ is $L^2(\Sigma)$-orthogonal to $\Lambda_h(\Sigma)$, another straightforward application of [16, Theorem 2] yields the required estimate. We omit further details here. $\square$

Our next goal is to bound the first two suprema on the right hand side of (3.4), for which we need several preliminary results. We begin with the following lemma showing the existence of stable Helmholtz decompositions for $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$ and $\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$.

Lemma 3.3

a) *For each* $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$ *there exist* $\mathbf{w} \in \mathbf{H}^1(\Omega_{\mathrm{D}})$ *and* $\beta \in H^1(\Omega_{\mathrm{D}})$, *with* $\displaystyle\int_{\Omega_{\mathrm{S}}} \beta = 0$, *such that there hold* $\mathbf{v}_{\mathrm{D}} = \mathbf{w} + \mathrm{curl}\,\beta$ *in* $\Omega_{\mathrm{D}}$, *and*

$$\|\mathbf{w}\|_{1,\Omega_{\mathrm{D}}} + \|\beta\|_{1,\Omega_{\mathrm{D}}} \le C_{\mathrm{D}}\, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}},$$

*where* $C_{\mathrm{D}}$ *is a positive constant independent of* $\mathbf{v}_{\mathrm{D}}$.

b) *For each* $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ *there exist* $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$ *and* $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ *such that there hold* $\boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\eta} + \mathbf{curl}\,\boldsymbol{\chi}$ *in* $\Omega_{\mathrm{S}}$, *and*

$$\|\boldsymbol{\eta}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \le C_{\mathrm{S}}\, \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}},$$

*where* $C_{\mathrm{S}}$ *is a positive constant independent of* $\boldsymbol{\tau}_{\mathrm{S}}$.

*Proof.* Given $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$, we let $G$ be a smooth convex domain containing $\Omega_{\mathrm{D}}$, and let $z \in H^1_0(G) \cap H^2(G)$ be the unique solution of

$$-\Delta z = \left\{ \begin{array}{ll} \mathrm{div}\,\mathbf{v}_{\mathrm{D}} & \text{in} \quad \Omega_{\mathrm{D}} \\ 0 \quad \text{in} & G \setminus \bar{\Omega}_{\mathrm{D}} \end{array} \right\} \quad \text{in} \quad G, \quad z = 0 \quad \text{on} \quad \partial G.$$

It follows that

$$\|z\|_{2,G} \le C\, \|\mathrm{div}\,\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \le C\, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}},$$

and hence, defining $\mathbf{w} := -\nabla z$ in $\Omega_{\mathrm{D}}$, we find that

$$\mathrm{div}\,\mathbf{w} = \mathrm{div}\,\mathbf{v}_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}} \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega_{\mathrm{D}}} \le \|z\|_{2,\Omega_{\mathrm{D}}} \le \|z\|_{2,G} \le C\, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}.$$

In addition, since $\mathrm{div}\,(\mathbf{v}_{\mathrm{D}} - \mathbf{w}) = 0$ and $\Omega_{\mathrm{D}}$ is connected, there exists $\beta \in H^1(\Omega_{\mathrm{D}})$, with $\displaystyle\int_{\Omega_{\mathrm{D}}} \beta = 0$, such that $\mathbf{v}_{\mathrm{D}} - \mathbf{w} = \mathrm{curl}\,\beta$ in $\Omega_{\mathrm{D}}$. In this way, using the generalized Poincare inequality and the above estimate for $\mathbf{w}$, we deduce that

$$\|\beta\|_{1,\Omega_{\mathrm{D}}} \le C\, |\beta|_{1,\Omega_{\mathrm{D}}} = C\, \|\mathrm{curl}\,\beta\|_{0,\Omega_{\mathrm{D}}} = C\, \|\mathbf{v}_{\mathrm{D}} - \mathbf{w}\|_{0,\Omega_{\mathrm{D}}} \le C \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}},$$

which completes the proof of a).

We now let $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$. Since $\Omega_{\mathrm{S}}$ is not necessarily connected, we first perform a suitable extension of $\boldsymbol{\tau}_{\mathrm{S}}$ to the domain $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, and then apply a) to each row of the resulting tensor. More precisely, let $\boldsymbol{\tau}_{\mathrm{S},i} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}})$ be the $i$-th row of $\boldsymbol{\tau}_{\mathrm{S}}$, $i \in \{1,2\}$, and let $\phi_i \in H^1(\Omega_{\mathrm{D}})$ be the unique solution of the Neumann problem:

$$\Delta \phi_i = -\frac{\langle \boldsymbol{\tau}_{\mathrm{S},i} \cdot \mathbf{n}, 1\rangle_\Sigma}{|\Omega_{\mathrm{D}}|} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \frac{\partial \phi_i}{\partial \mathbf{n}} = \boldsymbol{\tau}_{\mathrm{S},i} \cdot \mathbf{n} \quad \text{on} \quad \Sigma, \quad \int_{\Omega_{\mathrm{D}}} \phi_i = 0.$$

Then we define $\boldsymbol{\tau}_i^{ext} = \left\{ \begin{array}{ll} \boldsymbol{\tau}_{\mathrm{S},i} & \text{in} \quad \Omega_{\mathrm{S}} \\ \nabla \phi_i & \text{in} \quad \Omega_{\mathrm{D}} \end{array} \right.$, and notice that $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div}\,;\Omega)$ and

$$\begin{aligned} \|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div}\,;\Omega} &\le \|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}} + \|\nabla \phi_i\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}} \\ &\le \|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}} + C\, \|\boldsymbol{\tau}_{\mathrm{S},i} \cdot \mathbf{n}\|_{-1/2,\Sigma} \le C\, \|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}. \end{aligned}$$

Proceeding as in the proof of a), but now for $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div}\,;\Omega)$, we deduce the existence of $\mathbf{w}_i \in \mathbf{H}^1(\Omega)$ and $\beta_i \in H^1(\Omega)$, with $\int_\Omega \beta_i = 0$, such that $\boldsymbol{\tau}_i^{ext} = \mathbf{w}_i + \mathrm{curl}\,\beta_i$ in $\Omega$, and

$$\|\mathbf{w}_i\|_{1,\Omega} + \|\beta_i\|_{1,\Omega} \leq C\,\|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div}\,;\Omega} \leq C\,\|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}\,.$$

Hence, the proof of b) follows by defining $i$-th row of $\boldsymbol{\eta} := \mathbf{w}_i|_{\Omega_{\mathrm{S}}}$ and $\boldsymbol{\chi} := (\beta_1|_{\Omega_{\mathrm{S}}}, \beta_2|_{\Omega_{\mathrm{S}}})$. $\square$

The Raviart-Thomas interpolation operator $\Pi_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h(\Omega_\star)$ (cf. (2.10)), $\star \in \{\mathrm{S},\mathrm{D}\}$, which, given $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$, is characterized by

$$\Pi_h^\star(\mathbf{v}) \in \mathbf{H}_h(\Omega_\star) \quad \text{and} \quad \int_e \Pi_h^\star(\mathbf{v})\cdot\mathbf{n} = \int_e \mathbf{v}\cdot\mathbf{n} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star\,, \tag{3.10}$$

will also be needed in what follows. Note that as a consequence of (3.10), there holds

$$\mathrm{div}\,(\Pi_h^\star(\mathbf{v})) = \mathcal{P}_h^\star(\mathrm{div}\,\mathbf{v})\,, \tag{3.11}$$

where $\mathcal{P}_h^\star$, $\star \in \{\mathrm{S},\mathrm{D}\}$, is the $L^2(\Omega_\star)$-orthogonal projector onto the piecewise constant functions on $\Omega_\star$. A tensor version of $\Pi_h^\star$, say $\boldsymbol{\Pi}_h^\star : \mathbb{H}^1(\Omega_\star) \to \mathbb{H}_h(\Omega_\star)$, which is defined row-wise by $\Pi_h^\star$, and a vector version of $\mathcal{P}_h^\star$, say $\mathbf{P}_h^\star$, which is the $\mathbf{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant vectors on $\Omega_\star$, might also be required. The local approximation properties of $\Pi_h^\star$ (and hence of $\boldsymbol{\Pi}_h^\star$) are stated as follows.

Lemma 3.4 *For each $\star \in \{\mathrm{S},\mathrm{D}\}$ there exist constants $c_1$, $c_2 > 0$, independent of $h$, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$ there hold*

$$\|\mathbf{v} - \Pi_h^\star(\mathbf{v})\|_{0,T} \leq c_1\,h_T\,\|\mathbf{v}\|_{1,T} \qquad \forall\, T \in \mathcal{T}_h^\star\,,$$

*and*

$$\|\mathbf{v}\cdot\mathbf{n} - \Pi_h^\star(\mathbf{v})\cdot\mathbf{n}\|_{0,e} \leq c_2\,h_e^{1/2}\,\|\mathbf{v}\|_{1,T_e} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star\,,$$

*where $T_e$ is a triangle of $\mathcal{T}_h^\star$ containing $e$ on its boundary.*

*Proof.* See [12]. $\square$

We will also utilize the Clément interpolation operators $I_h^\star : H^1(\Omega_\star) \to X_{\star,h}$ (cf. [19]), where

$$X_{\star,h} := \{v \in C(\bar\Omega_\star) : \quad v|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^\star\} \qquad \text{for each } \star \in \{\mathrm{S},\mathrm{D}\}\,.$$

In addition, we will make use of a vector version of $I_h^\star$, say $\mathbf{I}_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{X}_{\star,h} := X_{\star,h} \times X_{\star,h}$, which is defined componentwise by $I_h^\star$. The following lemma establishes the local approximation properties of $I_h^\star$ (and hence of $\mathbf{I}_h^\star$).

Lemma 3.5 *For each $\star \in \{\mathrm{S},\mathrm{D}\}$ there exist constants $c_3$, $c_4 > 0$, independent of $h$, such that for all $v \in H^1(\Omega_\star)$ there hold*

$$\|v - I_h^\star(v)\|_{0,T} \leq c_3\,h_T\,\|v\|_{1,\Delta_\star(T)} \qquad \forall\, T \in \mathcal{T}_h^\star\,,$$

*and*

$$\|v - I_h^\star(v)\|_{0,e} \leq c_4\,h_e^{1/2}\,\|v\|_{1,\Delta_\star(e)} \qquad \forall\, e \in \mathcal{E}_h\,,$$

*where*

$$\Delta_\star(T) := \cup\{T' \in \mathcal{T}_h^\star : \quad T' \cap T \neq \mathbf{0}\} \quad \textit{and} \quad \Delta_\star(e) := \cup\{T' \in \mathcal{T}_h^\star : \quad T' \cap e \neq \mathbf{0}\}\,.$$

*Proof.* See [19]. ☐

Finally, we require the technical results given by the following two lemmas.

LEMMA 3.6 *Let $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_S)$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_S)$. Then there hold*

$$|R_1(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^S(\boldsymbol{\eta}))| \,\le\, c_1\,\nu^{-1} \sum_{T\in\mathcal{T}_h^S} h_T\,\|\boldsymbol{\sigma}_{S,h}^d\|_{0,T}\,\|\boldsymbol{\eta}\|_{1,T} \,+\, c_2 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\,\|\mathbf{u}_{S,h} + \boldsymbol{\varphi}_h\|_{0,e}\,\|\boldsymbol{\eta}\|_{1,T_e}\,,$$

*and*

$$\begin{aligned}
&|R_1(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h^S(\boldsymbol{\chi})))| \\
&\le\, c_3\,\nu^{-1} \sum_{T\in\mathcal{T}_h^S} h_T\,\|\mathbf{rot}\,(\boldsymbol{\sigma}_{S,h}^d)\|_{0,T}\,\|\boldsymbol{\chi}\|_{1,\Delta_S(T)} \,+\, c_4\,\nu^{-1} \sum_{e\in\mathcal{E}_h(\Omega_S)} h_e^{1/2}\,\|[\boldsymbol{\sigma}_{S,h}^d\mathbf{t}]\|_{0,e}\,\|\boldsymbol{\chi}\|_{1,\Delta_S(e)} \\
&+ c_4\,\nu^{-1} \sum_{e\in\mathcal{E}_h(\Gamma_S)} h_e^{1/2}\,\|\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\|_{0,e}\,\|\boldsymbol{\chi}\|_{1,\Delta_S(e)} + c_4 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\left\|\nu^{-1}\,\boldsymbol{\sigma}_{S,h}^d\mathbf{t} + \nabla\boldsymbol{\varphi}_h\,\mathbf{t}\right\|_{0,e}\|\boldsymbol{\chi}\|_{1,\Delta_S(e)}.
\end{aligned}$$

*Proof.* We first let $\boldsymbol{\zeta} := \boldsymbol{\eta} - \boldsymbol{\Pi}_h^S(\boldsymbol{\eta})$ and observe, according to (3.10) and (3.11), that

$$\int_e \mathbf{p}\cdot\boldsymbol{\zeta}\,\mathbf{n} \,=\, 0 \quad \forall\,\mathbf{p}\in[\mathbb{P}_0(e)]^2\,, \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^S\,, \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\zeta} \,=\, \mathbf{div}\,\boldsymbol{\eta} \,-\, \mathbf{P}_h^S(\mathbf{div}\,\boldsymbol{\eta})\,.$$

Then, since $\boldsymbol{\sigma}_{S,h}^d : \boldsymbol{\zeta}^d = \boldsymbol{\sigma}_{S,h}^d : \boldsymbol{\zeta}$ and $\mathbf{u}_{S,h}$ is a constant vector on each $T \in \mathcal{T}_h^S$, we deduce from the definition of $R_1$ and the above identities that

$$\begin{aligned}
R_1(\boldsymbol{\zeta}) &= \;-\,\nu^{-1} \sum_{T\in\mathcal{T}_h^S} \int_T \boldsymbol{\sigma}_{S,h}^d : \boldsymbol{\zeta}^d \,-\, \sum_{T\in\mathcal{T}_h^S} \int_T \mathbf{u}_{S,h}\cdot\mathbf{div}\boldsymbol{\zeta} \,-\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\varphi}_h\cdot\boldsymbol{\zeta}\,\mathbf{n} \\
&= \;-\,\nu^{-1} \sum_{T\in\mathcal{T}_h^S} \int_T \boldsymbol{\sigma}_{S,h}^d : \boldsymbol{\zeta} \,-\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\varphi}_h\cdot\boldsymbol{\zeta}\,\mathbf{n} \\
&= \;-\,\nu^{-1} \sum_{T\in\mathcal{T}_h^S} \int_T \boldsymbol{\sigma}_{S,h}^d : \boldsymbol{\zeta} \,-\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\mathbf{u}_{S,h} + \boldsymbol{\varphi}_h)\cdot\boldsymbol{\zeta}\,\mathbf{n}\,.
\end{aligned}$$

On the other hand, we now let $\boldsymbol{\rho} := \boldsymbol{\chi} - \mathbf{I}_h^S(\boldsymbol{\chi})$. Then, using that $\mathbf{div}(\mathbf{curl}\,(\boldsymbol{\rho})) = \mathbf{0}$, noting that $\mathbf{curl}\,(\boldsymbol{\rho})\,\mathbf{n} = \nabla\boldsymbol{\rho}\,\mathbf{t}$ on $\Sigma$, integrating by parts on each $T \in \mathcal{T}_h^S$ and on $\Sigma$, and observing that $\nabla\boldsymbol{\varphi}_h\,\mathbf{t} \in \mathbf{L}^2(\Sigma)$, we obtain

$$\begin{aligned}
&R_1(\mathbf{curl}\,(\boldsymbol{\rho})) \,=\, -\,\nu^{-1} \int_{\Omega_S} \boldsymbol{\sigma}_{S,h}^d : \mathbf{curl}\,(\boldsymbol{\rho}) \,-\, \langle \mathbf{curl}\,(\boldsymbol{\rho})\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma \\
&= \nu^{-1} \sum_{T\in\mathcal{T}_h^S} \left(-\int_T \boldsymbol{\rho}\cdot\mathbf{rot}\,(\boldsymbol{\sigma}_{S,h}^d) \,+\, \int_{\partial T} \boldsymbol{\rho}\cdot\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\right) \,+\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\rho}\cdot(\nabla\boldsymbol{\varphi}_h\,\mathbf{t}) \\
&= -\sum_{T\in\mathcal{T}_h^S} \nu^{-1} \int_T \boldsymbol{\rho}\cdot\mathbf{rot}\,(\boldsymbol{\sigma}_{S,h}^d) \,+\, \sum_{e\in\mathcal{E}_h(\Omega_S)} \nu^{-1} \int_e \boldsymbol{\rho}\cdot[\boldsymbol{\sigma}_{S,h}^d\,\mathbf{t}] \\
&\quad + \sum_{e\in\mathcal{E}_h(\Gamma_S)} \nu^{-1} \int_e \boldsymbol{\rho}\cdot\boldsymbol{\sigma}_{S,h}^d\,\mathbf{t} \,+\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\rho}\cdot\left(\nu^{-1}\,\boldsymbol{\sigma}_{S,h}^d \,+\, \nabla\boldsymbol{\varphi}_h\right)\mathbf{t}\,.
\end{aligned}$$

Hence, straighforward applications of the Cauchy-Schwarz inequality to the above equations, together with the approximation properties provided by Lemmas 3.4 and 3.5, namely

$$\|\boldsymbol{\eta} - \boldsymbol{\Pi}_h^{\mathrm{S}}(\boldsymbol{\eta})\|_{0,T} \,\le\, c_1\, h_T\, \|\boldsymbol{\eta}\|_{1,T}\,, \qquad \|\boldsymbol{\eta}\,\mathbf{n} - \boldsymbol{\Pi}_h^{\mathrm{S}}(\boldsymbol{\eta})\,\mathbf{n}\|_{0,e} \,\le\, c_2\, h_e^{1/2}\, \|\boldsymbol{\eta}\|_{1,T}$$

$$\|\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}(\boldsymbol{\chi})\|_{0,T} \,\le\, c_3\, h_T\, \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(T)}\,, \quad \text{and} \quad \|\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}(\boldsymbol{\chi})\|_{0,e} \,\le\, c_4\, h_e^{1/2}\, \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)}\,,$$

for each $T \in \mathcal{T}_h^{\mathrm{S}}$ and for each $e \in \mathcal{E}(T)$, imply the required estimates and finish the proof. □

Lemma 3.7 *Let* $\mathbf{w} \in \mathbf{H}^1(\Omega_{\mathrm{D}})$ *and* $\beta \in H^1(\Omega_{\mathrm{D}})$*. Then there hold*

$$|R_2(\mathbf{w} - \Pi_h^{\mathrm{D}}(\mathbf{w}))| \,\le\, c_1 \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T\, \|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}\, \|\mathbf{w}\|_{1,T} \,+\, c_2 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\, \|p_{\mathrm{D},h} \,-\, \lambda_h\|_{0,e}\, \|\mathbf{w}\|_{1,T_e}\,,$$

*and*

$$\begin{aligned} |R_2(\mathrm{curl}\,(\beta - I_h^{\mathrm{D}}(\beta)))| \,&\le\, c_3 \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T\, \|\mathrm{rot}\,(\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h})\|_{0,T}\, \|\beta\|_{1,\Delta_{\mathrm{D}}(T)} \\ &+\, c_4 \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e^{1/2}\, \|[\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}\, \|\beta\|_{1,\Delta_{\mathrm{D}}(e)} \\ &+\, c_4 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2} \left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \frac{d\lambda_h}{d\mathbf{t}}\right\|_{0,e} \|\beta\|_{1,\Delta_{\mathrm{D}}(e)}\,. \end{aligned}$$

*Proof.* Since $R_1$ and $R_2$ have analogue structures, the proof proceeds similarly as for Lemma 3.6. □

We are now in a position to bound the suprema depending on $R_1$ and $R_2$.

Lemma 3.8 *There exists* $C_1 > 0$*, independent of* $h$*, such that*

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S}}\neq\mathbf{0}}} \frac{|R_1(\boldsymbol{\tau}_{\mathrm{S}})|}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \,\le\, C_1 \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \widehat{\Theta}_{\mathrm{S},T}^2\right\}^{1/2}, \tag{3.12}$$

*where, for each* $T \in \mathcal{T}_h^{\mathrm{S}}$*:*

$$\begin{aligned} \widehat{\Theta}_{\mathrm{S},T}^2 \,:=&\, h_T^2\, \|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \,+\, h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\, \|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{h_e \left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \nabla\boldsymbol{\varphi}_h\,\mathbf{t}\right\|_{0,e}^2 \,+\, h_e\, \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2\right\} \end{aligned}$$

*Proof.* Given $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ we know from Lemma 3.3 that there exist $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ such that $\boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\eta} + \mathbf{curl}\,\boldsymbol{\chi}$ in $\Omega_{\mathrm{S}}$ and

$$\|\boldsymbol{\eta}\|_{1,\Omega_{\mathrm{S}}} \,+\, \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \,\le\, C\, \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}\,. \tag{3.13}$$

Then, since $R_1(\boldsymbol{\tau}_{\mathrm{S},h}) = 0 \quad \forall\, \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$, which follows from the first equation of the Galerkin scheme (2.11) taking $(\mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \boldsymbol{\xi}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, and thanks to the fact that $R_1$ is linear, we obtain

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) = R_1(\boldsymbol{\tau}_{\mathrm{S}} - \boldsymbol{\tau}_{\mathrm{S},h}) \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}})\,. \tag{3.14}$$

In particular, we let $\boldsymbol{\tau}_{\mathrm{S},h} := \mathbf{\Pi}_h^{\mathrm{S}}(\boldsymbol{\eta}) + \mathbf{curl}\,(\mathbf{I}_h^{\mathrm{S}}(\boldsymbol{\chi}))$, which can be seen as a discrete Helmholtz decomposition of $\boldsymbol{\tau}_{\mathrm{S},h}$, and obtain

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) = R_1(\boldsymbol{\eta} - \mathbf{\Pi}_h^{\mathrm{S}}(\boldsymbol{\eta})) + R_1(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}(\boldsymbol{\chi}))\,. \tag{3.15}$$

Hence, applying Lemma 3.6 and then the discrete Cauchy-Schwarz inequality to the resulting terms, noting that the numbers of triangles in $\Delta_{\mathrm{S}}(T)$ and $\Delta_{\mathrm{S}}(e)$ are bounded, and finally using the estimate (3.13), we conclude the upper bound (3.12). $\square$

LEMMA 3.9 *There exists $C_2 > 0$, independent of $h$, such that*

$$\sup_{\substack{\mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D}} \neq \mathbf{0}}} \frac{|R_2(\mathbf{v}_{\mathrm{D}})|}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}} \le C_2 \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \widehat{\Theta}_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{3.16}$$

*where, for each $T \in \mathcal{T}_h^{\mathrm{D}}$:*

$$\widehat{\Theta}_{\mathrm{D},T}^2 := h_T^2 \, \|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_T^2 \, \|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{D}})} h_e \, \left\|[\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t}]\right\|_{0,e}^2$$
$$+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \left\{ h_e \left\| \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t} + \frac{d\lambda_h}{d\mathbf{t}} \right\|_{0,e}^2 + h_e \, \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}.$$

*Proof.* It follows basically the same lines of the proof of Lemma 3.8. In fact, given $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$ we first apply Lemma 3.3 to deduce the existence of $\mathbf{w} \in \mathbf{H}^1(\Omega_{\mathrm{D}})$ and $\beta \in H^1(\Omega_{\mathrm{D}})$ such that $\mathbf{v}_{\mathrm{D}} = \mathbf{w} + \mathrm{curl}\,\beta$ and

$$\|\mathbf{w}\|_{1,\Omega_{\mathrm{D}}} + \|\beta\|_{1,\Omega_{\mathrm{D}}} \le C \, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}\,. \tag{3.17}$$

Then, since $R_2(\mathbf{v}_{\mathrm{D},h}) = 0 \quad \forall\, \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}})$, which corresponds to the first equation of the Galerkin scheme (2.11) with $(\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\psi}, \boldsymbol{\xi}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, and thanks to the fact that $R_2$ is linear, we obtain

$$R_2(\mathbf{v}_{\mathrm{D}}) = R_2(\mathbf{v}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D},h}) \qquad \forall\, \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}})\,. \tag{3.18}$$

Next, we choose $\mathbf{v}_{\mathrm{D},h} = \Pi_h^{\mathrm{D}}(\mathbf{w}) + \mathrm{curl}\,\left(I_h^{\mathrm{D}}(\beta)\right)$, notice that

$$R_2(\mathbf{v}_{\mathrm{D}}) = R_2(\mathbf{w} - \Pi_h^{\mathrm{D}}(\mathbf{w})) + R_2\left(\mathrm{curl}\,(\beta - I_h^{\mathrm{D}}(\beta))\right),$$

and apply Lemma 3.7. Thus, using again the discrete Cauchy-Schwarz inequality, noting that the numbers of triangles in $\Delta_{\mathrm{D}}(T)$ and $\Delta_{\mathrm{D}}(e)$ are bounded, and employing now the upper bound (3.17), we conclude (3.16). $\square$

We end this section by observing that the reliability estimate (3.2) (cf. Theorem 3.1) is a direct consequence of Lemmas 3.1, 3.2, 3.8, and 3.9.

### 3.2 Efficiency of the a posteriori error estimator

The main result of this section is stated as follows.

THEOREM 3.2 *There exists $C_{\mathtt{eff}} > 0$, independent of $h$, such that*

$$C_{\mathtt{eff}}\,\Theta \;\leq\; \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \,+\, \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \,+\, \text{h.o.t.}\,, \tag{3.19}$$

*where* h.o.t. *stands, eventually, for one or several terms of higher order.*

We remark in advance that the proof of (3.19) makes frequent use of the identities provided by Theorem 2.2. We begin with the estimates for the zero order terms appearing in the definition of $\Theta^2_{\mathrm{S},T}$ and $\Theta^2_{\mathrm{D},T}$.

LEMMA 3.10 *There hold*

$$\|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \,\leq\, \|\boldsymbol{\sigma}_{\mathrm{S}} \,-\, \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div};T} \qquad \forall\, T \,\in\, \mathcal{T}_{\mathrm{S},h}$$

*and*

$$\|f_{\mathrm{D}} \,-\, \mathrm{div}\; \mathbf{u}_{\mathrm{D},h}\|_{0,T} \,\leq\, \|\mathbf{u}_{\mathrm{D}} \,-\, \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div}\,;T} \qquad \forall\, T \,\in\, \mathcal{T}_{\mathrm{D},h}\,.$$

*Proof.* It suffices to recall, as established by Theorem 2.2, that $\mathbf{f}_{\mathrm{S}} \,=\, -\,\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}$ in $\Omega_{\mathrm{S}}$ and $f_{\mathrm{D}} \,=\, \mathrm{div}\;\mathbf{u}_{\mathrm{D}}$ in $\Omega_{\mathrm{D}}$. $\square$

In order to derive the upper bounds for the remaining terms defining the global a posteriori error estimator $\Theta$ (cf. (3.1)), we proceed similarly as in [8], using results from [15], [17] and [25], and apply Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. To this end, we now introduce further notations and preliminary results. Given $\mathcal{T} \,\in\, \{\mathcal{T}_h^{\mathrm{S}}, \mathcal{T}_h^{\mathrm{D}}\}$, $T \,\in\, \mathcal{T}$, and $e \,\in\, \mathcal{E}(T)$, we let $\phi_T$ and $\phi_e$ be the usual triangle-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [46]). In particular, $\phi_T$ satisfies $\phi_T \,\in\, \mathbb{P}_3(T)$, $\mathrm{supp}(\phi_T) \,\subseteq\, T$, $\phi_T \,=\, 0$ on $\partial T$, and $0 \,\leq\, \phi_T \,\leq\, 1$ in $T$. Similarly, $\phi_e|_T \,\in\, \mathbb{P}_2(T)$, $\mathrm{supp}(\phi_e) \,\subseteq\, w_e \,:=\, \cup\{T' \in \mathcal{T} \,:\, e \in \mathcal{E}(T')\}$, $\phi_e \,=\, 0$ on $\partial T\backslash e$, and $0 \,\leq\, \phi_e \,\leq\, 1$ in $w_e$. We also recall from [45] that, given $k \,\in\, \mathbb{N}\cup\{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \,\in\, \mathbb{P}_k(T)$ and $L(p)|_e \,=\, p \;\forall\, p \in \mathbb{P}_k(e)$. A corresponding vector version of $L$, that is the componentwise application of $L$, is denoted by $\mathbf{L}$. Additional properties of $\phi_T$, $\phi_e$, and $L$ are collected in the following lemma.

LEMMA 3.11 *Given $k \,\in\, \mathbb{N}\,\cup\,\{0\}$, there exist positive constants $c_1$, $c_2$ and $c_3$, depending only on $k$ and the shape regularity of the triangulations (minimum angle condition), such that for each triangle $T$ and $e \,\in\, \mathcal{E}(T)$, there hold*

$$\|q\|^2_{0,T} \,\leq\, c_1\, \|\phi_T^{1/2} q\|^2_{0,T} \qquad \forall\, q \,\in\, \mathbb{P}_k(T), \tag{3.20}$$

$$\|q\|^2_{0,e} \,\leq\, c_2\, \|\phi_e^{1/2} q\|^2_{0,e} \qquad \forall\, q \,\in\, \mathbb{P}_k(e), \tag{3.21}$$

*and*

$$\|\phi_e^{1/2} L(q)\|^2_{0,T} \,\leq\, c_3\, h_e \|q\|^2_{0,e} \qquad \forall\, q \,\in\, \mathbb{P}_k(e)\,. \tag{3.22}$$

*Proof.* See Lemma 1.3 in [45]. $\square$

The following inverse estimate for polynomials will also be used.

LEMMA 3.12 *Let $k$, $l$, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, there exists $c > 0$, depending only on $k$, $l$, $m$ and the shape regularity of the triangulations, such that for each triangle $T$ there holds*

$$|q|_{m,T} \,\leq\, c\, h_T^{l-m} |q|_{l,T}, \forall\, q \,\in\, \mathbb{P}_k(T)\,. \tag{3.23}$$

*Proof.* See Theorem 3.2.6 in [18]. □

In addition, we need to recall a discrete trace inequality, which establishes the existence of a positive constant $c$, depending only on the shape regularity of the triangulations, such that for each $T \,\in\, \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ and $e \in \mathcal{E}(T)$, there holds

$$\|v\|_{0,e}^2 \,\leq\, c \left\{ h_e^{-1}\, \|v\|_{0,T}^2 \,+\, h_e\, |v|_{1,T}^2 \right\} \qquad \forall\, v \in H^1(T)\,. \tag{3.24}$$

For a proof of inequality (3.24) we refer to Theorem 3.10 in [1] (see also eq. (2.4) in [6]).

The following lemma summarizes known efficiency estimates for ten terms defining $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$. In fact, their proofs, which apply the preliminary results described above, are already available in the literature (see, e.g. [8], [9], [15], [25], [26], [28]). From now on we assume, without loss of generality, that $\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h}$ is polynomial on each $T \in \mathcal{T}_h^{\mathrm{D}}$. Otherwise, additional higher order terms, given by the errors arising from suitable polynomial approximations, should appear in the corresponding bounds below, which explains the expression h.o.t. in (3.19).

LEMMA 3.13 *There exist positive constants $C_i$, $i \in \{1, ..., 10\}$, independent of $h$, such that*

a) $h_T^2\, \|\mathrm{rot}\, (\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \,\leq\, C_1\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \qquad \forall\, T \,\in\, \mathcal{T}_h^{\mathrm{D}}$,

b) $h_T^2\, \|\mathbf{rot}\, \boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \,\leq\, C_2\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall\, T \,\in\, \mathcal{T}_h^{\mathrm{S}}$,

c) $h_e\, \|[\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t}]\|_{0,e}^2 \,\leq\, C_3\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,w_e}^2 \qquad \forall\, e \,\in\, \mathcal{E}_h(\Omega_{\mathrm{D}})$, *where the set $w_e$ is given by* $w_e \,:=\, \cup \left\{ T' \in \mathcal{T}_h^{\mathrm{D}} \,:\, e \in \mathcal{E}(T') \right\}$,

d) $h_e\, \|[\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}]\|_{0,e}^2 \,\leq\, C_4\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,w_e}^2 \qquad \forall e \,\in\, \mathcal{E}_h(\Omega_{\mathrm{S}})$, *where the set $w_e$ is given by* $w_e \,:=\, \cup \left\{ T' \in \mathcal{T}_h^{\mathrm{S}} \,:\, e \in \mathcal{E}(T') \right\}$,

e) $h_e\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}\|_{0,e}^2 \,\leq\, C_5\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall e \,\in\, \mathcal{E}_h(\Gamma_{\mathrm{S}})$, *where $T$ is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having $e$ as an edge,*

f) $h_T^2\, \|\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \,\leq\, C_6 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 \,+\, h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \right\} \qquad \forall\, T \,\in\, \mathcal{T}_h^{\mathrm{D}}$,

g) $h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \,\leq\, C_7 \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \right\} \quad \forall\, T \,\in\, \mathcal{T}_h^{\mathrm{S}}$,

h) $h_e\, \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \,\leq\, C_8 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_e\, \|\lambda - \lambda_h\|_{0,e}^2 \right\} \;\forall e \in \mathcal{E}_h(\Sigma)$, *where $T$ is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having $e$ as an edge,*

i) $\displaystyle\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \left\| \mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t} \,+\, \frac{d\lambda_h}{d\mathbf{t}} \right\|_{0,e}^2 \,\leq\, C_9 \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 \,+\, \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \right\}$, *where, given $e \in \mathcal{E}_h(\Sigma)$, $T_e$ is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having $e$ as an edge, and*

j) $\displaystyle\sum_{e\in\mathcal{E}_h(\Gamma_{\rm S})} h_e \left\|\nu^{-1}\boldsymbol{\sigma}^d_{{\rm S},h}\mathbf{t} + \nabla\boldsymbol{\varphi}_h\mathbf{t}\right\|^2_{0,e} \le C_{10}\left\{\sum_{e\in\mathcal{E}_h(\Gamma_{\rm S})}\|\boldsymbol{\sigma}_{\rm S}-\boldsymbol{\sigma}_{{\rm S},h}\|^2_{0,T_e} + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{1/2,\Sigma}\right\}$,
*where, given $e\in\mathcal{E}_h(\Gamma_{\rm S})$, $T_e$ is the triangle of $\mathcal{T}_h^{\rm S}$ having $e$ as an edge.*

*Proof.* For a) and b) we refer to [15, Lemma 6.1]. Alternatively, a) and b) follow from straightforward applications of the technical result provided in [9, Lemma 4.3] (see also [28, Lemma 4.9]). Similarly, for c), d), and e) we refer to [15, Lemma 6.2] or apply the technical result given by [9, Lemma 4.4] (see also [28, Lemma 4.10]). Then, for f) and g) we refer to [15, Lemma 6.3] (see also [28, Lemma 4.13] or [25, Lemma 5.5]). On the other hand, the estimate given by h) corresponds to [8, Lemma 4.12]. In particular, its proof makes use of the discrete trace inequality (3.24). Finally, the proofs of i) and j) follow from very slight modifications of the proof of [25, Lemma 5.7]. Alternatively, an *elasticity version* of i) and j), which is provided in [26, Lemma 20], can also be adapted to our case. □

We find it important to remark that the estimates i) and j) in the previous lemma provide the only non-local bounds of the present efficiency analysis. However, under additional regularity assumptions on $\lambda$ and $\boldsymbol{\varphi}$, one is able to prove the following local bounds.

LEMMA 3.14 *Assume that $\lambda|_e\in H^1(e)$ for each $e\in\mathcal{E}_h(\Sigma)$, and that $\boldsymbol{\varphi}|_e\in\mathbf{H}^1(e)$ for each $e\in\mathcal{E}_h(\Gamma_{\rm S})$. Then there exist $\tilde C_9$, $\tilde C_{10}>0$, such that*

$$h_e\left\|\mathbf{K}^{-1}\mathbf{u}_{{\rm D},h}\cdot\mathbf{t} + \frac{d\lambda_h}{d\mathbf{t}}\right\|^2_{0,e} \le \tilde C_9\left\{\|\mathbf{u}_{\rm D}-\mathbf{u}_{{\rm D},h}\|^2_{0,T_e} + h_e\left\|\frac{d}{d\mathbf{t}}(\lambda-\lambda_h)\right\|^2_{0,e}\right\}\quad\forall\, e\in\mathcal{E}_h(\Sigma)\,,$$

*and*

$$h_e\left\|\nu^{-1}\boldsymbol{\sigma}^d_{{\rm S},h}\mathbf{t} + \nabla\boldsymbol{\varphi}_h\mathbf{t}\right\|^2_{0,e} \le \tilde C_{10}\left\{\|\boldsymbol{\sigma}_{\rm S}-\boldsymbol{\sigma}_{{\rm S},h}\|^2_{0,T_e} + h_e\left\|\frac{d}{d\mathbf{t}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\right\|^2_{0,e}\right\}\quad\forall\, e\in\mathcal{E}_h(\Gamma_{\rm S})\,.$$

*Proof.* Similarly as for i) and j) from Lemma 3.13, it follows by adapting the corresponding *elasticity version* from [26]. We omit details here and refer to [26, Lemma 21]. □

It remains to provide the efficiency estimates for three residual terms defined on the edges of the interface $\Sigma$. They have to do with the transmision conditions and with the trace equation $\mathbf{u}_{\rm S}+\boldsymbol{\varphi}=\mathbf{0}$ on $\Sigma$. More precisely, we have the following lemmas.

LEMMA 3.15 *There exists $C>0$, independent of $h$, such that for each $e\in\mathcal{E}_h(\Sigma)$, there holds*

$$h_e\|\mathbf{u}_{{\rm D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|^2_{0,e} \le C\left\{\|\mathbf{u}_{\rm D}-\mathbf{u}_{{\rm D},h}\|^2_{0,T} + h_T^2\|{\rm div}\,(\mathbf{u}_{\rm D}-\mathbf{u}_{{\rm D},h})\|^2_{0,T} + h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{0,e}\right\},$$

*where $T$ is the triangle of $\mathcal{T}_h^{\rm D}$ having $e$ as an edge.*

*Proof.* We proceed similarly as in [8, Lemma 4.7]. Given $e\in\mathcal{E}_h(\Sigma)$, we let $T$ be the triangle of $\mathcal{T}_h^{\rm D}$ having $e$ as an edge, and define $v_e := \mathbf{u}_{{\rm D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}$ on $e$. Then, applying (3.21), recalling that $\phi_e=0$ on $\partial T\backslash e$, extending $\phi_e\,L(v_e)$ by zero in $\Omega_{\rm D}\backslash T$ so that the resulting function belongs to $H^1(\Omega_{\rm D})$, and using that $\mathbf{u}_{\rm D}\cdot\mathbf{n}+\boldsymbol{\varphi}\cdot\mathbf{n}=0$ on $\Sigma$, we get

$$\begin{aligned}\|v_e\|^2_{0,e} &\le c_2\,\|\phi_e^{1/2}\,v_e\|^2_{0,e} = c_2\int_e\phi_e\,v_e\,(\mathbf{u}_{{\rm D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}) = c_2\,\langle\mathbf{u}_{{\rm D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n},\phi_e\,L(v_e)\rangle_\Sigma\\ &= c_2\,\langle\mathbf{u}_{{\rm D},h}\cdot\mathbf{n}-\mathbf{u}_{\rm D}\cdot\mathbf{n},\phi_e\,L(v_e)\rangle_\Sigma + c_2\,\langle\boldsymbol{\varphi}_h\cdot\mathbf{n}-\boldsymbol{\varphi}\cdot\mathbf{n},\phi_e\,L(v_e)\rangle_\Sigma\,,\end{aligned}\tag{3.25}$$

where $\langle\cdot,\cdot\rangle_\Sigma$ stands here for the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$. Next, integrating by parts in $\Omega_D$, and noting that $\big(\boldsymbol{\varphi}_h\cdot\mathbf{n}-\boldsymbol{\varphi}\cdot\mathbf{n}\big)\,\in\,L^2(\Sigma)$, we find, respectively, that

$$\langle\mathbf{u}_{D,h}\cdot\mathbf{n}\,-\,\mathbf{u}_D\cdot\mathbf{n},\phi_e\,L(v_e)\rangle_\Sigma\,=\,\int_T\nabla\big(\phi_e\,L(v_e)\big)\cdot(\mathbf{u}_{D,h}-\mathbf{u}_D)\,+\,\int_T\phi_e\,L(v_e)\,\mathrm{div}\,(\mathbf{u}_{D,h}-\mathbf{u}_D)\,,$$

and

$$\langle\boldsymbol{\varphi}_h\cdot\mathbf{n}\,-\,\boldsymbol{\varphi}\cdot\mathbf{n},\phi_e\,L(v_e)\rangle_\Sigma\,=\,\int_e\big(\boldsymbol{\varphi}_h\cdot\mathbf{n}\,-\,\boldsymbol{\varphi}\cdot\mathbf{n}\big)\,\phi_e\,v_e\,.$$

Thus, replacing the above expressions back into (3.25), applying the Cauchy-Schwarz inequality and the inverse estimate (3.23), and recalling that $0\le\phi_e\le 1$, we obtain

$$\|v_e\|_{0,e}^2\,\le\,C\Big\{h_T^{-1}\,\|\mathbf{u}_D-\mathbf{u}_{D,h}\|_{0,T}\,+\,\|\mathrm{div}\,(\mathbf{u}_D-\mathbf{u}_{D,h})\|_{0,T}\Big\}\|\phi_e L(v_e)\|_{0,T}\,+\,c\,\|v_e\|_{0,e}\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}.$$

But, using again that $0\le\phi_e\le 1$ and thanks to (3.22), we get

$$\|\phi_e\,L(v_e)\|_{0,T}\,\le\,\|\phi_e^{1/2}L(v_e)\|_{0,T}\,\le\,c_3^{1/2}\,h_e^{1/2}\,\|v_e\|_{0,e}\,, \tag{3.26}$$

whence the previous inequality yields

$$\|v_e\|_{0,e}\,\le\,C\,h_e^{1/2}\,\big\{h_T^{-1}\,\|\mathbf{u}_D-\mathbf{u}_{D,h}\|_{0,T}\,+\,\|\mathrm{div}\,(\mathbf{u}_D-\mathbf{u}_{D,h})\|_{0,T}\big\}\,+\,c\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}\,.$$

Finally, it is easy to see that this estimate and the fact that $h_e\le h_T$ imply the required upper bound for $h_e\,\|v_e\|_{0,e}^2$, which finishes the proof. □

LEMMA 3.16 *There exists $C>0$, independent of $h$, such that for each $e\,\in\,\mathcal{E}_h(\Sigma)$, there holds*

$$\begin{aligned}&h_e\,\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n}\,+\,\lambda_h\,\mathbf{n}\,-\,\frac{\nu}{\kappa}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\|_{0,e}^2\\&\quad\le\,C\,\Big\{\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0,T}^2\,+\,h_T^2\|\mathbf{div}(\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h})\|_{0,T}^2\,+\,h_e\,\|\lambda-\lambda_h\|_{0,e}^2\,+\,h_e\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2\Big\}\,,\end{aligned}$$

*where $T$ is the triangle of $\mathcal{T}_h^S$ having $e$ as an edge.*

*Proof.* We proceed as in the previous lemma (see also [8, Lemma 4.6]). Indeed, given $e\,\in\mathcal{E}_h(\Sigma)$, we let $T$ be the triangle of $\mathcal{T}_h^S$ having $e$ as an edge, and define $\mathbf{v}_e\,:=\,\boldsymbol{\sigma}_{S,h}\,\mathbf{n}+\lambda_h\,\mathbf{n}-\frac{\nu}{\kappa}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}$ on $e$. Then, applying (3.21), recalling that $\phi_e\,=\,0$ on $\partial T\backslash e$, extending $\phi_e\,\mathbf{L}(\mathbf{v}_e)$ by zero in $\Omega_S\backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega_S)$, using that $\boldsymbol{\sigma}_S\,\mathbf{n}\,+\,\lambda\,\mathbf{n}\,-\,\frac{\nu}{\kappa}\,(\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t}\,=\,0$ on $\Sigma$, and then integrating by parts in $\Omega_S$, we arrive at

$$\begin{aligned}\|\mathbf{v}_e\|_{0,e}^2\,\le\,c_2\,\|\phi_e^{1/2}\,\mathbf{v}_e\|_{0,e}^2\,&=\,c_2\,\int_e\phi_e\,\mathbf{v}_e\cdot\Big\{\boldsymbol{\sigma}_{S,h}\,\mathbf{n}\,+\,\lambda_h\,\mathbf{n}\,-\,\frac{\nu}{\kappa}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\Big\}\\&=\,c_2\,\int_T\nabla(\phi_e\,\mathbf{L}(\mathbf{v}_e)):(\boldsymbol{\sigma}_{S,h}-\boldsymbol{\sigma}_S)\,+\,c_2\,\int_T\phi_e\,\mathbf{L}(\mathbf{v}_e))\cdot\mathbf{div}(\boldsymbol{\sigma}_{S,h}-\boldsymbol{\sigma}_S)\\&+\,c_2\,\int_e\phi_e\,\mathbf{v}_e\cdot\Big\{(\lambda_h-\lambda)\,\mathbf{n}\,-\,\frac{\nu}{\kappa}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t}-\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t}\Big\}\,.\end{aligned}$$

Next, applying the Cauchy-Schwarz inequality and the inverse estimate (3.23), recalling that $0\le\phi_e\le 1$, and employing the vector version of (3.26), we deduce that

$$\begin{aligned}\|\mathbf{v}_e\|_{0,e}\,\le\,C\,h_e^{1/2}&\Big\{h_T^{-1}\,\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0.T}\,+\,\|\mathbf{div}(\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h})\|_{0,T}\Big\}\\&+\,C\,\Big\{\|\lambda-\lambda_h\|_{0,e}\,+\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}\Big\}\,,\end{aligned}$$

which easily yields the required estimate, thus finishing the proof. □

LEMMA 3.17 *There exists $C > 0$, independent of $h$, such that for each $e \in \mathcal{E}_h(\Sigma)$, there holds*

$$h_e \, \|\mathbf{u}_{\mathrm{S},h} \,+\, \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, C \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\},$$

*where $T$ is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having $e$ as an edge.*

*Proof.* Let $e \in \mathcal{E}_h(\Sigma)$ and let $T$ be the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having $e$ as an edge. We follow the proof of [8, Lemma 4.12] and obtain first an upper bound of $h_T^2 \, |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2$. Indeed, using that $\nabla \mathbf{u}_{\mathrm{S}} = \nu^{-1} \boldsymbol{\sigma}_{\mathrm{S}}^d$ in $\Omega_{\mathrm{S}}$ (cf. Theorem 2.2) and that $\mathbf{u}_{\mathrm{S},h}$ is constant in $T$, adding and substracting $\boldsymbol{\sigma}_{\mathrm{S},h}^d$, and then applying the estimate g) from Lemma 3.13, we deduce that

$$\begin{aligned} h_T^2 \, |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \,&=\, \frac{h_T^2}{\nu^2} \|\boldsymbol{\sigma}_{\mathrm{S}}^d\|_{0,T}^2 \,\leq\, C \, h_T^2 \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \right\} \\ &\leq\, C \, \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \right\}. \end{aligned} \tag{3.27}$$

Next, since $\boldsymbol{\varphi} \,=\, -\,\mathbf{u}_{\mathrm{S}}$ on $\Sigma$ (cf. Theorem 2.2), we find that

$$h_e \, \|\mathbf{u}_{\mathrm{S},h} \,+\, \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, 2 \, h_e \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,e}^2 \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\},$$

which, employing the discrete trace inequality (3.24) and the estimate (3.27), yields

$$\begin{aligned} h_e \, \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,&\leq\, C \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2 \, |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\} \\ &\leq\, C \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}, \end{aligned}$$

which completes the proof. □

We end this section by observing that the efficiency estimate (3.19) follows straightforwardly from Lemmas 3.10, 3.13, 3.15, 3.16, and 3.17. In particular, the terms $h_e \, \|\lambda - \lambda_h\|_{0,e}^2$ and $h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2$, which appear in Lemma 3.13 (item h)), 3.15, 3.16, and 3.17, are bounded as follows:

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\lambda - \lambda_h\|_{0,e}^2 \,\leq\, h \, \|\lambda - \lambda_h\|_{0,\Sigma}^2 \,\leq\, C \, h \, \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \,,$$

and

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, h \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \,\leq\, C \, h \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2 \,.$$

# 4 Numerical results

In [31, Section 5] we presented several numerical results illustrating the performance of the Galerkin scheme (2.11) with the subspaces $\mathbb{X}_h \,:=\, \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma)$ and $\mathbb{M}_h \,:=\, \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}})$ defined in Section 2.3. We now provide three examples confirming the reliability and efficiency of the respective a posteriori error estimator $\Theta$ derived in Section 3, and showing the behaviour of the associated adaptive algorithm.

In what follows, $N$ stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$, and the individual and total errors are defined by:

$$
\begin{array}{rclrcl}
\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) & := & \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}\,, & \mathsf{e}(\mathbf{u}_{\mathrm{S}}) & := & \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}\,, \\
\mathsf{e}(\mathbf{u}_{\mathrm{D}}) & := & \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}\,, & \mathsf{e}(p_{\mathrm{D}}) & := & \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,, \\
\mathsf{e}(\boldsymbol{\varphi}) & := & \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}\,, & \mathsf{e}(\lambda) & := & \|\lambda - \lambda_h\|_{1/2,\Sigma}\,,
\end{array}
$$

and

$$
\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,:=\, \Big\{ (\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{D}}))^2 + (\mathsf{e}(p_{\mathrm{D}}))^2 + (\mathsf{e}(\boldsymbol{\varphi}))^2 + (\mathsf{e}(\lambda))^2 \Big\}^{1/2}\,,
$$

whereas the effectivity index with respect to $\Theta$ is given by

$$
eff(\Theta) \,:=\, \mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})/\Theta\,,
$$

where

$$
(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,:=\, ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}})) \,\in\, \mathbb{X} \times \mathbb{M}
$$

and

$$
(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \,:=\, ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h), (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})) \,\in\, \mathbb{X}_h \times \mathbb{M}_h
$$

denote the unique solutions of (2.5) and (2.11), respectively.

Also, we let $r(\boldsymbol{\sigma}_{\mathrm{S}})$, $r(\mathbf{u}_{\mathrm{S}})$, $r(\mathbf{u}_{\mathrm{D}})$, $r(p_{\mathrm{D}})$, $r(\boldsymbol{\varphi})$, $r(\lambda)$, and $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ be the individual and global experimental rates of convergence given by

$$
r(\%) \,:=\, \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \,\in\, \big\{\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda\big\}\,,
$$

and

$$
r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,:=\, \frac{\log(\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})/\mathsf{e}'(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}))}{\log(h/h')}\,,
$$

where $h$ and $h'$ denote two consecutive meshsizes with errors $\mathsf{e}$ and $\mathsf{e}'$. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where $N$ and $N'$ denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose for simplicity $\nu = 1$, $\mathbf{K} = \mathrm{I}$, the identity matrix of $\mathbb{R}^{2\times 2}$, and $\kappa = 1$. Example 1 is employed to confirm the reliability and efficiency of the a posteriori error estimator $\Theta$. Then, Examples 2 and 3 are utilized to illustrate the behaviour of the associated adaptive algorithm, which applies the following procedure from [46]:

1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{D}} \cup \mathcal{T}_h^{\mathrm{S}}$.

2) Solve the discrete problem (2.11) for the actual mesh $\mathcal{T}_h$.

3) Compute $\Theta_{*,T}$ for each triangle $T \in \mathcal{T}_h^*$, $* \in \{\mathrm{D}, \mathrm{S}\}$.

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h^*$, $* \in \{\mathrm{D}, \mathrm{S}\}$, whose indicator $\Theta_{*,T'}$ satisfies

$$\Theta_{*T'} \geq \frac{1}{2} \max_{i \in \{\mathrm{D},\mathrm{S}\}} \left\{ \max \left\{ \Theta_{i,T} : \quad T \in \mathcal{T}_h^i \right\} \right\} .$$

6) Define resulting meshes as actual meshes $\mathcal{T}_h^{\mathrm{D}}$ and $\mathcal{T}_h^{\mathrm{S}}$, and go to step 2.

In Example 1 we consider the regions $\Omega_{\mathrm{D}} := ]-0.5, 0.5[^2$ and $\Omega_{\mathrm{S}} := ]-1, 1[^2 \setminus \bar{\Omega}_{\mathrm{D}}$, which yields a porous medium completely surrounded by a fluid, and choose the data $\mathbf{f}_{\mathrm{S}}$ and $f_{\mathrm{D}}$ so that the exact solution is given by the regular functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \begin{pmatrix} -2 \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_2) \\ \\ 2 \sin(\pi x_1) \sin^2(\pi x_2) \cos(\pi x_1) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}} ,$$

$$p_{\mathrm{S}}(\mathbf{x}) = x_1^3 e^{x_2} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}} ,$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = x_1^3 \sin(x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{D}} .$$

In Example 2 we consider $\Omega_{\mathrm{D}} := ]-1, 0[^2$ and let $\Omega_{\mathrm{S}}$ be the $L$-shaped domain given by $]-1, 1[^2 \setminus \bar{\Omega}_{\mathrm{D}}$, which yields a porous medium partially surrounded by a fluid. Then we choose the data $\mathbf{f}_{\mathrm{S}}$ and $f_{\mathrm{D}}$ so that the exact solution is given by

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \operatorname{curl}\left(0.1 \left(x_2^2 - 1\right)^2 \sin^2(\pi x_1)\right) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}} ,$$

$$p_{\mathrm{S}}(\mathbf{x}) = \frac{1}{100\,(x_1^2 + x_2^2) + 0.1} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}} ,$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = \left(\frac{x_1 + 1}{10}\right)^2 \sin^3(2\pi\,(x_2 + 0.5)) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{D}} .$$

Note that the fluid pressure $p_{\mathrm{S}}$ has high gradients around the origin.

Finally, in Example 3 we take $\Omega_{\mathrm{D}} := ]-1, 1[ \times ]-2, -1[$ and $\Omega_{\mathrm{S}} := ]-1, 1[^2 \setminus [0, 1]^2$, which yields a porous medium below a fluid, and choose the data $\mathbf{f}_{\mathrm{S}}$ and $f_{\mathrm{D}}$ so that the exact solution is given by

$$\mathbf{u}_{\mathrm{S}}(r, \theta) = \operatorname{curl}\left(0.1\, r^{5/3} \,(r^2 \cos^2(\theta) - 1)^2 \,(r \sin(\theta) - 1)^2 \sin^2\left(\frac{2\theta - \pi}{3}\right)\right) \quad \forall\, (r, \theta) \in \Omega_{\mathrm{S}} ,$$

$$p_{\mathrm{S}}(\mathbf{x}) = 0.1\, x_1 \sin(x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}} ,$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = 0.1\,(x_2 + 2)^2 \sin^3(\pi x_1) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{D}} .$$

Note that $\mathbf{u}_{\mathrm{S}}$ is defined in polar coordinates and that its derivatives are singular at the origin.

The numerical results shown below were obtained using a MATLAB code. In Table 4.1 we summarize the convergence history of the mixed finite element method (2.11), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain. We observe there,

Table 4.1: EXAMPLE 1, quasi-uniform scheme

| $N$ | $h$ | $\mathbf{e}(\boldsymbol{\sigma}_S)$ | $r(\boldsymbol{\sigma}_S)$ | $\mathbf{e}(\mathbf{u}_S)$ | $r(\mathbf{u}_S)$ | $\mathbf{e}(\mathbf{u}_D)$ | $r(\mathbf{u}_D)$ | $\mathbf{e}(p_D)$ | $r(p_D)$ |
|---|---|---|---|---|---|---|---|---|---|
| 321 | 0.5000 | 35.4015 | – | 0.6875 | – | 0.1996 | – | 0.0117 | – |
| 1201 | 0.2500 | 20.0107 | 0.8647 | 0.4266 | 0.7234 | 0.1121 | 0.8743 | 0.0057 | 1.0798 |
| 4641 | 0.1250 | 10.0700 | 1.0160 | 0.1615 | 1.4370 | 0.0531 | 1.1046 | 0.0023 | 1.3213 |
| 18241 | 0.0625 | 5.0492 | 1.0087 | 0.0801 | 1.0238 | 0.0259 | 1.0490 | 0.0011 | 1.0967 |
| 72321 | 0.0312 | 2.5268 | 1.0052 | 0.0401 | 1.0064 | 0.0129 | 1.0178 | 0.0005 | 1.0234 |
| 288001 | 0.0156 | 1.2637 | 1.0029 | 0.0200 | 1.0031 | 0.0064 | 1.0062 | 0.0003 | 1.0062 |

| $N$ | $h$ | $\mathbf{e}(\boldsymbol{\varphi})$ | $r(\boldsymbol{\varphi})$ | $\mathbf{e}(\lambda)$ | $r(\lambda)$ | $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $\Theta$ | $eff(\Theta)$ |
|---|---|---|---|---|---|---|---|---|---|
| 321 | 0.5000 | 4.2653 | – | 0.0981 | – | 35.6649 | – | 39.0015 | 0.9144 |
| 1201 | 0.2500 | 4.3919 | – | 0.0973 | 0.0124 | 20.4920 | 0.8399 | 22.6847 | 0.9033 |
| 4641 | 0.1250 | 1.7410 | 1.3690 | 0.0537 | 0.8781 | 10.2209 | 1.0292 | 11.1965 | 0.9129 |
| 18241 | 0.0625 | 0.8088 | 1.1202 | 0.0259 | 1.0670 | 5.1144 | 1.0117 | 5.5954 | 0.9140 |
| 72321 | 0.0312 | 0.3949 | 1.0408 | 0.0126 | 1.0516 | 2.5579 | 1.0060 | 2.7969 | 0.9145 |
| 288001 | 0.0156 | 0.1962 | 1.0123 | 0.0062 | 1.0266 | 1.2791 | 1.0031 | 1.3982 | 0.9148 |

looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 2.4 (when $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index $eff(\Theta)$ remains always in a neighborhood of 0.91, which illustrates the reliability and efficiency of $\Theta$ in the case of a regular solution.

Next, in Tables 4.2 - 4.5 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3. We observe that the errors of the adaptive procedures decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figures 4.1 and 4.3 where we display the total errors $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. the degrees of freedom $N$ for both refinements. As shown by the values of $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of $\Theta$ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figures 4.2 and 4.4. Note that the method is able to recognize the region with high gradients in Example 2, and the singularity of the solution in Example 3.

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Table 4.2: EXAMPLE 2, quasi-uniform scheme

| $N$ | $h$ | $\mathbf{e}(\boldsymbol{\sigma}_\mathrm{S})$ | $\mathbf{e}(\mathbf{u}_\mathrm{S})$ | $\mathbf{e}(\mathbf{u}_\mathrm{D})$ | $\mathbf{e}(p_\mathrm{D})$ | $\mathbf{e}(\boldsymbol{\varphi})$ | $\mathbf{e}(\lambda)$ |
|---|---|---|---|---|---|---|---|
| 608 | 0.3536 | 4.5187 | 0.1198 | 0.2649 | 0.0184 | 0.5760 | 0.1120 |
| 2332 | 0.1768 | 4.9963 | 0.0529 | 0.1520 | 0.0035 | 0.2653 | 0.0347 |
| 9140 | 0.0884 | 6.7481 | 0.0253 | 0.0778 | 0.0005 | 0.1485 | 0.0096 |
| 36196 | 0.0442 | 4.2857 | 0.0125 | 0.0392 | 0.0002 | 0.0771 | 0.0042 |
| 144068 | 0.0221 | 2.4834 | 0.0062 | 0.0196 | 0.0001 | 0.0348 | 0.0022 |

| $N$ | $h$ | $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $\Theta$ | $eff(\Theta)$ |
|---|---|---|---|---|---|
| 608 | 0.3536 | 4.5660 | – | 5.4033 | 0.8450 |
| 2332 | 0.1768 | 5.0060 | – | 5.2805 | 0.9480 |
| 9140 | 0.0884 | 6.7503 | – | 6.8230 | 0.9894 |
| 36196 | 0.0442 | 4.2866 | 0.6599 | 4.3158 | 0.9932 |
| 144068 | 0.0221 | 2.4837 | 0.7901 | 2.4958 | 0.9952 |

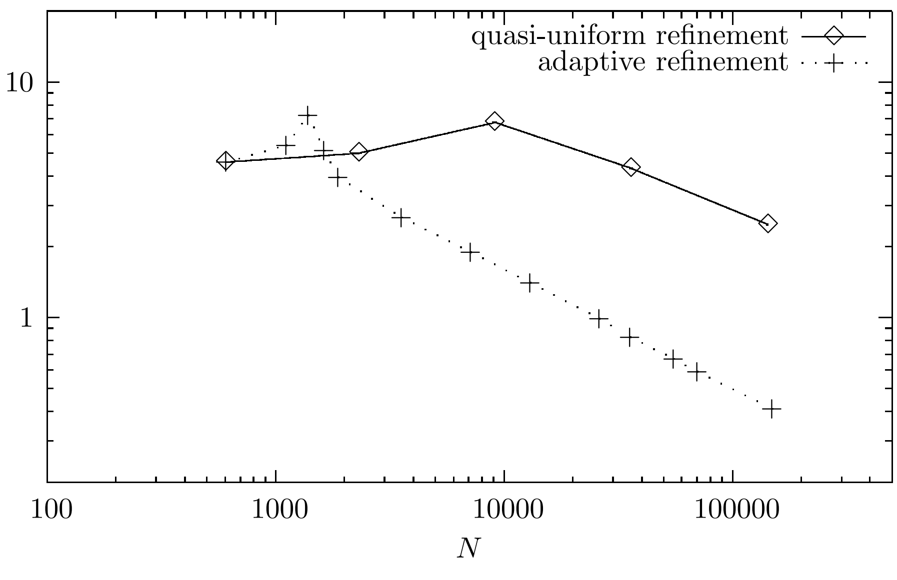


Figure 4.1: EXAMPLE 2, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. $N$ for quasi-uniform/adaptive schemes

Table 4.3: EXAMPLE 2, adaptive scheme

| $N$ | $\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{D}})$ | $\mathbf{e}(p_{\mathrm{D}})$ | $\mathbf{e}(\boldsymbol{\varphi})$ | $\mathbf{e}(\lambda)$ |
|---|---|---|---|---|---|---|
| 608 | 4.5188 | 0.1199 | 0.2649 | 0.0184 | 0.5760 | 0.1121 |
| 1118 | 5.3792 | 0.0709 | 0.2262 | 0.0091 | 0.3185 | 0.0402 |
| 1391 | 7.2290 | 0.0661 | 0.2098 | 0.0082 | 0.2846 | 0.0215 |
| 1636 | 5.1151 | 0.0657 | 0.2094 | 0.0110 | 0.2591 | 0.0236 |
| 1884 | 3.9177 | 0.0657 | 0.2093 | 0.0108 | 0.2577 | 0.0229 |
| 3558 | 2.6519 | 0.0491 | 0.2020 | 0.0037 | 0.1626 | 0.0128 |
| 7164 | 1.8814 | 0.0320 | 0.1751 | 0.0067 | 0.1160 | 0.0171 |
| 13073 | 1.3945 | 0.0237 | 0.1591 | 0.0034 | 0.0742 | 0.0109 |
| 26227 | 0.9771 | 0.0165 | 0.1222 | 0.0030 | 0.0730 | 0.0103 |
| 35611 | 0.8163 | 0.0140 | 0.1089 | 0.0018 | 0.0384 | 0.0075 |
| 55318 | 0.6608 | 0.0114 | 0.0808 | 0.0005 | 0.0375 | 0.0039 |
| 70434 | 0.5825 | 0.0099 | 0.0747 | 0.0005 | 0.0357 | 0.0038 |
| 149402 | 0.4052 | 0.0070 | 0.0548 | 0.0003 | 0.0208 | 0.0023 |

| $N$ | $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $\Theta$ | $eff(\Theta)$ |
|---|---|---|---|---|
| 608 | 4.5660 | – | 5.4033 | 0.8450 |
| 1118 | 5.3940 | – | 5.7977 | 0.9304 |
| 1391 | 7.2379 | – | 7.4956 | 0.9656 |
| 1636 | 5.1264 | 4.2524 | 5.4334 | 0.9435 |
| 1884 | 3.9324 | 3.7572 | 4.3145 | 0.9114 |
| 3558 | 2.6650 | 1.2238 | 2.9662 | 0.8985 |
| 7164 | 1.8934 | 0.9768 | 2.0913 | 0.9054 |
| 13073 | 1.4057 | 0.9902 | 1.5394 | 0.9132 |
| 26227 | 0.9876 | 1.0142 | 1.0951 | 0.9018 |
| 35611 | 0.8246 | 1.1796 | 0.9191 | 0.8972 |
| 55318 | 0.6669 | 0.9637 | 0.7388 | 0.9026 |
| 70434 | 0.5885 | 1.0359 | 0.6505 | 0.9046 |
| 149402 | 0.4095 | 0.9644 | 0.4550 | 0.8999 |

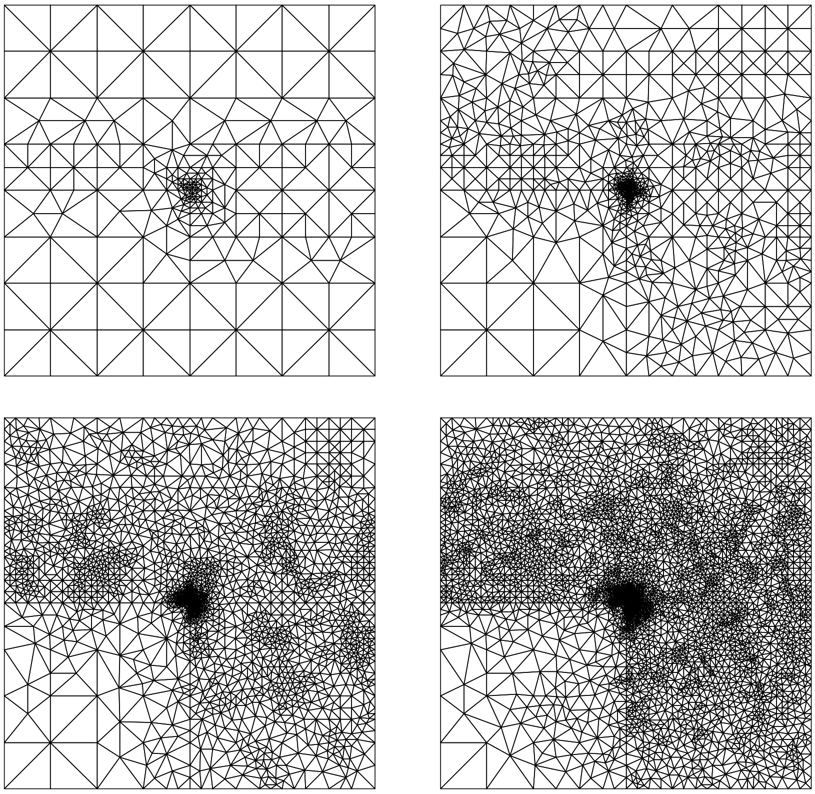

Figure 4.2: EXAMPLE 2, adapted meshes with 1884, 7164, 26227, and 55318 degrees of freedom

Table 4.4: EXAMPLE 3, quasi-uniform scheme

| $N$ | $h$ | $\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{D}})$ | $\mathbf{e}(p_{\mathrm{D}})$ | $\mathbf{e}(\boldsymbol{\varphi})$ | $\mathbf{e}(\lambda)$ |
|---|---|---|---|---|---|---|---|
| 344 | 0.5000 | 16.8563 | 0.4452 | 0.7130 | 0.0674 | 1.8109 | 0.1615 |
| 1324 | 0.2500 | 11.3317 | 0.3329 | 0.3846 | 0.0130 | 2.5160 | 0.0826 |
| 5204 | 0.1250 | 7.0011 | 0.0849 | 0.1980 | 0.0038 | 0.8665 | 0.0458 |
| 20644 | 0.0625 | 4.4530 | 0.0412 | 0.0992 | 0.0018 | 0.3859 | 0.0203 |
| 82244 | 0.0312 | 2.8037 | 0.0206 | 0.0496 | 0.0009 | 0.1877 | 0.0097 |

| $N$ | $h$ | $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $\Theta$ | $eff(\Theta)$ |
|---|---|---|---|---|---|
| 344 | 0.5000 | 16.9751 | – | 18.8901 | 0.8986 |
| 1324 | 0.2500 | 11.6191 | 0.5626 | 13.1132 | 0.8861 |
| 5204 | 0.1250 | 7.0579 | 0.7284 | 7.8041 | 0.9044 |
| 20644 | 0.0625 | 4.4711 | 0.6626 | 5.0014 | 0.8940 |
| 82244 | 0.0312 | 2.8105 | 0.6717 | 3.1653 | 0.8879 |

Table 4.5: EXAMPLE 3, adaptive scheme

| $N$ | $\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{S}})$ | $\mathbf{e}(\mathbf{u}_{\mathrm{D}})$ | $\mathbf{e}(p_{\mathrm{D}})$ | $\mathbf{e}(\boldsymbol{\varphi})$ | $\mathbf{e}(\lambda)$ |
|---|---|---|---|---|---|---|
| 344 | 16.8564 | 0.4453 | 0.7131 | 0.0675 | 1.8109 | 0.1616 |
| 684 | 11.8048 | 0.3406 | 0.5828 | 0.0177 | 2.5165 | 0.0863 |
| 1367 | 10.6242 | 0.1330 | 0.4426 | 0.0099 | 0.8682 | 0.0530 |
| 1625 | 10.4486 | 0.1314 | 0.4426 | 0.0099 | 0.8682 | 0.0530 |
| 1863 | 10.3440 | 0.1278 | 0.4426 | 0.0098 | 0.8678 | 0.0530 |
| 2291 | 9.2480 | 0.1173 | 0.4427 | 0.0097 | 0.8672 | 0.0526 |
| 3109 | 7.5456 | 0.1013 | 0.4425 | 0.0098 | 0.8670 | 0.0522 |
| 11719 | 3.9053 | 0.0530 | 0.3296 | 0.0072 | 0.3875 | 0.0271 |
| 34611 | 2.2713 | 0.0202 | 0.2614 | 0.0058 | 0.1901 | 0.0092 |
| 60159 | 1.7281 | 0.0153 | 0.1723 | 0.0034 | 0.1759 | 0.0083 |
| 79482 | 1.5031 | 0.0111 | 0.1644 | 0.0032 | 0.1154 | 0.0072 |
| 115241 | 1.2620 | 0.0167 | 0.1498 | 0.0019 | 0.1101 | 0.0055 |
| 182014 | 0.9954 | 0.0130 | 0.1226 | 0.0012 | 0.0900 | 0.0027 |

| $N$ | $\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ | $\Theta$ | $eff(\Theta)$ |
|---|---|---|---|---|
| 344 | 16.9751 | – | 18.8901 | 0.8986 |
| 684 | 12.0893 | 0.9877 | 13.6112 | 0.8882 |
| 1367 | 10.6698 | 0.3608 | 11.3264 | 0.9420 |
| 1625 | 10.4949 | 0.1912 | 11.1221 | 0.9436 |
| 1863 | 10.3907 | 0.1460 | 10.8244 | 0.9599 |
| 2291 | 9.3000 | 1.0724 | 9.9113 | 0.9383 |
| 3109 | 7.6090 | 1.3146 | 8.2092 | 0.9269 |
| 11719 | 3.9388 | 1.0924 | 4.2413 | 0.9362 |
| 34611 | 2.2943 | 0.9981 | 2.4691 | 0.9292 |
| 60159 | 1.7456 | 0.9889 | 1.8902 | 0.9235 |
| 79482 | 1.5165 | 1.0102 | 1.5941 | 0.9513 |
| 115241 | 1.2757 | 0.9309 | 1.3418 | 0.9507 |
| 182014 | 1.0070 | 1.0350 | 1.0817 | 0.9309 |

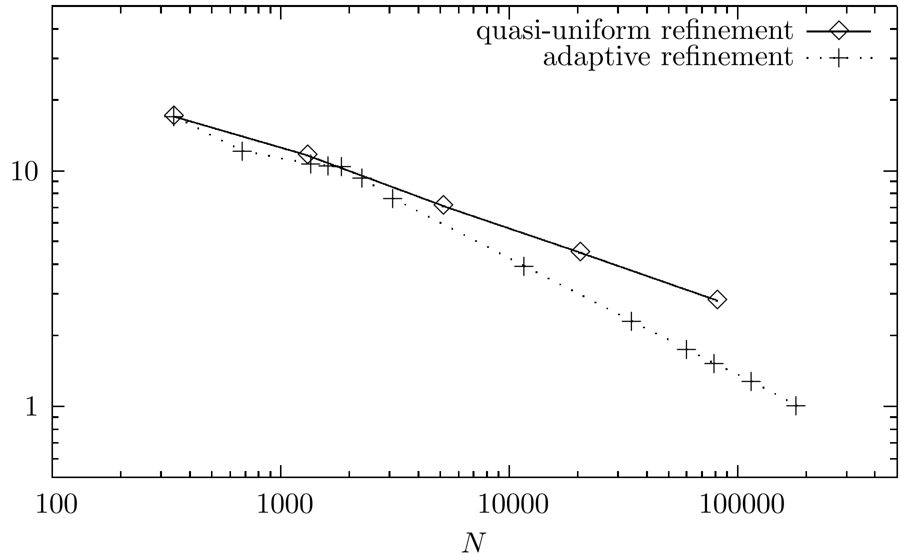


Figure 4.3: EXAMPLE 3, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. $N$ for quasi-uniform/adaptive schemes


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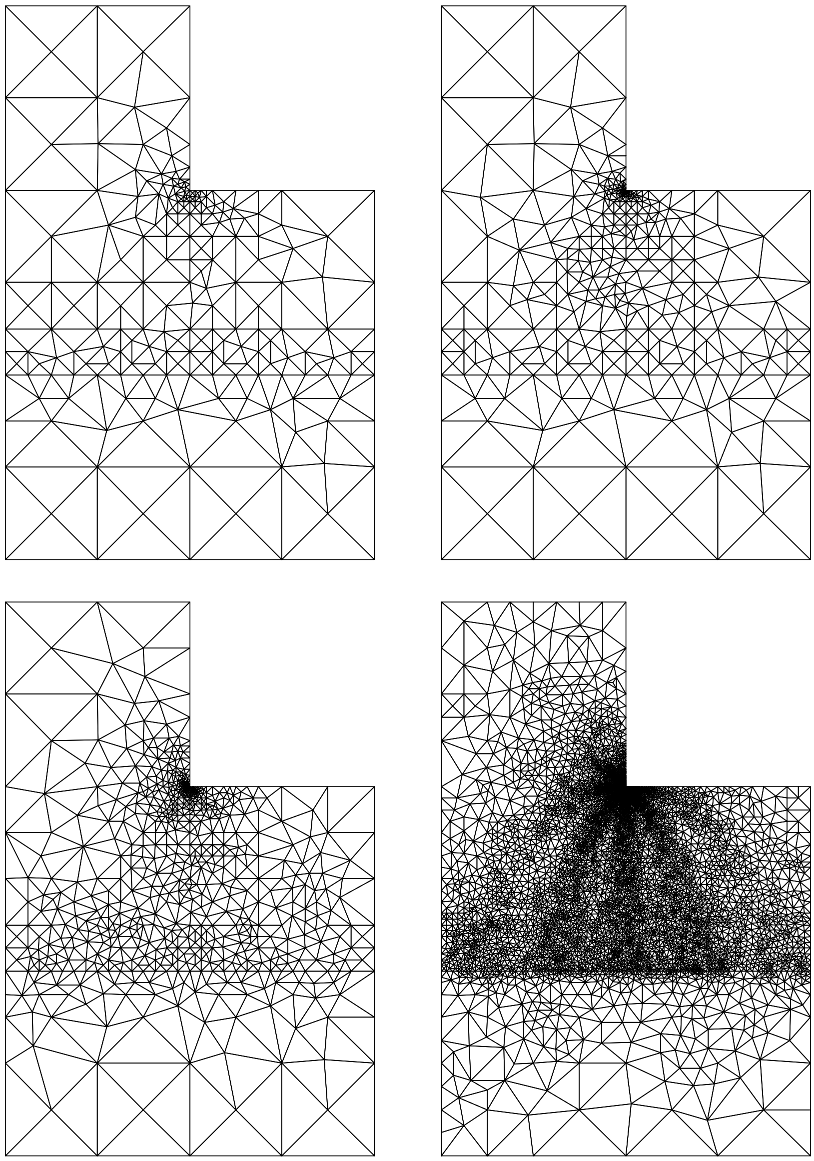

Figure 4.4: Example 3, adapted meshes with 1863, 3109, 11719, and 60159 degrees of freedom

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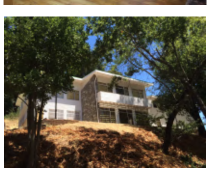
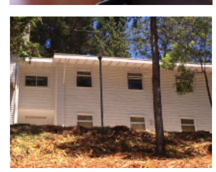


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