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A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow

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Abstract

In this paper we develop the a priori and a posteriori error analyses of a mixed finite element method for the coupling of fluid flow with nonlinear porous media flow. Flows are governed by the Stokes and nonlinear Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We consider dual-mixed formulations in both domains, and, in order to handle the nonlinearity involved, we introduce the pressure gradient in the Darcy region as an auxiliary unknown. In addition, the transmission conditions become essential, which leads to the introduction of the traces of the porous media pressure and the fluid velocity as the associated Lagrange multipliers. As a consequence, the resulting variational formulation can be written, conveniently, as a twofold saddle point operator equation. Thus, a well known generalization of the classical Babuška-Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations and to derive the corresponding a-priori error estimate. In particular, the set of feasible finite element subspaces includes Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with piecewise constant vectors for the Darcy pressure gradient and continuous piecewise linear elements for the traces. Then, we employ classical approaches and use known estimates to derive a reliable and efficient residual-based a posteriori error estimator for the coupled problem. Finally, several numerical results confirming the good performance of the method and the theoretical properties of the a posteriori error estimator, and illustrating the capability of the corresponding adaptive algorithm to localize the singularities of the solution, are reported.

Key words: mixed finite element, Stokes equation, nonlinear Darcy equation, a posteriori error analysis

Mathematics Subject Classifications (1991): 65N15, 65N30, 74F10, 74S05

1 Introduction

The development of appropriate numerical methods for the coupling of fluid flow (modeled by the Stokes equation) with porous media flow (modeled by the Darcy equation) has become a very

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active research area in recent years (see, e.g. [10], [15], [16], [17], [18], [23], [31], [32], [37], [38], [41] and the references therein). The above list includes porous media with cracks, the incorporation of the Brinkman equation in the model, and nonlinear problems. In particular, a mixed finite element method for a nonlinear Stokes-Darcy flow problem is introduced and analized in [17]. The fluid, being considered non-Newtonian in both domains, is modeled there by the generalized nonlinear Stokes equation in the free flow region and by the generalized nonlinear Darcy equation in the porous medium. In addition, the approach in [17] employs the primal method in the Stokes domain and the dual-mixed method in the Darcy region, which means that only the original velocity and pressure unknowns are considered in the fluid, whereas a further unknown (velocity) is added in the porous medium. The corresponding interface conditions are given, as usual lately, by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. Further, since one of these conditions becomes essential, the trace of the Darcy pressure on the interface needs also to be incorporated as an additional Lagrange multiplier.

On the other hand, in the recent paper [26] we have developed the a priori error analysis of a new fully-mixed variational formulation for the 2D Stokes-Darcy coupled problem. This approach allows, on the one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. More precisely, in [26] we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, and the traces of the porous media pressure and the fluid velocity on the interface, as the resulting unknowns. The pressure and the velocity gradient in the fluid can then be computed as a very simple postprocess, in which no numerical differentiation is applied, and hence no further sources of error arise.

Now, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local indicators θ_T defined on each element T of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\theta}$ is said to be efficient (resp. reliable) if there exists $C_{\mathtt{eff}} > 0$ (resp. $C_{\mathtt{rel}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|\text{error}\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. In spite of the many contributions available in the literature on the posteriori error analysis for variational formulations with saddle-point structure, the first results concerning the Stokes-Darcy coupled problem have been provided only in [8], where a reliable and efficient residual-based a posteriori error estimator for the variational formulation analyzed in [23] is derived. More recently, and following some of the techniques from [8] together with classical approaches, a reliable and efficient residual-based a posteriori error estimator for the fully-mixed variational method introduced in [26] was provided in [27].

The purpose of the present paper is to extend the results from [26] and [27] to the case of a nonlinear Stokes-Darcy coupled problem. More precisely, we develop the a priori and a posteriori error analyses of the fully mixed formulation from [26], as applied to the coupling of fluid flow with nonlinear porous media flow, where the nonlinearity in the latter region is given by the corresponding permeability. For this purpose, we consider a dual-mixed formulation in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity, the pressure and its gradient in the porous medium, as the main unknowns. Moreover, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous medium pressure and the fluid velocity as the corresponding Lagrange multipliers. As in [26], the remaining unknowns of physical interest can then be computed as a very simple postprocess that makes no use of any numerical differentiation procedure. Then, the corresponding variational formulation can be written as a two-fold saddle point operator equation, and hence the generalization of the Babuška-Brezzi theory developed in [20] is applied to prove the well-posedness of the continuous and discrete schemes. Furthermore, using some well known approaches (see, e.g. [1], [2], [3], [11], [13], [14], [22], [30], [33], [34], [36], [39], and the references therein), we derive a reliable and efficient residual-based a posteriori error estimator for our nonliner coupled problem. The proof of reliability makes use of a global inf-sup condition for a linearized version of the problem, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator.

The rest of this work is organized as follows. In Section 2 we introduce the model problem, show that the resulting variational formulation can be written as a two-fold saddle-point operator equation, introduce an equivalent formulation, which is easier to analyze, and collect the main results of the generalized Babuška-Brezzi theory developed in [20] (see also [28]). This abstract framework is then applied in Section 3 to prove the unique solvability of the equivalent formulation, which in turn yields the well posedness of our continuous problem. Next, in Section 4 we define the Galerkin scheme and derive general hypotheses on the finite element subspaces ensuring that the discrete scheme becomes well posed. A specific choice of finite element subspaces satisfying these assumptions, namely Raviart-Thomas of lowest order and piecewise constants on both domains, and piecewise linears on the interface, is described in Section 5. In Section 6 we derive the a posteriori error estimator and prove its reliability and efficiency. Finally, the numerical results are presented in Section 7.

We end this section with some notations to be used below. In particular, in what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) \,:=\, [H^r(\mathcal{O})]^2\,, \quad \mathbb{H}^r(\mathcal{O}) \,:=\, [H^r(\mathcal{O})]^{2\times 2}\,, \quad \text{and} \quad \mathbf{H}^r(\Gamma) \,:=\, [H^r(\Gamma)]^2\,.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) \,:=\, \big\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\;\mathbf{w} \in L^2(\mathcal{O})\big\}\,,$$

is standard in the realm of mixed problems (see, e.g. [12]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div}\,;\mathcal{O}}$, respectively. On the other hand, the symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda\rangle_\Gamma \,:=\, \int_\Gamma \xi\,\lambda \quad \forall\, \xi,\, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda}\rangle_\Gamma \,:=\, \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^2$, $\mathbb{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The continuous problem

2.1 Statement of the model problem

In order to describe the geometry, we let Ω_{S} and Ω_{D} be bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \backslash \bar{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \backslash \bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Figure 2.1 below).

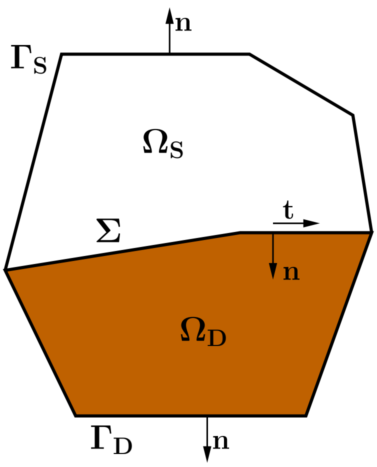

Figure 2.1: The domains for our 2D Stokes–Darcy model

The model consists of two separate groups of equations and a set of coupling terms. In Ω_{S}, the governing equations are those of the Stokes problem, which are written in the following velocity-pressure-pseudostress formulation:

$$\begin{gathered} \boldsymbol{\sigma}_{\mathrm{S}} = -p_{\mathrm{S}}\mathbf{I} + \nu\nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}, \\ \operatorname{div}\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}, \end{gathered} \tag{2.1}$$

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_{\mathrm{S}}$ is the fluid velocity, p_{S} is the pressure, $\boldsymbol{\sigma}_{\mathrm{S}}$ is the pseudostress tensor, $\mathbf{I}$ is the 2×2 identity matrix, $\mathbf{f}_{\mathrm{S}}$ are known source terms, and $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor. Now, using that $\operatorname{tr}(\nabla\mathbf{u}_{\mathrm{S}}) = \operatorname{div}\mathbf{u}_{\mathrm{S}} = 0$

in Ω_{S}, we notice that the equations in (2.1) can be rewritten equivalently as

$$\begin{gathered}\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S}}^{d} \,=\, \nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \,+\, \mathbf{f}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\ p_{\mathrm{S}} \,=\, -\,\tfrac{1}{2}\,\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,,\end{gathered} \tag{2.2}$$

where tr stands for the usual trace of tensors, that is $\mathrm{tr}\,\boldsymbol{\tau} \,:=\, \tau_{11} \,+\, \tau_{22}$, and

$$\boldsymbol{\tau}^{d} \,:=\, \boldsymbol{\tau} \,-\, \tfrac{1}{2}\,(\mathrm{tr}\,\boldsymbol{\tau})\,\mathbf{I}$$

is the deviatoric part of the tensor $\boldsymbol{\tau}$. On the other hand, in Ω_{D} we consider the following nonlinear Darcy model:

$$\begin{gathered}\mathbf{u}_{\mathrm{D}} \,=\, -\,\boldsymbol{\kappa}\,(\cdot, |\nabla\, p_{\mathrm{D}}|)\,\nabla\, p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \mathrm{div}\,\mathbf{u}_{\mathrm{D}} \,=\, f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \\ \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,,\end{gathered} \tag{2.3}$$

where $\mathbf{u}_{\mathrm{D}}$ and p_{D} denote the velocity and pressure, respectively, $\boldsymbol{\kappa} : \Omega_{\mathrm{D}}\times\mathbb{R}^{+} \to \mathbb{R}$ is a nonlinear operator representing the porous medium permeability, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2$, and f_{D} are known source terms satisfying $\displaystyle\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} \,=\, 0$. Throughout the paper we assume that $\boldsymbol{\kappa} \,\in\, C^1(\Omega_{\mathrm{D}}\times\mathbb{R}^{+})$ and that there exist constants $k_0,\ k_1 \,>\, 0$ such that for all $(x,\rho) \,\in\, \Omega_{\mathrm{D}}\times\mathbb{R}^{+}$:

$$\begin{aligned}& k_0 \,\le\, \boldsymbol{\kappa}(x,\rho) \,\le\, k_1\,, \\ & k_0 \,\le\, \boldsymbol{\kappa}(x,\rho) \,+\, \rho\frac{\partial}{\partial\rho}\boldsymbol{\kappa}(x,\rho) \,\le\, k_1\,, \quad \text{and} \\ & |\nabla_x\boldsymbol{\kappa}(x,\rho)| \,\le\, k_1\,.\end{aligned} \tag{2.4}$$

In order to handle the nonlinearity in Ω_{D} we proceed as in [20] (see also [25] and [28]), and introduce the additional unknown $\mathbf{t}_{\mathrm{D}} \,:=\, \nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}$. In this way, the Darcy model is rewritten as follows:

$$\begin{gathered}\mathbf{t}_{\mathrm{D}} \,=\, \nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \mathbf{u}_{\mathrm{D}} + \boldsymbol{\kappa}\,(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}} \,=\, \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \\ \mathrm{div}\,\mathbf{u}_{\mathrm{D}} \,=\, f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,.\end{gathered} \tag{2.5}$$

Finally, the transmission conditions on Σ are given by

$$\begin{aligned}\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} \,&=\, \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} && \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} \,+\, \nu\,\kappa_f^{-1}\,(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t})\,\mathbf{t} \,&=\, -\,p_{\mathrm{D}}\,\mathbf{n} && \text{on} \quad \Sigma\,,\end{aligned} \tag{2.6}$$

where κ_f, the friction coefficient, is assumed to be constant.

2.2 The dual-mixed formulation

Let us first introduce further notations. In what follows, given $\star \in \{\mathrm{S},\mathrm{D}\}$, we denote

$$(u,v)_\star \,:=\, \int_{\Omega_\star} u\,v, \qquad (\mathbf{u},\mathbf{v})_\star \,:=\, \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star \,:=\, \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}\,,$$

where $\boldsymbol{\sigma} : \boldsymbol{\tau} \,=\, \mathrm{tr}(\boldsymbol{\sigma}^t\boldsymbol{\tau}) \,=\, \displaystyle\sum_{ij=1}^{2} \sigma_{ij}\tau_{ij}$.

The unknows in the dual-mixed formulation will be the unknows of (2.2) without the pressure p_{S} and the three unknows in (2.5). Hence, the corresponding spaces will be:

$$\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}), \quad \mathbf{u}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}}), \quad \mathbf{t}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})\,, \quad \mathbf{u}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \quad p_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})\,,$$

where

$$\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) := \{\mathbf{v} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}}): \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\}\,.$$

In addition, we will need to define two unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1/2}_{00}(\Sigma)\,, \qquad \lambda := p_{\mathrm{D}} \in H^{1/2}(\Sigma)\,, \tag{2.7}$$

where $\mathbf{H}^{1/2}_{00}(\Sigma) := H^{1/2}_{00}(\Sigma) \times H^{1/2}_{00}(\Sigma)$ and

$$H^{1/2}_{00}(\Sigma) := \Big\{ v|_{\Sigma}: \quad v \in H^1(\Omega_{\mathrm{S}})\,, \quad v = 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \Big\}.$$

Equivalently, if $E_{0,\mathrm{S}} : H^{1/2}(\Sigma) \to L^2(\partial\Omega_{\mathrm{S}})$ is the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) := \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_{\mathrm{S}} \end{cases} \qquad \forall\,\psi \in H^{1/2}(\Sigma)\,,$$

we have that

$$H^{1/2}_{00}(\Sigma) = \Big\{ \psi \in H^{1/2}(\Sigma): \quad E_{0,\mathrm{S}}(\psi) \in H^{1/2}(\partial\Omega_{\mathrm{S}}) \Big\}\,,$$

endowed with the norm $\|\psi\|_{1/2,00,\Sigma} := \|E_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_{\mathrm{S}}}$. The dual space of $\mathbf{H}^{1/2}_{00}(\Sigma)$ is denoted by $\mathbf{H}^{-1/2}_{00}(\Sigma)$. Note that, in principle, the spaces for $\mathbf{u}_{\mathrm{S}}$ and p_{D} do not allow enough regularity for the traces $\boldsymbol{\varphi}$ and λ to exist. However, solutions of (2.2) and (2.5) have these unknowns in $\mathbf{H}^1(\Omega_{\mathrm{S}})$ and $H^1(\Omega_{\mathrm{D}})$ respectively.

Next, for the derivation of the weak formulation of (2.2)-(2.5)-(2.6), we begin by testing the first equations of (2.2) and (2.5) with arbitrary $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$, respectively. Thus, integrating by parts, and using the identity $\boldsymbol{\sigma}^d_{\mathrm{S}} : \boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\sigma}^d_{\mathrm{S}} : \boldsymbol{\tau}^d_{\mathrm{S}}$, we obtain

$$\nu^{-1}\,(\boldsymbol{\sigma}^d_{\mathrm{S}},\boldsymbol{\tau}^d_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})_{\mathrm{S}} + \langle\boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\varphi}\rangle_{\Sigma} = 0 \qquad \forall\,\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\,, \tag{2.8}$$

and

$$(\mathbf{t}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + (\mathrm{div}\,\mathbf{v}_{\mathrm{D}},p_{\mathrm{D}})_{\mathrm{D}} + \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_{\Sigma} = 0 \qquad \forall\,\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,. \tag{2.9}$$

In addition, the corresponding equilibrium equations become

$$(\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} = -\,(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \qquad \forall\,\mathbf{v}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})\,, \tag{2.10}$$

and

$$(\mathrm{div}\,\mathbf{u}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} = (f_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} \qquad \forall\,q_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})\,, \tag{2.11}$$

whereas the transmission conditions from (2.6), being essential due to the mixed nature of the coupled model, are imposed independently, which yields the introduction of the auxiliary unknowns (2.7) as the associated Lagrange multipliers. According to this, we get the equations

$$\langle\boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma} + \langle\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma} = 0 \qquad \forall\,\xi \in H^{1/2}(\Sigma) \tag{2.12}$$

and

$$\langle \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_\Sigma - \nu\,\kappa_f^{-1}\,\langle \boldsymbol{\psi}\cdot\mathbf{t}, \boldsymbol{\varphi}\cdot\mathbf{t}\rangle_\Sigma = 0 \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)\,. \tag{2.13}$$

Finally, the equation relating $\mathbf{u}_{\mathrm{D}}$ to the new unknown $\mathbf{t}_{\mathrm{D}}$ is incorporated by:

$$(\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}}, \mathbf{s}_{\mathrm{D}})_{\mathrm{D}} + (\mathbf{u}_{\mathrm{D}}, \mathbf{s}_{\mathrm{D}})_{\mathrm{D}} = 0 \qquad \forall\,\mathbf{s}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})\,. \tag{2.14}$$

As a consequence of the above, we find that the resulting variational formulation reduces to a nonlinear system of seven unknowns and seven equations given by the set (2.8) – (2.14). However, it is easy to see that this system is not uniquely solvable since, given any solution $((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}), (p_{\mathrm{D}}, \lambda))$ and $c \in \mathbb{R}$, $((\boldsymbol{\sigma}_{\mathrm{S}} - c\,\mathbf{I}, \mathbf{t}_{\mathrm{D}}), (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}), (p_{\mathrm{D}} + c, \lambda + c))$ also becomes a solution. In order to avoid this non-uniqueness from now on we require that the Darcy pressure p_{D} belongs to $L^2_0(\Omega_{\mathrm{D}}) := \left\{ v \in L^2(\Omega_{\mathrm{D}}) : \displaystyle\int_{\Omega_{\mathrm{D}}} v = 0 \right\}$.

Now, it is quite clear that there are many different ways of ordering the variational system (2.8) – (2.14). Throughout the rest of the paper, and for convenience of the analysis, we adopt one leading to a twofold saddle point structure. To this end, we group unknowns and spaces as follows:

$$\begin{aligned}
&(\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}) \in \mathbf{X} := \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}) \times \mathbf{L}^2(\Omega_{\mathrm{D}})\,,\\
&(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}) \in \mathbf{M} := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}; \Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)\,,\\
&(p_{\mathrm{D}}, \lambda) \in \mathbf{Q} := L^2_0(\Omega_{\mathrm{D}}) \times H^{1/2}(\Sigma)\,,
\end{aligned}$$

and consider the following product norms

$$\begin{aligned}
&\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}} + \|\mathbf{s}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{S}}} \qquad \forall\,\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}}) \in \mathbf{X}\,,\\
&\|\underline{\mathbf{v}}\|_{\mathbf{M}} := \|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}} + \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \qquad \forall\,\underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}) \in \mathbf{M}\,,\\
&\|\underline{\mathbf{q}}\|_{\mathbf{Q}} := \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\xi\|_{1/2,\Sigma} \qquad \forall\,\underline{\mathbf{q}} := (q_{\mathrm{D}}, \xi) \in \mathbf{Q}\,.
\end{aligned}$$

Next, we define the nonlinear operator $\mathbf{A} : \mathbf{X} \longrightarrow \mathbf{X}'$,

$$[\mathbf{A}(\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}})] := [\mathbf{A}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] + [\mathbf{A}_{\mathrm{D}}(\mathbf{t}_{\mathrm{D}}), \mathbf{s}_{\mathrm{D}}] \tag{2.15}$$

where $\mathbf{A}_{\mathrm{S}} : \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}) \rightarrow \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}})'$ and $\mathbf{A}_{\mathrm{D}} : \mathbf{L}^2(\Omega_{\mathrm{D}}) \rightarrow \mathbf{L}^2(\Omega_{\mathrm{D}})'$ are given, respectively, by

$$[\mathbf{A}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] := \nu^{-1}(\boldsymbol{\sigma}^d_{\mathrm{S}}, \boldsymbol{\tau}^d_{\mathrm{S}})_{\mathrm{S}}\,, \tag{2.16}$$

$$[\mathbf{A}_{\mathrm{D}}(\mathbf{t}_{\mathrm{D}}), \mathbf{s}_{\mathrm{D}}] := (\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}}, \mathbf{s}_{\mathrm{D}})_{\mathrm{D}}\,. \tag{2.17}$$

In addition, we define the bounded and linear operatos $\mathbf{B}_1 : \mathbf{X} \longrightarrow \mathbf{M}'$ and $\mathbf{B} : \mathbf{M} \longrightarrow \mathbf{Q}'$,

$$[\mathbf{B}_1(\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}}), (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi})] := (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{s}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma\,, \tag{2.18}$$

$$[\mathbf{B}(\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}), (q_{\mathrm{D}}, \xi)] := (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma\,, \tag{2.19}$$

the positive semi-definite and linear operator $\mathbf{S} : \mathbf{M} \longrightarrow \mathbf{M}'$,

$$[\mathbf{S}(\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}), (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi})] := \nu\kappa_f^{-1}\,\langle \boldsymbol{\psi}\cdot\mathbf{t}, \boldsymbol{\varphi}\cdot\mathbf{t}\rangle_\Sigma\,, \tag{2.20}$$

and the functionals $\mathbf{F} \in \mathbf{X}'$, $\mathbf{G}_1 \in \mathbf{M}'$, and $\mathbf{G} \in \mathbf{Q}'$, given by

$$[\mathbf{F}, (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}})] := 0\,, \quad [\mathbf{G}_1, (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi})] := (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\,, \quad \text{and} \quad [\mathbf{G}, (q_{\mathrm{D}}, \xi)] := (f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}\,. \tag{2.21}$$

Hereafter, $[\cdot, \cdot]$ denotes the duality pairing induced by the operators and functionals involved.

Hence, defining the global unknowns

$$\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}) \in \mathbf{X}\,, \quad \underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}) \in \mathbf{M}\,, \quad \text{and} \quad \underline{\mathbf{p}} := (p_{\mathrm{D}}, \lambda) \in \mathbf{Q}\,,$$

we realize that the variational system (2.8) – (2.14) can be stated as the twofold saddle point operator equation: Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ such that,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_1' & \mathbf{0} \\ \mathbf{B}_1 & -\mathbf{S} & \mathbf{B}' \\ \mathbf{0} & \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\mathbf{u}} \\ \underline{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G}_1 \\ \mathbf{G} \end{bmatrix}. \tag{2.22}$$

The abstract theory for this kind of continuous formulation is already available (see, e.g. [20]), and its main results are collected in the following subsection.

2.3 Abstract theory for twofold saddle point operator equations

Let X, M and Q be Hilbert spaces with duals X', M' and Q', and consider a nonlinear operator $A : X \to X'$, and linear bounded operators $S : M \to M'$, $B_1 : X \to M'$, and $B : M \to Q'$, with corresponding adjoints $B_1' : M \to X'$ and $B' : Q \to M'$. Then we are interested in the following nonlinear variational problem: Given $(F, G_1, G) \in X' \times M' \times Q'$, find $(\sigma, u, p) \in X \times M \times Q$ such that

$$\begin{bmatrix} A & B_1' & O \\ B_1 & -S & B' \\ O & B & O \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ p \end{bmatrix} = \begin{bmatrix} F_1 \\ G_1 \\ G \end{bmatrix} \tag{2.23}$$

We have the following theorem.

Theorem 2.1 *Let V be the kernel of B, that is*

$$V := \big\{ v \in M \,:\, [B(v), q] = 0 \qquad \forall\, q \in Q \big\}\,.$$

Assume that:

i) *A is strongly monotone and Lipschitz continuous , that is, there exist α, $\gamma > 0$ such that*

$$[A(\tau) - A(\zeta), \tau - \zeta] \geq \alpha \, \|\tau - \zeta\|_X^2 \qquad \forall\, \tau, \zeta \in X\,,$$

and

$$\|A(\tau) - A(\zeta)\|_{X'} \leq \gamma \, \|\tau - \zeta\|_X \qquad \forall\, \tau, \zeta \in X\,;$$

ii) *S is positive semi-definite on V, that is*

$$[S(v), v] \geq 0 \qquad \forall\, v \in V\,;$$

iii) B_1 *satisfies the inf-sup condition on* $X \times V$*, that is, there exists* $\beta_1 > 0$ *such that*

$$\sup_{\substack{\tau \in X \\ \tau \neq \mathbf{0}}} \frac{[B_1(\tau), v]}{\|\tau\|_X} \;\geq\; \beta_1 \, \|v\|_M \qquad \forall \, v \, \in \, V\,;$$

iv) B *satisfies the inf-sup condition on* $M \times Q$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{v \in M \\ v \neq \mathbf{0}}} \frac{[B(v), q]}{\|v\|_M} \;\geq\; \beta \, \|q\|_Q \qquad \forall \, q \, \in \, Q\,.$$

Then, for each $(F, G_1, G) \in X_1' \times M' \times Q'$*, there exists a unique* $(\sigma, u, p) \in X \times M \times Q$ *solution of* (2.23)*. In addition, there exists* $C > 0$*, depending only on* γ*,* α*,* β_1*,* β*,* $\|B_1\|$*, and* $\|S\|$*, such that*

$$\|\sigma\|_X \,+\, \|u\|_M \,+\, \|p\|_Q \leq C \Big\{ \|F_1\|_{X'} \,+\, \|G_1\|_{M'} \,+\, \|G\|_{Q'} \,+\, \|A(\mathbf{0})\|_{X'} \Big\}\,. \tag{2.24}$$

Proof. See Theorem 2.1 in [20]. □

Now, let X_h, M_h and Q_h be finite-dimensional subspaces of X, M and Q, respectively. Then the Galerkin scheme associated with (2.23) reads as follows: Given $(F, G_1, G) \in X' \times M' \times Q'$, find $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ such that

$$\begin{aligned}
[A(\sigma_h), \tau_h] \,+\, [B_1(\tau_h), u_h] \qquad\qquad\qquad &= [F, \tau_h] \quad \forall \, \tau_h \in X_h\,, \\
[B_1(\sigma_h), v_h] \,-\, [S(u_h), v_h] \,+ [B(v_h), p_h] &= [G_1, v_h] \quad \forall \, v_h \in M_h\,, \\
[B(u_h), q_h] \qquad\qquad &= [G, q_h] \quad \forall \, q_h \in Q_h\,.
\end{aligned} \tag{2.25}$$

The discrete analogue of Theorem 2.1 is established next.

Theorem 2.2 *Let* V_h *be the discrete kernel of* B*, that is*

$$V_h \;:=\; \big\{ v_h \in M_h : \quad [B(v_h), q_h] \,=\, 0 \quad \forall \, q_h \, \in \, Q_h \big\}\,.$$

Assume that

i) A *is strongly monotone and Lipschitz continuous (cf. hypothesis* i) *in Theorem* 2.1*);*

ii) S *is positive semi-definite on* V_h*, that is*

$$[S(v_h), v_h] \;\geq\; 0 \qquad \forall \, v_h \, \in \, V_h\,;$$

iii) B_1 *satisfies the inf-sup condition on* $X_h \times V_h$*, that is, there exists* $\beta_1^* > 0$ *such that*

$$\sup_{\substack{\tau_h \in X_h \\ \tau_h \neq \mathbf{0}}} \frac{[B_1(\tau_h), v_h]}{\|\tau_h\|_X} \;\geq\; \beta_1^* \, \|v_h\|_M \qquad \forall \, v_h \, \in \, V_h\,;$$

iv) B *satisfies the inf-sup condition on* $M_h \times Q_h$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{v_h \in M_h \\ v_h \neq \mathbf{0}}} \frac{[B(v_h), q_h]}{\|v_h\|_M} \;\geq\; \beta^* \, \|q_h\|_Q \qquad \forall \, q_h \, \in \, Q_h\,.$$

Then, there exists a unique $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ *solution of* (2.25). *In addition, there exists* $C > 0$, *depending only on* γ, α, β_1^*, β^*, $\|B_1\|$, *and* $\|S\|$, *such that*

$$\|\sigma_h\|_X + \|u_h\|_M + \|p_h\|_Q \le C \left\{ \|F_h\|_{X_h'} + \|G_{1,h}\|_{M_h'} + \|G_h\|_{Q_h'} + \|A_h(\mathbf{0})\|_{X_h'} \right\},$$

where $F_h := F|_{X_h}$, $G_{1,h} := G_1|_{X_h}$, $G_h := G|_{Q_h}$, *and* $A_h(\mathbf{0}) := A(\mathbf{0})|_{X_h}$.

Proof. See Theorem 3.1 in [20]. □

Finally, concerning the error analysis, we have the following result.

Theorem 2.3 *Assume that the hypotheses of Theorem* 2.1 *and Theorem* 2.2 *hold and that the operator* $A : X \to X'$ *has a hemi-continuous first order Gâteaux derivative* $DA : X \to \mathcal{L}(X, X')$, *that is, for any* τ, $\zeta \in X$, *the mapping* $\mathbb{R} \ni \mu \to DA(\zeta + \mu\,\tau)(\tau, \cdot) \in X'$ *is continuous. Let* $(\sigma, u, p) \in X \times M \times Q$ *and* $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ *be the unique solutions of* (2.23) *and* (2.25), *respectively. Then there exists* $C > 0$, *independent of* h, *such that*

$$\|(\sigma, u, p) - (\sigma_h, u_h, p_h)\| \le C \inf_{\substack{(\tau_h, v_h, q_h) \\ \in X_h \times M_h \times Q_h}} \|(\sigma, u, p) - (\tau_h, v_h, q_h)\| . \tag{2.26}$$

Proof. See Theorem 3.3 in [20]. □

2.4 An equivalent twofold saddle point formulation

In order to apply the abstract theory from Section 2.3 to our problem (2.22), we need first to introduce an equivalent formulation. To this end, we now reutilize the equilibrium equation of the Stokes problem in the form of the following Galerkin least squares-type term

$$(\mathbf{div}\,\boldsymbol{\sigma}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S = -(\mathbf{f}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S \qquad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}\,; \Omega_S) , \tag{2.27}$$

which is then added to the formulation (2.22) and placed within the operator $\mathbf{A}$, thus giving rise to a modified operator $\tilde{\mathbf{A}}$ (see (2.34), (2.35) below). In addition, we consider the decomposition

$$\mathbb{H}(\mathbf{div}\,; \Omega_S) = \mathbb{H}_0(\mathbf{div}\,; \Omega_S) \oplus \mathbb{P}_0(\Omega_S)\mathbf{I} , \tag{2.28}$$

where

$$\mathbb{H}_0(\mathbf{div}\,; \Omega_S) := \left\{ \boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}\,; \Omega_S) : \int_{\Omega_S} \mathrm{tr}\,\boldsymbol{\sigma} = 0 \right\} ,$$

and set $\boldsymbol{\sigma}_S = \tilde{\boldsymbol{\sigma}}_S + c\mathbf{I}$, with the new unknowns $\tilde{\boldsymbol{\sigma}}_S \in \mathbb{H}_0(\mathbf{div}\,; \Omega_S)$ and $c \in \mathbb{R}$.

In this way, the equations (2.8), (2.13) and (2.27) are rewritten, equivalently as

$$\nu^{-1}(\tilde{\boldsymbol{\sigma}}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{div}\,\boldsymbol{\tau}_S, \mathbf{u}_S)_S + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,; \Omega_S , \tag{2.29}$$

$$d\,\langle \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall\, d \in \mathbb{R} , \tag{2.30}$$

$$\langle \tilde{\boldsymbol{\sigma}}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma - \nu\,\kappa_f^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\varphi} \cdot \mathbf{t} \rangle_\Sigma + c\,\langle \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) , \tag{2.31}$$

and

$$(\mathbf{div}\,\tilde{\boldsymbol{\sigma}}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S = -(\mathbf{f}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S \qquad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,; \Omega_S) . \tag{2.32}$$

Then, we define the global unknowns

$$\underline{\tilde{\boldsymbol{\sigma}}} := (\tilde{\boldsymbol{\sigma}}_S, \mathbf{t}_D) \in \tilde{\mathbf{X}} := \mathbb{H}_0(\mathbf{div}\,;\Omega_S) \times \mathbf{L}^2(\Omega_D)\,, \quad \underline{\tilde{\mathbf{p}}} := (\underline{\mathbf{p}}, c) \in \tilde{\mathbf{Q}} := \mathbf{Q} \times \mathbb{R}\,,$$

and group the equations (2.9)–(2.12), (2.14), (2.29)–(2.32), which yields the following variational formulation: Find $(\underline{\tilde{\boldsymbol{\sigma}}}, \underline{\mathbf{u}}, \underline{\tilde{\mathbf{p}}}) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ such that

$$\begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{B}_1' & \mathbf{0} \\ \mathbf{B}_1 & -\mathbf{S} & \tilde{\mathbf{B}}' \\ \mathbf{0} & \tilde{\mathbf{B}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{\sigma}}} \\ \underline{\mathbf{u}} \\ \underline{\tilde{\mathbf{p}}} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}} \\ \mathbf{G}_1 \\ \tilde{\mathbf{G}} \end{bmatrix}. \tag{2.33}$$

Hereafter, the nonlinear operator $\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}'$ is given by

$$[\tilde{\mathbf{A}}(\boldsymbol{\sigma}_S, \mathbf{t}_D), (\boldsymbol{\tau}_S, \mathbf{s}_D)] := [\tilde{\mathbf{A}}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] + [\mathbf{A}_D(\mathbf{t}_D), \mathbf{s}_D]\,, \tag{2.34}$$

with $\tilde{\mathbf{A}}_S : \mathbb{H}_0(\mathbf{div}\,;\Omega_S) \longrightarrow \mathbb{H}_0(\mathbf{div}\,;\Omega_S)'$ the linear and bounded operator defined by

$$[\tilde{\mathbf{A}}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] := [\mathbf{A}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] + (\mathbf{div}\,\boldsymbol{\sigma}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S\,,$$

which, according to the definition of $\mathbf{A}_S$ (cf. (2.16)), yields

$$[\tilde{\mathbf{A}}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] := \nu^{-1}\,(\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{div}\,\boldsymbol{\sigma}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S\,. \tag{2.35}$$

In addition, the linear and bounded operator $\tilde{\mathbf{B}} : \mathbf{M} \to \tilde{\mathbf{Q}}'$, and the functionals $\tilde{\mathbf{F}} \in \tilde{\mathbf{X}}'$ and $\tilde{\mathbf{G}} \in \tilde{\mathbf{Q}}'$, are given, respectively, by

$$\begin{aligned} [\tilde{\mathbf{B}}(\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}), (q_D, \xi, d)] &:= [\mathbf{B}(\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}), (q_D, \xi)] + d\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma \\ &= (\operatorname{div} \mathbf{v}_D, q_D)_D + \langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi\rangle_\Sigma + d\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma\,, \end{aligned} \tag{2.36}$$

$$[\tilde{\mathbf{F}}, (\boldsymbol{\tau}_S, \mathbf{s}_D)] = [\mathbf{F}, (\boldsymbol{\tau}_S, \mathbf{s}_D)] - (\mathbf{f}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S = -(\mathbf{f}_S, \mathbf{div}\,\boldsymbol{\tau}_S)_S\,,$$

and

$$[\tilde{\mathbf{G}}, (q_D, \xi, d)] = [\mathbf{G}, (q_D, \xi)] = (f_D, q_D)_D\,.$$

The following theorem establishes the equivalence between (2.22) and (2.33).

Theorem 2.4 *If* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := ((\boldsymbol{\sigma}_S, \mathbf{t}_D), \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *is a solution of* (2.22)*, where* $\boldsymbol{\sigma}_S = \tilde{\boldsymbol{\sigma}}_S + c\mathbf{I}$*, with* $\tilde{\boldsymbol{\sigma}}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ *and* $c \in \mathbb{R}$*, then* $(\underline{\tilde{\boldsymbol{\sigma}}}, \underline{\mathbf{u}}, \underline{\tilde{\mathbf{p}}}) := ((\tilde{\boldsymbol{\sigma}}_S, \mathbf{t}_D), \underline{\mathbf{u}}, (\underline{\mathbf{p}}, c)) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ *is a solution of* (2.33)*. Conversely, if* $((\tilde{\boldsymbol{\sigma}}_S, \mathbf{t}_D), \underline{\mathbf{u}}, (\underline{\mathbf{p}}, c)) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ *is solution of* (2.33)*, then* $((\tilde{\boldsymbol{\sigma}}_S + c\mathbf{I}, \mathbf{t}_D), \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *is a solution of* (2.22)*.*

Proof. It suffices to apply the decomposition (2.28) and observe that in either direction one deduces that $\mathbf{div}\,\boldsymbol{\sigma}_S = \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_S = -\mathbf{f}_S$ in Ω_S. We omit futher details. □

3 Analysis of the continuous problem

In this section we analyze the well posedness of (2.22) (equivalently (2.33)). To this end, we prove below in Section 3.2 that the formulation (2.33) satisfies the hypotheses of Theorem 2.1.

3.1 Preliminaries

Here we group some merely technical results and further notations that we will serve for the forthcoming analyis. The following lemma is already well known.

Lemma 3.1 *There exists $C > 0$, depending only on Ω_S, such that*

$$C\,\|\boldsymbol{\tau}_S\|_{0,\Omega_S}^2 \,\leq\, \|\boldsymbol{\tau}_S^d\|_{0,\Omega_S}^2 \,+\, \|\mathbf{div}\,\boldsymbol{\tau}_S\|_{0,\Omega_S}^2 \qquad \forall\,\boldsymbol{\tau}_S \,\in\, \mathbb{H}_0(\mathbf{div}\,;\Omega_S)\,. \tag{3.1}$$

Proof. See [5, Lemma 3.1] or [12, Proposition 3.1, Chapter IV]. □

We also recall that, given $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$, the boundary condition $\mathbf{v}_D\cdot\mathbf{n} = 0$ on Γ_D means $\langle \mathbf{v}_D\cdot\mathbf{n}, E_{0,D}(\mu)\rangle_{\partial\Omega_D} = 0 \quad \forall\,\mu \in H_{00}^{1/2}(\Gamma_D)$, where $\langle\cdot,\cdot\rangle_{\partial\Omega_D}$ stands for the duality pairing of $H^{-1/2}(\partial\Omega_D)$ and $H^{1/2}(\partial\Omega_D)$ with respect to the $L^2(\partial\Omega_D)$-inner product, $E_{0,D} : H^{1/2}(\Gamma_D) \to L^2(\partial\Omega_D)$ is the extension operator defined by

$$E_{0,D}(\mu) \,:=\, \begin{cases} \mu & \text{on} \quad \Gamma_D \\ 0 & \text{on} \quad \Sigma \end{cases} \qquad \forall\,\mu \,\in\, H^{1/2}(\Gamma_D)\,,$$

and

$$H_{00}^{1/2}(\Gamma_D) \,=\, \Big\{ \mu \,\in\, H^{1/2}(\Gamma_D) : \quad E_{0,D}(\mu) \,\in\, H^{1/2}(\partial\Omega_D) \Big\}\,,$$

endowed with the norm $\|\mu\|_{1/2,00,\Gamma_D} \,:=\, \|E_{0,D}(\mu)\|_{1/2,\partial\Omega_D}$.

As a consequence, it is not difficult to show (see e.g. Section 2 in [18]) that the restriction of $\mathbf{v}_D\cdot\mathbf{n}$ to Σ can be identified with an element of $H^{-1/2}(\Sigma)$, namely

$$\langle \mathbf{v}_D\cdot\mathbf{n}, \xi\rangle_\Sigma \,:=\, \langle \mathbf{v}_D\cdot\mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} \qquad \forall\,\xi \,\in\, H^{1/2}(\Sigma)\,, \tag{3.2}$$

where $E_D : H^{1/2}(\Sigma) \to H^{1/2}(\partial\Omega_D)$ is any bounded extension operator. In particular, given $\xi \in H^{1/2}(\Sigma)$, one could define $E_D(\xi) := z|_{\partial\Omega_D}$, where $z \in H^1(\Omega_D)$ is the unique solution of the boundary value problem: $\Delta z = 0$ in Ω_D, $z = \xi$ on Σ, $\nabla z\cdot\mathbf{n} = 0$ on Γ_D. In addition, one can show that for all $\mu \in H^{1/2}(\partial\Omega_D)$, there exist unique elements $\mu_\Sigma \in H^{1/2}(\Sigma)$ and $\mu_{\Gamma_D} \in H_{00}^{1/2}(\Gamma_D)$ such that

$$\mu \,=\, E_D(\mu_\Sigma) \,+\, E_{0,D}(\mu_{\Gamma_D})\,, \tag{3.3}$$

and

$$C_1\,\big(\|\mu_\Sigma\|_{1/2,\Sigma} \,+\, \|\mu_{\Gamma_D}\|_{1/2,00,\Gamma_D}\big) \,\leq\, \|\mu\|_{1/2,\partial\Omega_D} \,\leq\, C_2\,\big(\|\mu_\Sigma\|_{1/2,\Sigma} \,+\, \|\mu_{\Gamma_D}\|_{1/2,00,\Gamma_D}\big)\,.$$

3.2 The main results

We begin by proving the continuous inf-sup condition for $\tilde{\mathbf{B}}$ (cf. (2.36)), which will follow from the next two lemmas that separate the required estimate into two parts.

Lemma 3.2 *There exist C_1, $C_2 > 0$ such that*

$$S_1(\xi,d) \,:=\, \sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}_{00}^{1/2}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}} \frac{d\,\langle\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma \,+\, \langle\boldsymbol{\psi}\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00,\Sigma}} \,\geq\, C_1\,|d| \,-\, C_2\,\|\xi\|_{1/2,\Sigma}\,, \tag{3.4}$$

for all $(\xi,d) \in H^{1/2}(\Sigma)\times\mathbb{R}$.

Proof. Let $\boldsymbol{\psi}_0$ be a fixed element in $\mathbf{H}^{1/2}_{00}(\Sigma)$ such that $\langle \mathbf{n}, \boldsymbol{\psi}_0 \rangle_\Sigma \neq 0$. Hence, given $(\xi, d) \in H^{1/2}(\Sigma) \times \mathbb{R}$, we find that

$$S_1(\xi, d) \;\geq\; \frac{\Big| d\, \langle \mathbf{n}, \boldsymbol{\psi}_0 \rangle_\Sigma \,+\, \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, \xi \rangle_\Sigma \Big|}{\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}} \;\geq\; C_1\, |d| \,-\, C_2\, \|\xi\|_{1/2,\Sigma}\,, \tag{3.5}$$

where $C_1 \,:=\, \frac{|\langle \mathbf{n}, \boldsymbol{\psi}_0 \rangle_\Sigma|}{\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}}$, and C_2 satisfies $|\,\langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, \xi \rangle_\Sigma\,| \;\leq\; C_2\, \|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}\, \|\xi\|_{1/2,\Sigma}$. □

Note that there is a very simple way of defining such an element $\boldsymbol{\psi}_0$. In fact, as explained in [26, Section 3.2], we pick one interior corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex, and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 \,:=\, v\,(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies that property. If the interface Σ were a line segment (without interior corners), we pick v as the continuous linear function on Σ, equal to one in any interior point and zero in the extreme points, and define $\boldsymbol{\psi}_0 \,:=\, v\,\mathbf{n}$.

Lemma 3.3 *There exists $C_3 \,>\, 0$ such that*

$$S_2(q_{\rm D}, \xi) \,:=\, \sup_{\substack{\mathbf{v}_{\rm D} \in \mathbf{H}_{\Gamma_{\rm D}}(\mathrm{div};\Omega_{\rm D}) \\ \mathbf{v}_{\rm D} \neq \mathbf{0}}} \frac{(\mathrm{div}\, \mathbf{v}_{\rm D}, q_{\rm D})_{\rm D} \,+\, \langle \mathbf{v}_{\rm D} \cdot \mathbf{n}, \xi \rangle_\Sigma}{\|\mathbf{v}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}}} \;\geq\; C_3 \,\Big\{ \|q_{\rm D}\|_{0,\Omega_{\rm D}} \,+\, \|\xi\|_{1/2,\Sigma} \Big\}\,, \tag{3.6}$$

for all $(q_{\rm D}, \xi) \,\in\, L_0(\Omega_{\rm D}) \times H^{1/2}(\Sigma)$.

Proof. Let $(q_{\rm D}, \xi) \,\in\, L_0(\Omega_{\rm D}) \times H^{1/2}(\Sigma)$. Then, we define $\mathbf{w}_{\rm D} \,:=\, \nabla z$ in $\Omega_{\rm D}$, where $z \in H^1(\Omega_{\rm D})$ is the unique solution of the boundary value problem:

$$\Delta\, z \,=\, q_{\rm D} \quad \text{in} \quad \Omega_{\rm D}\,, \quad \nabla z \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \partial\Omega_{\rm D}\,, \quad \int_{\Omega_{\rm D}} z \,=\, 0\,.$$

It is clear that $\mathrm{div}\, \mathbf{w}_{\rm D} \,=\, q_{\rm D}$ in $\Omega_{\rm D}$, $\mathbf{w}_{\rm D} \in \mathbf{H}_{\Gamma_{\rm D}}(\mathrm{div};\Omega_{\rm D})$ (since actually $\mathbf{w}_{\rm D} \cdot \mathbf{n} \,=\, 0$ on $\partial\Omega_{\rm D}$), and $\|\mathbf{w}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}} \,\leq\, C\, \|q_{\rm D}\|_{0,\Omega_{\rm D}}$. Hence, using from (3.2) that $\langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, \xi \rangle_\Sigma = \langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, E_{\rm D}(\xi) \rangle_{\partial\Omega_{\rm D}} = 0$, we deduce that

$$S_2(q_{\rm D}, \xi) \;\geq\; \frac{(\mathrm{div}\, \mathbf{w}_{\rm D}, q_{\rm D})_{\rm D}}{\|\mathbf{w}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}}} \;\geq\; C_3\, \|q_{\rm D}\|_{0,\Omega_{\rm D}}\,. \tag{3.7}$$

On the other hand, given $\phi \in H^{-1/2}(\Sigma)$, we define $\eta \in H^{-1/2}(\partial\Omega_{\rm D})$ as

$$\langle \eta, \mu \rangle_{\partial\Omega_{\rm D}} \,:=\, \langle \phi, \mu_\Sigma \rangle_\Sigma \qquad \forall\, \mu \,\in\, H^{1/2}(\partial\Omega_{\rm D})\,, \tag{3.8}$$

where μ_Σ is given by the decomposition (3.3). It is not difficult to see that

$$\langle \eta, E_{0,{\rm D}}(\rho) \rangle_{\partial\Omega_{\rm D}} \,=\, 0 \qquad \forall\, \rho \,\in\, H^{1/2}_{00}(\Gamma_{\rm D})\,, \tag{3.9}$$

$$\langle \eta, E_{\rm D}(\xi) \rangle_{\partial\Omega_{\rm D}} \,=\, \langle \phi, \xi \rangle_\Sigma\,, \tag{3.10}$$

and

$$\|\eta\|_{-1/2,\partial\Omega_{\rm D}} \,\leq\, C\, \|\phi\|_{-1/2,\Sigma}\,. \tag{3.11}$$

Hence, we now define $\mathbf{w}_{\rm D} := \nabla z$ in $\Omega_{\rm D}$, where $z \in H^1(\Omega_{\rm D})$ is the unique solution of the boundary value problem:

$$\Delta z = \frac{1}{|\Omega_{\rm D}|} \langle \eta, 1 \rangle_{\partial\Omega_{\rm D}} \quad \text{in} \quad \Omega_{\rm D}, \quad \nabla z \cdot \mathbf{n} = \eta \quad \text{on} \quad \partial\Omega_{\rm D}, \quad \int_{\Omega_{\rm D}} z = 0 .$$

It follows that $\operatorname{div} \mathbf{w}_{\rm D} = \frac{1}{|\Omega_{\rm D}|} \langle \eta, 1 \rangle_{\partial\Omega_{\rm D}} \in \mathbb{P}_0(\Omega_{\rm D})$, $\mathbf{w}_{\rm D} \cdot \mathbf{n} = \eta$ on $\partial\Omega_{\rm D}$, and, using the estimate (3.11), $\|\mathbf{w}_{\rm D}\|_{{\rm div};\Omega_{\rm D}} \le C \, \|\eta\|_{-1/2,\partial\Omega_{\rm D}} \le C \, \|\phi\|_{-1/2,\Sigma}$. In addition, according to (3.2) and (3.10), and (3.9), we find, respectively, that

$$\langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, \xi \rangle_\Sigma = \langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, E_{\rm D}(\xi) \rangle_{\partial\Omega_{\rm D}} = \langle \eta, E_{\rm D}(\xi) \rangle_{\partial\Omega_{\rm D}} = \langle \phi, \xi \rangle_\Sigma \,,$$

and

$$\langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, E_{0,{\rm D}}(\rho) \rangle_{\partial\Omega_{\rm D}} = \langle \eta, E_{0,{\rm D}}(\rho) \rangle_{\partial\Omega_{\rm D}} = 0 \qquad \forall \rho \in H^{1/2}_{00}(\Gamma_{\rm D}) \,,$$

which implies that $\mathbf{w}_{\rm D} \in \mathbf{H}_{\Gamma_{\rm D}}({\rm div};\Omega_{\rm D})$. In this way, since $q_{\rm D} \in L^2_0(\Omega_{\rm D})$, we conclude that

$$S_2(q_{\rm D}, \xi) \ge \frac{|\langle \mathbf{w}_{\rm D} \cdot \mathbf{n}, \xi \rangle_\Sigma|}{\|\mathbf{w}_{\rm D}\|_{{\rm div};\Omega_{\rm D}}} \ge C \, \frac{|\langle \phi, \xi \rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} \qquad \forall \, \phi \in H^{-1/2}(\Sigma) \,,$$

and therefore

$$S_2(q_{\rm D}, \xi) \ge C \sup_{\substack{\phi \in H^{-1/2}(\Sigma) \\ \phi \neq \mathbf{0}}} \frac{|\langle \phi, \xi \rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} = C \, \|\xi\|_{1/2,\Sigma} \,.$$

This estimate and (3.7) imply (3.6), which finishes the proof. □

The continuous inf-sup condition for $\tilde{\mathbf{B}}$ follows straightforwardly from the previous lemmas.

Lemma 3.4 *There exists $\beta > 0$ such that*

$$\sup_{\substack{\underline{\mathbf{v}} \in \mathbf{M} \\ \underline{\mathbf{v}} \neq \mathbf{0}}} \frac{[\tilde{\mathbf{B}}(\underline{\mathbf{v}}), (\underline{\mathbf{q}}, d)]}{\|\underline{\mathbf{v}}\|_{\mathbf{M}}} \ge \beta \left\{ \|\underline{\mathbf{q}}\|_{\mathbf{Q}} + |d| \right\} \qquad \forall \, (\underline{\mathbf{q}}, d) \in \tilde{\mathbf{Q}} := \mathbf{Q} \times \mathbb{R} . \tag{3.12}$$

Proof. It suffices to observe, recalling that $\mathbf{M} := \mathbf{L}^2(\Omega_{\rm S}) \times \mathbf{H}_{\Gamma_{\rm D}}({\rm div};\Omega_{\rm D}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$, that

$$\sup_{\substack{\underline{\mathbf{v}} \in \mathbf{M} \\ \underline{\mathbf{v}} \neq \mathbf{0}}} \frac{[\tilde{\mathbf{B}}(\underline{\mathbf{v}}), (\underline{\mathbf{q}}, d)]}{\|\underline{\mathbf{v}}\|_{\mathbf{M}}} \ge \max \left\{ S_1(\xi, d), S_2(q_{\rm D}, \xi) \right\} \qquad \forall \, (\underline{\mathbf{q}}, d) := ((q_{\rm D}, \xi), d) \in \tilde{\mathbf{Q}} \,,$$

and then perform a suitable linear combination of (3.4) and (3.6) (cf. Lemmas 3.2 and 3.3). □

We continue the analysis with the continuous inf-sup condition for $\mathbf{B}_1$ on $\tilde{\mathbf{X}} \times \mathbf{V}$, where $\mathbf{V}$, the kernel of $\tilde{\mathbf{B}}$, is given by

$$\mathbf{V} := \left\{ \underline{\mathbf{v}} \in \mathbf{M} : \quad [\tilde{\mathbf{B}}(\underline{\mathbf{v}}), (\underline{\mathbf{q}}, d)] = 0, \quad \forall \, (\underline{\mathbf{q}}, d) \in \tilde{\mathbf{Q}} \right\} .$$

More precisely, according to the definition of $\tilde{\mathbf{B}}$ (cf. (2.36)), we find that

$$\mathbf{V} := \left\{ (\mathbf{v}_{\rm S}, \mathbf{v}_{\rm D}, \boldsymbol{\psi}) \in \mathbf{M} : \ {\rm div} \, \mathbf{v}_{\rm D} = 0 \text{ in } \Omega_{\rm D} \,, \ \mathbf{v}_{\rm D} \cdot \mathbf{n} = -\boldsymbol{\psi} \cdot \mathbf{n} \text{ on } \Sigma \,, \ \langle \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma = 0 \right\} .$$

Then, similarly as for $\tilde{\mathbf{B}}$, we recall from (2.18) the definition of $\mathbf{B}_1$, and separate the required estimate into the following two parts.

Lemma 3.5 *There holds*

$$S_3(\mathbf{v}_{\rm D}) \,:=\, \sup_{\substack{\mathbf{s}_{\rm D}\in \mathbf{L}^2(\Omega_{\rm D})\\ \mathbf{s}\neq \mathbf{0}}} \frac{(\mathbf{s}_{\rm D},\mathbf{v}_{\rm D})_{\rm D}}{\|\mathbf{s}_{\rm D}\|_{0,\Omega_{\rm D}}} \,\geq\, \|\mathbf{v}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}} \quad \forall\,(\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \,\in\, \mathbf{V}\,. \tag{3.13}$$

Proof. Given $(\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \in \mathbf{V}$, it suffices to bound $S_3(\mathbf{v}_{\rm D})$ by taking in particular $\mathbf{s}_{\rm D} = \mathbf{v}_{\rm D}$, and then use that $\|\mathbf{v}_{\rm D}\|_{0,\Omega_{\rm D}} = \|\mathbf{v}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}}$. □

Lemma 3.6 *There exists $C_4 > 0$ such that*

$$S_4(\mathbf{v}_{\rm S},\boldsymbol{\psi}) \,:=\, \sup_{\substack{\boldsymbol{\tau}_{\rm S}\,\in\,\mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S})\\ \boldsymbol{\tau}_{\rm S}\neq\mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\rm S},\mathbf{v}_{\rm S})_{\rm S} \,+\, \langle\boldsymbol{\tau}_{\rm S}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\tau}_{\rm S}\|_{\mathbf{div}\,;\Omega_{\rm S}}} \,\geq\, C_4\left\{\|\mathbf{v}_{\rm S}\|_{0,\Omega_{\rm S}} + \|\boldsymbol{\psi}\|_{1/2,00,\Sigma}\right\} \tag{3.14}$$

for all $(\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \in \mathbf{V}$.

Proof. Given $(\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \in \mathbf{V}$ and $\boldsymbol{\tau}_{\rm S} := \boldsymbol{\tau}_{{\rm S},0} + c\,\mathbf{I} \in \mathbb{H}(\mathbf{div};\Omega_{\rm S})$ with $\boldsymbol{\tau}_{{\rm S},0} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S})$ and $c \in \mathbb{P}_0(\Omega_{\rm S})$ (cf. (2.28)), we notice that $(\mathbf{div}\,\boldsymbol{\tau}_{\rm S},\mathbf{v}_{\rm S})_{\rm S} = (\mathbf{div}\,\boldsymbol{\tau}_{{\rm S},0},\mathbf{v}_{\rm S})_{\rm S}$, $\langle\boldsymbol{\tau}_{\rm S}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma = \langle\boldsymbol{\tau}_{{\rm S},0}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma$, and $\|\boldsymbol{\tau}_{\rm S}\|^2_{\mathbf{div}\,;\Omega_{\rm S}} = \|\boldsymbol{\tau}_{{\rm S},0}\|^2_{\mathbf{div}\,;\Omega_{\rm S}} + 2\,c^2\,|\Omega_{\rm S}|$. Hence, the supremum S_4 remains the same if taken on $\mathbb{H}(\mathbf{div};\Omega_{\rm S})$ instead of $\mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S})$. The rest proceeds exactly as in the proof of [7, Theorem 2.1] by defining suitable auxiliary problems. We omit further details. □

As a consequence of the previous lemmas, and recalling that $\tilde{\mathbf{X}} := \mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S}) \times \mathbf{L}^2(\Omega_{\rm D})$, we are able to establish the following result.

Lemma 3.7 *There exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\underline{\tilde{\boldsymbol{\tau}}}\in\tilde{\mathbf{X}}\\ \underline{\tilde{\boldsymbol{\tau}}}\neq\mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\tilde{\boldsymbol{\tau}}}),\underline{\mathbf{v}}]}{\|\underline{\tilde{\boldsymbol{\tau}}}\|_{\tilde{\mathbf{X}}}} \,\geq\, \beta_1\,\|\underline{\mathbf{v}}\|_{\mathbf{M}} \qquad \forall\,\underline{\mathbf{v}} \,:=\, (\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \,\in\, \mathbf{V}\,. \tag{3.15}$$

Proof. It suffices to observe that

$$\sup_{\substack{\underline{\tilde{\boldsymbol{\tau}}}\in\tilde{\mathbf{X}}\\ \underline{\tilde{\boldsymbol{\tau}}}\neq\mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\tilde{\boldsymbol{\tau}}}),\underline{\mathbf{v}}]}{\|\underline{\tilde{\boldsymbol{\tau}}}\|_{\tilde{\mathbf{X}}}} \,\geq\, \max\left\{S_3(\mathbf{v}_{\rm D}), S_4(\mathbf{v}_{\rm S},\boldsymbol{\psi})\right\} \quad \forall\,(\mathbf{v}_{\rm S},\mathbf{v}_{\rm D},\boldsymbol{\psi}) \,\in\, \mathbf{V}\,,$$

and then apply the estimates (3.5) and (3.6) (cf. Lemmas 3.5 and 3.6). □

We now come to the strong monotonicity and Lipschitz-continuity of $\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}'$.

Lemma 3.8 *There exist constants $\alpha\,,\ \gamma\,>\,0$ such that*

$$[\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\tau}}}) \,-\, \tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\zeta}}}),\underline{\tilde{\boldsymbol{\tau}}} \,-\, \underline{\tilde{\boldsymbol{\zeta}}}] \,\geq\, \alpha\,\|\underline{\tilde{\boldsymbol{\tau}}} \,-\, \underline{\tilde{\boldsymbol{\zeta}}}\|^2_{\tilde{\mathbf{X}}}$$

and

$$\|\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\tau}}}) \,-\, \tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\zeta}}})\|_{\tilde{\mathbf{X}}'} \,\leq\, \gamma\,\|\underline{\tilde{\boldsymbol{\tau}}} - \underline{\tilde{\boldsymbol{\zeta}}}\|_{\tilde{\mathbf{X}}}\,,$$

for all $\underline{\tilde{\boldsymbol{\tau}}}\,,\ \underline{\tilde{\boldsymbol{\zeta}}} \in \tilde{\mathbf{X}}$.

Proof. Let us have in mind the definition of $\tilde{\mathbf{A}}$ from (2.34). Then, thanks to the assumptions (2.4), one can show (see e.g. [28, Theorem 3.8] for details) that the nonlinear operator $\mathbf{A}_{\mathrm{D}}$ (cf. (2.17)) is strongly monotone and Lipschitz continuous on $\mathbf{L}^2(\Omega_{\mathrm{D}})$. In addition, it is easy to see, using Lemma 3.1, that the bounded linear operator $\tilde{\mathbf{A}}_{\mathrm{S}}$ (cf. (2.35)) is elliptic on $\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$. These results yield the required estimates for $\tilde{\mathbf{A}}$. $\square$

We are now in a position to establish the well-posedness of (2.22).

Theorem 3.1 *For each* $(\mathbf{F}, \mathbf{G}_1, \mathbf{G}) \in \mathbf{X}' \times \mathbf{M}' \times \mathbf{Q}'$ *there exists a unique* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *solution of* (2.22). *Moreover, there exists a constant* $C > 0$, *independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \,\le\, C\,\left\{\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{G}_1\|_{\mathbf{M}'} + \|\mathbf{G}\|_{\mathbf{Q}'}\right\}. \tag{3.16}$$

Proof. It follows from Lemmas 3.4, 3.7 and 3.8, and a direct application of the abstract result given by Theorem 2.1, that problem (2.33) is well-posed and the analogue estimate (3.16) holds. Then, the equivalence result provided by Theorem 2.4 completes the proof. $\square$

We end this section with the converse of the derivation of (2.22). More precisely, the following theorem establishes that the unique solution of (2.22) solves the original transmission problem described in Section 2.1. We remark that no extra regularity assumptions on the data, but only $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $f_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$, are required here.

Theorem 3.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *be the unique solution of the variational formulation* (2.22) *with* $\mathbf{F}$, $\mathbf{G}_1$ *and* $\mathbf{G}$ *given by* (2.21). *Then* $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = -\mathbf{f}_{\mathrm{S}}$ *in* Ω_{S}, $\nu^{-1}\,\boldsymbol{\sigma}^d_{\mathrm{S}} = \nabla \mathbf{u}_{\mathrm{S}}$ *in* Ω_{S}, $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$, $\mathrm{div}\,\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}}$ *in* Ω_{D}, $\mathbf{u}_{\mathrm{D}} = -\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\,\mathbf{t}_{\mathrm{D}}$ *in* Ω_{D}, $\mathbf{t}_{\mathrm{D}} = \nabla p_{\mathrm{D}}$ *in* Ω_{D}, $p_{\mathrm{D}} \in H^1(\Omega_{\mathrm{D}})$, $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} + \boldsymbol{\varphi} \cdot \mathbf{n} = 0$ *on* Σ, $\boldsymbol{\sigma}\,\mathbf{n} + \lambda \mathbf{n} - \nu\,\kappa_f^{-1}\,(\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t} = 0$ *on* Σ, $\lambda = p_{\mathrm{D}}$ *on* Σ, $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}$ *on* Σ, $\mathbf{u}_{\mathrm{S}} = 0$ *on* Γ_{S}, *and* $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} = 0$ *on* Γ_{D}.

Proof. It basically follows by applying integration by parts backwardly in (2.22) and using suitable test functions. We omit further details. $\square$

4 The mixed finite element scheme

In this section we analyze the well-posedness of the Galerkin scheme of (2.22). For this purpose, we also introduce the Galerkin scheme of the auxiliary problem (2.33), and establish suitable assumptions on the finite element subspaces ensuring that both discrete schemes are equivalent and that the latter is well-posed.

4.1 Preliminaries

We begin by selecting two collections of discrete spaces:

$$\begin{aligned} &\mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}})\,, \quad L_h(\Omega_{\mathrm{S}}) \subseteq L^2(\Omega_{\mathrm{S}})\,, \quad \Lambda^{\mathrm{S}}_h(\Sigma) \subseteq H^{1/2}_{00}(\Sigma)\,, \\ &\mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}})\,, \quad T_h(\Omega_{\mathrm{D}})\,, L_h(\Omega_{\mathrm{D}}) \subseteq L^2(\Omega_{\mathrm{D}})\,, \quad \Lambda^{\mathrm{D}}_h(\Sigma) \subseteq H^{1/2}(\Sigma)\,. \end{aligned} \tag{4.1}$$

According to this, for the Stokes domain we define the subspaces

$$\mathbf{L}_h(\Omega_{\mathrm{S}}) := L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}}), \qquad \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) := \Lambda^{\mathrm{S}}_h(\Sigma) \times \Lambda^{\mathrm{S}}_h(\Sigma)\,,$$

$$\mathbb{H}_h(\Omega_S) := \left\{ \boldsymbol{\tau} : \Omega_S \to \mathbb{R}^{2\times 2} : \quad \mathbf{a}^t \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall\, \mathbf{a} \in \mathbb{R}^2 \right\},$$
$$\mathbb{H}_{h,0}(\Omega_S) := \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_S)\,,$$

and for the Darcy domain we set

$$\begin{aligned} \mathbf{T}_h(\Omega_D) &:= T_h(\Omega_D) \times T_h(\Omega_D)\,, \\ \mathbf{H}_{h,\Gamma_D}(\Omega_D) &:= \left\{ \mathbf{v} \in \mathbf{H}_h(\Omega_D) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D \right\}, \\ L_{h,0}(\Omega_D) &:= L_h(\Omega_D) \cap L^2_0(\Omega_D)\,. \end{aligned} \tag{4.2}$$

Then, the global unknowns and corresponding finite element subspaces are given by:

$$\begin{aligned} \underline{\boldsymbol{\sigma}}_h &:= (\boldsymbol{\sigma}_{S,h}, \mathbf{t}_{D,h}) \in \mathbf{X}_h := \mathbb{H}_h(\Omega_S) \times \mathbf{T}_h(\Omega_D)\,, \\ \underline{\tilde{\boldsymbol{\sigma}}}_h &:= (\tilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{t}_{D,h}) \in \tilde{\mathbf{X}}_h := \mathbb{H}_{h,0}(\Omega_S) \times \mathbf{T}_h(\Omega_D)\,, \\ \underline{\mathbf{u}}_h &:= (\mathbf{u}_{S,h}, \mathbf{u}_{D,h}, \boldsymbol{\varphi}_h) \in \mathbf{M}_h := \mathbf{L}_h(\Omega_S) \times \mathbf{H}_{h,\Gamma_D}(\Omega_D) \times \boldsymbol{\Lambda}^S_h(\Sigma)\,, \\ \underline{\mathbf{p}}_h &:= (p_{D,h}, \lambda_h) \in \mathbf{Q}_h := L_{h,0}(\Omega_D) \times \Lambda^D_h(\Sigma)\,, \\ \underline{\tilde{\mathbf{p}}}_h &:= (\underline{\mathbf{p}}_h, c_h) \in \tilde{\mathbf{Q}}_h := \mathbf{Q}_h \times \mathbb{R}\,. \end{aligned}$$

In this way, the Galerkin schemes for (2.22) and (2.33) read, respectively: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\underline{\boldsymbol{\sigma}}_h), \underline{\boldsymbol{\tau}}] + [\mathbf{B}_1(\underline{\boldsymbol{\tau}}), \underline{\mathbf{u}}_h] &= [\mathbf{F}, \underline{\boldsymbol{\tau}}] \quad \forall\, \underline{\boldsymbol{\tau}} \in \mathbf{X}_h\,, \\ [\mathbf{B}_1(\underline{\boldsymbol{\sigma}}_h), \underline{\mathbf{v}}] - [\mathbf{S}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}] + [\mathbf{B}(\underline{\mathbf{v}}), \underline{\mathbf{p}}_h] &= [\mathbf{G}_1, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M}_h\,, \\ [\mathbf{B}(\underline{\mathbf{u}}_h), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \quad \forall\, \underline{\mathbf{q}} \in \mathbf{Q}_h\,, \end{aligned} \tag{4.3}$$

and: Find $(\underline{\tilde{\boldsymbol{\sigma}}}_h, \underline{\mathbf{u}}_h, \underline{\tilde{\mathbf{p}}}_h) \in \tilde{\mathbf{X}}_h \times \mathbf{M}_h \times \tilde{\mathbf{Q}}_h$ such that

$$\begin{aligned} [\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\sigma}}}_h), \underline{\boldsymbol{\tau}}] + [\mathbf{B}_1(\underline{\boldsymbol{\tau}}), \underline{\mathbf{u}}_h] &= [\tilde{\mathbf{F}}, \underline{\boldsymbol{\tau}}] \quad \forall\, \underline{\boldsymbol{\tau}} \in \tilde{\mathbf{X}}_h\,, \\ [\mathbf{B}_1(\underline{\tilde{\boldsymbol{\sigma}}}_h), \underline{\mathbf{v}}] - [\mathbf{S}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}] + [\tilde{\mathbf{B}}(\underline{\mathbf{v}}), \underline{\tilde{\mathbf{p}}}_h] &= [\mathbf{G}_1, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M}_h\,, \\ [\tilde{\mathbf{B}}(\underline{\mathbf{u}}_h), \underline{\tilde{\mathbf{q}}}] &= [\tilde{\mathbf{G}}, \underline{\tilde{\mathbf{q}}}] \quad \forall\, \underline{\tilde{\mathbf{q}}} \in \tilde{\mathbf{Q}}_h\,. \end{aligned} \tag{4.4}$$

4.2 The main results

In what follows, we proceed analogously to [26, Section 4] and derive general hypotheses on the subspaces (4.1) that allow us to show that (4.3) and (4.4) are equivalent, and that (4.4) is well posed. Our approach consists of adapting to the present discrete setting the arguments employed in the corresponding continuous analyses (cf. Theorem 2.4 and Lemmas 3.2, 3.3, 3.5 and 3.6).

We observe first that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_S)$ and $L_{h,0}(\Omega_D)$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h(\Omega_S)$ and constant polynomials from $L_h(\Omega_D)$. This request is certainly satisfied if we assume the following:

(H.0) $\quad [\mathbb{P}_0(\Omega_S)]^2 \subseteq \mathbf{H}_h(\Omega_S)$ and $\mathbb{P}_0(\Omega_D) \subseteq L_h(\Omega_D)$.

In particular, it follows that $\mathbf{I} \in \mathbb{H}_h(\Omega_S)$ for all h, and hence there holds the decomposition:

$$\mathbb{H}_h(\Omega_S) \,=\, \mathbb{H}_{h,0}(\Omega_S) \,\oplus\, \mathbb{P}_0(\Omega_S)\,\mathbf{I}\,. \tag{4.5}$$

Next, in order to prove the equivalence between (4.3) and (4.4), we assume that:

(H.1) $\operatorname{div} \mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S)$.

As a consequence, we have the following theorem.

Theorem 4.1 *If* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \,:=\, ((\boldsymbol{\sigma}_{S,h}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \,\in\, \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *is a solution of* (4.3)*, where* $\boldsymbol{\sigma}_{S,h} \,=\, \tilde{\boldsymbol{\sigma}}_{S,h} + c_h\mathbf{I}$*, with* $\tilde{\boldsymbol{\sigma}}_{S,h} \,\in\, \mathbb{H}_{h,0}(\Omega_S)$ *and* $c_h \,\in\, \mathbb{R}$*, then* $(\underline{\tilde{\boldsymbol{\sigma}}}_h, \underline{\mathbf{u}}_h, \underline{\tilde{\mathbf{p}}}_h) \,:=\, ((\tilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, (\underline{\mathbf{p}}_h, c_h)) \,\in\, \tilde{\mathbf{X}}_h \times \mathbf{M}_h \times \tilde{\mathbf{Q}}_h$ *is a solution of* (4.4)*. Conversely, if* $(\underline{\tilde{\boldsymbol{\sigma}}}_h, \underline{\mathbf{u}}_h, \underline{\tilde{\mathbf{p}}}_h) \in \tilde{\mathbf{X}}_h \times \mathbf{M}_h \times \tilde{\mathbf{Q}}_h$ *is a solution of* (4.4)*, with* $\underline{\tilde{\boldsymbol{\sigma}}}_h \,=\, (\tilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{t}_{D,h})$ *and* $\underline{\tilde{\mathbf{p}}}_h \,:=\, (\underline{\mathbf{p}}_h, c_h)$*, then* $((\tilde{\boldsymbol{\sigma}}_{S,h} + c_h\mathbf{I}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \,\in\, \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *is a solution of* (4.3)*.*

Proof. Thanks to **(H.1)**, it suffices to apply the decomposition (4.5) and observe that in either direction one deduces that $\mathbf{div}\,\boldsymbol{\sigma}_{S,h} \,=\, \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_{S,h} \,=\, -\mathbf{f}_S$. We omit futher details. □

As already announced, we now analyze the well-posedness of the Galerkin scheme (4.4), thanks to which we will conclude the well-posedness of the equivalent scheme (4.3). To this end, and in order to apply the abstract result given by Theorem 2.2, we need to introduce further hypotheses. We begin with the following:

(H.2) There exists $\boldsymbol{\psi}_0 \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma)$ such that

$$\boldsymbol{\psi}_0 \,\in\, \boldsymbol{\Lambda}^S_h(\Sigma) \quad \forall\, h \quad \text{and} \quad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \,\neq\, 0\,. \tag{4.6}$$

It is easy to see that **(H.2)** yields the following inf-sup condition, which constitutes the discrete version of Lemma 3.2: There exist $\tilde{C}_1\,,\ \tilde{C}_2 \,>\, 0$, independent of h, such that

$$S_{1,h}(\xi_h, d_h) \,:=\, \sup_{\substack{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^S_h(\Sigma) \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{d_h\,\langle \mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma \,+\, \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, \xi_h\rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma}} \,\geq\, \tilde{C}_1\,|d_h| \,-\, \tilde{C}_2\,\|\xi_h\|_{1/2,\Sigma}\,, \tag{4.7}$$

for all $(\xi_h, d_h) \,\in\, \Lambda^D_h(\Sigma) \times \mathbb{R}$.

Next, we assume that the discrete version of Lemma 3.3 holds, that is:

(H.3) There exist $\tilde{C}_3 \,>\, 0$, independent of h, such that

$$S_{2,h}(q_h, \xi_h) \,:=\, \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_{h,\Gamma_D}(\Omega_D) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)_D \,+\, \langle \mathbf{v}_h \cdot \mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_D}} \,\geq\, \tilde{C}_3 \left\{ \|q_h\|_{0,\Omega_D} \,+\, \|\xi_h\|_{1/2,\Sigma} \right\} \tag{4.8}$$

$\forall\,(q_h, \xi_h) \,\in\, L_{h,0}(\Omega_D) \times \Lambda^D_h(\Sigma)$.

On the other hand, we now look at the discrete kernel of $\tilde{\mathbf{B}}$, which is defined by

$$\mathbf{V}_h \,:=\, \left\{ \underline{\mathbf{v}}_h \in \mathbf{M}_h : \quad [\tilde{\mathbf{B}}(\underline{\mathbf{v}}_h), (\underline{\mathbf{q}}_h, d_h)] \,=\, 0 \quad \forall\,(\underline{\mathbf{q}}_h, d_h) \,\in\, (\mathbf{Q}_h \times \mathbb{R}) \right\}.$$

Moreover, in order to deduce a more explicit definition of $\mathbf{V}_h$, we introduce the hypothesis:

(H.4) $\operatorname{div} \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$ and $\mathbb{P}_0(\Sigma) \subseteq \Lambda_h^{\mathrm{D}}(\Sigma)$.

It follows, according to the definition of $\tilde{\mathbf{B}}$ (cf. (2.36)) and **(H.4)**, that $\underline{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h)$ belongs to $\mathbf{V}_h$ if and only if

$$\operatorname{div} \mathbf{v}_{\mathrm{D},h} \in \mathbb{P}_0(\Omega_{\mathrm{D}}), \ \langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle = - \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma \ \ \forall \xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma), \text{ and } \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 .$$

In particular, taking $\xi_h := 1 \in \Lambda_h^{\mathrm{D}}(\Sigma)$ we find that $\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, 1 \rangle_\Sigma = 0$, which implies that $\operatorname{div} \mathbf{v}_{\mathrm{D},h} = 0$ in Ω_{D}, and hence

$$\begin{aligned} \mathbf{V}_h := \Big\{ & (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h) \in \mathbf{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) : \ \operatorname{div} \mathbf{v}_{\mathrm{D},h} = 0 \text{ on } \Omega_{\mathrm{D}}, \\ & \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma = - \langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle \quad \forall\, \xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma), \quad \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \Big\} . \end{aligned} \tag{4.9}$$

In virtue of the above, and aiming now to establish the discrete versions of Lemmas 3.5 and 3.6, we define

$$\mathbf{V}_h(\Omega_{\mathrm{D}}) := \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}}) : \quad \operatorname{div} \mathbf{v}_{\mathrm{D},h} = 0 \Big\}, \tag{4.10}$$

and consider the following hypothesis:

(H.5) $\mathbf{V}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{T}_h(\Omega_{\mathrm{D}})$, and there exists $c_4 > 0$, independent of h, such that

$$S_{4,h}(v_h, \psi_h) := \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\operatorname{div} \boldsymbol{\tau}_h, v_h)_{\mathrm{S}} + \langle \boldsymbol{\tau}_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\operatorname{div};\Omega_{\mathrm{S}}}} \geq c_4 \Big\{ \|v_h\|_{0,\Omega_{\mathrm{S}}} + \|\psi_h\|_{1/2,00,\Sigma} \Big\} \tag{4.11}$$

for all $(v_h, \psi_h) \in L_h(\Omega_{\mathrm{S}}) \times \Lambda_h^{\mathrm{S}}(\Sigma)$.

Hence, it is easy to see that the condition $\mathbf{V}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{T}_h(\Omega_{\mathrm{D}})$ allows to extend the simple argument employed in the proof of Lemma 3.5 to the present discrete case, which yields

$$S_{3,h}(\mathbf{v}_{\mathrm{D},h}) := \sup_{\substack{\mathbf{s}_{\mathrm{D},h} \in \mathbf{L}_h(\Omega_{\mathrm{D}}) \\ \mathbf{s}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{(\mathbf{s}_{\mathrm{D},h}, \mathbf{v}_{\mathrm{D},h})}{\|\mathbf{s}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}} \geq \|\mathbf{v}_{\mathrm{D},h}\|_{\operatorname{div};\Omega_{\mathrm{D}}} \quad \forall\, (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h) \in \mathbf{V}_h . \tag{4.12}$$

Furthermore, since $\mathbf{div}\, \mathbb{H}_h(\Omega_{\mathrm{S}}) = \mathbf{div}\, \mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$ (cf. 4.5), the inf-sup condition (4.11) implies the existence of $\tilde{C}_4 > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_{\mathrm{S},h} \neq \mathbf{0}}} \frac{(\mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S},h}\, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \geq \tilde{C}_4 \Big\{ \|\mathbf{v}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \Big\} \tag{4.13}$$

for all $(\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h) \in \mathbf{V}_h$.

We are now in a position to establish, under the hypotheses specified throughout this section, the well posedness of (4.3) and the associated Cea estimate, which follows straightforwardly from the corresponding results for the equivalent scheme (4.4).

Theorem 4.2 *Assume that* **(H.0)** – **(H.5)** *hold. Then the Galerkin scheme* (4.3) *has a unique solution* $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$. *In addition, there exist* $C, \tilde{C} > 0$, *independent of* h, *such that*

$$\|(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \le C\left\{\|\mathbf{F}|_{\mathbf{X}_h}\|_{\mathbf{X}_h'} + \|\mathbf{G}_1|_{\mathbf{M}_h}\|_{\mathbf{M}_h'} + \|\mathbf{G}|_{\mathbf{Q}_h}\|_{\mathbf{Q}_h'}\right\}, \tag{4.14}$$

and

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbf{M}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{Q}} \\ &\quad\le \tilde{C}\left\{\inf_{\boldsymbol{\tau}_h\in\mathbf{X}_h}\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{X}} + \inf_{\underline{\mathbf{v}}_h\in\mathbf{M}_h}\|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbf{M}} + \inf_{\underline{\mathbf{q}}_h\in\mathbf{Q}_h}\|\underline{\mathbf{p}} - \underline{\mathbf{q}}_h\|_{\mathbf{Q}}\right\}. \end{aligned} \tag{4.15}$$

Proof. We first observe, thanks to (4.7) (which follows from **(H.2)**) and **(H.3)**, and proceeding analogously to the proof of Lemma 3.4, that $\tilde{\mathbf{B}}$ satisfies the discrete inf-sup condition on $\mathbf{M}_h\times\tilde{\mathbf{Q}}_h$. Similarly, using (4.12) and (4.13) (which follows from **(H.4)** and **(H.5)**), and proceeding as in the proof of Lemma 3.7, one can easily show that $\mathbf{B}_1$ satisfies the discrete inf-sup condition on $\tilde{\mathbf{X}}_h \times \mathbf{V}_h$. In addition, we recall that the nonlinear operator $\tilde{\mathbf{A}}$ is strongly monotone and Lipschitz-continuous (cf. Lemma 3.8), and that $\mathbf{S}$ is positive semidefinite on $\mathbf{M}$ (cf. (2.20)). On the other hand, it is known from [4, Lemma 3] that the operator $\mathbf{A}_{\mathrm{D}}$ (cf. (2.17)) has a continuous first order Gâteaux derivative $\mathcal{D}\mathbf{A}_{\mathrm{D}} : \mathbf{L}^2(\Omega_{\mathrm{D}}) \to \mathcal{L}(\mathbf{L}^2(\Omega_{\mathrm{D}}), \mathbf{L}^2(\Omega_{\mathrm{D}})')$. Hence, due also to the boundedness of the linear operator $\tilde{\mathbf{A}}_{\mathrm{S}}$ (cf. (2.35)), we conclude that $\tilde{\mathbf{A}}$ (cf. (2.34)) has a continuous first order Gâteaux derivative $\mathcal{D}\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \mathcal{L}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}')$ as well. Consequently, straightforward applications of Theorems 2.2 and 2.3 imply the well-posedness of the auxiliary Galerkin scheme (4.4) and the associated Cea estimate. Finally, the equivalence results provided by Theorems 2.4 and 4.1 yield the unique solvability of the original Galerkin scheme (4.3) and the required estimates (4.14) and (4.15). □

5 A particular mixed finite element scheme

In this section we follow very closely the analysis and results from [26, Section 5] to define specific finite element subspaces verifying the hypotheses **(H.0)** – **(H.5)**. In this way, a particular mixed finite element scheme (4.3) satisfying the estimates (4.14), (4.15), and the corresponding rate of convergence, is derived.

5.1 The finite element subspaces

Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. In addition, $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ are supposed to be quasiuniform in a neighborhood of Σ. Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of the lowest order

$$\mathrm{RT}_0(T) := \mathrm{span}\left\{(1,0),\ (0,1),\ (x_1,x_2)\right\}.$$

We also define one Raviart–Thomas space on each subdomain and their usual companion spaces of piecewise constant functions: for $\star \in \{\mathrm{S},\mathrm{D}\}$

$$\begin{aligned} \mathbf{H}_h(\Omega_\star) &:= \left\{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\right\}, \\ L_h(\Omega_\star) &:= \left\{q_h : \Omega_\star \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\right\}. \end{aligned} \tag{5.1}$$

It is clear that **(H.0)**, **(H.1)**, and the condition $\operatorname{div}\mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D)$ in **(H.4)** are satisfied. In addition, it is easy to see that in this case $\mathbf{V}_h(\Omega_D)$ (cf. (4.10)) is contained in $L_h(\Omega_D)\times L_h(\Omega_D)$, and hence, in order to have the condition $\mathbf{V}_h(\Omega_D) \subseteq \mathbf{T}_h(\Omega_D)$ in **(H.5)**, it suffices to choose $T_h(\Omega_D) = L_h(\Omega_D)$, that is

$$T_h(\Omega_D) := \Big\{ q_h : \Omega_D \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^D \Big\}. \tag{5.2}$$

Furthermore, it is well known (see, e.g. [12, Chapter IV] or [35, Chapter 7]) that the pairs of subspaces $(\mathbf{H}_h(\Omega_S), L_h(\Omega_S))$ and $(\mathbf{H}_{h,\Gamma_D}(\Omega_D), L_{h,0}(\Omega_D))$ (cf. (4.2) and (5.1)) satisfy the usual discrete inf-sup conditions, that is there exist $\tilde{\beta}_S$, $\tilde{\beta}_D > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbf{H}_h(\Omega_S)\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\boldsymbol{\tau}_h, v_h)_S}{\|\boldsymbol{\tau}_h\|_{\operatorname{div};\Omega_S}} \geq \tilde{\beta}_S\, \|v_h\|_{0,\Omega_S} \qquad \forall\, v_h \in L_h(\Omega_S)\,, \tag{5.3}$$

and

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{h,\Gamma_D}(\Omega_D)\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_h, q_h)_D}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_D}} \geq \tilde{\beta}_D\, \|q_h\|_{0,\Omega_D} \qquad \forall\, q_h \in L_{h,0}(\Omega_D)\,. \tag{5.4}$$

In addition, the set of discrete normal traces on Σ of $\mathbf{H}_h(\Omega_S)$ and $\mathbf{H}_h(\Omega_D)$ is given by

$$\Phi_h(\Sigma) := \Big\{ \phi_h : \Sigma \to \mathbb{R} : \quad \phi_h|_e \in \mathbb{P}_0(e) \qquad \forall \text{ edge } e \in \Sigma_h \Big\}, \tag{5.5}$$

where, hereafter, Σ_h denotes the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$). Note that the local quasiuniformity around Σ and the shape regularity property of the triangulations imply that Σ_h is also quasiuniform, which yields a classical inverse inequality for $\Phi_h(\Sigma)$ (see [27, eq. (5.3)]).

Next, in order to introduce the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that because Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we define

$$\Lambda_h^S(\Sigma) := \Big\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \in \Sigma_{2h}\,, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \Big\}, \tag{5.6}$$

$$\Lambda_h^D(\Sigma) = \Big\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \in \Sigma_{2h} \Big\}. \tag{5.7}$$

It is clear from (5.7) that $\mathbb{P}_0(\Sigma) \subseteq \Lambda_h^D(\Sigma)$, which completes the requirements of **(H.4)**. In addition, if we assume that the elements of Σ_{2h} are segments, that is no element of Σ_{2h} crosses a corner point, then we can construct $\boldsymbol{\psi}_0$ satisfying **(H.2)**, exactly as explained at the end of the proof of Lemma 3.2.

Furthermore, at this point we recall from [26, Lemma 5.2] that there exist $\widehat{\beta}_S$, $\widehat{\beta}_D > 0$, independent of h, such that the pairs of subspaces $(\Phi_h(\Sigma), \Lambda_h^S(\Sigma))$ and $(\Phi_h(\Sigma), \Lambda_h^D(\Sigma))$ satisfy, respectively, the following discrete inf-sup conditions:

$$\sup_{\substack{\phi_h\in\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, \psi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geq \widehat{\beta}_S\, \|\psi_h\|_{1/2,00,\Sigma} \qquad \forall\, \psi_h \in \Lambda_h^S(\Sigma)\,, \tag{5.8}$$

and

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \;\geq\; \widehat{\beta}_{\rm D}\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h^{\rm D}(\Sigma)\,. \tag{5.9}$$

5.2 The discrete inf-sup conditions

In what follows we complete the verification of the hypotheses required by Theorem 4.2. More precisely, according to our previous analysis, it only remains to show the discrete inf-sup conditions (4.8) and (4.11), which yield **(H.3)** and **(H.5)**, respectively. This is the purpose of the following two lemmas.

Lemma 5.1 *Let us recall from* (4.2) *that* $\mathbf{H}_{h,\Gamma_{\rm D}}(\Omega_{\rm D}) \,:=\, \big\{ \mathbf{v} \in \mathbf{H}_h(\Omega_{\rm D}) : \quad \mathbf{v}\cdot\mathbf{n} \,=\, 0 \quad on \quad \Gamma_{\rm D} \big\}$ *and* $L_{h,0}(\Omega_{\rm D}) \,:=\, L_h(\Omega_{\rm D}) \cap L_0(\Omega_{\rm D})$*, with* $\mathbf{H}_h(\Omega_{\rm D})$ *and* $L_h(\Omega_{\rm D})$ *given by* (5.1)*, and let* $\Lambda_h^{\rm D}(\Sigma)$ *be defined by* (5.7)*. Then, there exists* $\tilde{C}_3 \,>\, 0$*, independent of* h*, such that*

$$S_{2,h}(q_h,\xi_h) \,:=\, \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_{h,\Gamma_{\rm D}}(\Omega_{\rm D}) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{({\rm div}\, \mathbf{v}_h, q_h)_{\rm D} \,+\, \langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{{\rm div}\,;\Omega_{\rm D}}} \,\geq\, \tilde{C}_3 \Big\{ \|q_h\|_{0,\Omega_{\rm D}} \,+\, \|\xi_h\|_{1/2,\Sigma} \Big\}$$

$\forall\, (q_h,\xi_h) \,\in\, L_{h,0}(\Omega_{\rm D}) \times \Lambda_h^{\rm D}(\Sigma)$.

Proof. Let $(q_h,\xi_h) \,\in\, L_{h,0}(\Omega_{\rm D}) \times \Lambda_h^{\rm D}(\Sigma)$. It is easy to see, using the estimate (5.4) and the boundedness of the normal trace of $\mathbf{H}({\rm div}\,;\Omega_{\rm D})$, that

$$S_{2,h}(q_h,\xi_h) \,\geq\, \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_{h,\Gamma_{\rm D}}(\Omega_{\rm D}) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{({\rm div}\, \mathbf{v}_h, q_h)_{\rm D}}{\|\mathbf{v}_h\|_{{\rm div}\,;\Omega_{\rm D}}} \,-\, \|\xi_h\|_{1/2,\Sigma} \,\geq\, \tilde{\beta}_{\rm D}\, \|q_h\|_{0,\Omega_{\rm D}} \,-\, \|\xi_h\|_{1/2,\Sigma} \tag{5.10}$$

On the other hand, given $\phi_h \,\in\, \Phi_h(\Sigma)$, we proceed similarly to the proof of Lemma 3.3 and define $\eta_h \,\in\, H^{-1/2}(\partial\Omega_{\rm D})$ as

$$\langle \eta_h, \mu\rangle_{\partial\Omega_{\rm D}} \,=\, \langle \phi_h, \mu_\Sigma\rangle_\Sigma \qquad \forall\, \mu \,\in\, H^{1/2}(\partial\Omega_{\rm D})\,, \tag{5.11}$$

which satisfies

$$\langle \eta_h, E_{0,{\rm D}}(\rho)\rangle_{\partial\Omega_{\rm D}} \,=\, 0 \qquad \forall\, \rho \,\in\, H_{00}^{1/2}(\Gamma_{\rm D})\,, \tag{5.12}$$

$$\langle \eta_h, E_{\rm D}(\xi_h)\rangle_{\partial\Omega_{\rm D}} \,=\, \langle \phi_h, \xi_h\rangle_\Sigma\;, \tag{5.13}$$

and

$$\|\eta_h\|_{-1/2,\partial\Omega_{\rm D}} \,\leq\, C\, \|\phi_h\|_{-1/2,\Sigma}\,. \tag{5.14}$$

Then, according to the result provided by [26, Lemma 5.1] for the Darcy domain $\Omega_{\rm D}$, we deduce the existence of $\bar{\mathbf{v}}_h \in \mathbf{H}_h(\Omega_{\rm D})$ such that

$${\rm div}\, \bar{\mathbf{v}}_h \,\in\, \mathbb{P}_0(\Omega_{\rm D}) \quad \text{in} \quad \Omega_{\rm D}\,, \quad \bar{\mathbf{v}}_h\cdot\mathbf{n} \,=\, \eta_h \quad \text{on} \quad \partial\Omega_{\rm D}\,, \tag{5.15}$$

and

$$\|\bar{\mathbf{v}}_h\|_{{\rm div}\,;\Omega_{\rm D}} \,\leq\, C\, \|\eta_h\|_{-1/2,\partial\Omega_{\rm D}}\,. \tag{5.16}$$

In this way, thanks to (3.2) and (5.13), and (5.12), we find, respectively, that

$$\langle \bar{\mathbf{v}}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma \,=\, \langle \bar{\mathbf{v}}_h\cdot\mathbf{n}, E_{\rm D}(\xi_h)\rangle_{\partial\Omega_{\rm D}} \,=\, \langle \eta_h, E_{\rm D}(\xi_h)\rangle_{\partial\Omega_{\rm D}} \,=\, \langle \phi_h, \xi_h\rangle_\Sigma\;,$$

and

$$\langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, E_{0,\mathrm{D}}(\rho) \rangle_{\partial\Omega_\mathrm{D}} \,=\, \langle \eta_h, E_{0,\mathrm{D}}(\rho) \rangle_{\partial\Omega_\mathrm{D}} \,=\, 0 \qquad \forall \rho \in H^{1/2}_{00}(\Gamma_\mathrm{D}),$$

which implies that $\bar{\mathbf{v}}_h \in \mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})$. Moreover, it is clear from (5.14) and (5.16) that

$$\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{D}} \,\le\, C\,\|\phi_h\|_{-1/2,\Sigma}\,. \tag{5.17}$$

Hence, bounding from below with $\mathbf{v}_h \,=\, \bar{\mathbf{v}}_h$, and recalling that $q_h \in L^2_0(\Omega_\mathrm{D})$, we deduce that

$$S_{2,h}(q_h,\xi_h) \,\ge\, \frac{|\,(\mathrm{div}\,\bar{\mathbf{v}}_h, q_h)_\mathrm{D} \,+\, \langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma\,|}{\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{D}}} \,=\, \frac{|\,\langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma\,|}{\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{D}}} \,\ge\, \bar{C}\,\frac{|\,\langle \phi_h, \xi_h \rangle_\Sigma\,|}{\|\phi_h\|_{-1/2,\Sigma}}\,,$$

which, noting that ϕ_h is arbitrary in $\Phi_h(\Sigma)$, yields

$$S_{2,h}(q_h,\xi_h) \,\ge\, C \sup_{\substack{\phi_h \,\in\, \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}}\,.$$

This inequality and (5.9) imply that $S_{2,h}(q_h,\xi_h) \,\ge\, C\,\|\xi_h\|_{1/2,\Sigma}$, which, combined with (5.10), completes the proof. □

Lemma 5.2 *Let* $\mathbf{H}_h(\Omega_\mathrm{S})$ *and* $L_h(\Omega_\mathrm{S})$ *be given by* (5.1)*, and let* $\Lambda^\mathrm{S}_h(\Sigma)$ *be defined by* (5.6)*. Then there exists* $c_4 > 0$*, independent of* h*, such that*

$$S_{4,h}(v_h,\boldsymbol{\psi}_h) \,:=\, \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_\mathrm{S})\\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)_\mathrm{S} \,+\, \langle \boldsymbol{\tau}_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\mathrm{div}\,;\Omega_\mathrm{S}}} \,\ge\, c_4 \Big\{ \|v_h\|_{0,\Omega_\mathrm{S}} \,+\, \|\psi_h\|_{1/2,00,\Sigma} \Big\}$$

for all $(v_h,\psi_h) \,\in\, L_h(\Omega_\mathrm{S}) \times \Lambda^\mathrm{S}_h(\Sigma)$*.*

Proof. Let $(v_h,\psi_h) \,\in\, L_h(\Omega_\mathrm{S}) \times \Lambda^\mathrm{S}_h(\Sigma)$. We first observe, using (5.3) and the boundedness of the normal trace of $\mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{S})$, that

$$S_{4,h}(v_h,\boldsymbol{\psi}_h) \,\ge\, \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_\mathrm{S})\\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)_\mathrm{S}}{\|\boldsymbol{\tau}_h\|_{\mathrm{div}\,;\Omega_\mathrm{S}}} \,-\, \|\psi_h\|_{1/2,00,\Sigma} \,\ge\, \tilde{\beta}_\mathrm{S}\,\|v_h\|_{0,\Omega_\mathrm{S}} \,-\, \|\psi_h\|_{1/2,00,\Sigma}\,. \tag{5.18}$$

Next, given $\phi_h \in \Phi_h(\Sigma)$, we apply a slight modification of [26, Lemma 5.1] for the Stokes domain Ω_S, and deduce the existence of $\bar{\boldsymbol{\tau}}_h \in \mathbf{H}_h(\Omega_\mathrm{S})$ such that

$$\mathrm{div}\,\bar{\boldsymbol{\tau}}_h \,=\, 0 \quad \text{in} \quad \Omega_\mathrm{S}\,, \quad \bar{\boldsymbol{\tau}}_h \cdot \mathbf{n} \,=\, \phi_h \quad \text{on} \quad \Sigma\,, \tag{5.19}$$

and

$$\|\bar{\boldsymbol{\tau}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{S}} \,\le\, C\,\|\phi_h\|_{-1/2,\Sigma}\,. \tag{5.20}$$

Therefore, bounding from below with $\boldsymbol{\tau}_h \,=\, \bar{\boldsymbol{\tau}}_h$, we deduce in this case that

$$S_{4,h}(v_h,\psi_h) \,\ge\, \frac{|\,(\mathrm{div}\,\bar{\boldsymbol{\tau}}_h, v_h)_\mathrm{S} \,+\, \langle \bar{\boldsymbol{\tau}}_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma\,|}{\|\bar{\boldsymbol{\tau}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{S}}} \,=\, \frac{|\,\langle \bar{\boldsymbol{\tau}}_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma\,|}{\|\bar{\boldsymbol{\tau}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{S}}} \,\ge\, \bar{C}\,\frac{|\,\langle \phi_h, \psi_h \rangle_\Sigma\,|}{\|\phi_h\|_{-1/2,\Sigma}}\,,$$

which, noting that ϕ_h is arbitrary in $\Phi_h(\Sigma)$, yields

$$S_{4,h}(v_h,\psi_h) \,\ge\, C \sup_{\substack{\phi_h \,\in\, \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \psi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}}\,.$$

This inequality and (5.8) imply that $S_{4,h}(v_h,\psi_h) \,\ge\, C\,\|\psi_h\|_{1/2,00,\Sigma}$, which, combined with (5.18), completes the proof. □

5.3 The main results

In this section we prove the unique solvability of (4.3) for the subspaces introduced in Section 5.1, and establish the associated rate of convergence.

Theorem 5.1 *Assume that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ are quasiuniform in a neighborhood of Σ. Let $\mathbf{H}_h(\Omega_{\mathrm{S}})$, $\mathbf{H}_h(\Omega_{\mathrm{D}})$, $L_h(\Omega_{\mathrm{S}})$, $L_h(\Omega_{\mathrm{D}})$, $T_h(\Omega_{\mathrm{D}})$, $\Lambda_h^{\mathrm{S}}(\Sigma)$, and $\Lambda_h^{\mathrm{D}}(\Sigma)$ be the finite element subspaces defined in* (5.1), (5.2), (5.6), *and* (5.7), *respectively, and let*

$$
\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \left\{ \boldsymbol{\tau} : \Omega_{\mathrm{S}} \to \mathbb{R}^{2\times 2} : \quad \mathbf{a}^{\mathrm{t}}\,\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{a} \in \mathbb{R}^2 \right\},\\
\mathbf{T}_h(\Omega_{\mathrm{D}}) &:= T_h(\Omega_{\mathrm{D}}) \times T_h(\Omega_{\mathrm{D}})\,,\\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}})\,,\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \left\{ \mathbf{v} \in \mathbf{H}_h(\Omega_{\mathrm{D}}) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \textit{on} \quad \Gamma_{\mathrm{D}} \right\},\\
L_{h,0}(\Omega_{\mathrm{D}}) &:= L_h(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}})\,,\\
\boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) &:= \Lambda_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{S}}(\Sigma)\,.
\end{aligned}
$$

Then the Galerkin scheme (4.3) *with the discrete spaces $\mathbf{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{T}_h(\Omega_{\mathrm{D}})$, $\mathbf{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)$, and $\mathbf{Q}_h := L_{h,0}(\Omega_{\mathrm{D}}) \times \Lambda_h^{\mathrm{D}}(\Sigma)$ has a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$, which satisfies the estimates* (4.14) *and* (4.15).

Proof. Since the hypotheses **(H.0)** – **(H.5)** are satisfied by the specific finite element subspaces $\mathbf{X}_h$, $\mathbf{M}_h$, and $\mathbf{Q}_h$, the conclusion follows from a straightforward application of Theorem 4.2. □

Our next goal is to provide the rate of convergence of the Galerkin scheme (4.3). To this end, we now recall the approximation properties of the subspaces involved (see, e.g. [6], [12], [29]). Note that each one of them is named after the unknown to which it is applied later on.

($\mathbf{AP}_h^{\boldsymbol{\sigma}_{\mathrm{S}}}$) For each $\delta \in (0,1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$ with $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, there exists $\boldsymbol{\tau}_h \in \mathbb{H}_h(\Omega_{\mathrm{S}})$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div};\Omega_{\mathrm{S}}} \le C\,h^\delta \left\{ \|\boldsymbol{\tau}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\delta,\Omega_{\mathrm{S}}} \right\}.$$

($\mathbf{AP}_h^{\mathbf{t}_{\mathrm{D}}}$) For each $\delta \in [0,1]$, and for each $\mathbf{s} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, there exists $\mathbf{s}_h \in \mathbf{T}_h(\Omega_{\mathrm{D}})$ such that

$$\|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega_{\mathrm{D}}} \le C\,h^\delta\,\|\mathbf{s}\|_{\delta,\Omega_{\mathrm{D}}}\,.$$

($\mathbf{AP}_h^{\mathbf{u}_{\mathrm{S}}}$) For each $\delta \in [0,1]$, and for each $\mathbf{v} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, there exists $\mathbf{v}_h \in \mathbf{L}_h(\Omega_{\mathrm{S}})$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega_{\mathrm{S}}} \le C\,h^\delta\,\|\mathbf{v}\|_{\delta,\Omega_{\mathrm{S}}}\,.$$

($\mathbf{AP}_h^{\mathbf{u}_{\mathrm{D}}}$) For each $\delta \in (0,1]$, and for each $\mathbf{v} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ with $\mathrm{div}\,\mathbf{v} \in H^\delta(\Omega_{\mathrm{D}})$, there exists $\mathbf{v}_h \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathrm{div};\Omega_{\mathrm{D}}} \le C\,h^\delta \left\{ \|\mathbf{v}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathrm{div}\,\mathbf{v}\|_{\delta,\Omega_{\mathrm{D}}} \right\}.$$

$(\mathbf{AP}_h^{p_{\mathrm{D}}})$ For each $\delta \in [0,1]$, and for each $q \in H^\delta(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}})$, there exists $q_h \in L_{h,0}(\Omega_{\mathrm{D}})$ such that

$$\|q - q_h\|_{0,\Omega_{\mathrm{D}}} \,\leq\, C\, h^\delta\, \|q\|_{\delta,\Omega_{\mathrm{D}}}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\varphi}})$ For each $\delta \in [0,1]$ and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2+\delta}(\Sigma) \cap \mathbf{H}_{00}^{1/2}(\Sigma)$, there exists $\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \,\leq\, C\, h^\delta\, \|\boldsymbol{\psi}\|_{1/2+\delta,\Sigma}\,.$$

$(\mathbf{AP}_h^{\lambda})$ For each $\delta \in [0,1]$ and for each $\xi \in H^{1/2+\delta}(\Sigma)$, there exists $\xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \,\leq\, C\, h^\delta\, \|\xi\|_{1/2+\delta,\Sigma}\,.$$

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (4.3) under suitable regularity assumptions on the exact solution.

Theorem 5.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete formulations* (2.22) *and* (4.3)*, respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{t}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, *and* $\operatorname{div} \mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$*. Then,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \,-\, (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \,\leq\, C\, h^\delta \Big\{ \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \,+\, \|\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \\ +\,\, \|\mathbf{t}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\operatorname{div} \mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}\,. \end{aligned} \tag{5.21}$$

Proof. We first recall from Theorem 3.2 that $\nabla \mathbf{u}_{\mathrm{S}} = \nu^{-1} \boldsymbol{\sigma}_{\mathrm{S}}^d$ and $\nabla p_{\mathrm{D}} = \mathbf{t}_{\mathrm{D}}$, which implies that $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$ and $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, whence $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}|_\Sigma \in \mathbf{H}^{1/2+\delta}(\Sigma)$ and $\lambda = p_{\mathrm{D}}|_\Sigma \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the corresponding Cea estimate, the above approximation properties, and the fact that, thanks to the trace theorem in Ω_{S} and Ω_{D}, respectively, there holds

$$\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \,\leq\, c\, \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \,\leq\, c\, \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}}\,.$$

□

6 The a-posteriori error analysis

In this section we derive a reliable and efficient residual-based a-posteriori error estimate for our mixed finite element scheme (4.3) with the discrete spaces introduced in Section 5. Most of our analysis makes extensive use of the estimates derived in [27] and [8] for the corresponding linear case. We begin with some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathcal{E}(T)$ be the set of edges of T, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h \,=\, \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\mathcal{E}_h(\Gamma_{\mathrm{S}}) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Gamma_{\mathrm{S}} \,\}$, $\mathcal{E}_h(\Omega_\star) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Omega_\star \,\}$ for each $\star \in \{\mathrm{S}, \mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Sigma \,\}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition

Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{D}, \mathrm{S}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v}\cdot\mathbf{t}_e]$ and $[\boldsymbol{\tau}\,\mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v}\cdot\mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\}\cdot\mathbf{t}_e$ and $[\boldsymbol{\tau}\,\mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\,\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for sufficiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}\, q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathtt{t}},$$

$$\mathrm{rot}\,\mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathtt{t}}.$$

In what follows, $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{t}_{\mathrm{D},h}), (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h), (p_{\mathrm{D},h}, \lambda_h)) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ and $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ denote the solutions of (4.3) and (2.22), respectively. Then we introduce the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T\in\mathcal{T}_h^S} \Theta_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^D} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{6.1}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$

$$\begin{aligned} \Theta_{\mathrm{S},T}^2 &:= \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 + h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\,\|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa_f^{-1}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2 + h_e\,\left\|\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \right\}, \end{aligned}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$

$$\begin{aligned} \Theta_{\mathrm{D},T}^2 &:= \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{t}_{\mathrm{D},h}\|_{0,T}^2 + \|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\,\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}}))} h_e\,\|[\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e\,\|\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t} - \lambda_h'\right\|_{0,e}^2 + h_e\,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 + h_e\,\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}. \end{aligned}$$

Here, φ_h' andλ_h' have to be understood as tangential derivatives, that is in the direction imposed by the tangential vector field $\mathbf{t}$ on Σ. In addition, it is important to remark, as announced at the beginning of this section, that some components of the a posteriori error estimator (6.1) coincide with those obtained in [27] and [8]. In particular, $\Theta_{\mathrm{S},T}$ is exactly the same estimator for the Stokes domain provided in [27].

The main result of this section is stated as follows.

Theorem 6.1 *There exist positive constants $C_{\texttt{rel}}$ and $C_{\texttt{eff}}$, independent of h, such that*

$$C_{\texttt{eff}}\,\Theta \;\le\; \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbf{M}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{Q}} \;\le\; C_{\texttt{rel}}\,\Theta\,. \tag{6.2}$$

The efficiency of Θ (lower bound in (6.2)) is proved below in Section 6.2, whereas the corresponding reliability estimate (upper bound in (6.2)) is proved next in Section 6.1.

6.1 Reliability of the a posteriori error estimator

We begin by noticing, thanks to the assumptions (2.4), that the Gâteaux derivative of $\mathbf{A}_{\mathrm{D}}$ at any $\mathbf{r}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, say $\mathcal{D}\mathbf{A}_{\mathrm{D}}(\mathbf{r}_{\mathrm{D}})$, is a uniformly bounded and uniformly elliptic bilinear form on $\mathbf{L}^2(\Omega_{\mathrm{D}}) \times \mathbf{L}^2(\Omega_{\mathrm{D}})$ (see, e.g. [28, Theorem 3.8] for details). Hence, as a consequence of the continuous dependence result provided by the linear version of Theorem 2.1 (cf. (2.24) with A linear), we conclude that the linear operator obtained by adding the three equations of the left hand side of (2.22), after replacing $\mathbf{A}_{\mathrm{D}}$ by $\mathcal{D}\mathbf{A}_{\mathrm{D}}(\mathbf{r}_{\mathrm{D}})$, satisfies a global inf-sup condition. Furthermore, we observe that the continuity of $\mathcal{D}\mathbf{A}_{\mathrm{D}}$ guarantees that there exists a particular $\tilde{\mathbf{r}}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, which is a convex combination of $\mathbf{t}_{\mathrm{D}}$ and $\mathbf{t}_{\mathrm{D},h}$, such that

$$[\mathcal{D}\mathbf{A}_{\mathrm{D}}(\tilde{\mathbf{r}}_{\mathrm{D}})(\mathbf{t}_{\mathrm{D}} - \mathbf{t}_{\mathrm{D},h}), \mathbf{s}_{\mathrm{D}}] \;=\; [\mathbf{A}_{\mathrm{D}}(\mathbf{t}_{\mathrm{D}}) - \mathbf{A}_{\mathrm{D}}(\mathbf{t}_{\mathrm{D},h}), \mathbf{s}_{\mathrm{D}}] \qquad \forall\,\mathbf{s}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})\,. \tag{6.3}$$

Hence, applying the above described inf-sup estimate (with $\mathbf{r}_{\mathrm{D}} = \tilde{\mathbf{r}}_{\mathrm{D}}$) to our Galerkin error $(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$, we find that

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \le C \sup_{\substack{(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}},\underline{\mathbf{q}}) \in \mathbf{X}\times\mathbf{M}\times\mathbf{Q} \\ (\underline{\boldsymbol{\tau}},\underline{\mathbf{v}},\underline{\mathbf{q}}) \neq \mathbf{0}}} \frac{R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}})}{\|(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}})\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}}}\,, \tag{6.4}$$

where, according to (2.22), (6.3), and the definitions of $\mathbf{B}_1$, $\mathbf{B}$ and $\mathbf{S}$, the residual functional $R : \mathbf{X} \times \mathbf{M} \times \mathbf{Q} \to \mathbb{R}$ is given by

$$R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}}) \;:=\; R_1(\boldsymbol{\tau}_{\mathrm{S}}) + R_2(\mathbf{s}_{\mathrm{D}}) + R_3(\mathbf{v}_{\mathrm{S}}) + R_4(\mathbf{v}_{\mathrm{D}}) + R_5(\boldsymbol{\psi}) + R_6(q_{\mathrm{D}}) + R_7(\xi)\,,$$

for each $\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}}) \in \mathbf{X}$, $\underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}) \in \mathbf{M}$, and $\underline{\mathbf{q}} := (q_{\mathrm{D}}, \xi) \in \mathbf{Q}$, with

$$\begin{aligned}
R_1(\boldsymbol{\tau}_{\mathrm{S}}) &:= -\nu^{-1} \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\tau}_{\mathrm{S}}^d - \int_{\Omega_{\mathrm{S}}} \mathbf{u}_{\mathrm{S},h} \cdot \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}} - \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}_h \rangle_{\Sigma}\,, \\
R_2(\mathbf{s}_{\mathrm{D}}) &:= -\int_{\Omega_{\mathrm{D}}} (\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\,\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}) \cdot \mathbf{s}_{\mathrm{D}}\,, \\
R_3(\mathbf{v}_{\mathrm{S}}) &:= -\int_{\Omega_{\mathrm{S}}} \mathbf{v}_{\mathrm{S}} \cdot (\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h})\,,
\end{aligned}$$

$$
\begin{aligned}
R_4(\mathbf{v}_\mathrm{D}) &:= -\int_{\Omega_\mathrm{D}} \mathbf{t}_{\mathrm{D},h}\cdot\mathbf{v}_\mathrm{D} - \int_{\Omega_\mathrm{D}} p_{\mathrm{D},h}\,\mathrm{div}\,\mathbf{v}_\mathrm{D} - \langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n},\lambda_h\rangle_\Sigma\,,\\
R_5(\boldsymbol{\psi}) &:= -\langle \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda_h\rangle_\Sigma + \nu\,\kappa_f^{-1}\,\langle \boldsymbol{\psi}\cdot\mathbf{t},\boldsymbol{\varphi}_h\cdot\mathbf{t}\rangle_\Sigma\,,\\
R_6(q_\mathrm{D}) &:= \int_{\Omega_\mathrm{D}} q_\mathrm{D}\,(f_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h})\,,\\
R_7(\xi) &:= \langle \mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n},\xi\rangle_\Sigma + \langle \boldsymbol{\varphi}_h\cdot\mathbf{n},\xi\rangle_\Sigma\,.
\end{aligned}
$$

Hence, the supremum in (6.4) can be bounded in terms of R_i, $i\in\{1,...,7\}$, which yields

$$
\begin{aligned}
&\|(\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\sigma}}_h,\underline{\mathbf{u}}-\underline{\mathbf{u}}_h,\underline{\mathbf{p}}-\underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \le C\Big\{\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_\mathrm{S})'} + \|R_2\|_{\mathbf{L}^2(\Omega_\mathrm{D})'}\\
&+\|R_3\|_{\mathbf{L}^2(\Omega_\mathrm{S})'} + \|R_4\|_{\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})'} + \|R_5\|_{\mathbf{H}_{00}^{1/2}(\Sigma)'} + \|R_6\|_{L_0^2(\Omega_\mathrm{D})'} + \|R_7\|_{H^{1/2}(\Sigma)'}\Big\}\,.
\end{aligned}
\tag{6.5}
$$

Throughout the rest of this section we provide suitable upper bounds for each one of the terms on the right hand side of (6.5). The following lemma, whose proof follows from straightforward applications of the Cauchy-Schwarz inequality, is stated first (see also [27, Lemma 3.1] for the estimates (6.7) and (6.8) below).

Lemma 6.1 *There hold*

$$
\|R_2\|_{\mathbf{L}^2(\Omega_\mathrm{D})'} = \|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\,\mathbf{t}_{\mathrm{D},h}+\mathbf{u}_{\mathrm{D},h}\|_{0,\Omega_\mathrm{D}} = \left\{\sum_{T\in\mathcal{T}_h^\mathrm{D}} \|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\,\mathbf{t}_{\mathrm{D},h}+\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\right\}^{1/2}, \tag{6.6}
$$

$$
\|R_3\|_{\mathbf{L}^2(\Omega_\mathrm{S})'} = \|\mathbf{f}_\mathrm{S} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}} = \left\{\sum_{T\in\mathcal{T}_h^\mathrm{S}} \|\mathbf{f}_\mathrm{S} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2\right\}^{1/2}, \tag{6.7}
$$

$$
\|R_6\|_{L_0^2(\Omega_\mathrm{D})'} \le \|f_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,\Omega_\mathrm{D}} = \left\{\sum_{T\in\mathcal{T}_h^\mathrm{D}} \|f_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\right\}^{1/2}. \tag{6.8}
$$

Next, we give the estimates for the suprema on the spaces defined in the interface Σ.

Lemma 6.2 *There exist C_5, $C_7>0$, independent of h, such that*

$$
\|R_5\|_{\mathbf{H}_{00}^{1/2}(\Sigma)'} \le C_5\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n}+\lambda_h\,\mathbf{n}-\nu\,\kappa_f^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2\right\}^{1/2}, \tag{6.9}
$$

and

$$
\|R_7\|_{H^{1/2}(\Sigma)'} \le C_7\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2\right\}^{1/2}. \tag{6.10}
$$

Proof. See [27, Lemma 3.2] for details. $\square$

It remains to provide the upper bounds for $\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$ and $\|R_4\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'}$. For this purpose, we also proceed as in [27] and apply Helmholtz decompositions of $\mathbb{H}(\mathbf{div};\Omega_S)$ and $\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$ (see, e.g. [27, Lemma 3.3]), the usual integration by parts on each element, and the approximation properties of the Clément and Raviart-Thomas interpolation operators in both domains. More precisely, applying the same analysis suggested by [27, Lemmas 3.6 and 3.7], we observe that the estimate for $\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$ is exactly the one provided by [27, Lemma 3.8], whereas the estimate for $\|R_4\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'}$ arises from a slight modification of the proof of [27, Lemma 3.9]. These results are established as follows.

Lemma 6.3 *There exists $C_1 > 0$, independent of h, such that*

$$\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} \;\le\; C_1 \left\{ \sum_{T\in\mathcal{T}_h^{S}} \widehat{\Theta}_{S,T}^2 \right\}^{1/2}, \tag{6.11}$$

where, for each $T \in \mathcal{T}_h^{S}$

$$\widehat{\Theta}_{S,T}^2 \;:=\; h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 \;+\; h_T^2\,\|\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e\,\|[\boldsymbol{\sigma}_{S,h}^d\mathbf{t}]\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e\,\|\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\nu^{-1}\boldsymbol{\sigma}_{S,h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 + h_e\,\|\mathbf{u}_{S,h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}$$

Proof. See [27, Lemma 3.8]. $\square$

Lemma 6.4 *There exists $C_4 > 0$, independent of h such that*

$$\|R_4(\mathbf{v}_D)\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'} \;\le\; C_4 \left\{ \sum_{T\in\mathcal{T}_h^{D}} \widehat{\Theta}_{D,T}^2 \right\}^{1/2}, \tag{6.12}$$

where, for each $T \in \mathcal{T}_h^{D}$

$$\hat{\Theta}_{D,T}^2 \;:=\; h_T^2\,\|\mathbf{t}_{D,h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)} h_e\,\|[\mathbf{t}_{D,h}\cdot\mathbf{t}]\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_D)} h_e\,\|\mathbf{t}_{D,h}\cdot\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\|\mathbf{t}_{D,h}\cdot\mathbf{t} \,-\, \lambda_h'\|_{0,e}^2 + h_e\,\|p_{D,h} - \lambda_h\|_{0,e}^2 \right\}.$$

Proof. It suffices to apply [27, Lemma 3.9] with $\mathbf{t}_{D,h}$ instead of $\mathbf{K}^{-1}\,\mathbf{u}_{D,h}$, noting that rot $(\mathbf{t}_{D,h})$ vanishes since $\mathbf{t}_{D,h}$ is piecewise constant, and then recalling that in the present geometry the boundary of Ω_D includes also the additional part given by Γ_D. $\square$

We end this section by observing that the reliability estimate (upper bound in (6.2)) is a direct consequence of Lemmas 6.1, 6.2, 6.3, and 6.4.

6.2 Efficiency of the a posteriori error estimator

We now aim to prove the eficiency of Θ, that is the lower bound in (6.2). We begin with the estimates for the zero order terms appearing in the definition of $\Theta^2_{\mathrm{S},T}$ and $\Theta^2_{\mathrm{D},T}$.

Lemma 6.5 *There hold*

$$\|\mathbf{f}_\mathrm{S} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \,\le\, \|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div};T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}},$$

$$\|f_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T} \,\le\, \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}},$$

and there exists $c > 0$, depending on κ_1 (cf. (2.4)), such that

$$\|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T} \,\le\, c\Big\{\|\mathbf{t}_\mathrm{D} - \mathbf{t}_{\mathrm{D},h}\|_{0,T} + \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};T}\Big\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}.$$

Proof. For the first two estimates it suffices to recall, as established by Theorem 3.2, that $\mathbf{f}_\mathrm{S} = -\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{S}$ in Ω_S and $f_\mathrm{D} = \mathrm{div}\,\mathbf{u}_\mathrm{D}$ in Ω_D. Next, adding and subtracting $\mathbf{u}_\mathrm{D}$, and using also from Theorem 3.2 that $\mathbf{u}_\mathrm{D} = -\boldsymbol{\kappa}(\cdot,|\mathbf{t}_\mathrm{D}|)\mathbf{t}_\mathrm{D}$, we find that

$$\|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T} \,\le\, \|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} - \boldsymbol{\kappa}(\cdot,|\mathbf{t}_\mathrm{D}|)\mathbf{t}_\mathrm{D}\|_{0,T} + \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};T}.$$

Then, proceeding similarly as in the proof of [4, Lemma 3] and using the assumptions on $\boldsymbol{\kappa}$ (cf. (2.4)), we deduce that

$$\|\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} - \boldsymbol{\kappa}(\cdot,|\mathbf{t}_\mathrm{D}|)\mathbf{t}_\mathrm{D}\|_{0,T} \,\le\, 3\,k_1\,\|\mathbf{t}_\mathrm{D} - \mathbf{t}_{\mathrm{D},h}\|_{0,T},$$

which, replaced back into the previous estimate, completes the proof. $\square$

The derivation of the upper bounds for the remaining terms defining the global a posteriori error estimator proceeds similarly as in [27] (see also [8]), using known results from [13], [14], and [19], and applying Helmholtz decompositions, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions. We omit further details and just provide the following lemma that summarizes known efficiency estimates for thirteen terms defining $\Theta^2_{\mathrm{S},T}$ and $\Theta^2_{\mathrm{D},T}$. The corresponding proofs, as detailed below, can be found in [8], [9], [13], [19], [21], [24], and [27]).

Lemma 6.6 *There exist positive constants c_i, $i \in \{1,...,13\}$, independent of h, such that*

a) $h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}^d_{\mathrm{S},h}\|^2_{0,T} \,\le\, c_1\,\|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}$,

b) $h_e\,\|[\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}]\|^2_{0,e} \,\le\, c_2\,\|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,w_e} \qquad \forall\, e \in \mathcal{E}_h(\Omega_\mathrm{D})$, *where the set w_e is given by* $w_e := \cup\Big\{T' \in \mathcal{T}_h^{\mathrm{D}} :\quad e \in \mathcal{E}(T')\Big\}$,

c) $h_e\,\|[\boldsymbol{\sigma}^d_{\mathrm{S},h}\mathbf{t}]\|^2_{0,e} \,\le\, c_3\,\|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,w_e} \qquad \forall e \in \mathcal{E}_h(\Omega_\mathrm{S})$, *where the set w_e is given by* $w_e := \cup\Big\{T' \in \mathcal{T}_h^{\mathrm{S}} :\quad e \in \mathcal{E}(T')\Big\}$,

d) $h_e\,\|\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}\|^2_{0,e} \,\le\, c_4\,\|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T} \qquad \forall e \in \mathcal{E}_h(\Gamma_\mathrm{D})$, *where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,*

e) $h_e \, \|\boldsymbol{\sigma}^d_{\mathrm{S},h}\mathbf{t}\|^2_{0,e} \,\le\, c_5 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} \qquad \forall e \,\in\, \mathcal{E}_h(\Gamma_{\mathrm{S}})$, *where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{S}}$ *having* e *as an edge,*

f) $h_T^2 \, \|\mathbf{t}_{\mathrm{D},h}\|^2_{0,T} \,\le\, c_6 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|^2_{0,T} \,+\, h_T^2 \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T} \right\} \qquad \forall T \,\in\, \mathcal{T}_h^{\mathrm{D}}$,

g) $h_T^2 \, \|\boldsymbol{\sigma}^d_{\mathrm{S},h}\|^2_{0,T} \,\le\, c_7 \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|^2_{0,T} \,+\, h_T^2 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} \right\} \quad \forall T \,\in\, \mathcal{T}_h^{\mathrm{S}}$,

h) $h_e \, \|p_{\mathrm{D},h} - \lambda_h\|^2_{0,e} \,\le\, c_8 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|^2_{0,T} + h_T^2 \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T} + h_e \, \|\lambda - \lambda_h\|^2_{0,e} \right\} \;\; \forall e \,\in\, \mathcal{E}_h(\Sigma)$, *where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{D}}$ *having* e *as an edge,*

i) $$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \left\| \mathbf{t}_{\mathrm{D},h} \cdot \mathbf{t} \,-\, \lambda'_h \right\|^2_{0,e} \,\le\, c_9 \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T_e} \,+\, \|\lambda - \lambda_h\|^2_{1/2,\Sigma} \right\},$$
where, given $e \in \mathcal{E}_h(\Sigma)$, T_e *is the triangle of* $\mathcal{T}_h^{\mathrm{D}}$ *having* e *as an edge.*

j) $$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \left\| \nu^{-1} \boldsymbol{\sigma}^d_{\mathrm{S},h}\mathbf{t} \,+\, \boldsymbol{\varphi}'_h \right\|^2_{0,e} \,\le\, c_{10} \left\{ \sum_{e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})} \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T_e} \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{1/2,00,\Sigma} \right\},$$
where, given $e \,\in\, \mathcal{E}_h(\Sigma)$, T_e *is the triangle of* $\mathcal{T}_h^{\mathrm{S}}$ *having* e *as an edge.*

k) $h_e \, \|\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}\|^2_{0,e} \,\le\, c_{11} \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T} + h_T^2 \, \|\mathrm{div}\,(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|^2_{0,T} + h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \right\}$, *for all* $e \,\in\, \mathcal{E}_h(\Sigma)$, *where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{D}}$ *having* e *as an edge,*

l) $h_e \, \|\boldsymbol{\sigma}_{\mathrm{S},h} \, \mathbf{n} \,+\, \lambda_h \, \mathbf{n} \,-\, \nu \, \kappa_f^{-1} \, (\boldsymbol{\varphi}_h \cdot \mathbf{t}) \, \mathbf{t}\|^2_{0,e}$
$\le\, c_{12} \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} \,+\, h_T^2 \, \|\mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|^2_{0,T} \,+\, h_e \, \|\lambda - \lambda_h\|^2_{0,e} \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \right\}$, *for all* $e \,\in\, \mathcal{E}_h(\Sigma)$, *where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{S}}$ *having* e *as an edge, and*

m) $h_e \, \|\mathbf{u}_{\mathrm{S},h} \,+\, \boldsymbol{\varphi}_h\|^2_{0,e} \,\le\, c_{13} \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|^2_{0,T} \,+\, h_T^2 \, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \right\}$, *for all* $e \,\in\, \mathcal{E}_h(\Sigma)$, *where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{S}}$ *having* e *as an edge.*

Proof. For a) we refer to [13, Lemma 6.1]. Alternatively, a) follows from straightforward applications of the technical result provided in [9, Lemma 4.3] (see also [24, Lemma 4.9]). Similarly, for b), c), d), and e) we refer to [13, Lemma 6.2] or apply the technical result given by [9, Lemma 4.4] (see also [24, Lemma 4.10]). Then, for f) and g) we refer to [13, Lemma 6.3] (see also [24, Lemma 4.13] or [19, Lemma 5.5]). On the other hand, the estimate given by h) corresponds to [8, Lemma 4.12]. The proofs of i) and j) follow from very slight modifications of the proof of [19, Lemma 5.7]. Alternatively, an *elasticity version* of i) and j), which is provided in [21, Lemma 20], can also be adapted to our case. Finally, for k), l) and m) we refer to [27, Lemmas 3.15, 3.16 and 3.17]. □

The estimates i) and j) in the previous lemma provide the only non-local bounds of the present efficiency analysis. However, under additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can give the following local bounds instead.

Lemma 6.7 *Assume that $\lambda|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$, and that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$. Then there exist $\tilde{c}_9$, $\tilde{c}_{10} > 0$, such that*

$$h_e \left\|\mathbf{t}_{\mathrm{D},h} \cdot \mathbf{t} + \lambda_h'\right\|_{0,e}^2 \leq \tilde{c}_9 \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 + h_e \left\|\lambda' - \lambda_h'\right\|_{0,e}^2 \right\} \quad \forall\, e \in \mathcal{E}_h(\Sigma)\,,$$

and

$$h_e \left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \leq \tilde{c}_{10} \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 + h_e \left\|\boldsymbol{\varphi}' - \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \right\} \quad \forall\, e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})\,.$$

Proof. Similarly as for i) and j) from Lemma 6.6, it follows by adapting the corresponding *elasticity version* from [21]. We omit details here and refer to [21, Lemma 21]. □

We end this section by observing that the required efficiency estimate follows straightforwardly from Lemmas 6.5, 6.6, and 6.7. In particular, the terms $h_e \|\lambda - \lambda_h\|_{0,e}^2$ and $h_e \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2$, which appear in Lemma 6.6 (items h), k), l), and m)), are bounded as follows:

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \|\lambda - \lambda_h\|_{0,e}^2 \leq h \|\lambda - \lambda_h\|_{0,\Sigma}^2 \leq C\, h \|\lambda - \lambda_h\|_{1/2,\Sigma}^2\,,$$

and

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \leq h \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \leq C\, h \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}^2\,.$$

7 Numerical results

In this section we provide three examples illustrating the performance of the Galerkin scheme (4.3) with the subspaces $\mathbf{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{T}_h(\Omega_{\mathrm{D}})$, $\mathbf{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)$ and $\mathbf{Q}_h := L_{h,0}(\Omega_{\mathrm{D}}) \times \Lambda_h^{\mathrm{D}}(\Sigma)$ defined in Section 5, confirming the reliability and efficiency of the a posteriori error estimator Θ, and showing the behaviour of the associated adaptive algorithm.

In what follows, N stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$. The solution of (2.22) and (4.3) are denoted

$$(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}), (p_{\mathrm{D}}, \lambda)) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$$

and

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{t}_{\mathrm{D},h}), (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h), (p_{\mathrm{D},h}, \lambda_h)) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h\,.$$

The individual and global errors are defined by:

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}}\,, \qquad \mathsf{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{\mathrm{div};\Omega_{\mathrm{S}}}\,,$$

$$\mathsf{e}(\mathbf{t}_{\mathrm{D}}) := \|\mathbf{t}_{\mathrm{D}} - \mathbf{t}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,, \qquad \mathsf{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\,, \qquad \mathsf{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}\,, \qquad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}\,,$$

and

$$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := \left\{ (\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{t}_{\mathrm{D}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{D}}))^2 + (\mathsf{e}(p_{\mathrm{D}}))^2 + (\mathsf{e}(\boldsymbol{\varphi}))^2 + (\mathsf{e}(\lambda))^2 \right\}^{1/2},$$

whereas the effectivity index with respect to Θ is given by

$$\mathrm{eff}(\Theta) := \mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})/\Theta .$$

Also, we let $r(\boldsymbol{\sigma}_{\rm S})$, $r(\mathbf{u}_{\rm S})$, $r(\mathbf{t}_{\rm D})$, $r(\mathbf{u}_{\rm D})$, $r(p_{\rm D})$, $r(\boldsymbol{\varphi})$, $r(\lambda)$, and $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ be the individual and global experimental rates of convergence given by

$$r(\%) := \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \in \left\{\boldsymbol{\sigma}_{\rm S}, \mathbf{u}_{\rm S}, \mathbf{t}_{\rm D}, \mathbf{u}_{\rm D}, p_{\rm D}, \boldsymbol{\varphi}, \lambda\right\},$$

and

$$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := \frac{\log(\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})/\mathsf{e}'(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e'. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose $\nu = 1$, $\kappa_f = 1$, and $\boldsymbol{\kappa}(\cdot, s) = 2 + 1/(1+s)$. It is easy to check that $\boldsymbol{\kappa}$ satisfies the assumptions (2.4) with $k_0 = 1$ and $k_1 = 3$. Example 1 is used to illustrate the performance of the Galerkin scheme (4.3) and to corroborate the reliability and efficiency of the a posteriori error estimator Θ. Then, Examples 2 and 3 are utilized to illustrate the behavior of the associated adaptive algorithm, which applies the following procedure from [40]:

1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\rm D} \cup \mathcal{T}_h^{\rm S}$.

2) Solve the discrete problem (4.3) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T \in \mathcal{T}_h^\star$, $\star \in \{\mathrm{D}, \mathrm{S}\}$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2} \max_{T\in\mathcal{T}_h} \{\Theta_T : \quad T \in \mathcal{T}_h\}.$$

6) Define resulting meshes as current meshes $\mathcal{T}_h^{\rm D}$ and $\mathcal{T}_h^{\rm S}$, and go to step 2.

In Example 1 we consider the regions $\Omega_{\rm S} := (-1,1)\times(0,1)$ and $\Omega_{\rm D} := (-1,1)\times(-1,0)$, and choose the data $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ so that the exact solution is given by the smooth functions

$$\mathbf{u}_{\rm S}(\mathbf{x}) = \mathbf{curl}\left(x_2^2 \sin(\pi\, x_1)\right) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm S},$$

$$p_{\rm S}(\mathbf{x}) = x_1^3 + x_2^3 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm S},$$

and

$$p_{\rm D}(\mathbf{x}) = x_1\, \left(x_1^2 - 1\right)^2 (x_2 + 1)^2 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm D}.$$

In Example 2 we consider $\Omega_{\rm D} := (-1,1)\times(-2,-1)$ and let $\Omega_{\rm S}$ be the L-shaped domain given by $(-1,1)^2\setminus[0,1]^2$. Then we choose $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ so that the exact solution is given by

$$\mathbf{u}_{\rm S}(\mathbf{x}) \,=\, \operatorname{curl}\Big(3\,(x_1^2+x_2^2)^{4/3}\,(x_2+1)^2\Big) \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm S}\,,$$

$$p_{\rm S}(\mathbf{x}) \,=\, (x_2+1)^2\,e^{x_1} \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm S}\,,$$

and

$$p_{\rm D}(\mathbf{x}) \,=\, \frac{1}{5}\,(x_1^3-3x_1)\,\cos(\pi\,x_2) \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm D}\,.$$

Note that $\nabla\mathbf{u}_{\rm S}$ and $\boldsymbol{\sigma}_{\rm S}$ have a singularity at the origin.

In Example 3 we consider the same geometry of Example 1 and choose the data $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ so that the exact solution is given by the smooth functions

$$\mathbf{u}_{\rm S}(\mathbf{x}) \,=\, \operatorname{curl}\big(0.2\,x_2^3\,e^{x_1+x_2}\big) \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm S}\,,$$

$$p_{\rm S}(\mathbf{x}) \,=\, x_2^2\,e^{x_1} \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm S}\,,$$

and

$$p_{\rm D}(\mathbf{x}) \,=\, \frac{x_1\,\left(x_1^2-1\right)^2}{\left(x_1^2+(x_2+1)^2+0.05\right)} \quad \forall\,\mathbf{x} := (x_1,x_2)\in\Omega_{\rm D}\,,$$

In this case, $p_{\rm D}$ and hence $\mathbf{t}_{\rm D} = \nabla p_{\rm D}$ and $\mathbf{u}_{\rm D} = -\boldsymbol{\kappa}\,(\cdot,|\nabla p_{\rm D}|)\,\nabla p_{\rm D}$ show a numerical singularity in a neighborhood of the point $(0,-1)$.

The numerical results shown below were obtained using a MATLAB code. In Table 7.1 we summarize the convergence history of the mixed finite element method (4.3), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 5.2 (here $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index $\mathrm{eff}(\Theta)$ remains always in a neighborhood of 0.87, which illustrates the reliability and efficiency of Θ in the case of a regular solution.

Next, in Tables 7.2 - 7.5 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3. We observe that the errors of the adaptive procedures decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figures 7.1 and 7.3 where we display the total errors $\mathsf{e}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}},\underline{\mathbf{p}})$ vs. the number of degrees of freedom N for both refinements. As shown by the values of $r(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}},\underline{\mathbf{p}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figures 7.2 and 7.4. Note that the method is able to recognize the singularity of the solution in Example 2 and the region with high gradients in Example 3.

Table 7.1: EXAMPLE 1, quasi-uniform scheme

N	h	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{t}_{\mathrm{D}})$	$r(\mathbf{t}_{\mathrm{D}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$
168	0.707	7.359	–	0.865	–	0.489	–	1.694	–
640	0.354	4.312	0.799	0.457	0.953	0.220	1.196	1.375	0.312
2496	0.177	2.195	0.992	0.230	1.008	0.106	1.078	0.829	0.743
9856	0.088	1.103	1.002	0.115	1.007	0.052	1.036	0.476	0.810
39168	0.044	0.552	1.003	0.058	1.004	0.026	1.011	0.260	0.878
156160	0.022	0.276	1.002	0.029	1.002	0.013	1.004	0.137	0.929

N	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\lambda)$	$r(\lambda)$	$\mathsf{e}(\varphi)$	$r(\varphi)$	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$\mathrm{eff}(\Theta)$
168	0.126	–	0.683	–	0.037	–	7.648	–	0.862
640	0.045	1.524	0.497	0.475	0.139	–	4.583	0.765	0.879
2496	0.018	1.371	0.244	1.041	0.042	1.734	2.373	0.967	0.898
9856	0.008	1.179	0.120	1.030	0.014	1.549	1.213	0.976	0.845
39168	0.004	1.061	0.060	1.011	0.005	1.513	0.616	0.982	0.875
156160	0.002	1.018	0.030	1.004	0.001	1.505	0.311	0.988	0.871

Table 7.2: EXAMPLE 2, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{t}_{\mathrm{D}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathbf{e}(p_{\mathrm{D}})$
404	0.5000	29.3565	5.8914	0.3784	2.2553	0.0806
1576	0.2500	19.7820	3.0327	0.1895	1.2565	0.0409
6224	0.1250	13.2561	1.5276	0.0948	0.6588	0.0204
24736	0.0625	8.4281	0.7652	0.0474	0.3369	0.0102
98624	0.0312	5.5354	0.3828	0.0237	0.1703	0.0051

N	$\mathbf{e}(\lambda)$	$\mathbf{e}(\varphi)$	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$\mathrm{eff}(\Theta)$
404	0.3325	0.2636	30.0322	–	0.5258
1576	0.1713	0.1226	20.0546	0.5933	0.5631
6224	0.0887	0.0477	13.3608	0.5914	0.5627
24736	0.0450	0.0172	8.4697	0.6607	0.5986
98624	0.0226	0.0060	5.5513	0.6109	0.5598

Table 7.3: EXAMPLE 2, adaptive scheme

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	Θ	$\mathrm{eff}(\Theta)$
404	30.0322	–	57.1171	0.5258
548	22.2145	1.9781	34.9558	0.6355
784	18.8764	0.9093	27.8872	0.6769
1544	12.4998	1.2164	18.5554	0.6736
2026	10.6033	1.2113	15.9807	0.6635
4373	7.8376	0.7856	11.0736	0.7078
4781	7.2224	1.8328	10.3884	0.6952
7105	5.9397	0.9872	8.4901	0.6996
9673	5.2169	0.8411	7.3908	0.7059
20712	3.6174	0.9618	5.0386	0.7179
29906	2.9286	1.1501	4.1342	0.7084
36304	2.6731	0.9416	3.7189	0.7188
53634	2.2272	0.9353	3.0884	0.7212
67436	1.9670	1.0850	2.7358	0.7190
71449	1.9011	1.1802	2.6419	0.7196
96176	1.6508	0.9499	2.2885	0.7213
126900	1.4424	0.9737	2.0029	0.7201

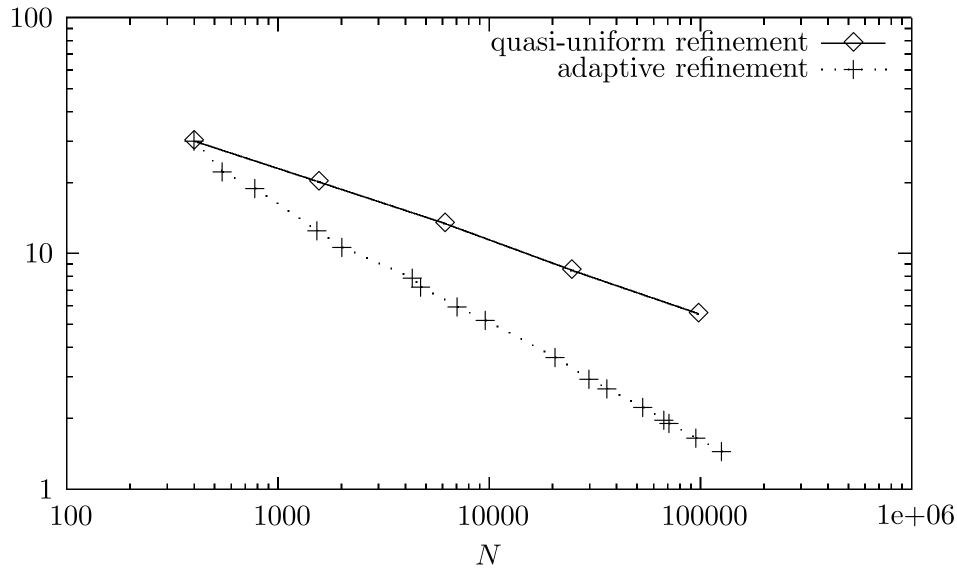

Figure 7.1: EXAMPLE 2, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ vs. N for the quasi-uniform and adaptive schemes

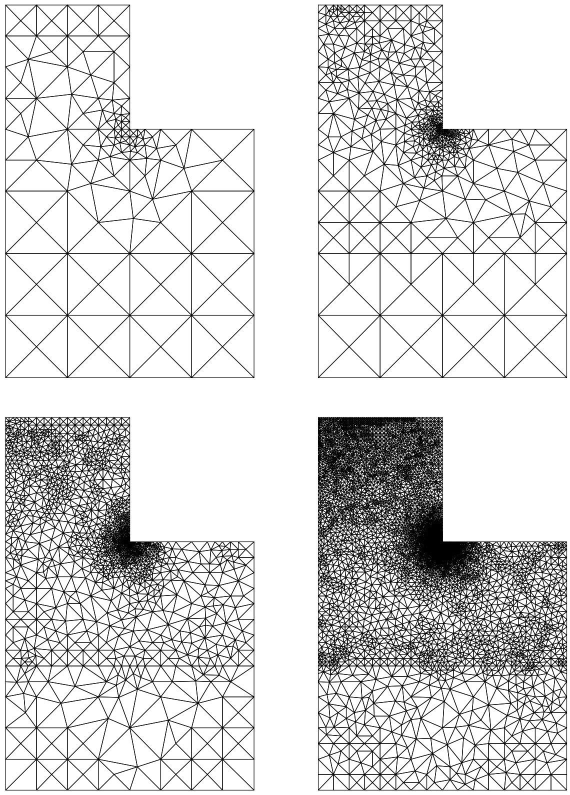

Figure 7.2: Example 2, adapted meshes with 1544, 4781, 20712, and 67436 degrees of freedom

Table 7.4: Example 3, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_\mathrm{S})$	$\mathbf{e}(\mathbf{u}_\mathrm{S})$	$\mathbf{e}(\mathbf{t}_\mathrm{D})$	$\mathbf{e}(\mathbf{u}_\mathrm{D})$	$\mathbf{e}(p_\mathrm{D})$
168	0.7071	1.6203	0.1781	2.7631	5.7862	0.4991
640	0.3536	0.8166	0.0920	2.5015	6.6021	0.2920
2496	0.1768	0.4069	0.0446	1.3345	10.7401	0.1288
9856	0.0884	0.2035	0.0222	0.6524	7.9304	0.0642
39168	0.0442	0.1017	0.0111	0.3264	4.9954	0.0322
156160	0.0221	0.0509	0.0055	0.1632	2.7788	0.0161

N	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathrm{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$\mathrm{eff}(\Theta)$
168	0.4683	0.0570	6.6515	–	0.9877
640	0.6154	0.0736	7.1407	–	1.0202
2496	0.2349	0.0159	10.8337	–	1.0145
9856	0.1023	0.0044	7.9607	0.4487	1.0077
39168	0.0478	0.0014	5.0074	0.6720	1.0050
156160	0.0253	0.0005	2.7842	0.8488	1.0041

Table 7.5: EXAMPLE 3, adaptive scheme

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	Θ	$\mathrm{eff}(\Theta)$
1346	8.5936	–	8.5943	0.9999
1866	6.9966	1.2588	7.0143	0.9975
3633	5.3139	0.8258	5.3029	1.0021
5069	4.4949	1.0051	4.4942	1.0001
5146	4.4662	0.8474	4.4546	1.0026
8042	3.6365	0.9207	3.6203	1.0045
13148	2.8766	0.9538	2.8588	1.0062
15921	2.5961	1.0722	2.5742	1.0085
23197	2.1824	0.9225	2.1712	1.0051
28262	1.9700	1.0365	1.9556	1.0074
43218	1.6240	0.9096	1.6176	1.0039
50762	1.4914	1.0589	1.4833	1.0055
62798	1.3415	0.9958	1.3341	1.0055
76352	1.2116	1.0424	1.2053	1.0052
88422	1.1253	1.0064	1.1186	1.0060
133093	0.9381	0.8898	0.9318	1.0068
144737	0.8932	1.1703	0.8877	1.0062
191228	0.7814	0.9597	0.7767	1.0062

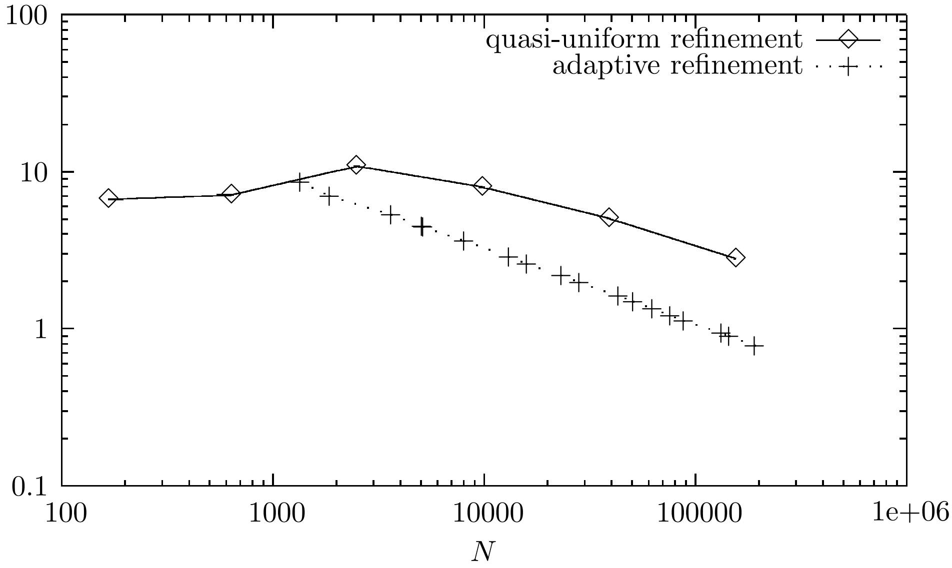

Figure 7.3: EXAMPLE 3, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ vs. N for the quasi-uniform and adaptive schemes

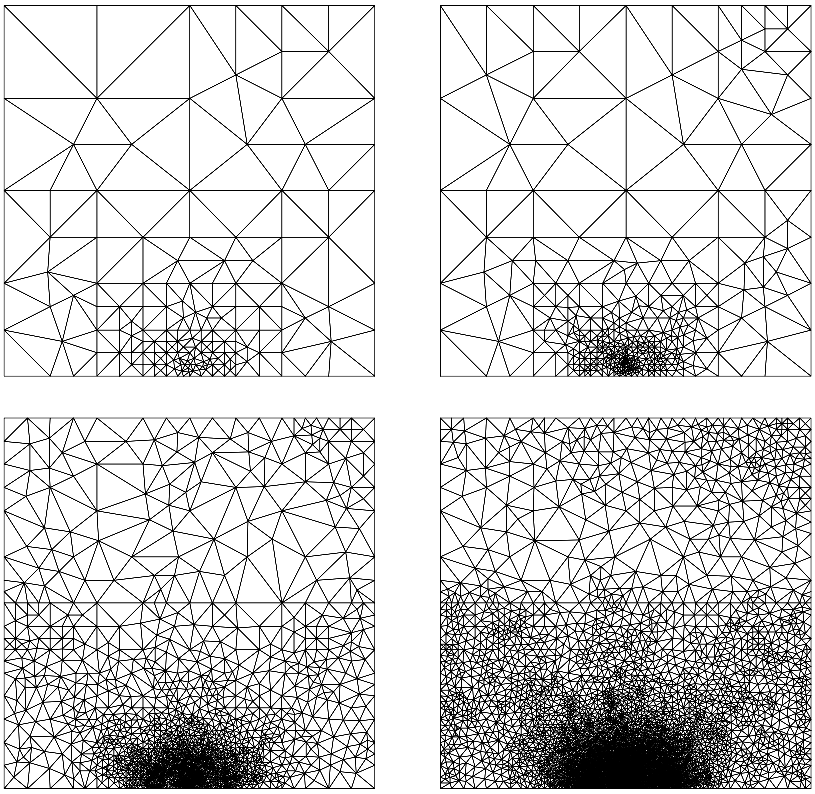

Figure 7.4: Example 3, adapted meshes with 1346, 3633, 15921, and 62798 degrees of freedom

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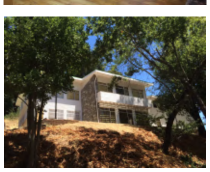

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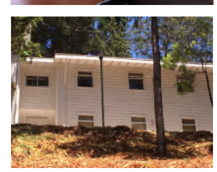

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