UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI²MA)

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PREPRINT 2010-20

SERIE DE PRE-PUBLICACIONES

A priori and a posteriori error analyses of a velocity-pseudostress formulation for a class of quasi-Newtonian Stokes flows*

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Abstract

In this paper we introduce and analyze new mixed finite element schemes for a class of nonlinear Stokes models arising in quasi-Newtonian fluids. The methods are based on a non-standard mixed approach in which the velocity, the pressure, and the pseudostress are the original unknowns. However, we use the incompressibility condition to eliminate the pressure, and set the velocity gradient as an auxiliary unknown, which yields a twofold saddle point operator equation as the resulting dual-mixed variational formulation. In addition, a suitable augmented version of the latter showing a single saddle point structure is also considered. We apply known results from nonlinear functional analysis to prove that the corresponding continuous and discrete schemes are well-posed. In particular, we show that Raviart-Thomas elements of order $k \geq 0$ for the pseudostress, and piecewise polynomials of degree k for the velocity and its gradient, ensure the well-posedness of the associated Galerkin schemes. Moreover, we prove that any finite element subspace of the square integrable tensors can be employed to approximate the velocity gradient in the case of the augmented formulation. Then, we derive a reliable and efficient residual-based a posteriori error estimator for each scheme. Finally, we provide several numerical results illustrating the good performance of the resulting mixed finite element methods, confirming the theoretical properties of the estimator, and showing the behaviour of the associated adaptive algorithms.

Key words: nonlinear Stokes model, twofold saddle point equation, mixed finite element method, a posteriori error estimator

Mathematics Subject Classifications (1991): 65N15, 65N30, 65N50, 74B05

1 Introduction

The derivation of new mixed finite element methods for linear and nonlinear Stokes and related problems has become a very active research area during the last decade. In particular, the velocity-pressure-stress formulation for incompressible flows has gained considerable attention in recent years due to its natural applicability to non-Newtonian flows, where the corresponding constitutive equations are nonlinear. In general, an interesting feature of the mixed approaches

*This research was partially supported by FONDAP and BASAL projects CMM, Universidad de Chile, and by Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción.

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is given by the fact that, besides the original unknowns, they yield direct approximations of several other quantities of physical interest. For instance, an accurate direct calculation of the stresses is very desirable for flow problems involving interaction with solid structures. However, the symmetry requirement for the stress tensor is one of the main drawbacks of the formulations involving this unknown. Two main ideas circumventing this disadvantage are available in the literature. The first one, which goes back to [1], consists of imposing the symmetry of the stress in a weak sense through the introduction of a suitable Lagrange multiplier (rotation in elasticity and vorticity in fluid mechanics), whereas the second one makes use of the pseudostress instead of the stress in the corresponding setting of the Stokes equations. As a consequence, two new approaches for incompressible flows, specially in the context of least-squares and augmented methods, arise: the velocity-pressure-pseudostress and velocity-pseudostress formulations (see, e.g. [7], [9], [18]). Other least-squares methods for the steady Stokes problem, based on formulations with two or three fields among velocity, velocity gradient, pressure, vorticity, stress, and pseudostress, can be found in [4], [5], [8], [11], [13], and the references therein. Similarly, augmented mixed finite element methods for both pseudostress-based formulations of the stationary Stokes equations, which extends analogue results for linear elasticity problems (see [21], [22], [25]), are introduced and analyzed in [18]. The corresponding augmented mixed finite element schemes for the velocity-pressure-stress formulation of the Stokes problem, in which the vorticity is introduced as the Lagrange multiplier taking care of the weak symmetry of the stress, are studied in [17]. On the other hand, the velocity-pressure-pseudostress formulation has also been applied to nonlinear Stokes problems. In particular, a new mixed finite element method for a class of models arising in quasi-Newtonian fluids, is introduced in [23]. The associated variational formulation has a twofold saddle point structure, and the abstract theory needed for its analysis, which constitutes a generalization of the Babuška-Brezzi theory, can be found in [19] and [24]. The results in [23] are extended in [16] to a setting in reflexive Banach spaces, thus allowing other nonlinear models such as the Carreau law for viscoplastic flows. In addition, the dual-mixed approach from [23] and [16] is reformulated in [30] by restricting the space for the velocity gradient to that of trace-free tensors. In this way, the pressure is eliminated and a three-field formulation with the pseudostress, the velocity, and the velocity gradient as unknowns, is obtained.

Now, it is surprising to realize that mixed finite element methods for the pure velocity-pseudostress formulation of the Stokes equations, that is without augmenting or employing least-squares terms, had not been studied until [10]. It is shown there that Raviart-Thomas elements of index $k \geq 0$ for the pseudostress and piecewise discontinuous polynomials of degree k for the velocity lead to a stable Galerkin scheme with quasi-optimal accuracy. The pressure and other physical quantities (if needed) can be computed in a postprocessing procedure without affecting the accuracy of approximation. More recently, in [26] we reconsider the pure velocity-pseudostress formulation from [10] and provide further related results. Indeed, we incorporate the pressure unknown into the discrete analysis, which does not necessarily yield an equivalent formulation at that level, and derive reliable and efficient residual-based a posteriori error estimators for both Galerkin schemes. The idea of reintroducing the pressure in [26] is to allow further flexibility in approximating this unknown. To this respect, we show there that a Galerkin scheme for the velocity-pressure-pseudostress formulation only makes sense for pressure finite element subspaces not containing the traces of the pseudostresses subspace. In particular, this is the case when Raviart-Thomas elements of index $k \geq 0$ for the pseudostress, and piecewise discontinuous polynomials of degree k for the velocity and the pressure, are utilized. Otherwise, both discrete schemes coincide and hence one obviously stays with the simplest one.

The purpose of the present paper is to additionally contribute in the direction suggested by [10], [16], [23], [26], and [30], by extending the results from [26] to the class of nonlinear problems studied in [23] and [30]. More precisely, we develop the a priori and a posteriori error analyses of the velocity-pseudostress formulation as applied to quasi-Newtonian Stokes flows whose kinematic viscosities are a nonlinear monotone function of the velocity gradient of the fluid. The latter is introduced as an auxiliary unknown, and the pressure is eliminated using the incompressibility condition, whence the resulting variational formulation shows a twofold saddle point structure (as in [23] and [30]). In addition, an augmented version of this formulation, which, thanks to its single saddle point structure, simplifies the requirements for well-posedness of the associated Galerkin scheme, is also introduced and analyzed. We remark in advance that the twofold saddle point operator equation to be derived here, though written slightly different, is equivalent to the one obtained in [30, Section 3]. However, neither the a posteriori error analysis nor the augmented formulation are considered in [30].

In order to describe the boundary value problem of interest, we now let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with boundary Γ. As in [26], our goal is to determine the velocity $\mathbf{u}$, the pseudostress tensor $\boldsymbol{\sigma}$, and the pressure p of a steady flow occupying the region Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$, and a scalar field p such that

$$\begin{gathered}\boldsymbol{\sigma} \,=\, 2\,\mu(|\nabla\mathbf{u}|)\,\nabla\mathbf{u} \,-\, p\,\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,,\end{gathered} \tag{1.1}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is the non-linear kinematic viscosity function of the fluid, $\mathbf{div}$ stands for the usual divergence operator div acting along each row of the tensor, $\nabla\mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $\mathbb{R}^{2\times 2}$, and $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$. As required by the incompressibility condition, we assume from now on that the datum $\mathbf{g}$ satisfies the compatibility condition

$$\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} \,=\, 0\,, \tag{1.2}$$

where $\boldsymbol{\nu}$ stands for the unit outward normal at Γ. The kind of nonlinear Stokes problem given by (1.1) appears in the modeling of a large class of non-Newtonian fluids (see, e.g. [3], [31], [32], [35]). In particular, the Ladyzhenskaya law for fluids with large stresses (see [31]), also known as power law, is given by $\mu(t) := \kappa_0 + \kappa_1 t^{\beta-2}\ \forall\, t \in \mathbb{R}^+$, with $\kappa_0 \geq 0$, $\kappa_1 > 0$, and $\beta > 1$, and the Carreau law for viscoplastic flows (see, e.g. [32], [35]) reads $\mu(t) := \kappa_0 + \kappa_1\,(1+t^2)^{(\beta-2)/2}$ $\forall\, t \in \mathbb{R}^+$, with $\kappa_0 \geq 0$, $\kappa_1 > 0$, and $\beta \geq 1$.

In what folows we let $\mu_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the mapping given by $\mu_{ij}(\mathbf{r}) := \mu(|\mathbf{r}|)\,r_{ij}$ for all $\mathbf{r} := (r_{ij}) \in \mathbb{R}^{2\times 2}$, for all $i,j \in \{1,2\}$. Then, throughout this paper we assume that μ is of class C^1 and that there exist γ_0, $\alpha_0 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$, there holds

$$|\,\mu_{ij}(\mathbf{r})\,| \,\leq\, \gamma_0\,\|\mathbf{r}\|_{\mathbb{R}^{2\times 2}}\,, \qquad \left|\frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})\right| \,\leq\, \gamma_0 \quad \forall\, i,j,k,l \,\in\, \{1,2\}\,, \tag{1.3}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})\, s_{ij}\, s_{kl} \,\geq\, \alpha_0\,\|\mathbf{s}\|^2_{\mathbb{R}^{2\times 2}}\,. \tag{1.4}$$

It is easy to check that the Carreau law satisfies (1.3) and (1.4) for all $\kappa_0 > 0$, and for all $\beta \in [1,2]$. In particular, with $\beta = 2$ we recover the usual linear Stokes model.

The rest of this work is organized as follows. In Section 2 we establish and analyze the variational formulation involving the velocity, the pseudostress, and the velocity gradient, as unknowns. In addition, we define the associated Galerkin scheme and show its well-posedness under the assumption that the finite element subspace for the velocity gradient contains the free divergence tensors of the finite element subspace for the pseudostress. In particular, Raviart-Thomas elements of order $k \geq 0$ for the pseudostress, and piecewise polynomials of degree k for the velocity and its gradient become feasible choices. Some aspects of the analysis in Section 2, though developed differently, coincide with the corresponding discussion in [30]. An equivalent augmented variational formulation, which arises from the further introduction of the constitutive equation multiplied by a stabilization parameter, is proposed in Section 3. The resulting operator equation shows a single saddle point structure, and the well-posedness of the associated Galerkin scheme does not require any restriction on the finite element subspace for the velocity gradient, but being only a finite dimensional subspace of the square integrable tensors. Next, in Section 4 we derive a reliable and efficient residual-based a posteriori error estimator for each scheme. Our analysis here benefits strongly from the general results and estimates available in [26] for the linear case. Finally, several numerical results illustrating the performance of both mixed finite element methods, confirming the reliability and efficiency of the a posteriori estimators, and showing the good behaviour of the associated adaptive algorithms, are given in Section 5.

We end this section with some notations to be used below. Given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In addition, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}\,.$$

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector or null operator, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The continuous and discrete formulations

2.1 The mixed variational formulation

We begin by eliminating the pressure. Indeed, it follows from the first equation in (1.1), using that $\mathrm{tr}(\nabla \mathbf{u}) = \mathrm{div}(\mathbf{u})$ in Ω, that the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω can be stated in terms of the pseudostress tensor and the pressure as follows

$$p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,. \tag{2.1}$$

Conversely, starting from (2.1), and using the first equation in (1.1), we recover the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω. In other words, the pair of equations given by

$$\boldsymbol{\sigma} = 2\,\mu(|\nabla \mathbf{u}|)\,\nabla \mathbf{u} - p\,\mathbf{I} \ \text{ in } \ \Omega \quad \text{and} \quad \mathrm{div}(\mathbf{u}) = 0 \ \text{ in } \ \Omega\,, \tag{2.2}$$

is equivalent to

$$\boldsymbol{\sigma} \,=\, 2\,\mu(|\nabla\mathbf{u}|)\,\nabla\mathbf{u} \,-\, p\,\mathbf{I} \;\text{ in }\; \Omega \quad\text{and}\quad p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0 \;\text{ in }\; \Omega\,, \tag{2.3}$$

and therefore, instead of (1.1), we now consider:

$$\begin{gathered} \boldsymbol{\sigma} \,=\, 2\,\mu(|\nabla\mathbf{u}|)\,\nabla\mathbf{u} \,-\, p\,\mathbf{I} \quad\text{in}\quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad\text{in}\quad \Omega\,, \\ p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0 \;\text{ in }\; \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad\text{on}\quad \Gamma\,. \end{gathered} \tag{2.4}$$

Moreover, replacing p by $-\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$ in the first equation of (2.4), and introducing the auxiliary unknown $\mathbf{t} := \nabla\mathbf{u}$ in Ω, we arrive at the following system

$$\begin{gathered} \boldsymbol{\sigma}^{\mathrm{d}} \,=\, 2\,\mu(|\mathbf{t}|)\,\mathbf{t} \quad\text{in}\quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad\text{in}\quad \Omega\,, \\ \mathbf{t} \,=\, \nabla\mathbf{u} \quad\text{in}\quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad\text{on}\quad \Gamma\,. \end{gathered} \tag{2.5}$$

Now, we adopt the usual procedure and test the three field equations of (2.5) with $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$, $\mathbf{v} \in [L^2(\Omega)]^2$, and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, respectively. In this way, integrating by parts the expression $\displaystyle\int_\Omega \nabla\mathbf{u} : \boldsymbol{\tau}$ and using the Dirichlet boundary condition, we arrive initially at the formulation: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in [L^2(\Omega)]^{2\times 2} \times \mathbb{H}(\mathbf{div};\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{array}{llllll} 2\displaystyle\int_\Omega \mu(|\mathbf{t}|)\,\mathbf{t} : \mathbf{s} & -\displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} & & = & 0 & \forall\, \mathbf{s} \in [L^2(\Omega)]^{2\times 2}\,, \\ -\displaystyle\int_\Omega \mathbf{t} : \boldsymbol{\tau} & & -\displaystyle\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) & = & -\,\langle \boldsymbol{\tau}\,\nu, \mathbf{g}\rangle_\Gamma & \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\,, \\ & -\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) & & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall\, \mathbf{v} \in [L^2(\Omega)]^2\,. \end{array} \tag{2.6}$$

Hereafter, $\mathbb{H}(\mathbf{div};\Omega) := \big\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\big\}$, and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$, with respect to the $[L^2(\Gamma)]^2$-inner product.

We now observe that (2.6) is not uniquely solvable since adding $(0, c\mathbf{I}, 0)$ to $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$, for any $c \in \mathbb{R}$, yields further solutions of this problem. Therefore, in order to guarantee uniqueness, we proceed as in [26] and require additionally that $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$, where

$$\mathbb{H}_0(\mathbf{div};\Omega) \,:=\, \left\{ \boldsymbol{\tau} \,\in\, \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}.$$

Alternatively, we observe that (2.6) remains unchanged if the unknown $\boldsymbol{\sigma}$ is replaced there by its $\mathbb{H}_0(\mathbf{div};\Omega)$-component according to the decomposition $\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathbb{R}\,\mathbf{I}$. Moreover, using that $\mathrm{tr}(\mathbf{t}) = 0$ and thanks to the compatibility condition (1.2), the first two equations of (2.6) can be stated, equivalently, for each $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$ such that $\mathrm{tr}(\mathbf{s}) = 0$ and for each $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$, respectively, whence the expression $\int_\Omega \mathbf{t} : \boldsymbol{\tau}$ appearing in the second equation can be replaced by $\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}}$. Consequently, we introduce the spaces

$$X_1 \,:=\, \left\{ \mathbf{s} \,\in\, [L^2(\Omega)]^{2\times 2} : \quad \mathrm{tr}(\mathbf{s}) \,=\, 0 \right\}, \quad M_1 \,:=\, \mathbb{H}_0(\mathbf{div};\Omega)\,, \quad M \,:=\, [L^2(\Omega)]^2\,,$$

endowed, respectively, with the usual norms of $[L^2(\Omega)]^{2\times 2}$, $\mathbb{H}(\mathbf{div};\Omega)$, and $[L^2(\Omega)]^2$, and from now on consider, instead of (2.6), the following variational formulation: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X_1 \times M_1 \times M$ such that

$$
\begin{array}{lllcll}
2\displaystyle\int_\Omega \mu(|\mathbf{t}|)\,\mathbf{t}:\mathbf{s} & -\displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}:\mathbf{s} & & = & 0 & \forall\,\mathbf{s}\in X_1\,,\\
-\displaystyle\int_\Omega \mathbf{t}:\boldsymbol{\tau}^{\mathrm{d}} & & -\displaystyle\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) & = & -\,\langle\boldsymbol{\tau}\,\nu,\mathbf{g}\rangle_\Gamma & \forall\,\boldsymbol{\tau}\in M_1\,,\\
 & -\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) & & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall\,\mathbf{v}\in M\,.
\end{array}
\tag{2.7}
$$

Next, we notice that (2.7) has a typical twofold saddle point structure (see [19], [20], [24], [27]). In fact, let us define the non-linear operator $\mathbf{A}_1 : X_1 \to X_1'$, the bounded linear operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : M_1 \to M'$, with transposes $\mathbf{B}_1' : M_1 \to X_1'$ and $\mathbf{B}' : M \to M_1'$, and the functionals $\mathbf{H} \in X_1'$, $\mathbf{G} \in M_1'$ and $\mathbf{F} \in M'$, as follows:

$$
\begin{aligned}
[\mathbf{A}_1(\mathbf{r}),\mathbf{s}] &:= 2\int_\Omega \mu(|\mathbf{r}|)\,\mathbf{r}:\mathbf{s}\,,\\
[\mathbf{B}_1(\mathbf{r}),\boldsymbol{\tau}] &:= -\int_\Omega \mathbf{r}:\boldsymbol{\tau}^{\mathrm{d}}\,,\\
[\mathbf{B}(\boldsymbol{\zeta}),\mathbf{v}] &:= -\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta})\,,
\end{aligned}
\tag{2.8}
$$

and

$$
[\mathbf{H},\mathbf{s}] := 0\,,\qquad [\mathbf{G},\boldsymbol{\tau}] := -\langle\boldsymbol{\tau}\nu,\,\mathbf{g}\rangle_\Gamma\,,\qquad [\mathbf{F},\mathbf{v}] := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,,
\tag{2.9}
$$

for all $(\mathbf{r},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\tau}) \in X_1 \times M_1$ and for all $\mathbf{v} \in M$, where $[\cdot,\cdot]$ stands in each case for the duality pairing induced by the correspondig operators and functionals. Then, it is easy to see that the variational formulation (2.7) can be rewritten as: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X_1 \times M_1 \times M$ such that

$$
\begin{array}{lllcll}
[\mathbf{A}_1(\mathbf{t}),\mathbf{s}] & +\ [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\sigma}] & & = & [\mathbf{H},\mathbf{s}] & \forall\,\mathbf{s}\in X_1\,,\\
[\mathbf{B}_1(\mathbf{t}),\boldsymbol{\tau}] & & +\ [\mathbf{B}(\boldsymbol{\tau}),\mathbf{u}] & = & [\mathbf{G},\boldsymbol{\tau}] & \forall\,\boldsymbol{\tau}\in M_1\,,\\
 & [\mathbf{B}(\boldsymbol{\sigma}),\mathbf{v}] & & = & [\mathbf{F},\mathbf{v}] & \forall\,\mathbf{v}\in M\,.
\end{array}
\tag{2.10}
$$

The abstract theory for this kind of twofold saddle point operator equation is already available (see [19], [20], [24]), and their main results are collected in the following Section.

2.2 Abstract theory for two-fold saddle point equations

Let X_1, M_1, and M be Hilbert spaces, and consider a nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : M_1 \to M'$, with transposes $\mathbf{B}_1' : M_1 \to X_1'$ and $\mathbf{B}' : M \to M_1'$, respectively. Then, given $(\mathbf{H},\mathbf{G},\mathbf{F}) \in X_1' \times M_1' \times M'$, we are interested in the following nonlinear variational problem (written as a matrix operator equation): Find $(\mathbf{t},\boldsymbol{\sigma},u) \in X_1 \times M_1 \times M$ such that

$$
\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O}\\ \mathbf{B}_1 & \mathbf{O} & \mathbf{B}'\\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{pmatrix}
\begin{pmatrix} \mathbf{t}\\ \boldsymbol{\sigma}\\ u \end{pmatrix}
=
\begin{pmatrix} \mathbf{H}\\ \mathbf{G}\\ \mathbf{F} \end{pmatrix}.
\tag{2.11}
$$

We have the following theorem.

THEOREM 2.1 *Let* $V := \mathrm{Ker}\,(\mathbf{B})$, *define* $V_1 := \{\mathbf{s} \in X_1 : [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \ \forall\, \boldsymbol{\tau} \in V\}$, *and let* $\Pi_1 : X_1' \to V_1'$ *be the canonical imbedding defined by* $\Pi_1(\mathbf{H}) = \mathbf{H}|_{V_1}$ *for all* $\mathbf{H} \in X_1'$. *Assume that*

i) *the nonlinear operator* $\mathbf{A}_1 : X_1 \to X_1'$ *is Lipschitz continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\tilde{\mathbf{t}} \in X_1$, *the nonlinear operator* $\Pi_1 \mathbf{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone with a monotonicity constant* $\alpha > 0$ *independent of* $\tilde{\mathbf{t}}$.

ii) *there exists* $\beta > 0$ *such that for all* $v \in M$

$$\sup_{\boldsymbol{\tau} \in M_1 \setminus \{\mathbf{0}\}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{||\boldsymbol{\tau}||_{M_1}} \geq \beta\, ||v||_M \,; \tag{2.12}$$

iii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in V$

$$\sup_{\mathbf{s} \in X_1 \setminus \{\mathbf{0}\}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \geq \beta_1\, ||\boldsymbol{\tau}||_{M_1} \,; \tag{2.13}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *solution of* (2.11). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u)\|_{X_1 \times M_1 \times M} \leq C \Big\{ \|\mathbf{H}\| + \|\mathbf{G}\| + \|\mathbf{F}\| + \|\mathbf{A}_1(\mathbf{0})\| \Big\} . \tag{2.14}$$

Proof. See [19, Theorem 2.4] (see also [24, Theorem 2.1], [20, Theorem 1], or [27, Theorem 4.1]). □

Now, let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively. Then the Galerkin scheme associated with (2.11) reads as follows: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned}
&[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &&= [\mathbf{H}, \mathbf{s}_h] \quad \forall\, \mathbf{s}_h \in X_{1,h} \,,\\
&[\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\tau}_h), u_h] &&= [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in M_{1,h} \,,\\
&[\mathbf{B}(\boldsymbol{\sigma}_h), v_h] &&= [\mathbf{F}, v_h] \quad \forall\, v_h \in M_h \,.
\end{aligned} \tag{2.15}$$

The discrete analogue of Theorem 2.1 is established next.

THEOREM 2.2 *Let* $V_h := \{\boldsymbol{\tau}_h \in M_{1,h} : \ [\mathbf{B}(\boldsymbol{\tau}_h), v_h] = 0 \ \ \forall\, v_h \in M_h\}$, *define the space* $V_{1,h} := \{\mathbf{s}_h \in X_{1,h} : \ [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \ \ \forall\, \boldsymbol{\tau}_h \in V_h\}$ *and let* $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ *be the canonical imbedding. Further, let* $\mathbf{A}_{1,h} := p_h' \mathbf{A}_1 : X_1 \to X_{1,h}'$ *where* $p_h : X_{1,h} \to X_1$ *is the canonical injection with adjoint* $p_h' : X_1' \to X_{1,h}'$. *Assume that*

i) *the nonlinear operator* $\mathbf{A}_{1,h} : X_1 \to X_{1,h}'$ *is Lipschitz-continuous with a Lipschitz constant* $\gamma_h > 0$, *and for any* $\tilde{\mathbf{t}} \in X_{1,h}$, *the nonlinear operator* $\Pi_{1,h} \mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V_{1,h}'$ *is strongly monotone with a monotonicity constant* $\alpha_h > 0$ *independent of* $\tilde{\mathbf{t}}$.

ii) *there exists* $\beta_h > 0$ *such that for all* $v_h \in M_h$

$$\sup_{\boldsymbol{\tau}_h \in M_{1,h} \setminus \{\mathbf{0}\}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), v_h]}{||\boldsymbol{\tau}_h||_{M_1}} \geq \beta_h\, ||v_h||_M; \tag{2.16}$$

iii) *there exists* $\beta_{1,h} > 0$ *such that for all* $\boldsymbol{\tau}_h \in V_h$

$$\sup_{\mathbf{s}_h \in X_{1,h}\backslash\{\mathbf{0}\}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \geq \beta_{1,h} ||\boldsymbol{\tau}_h||_{M_1}; \tag{2.17}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *solution of* (2.15). *Moreover, there exists* $C_h > 0$*, independent of the solution, but depending on* h*, such that*

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{X_1 \times M_1 \times M} \leq C_h \left\{ \|\mathbf{H}_h\| + \|\mathbf{G}_h\| + \|\mathbf{F}_h\| + \|\mathbf{A}_{1,h}(\mathbf{0})\| \right\},$$

where $\mathbf{H}_h := \mathbf{H}|_{X_{1,h}}$, $\mathbf{G}_h := \mathbf{G}|_{M_{1,h}}$, *and* $\mathbf{F}_h := \mathbf{F}|_{M_h}$.

Proof. See [19, Theorem 3.2] (see also [24, Theorem 3.1], [20, Theorem 3], or [27, Theorem 4.2]). □

Finally, concerning the error analysis, we have the following result.

THEOREM 2.3 *Assume that the hypotheses of Theorems* 2.1 *and* 2.2 *are satisfied, and let* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *be the unique solutions of* (2.11) *and* (2.15)*, respectively. In addition, suppose that there exist positive constants* $\tilde{\gamma}$, $\tilde{\alpha}$, $\tilde{\beta}$, *and* $\tilde{\beta}_1$ *such that* $\gamma_h \leq \tilde{\gamma}$, $\alpha_h \geq \tilde{\alpha}$, $\beta_h \geq \tilde{\beta}$, *and* $\beta_{1,h} \geq \tilde{\beta}_1$ *for all* h. *Then, there exists* $C > 0$*, independent of* h*, such that the following Céa error estimate holds:*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\| \leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in X_{1,h} \times M_{1,h} \times M_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)\|. \tag{2.18}$$

Proof. See [19, Section 4] (see also [24, Theorem 3.3] or [20, Theorem 5]). □

2.3 Analysis of the mixed variational formulation

In this section we apply Theorem 2.1 to prove the well-posedness of (2.7) (equivalently (2.10)). We begin with the following lemma showing the Lipschitz-continuity and strong monotonicity of the nonlinear operator $\mathbf{A}_1$, which provides the hypothesis i) of Theorem 2.1.

LEMMA 2.1 *Let* γ_0 *and* α_0 *be the positive constants from* (1.3) *and* (1.4)*, respectively. Then, for each* $\mathbf{t}$, $\mathbf{r} \in X_1$ *there hold*

$$\|\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{r})\|_{X_1'} \leq 2\,\gamma_0\, \|\mathbf{t} - \mathbf{r}\|_{X_1}, \tag{2.19}$$

and

$$[\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{r}), \mathbf{t} - \mathbf{r}] \geq 2\,\alpha_0\, \|\mathbf{t} - \mathbf{r}\|_{X_1}^2. \tag{2.20}$$

Proof. We first observe that for each $\tilde{\mathbf{r}} \in X_1$ the Gâteaux derivative $\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ is a bilinear form on $X_1 \times X_1$, which is uniformly bounded and uniformly X_1-elliptic. In fact, using the definitions of $\mathbf{A}_1$ and μ_{ij}, we find that

$$\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) = 2 \int_\Omega \left\{ \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \mu_{ij}(\tilde{\mathbf{r}})\, r_{kl}\, s_{ij} \right\} \qquad \forall\, \mathbf{r}, \mathbf{s} \in X_1, \tag{2.21}$$

which, according to (1.3) and (1.4), implies that

$$|\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})| \;\le\; 2\,\gamma_0\,\|\mathbf{r}\|_{X_1}\,\|\mathbf{s}\|_{X_1} \qquad \forall\,\mathbf{r},\,\mathbf{s}\,\in\,X_1\,, \tag{2.22}$$

and

$$\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \;\ge\; 2\,\alpha_0\,\|\mathbf{s}\|^2_{X_1} \qquad \forall\,\mathbf{s}\,\in\,X_1\,. \tag{2.23}$$

Now, given $\mathbf{t},\,\mathbf{r}\,\in\,X_1$, a straightforward application of the mean value theorem yields the existence of a convex combination of $\mathbf{t}$ and $\mathbf{r}$, say $\tilde{\mathbf{r}}\,\in\,X_1$, such that

$$[\mathbf{A}_1(\mathbf{t})-\mathbf{A}_1(\mathbf{r}),\mathbf{s}] \;=\; \mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{t}-\mathbf{r},\mathbf{s}) \quad \forall\,\mathbf{s}\,\in\,X_1\,. \tag{2.24}$$

Hence, (2.19) and (2.20) follow easily from (2.24) and the estimates (2.22) and (2.23). □

We now show the continuous inf-sup condition for $\mathbf{B}$, which gives the hypothesis ii) of Theorem 2.1.

LEMMA 2.2 *There exists $\beta > 0$ such that*

$$\sup_{\boldsymbol{\tau}\,\in\,M_1\backslash\{\mathbf{0}\}} \frac{[\mathbf{B}(\boldsymbol{\tau}),\mathbf{v}]}{||\boldsymbol{\tau}||_{M_1}} \;\ge\; \beta\,||\mathbf{v}||_M \qquad \forall\,\mathbf{v}\,\in\,M\,. \tag{2.25}$$

Proof. We first notice from (2.8) that actually $\mathbf{B}: M_1 \to M'$ is defined by $\mathbf{B}(\boldsymbol{\tau}) \,:=\, -\,\mathbf{div}(\boldsymbol{\tau})$ $\forall\,\boldsymbol{\tau}\,\in\,M_1$. Thus, given $\mathbf{v}\,\in\,M$, we proceed as in the proof of [26, Theorem 2.1] and let $\mathbf{z}\,\in\,[H^1_0(\Omega)]^2$ be the unique weak solution of the boundary value problem:

$$-\,\Delta\mathbf{z} \;=\; \mathbf{v} \quad \text{in} \quad \Omega\,, \quad \mathbf{z} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{2.26}$$

Then, defining $\bar{\boldsymbol{\tau}}$ as the $\mathbb{H}_0(\mathbf{div};\Omega)$-component of $\nabla\mathbf{z}\,\in\,\mathbb{H}(\mathbf{div};\Omega)$, we find that $\bar{\boldsymbol{\tau}}\,\in\,M_1$ and $\mathbf{B}(\bar{\boldsymbol{\tau}}) \,=\, -\,\mathbf{div}(\bar{\boldsymbol{\tau}}) \,=\, -\,\mathbf{div}(\nabla\mathbf{z}) \,=\, \mathbf{v}$, which proves that $\mathbf{B}$ is surjective, that is (2.25). □

The continuous inf-sup condition for $\mathbf{B}_1$, which constitutes the hypothesis iii) of Theorem 2.1, is established next. Before it, we need to recall the following technical result.

LEMMA 2.3 *There exists $c_1 > 0$, depending only on Ω, such that*

$$c_1\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \;\le\; \left\|\boldsymbol{\tau}^{\mathtt{d}}\right\|^2_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega} \quad \forall\,\boldsymbol{\tau}\,\in\,\mathbb{H}_0(\mathbf{div};\Omega)\,. \tag{2.27}$$

Proof. See [2, Lemma 3.1] or [6, Proposition 3.1, Chapter IV]. □

LEMMA 2.4 *Let $V \,:=\, \mathrm{Ker}(\mathbf{B})$. Then, there exists $\beta_1 > 0$ such that*

$$\sup_{\mathbf{s}\in X_1\backslash\{\mathbf{0}\}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \;\ge\; \beta_1\,||\boldsymbol{\tau}||_{M_1} \qquad \forall\,\boldsymbol{\tau}\,\in\,V\,. \tag{2.28}$$

Proof. It is easy to see that $V \,:=\, \mathrm{Ker}(\mathbf{B}) \,=\, \big\{\boldsymbol{\tau}\in M_1 : \quad \mathbf{div}(\boldsymbol{\tau}) \,=\, \mathbf{0}\big\}$. Then, given $\boldsymbol{\tau}\,\in\,V$, we notice that $\boldsymbol{\tau}^{\mathtt{d}}\,\in\,X_1$, and hence, bounding below the supremum with $s \,=\, -\,\boldsymbol{\tau}^{\mathtt{d}}$, and applying Lemma 2.3, we find that

$$\sup_{\mathbf{s}\in X_1\backslash\{\mathbf{0}\}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \;=\; \sup_{\mathbf{s}\in X_1\backslash\{\mathbf{0}\}} \frac{-\displaystyle\int_\Omega \mathbf{s}:\boldsymbol{\tau}^{\mathtt{d}}}{||\mathbf{s}||_{X_1}} \;\ge\; \frac{\displaystyle\int_\Omega \boldsymbol{\tau}^{\mathtt{d}}:\boldsymbol{\tau}^{\mathtt{d}}}{||\boldsymbol{\tau}^{\mathtt{d}}||_{X_1}}$$

$$=\; \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega} \;\ge\; \sqrt{c_1}\,\|\boldsymbol{\tau}\|_{0,\Omega} \;=\; \sqrt{c_1}\,\|\boldsymbol{\tau}\|_{M_1}\,,$$

which finishes the proof. □

We are now in a position to establish the main result of this section.

THEOREM 2.4 *There exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X_1 \times M_1 \times M$ *solution of problem* (2.10). *Moreover, there exists* $C > 0$*, independent of the solution, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{X_1 \times M_1 \times M} \le C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\}. \tag{2.29}$$

Proof. Thanks to Lemmas 2.1, 2.2, and 2.4, the proof follows from a straightforward application of Theorem 2.1, taking also into account that $\mathbf{A}_1(\mathbf{0})$ becomes the null functional $\mathbf{0}$. □

2.4 The mixed finite element method

In this section we apply Theorems 2.2 and 2.3 to derive finite element subspaces $X_{1,h}$, $M_{1,h}$ and M_h of X_1, M_1 and M, respectively, yielding the following Galerkin scheme for (2.7) (equivalently (2.10)) to be well-posed and convergent: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned}
&[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &&= [\mathbf{H}, \mathbf{s}_h] \quad \forall\, \mathbf{s}_h \in X_{1,h}\,,\\
&[\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] &&= [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in M_{1,h}\,,\\
&[\mathbf{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= [\mathbf{F}, \mathbf{v}_h] \quad \forall\, \mathbf{v}_h \in M_h\,.
\end{aligned} \tag{2.30}$$

We first realize, thanks to the analysis provided in the proof of Lemma 2.1, that the hypothesis i) of Theorem 2.2 is automatically satisfied by any subspace $X_{1,h}$. In fact, it is clear from (2.24) that, given $\mathbf{t}_h$, $\mathbf{r}_h \in X_{1,h}$, there holds

$$\begin{aligned}
[\mathbf{A}_{1,h}(\mathbf{t}_h) - \mathbf{A}_{1,h}(\mathbf{r}_h), \mathbf{s_h}] &= [p_h' \,\mathbf{A}_1(\mathbf{t}_h) - p_h' \,\mathbf{A}_1(\mathbf{r}_h), \mathbf{s_h}]\\
= [\mathbf{A}_1(\mathbf{t}_h) - \mathbf{A}_1(\mathbf{r}_h), \mathbf{s_h}] &= \mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}}_h)(\mathbf{t}_h - \mathbf{r}_h, \mathbf{s}_h) \quad \forall\, \mathbf{s}_h \in X_{1,h}\,,
\end{aligned} \tag{2.31}$$

where $\tilde{\mathbf{r}}_h \in X_{1,h}$ is a convex combination of $\mathbf{t}_h$ and $\mathbf{r}_h$. Hence, the Lipschitz-continuity and strong monotonicity of $\mathbf{A}_{1,h} : X_{1,h} \to X_{1,h}'$ follow straightforwardly from the identity (2.31) and the estimates (2.22) and (2.23). Moreover, the corresponding constants γ_h and α_h are actually independent of h since they coincide with those provided by Lemma 2.1.

Next, we recall from our analysis for the linear case (cf. [26, Lemma 3.2]) that, given a non-negative integer k, the hypothesis ii) of Theorem 2.2, that is the discrete inf-sup condition for $\mathbf{B}$, holds true for Raviart-Thomas elements of order k and piecewise polynomials of degree $\le k$. More precisely, let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T such that $\bar{\Omega} = \cup \{T : T \in \mathcal{T}_h\}$, and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $\ell \ge 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials of total degree at most ℓ defined on S. Then, for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order k (see, e.g. [34], [6])

$$\mathbb{RT}_k(T) = [\mathbb{P}_k(T)]^2 \oplus \mathbb{P}_k(T)\,\mathbf{x}, \tag{2.32}$$

where $\mathbf{x}$ is a generic vector of $\mathbb{R}^2$, and let $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global tensor space, that is

$$\mathbb{RT}_k(\mathcal{T}_h) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega) : \quad (\tau_{i1}, \tau_{i2})^{\mathbf{t}}|_T \in \mathbb{RT}_k(T) \quad \forall\, i \in \{1, 2\}, \quad \forall\, T \in \mathcal{T}_h \right\}. \tag{2.33}$$

We also let $\mathbb{P}_k(\mathcal{T}_h)$ be the global space of piecewise polynomials of degree $\leq k$, that is

$$\mathbb{P}_k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) : \quad v|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h \right\}. \tag{2.34}$$

In this way, we introduce the following subspaces of M_1 and M, respectively,

$$M_{1,h} := \mathbb{RT}_k(\mathcal{T}_h) \cap \mathbb{H}_0(\mathbf{div};\Omega) = \left\{ \boldsymbol{\tau}_h \in \mathbb{RT}_k(\mathcal{T}_h) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) = 0 \right\}, \tag{2.35}$$

and

$$M_h := [\mathbb{P}_k(\mathcal{T}_h)]^2 . \tag{2.36}$$

Hence, the following result confirms the verification of the hypothesis ii) of Theorem 2.2.

Lemma 2.5 *Let k be a non-negative integer and let $M_{1,h}$ and M_h be given by* (2.35) *and* (2.36)*. Then there exists $\beta > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_h \in M_{1,h} \setminus \{\mathbf{0}\}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \geq \beta \, \|\mathbf{v}_h\|_M \qquad \forall \, \mathbf{v}_h \in M_h . \tag{2.37}$$

Proof. See [26, Lemma 3.2]. □

We now aim to satisfy the hypothesis iii) of Theorem 2.2, that is the discrete inf-sup condition for $\mathbf{B}_1$. To this end, we notice that the proof of Lemma 2.4 can be applied verbatim to the present discrete case if we assume that

$$\boldsymbol{\tau}_h^{\mathtt{d}} \in X_{1,h} \qquad \forall \, \boldsymbol{\tau}_h \in V_h , \tag{2.38}$$

where $V_h := \{\boldsymbol{\tau}_h \in M_{1,h} : \quad [\mathbf{B}(\boldsymbol{\tau}_h), \mathbf{v}_h] = 0 \quad \forall \, \mathbf{v}_h \in M_h\}$ is the discrete kernel of $\mathbf{B}$. Now, it is quite clear that, given $M_{1,h}$ and M_h, the assumption (2.38) restricts the feasible choices for the finite element subspace $X_{1,h}$. For instance, it is easy to see that, with the subspaces (2.35) and (2.36), there holds

$$V_h = \{\boldsymbol{\tau}_h \in M_{1,h} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}\} \subseteq [\mathbb{P}_k(\mathcal{T}_h)]^{2\times 2} ,$$

and hence, noting that certainly $\boldsymbol{\tau}_h^{\mathtt{d}} \in X_1$, a sufficient condition for (2.38) to hold is that $[\mathbb{P}_k(\mathcal{T}_h)]^{2\times 2} \cap X_1$ be contained in $X_{1,h}$. Therefore, defining exactly

$$X_{1,h} := [\mathbb{P}_k(\mathcal{T}_h)]^{2\times 2} \cap X_1 := \left\{ \mathbf{s}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^{2\times 2} : \quad \operatorname{tr}(\mathbf{s}_h) = 0 \right\}, \tag{2.39}$$

we are able to prove the following result.

Lemma 2.6 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (2.39)*,* (2.35)*, and* (2.36)*. Then there exists $\beta_1 > 0$, independent of h, such that*

$$\sup_{\mathbf{s}_h \in X_{1,h} \setminus \{\mathbf{0}\}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{X_1}} \geq \beta \, \|\boldsymbol{\tau}_h\|_{M_1} \qquad \forall \, \boldsymbol{\tau}_h \in V_h . \tag{2.40}$$

Proof. By virtue of the previous analysis, it follows analogously to the proof of Lemma 2.4. □

We are now in a position to establish the following main theorem.

THEOREM 2.5 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (2.39), (2.35), *and* (2.36). *Then there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$ solution of the Galerkin scheme* (2.30). *Moreover, there exist constants c, $C > 0$, independent of h, such that*

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}} \leq c \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\}, \tag{2.41}$$

and

$$||(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)||_{\mathbf{X}} \leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h) \\ \in X_{1,h} \times M_{1,h} \times M_h}} ||(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h)||_{\mathbf{X}}, \tag{2.42}$$

where $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X} := X_1 \times M_1 \times M$ is the unique solution of (2.10).

Proof. Thanks to the previous results in this section, the proof follows from straightforward applications of Theorems 2.2 and 2.3, taking also into account that $\mathbf{A}_{1,h}(\mathbf{0})$ becomes the null functional $\mathbf{0}$. □

Next, in order to derive the rate of convergence of the Galerkin solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$, we need the approximation properties of the finite element subspaces involved (cf. (2.39), (2.35), and (2.36)). For this purpose, we now let $\mathcal{E}_h^k : [H^1(\Omega)]^{2\times 2} \longrightarrow \mathbb{RT}_k(\mathcal{T}_h)$ be the usual equilibrium interpolation operator (see, e.g. [34], [6]), which, given $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$, is characterized by the following identities:

$$\int_e \mathcal{E}_h^k(\boldsymbol{\tau})\boldsymbol{\nu} \cdot \boldsymbol{\psi} = \int_e \boldsymbol{\tau}\boldsymbol{\nu} \cdot \boldsymbol{\psi} \qquad \forall \text{ edge } e \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\psi} \in [\mathbb{P}_k(e)]^2, \quad \text{when} \quad k \geq 0, \tag{2.43}$$

and

$$\int_T \mathcal{E}_h^k(\boldsymbol{\tau}) : \boldsymbol{\psi} = \int_T \boldsymbol{\tau} : \boldsymbol{\psi} \qquad \forall\, T \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\psi} \in [\mathbb{P}_{k-1}(T)]^{2\times 2}, \quad \text{when} \quad k \geq 1. \tag{2.44}$$

It is easy to show, using (2.43) and (2.44), that

$$\mathbf{div}(\mathcal{E}_h^k(\boldsymbol{\tau})) = \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau})), \tag{2.45}$$

where $\mathcal{P}_h^k$ is the orthogonal projector from $[L^2(\Omega)]^2$ into $[\mathbb{P}_k(\mathcal{T}_h)]^2$. Note that $\mathcal{P}_h^k$ can also be identified with $(\mathbf{P}_h^k, \mathbf{P}_h^k)$, where $\mathbf{P}_h^k$ is the orthogonal projector from $L^2(\Omega)$ into $\mathbb{P}_k(\mathcal{T}_h)$. It is well known (see, e.g. [14]) that for each $v \in H^m(\Omega)$, with $0 \leq m \leq k+1$, there holds

$$\|v - \mathbf{P}_h^k(v)\|_{0,T} \leq C\, h_T^m\, |v|_{m,T} \qquad \forall T \in \mathcal{T}_h. \tag{2.46}$$

In addition, the operator $\mathcal{E}_h^k$ satisfies the following approximation properties (see, e.g. [6], [34]):

$$\|\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau})\|_{0,T} \leq C\, h_T^m\, |\boldsymbol{\tau}|_{m,T} \qquad \forall T \in \mathcal{T}_h, \tag{2.47}$$

for each $\boldsymbol{\tau} \in [H^m(\Omega)]^{2\times 2}$, with $1 \leq m \leq k+1$,

$$\|\mathbf{div}(\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau}))\|_{0,T} \leq C\, h_T^m\, |\mathbf{div}(\boldsymbol{\tau})|_{m,T} \qquad \forall T \in \mathcal{T}_h, \tag{2.48}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$ such that $\mathbf{div}(\boldsymbol{\tau}) \in [H^m(\Omega)]^2$, with $0 \leq m \leq k+1$, and

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu} - \mathcal{E}_h^k(\boldsymbol{\tau})\,\boldsymbol{\nu}\|_{0,e} \leq C\, h_e^{1/2}\, \|\boldsymbol{\tau}\|_{1,T_e} \qquad \forall \text{ edge } e \in \mathcal{T}_h, \tag{2.49}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$, where $T_e \in \mathcal{T}_h$ contains e on its boundary. In particular, note that (2.48) follows easily from (2.45) and (2.46). Moreover, it turns out (see, e.g. Theorem 3.16 in [29]) that $\mathcal{E}_h^k$ can also be defined as a bounded linear operator from the larger space $[H^\delta(\Omega)]^{2\times 2} \cap \mathbb{H}(\mathbf{div};\Omega)$ into $\mathbb{RT}_k(\mathcal{T}_h)$ for all $\delta \in (0,1]$, and that in this case there holds the following interpolation error estimate

$$\|\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau})\|_{0,T} \,\le\, C\, h_T^\delta \Big\{ \|\boldsymbol{\tau}\|_{\delta,T} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,T} \Big\} \qquad \forall\, T \,\in\, \mathcal{T}_h\,. \tag{2.50}$$

Then, as a consequence of (2.46), (2.47), (2.48), (2.49), (2.50), and the usual interpolation estimates, we find that the finite element subspaces $X_{1,h}$, $M_{1,h}$, and M_h given by (2.39), (2.35), and (2.36), satisfy the following approximation properties:

($\mathrm{AP}_{1,h}^{\mathbf{t}}$) *For each $\delta \in [0,k+1]$ and for each $\mathbf{s} \in [H^\delta(\Omega)]^{2\times 2} \cap X_1$ there exists $\mathbf{s}_h \in X_{1,h}$ such that*

$$\|\mathbf{s} - \mathbf{s}_h\|_{X_1} \,=\, \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \,\le\, C\, h^\delta\, \|\mathbf{s}\|_{\delta,\Omega}\,.$$

($\mathrm{AP}_{1,h}^{\boldsymbol{\sigma}}$) *For each $\delta \in (0,k+1]$ and for each $\boldsymbol{\tau} \in [H^\delta(\Omega)]^{2\times 2} \cap \mathbb{H}_0(\mathbf{div};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in [H^\delta(\Omega)]^2$ there exists $\boldsymbol{\tau}_h \in M_{1,h}$ such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{M_1} \,=\, \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathrm{div},\Omega} \,\le\, C\, h^\delta \Big\{ \|\boldsymbol{\tau}\|_{\delta,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{\delta,\Omega} \Big\}\,.$$

($\mathrm{AP}_h^{\mathbf{u}}$) *For each $\delta \in [0,k+1]$ and for each $\mathbf{v} \in [H^\delta(\Omega)]^2$ there exists $\mathbf{v}_h \in M_h$ such that*

$$\|\mathbf{v} - \mathbf{v}_h\|_{M} \,=\, \|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega} \,\le\, C\, h^\delta\, \|\mathbf{v}\|_{\delta,\Omega}\,.$$

The following theorem establishes the rate of convergence of the Galerkin scheme (2.30).

THEOREM 2.6 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (2.39), (2.35), *and* (2.36). *Let $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X_1 \times M_1 \times M$ and $(\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$ be the unique solutions of the continuous and discrete formulations* (2.10) *and* (2.30), *respectively. Assume that $\mathbf{t} \in [H^\delta(\Omega)]^{2\times 2}$, $\boldsymbol{\sigma} \in [H^\delta(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^\delta(\Omega)]^2$ and $\mathbf{u} \in [H^\delta(\Omega)]^2$, for some $\delta \in (0,k+1]$. Then there exists $C > 0$, independent of h, such that*

$$\|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}} \,\le\, C\, h^\delta \Big\{ \|\mathbf{t}\|_{\delta,\Omega} \,+\, \|\boldsymbol{\sigma}\|_{\delta,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{\delta,\Omega} \,+\, \|\mathbf{u}\|_{\delta,\Omega} \Big\}\,. \tag{2.51}$$

Proof. It follows from the Céa estimate (2.42) (cf. Theorem 2.5) and the above approximation properties. □

3 The augmented variational formulation

In this section we propose an augmented formulation for (2.10) and a corresponding discrete scheme whose main advantage is the elimination of the assumption (2.38) on the finite element subspace for the velocity gradient $\mathbf{t}$. More precisely, we show that a suitable enrichment of (2.10) yields an associated Galerkin scheme whose well-posedness is guaranteed by any finite dimensional subspace $X_{1,h}$ of X_1 and any pair $(M_{1,h}, M_h)$ satisfying $\mathbf{div}(M_{1,h}) \subseteq M_h$ and the discrete inf-sup condition for $\mathbf{B}$.

3.1 The continuous augmented formulation

As mentioned above, we now enrich the formulation (2.10) with the further introduction of the constitutive law relating $\boldsymbol{\sigma}$ and $\mathbf{t}$ (written as in the first equation of (2.5)) multiplied by a stabilization parameter. More precisely, given $\kappa > 0$, to be chosen later, we add

$$\kappa \int_\Omega (\boldsymbol{\sigma}^{\mathrm{d}} - 2\,\mu(|\mathbf{t}|)\,\mathbf{t}) : \boldsymbol{\tau}^{\mathrm{d}} \;=\; 0 \qquad \forall\,\boldsymbol{\tau} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega) \tag{3.1}$$

to the first equation of (2.10), and subtract the second equation of (2.10) to the resulting expression. In addition, we keep the third equation of (2.10) as it is, but multiplied by -1. In this way, denoting $X := X_1 \times M_1$, we arrive at the following augmented formulation (written as a single saddle point system): Find $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in X \times M$ such that

$$\begin{aligned} [\mathcal{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})] \,+\, [\mathcal{B}(\mathbf{s},\boldsymbol{\tau}),\mathbf{u}] \;&=\; [\mathcal{F},(\mathbf{s},\boldsymbol{\tau})] \quad \forall\,(\mathbf{s},\boldsymbol{\tau}) \in X\,, \\ [\mathcal{B}(\mathbf{t},\boldsymbol{\sigma}),\mathbf{v}] \;&=\; [\mathcal{G},\mathbf{v}] \qquad\quad \forall\,\mathbf{v} \in M\,, \end{aligned} \tag{3.2}$$

where the nonlinear operator $\mathcal{A} : X \to X'$, the linear operator $\mathcal{B} : X \to M'$, and the functionals $\mathcal{F} \in X'$ and $\mathcal{G} \in M'$, are defined by:

$$[\mathcal{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})] \,:=\, [\mathbf{A}_1(\mathbf{t}),\mathbf{s}] \,+\, [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\sigma}] \,-\, [\mathbf{B}_1(\mathbf{t}),\boldsymbol{\tau}] \,+\, \kappa \int_\Omega (\boldsymbol{\sigma}^{\mathrm{d}} - 2\,\mu(|\mathbf{t}|)\,\mathbf{t}) : \boldsymbol{\tau}^{\mathrm{d}}\,, \tag{3.3}$$

$$[\mathcal{B}(\mathbf{s},\boldsymbol{\tau}),\mathbf{v}] \,:=\, -\,[\mathbf{B}(\boldsymbol{\tau}),\mathbf{v}] \,=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{3.4}$$

$$[\mathcal{F},(\mathbf{s},\boldsymbol{\tau})] \,:=\, [\mathbf{H},\mathbf{s}] \,-\, [\mathbf{G},\boldsymbol{\tau}] \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma\,, \tag{3.5}$$

and

$$[\mathcal{G},\mathbf{v}] \,:=\, -\,[\mathbf{F},\mathbf{v}] \,=\, -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,. \tag{3.6}$$

Our next goal is to show the unique solvability of the variational formulation (3.2), whence (2.10) and (3.2) share the same unique solution (denoted either $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X_1 \times M_1 \times M$ or $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in X \times M$).

We first recall from [36] the following abstract theorem.

Theorem 3.1 *Let X, M be Hilbert spaces and let $\mathcal{A} : X \to X'$ and $\mathcal{B} : X \to M'$ be nonlinear and linear operators, respectively. Let $V := \mathrm{Ker}\,(\mathcal{B}) = \{\, x \in X : \; [\mathcal{B}(x),q] = 0 \;\; \forall\, q \in M \,\}$. Assume that $\mathcal{A}$ is Lipschitz-continuous on X and that for all $\tilde{z} \in X$, $\mathcal{A}(\tilde{z} + \cdot)$ is uniformly strongly monotone on V, that is, there exist constants γ, $\alpha > 0$ such that*

$$||\mathcal{A}(x) \,-\, \mathcal{A}(y)||_{X'} \,\le\, \gamma\,\|x - y\|_X \quad \forall\, x,\, y \in X\,,$$

and

$$[\mathcal{A}(\tilde{z} + x) - \mathcal{A}(\tilde{z} + y), x - y] \,\ge\, \alpha\,\|x - y\|_X^2$$

for all $\tilde{z} \in X$ and for all x, $y \in V$. In addition, assume that there exists $\beta > 0$ such that for all $q \in M$

$$\sup_{x \in X\setminus\{\mathbf{0}\}} \frac{[\mathcal{B}(x),q]}{\|x\|_X} \,\ge\, \beta\,\|q\|_M\,.$$

Then, given $(\mathcal{F}, \mathcal{G}) \in X' \times M'$, *there exists a unique* $(x, p) \in X \times M$ *such that*

$$
\begin{aligned}
[\mathcal{A}(x), y] \,+\, [\mathcal{B}(y), p] \;&=\; [\mathcal{F}, y] \quad \forall\, y \in X\,, \\
[\mathcal{B}(x), q] \;&=\; [\mathcal{G}, q] \quad \forall\, q \in M\,.
\end{aligned}
$$

Further, the following estimates hold

$$
\|x\|_X \;\le\; \frac{1}{\alpha}\,\|\mathcal{F}\| + \frac{1}{\beta}\left(1 + \frac{\gamma}{\alpha}\right)\,\|\mathcal{G}\|\,, \tag{3.7}
$$

$$
\|p\|_M \;\le\; \frac{1}{\beta}\left(1 + \frac{\gamma}{\alpha}\right)\left(\|\mathcal{F}\| \,+\, \frac{\gamma}{\beta}\|\mathcal{G}\|\right)\,. \tag{3.8}
$$

Proof. It is a particular case of Proposition 2.3 in [36]. $\square$

In order to apply Theorem 3.1 to the augmented formulation (3.2), we need the following two lemmas establishing the required properties for our nonlinear operator $\mathcal{A}$ (cf. (3.3)).

LEMMA 3.1 *Let* $\mathcal{A}$ *be the nonlinear operator defined by* (3.3). *Then, there exists a constant* $\gamma > 0$ *such that*

$$
||\mathcal{A}(\mathbf{t}, \boldsymbol{\sigma}) \,-\, \mathcal{A}(\mathbf{s}, \boldsymbol{\tau})||_{X'} \;\le\; \gamma\,\|(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}, \boldsymbol{\tau})\|_X \qquad \forall\,(\mathbf{t}, \boldsymbol{\sigma}),\, (\mathbf{s}, \boldsymbol{\tau}) \,\in\, X\,.
$$

Proof. Given $(\mathbf{t}, \boldsymbol{\sigma})$, $(\mathbf{s}, \boldsymbol{\tau})$, $(\mathbf{r}, \boldsymbol{\zeta}) \,\in\, X$, we obtain, according to (3.3) and the definition of $\mathbf{A}_1$ (cf. (2.8)), that

$$
\begin{aligned}
[\mathcal{A}(\mathbf{t}, \boldsymbol{\sigma}) - \mathcal{A}(\mathbf{s}, \boldsymbol{\tau}), (\mathbf{r}, \boldsymbol{\zeta})] \;&=\; [\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s}), \mathbf{r}] \,+\, [\mathbf{B}_1(\mathbf{r}), \boldsymbol{\sigma} - \boldsymbol{\tau}] \,-\, [\mathbf{B}_1(\mathbf{t} - \mathbf{s}), \boldsymbol{\zeta}] \\
&\quad + \kappa \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\tau})^{\mathrm{d}} : \boldsymbol{\zeta}^{\mathrm{d}} \,-\, \kappa\,[\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s}), \boldsymbol{\zeta}^{\mathrm{d}}]\,.
\end{aligned} \tag{3.9}
$$

Hence, it follows easily from (3.9), Lemma 2.1, and the boundedness of $\mathbf{B}_1$, that $\mathcal{A}$ is Lipschitz continuous on X with a constant γ depending on γ_0, $\|\mathbf{B}_1\|$, and κ. $\square$

LEMMA 3.2 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be the operators defined by* (3.3) *and* (3.4), *assume that the parameter* κ *lies in* $\left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$, *where* γ_0 *and* α_0 *are the positive constants from* (1.3) *and* (1.4), *and let* $V \,:=\, \mathrm{Ker}\,(\mathcal{B})$. *Then, there exists a constant* $\alpha > 0$ *such that*

$$
[\mathcal{A}((\mathbf{r}, \boldsymbol{\zeta}) + (\mathbf{t}, \boldsymbol{\sigma})) - \mathcal{A}((\mathbf{r}, \boldsymbol{\zeta}) + (\mathbf{s}, \boldsymbol{\tau})), (\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}, \boldsymbol{\tau})] \;\ge\; \alpha\,\|(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}, \boldsymbol{\tau})\|_X^2
$$

for all $(\mathbf{r}, \boldsymbol{\zeta}) \in X$, *and for all* $(\mathbf{t}, \boldsymbol{\sigma})$, $(\mathbf{s}, \boldsymbol{\tau}) \in V$.

Proof. We first observe from the definition of $\mathcal{B}$ (cf. (3.4)) that

$$
V \;:=\; \mathrm{Ker}\,(\mathcal{B}) \;=\; X_1 \,\times\, \Big\{ \boldsymbol{\tau} \,\in\, M_1 : \quad \mathbf{div}(\boldsymbol{\tau}) \;=\; \mathbf{0} \Big\}\,. \tag{3.10}
$$

Now, given $(\mathbf{r}, \boldsymbol{\zeta}) \in X$ and $(\mathbf{t}, \boldsymbol{\sigma})$, $(\mathbf{s}, \boldsymbol{\tau}) \,\in\, V$, we find, using the identity (3.9) and noting that the terms involving $\mathbf{B}_1$ cancell out, that

$$
\begin{aligned}
[\mathcal{A}((\mathbf{r}, \boldsymbol{\zeta}) + (\mathbf{t}, \boldsymbol{\sigma})) - \mathcal{A}((\mathbf{r}, \boldsymbol{\zeta}) + (\mathbf{s}, \boldsymbol{\tau})), (\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}, \boldsymbol{\tau})] \;&=\; [\mathbf{A}_1(\mathbf{r} + \mathbf{t}) - \mathbf{A}_1(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{s}] \\
+ \quad \kappa\,\|(\boldsymbol{\sigma} - \boldsymbol{\tau})^{\mathrm{d}}\|_{0,\Omega}^2 \,-\, \kappa\,[\mathbf{A}_1(\mathbf{r} + \mathbf{t}) - \mathbf{A}_1(\mathbf{r} + \mathbf{s}), (\boldsymbol{\sigma} - \boldsymbol{\tau})^{\mathrm{d}}]\,.&
\end{aligned}
$$

Then, using that $[\mathbf{A}_1(\mathbf{r}+\mathbf{t}) - \mathbf{A}_1(\mathbf{r}+\mathbf{s}), \mathbf{t}-\mathbf{s}] = [\mathbf{A}_1(\mathbf{r}+\mathbf{t}) - \mathbf{A}_1(\mathbf{r}+\mathbf{s}), (\mathbf{r}+\mathbf{t}) - (\mathbf{r}+\mathbf{s})]$, and applying the strong monotonicity and Lipschitz-continuity of $\mathbf{A}_1$ (cf. Lemma 2.1), we deduce from the above equation that

$$
\begin{aligned}
&[\mathcal{A}((\mathbf{r},\boldsymbol{\zeta}) + (\mathbf{t},\boldsymbol{\sigma})) - \mathcal{A}((\mathbf{r},\boldsymbol{\zeta}) + (\mathbf{s},\boldsymbol{\tau})), (\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s},\boldsymbol{\tau})] \\
&\quad \geq\; 2\,\alpha_0\,\|\mathbf{t}-\mathbf{s}\|_{X_1}^2 \;+\; \kappa\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega}^2 \;-\; 2\,\kappa\,\gamma_0\,\|\mathbf{t}-\mathbf{s}\|_{X_1}\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega} \\
&\quad \geq\; 2\,\alpha_0\,\|\mathbf{t}-\mathbf{s}\|_{X_1}^2 \;+\; \kappa\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega}^2 \;-\; \kappa\,\gamma_0\,\left\{\frac{\|\mathbf{t}-\mathbf{s}\|_{X_1}^2}{\delta} \;+\; \delta\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega}^2\right\} \\
&\quad =\; \left(2\,\alpha_0 - \frac{\kappa\,\gamma_0}{\delta}\right)\,\|\mathbf{t}-\mathbf{s}\|_{X_1}^2 \;+\; \kappa\,(1-\gamma_0\,\delta)\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega}^2 \qquad \forall\,\delta>0\,.
\end{aligned}
$$

It follows that the constants multiplying the norms above become positive if $\delta \in \left(0, \dfrac{1}{\gamma_0}\right)$ and $\kappa \in \left(0, \dfrac{2\,\alpha_0\,\delta}{\gamma_0}\right)$. In particular, for $\delta = \dfrac{1}{2\,\gamma_0}$ we require $\kappa \in \left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$, whence we find that

$$
\begin{aligned}
&[\mathcal{A}((\mathbf{r},\boldsymbol{\zeta}) + (\mathbf{t},\boldsymbol{\sigma})) - \mathcal{A}((\mathbf{r},\boldsymbol{\zeta}) + (\mathbf{s},\boldsymbol{\tau})), (\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s},\boldsymbol{\tau})] \\
&\quad \geq\; 2\,(\alpha_0 - \kappa\,\gamma_0^2)\,\|\mathbf{t}-\mathbf{s}\|_{X_1}^2 \;+\; \frac{\kappa}{2}\,\|(\boldsymbol{\sigma}-\boldsymbol{\tau})^{\mathtt{d}}\|_{0,\Omega}^2\,.
\end{aligned}
$$

Finally, this inequality and Lemma 2.3 imply the required estimate with a constant α depending on α_0, γ_0, κ, and c_1 (cf. Lemma 2.3). □

On the other hand, it is clear from the definition of our linear operator $\mathcal{B}$ (cf. (3.4)) that

$$
\sup_{(\mathbf{s},\boldsymbol{\tau})\in X\backslash\{\mathbf{0}\}} \frac{[\mathcal{B}(\mathbf{s},\boldsymbol{\tau}),\mathbf{v}]}{\|(\mathbf{s},\boldsymbol{\tau})\|_X} \;=\; \sup_{\boldsymbol{\tau}\in M_1\backslash\{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}),\mathbf{v}]}{\|\boldsymbol{\tau}\|_{M_1}} \qquad \forall\,\mathbf{v}\in M\,, \tag{3.11}
$$

which implies that the continuous inf-sup conditions for $\mathcal{B}$ and $\mathbf{B}$ coincide. Consequently, the well-posedness of the augmented formulation (3.2) can be established as follows.

Theorem 3.2 *Assume that the parameter κ defining the nonlinear operator $\mathcal{A}$ (cf.* (3.3)*) lies in $\left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$, where γ_0 and α_0 are the positive constants from* (1.3) *and* (1.4)*. Then, there exists a unique $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in X\times M$ solution of* (3.2)*. Moreover, there exists $C>0$, depending on β (cf. Lemma* 2.2*), γ (cf. Lemma* 3.1*), and α (cf. Lemma* 3.2*), such that*

$$
\|((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u})\|_{X\times M} \;\leq\; C\left\{\|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{g}\|_{1/2,\Gamma}\right\}.
$$

Proof. By virtue of the previous remark and Lemmas 2.2, 3.1, and 3.2, the proof follows from a direct application of Theorem 3.1. □

3.2 The discrete augmented formulation

We now come to the analysis of the Galerkin scheme associated with the augmented formulation (3.2). To this end, we now let $X_{1,h}$, $M_{1,h}$, and M_h be finite dimensional subspaces of X_1, M_1, and M, respectively, and define $X_h := X_{1,h} \times M_{1,h}$. Then, we are interested in the following discrete scheme: Find $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in X_h \times M_h$ such that

$$\begin{aligned} [\mathcal{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\tau})] + [\mathcal{B}(\mathbf{s}, \boldsymbol{\tau}), \mathbf{u}_h] &= [\mathcal{F}, (\mathbf{s}, \boldsymbol{\tau})] \quad \forall\, (\mathbf{s}, \boldsymbol{\tau}) \in X_h\,, \\ [\mathcal{B}(\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{v}] &= [\mathcal{G}, \mathbf{v}] \qquad\quad \forall\, \mathbf{v} \in M_h\,. \end{aligned} \tag{3.12}$$

In order to analyze the solvability of (3.12), we first notice from (3.4) that the discrete kernel of $\mathcal{B}$, that is $V_h := \Big\{ (\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h : \quad [\mathcal{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), \mathbf{v}_h] = 0 \quad \forall\, \mathbf{v}_h \in M_h \Big\}$, reduces to

$$V_h = X_{1,h} \times \Big\{ \boldsymbol{\tau}_h \in M_{1,h} : \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in M_h \Big\}.$$

In addition, as in (3.11), we realize that

$$\sup_{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h \setminus \{\mathbf{0}\}} \frac{[\mathcal{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), \mathbf{v}_h]}{\|(\mathbf{s}_h, \boldsymbol{\tau}_h)\|_X} = \sup_{\boldsymbol{\tau}_h \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \qquad \forall\, \mathbf{v}_h \in M_h\,, \tag{3.13}$$

which implies that the discrete inf-sup conditions for $\mathcal{B}$ and $\mathbf{B}$ coincide. Hence, we are in a position to establish the following result.

THEOREM 3.3 *Besides the hypotheses of Theorem* 3.2*, assume that* V_h *is contained in* V *(cf.* (3.10)*), that is*

$$\boldsymbol{\tau}_h \in M_{1,h} \quad \textit{and} \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in M_h \quad \Rightarrow \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}\,, \tag{3.14}$$

and that $\mathbf{B}$ *satisfies the discrete inf-sup condition on* $M_{1,h} \times M_h$*, that is, there exists* $\tilde{\beta} > 0$*, independent of* h*, such that*

$$\sup_{\boldsymbol{\tau}_h \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \geq \tilde{\beta}\, \|\mathbf{v}_h\|_M \qquad \forall\, \mathbf{v}_h \in M_h\,. \tag{3.15}$$

Then there exists a unique $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in X_h \times M_h$ *solution of* (3.12)*. Moreover, there exist* C_1, $C_2 > 0$*, independent of* h*, such that*

$$\|((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h)\|_{X \times M} \leq C_1 \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\}\,, \tag{3.16}$$

and

$$\begin{aligned} &\|((\mathbf{t}, \boldsymbol{\sigma}), \mathbf{u}) - ((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h)\|_{X \times M} \\ &\leq \quad C_2 \Big\{ \inf_{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h} \|(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|_X + \inf_{\mathbf{v}_h \in M_h} \|\mathbf{u} - \mathbf{v}_h\|_M \Big\}\,. \end{aligned} \tag{3.17}$$

Proof. It is clear that the Lipschitz-continuity of $\mathcal{A}$ (cf. Lemma 3.1) is also valid on $X_h \times X_h'$, which means that, with the same constant γ from Lemma 3.1, there holds

$$||\mathcal{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}(\mathbf{s}_h, \boldsymbol{\tau}_h)||_{X_h'} \leq \gamma\, \|(\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|_X \qquad \forall\, (\mathbf{t}_h, \boldsymbol{\sigma}_h),\, (\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h\,.$$

In addition, since $V_h \subseteq V$, the strong monotonicity of $\mathcal{A}$ provided by Lemma 3.2 also holds for all $(\mathbf{r}_h, \boldsymbol{\zeta}_h) \in X_h$, and for all $(\mathbf{t}_h, \boldsymbol{\sigma}_h)$, $(\mathbf{s}_h, \boldsymbol{\tau}_h) \in V_h$, with the same constant α. Therefore, the unique solvability of (3.12) and the estimate (3.16) are again consequence of Theorem 3.1 (see also the discrete analogue given by [36, Proposition 2.6]). Furthermore, the Céa estimate (3.17) constitutes a particular application of the general result given by [36, Theorem 2.1]. $\square$

It is interesting to notice at this point that, on the contrary to (2.38), Theorem 3.3 does not impose any additional restriction on $X_{1,h}$, but being only a finite dimensional subspace of X_1. Also, it is easy to see that a sufficient condition for (3.14) to hold is that $\mathbf{div}(M_{1,h}) \subseteq M_h$. On the other hand, we find it important to remark that, though they are denoted in the same way, the unique solutions of the discrete schemes (2.30) and (3.12) not necessarily coincide.

Now, for a concrete example of subspaces verifying (3.14) and (3.15), we recall from Lemma 2.5 that $\mathbf{B}$ satisfies precisely the discrete inf-sup condition with $M_{1,h}$ and M_h given by (2.35) and (2.36), respectively, that is

$$M_{1,h} = \left\{ \boldsymbol{\tau}_h \in \mathbb{RT}_k(\mathcal{T}_h) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) = 0 \right\} \quad \text{and} \quad M_h := [\mathbb{P}_k(\mathcal{T}_h)]^2 .$$

Moreover, since in this case one can easily prove that $\mathbf{div}(M_{1,h}) = M_h$, the assumption (3.14) is automatically satisfied, as well. Therefore, we can state the following result.

THEOREM 3.4 *Besides the hypotheses of Theorem* 3.2, *assume that $X_{1,h}$ is any finite element subspace of X_1, and that $M_{1,h}$ and M_h are given by* (2.35) *and* (2.36). *Then there exists a unique $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in X_h \times M_h$ solution of* (3.12). *Moreover, there exist C_1, $C_2 > 0$, independent of h, such that*

$$\|((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h)\|_{X \times M} \leq C_1 \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\}, \tag{3.18}$$

and

$$\begin{aligned} &\|((\mathbf{t}, \boldsymbol{\sigma}), \mathbf{u}) - ((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h)\|_{X \times M} \\ &\leq \quad C_2 \left\{ \inf_{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h} \|(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|_X + \inf_{\mathbf{v}_h \in M_h} \|\mathbf{u} - \mathbf{v}_h\|_M \right\}. \end{aligned} \tag{3.19}$$

Proof. It is a direct consequence of the previous analysis and Theorem 3.3. $\square$

Next, for the rate of convergence of (3.12) we proceed similarly as we did for Theorem 2.6, using now the Céa estimate (3.17) (or (3.19)), and the approximation properties of the subspaces involved. In particular, for the subspaces from Theorem 2.6 we obtain the same rate provided there. This result is stated as follows.

THEOREM 3.5 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (2.39), (2.35), *and* (2.36). *Let $((\mathbf{t}, \boldsymbol{\sigma}), \mathbf{u}) \in X \times M$ and $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in X_h \times M_h$ be the unique solutions of the continuous and discrete augmented formulations* (3.2) *and* (3.12), *respectively. Assume that $\mathbf{t} \in [H^\delta(\Omega)]^{2\times2}$, $\boldsymbol{\sigma} \in [H^\delta(\Omega)]^{2\times2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^\delta(\Omega)]^2$ and $\mathbf{u} \in [H^\delta(\Omega)]^2$, for some $\delta \in (0, k+1]$. Then there exists $C > 0$, independent of h, such that*

$$\|((\mathbf{t}, \boldsymbol{\sigma}), \mathbf{u}) - ((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h)\|_{X \times M} \leq C\, h^\delta \left\{ \|\mathbf{t}\|_{\delta,\Omega} + \|\boldsymbol{\sigma}\|_{\delta,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\delta,\Omega} + \|\mathbf{u}\|_{\delta,\Omega} \right\}. \tag{3.20}$$

Proof. It follows from the Céa estimate (3.19) and the approximation properties ($\mathrm{AP}^{\mathbf{t}}_{1,h}$), ($\mathrm{AP}^{\boldsymbol{\sigma}}_{1,h}$), and ($\mathrm{AP}^{\mathbf{u}}_{h}$) provided in Section 2.4. $\square$

Alternatively, we could keep $M_{1,h}$ and M_h as given by (2.35) and (2.36), but use a lower polynomial degree for approximating $\mathbf{t}$ in the case $k \geq 1$. For example, instead of (2.39), we could consider:

$$\widetilde{X}_{1,h} := \left\{ \mathbf{s}_h \in [\mathbb{P}_{k-1}(\mathcal{T}_h)]^{2\times 2} : \quad \operatorname{tr}(\mathbf{s}_h) = 0 \right\}, \tag{3.21}$$

which clearly does not satisfy (2.38), but according to Theorem 3.3 (or Theorem 3.4) does constitute a feasible choice for the present augmented scheme. It follows, applying (2.46), that the approximation property of $\widetilde{X}_{1,h}$ becomes as $(\mathrm{AP}^{\mathbf{t}}_{1,h})$, but with regularity range $[0,k]$ instead of $[0,k+1]$, that is:

$(\widetilde{\mathrm{AP}}^{\mathbf{t}}_{1,h})$ *For each* $\delta \in [0,k]$ *and for each* $\mathbf{s} \in [H^\delta(\Omega)]^{2\times 2} \cap X_1$ *there exists* $\mathbf{s}_h \in \widetilde{X}_{1,h}$ *such that*

$$\|\mathbf{s} - \mathbf{s}_h\|_{X_1} = \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leq C\, h^\delta\, \|\mathbf{s}\|_{\delta,\Omega}.$$

Hence, thanks to the approximation properties of $M_{1,h}$ and M_h (cf. $(\mathrm{AP}^{\boldsymbol{\sigma}}_{1,h})$ and $(\mathrm{AP}^{\mathbf{u}}_{h})$ in Section 2.4), we also obtain in this case the same rate of convergence provided by Theorem 3.5, but limited to $\delta \in (0,k]$.

Another feasible choice is to approximate $\mathbf{t}$ by continuous piecewise polynomial tensors. For instance, we could keep again $M_{1,h}$ and M_h as given by (2.35) and (2.36), and consider now:

$$\widehat{X}_{1,h} := \left\{ \mathbf{s}_h \in [C(\overline{\Omega}) \cap \mathbb{P}_{k+1}(\mathcal{T}_h)]^{2\times 2} : \quad \operatorname{tr}(\mathbf{s}_h) = 0 \right\}, \tag{3.22}$$

which, due to the continuity requirement, does not verify (2.38) either. In this case, assuming a convex domain Ω, one can show (cf. [33, eq. (3.5.15) and Remark 6.2.1]) that $\widehat{X}_{1,h}$ satisfies the following approximation property:

$(\widehat{\mathrm{AP}}^{\mathbf{t}}_{1,h})$ *For each* $\delta \in [0,k+1]$ *and for each* $\mathbf{s} \in [H^\delta(\Omega)]^{2\times 2} \cap X_1$ *there exists* $\mathbf{s}_h \in \widehat{X}_{1,h}$ *such that*

$$\|\mathbf{s} - \mathbf{s}_h\|_{X_1} = \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leq C\, h^\delta\, \|\mathbf{s}\|_{\delta,\Omega}.$$

Hence, the rate of convergence of the resulting augmented scheme is the same of Theorem 3.5.

4 A posteriori error analysis

In this section we derive reliable and efficient residual-based a posteriori error estimators for the Galerkin schemes (2.30) and (3.12) with $M_{1,h}$ and M_h given by (2.35) and (2.36).

4.1 Preliminaries and main results

We begin by introducing several notations. We let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, and given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the length of the edge e. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathtt{t}}$, and let $s_e := (-\nu_2, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $e \in \mathcal{E}_h(\Omega)$ and $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$ on each $T \in \mathcal{T}_h$, we let $[\boldsymbol{\tau}\, s_e]$ be the corresponding jump across e, that is $[\boldsymbol{\tau}\, s_e] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\, s_e$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Abusing notation, when $e \in \mathcal{E}_h(\Gamma)$, we also write

$[\boldsymbol{\tau}\, s_e] := \boldsymbol{\tau}|_e\, s_e$. Similar definitions hold for the tangential jumps of scalar fields $v \in L^2(\Omega)$ such that $v|_T \in C(T)$ on each $T \in \mathcal{T}_h$. From now on, when no confusion arises, we simply write s and $\boldsymbol{\nu}$ instead of s_e and $\boldsymbol{\nu}_e$, respectively. Finally, given scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \ \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix}, \text{ and } \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, letting $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X_1 \times M_1 \times M$ and $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$ be the unique solutions of the continuous and discrete formulations (2.10) and (2.30), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$\begin{aligned} \theta_T^2 \,:=\, & \|\,\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,T}^2 \,+\, \big\|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big\|_{0,T}^2 \,+\, h_T^2\, \big\|\operatorname{curl}\{\mathbf{t}_h\}\big\|_{0,T}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e\, \|\,[\mathbf{t}_h\, s]\,\|_{0,e}^2 \,+\, h_T^2\, \|\,\nabla \mathbf{u}_h - \mathbf{t}_h\,\|_{0,T}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\{ \left\| \frac{d\mathbf{g}}{ds} - \mathbf{t}_h\, s \right\|_{0,e}^2 \,+\, \|\mathbf{g} - \mathbf{u}_h\,\|_{0,e}^2 \right\}. \end{aligned} \tag{4.1}$$

Note that the above requires that $\left.\frac{d\mathbf{g}}{ds}\right|_e \in [L^2(e)]^2$ for each $e \in \mathcal{E}_h(\Gamma)$. This is fixed below by assuming that $\mathbf{g} \in [H^1(\Gamma)]^2$. Similarly, letting $((\mathbf{t}, \boldsymbol{\sigma}), \mathbf{u}) \in X \times M$ and $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in X_h \times M_h$ be the unique solutions of the continuous and discrete formulations (3.2) and (3.12), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$\eta_T^2 \,:=\, \theta_T^2 \,+\, h_T^2\, \big\|\operatorname{curl}\{\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\}\big\|_{0,T}^2 \,+\, \sum_{e \in \mathcal{E}(T)} h_e\, \big\|\big[\big(\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big)\, s\big]\big\|_{0,e}^2\,. \tag{4.2}$$

The residual character of each term on the right hand sides of (4.1) and (4.2) is quite clear. As usual the expressions

$$\boldsymbol{\theta} \,:=\, \left\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \right\}^{1/2} \quad \text{and} \quad \boldsymbol{\eta} \,:=\, \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}$$

are employed as the respective global residual error estimators.

The following theorems constitute the main results of this section.

Theorem 4.1 *Let* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X} := X_1 \times M_1 \times M$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$ *be the unique solutions of the continuous and discrete formulations* (2.10) *and* (2.30)*, respectively, and assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$*. Then there exist positive constants* $C_{\mathtt{eff}}$ *and* $C_{\mathtt{rel}}$*, independent of* h*, such that*

$$C_{\mathtt{eff}}\, \boldsymbol{\theta} \,+\, \text{h.o.t.} \,\leq\, \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}} \,\leq\, C_{\mathtt{rel}}\, \boldsymbol{\theta}\,, \tag{4.3}$$

where h.o.t. *stands for one or several terms of higher order.*

THEOREM 4.2 *Let* $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in X\times M$ *and* $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \in X_h\times M_h$ *be the unique solutions of the continuous and discrete augmented formulations* (3.2) *and* (3.12)*, respectively, and assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$*. Then there exist positive constants* $\tilde{C}_{\texttt{eff}}$ *and* $\tilde{C}_{\texttt{rel}}$*, independent of* h*, such that*

$$\tilde{C}_{\texttt{eff}}\,\boldsymbol{\eta} + \text{h.o.t.} \;\le\; \|((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) - ((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h)\|_{X\times M} \;\le\; \tilde{C}_{\texttt{rel}}\,\boldsymbol{\eta}\,, \tag{4.4}$$

where h.o.t. *stands for one or several terms of higher order.*

The efficiency of the global error estimators (lower bounds in (4.3) and (4.4)) is proved below in Section 4.4, whereas the corresponding reliability (upper bounds in (4.3) and (4.4)) are derived next in Sections 4.2 and 4.3.

4.2 Reliability of the a posteriori error estimator $\boldsymbol{\theta}$

We begin by recalling from the analysis in Section 2.3 that $\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ is a uniformly bounded and uniformly elliptic bilinear form on $X_1\times X_1$ for all $\tilde{\mathbf{r}} \in X_1$ (see (2.22) and (2.23) in the proof of Lemma 2.1), and that the operators $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the corresponding continuous inf-sup conditions (cf. Lemmas 2.2 and 2.4). Hence, as a consequence of the continuous dependence result provided by the linear version of Theorem 2.1 (cf. (2.14) with $\mathbf{A}_1$ linear), we conclude that the linear operator $\mathcal{L}$ obtained by adding the three equations of the left hand side of (2.10), after replacing $\mathbf{A}_1$ by the Gâteaux derivative $\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ at any $\tilde{\mathbf{r}} \in X_1$, satisfies a global inf-sup condition. More precisely, there exists a constant $\tilde{C} > 0$ such that

$$\tilde{C}\,\|(\mathbf{r},\boldsymbol{\zeta},\mathbf{w})\|_{\mathbf{X}} \le \sup_{(\mathbf{s},\boldsymbol{\tau},\mathbf{v})\in\mathbf{X}\setminus\{\mathbf{0}\}} \frac{[\mathcal{L}(\mathbf{s},\boldsymbol{\tau},\mathbf{v}),(\mathbf{r},\boldsymbol{\zeta},\mathbf{w})]}{\|(\mathbf{s},\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{X}}} \tag{4.5}$$

for all $(\tilde{\mathbf{r}},(\mathbf{r},\boldsymbol{\zeta},\mathbf{w})) \in X_1\times\mathbf{X}$, where

$$[\mathcal{L}(\mathbf{s},\boldsymbol{\tau},\mathbf{v}),(\mathbf{r},\boldsymbol{\zeta},\mathbf{w})] := \mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) + [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\zeta}] + [\mathbf{B}_1(\mathbf{r}),\boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}),\mathbf{w}] + [\mathbf{B}(\boldsymbol{\zeta}),\mathbf{v}]\,. \tag{4.6}$$

We now have the following preliminary result.

LEMMA 4.1 *Let* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{X} := X_1\times M_1\times M$ *and* $(\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h) \in X_{1,h}\times M_{1,h}\times M_h$ *be the unique solutions of the continuous and discrete formulations* (2.10) *and* (2.30)*, respectively. Then there exists* $C>0$*, independent of* h*, such that*

$$\begin{aligned} C\,\|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}} \;&\le\; \left\|\boldsymbol{\sigma}_h^{\mathtt{d}} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\right\|_{0,\Omega} \\ &+ \sup_{\boldsymbol{\tau}\in M_1\setminus\{\mathbf{0}\}} \frac{R(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{M_1}} + \|f + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \end{aligned} \tag{4.7}$$

where, given any $\boldsymbol{\tau}_h \in M_{1,h}$,

$$R(\boldsymbol{\tau}) := -\langle(\boldsymbol{\tau}-\boldsymbol{\tau}_h)\,\nu,\mathbf{g}\rangle_\Gamma + \int_\Omega \mathbf{t}_h : (\boldsymbol{\tau}-\boldsymbol{\tau}_h) + \int_\Omega \mathbf{u}_h\cdot\mathbf{div}(\boldsymbol{\tau}-\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau} \in M_1\,. \tag{4.8}$$

Proof. As in (2.24), and thanks to the mean value theorem, there exists a convex combination of $\mathbf{t}$ and $\mathbf{t}_h$, say $\tilde{\mathbf{r}}_h \in X_1$, such that

$$\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}}_h)(\mathbf{t}-\mathbf{t}_h,\mathbf{s}) = [\mathbf{A}_1(\mathbf{t}),\mathbf{s}] - [\mathbf{A}_1(\mathbf{t}_h),\mathbf{s}] \qquad \forall\,\mathbf{s} \in X_1\,. \tag{4.9}$$

Then, applying (4.5)-(4.6) to the error $(\mathbf{r}, \boldsymbol{\zeta}, \mathbf{w}) := (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)$, and using the identity (4.9) and the fact that $\|(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}} \geq \max\left\{\|\mathbf{s}\|_{X_1}, \|\boldsymbol{\tau}\|_{M_1}, \|\mathbf{v}\|_M\right\}$, we find that

$$\begin{aligned}\tilde{C}\,\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}} \;&\leq \sup_{(\mathbf{s},\boldsymbol{\tau},\mathbf{v})\in\mathbf{X}\backslash\{\mathbf{0}\}} \left\{\frac{Q(\mathbf{s}) \,+\, R(\boldsymbol{\tau}) \,+\, S(\mathbf{v})}{\|(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}}}\right\} \\ &\leq \|Q\|_{X_1'} \,+\, \|R\|_{M_1'} \,+\, \|S\|_{M'}\,,\end{aligned} \tag{4.10}$$

where $Q \in X_1'$, $R \in M_1'$, and $S \in M'$, are given by

$$\begin{aligned} Q(\mathbf{s}) \,&:=\, [\mathbf{A}_1(\mathbf{t}), \mathbf{s}] \,-\, [\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}] \,+\, [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h] \qquad && \forall\, \mathbf{s} \in X_1\,, \\ R(\boldsymbol{\tau}) \,&:=\, [\mathbf{B}_1(\mathbf{t} - \mathbf{t}_h), \boldsymbol{\tau}] \,+\, [\mathbf{B}(\boldsymbol{\tau}), \mathbf{u} - \mathbf{u}_h] && \forall\, \boldsymbol{\tau} \in M_1\,, \\ S(\mathbf{v}) \,&:=\, [\mathbf{B}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \mathbf{v}] && \forall\, \mathbf{v} \in M\,. \end{aligned}$$

According to the three equations of (2.10) and the definitions of the operators $\mathbf{A}_1$, $\mathbf{B}_1$, and $\mathbf{B}$ (cf. (2.8)), and noting that $\int_\Omega \mathbf{t}_h : \boldsymbol{\tau}^{\mathrm{d}} = \int_\Omega \mathbf{t}_h : \boldsymbol{\tau}$ since $\mathrm{tr}(\mathbf{t}_h) = 0$, the above functionals become

$$\begin{aligned} Q(\mathbf{s}) \,&=\, [\mathbf{H}, \mathbf{s}] \,-\, [\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}] \,-\, [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma}_h] \,=\, \int_\Omega \left(\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\right) : \mathbf{s}\,, \\ R(\boldsymbol{\tau}) \,&=\, [\mathbf{G}, \boldsymbol{\tau}] \,-\, [\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}] \,-\, [\mathbf{B}(\boldsymbol{\tau}), \mathbf{u}_h] \,=\, -\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \,+\, \int_\Omega \mathbf{t}_h : \boldsymbol{\tau} \,+\, \int_\Omega \mathbf{u}_h \cdot \mathbf{div}(\boldsymbol{\tau})\,, \\ S(\mathbf{v}) \,&=\, [\mathbf{F}, \mathbf{v}] \,-\, [\mathbf{B}(\boldsymbol{\sigma}_h), \mathbf{v}] \,=\, \int_\Omega \left(\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\right) \cdot \mathbf{v}\,. \end{aligned}$$

It follows easily, using the Cauchy-Schwarz inequality, that

$$\|Q'\|_{M_1'} \,\leq\, \left\|\boldsymbol{\sigma}_h^{\mathrm{d}} \,-\, 2\,\mu(\|\mathbf{t}_h\|)\,\mathbf{t}_h\right\|_{0,\Omega} \quad \text{and} \quad \|S'\|_{M'} \,\leq\, \|f + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega}\,, \tag{4.11}$$

whereas the second equation of (2.30) gives

$$R(\boldsymbol{\tau}_h) \,=\, -\,\langle \boldsymbol{\tau}_h\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \,+\, \int_\Omega \mathbf{t}_h : \boldsymbol{\tau}_h \,+\, \int_\Omega \mathbf{u}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) \,=\, 0 \qquad \forall\, \boldsymbol{\tau}_h \in M_{1,h}\,.$$

This identity and the original expression for R yield (4.8), which together with (4.10) and (4.11), completes the proof. □

We now follow very closely the analysis from the linear case (cf. [26, Section 4.1]) and derive an upper bound for the supremum in (4.7). To this end, and in order to choose below a suitable $\boldsymbol{\tau}_h \in M_{1,h}$ for the computation of $R(\boldsymbol{\tau})$ (cf. (4.8)), we now let $I_h : H^1(\Omega) \longrightarrow X_h$ be the Clément interpolation operator (cf. [15]), where

$$X_h \,:=\, \left\{v_h \in C(\bar{\Omega}) : \quad v_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\right\}\,. \tag{4.12}$$

The following lemma establishes the local approximation properties of I_h.

Lemma 4.2 *There exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there hold*

$$\|v - I_h(v)\|_{0,T} \,\leq\, C_1\, h_T\, \|v\|_{1,\Delta(T)} \quad \forall\, T \in \mathcal{T}_h, \tag{4.13}$$

and

$$\|v - I_h(v)\|_{0,e} \,\leq\, C_2\, h_e^{1/2}\, \|v\|_{1,\Delta(e)} \quad \forall\, e \in \mathcal{E}_h\,, \tag{4.14}$$

where $\Delta(T)$ and $\Delta(e)$ are the union of all elements intersecting with T and e, respectively.

Proof. See [15]. □

Next, proceeding exactly as in [26, Section 4.1], we consider a Helmholtz decomposition of M_1. Indeed, given $\boldsymbol{\tau} \in M_1$, we let $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)^{\mathrm{t}} \in [H^1(\Omega)]^2$, with $\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$, and $\mathbf{z} \in [H^2(\Omega)]^2$, such that

$$\boldsymbol{\tau} = \underline{\mathbf{curl}}(\boldsymbol{\varphi}) + \nabla \mathbf{z}, \tag{4.15}$$

and

$$\|\boldsymbol{\varphi}\|_{1,\Omega} + \|\mathbf{z}\|_{2,\Omega} \leq C \, \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}. \tag{4.16}$$

Then, we let $\boldsymbol{\varphi}_h := (I_h(\varphi_1), I_h(\varphi_2))^{\mathbf{t}}$ and define

$$\boldsymbol{\tau}_h := \underline{\mathbf{curl}}(\boldsymbol{\varphi}_h) + \mathcal{E}_h^k(\nabla \mathbf{z}) + c\,\mathbf{I}, \tag{4.17}$$

where $\mathcal{E}_h^k$ is the Raviart-Thomas interpolation operator (cf. (2.43), (2.44)), and the constant c is chosen so that $\boldsymbol{\tau}_h$, which is already in $\mathbb{RT}_k(\mathcal{T}_h)$, belongs to $M_{1,h} := \mathbb{RT}_k(\mathcal{T}_h) \cap \mathbb{H}_0(\mathbf{div};\Omega)$. Equivalently, $\boldsymbol{\tau}_h$ is the $\mathbb{H}_0(\mathbf{div};\Omega)$-component of $\underline{\mathbf{curl}}(\boldsymbol{\varphi}_h) + \mathcal{E}_h^k(\nabla \mathbf{z}) \in \mathbb{RT}_k(\mathcal{T}_h)$. We refer to (4.17) as a discrete Helmholtz descomposition of $\boldsymbol{\tau}_h$.

Then, replacing $\boldsymbol{\tau}$ (cf. (4.15)) and $\boldsymbol{\tau}_h$ (cf. (4.17)) into (4.8), observing that the expression $c\,\mathbf{I}$ cancells out from the three terms defining R, and noting, according to (2.34) and (2.45) and the fact that $\mathbf{div}(\nabla \mathbf{z}) = \mathbf{div}(\boldsymbol{\tau})$, that

$$\int_\Omega \mathbf{u}_h \cdot \mathbf{div}(\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z})) = \int_\Omega \mathbf{u}_h \cdot (\mathbf{div}(\boldsymbol{\tau}) - \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau}))) = 0,$$

we find that R can be decomposed as $R(\boldsymbol{\tau}) = R_1(\boldsymbol{\varphi}) + R_2(\mathbf{z})$, where

$$R_1(\boldsymbol{\varphi}) := -\langle \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma + \int_\Omega \mathbf{t}_h : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h), \tag{4.18}$$

and

$$R_2(\mathbf{z}) := -\langle (\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z}))\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma + \int_\Omega \mathbf{t}_h : (\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z})). \tag{4.19}$$

The following two lemmas provide upper bounds for $|R_1(\varphi)|$ and $|R_2(\mathbf{z})|$.

Lemma 4.3 *Assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$|R_1(\varphi)| \leq C \left\{ \sum_{T \in \mathcal{T}_h} \theta_{1,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \tag{4.20}$$

where

$$\theta_{1,T}^2 := h_T^2 \, \|\mathrm{curl}\{\mathbf{t}_h\}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e \, \|[\mathbf{t}_h\, s]\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\| \frac{d\mathbf{g}}{ds} - \mathbf{t}_h\, s \right\|_{0,e}^2.$$

Proof. It follows analogously to the proof of [26, Lemma 4.3]. The main tools employed are integration by parts, the Cauchy-Schwarz inequality, the approximation properties provided by Lemma 4.2 , the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and the estimate (4.16). We omit further details here. $\square$

LEMMA 4.4 *There exists $C > 0$, independent of h, such that*

$$|R_2(\mathbf{z})| \,\leq\, C \left\{ \sum_{T\in\mathcal{T}_h} \theta_{2,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}\,, \tag{4.21}$$

where

$$\theta_{2,T}^2 \,=\, h_T^2\, \|\nabla\mathbf{u}_h - \mathbf{t}_h\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g}-\mathbf{u}_h\|_{0,e}^2\,.$$

Proof. It follows analogously to the proof of [26, Lemma 4.4]. In this case the main tools are given by the identities (2.43) and (2.44) characterizing $\mathcal{E}_h^k$, the Cauchy-Schwarz inequality, the approximation properties (2.49) and (2.47) (with $m = 1$), and the estimate (4.16). Further details are omitted here. $\square$

Finally, it follows from the decomposition of R and Lemmas 4.3 and 4.4 that

$$|\,R(\boldsymbol{\tau})\,| \,\leq\, \left\{ \sum_{T\in\mathcal{T}_h} \left(\theta_{1,T}^2 + \theta_{2,T}^2\right) \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \qquad \forall\,\boldsymbol{\tau}\,\in\, M_1\,, \tag{4.22}$$

which, together with the estimate (4.7) (cf. Lemma 4.1), yields the reliability of $\boldsymbol{\theta}$.

4.3 Reliability of the a posteriori error estimator $\boldsymbol{\eta}$

We first observe from (3.3) that the nonlinear operator $\mathcal{A}$ can be rewritten as:

$$[\mathcal{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})] \,:=\, [\mathbf{A}_1(\mathbf{t}),\mathbf{s}\,-\,\kappa\,\boldsymbol{\tau}^{\mathrm{d}}] \,+\, [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\sigma}] \,-\, [\mathbf{B}_1(\mathbf{t}),\boldsymbol{\tau}] \,+\, \kappa \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}\,. \tag{4.23}$$

Hence, following the same reasoning that yielded (4.5) (cf. Section 4.2), we now apply the continuous dependence result provided by the linear version of Theorem 3.1 (cf. (3.7)-(3.8) with $\mathcal{A}$ linear), which is actually the usual estimate provided by the Babuška-Brezzi theory (see, e.g. [28, Theorem 4.1, Chapter I]). In this way, we conclude that the linear operator $\mathcal{M}$ obtained by adding the two equations of the left hand side of (3.2), after replacing $\mathbf{A}_1$ within $\mathcal{A}$ (see (4.23)) by the Gâteaux derivative $\mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})$ at any $\tilde{\mathbf{r}} \in X_1$, satisfies a global inf-sup condition. More precisely, there exists a constant $\tilde{C} > 0$ such that

$$\tilde{C}\,\|((\mathbf{r},\boldsymbol{\zeta}),\mathbf{w})\|_{X\times M} \,\leq \sup_{((\mathbf{s},\boldsymbol{\tau}),\mathbf{v})\in X\times M\setminus\{\mathbf{0}\}} \frac{[\mathcal{M}((\mathbf{s},\boldsymbol{\tau}),\mathbf{v}),((\mathbf{r},\boldsymbol{\zeta}),\mathbf{w})]}{\|((\mathbf{s},\boldsymbol{\tau}),\mathbf{v})\|_{X\times M}} \tag{4.24}$$

for all $(\tilde{\mathbf{r}},((\mathbf{r},\boldsymbol{\zeta}),\mathbf{w})) \,\in\, X_1\times(X\times M)$, where

$$\begin{aligned} [\mathcal{M}((\mathbf{s},\boldsymbol{\tau}),\mathbf{v}),((\mathbf{r},\boldsymbol{\zeta}),\mathbf{w})] \,&:=\, \mathcal{D}\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}-\kappa\,\boldsymbol{\tau}^{\mathrm{d}}) \,+\, [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\zeta}] \,-\, [\mathbf{B}_1(\mathbf{r}),\boldsymbol{\tau}] \\ &+\;\; \kappa\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \,+\, [\mathcal{B}(\mathbf{s},\boldsymbol{\tau}),\mathbf{w}] \,+\, [\mathcal{B}(\mathbf{r},\boldsymbol{\zeta}),\mathbf{v}]\,. \end{aligned} \tag{4.25}$$

The analogue of Lemma 4.1 is established as follows.

Lemma 4.5 *Let* $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in X \times M$ *and* $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \in X_h \times M_h$ *be the unique solutions of the continuous and discrete augmented formulations* (3.2) *and* (3.12)*, respectively. Then there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned} C\,\|((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) - ((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h)\|_{X\times M} \;&\leq\; \left\|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\right\|_{0,\Omega} \\ &+ \sup_{\boldsymbol{\tau}\in M_1\backslash\{\mathbf{0}\}} \frac{R(\boldsymbol{\tau}) + \widetilde{R}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{M_1}} \;+\; \|f + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \end{aligned} \tag{4.26}$$

where, given any $\boldsymbol{\tau}_h \in M_{1,h}$*,*

$$\widetilde{R}(\boldsymbol{\tau}) \;:=\; k\int_\Omega (\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h) : (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau} \in M_1\,, \tag{4.27}$$

and R *is defined by* (4.8)*.*

Proof. We proceed analogously to the proof of Lemma 4.1, though we omit several similar details. Indeed, applying now (4.24) and (4.25), we easily deduce that

$$\tilde{C}\,\|((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) - ((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h)\|_{X\times M} \;\leq\; \sup_{((\mathbf{s},\boldsymbol{\tau}),\mathbf{v})\in X\times M\backslash\{\mathbf{0}\}} \left\{\frac{Q(\mathbf{s},\boldsymbol{\tau}) \,+\, S(\mathbf{v})}{\|((\mathbf{s},\boldsymbol{\tau}),\mathbf{v})\|_{X\times M}}\right\}, \tag{4.28}$$

where $Q \in X'$ and $S \in M'$ are given by

$$\begin{aligned} Q(\mathbf{s},\boldsymbol{\tau}) \;&:=\; [\mathcal{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})] - [\mathcal{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})] + [\mathcal{B}(\mathbf{s},\boldsymbol{\tau}),\mathbf{u}-\mathbf{u}_h] \quad \forall\,(\mathbf{s},\boldsymbol{\tau}) \in X\,, \\ S(\mathbf{v}) \;&:=\; [\mathcal{B}(\mathbf{t}-\mathbf{t}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathbf{v}] \qquad\qquad \forall\,\mathbf{v}\in M\,. \end{aligned}$$

Then, according to the formulations (3.2) and (3.12), noting in particular that $Q(\mathbf{0},\boldsymbol{\tau}_h) = 0$ for all $\boldsymbol{\tau}_h \in M_{1,h}$, and performing some algebraic manipulations, we find that

$$Q(\mathbf{s},\boldsymbol{\tau}) \;=\; -\,R(\boldsymbol{\tau}) - \widetilde{R}(\boldsymbol{\tau}) + \int_\Omega (\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h) : \mathbf{s} \quad \text{and} \quad S(\mathbf{v}) \;=\; -\int_\Omega \big(\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\big)\cdot\mathbf{v}\,.$$

These identities, the Cauchy-Schwarz inequality, and (4.28) complete the proof of (4.26). □

We now aim to derive an upper bound for the supremum on the right hand side of (4.26). For this purpose, we proceed as in the second part of Section 4.2 and employ again the Helmholtz decompositions (4.15) and (4.17). More precisely, since R is already bounded by (4.22), it only remains to estimate the extra-term given by $\widetilde{R}$, which, according to (4.15) and (4.17) and the fact that obviously $\boldsymbol{\sigma}_h^{\mathtt{d}} : \mathbf{I} \,=\, \mathbf{t}_h : \mathbf{I} \,=\, 0$, becomes

$$\begin{aligned} \widetilde{R}(\boldsymbol{\tau}) \;&:=\; k\int_\Omega \big(\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big) : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \\ &+\; k\int_\Omega \big(\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big) : (\nabla\mathbf{z} \,-\, \mathcal{E}_h^k(\nabla\mathbf{z}))\,. \end{aligned} \tag{4.29}$$

The following lemma provides the required estimate for $\widetilde{R}$.

Lemma 4.6 *There exists $C > 0$, independent of h, such that*

$$|\widetilde{R}(\boldsymbol{\tau})| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \eta_{1,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}\,, \tag{4.30}$$

where

$$\eta_{1,T}^2 \,=\, h_T^2 \,\big\|\mathrm{curl}\{\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\}\big\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}(T)} h_e \,\big\|\big[\big(\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big)\,s\big]\big\|_{0,e}^2$$
$$+\quad h_T^2 \,\big\|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big\|_{0,T}^2\,.$$

Proof. It follows by applying the same techniques employed to prove Lemmas 4.3 and 4.4 (see also [26, Lemmas 4.3 and 4.4] for further details). $\square$

As a consequence of (4.22), (4.27) and (4.30) we deduce that

$$|\,R(\boldsymbol{\tau}) + \widetilde{R}(\boldsymbol{\tau})\,| \,\le\, \left\{ \sum_{T\in\mathcal{T}_h} \left(\theta_{1,T}^2 \,+\, \theta_{2,T}^2 \,+\, \eta_{1,T}^2\right) \right\}^{1/2} \|\boldsymbol{\tau}\|_{M_1} \qquad \forall\,\boldsymbol{\tau}\in M_1\,,$$

which, replaced back into (4.26), and having also in mind that $h_T^2 \,\big\|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big\|_{0,T}^2$ is dominated by $\big\|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\big\|_{0,T}^2$, yields the reliability of the a posteriori error estimator $\boldsymbol{\eta}$.

4.4 Efficiency of the a posteriori error estimators $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$

In this section we establish the efficiency of our a posteriori error estimators $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ (lower bounds in (4.3) and (4.4), respectively). In other words, we provide suitable upper bounds for the seven terms defining the local error indicator θ_T^2 (cf. (4.1)), and for the remaining two terms completing the definition of the local error indicator η_T^2 (cf. (4.2)). For this purpose, we first notice that the converses of the derivations of (2.10) and (3.2) from (2.5) hold true. Indeed, it is easy to show, applying integration by parts backwardly and using appropriate test functions, that the unique solution $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X_1 \times M_1 \times M$ of (2.10) (which coincides with that of (3.2)) solves the original problem (2.5).

We begin with two simple estimates. Since $\mathbf{f} \,=\, -\,\mathbf{div}(\boldsymbol{\sigma})$ in Ω, it is clear that

$$\|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,T} \,=\, \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}\,. \tag{4.31}$$

Then, using that $\boldsymbol{\sigma}^{\mathtt{d}} \,=\, 2\,\mu(|\mathbf{t}|)\,\mathbf{t}$ in Ω and applying the Lipschitz-continuity of $\mathbf{A}_1$ (cf. Lemma 2.1), but restricted to the triangle $T \in \mathcal{T}_h$ instead of Ω, we deduce that

$$\begin{aligned} \|\boldsymbol{\sigma}_h^{\mathtt{d}} \,-\, 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\|_{0,T} \,&\le\, \Big\{\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^{\mathtt{d}}\|_{0,T} \,+\, \|2\,\mu(|\mathbf{t}|)\,\mathbf{t} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\|_{0,T}\Big\} \\ &\le\, \Big\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \,+\, 2\,\gamma_0\,\|\mathbf{t} - \mathbf{t}_h\|_{0,T}\Big\}\,. \end{aligned} \tag{4.32}$$

Next, in order to bound the terms involving the mesh parameters h_T and h_e, we make use of the general results and estimates available in the analysis for the linear case (cf. [26, Section 4.2]).

The technique applied there is based on triangle-bubble and edge-bubble functions, extension operators, and discrete trace and inverse inequalities. For further details on these tools we refer particularly to [26, Lemmas 4.7 and 4.8, and eq. (4.34)].

Hence, the estimates of the remaining five terms defining θ_T^2 (cf. (4.1)) are given as follows.

LEMMA 4.7 *There exist C_1, $C_2 > 0$, independent of h, such that*

$$h_T^2 \left\|\mathrm{curl}\{\mathbf{t}_h\}\right\|_{0,T}^2 \leq C_1 \|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h\,, \tag{4.33}$$

$$h_e \,\|\,[\mathbf{t}_h\, s]\,\|_{0,e}^2 \leq C_2 \,\|\,\mathbf{t}-\mathbf{t}_h\,\|_{0,\omega_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Omega)\,, \tag{4.34}$$

where $\omega_e := \cup\left\{T \in \mathcal{T}_h : \quad e \in \mathcal{E}(T)\right\}$.

Proof. It suffices to apply the general results stated in [26, Lemmas 4.9 and 4.10] to $\boldsymbol{\rho}_h = \mathbf{t}_h$ and $\boldsymbol{\rho} = \mathbf{t}$, noting that $\mathrm{curl}(\boldsymbol{\rho}) = \mathrm{curl}(\nabla\mathbf{u}) = \mathbf{0}$ in Ω. □

LEMMA 4.8 *There exists $C_3 > 0$, independent of h, such that*

$$h_T^2 \,\|\,\nabla\mathbf{u}_h-\mathbf{t}_h\,\|_{0,T}^2 \leq C_3\left\{\,\|\,\mathbf{u}-\mathbf{u}_h\,\|_{0,T}^2 + h_T^2\,\|\,\mathbf{t}-\mathbf{t}_h\,\|_{0,T}^2\right\} \qquad \forall\, T \in \mathcal{T}_h\,. \tag{4.35}$$

Proof. It follows from the proof of [26, Lemma 4.13], which itself is a slight modification of the proof of [12, Lemma 6.3], by replacing there $\dfrac{1}{2\,\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}$ by our $\mathbf{t}_h$ and using that $\mathbf{t} = \nabla\mathbf{u}$ in Ω. □

LEMMA 4.9 *Assume that $\mathbf{g}$ is piecewise polynomial. Then there exists $C_4 > 0$, independent of h, such that*

$$h_e\left\|\frac{d\mathbf{g}}{ds}-\mathbf{t}_h\,s\right\|_{0,e}^2 \leq C_4\,\|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2 \qquad \forall\, e \in \mathcal{E}_h(\Gamma)\,, \tag{4.36}$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. It is a slight modification of the proof of [26, Lemma 4.15]. In fact, it suffices to replace there $\dfrac{1}{2\,\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}$ by our $\mathbf{t}_h$ and use now that $\dfrac{d\mathbf{g}}{ds} = \nabla\mathbf{u}\,s = \mathbf{t}\,s$ on Γ. □

LEMMA 4.10 *There exists $C_5 > 0$, independent of h, such that*

$$h_e\,\|\mathbf{g}-\mathbf{u}_h\|_{0,e}^2 \leq C_5\left\{\|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 + h_T^2\,\|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2\right\} \qquad \forall\, e \in \mathcal{E}_h(\Gamma)\,, \tag{4.37}$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Similarly to the previous lemmas, it follows by replacing $\dfrac{1}{2\,\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}$ by our $\mathbf{t}_h$ in the proof of [26, Lemma 4.14], and then using that $\nabla\mathbf{u} = \mathbf{t}$ in Ω and $\mathbf{u} = \mathbf{g}$ on Γ. At the end, the above estimate (4.35) for $h_T^2\,\|\,\nabla\mathbf{u}_h-\mathbf{t}_h\,\|_{0,T}^2$ is also employed. □

We remark here that if $\mathbf{g}$ were not piecewise polynomial but sufficiently smooth, then higher order terms given by the errors arising from suitable polynomial approximations would appear in (4.36). This explains the eventual expression h.o.t. in (4.3).

In this way, the efficiency of $\boldsymbol{\theta}$ follows straightforwardly from estimates (4.31) and (4.32), together with Lemmas 4.7 throughout 4.10, after summing up over $T \in \mathcal{T}_h$ and using that the number of triangles on each domain ω_e is bounded by two.

Finally, for the efficiency of $\boldsymbol{\eta}$ it only remains to provide upper bounds for the two terms completing the definition of the local error indicator η_T^2 (cf. (4.2)), which is established in the following lemma.

Lemma 4.11 *There exist C_6, $C_7 > 0$, independent of h, such that*

$$h_T^2 \, \left\| \operatorname{curl}\{\boldsymbol{\sigma}_h^{\mathtt{d}} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h\}\right\|_{0,T}^2 \,\le\, C_6 \left\{ \|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \right\} \qquad \forall\, T \in \mathcal{T}_h\,, \quad (4.38)$$

$$h_e \, \left\| \left[(\boldsymbol{\sigma}_h^{\mathtt{d}} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h)\, s \right] \right\|_{0,e}^2 \,\le\, C_7 \left\{ \|\mathbf{t}-\mathbf{t}_h\|_{0,\omega_e}^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \right\} \qquad \forall\, e \in \mathcal{E}_h\,. \quad (4.39)$$

Proof. As in the proof of Lemma 4.7, it suffices now to apply the general results stated in [26, Lemmas 4.9 and 4.10] to $\boldsymbol{\rho}_h = \boldsymbol{\sigma}_h^{\mathtt{d}} - 2\,\mu(|\mathbf{t}_h|)\,\mathbf{t}_h$ and $\boldsymbol{\rho} = \boldsymbol{\sigma}^{\mathtt{d}} - 2\,\mu(|\mathbf{t}|)\,\mathbf{t} = 0$ in Ω, and then use the Lipschitz-continuity of $\mathbf{A}_1$ (cf. Lemma 2.1) restricted to T and ω_e. $\square$

5 Numerical results

In this section we present numerical examples illustrating the performance of the Galerkin schemes (2.30) and (3.12), confirming the reliability and efficiency of the a posteriori error estimators $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ derived in Section 4, and showing the behaviour of the associated adaptive algorithms. We consider the specific finite element subspaces $X_{1,h}$, $M_{1,h}$, and M_h given by (2.39), (2.35), and (2.36) with $k = 0$ for the computational implementation of both schemes. In addition, we also utilize $\widehat{X}_{1,h}$ (cf. (3.22)) with $k = 0$ to illustrate the main features of the augmented scheme (3.12). All the nonlinear algebraic systems arising from (2.30) and (3.12) are solved by the Newton method with a tolerance of 1E-05 and taking as initial iteration the solution of the associated linear problem with $\mu = 1$.

In what follows, N stands for the total number of degrees of freedom (unknowns) of each Galerkin scheme, which, for $X_{1,h}$ (resp. $\widehat{X}_{1,h}$), $M_{1,h}$, and M_h with $k = 0$, can be proved to behave asymptotically as 8 (resp. 6.5) times the number of elements of each triangulation. Also, the individual and total errors are given by

$$\mathtt{e}(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}\,, \quad \mathtt{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathrm{div},\Omega}\,, \quad \mathtt{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\,,$$

and

$$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) := \left\{ (\mathtt{e}(\mathbf{t}))^2 + (\mathtt{e}(\boldsymbol{\sigma}))^2 + (\mathtt{e}(\mathbf{u}))^2 \right\}^{1/2},$$

whereas the effectivity indexes with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are defined, respectively, by

$$\mathtt{ef}(\boldsymbol{\theta}) := \mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})/\boldsymbol{\theta} \quad \text{and} \quad \mathtt{ef}(\boldsymbol{\eta}) := \mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})/\boldsymbol{\eta}\,.$$

Then, we introduce the experimental rates of convergence

$$\mathtt{r}(\mathbf{t}) := \frac{\log(\mathtt{e}(\mathbf{t})/\mathtt{e}'(\mathbf{t}))}{\log(h/h')}\,, \quad \mathtt{r}(\boldsymbol{\sigma}) := \frac{\log(\mathtt{e}(\boldsymbol{\sigma})/\mathtt{e}'(\boldsymbol{\sigma}))}{\log(h/h')}\,, \quad \mathtt{r}(\mathbf{u}) := \frac{\log(\mathtt{e}(\mathbf{u})/\mathtt{e}'(\mathbf{u}))}{\log(h/h')}\,,$$

and

$$\mathtt{r}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) := \frac{\log(\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})/\mathtt{e}'(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}))}{\log(h/h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the corresponding errors at two consecutive triangulations with mesh sizes h and h', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by

$-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. Examples 1 and 2 (linear and nonlinear, respectively) are employed to illustrate the performance of the discrete schemes and to confirm the reliability and efficiency of the a posteriori error estimators $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ when a sequence of quasi-uniform meshes is considered. Then, Example 3 (nonlinear) is utilized to show the behavior of the associated adaptive algorithms, which apply the following procedure from [37]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (2.30) (resp.(3.12)) for the actual mesh $\mathcal{T}_h$.

3) Compute θ_T (resp. η_T) for each triangle $T \in \mathcal{T}_h$.

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\theta_{T'}$ (resp. $\eta_{T'}$) satisfies

$$\theta_{T'} \geq \frac{1}{2} \max\Big\{\theta_T : \quad T \in \mathcal{T}_h\Big\} \quad \left(\text{resp. } \eta_{T'} \geq \frac{1}{2} \max\Big\{\eta_T : \quad T \in \mathcal{T}_h\Big\}\right)$$

6) Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

For Example 1 we take $\mu = 1$ and for the remaining two examples we consider the nonlinear function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ given by the Carreau law

$$\mu(t) := \kappa_0 + \kappa_1 (1 + t^2)^{(\beta-2)/2} \qquad \forall\, t \in \mathbb{R}^+ ,$$

with $\kappa_0 = \kappa_1 = 0.5$ and $\beta = 1.5$. It is easy to check that the assumptions (1.3) and (1.4) are satisfied with

$$\gamma_0 = \kappa_0 + \kappa_1 \left\{\frac{|\beta - 2|}{2} + 1\right\} \quad \text{and} \quad \alpha_0 = \kappa_0 .$$

Hence, for the implementation of the augmented scheme (3.12) we use the stabilization parameter $\kappa = \dfrac{\alpha_0}{2\,\gamma_0^2}$, which certainly satisfies the required hypothesis $\kappa \in \left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$.

In Example 1 we consider $\Omega =]0,1[^2$ and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := \frac{1}{8\pi\mu}\left\{-\log \mathbf{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\mathbf{r}^2}\begin{pmatrix} (x_1-2)^2 \\ (x_1-2)(x_2-2) \end{pmatrix}\right\} \quad \text{and} \quad p(\mathbf{x}) = \frac{(x_1-2)}{2\pi\mathbf{r}^2} - p_0 ,$$

with $\mathbf{r} = \sqrt{(x_1-2)^2 + (x_2-2)^2}$, for all $\mathbf{x} := (x_1, x_2) \in \Omega$, where $p_0 \in \mathbb{R}$ is such that $\int_\Omega p = 0$ holds. At this point we recall from (2.3) and the fact that $\boldsymbol{\sigma} \in H_0$, that an admissible solution p must satisfy $\int_\Omega p = 0$. Note in this example that $(\mathbf{u}, p)$ corresponds to the fundamental solution located at the point $(2,2)$, which is outside $\bar{\Omega}$. Hence, $\mathbf{f} = \mathbf{0}$, $\mathbf{u}$ is divergence free, and $(\mathbf{u}, p)$ is regular in the whole domain Ω.

In Example 2 we consider the same geometry of Example 1 and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\boldsymbol{x}) := \begin{pmatrix} \sin x_1 \, \cos x_2 \, \exp(-x_1) \\ (\sin x_1 - \cos x_1) \, \sin x_2 \, \exp(-x_1) \end{pmatrix} \quad \text{and} \quad p(\boldsymbol{x}) := \cos x_1 \, \cos x_2 \, \exp(-x_1) - p_0 \,,$$

where $p_0 \in \mathbb{R}$ is such that $\int_\Omega p = 0$. Note that $\mathbf{u}$ is divergence free and $(\mathbf{u}, p)$ is regular in the whole domain Ω.

Finally, in Example 3 we consider the L-shaped domain $\Omega :=]-1,1[^2 \setminus [0,1]^2$ and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\boldsymbol{x}) := \left[(x_1 - 0.1)^2 + (x_2 - 0.1)^2\right]^{-1/2} \begin{pmatrix} 0.1 - x_2 \\ x_1 - 0.1 \end{pmatrix} \quad \text{and} \quad p(\boldsymbol{x}) := \frac{1}{x_1 + 1.1} - p_0 \,,$$

where $p_0 \in \mathbb{R}$ is such that $\int_\Omega p = 0$. Note that $\mathbf{u}$ is divergence free. In addition, $\mathbf{u}$ and p are singular at $(0.1, 0.1)$ and along the line $x_1 = -1.1$, respectively. Hence, we should expect regions of high gradients around the origin and along the line $x_1 = -1$.

The numerical results shown below were obtained using a MATLAB code. In Tables 5.1 up to 5.6 we summarize the convergence history of the mixed finite element method (2.30) and its augmented version (3.12), as applied to Examples 1 and 2 for a sequence of quasi-uniform triangulations of the domains. The number of Newton iterations required in Example 2, for the tolerance given, ranges between 1 and 3. We observe in these tables, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorems 2.6 and 3.5 (with $\delta = 1$ in both cases) is attained in all the unknowns. The above includes the augmented scheme with piecewise constant and continuous piecewise linear approximations for the unknown $\mathbf{t}$, that is with $\mathbf{t}_h \in X_{1,h}$ (see Tables 5.2 and 5.5) and $\mathbf{t}_h \in \widehat{X}_{1,h}$ (see Tables 5.3 and 5.6). In particular, in the latter case the experimental rate of convergence of $\mathbf{t}$ is a bit higher than expected (around 1.5), which could mean either a superconvergence phenomenon or a special behavior of the particular solutions involved. We will investigate this issue in a separate work. On the other hand, we notice that the effectivity indexes $\texttt{ef}(\boldsymbol{\theta})$ and $\texttt{ef}(\boldsymbol{\eta})$ remain bounded in both examples, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ in the case of regular solutions. Finally, in order to emphasize the good performance of our schemes, in Figures 5.1 and 5.2 we display components of the approximate and exact solutions for Examples 1 and 2, respectively, with the largest value of N. In the case of the augmented scheme (3.12) (see Figure 5.2), the pictures look the same with $\mathbf{t}_h \in X_{1,h}$ and $\mathbf{t}_h \in \widehat{X}_{1,h}$.

Next, in Tables 5.7 - 5.8 and Figures 5.3 - 5.4 we provide the convergence history of the quasi-uniform and adaptive schemes (2.30) and (3.12) (with $\mathbf{t}_h \in \widehat{X}_{1,h}$), as applied to Example 3. In this example the number of Newton iterations required ranges between 5 and 10. We observe from the figures that the errors of the adaptive procedures decrease faster than those obtained by the quasi-uniform ones. Furthermore, the effectivity indexes remain bounded, which confirms the reliability and efficiency of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ in this case of a non-smooth solution. We remark that, though we do not display the corresponding tables, the values of $\texttt{ef}(\boldsymbol{\theta})$ and $\texttt{ef}(\boldsymbol{\eta})$ are also bounded for the adaptive refinements. The good performance of the schemes is now illustrated with Figures 5.5 and 5.6. Finally, adaptive intermediate meshes are displayed in Figures 5.7 and 5.8 (with $\mathbf{t}_h \in \widehat{X}_{1,h}$ for (3.12)). Note that the method is able to recognize the regions with high gradients.

Table 5.1: Example 1, quasi-uniform scheme (2.30)

N	h	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}$)	ef($\boldsymbol{\theta}$)
9313	1/24	4.66E-04	–	1.15E-03	–	2.65E-04	–	1.27E-03	0.453
10921	1/26	4.31E-04	0.988	1.06E-03	1.021	2.45E-04	1.001	1.17E-03	0.452
12657	1/28	4.00E-04	0.989	9.86E-04	1.019	2.27E-04	1.001	1.08E-03	0.451
14521	1/30	3.74E-04	0.990	9.19E-04	1.018	2.12E-04	1.000	1.01E-03	0.450
16513	1/32	3.51E-04	0.991	8.61E-04	1.016	1.99E-04	1.000	9.51E-04	0.450
18633	1/34	3.30E-04	0.992	8.09E-04	1.015	1.87E-04	1.000	8.94E-04	0.449
20881	1/36	3.12E-04	0.993	7.64E-04	1.014	1.77E-04	1.000	8.44E-04	0.449
25761	1/40	2.81E-04	0.994	6.86E-04	1.012	1.59E-04	1.000	7.59E-04	0.448
37057	1/48	2.34E-04	0.995	5.71E-04	1.010	1.32E-04	1.000	6.31E-04	0.447
50401	1/56	2.01E-04	0.996	4.89E-04	1.008	1.13E-04	1.000	5.41E-04	0.446
65793	1/64	1.76E-04	0.997	4.27E-04	1.006	9.96E-05	1.000	4.73E-04	0.446
102721	1/80	1.40E-04	0.998	3.41E-04	1.005	7.97E-05	1.000	3.78E-04	0.445
147841	1/96	1.17E-04	0.999	2.84E-04	1.003	6.64E-05	1.000	3.15E-04	0.445
201153	1/112	1.00E-04	0.999	2.43E-04	1.003	5.69E-05	1.000	2.69E-04	0.445
262657	1/128	8.81E-05	0.999	2.13E-04	1.002	4.98E-05	1.000	2.36E-04	0.445
332353	1/144	7.83E-05	0.999	1.89E-04	1.002	4.43E-05	1.000	2.09E-04	0.445

Table 5.2: Example 1, quasi-uniform scheme (3.12) with $\mathbf{t}_h \in X_{1,h}$

N	h	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}$)	ef($\boldsymbol{\eta}$)
9313	1/24	4.66E-04	–	1.15E-03	–	2.65E-04	–	1.27E-03	0.454
10921	1/26	4.31E-04	0.988	1.06E-03	1.021	2.45E-04	1.001	1.17E-03	0.453
12657	1/28	4.00E-04	0.989	9.86E-04	1.019	2.27E-04	1.001	1.08E-03	0.452
14521	1/30	3.74E-04	0.990	9.19E-04	1.018	2.12E-04	1.000	1.01E-03	0.451
16513	1/32	3.51E-04	0.991	8.61E-04	1.016	1.99E-04	1.000	9.51E-04	0.450
18633	1/34	3.30E-04	0.992	8.09E-04	1.015	1.87E-04	1.000	8.94E-04	0.450
20881	1/36	3.12E-04	0.993	7.64E-04	1.014	1.77E-04	1.000	8.44E-04	0.449
25761	1/40	2.81E-04	0.994	6.86E-04	1.012	1.59E-04	1.000	7.59E-04	0.448
37057	1/48	2.34E-04	0.995	5.71E-04	1.010	1.32E-04	1.000	6.31E-04	0.447
50401	1/56	2.01E-04	0.996	4.89E-04	1.008	1.13E-04	1.000	5.41E-04	0.447
65793	1/64	1.76E-04	0.997	4.27E-04	1.006	9.96E-05	1.000	4.73E-04	0.446
102721	1/80	1.40E-04	0.998	3.41E-04	1.005	7.97E-05	1.000	3.78E-04	0.445
147841	1/96	1.17E-04	0.999	2.84E-04	1.003	6.64E-05	1.000	3.15E-04	0.445
201153	1/112	1.00E-04	0.999	2.43E-04	1.003	5.69E-05	1.000	2.69E-04	0.445
262657	1/128	8.81E-05	0.999	2.13E-04	1.002	4.98E-05	1.000	2.36E-04	0.445
332353	1/144	7.83E-05	0.999	1.89E-04	1.002	4.43E-05	1.000	2.09E-04	0.445

Table 5.3: Example 1, quasi-uniform scheme (3.12) with $\mathbf{t}_h \in \widehat{X}_{1,h}$

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\eta})$
7732	1/24	1.36E-04	–	1.22E-03	–	2.66E-04	–	1.25E-03	0.258
9052	1/26	1.19E-04	1.631	1.11E-03	1.087	2.45E-04	1.002	1.15E-03	0.256
10476	1/28	1.06E-04	1.634	1.03E-03	1.081	2.28E-04	1.002	1.06E-03	0.253
12004	1/30	9.48E-05	1.635	9.59E-04	1.075	2.12E-04	1.002	9.86E-04	0.251
13636	1/32	8.53E-05	1.637	8.95E-04	1.070	1.99E-04	1.002	9.21E-04	0.249
15372	1/34	7.72E-05	1.637	8.39E-04	1.065	1.87E-04	1.001	8.63E-04	0.248
17212	1/36	7.03E-05	1.638	7.89E-04	1.061	1.77E-04	1.001	8.12E-04	0.247
21204	1/40	5.91E-05	1.638	7.06E-04	1.055	1.59E-04	1.001	7.26E-04	0.245
30436	1/48	4.39E-05	1.636	5.84E-04	1.046	1.32E-04	1.001	6.00E-04	0.241
41332	1/56	3.41E-05	1.633	4.97E-04	1.037	1.13E-04	1.001	5.11E-04	0.239
53892	1/64	2.74E-05	1.629	4.33E-04	1.030	9.96E-05	1.000	4.45E-04	0.238
84004	1/80	1.91E-05	1.623	3.45E-04	1.023	7.97E-05	1.000	3.54E-04	0.236
120772	1/96	1.42E-05	1.615	2.86E-04	1.017	6.64E-05	1.000	2.94E-04	0.235
164196	1/112	1.11E-05	1.608	2.45E-04	1.013	5.69E-05	1.000	2.52E-04	0.235
214276	1/128	8.98E-06	1.601	2.14E-04	1.011	4.98E-05	1.000	2.20E-04	0.234
271012	1/144	7.44E-06	1.596	1.90E-04	1.009	4.43E-05	1.000	1.95E-04	0.234

Table 5.4: Example 2, quasi-uniform scheme (2.30)

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\theta})$
9313	1/24	2.61E-02	–	4.99E-02	–	9.30E-03	–	5.71E-02	0.410
10921	1/26	2.41E-02	0.993	4.60E-02	1.016	8.58E-03	1.000	5.26E-02	0.409
12657	1/28	2.24E-02	0.994	4.27E-02	1.014	7.97E-03	1.000	4.88E-02	0.408
14521	1/30	2.09E-02	0.994	3.98E-02	1.013	7.44E-03	1.000	4.56E-02	0.407
16513	1/32	1.96E-02	0.995	3.73E-02	1.013	6.97E-03	1.000	4.27E-02	0.407
18633	1/34	1.84E-02	0.995	3.50E-02	1.012	6.56E-03	1.000	4.02E-02	0.406
20881	1/36	1.74E-02	0.996	3.31E-02	1.011	6.20E-03	1.000	3.79E-02	0.406
25761	1/40	1.57E-02	0.996	2.97E-02	1.010	5.58E-03	1.000	3.41E-02	0.405
37057	1/48	1.31E-02	0.997	2.47E-02	1.008	4.65E-03	1.000	2.84E-02	0.404
50401	1/56	1.12E-02	0.998	2.12E-02	1.006	3.98E-03	1.000	2.43E-02	0.404
65793	1/64	9.83E-03	0.998	1.85E-02	1.005	3.48E-03	1.000	2.12E-02	0.403
102721	1/80	7.87E-03	0.998	1.48E-02	1.004	2.79E-03	1.000	1.70E-02	0.403
147841	1/96	6.56E-03	0.999	1.23E-02	1.003	2.32E-03	1.000	1.41E-02	0.402
201153	1/112	5.62E-03	0.999	1.05E-02	1.002	1.99E-03	1.000	1.21E-02	0.402
262657	1/128	4.92E-03	0.999	9.25E-03	1.002	1.74E-03	1.000	1.06E-02	0.402
332353	1/144	4.37E-03	1.000	8.22E-03	1.002	1.55E-03	1.000	9.44E-03	0.402

Table 5.5: EXAMPLE 2, quasi-uniform scheme (3.12) with $\mathbf{t}_h \in X_{1,h}$

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\eta})$
9313	1/24	2.61E-02	–	4.99E-02	–	9.30E-03	–	5.71E-02	0.394
10921	1/26	2.41E-02	0.993	4.60E-02	1.016	8.58E-03	1.000	5.26E-02	0.394
12657	1/28	2.24E-02	0.994	4.27E-02	1.014	7.97E-03	1.000	4.88E-02	0.393
14521	1/30	2.09E-02	0.994	3.98E-02	1.013	7.44E-03	1.000	4.56E-02	0.392
16513	1/32	1.96E-02	0.995	3.73E-02	1.013	6.97E-03	1.000	4.27E-02	0.392
18633	1/34	1.84E-02	0.995	3.50E-02	1.012	6.56E-03	1.000	4.02E-02	0.391
20881	1/36	1.74E-02	0.996	3.31E-02	1.011	6.20E-03	1.000	3.79E-02	0.391
25761	1/40	1.57E-02	0.996	2.97E-02	1.010	5.58E-03	1.000	3.41E-02	0.390
37057	1/48	1.31E-02	0.997	2.47E-02	1.008	4.65E-03	1.000	2.84E-02	0.389
50401	1/56	1.12E-02	0.998	2.12E-02	1.006	3.98E-03	1.000	2.43E-02	0.389
65793	1/64	9.83E-03	0.998	1.85E-02	1.005	3.48E-03	1.000	2.12E-02	0.388
102721	1/80	7.87E-03	0.998	1.48E-02	1.004	2.79E-03	1.000	1.70E-02	0.388
147841	1/96	6.56E-03	0.999	1.23E-02	1.003	2.32E-03	1.000	1.41E-02	0.388
201153	1/112	5.62E-03	0.999	1.05E-02	1.002	1.99E-03	1.000	1.21E-02	0.387
262657	1/128	4.92E-03	0.999	9.25E-03	1.002	1.74E-03	1.000	1.06E-02	0.387
332353	1/144	4.37E-03	1.000	8.22E-03	1.002	1.55E-03	1.000	9.44E-03	0.387

Table 5.6: EXAMPLE 2, quasi-uniform scheme (3.12) with $\mathbf{t}_h \in \widehat{X}_{1,h}$

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\eta})$
7732	1/24	5.52E-03	–	5.08E-02	–	9.30E-03	–	5.19E-02	0.238
9052	1/26	4.86E-03	1.584	4.67E-02	1.035	8.58E-03	1.001	4.78E-02	0.237
10476	1/28	4.32E-03	1.584	4.33E-02	1.033	7.97E-03	1.001	4.42E-02	0.236
12004	1/30	3.88E-03	1.585	4.03E-02	1.031	7.44E-03	1.001	4.12E-02	0.235
13636	1/32	3.50E-03	1.585	3.77E-02	1.029	6.97E-03	1.001	3.85E-02	0.234
15372	1/34	3.18E-03	1.585	3.54E-02	1.027	6.56E-03	1.000	3.62E-02	0.233
17212	1/36	2.90E-03	1.584	3.34E-02	1.025	6.20E-03	1.000	3.41E-02	0.232
21204	1/40	2.46E-03	1.584	3.00E-02	1.023	5.58E-03	1.000	3.06E-02	0.231
30436	1/48	1.84E-03	1.582	2.49E-02	1.019	4.65E-03	1.000	2.54E-02	0.230
41332	1/56	1.44E-03	1.579	2.13E-02	1.015	3.98E-03	1.000	2.17E-02	0.229
53892	1/64	1.17E-03	1.576	1.86E-02	1.013	3.48E-03	1.000	1.90E-02	0.228
84004	1/80	8.24E-04	1.571	1.48E-02	1.010	2.79E-03	1.000	1.51E-02	0.227
120772	1/96	6.19E-04	1.566	1.23E-02	1.007	2.32E-03	1.000	1.26E-02	0.227
164196	1/112	4.87E-04	1.561	1.06E-02	1.006	1.99E-03	1.000	1.08E-02	0.226
214276	1/128	3.95E-04	1.555	9.27E-03	1.005	1.74E-03	1.000	9.44E-03	0.226
271012	1/144	3.29E-04	1.554	8.23E-03	1.004	1.55E-03	1.000	8.38E-03	0.226

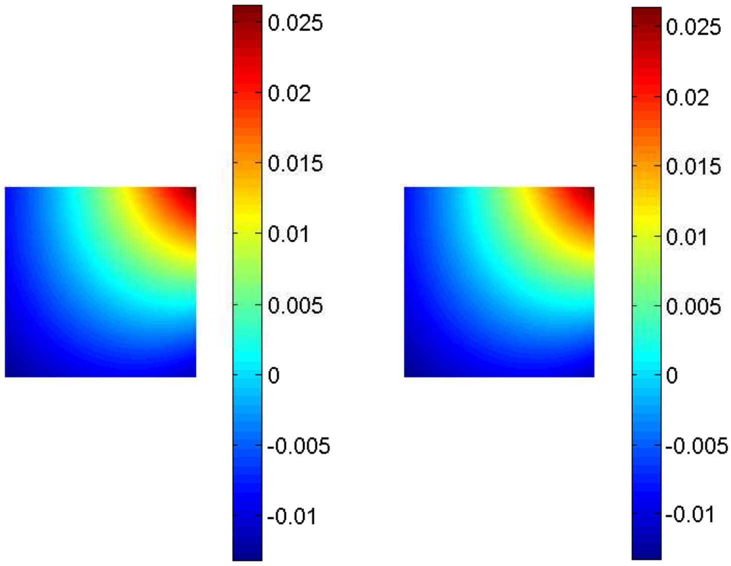

Figure 5.1: Example 1, approximate and exact σ_{22} for quasi-uniform scheme (2.30)

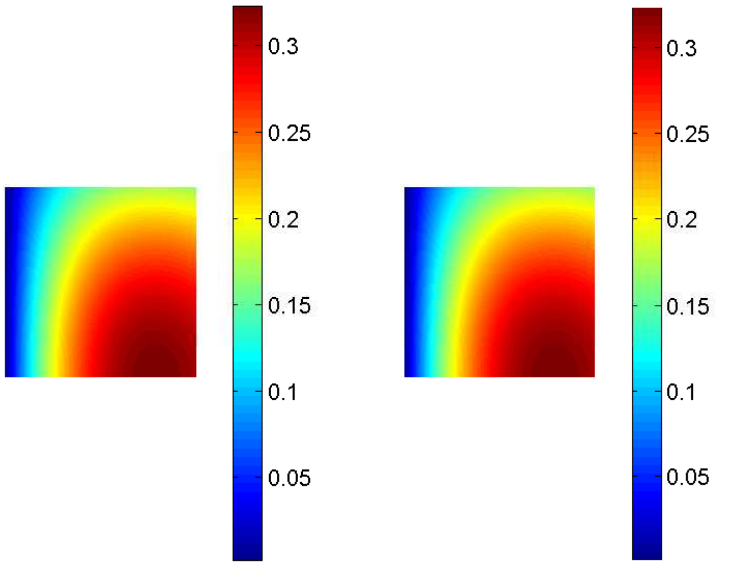

Figure 5.2: Example 2, approximate and exact u_1 for quasi-uniform scheme (3.12)

Table 5.7: EXAMPLE 3, quasi-uniform scheme (2.30)

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\theta})$
57	1/1	2.21E-00	–	1.13E+01	–	6.29E-01	–	1.16E+01	0.828
713	1/3	1.39E-00	0.429	1.28E+01	0.168	1.99E-01	0.993	1.29E+01	0.972
1929	1/5	9.76E-01	0.736	1.05E+01	0.427	1.15E-01	1.082	1.06E+01	0.974
5361	1/8	6.15E-01	1.083	8.17E+00	0.693	6.90E-02	1.205	8.19E+00	0.973
6505	1/9	5.63E-01	0.759	7.58E+00	0.632	6.31E-02	0.758	7.61E+00	0.973
7825	1/10	5.02E-01	1.084	7.16E+00	0.544	5.53E-02	1.248	7.18E+00	0.976
11809	1/12	4.44E-01	0.806	6.35E+00	0.713	4.67E-02	1.024	6.37E+00	0.974
14041	1/13	4.07E-01	1.088	5.71E+00	1.343	4.32E-02	0.961	5.72E+00	0.972
18265	1/15	3.54E-01	1.019	5.23E+00	0.511	3.72E-02	1.076	5.24E+00	0.975
20577	1/16	3.33E-01	0.967	4.91E+00	0.964	3.47E-02	1.059	4.92E+00	0.974
23593	1/17	3.06E-01	1.347	4.64E+00	0.931	3.22E-02	1.218	4.65E+00	0.976
26529	1/18	2.88E-01	1.084	4.40E+00	0.926	3.02E-02	1.153	4.41E+00	0.976
39569	1/22	2.37E-01	1.081	3.64E+00	1.130	2.46E-02	1.174	3.64E+00	0.975
52217	1/25	2.10E-01	0.956	3.20E+00	1.005	2.18E-02	0.951	3.20E+00	0.973
68489	1/29	1.77E-01	1.139	2.78E+00	0.935	1.88E-02	0.992	2.79E+00	0.975
100793	1/35	1.50E-01	0.878	2.40E+00	0.794	1.56E-02	0.982	2.40E+00	0.975
145761	1/42	1.23E-01	1.070	1.98E+00	1.040	1.28E-02	1.071	1.99E+00	0.976
203889	1/50	1.06E-01	0.895	1.67E+00	0.993	1.09E-02	0.945	1.67E+00	0.974
257633	1/56	9.48E-02	0.979	1.50E+00	0.895	9.68E-03	1.065	1.51E+00	0.975
328537	1/63	8.22E-02	1.211	1.31E+00	1.194	8.61E-03	0.997	1.31E+00	0.974

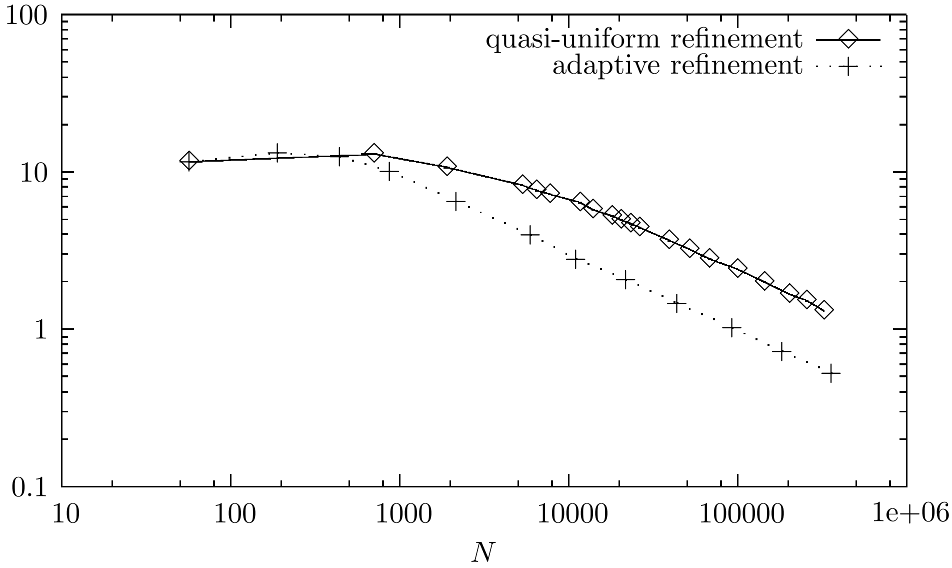

Figure 5.3: EXAMPLE 3, $\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ vs. N for the scheme (2.30)

Table 5.8: EXAMPLE 3, quasi-uniform scheme (3.12) with $\mathbf{t}_h \in \widehat{X}_{1,h}$

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$	$\mathtt{ef}(\boldsymbol{\eta})$
63	1/1	2.17E-00	–	1.13E+01	–	7.04E-01	–	1.15E+01	0.701
291	1/2	1.54E-00	0.493	1.38E+01	–	3.26E-01	1.111	1.38E+01	0.868
1124	1/4	9.61E-01	0.907	1.16E+01	0.353	1.55E-01	1.117	1.16E+01	0.901
1638	1/5	8.02E-01	0.809	1.05E+01	0.426	1.19E-01	1.164	1.06E+01	0.910
3394	1/7	5.19E-01	1.340	8.96E+00	0.573	8.29E-02	1.214	8.98E+00	0.917
4467	1/8	4.15E-01	1.658	8.17E+00	0.693	7.04E-02	1.233	8.18E+00	0.921
8180	1/11	2.87E-01	0.788	6.76E+00	0.603	5.17E-02	0.877	6.77E+00	0.927
9760	1/12	2.55E-01	1.331	6.35E+00	0.712	4.72E-02	1.037	6.36E+00	0.926
11587	1/13	2.23E-01	1.696	5.71E+00	1.342	4.36E-02	1.002	5.71E+00	0.922
19402	1/17	1.54E-01	1.624	4.64E+00	0.931	3.24E-02	1.223	4.65E+00	0.930
27236	1/20	1.23E-01	1.218	4.05E+00	0.792	2.76E-02	0.885	4.05E+00	0.930
32450	1/22	1.07E-01	1.482	3.64E+00	1.130	2.47E-02	1.182	3.64E+00	0.927
42767	1/25	8.91E-02	1.458	3.20E+00	1.005	2.18E-02	0.959	3.20E+00	0.927
56042	1/29	6.79E-02	1.832	2.78E+00	0.935	1.88E-02	0.996	2.78E+00	0.929
82370	1/35	5.72E-02	0.906	2.40E+00	0.794	1.56E-02	0.985	2.40E+00	0.930
119001	1/42	4.50E-02	1.315	1.98E+00	1.040	1.29E-02	1.073	1.98E+00	0.930
166338	1/50	3.56E-02	1.346	1.67E+00	0.993	1.09E-02	0.947	1.67E+00	0.928
210086	1/56	3.16E-02	1.055	1.50E+00	0.895	9.69E-03	1.066	1.51E+00	0.930
267790	1/63	2.41E-02	2.293	1.31E+00	1.194	8.61E-03	0.998	1.31E+00	0.928

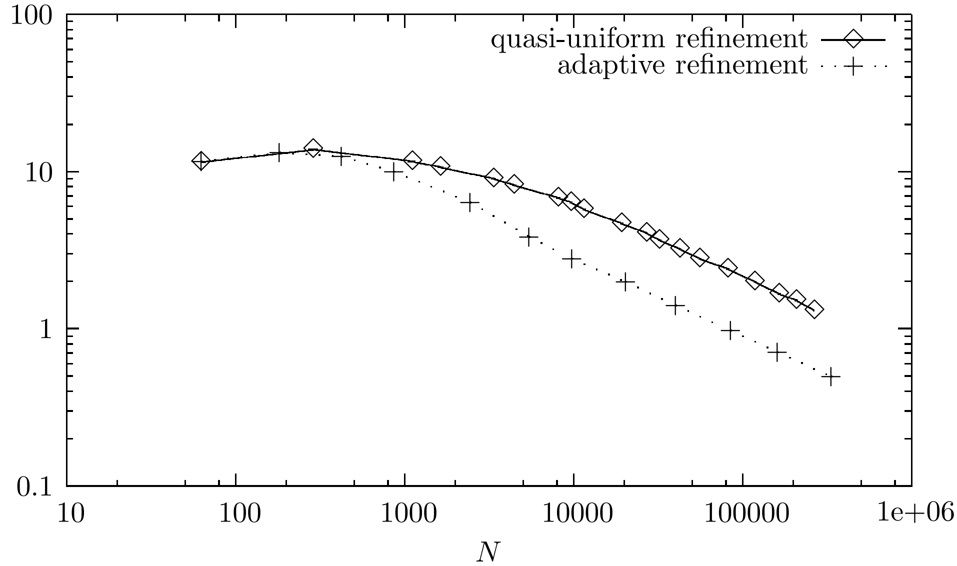

Figure 5.4: EXAMPLE 3, $\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ vs. N for the scheme (3.12) with $\mathbf{t}_h \in \widehat{X}_{1,h}$

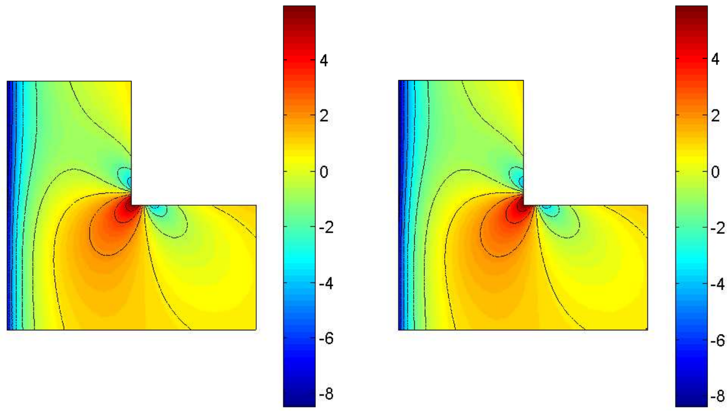

Figure 5.5: Example 3, approximate and exact σ_{11} for adaptive scheme (2.30)

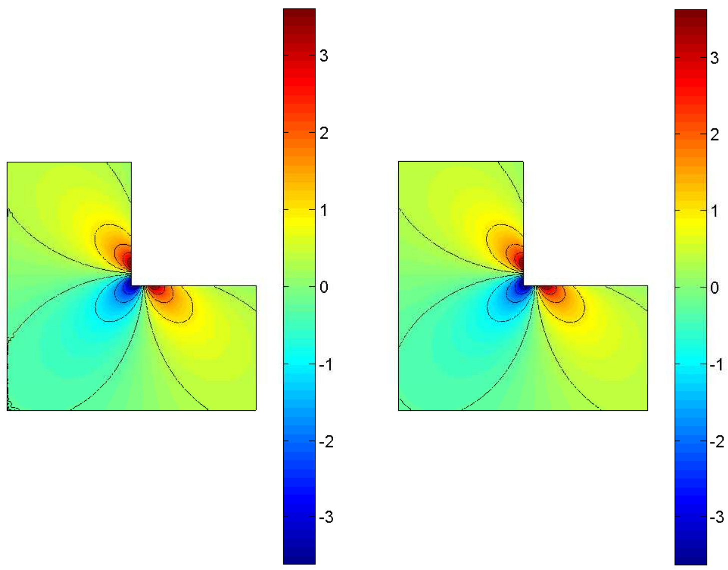

Figure 5.6: Example 3, approximate and exact t_{22} for adaptive scheme (3.12)

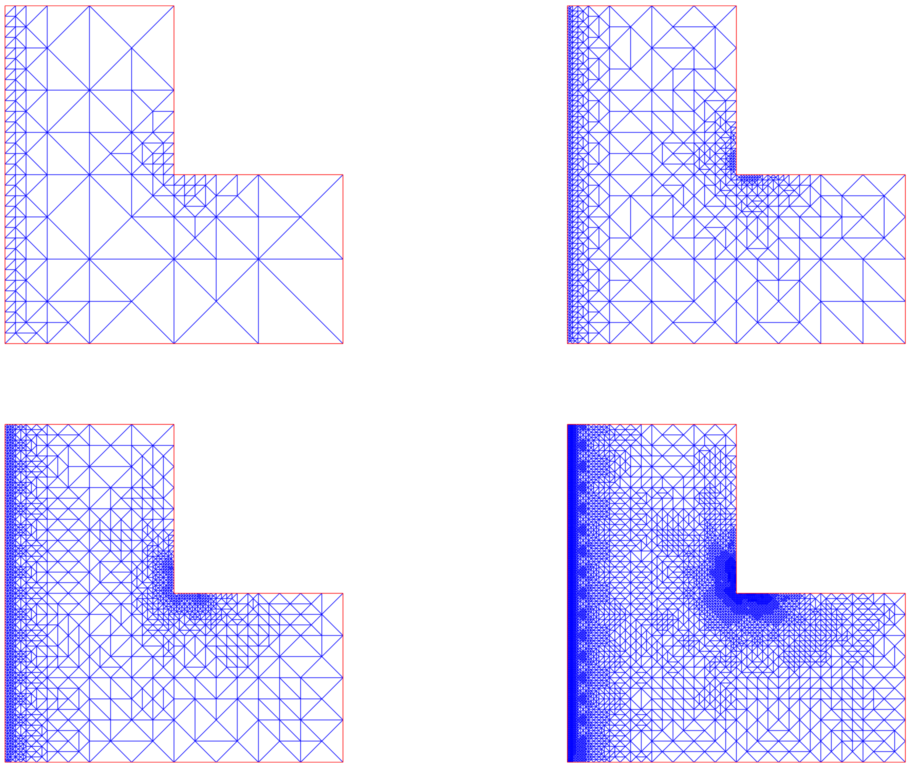

Figure 5.7: Example 3, adapted meshes with $N \in \{2164, 11062, 21990, 93351\}$ for scheme (2.30)

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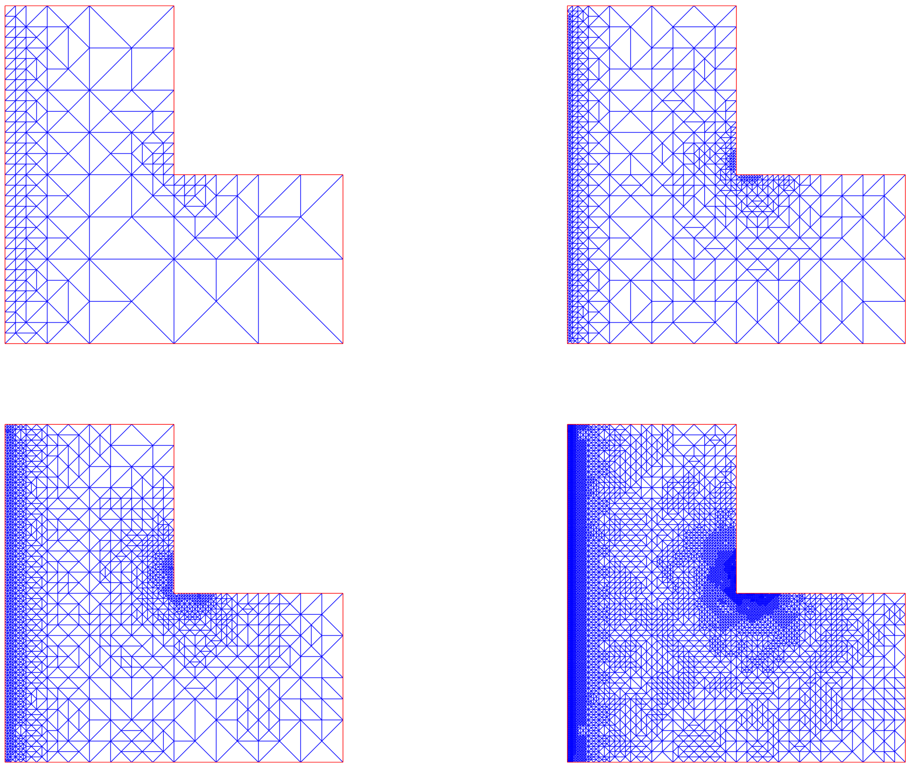

Figure 5.8: EXAMPLE 3, adapted meshes with $N \in \{2451, 9800, 20403, 85683\}$ for scheme (3.12)

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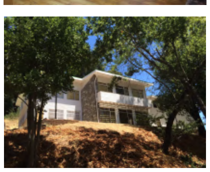

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