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Abstract

We use asymptotic analysis to develop finer estimates for the efficient, weak efficient and proper efficient solution sets (and for their asymptotic cones) to a convex/quasiconvex vector optimization problems. We also provide a new representation for the efficient solution set without any convexity assumption, and the estimates involve the minima of the linear scalarization of the original vector problem. Some new necessary conditions for a point to be efficient or weak efficient solution for general convex vector optimization problems, as well as for the nonconvex quadratic multiobjective optimization problem, are established.

Key words. nonconvex vector optimization, efficiency, weak efficiency, proper efficiency, linear scalarization, asymptotic functions and cones, necessary conditions

Mathematics subject classification 2000. 90C25, 90C26, 90C29, 90C30, 90C46, 90C48

1 Introduction

When dealing with multiobjective optimization problems we have to specify what we mean by a solution to such a problem. Usually, the notion of efficient or weakly efficient soluion is considered. A point is called efficient or Pareto-optimal, if there does not exist a different point with smaller or equal objective functions values, such that there is a decrease in at least one objective function value; a point is called weakly efficient or weakly Pareto-optimal, if there exists no other point with strictly smaller objective function value. Another notion of solution is that of proper efficient. This notion was

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introduced in order to avoid efficient points satisfying certain abnormal properties: for instance, efficient points for which at least one objective function exists for which the marginal trade off between it and each of the other objective functions is infinitely large [14, 2].
Mathematically, given normed vector spaces X, Y, a convex cone $P \subsetneq Y$ and a vector function $F : K \subseteq X \to Y$, we say that $\bar{x} \in K$ is a

- (int $P \neq \emptyset$) "weakly efficient" point of F (on K) if

$$F(x) - F(\bar{x}) \notin -\text{int } P, \quad \forall\, x \in K,$$

  or equivalently, $(F(K) - F(\bar{x})) \cap (-\text{int } P) = \emptyset$, or equivalently, $\overline{\text{cone}}(F(K) - F(\bar{x}) + P) \cap (-\text{int } P) = \emptyset$;

- "efficient" point of F (on K) if

$$F(x) - F(\bar{x}) \notin -P \setminus l(P), \quad \forall\, x \in K,$$

  or equivalently, $(F(K) - F(\bar{x})) \cap (-P \setminus l(P)) = \emptyset$, or equivalently, $\text{cone}(F(K) - F(\bar{x}) + P) \cap (-P \setminus l(P)) = \emptyset$;

- "proper efficient" point of F (on K) if

$$\overline{\text{cone}}(F(K) - F(\bar{x}) + P) \cap (-P) = \{0\}.$$

Here, $l(P) \doteq P \cap (-P)$. Notice that every proper efficient point is efficient and every efficient is weakly efficient. The set of weakly efficient points is denoted by E_W, that of efficient by E, and the set of proper efficient by E_{pr}. It is easy to see that $E_{\text{pr}} \neq \emptyset$ implies the pointedness of P, that is, $l(P) = \{0\}$.

It is well known that asymptotic tools have proved to be very useful in the study of minimization problems even in absence of convexity, and have a long history. Roughly speaking, it serves to describe the asymptotic behaviour of the objective function along directions that are limit of the normalized of unbounded minimizing sequences. In vector optimization, it has been used in [4] to characterize the non-emptiness and compactness of weakly efficient solutions set of convex problems, and for efficiency in [5, 6]. Further developments in vector optimization can be found in [7, 8, 9, 10, 12, 13], and references therein.

One of the main goals of the present paper is to find finer outer and inner estimates for E_W, E, and E_{pr} and for their asymptotic cones. These estimates involve the minima of the linear scalarization of the original vector problem, and will be carried out under convexity and quasiconvexity assumptions. We also provide two representations for E

in absence of convexity, see Lemma 3.1. Furthermore, we revise some characterizations of the nonemptiness and boundedness of E and E_W in terms of some cones of critical directions, which have been used elsewhere by one of the authors.

The paper is organized as follows. In Section 2, we present the necessary basic definitions and some preliminaries. Section 3 is devoted to find new representations for the solution set E in absence of convexity (Lemma 3.1) along with some estimates for E_W and E. In Section 4, we present some inner and outer estimates for $(E_W)^\infty$, E^∞ and $(E_{\rm pr})^\infty$. Section 5 establishes some characterization of the nonemptiness and boundedness of E_W and E. Finally, Section 6 developes new necessary conditions for a point to be efficient or weak efficient solution for general convex vector optimization problems, and also for the nonconvex (nonhomogeneous) quadratic multiobjective optimization problem.

2 Some basic definitions and preliminaries

Let Y^* denote the topological dual space of Y and let the duality pairing between Y^* and Y be denoted by $\langle \cdot, \cdot \rangle$. The set $P^* \subseteq Y^*$ is the polar (positive) cone of P defined by

$$P^* \doteq \{q \in Y^* : \ \langle q, p \rangle \geq 0, \quad \forall \, p \in P\}.$$

For any given function $h : X \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of h is defined as the function h^∞ such that

$$\text{epi } h^\infty = (\text{epi } h)^\infty.$$

Here, epi $h = \{(x, t) \in X \times \mathbb{R} : \ h(x) \leq t\}$ is the epigraph of h. Consequently, when h is a convex and lower semicontinuous function, we have

$$h^\infty(v) = \lim_{\lambda \to +\infty} \frac{h(x_0 + \lambda v) - h(x_0)}{\lambda} = \sup_{\lambda > 0} \frac{h(x_0 + \lambda v) - h(x_0)}{\lambda} \quad , \forall \, x_0 \in h^{-1}(\mathbb{R}).$$

We notice the independence of h^∞ on the choice of x_0. If $f : K \subseteq X \to \mathbb{R}$, f^∞ denotes the asymptotic function of f, where we extend f to the whole X by setting $f(x) = +\infty$ if $x \in X \setminus K$. More detailed information on asymptotic sets and functions may be found in [18].

For a given closed convex cone $P \subsetneq Y$, we have by the bipolar theorem

$$p \in P \Longleftrightarrow \langle q, p \rangle \geq 0 \ \ \forall \, q \in P^*,$$

and if int $P \neq \emptyset$,

$$p \in \text{int } P \Longleftrightarrow \langle q, p \rangle > 0 \ \ \forall \, q \in P^* \setminus \{0\}.$$

We say that a convex cone P is pointed if $P\cap(-P) = \{0\}$. We denote $l(P) \doteq P\cap(-P)$.

Given $q \in P^*$, let the function $h_q : K \to \mathbb{R}$ defined as $h_q(x) = \langle q, F(x)\rangle$, and consider the minimization problem

$$\min_{x\in K(z)} h_q(x), \qquad (P(q,z))$$

where $z \in K$ and

$$K(z) \doteq \{x \in K : F(x) - F(z) \in -P\}.$$

As usual, we denote the set of solution to $(P(q, z))$ by $\mathrm{argmin}_{K(z)} h_q$.

3 Representations for the efficient solution set

By extrd P^* we mean the set of extreme directions of P^*: here $q^* \in \mathrm{extrd}\ P^*$ if and only if $q^* \in P^* \setminus \{0\}$ and for all $q_1^*, q_2^* \in P^*$ such that $q^* = q_1^* + q_2^*$ we actually have $q_1^*, q_2^* \in \mathbb{R}_{++}q^*$.

We start by establishing a characterization of efficiency in terms of existence to special scalar minimization problems. Part (a) is new, and particular cases of Part (b) (P polyhedral) may be found in [5, 15] .

Lemma 3.1. *Let $P \subsetneq Y$ be a closed convex cone.*

(a) *Assume that P^* is the weak-star closed convex hull of* extrd P^*, *then $\bar{x} \in E$ if, and only if there exists $z \in K$ such that $\bar{x} \in \mathrm{argmin}_{K(z)} h_q$ for all $q \in \mathrm{extrd}\ P^*$, that is,*

$$E = \bigcup_{z\in K} \bigcap_{q\in \mathrm{extrd}\ P^*} \mathrm{argmin}_{K(z)} h_q. \qquad (1)$$

(b) *Assume that* int $P^* \neq \emptyset$, *then $\bar{x} \in E$ if, and only if for all $q \in \mathrm{int}\ P^*$ there exists $z \in K$ such that $\bar{x} \in \mathrm{argmin}_{K(z)} h_q$, that is,*

$$E = \bigcup_{z\in K} \mathrm{argmin}_{K(z)} h_q, \quad \forall\ q \in \mathrm{int}\ P^*. \qquad (2)$$

Proof. (a): $\Rightarrow$ Let $\bar{x} \in E$ and suppose that for all $z \in K$ there exists $q_z \in \mathrm{extrd}\ P^*$ such that $\bar{x} \notin \mathrm{argmin}_{K(z)} h_{q_z}$. Thus, setting $z = \bar{x}$, there exist $\bar{q} \in \mathrm{extrd}\ P^*$ and $x \in K(\bar{x})$ satisfying $h_{\bar{q}}(x) < h_{\bar{q}}(\bar{x})$. This implies that $F(x) - F(\bar{x}) \notin P$. Since $x \in K(\bar{x})$, we also get $F(x) - F(\bar{x}) \in -P$. Both relations give $F(x) - F(\bar{x}) \in (-P) \setminus P$, contradicting the fact that $\bar{x} \in E$.

$\Leftarrow$ Let $z \in K$ such that $\bar{x} \in \mathrm{argmin}_{K(z)} h_q$ for all $q \in \mathrm{extrd}\ P^*$. Suppose that $\bar{x} \notin E$; then there exists $x \in K$ such that

$$F(x) - F(\overline{x}) \in (-P) \setminus l(P) = (-P) \setminus P.$$

Thus, $x \in K(z)$ and there exits $\bar{q} \in P^* \setminus \{0\}$ such that $\langle \bar{q}, F(x) - F(\bar{x}) \rangle < 0$. By assumption, the latter implies the existence of $q \in \text{extrd } P^*$, $h_q(x) < h_q(\bar{x})$, which cannot happen if $\bar{x} \in \text{argmin}_{K(z)} h_q$.
(b): $\Rightarrow$ It is not difficult to prove that $\bar{x} \in E$ implies $\bar{x} \in \text{argmin}_{K(\bar{x})} h_q$.
$\Leftarrow$ Let $q \in \text{int } P^*$. Let $z \in K$ and $\bar{x} \in \text{argmin}_{K(z)} h_q$. Suppose to the contrary that $\bar{x} \notin E$; then there exists $x \in K$ satisfying $F(x) - F(\bar{x}) \in -P \setminus l(P) \subseteq -P \setminus \{0\}$. By the choice of q, we obtain $\langle q, F(x) - F(\bar{x}) \rangle < 0$, and as $x \in K(z)$, a contradiction is obtained. $\square$

Remark 3.2. *Conditions ensuring that P^* is the weak-star closed convex hull of* extrd P^**, may be found in Remark 2.2 of [1]. It is true, in particular, when* int $P \neq \emptyset$*. On the other hand, in finite dimensional spaces* int $P^* \neq \emptyset$ *is equivalent to pointedness of P.*

Next example is an instance where Part (a) is applicable whereas (b) is not.

Example 3.3. *Consider $P = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, $K = \mathbb{R}$ and $F = (f_1, f_2)$, $f_1(x) = x$, $f_2(x) = 1$. Here, $E = \mathbb{R}$; indeed, given any $\bar{x} \in \mathbb{R}$, one obtains*

$$F(y) - F(\bar{x}) = (y, 1) - (\bar{x}, 1) = (y - \bar{x}, 0) \notin -P \setminus l(P), \ \forall \ y \in \mathbb{R}.$$

Moreover, extrd $P^* = \{(0, 1)\}$*, $K(z) = \{x \in \mathbb{R} : \ F(x) - F(z) \in -P\} = \mathbb{R}$, and therefore,*

$$\bigcup_{z \in \mathbb{R}} \bigcap_{q \in \text{extrd } P^*} \text{argmin}_{K(z)} h_q = \bigcup_{z \in \mathbb{R}} \text{argmin}_{\mathbb{R}} f_2 = \mathbb{R}.$$

Part (a) of the preceding lemma can be applied to situations where $\text{argmin}_K h_q = \emptyset$ for all $q \in \text{extrd } P^*$, as the next example shows.

Example 3.4. *Consider $P = \mathbb{R}^2_+$, $K = \mathbb{R}$, and $F = (f_1, f_2)$, $f_1(x) = x$, $f_2(x) = -x$. Here, $E = \mathbb{R}$ and $\text{argmin}_K h_q = \emptyset$ for $q \in \text{extrd } P^*$. We claim that $\bigcup_{z \in K} \bigcap_{q \in \text{extrd } P^*} \text{argmin}_{K(z)} h_q = \mathbb{R}$, where* extrd $P^* = \{(t, 0), (0, t) : \ t > 0\}$*. Indeed, $K(z) = \{z\}$ for all $z \in \mathbb{R}$, and so $\text{argmin}_{\{z\}} h_q = \{z\}$ for all $q \in P^* = P$. This provex the claim.*

A relationship between $\text{argmin}_K h_q$ and $\text{argmin}_{K(z)} h_q$ is given below.

Proposition 3.5. *For every $q \in P^*$, we have*

$$\text{argmin}_K h_q \subseteq \bigcup_{z \in K} \text{argmin}_{K(z)} h_q.$$

Proof. It is straightforward since, $x \in \text{argmin}_K h_q \Longrightarrow x \in \text{argmin}_{K(x)} h_q$. $\square$

The other inclusion may be false as the next example shows.

Example 3.6. *Consider* $K = \mathbb{R}$, $F = (f_1, f_2)$, $f_1(x) = |x|$, $f_2(x) = |x-1|$, $P = \mathbb{R}^2_+$. *Then* $K(0) = \{0\}$, $K(1) = \{1\}$. *Take* $q = (1,0) \in P^* = \mathbb{R}^2_+$, *we obtain*

$$\mathrm{argmin}_K h_q = \mathrm{argmin}_{\mathbb{R}} f_1 = \{0\} \subseteq \{0,1\} \subseteq \bigcup_{z\in\mathbb{R}} \mathrm{argmin}_{K(z)} f_1.$$

It is well known that $\mathrm{argmin}_K h_q \subseteq E_W$ for all $q \in P^* \setminus \{0\}$. By taking into account the previous proposition, next lemma says a finer inclusion holds.

Lemma 3.7. *Let* P *be a closed convex cone with* $\mathrm{int}\ P \neq \emptyset$. *Then,*

$$\bigcup_{z\in K} \mathrm{argmin}_{K(z)} h_q \subseteq E_W, \ \forall\ q \in P^* \setminus \{0\}.$$

Proof. Take any $q \in P^*$, $q \neq 0$, and let $\bar{x} \in \mathrm{argmin}_{K(z)} h_q$ for some $z \in K$. If on the contrary, $\bar{x} \notin E_W$, there exists $x \in K$ satisfying $F(x) - F(\bar{x}) \in -\mathrm{int}\ P$, we obtain $x \in K(z)$ and $h_q(x) < h_q(\bar{x})$, yielding a contradiction, and therefore $\bar{x} \in E_W$. □

We are in a position to establish a chain of inclusions involving E_{pr}, E, E_W and $\mathrm{argmin}_{K(z)} h_q$, where the first inclusion was already appeared in [19].

Remark 3.8. *Let* P *be a closed convex cone with* $\mathrm{int}\ P \neq \emptyset \neq \mathrm{int}\ P^*$. *Then,*

$$\mathrm{argmin}_K \langle q_0, F(\cdot)\rangle \subseteq E_{\mathrm{pr}} \subseteq E = \bigcup_{z\in K} \mathrm{argmin}_{K(z)} \langle q_0, F(\cdot)\rangle \subseteq E_W, \ \forall \ \ q_0 \in \mathrm{int}\ P^*.$$

We present the next lemma just for completeness: part of it gives a complete characterization of E_W in terms of scalar minimization problems. This was established in [16, Theorem 2.1], see also [11]. In what follows

$$P^{*i} \doteq \{q \in Y^* :\ \langle q, p\rangle > 0 \quad \forall\ p \in P \setminus \{0\}\}.$$

Lemma 3.9. *Let* $P \subsetneq Y$ *be a closed convex cone and* h_q *be quasiconvex and lower semicontinuous for all* $q \in P^* \setminus \{0\}$.

(*a*) *If* $\mathrm{int}\ P \neq \emptyset$, *then*

$$E_W = \bigcup_{q\in P^*\setminus\{0\}} \mathrm{argmin}_K h_q. \tag{3}$$

(*b*) *Assume that* $Y = \mathbb{R}^m$, *then*

$$E \subseteq \bigcup_{q\in P^*\setminus\{0\}} \mathrm{argmin}_K h_q. \tag{4}$$

(c) *Assume that P is locally compact and pointed, then*

$$E_{\rm pr} = \bigcup_{q \in P^{*i}} \mathrm{argmin}_K h_q. \tag{5}$$

Proof. Since

$$\bar{x} \in E_W \Longleftrightarrow (F(K) - F(\bar{x}) + P) \cap (-\mathrm{int}\ P) = \emptyset;$$

$$\bar{x} \in E \Longleftrightarrow (F(K) - F(\bar{x}) + P) \cap (-P \setminus l(P)) = \emptyset;$$

$$\bar{x} \in E_{\rm pr} \Longleftrightarrow \overline{\mathrm{cone}}(F(K) - F(\bar{x}) + P) \cap (-P) = \{0\}$$

and $F(K) + P$ is convex by [11, Corollary 3.11], we conclude with (a) and (b) after applying a standard convex separation theorem, since $P \setminus l(P)$ is also convex. For (c) we use the separation result for convex cones [3, Proposition 3]. ☐

Remark 3.10. (a) *Since* int $P \neq \emptyset$ *implies that* $B^* = \{q \in P^* : \langle q, p\rangle = 1\}$ *is a base for* P^*, *i.e.,* $P^* = \bigcup_{t \geq 0} tB^*$, *(3) reduces to*

$$E_W = \bigcup_{q \in B^*} \mathrm{argmin}_K h_q.$$

(b) *If, in addition,* $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ *(which implies the pointedness of* P^**), we can substitute* $P^* \setminus \{0\}$ *by* extrd P^* *in (3) and (4).*

Next example shows the inclusion in (4) may be strict.

Example 3.11. *Consider* $K = \mathbb{R}$, $P = \mathbb{R}^2_+$, $F = (f_1, f_2)$, *where* $f_1(x) = \sqrt{|x|}$ *and* $f_2(x) = 1$. *Here,* $E = \{0\}$ *whereas*

$$\bigcup_{q \in B^*} \mathrm{argmin}_K h_q = \mathbb{R}.$$

4 Characterizations and asymptotic estimates

Throughout this section we consider $Y = \mathbb{R}^m$, $X = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a closed convex set.

The next result, which provides some estimates for the asymptotic cone of the efficient set E, is the starting point for our further analysis. The case $P = \mathbb{R}^n_+$ was proved in [6]. We present its proof for convenience of the reader.

We recall that $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ implies the pointedness of P^*, and therefore int $P \neq \emptyset$.

Lemma 4.1. *Let $P \subsetneq \mathbb{R}^m$ be a closed convex cone such that $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ with* extrd P^* *being a finite set, that is, P is a polyhedral cone. Assume that* $\mathrm{argmin}_{K(z)} h_{q_0} \neq \emptyset$ *for some $z \in K$ and $q_0 \in \mathrm{int}\ P^*$. Then,*

$$(\mathrm{argmin}_{K(z)} h_{q_0})^\infty \subseteq E^\infty \subseteq \bigcup_{q \in \mathrm{extrd}\ P^*} \{v \in K^\infty : \quad h_q^\infty(v) \leq 0\}.$$

Proof. The first inclusion follows from Lemma 3.1. We now check the remaining inclusion. In case E is bounded such an inclusion is trivial.
Assume that extrd $P^* = \{q_i : \ i = 1, \dots, p\}$ and that E is unbounded. Let $v \in E^\infty$, with $\|v\| = 1$; there exists $\{x_k\}_{k \in \mathbb{N}} \subseteq E$ satisfying $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to v$.
(i): If $\sup\limits_{k \in \mathbb{N}} h_{q_{i_0}}(x_k) \leq \alpha < +\infty$ for some $i_0 \in \{1, 2, \dots, p\}$, then

$$\{x \in K : h_{q_{i_0}}(x) \leq \alpha\}^\infty \subseteq \{v \in K^\infty : h_{q_{i_0}}^\infty(v) \leq 0\} \subseteq \bigcup_{i=1}^{p} \{v \in K^\infty : h_{q_i}^\infty(v) \leq 0\},$$

where the first inclusion is easily obtained.
(ii): If on the contrary, for all $i = 1, 2, \dots, p$, we have

$$\sup_{k \in \mathbb{N}} h_{q_i}(x_k) = +\infty,$$

then there exists $\{x_k^1\}_{k \in \mathbb{N}} \subseteq \{x_k\}_{k \in \mathbb{N}}$ such that $\|x_k^1\| \to +\infty$, $\dfrac{x_k^1}{\|x_k^1\|} \to v$ and in addition

$$\sup_{k \in \mathbb{N}} h_{q_i}(x_k^1) = +\infty, \quad \forall\ i = 1, 2, \dots, p.$$

For $\bar{k} \in \mathbb{N}$ fixed and any $i \in \{1, 2, \dots, p\}$ there exists $k_i \in \mathbb{N}$ such that $h_{q_i}(x_k^1) > h_{q_i}(x_{\bar{k}}^1), \ \ \forall\ k \geq k_i$. That is, $\langle q_i, F(x_{\bar{k}}^1) - F(x_k^1) \rangle < 0, \ \ \forall \ \ k \geq k_i, \ \ \forall \ \ i = 1, 2, \dots, p$.
Set $k_0 = \max\limits_{1 \leq i \leq p} k_i$. Then, for all $i = 1, 2, \dots, p$,

$$\langle q_i, F(x_{\bar{k}}^1) - F(x_k^1) \rangle < 0, \ \ \forall \ \ k \geq k_0.$$

This implies that $F(x_{\bar{k}}^1) - F(x_k^1) \in -\mathrm{int}\ P \subseteq -P \setminus l(P)$, which cannot happen if $x_k^1 \in E$, proving that case (i) is only possible. $\square$

Next example shows the outer estimate for E^∞ is very large when each h_{q_i} is non convex.

Example 4.2. *Take $K = \mathbb{R}$, $P = \mathbb{R}_+^2$ and consider $f_1(x) = \sqrt{|x|}$, $f_2(x) = \sqrt{|x - 2|}$. Then $f_1^\infty(v) = 0 = f_2^\infty(v), \ \ \forall \ \ v \in \mathbb{R}$. Thus,*

$$\bigcup_{i=1}^{2} \{v \in K^\infty : \quad f_i^\infty(v) \leq 0\} = \mathbb{R},$$

whereas $E = [0, 2] = E_W$ and therefore $E^\infty = \{0\}$.

In order to find finer outer estimates to E^∞, we consider closed convex cones P^* such that

$$P^* = \bigcup_{t>0} tB^*, \quad B^* = \mathrm{co}(S_0), \quad S_0 \text{ is compact,} \tag{6}$$

where $0 \notin B^*$ is convex and compact, and S_0 is the set of extreme points of B^*. Under assumption (6), P^* is pointed, which is equivalent to $\mathrm{int}\, P \neq \emptyset$.
Like in [7, 9], we consider the following cones

$$R_P \doteq \bigcap_{y\in K} \{v \in K^\infty : \quad F(y+\lambda v) - F(y) \in -P, \quad \forall\, \lambda > 0\}. \tag{7}$$

$$R_W \doteq \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcup_{q\in S_0} \{v \in K^\infty : \ \langle q, F(y+\lambda v) - F(y)\rangle \leq 0\}. \tag{8}$$

$$\widetilde{R}_W \doteq \bigcap_{y\in K} \bigcup_{q\in S_0} \{v \in K^\infty : \quad \langle q, F(y+\lambda v) - F(y)\rangle \leq 0, \quad \forall\, \lambda > 0\}. \tag{9}$$

$$\widehat{R}_W \doteq \bigcup_{q\in S_0} \bigcap_{y\in K} \{v \in K^\infty : \ \langle q, F(y+\lambda v) - F(y)\rangle \leq 0, \quad \forall\, \lambda > 0\}. \tag{10}$$

It is obvious that

$$R_P \subseteq \widehat{R}_W \subseteq \widetilde{R}_W \subseteq R_W. \tag{11}$$

Remark 4.3. *For the example above, we obtain that* $E^\infty = (E_W)^\infty = R_P = \{0\}$*.*

Conditions ensuring some equalities in (11) are given in the next proposition which is taken from [12].

Proposition 4.4. *Let P be a closed convex cone and h_q are quasiconvex for all $q \in B^*$. Then, $R_W \subseteq \widetilde{R}_W$, that is, $R_W = \widetilde{R}_W$,*

In what follows we need the following notion. The function $F : K \to Y$ is said to be P-lower semicontinuous (P-lsc) at $x_0 \in K$ ([17]) if for any neighborhood V of $F(x_0)$ in Y there exists a neighborhood U of x_0 in X such that $F(U \cap K) \subseteq V + P$. The function $F : K \to Y$ is said to be P-lsc if it is at every point $x_0 \in K$.
It is easy to check that $\mathbb{R}^m_+$-lower semicontinuity of $F = (f_1, \dots, f_m)$ is equivalent to the (usual) lower semicontinuity of each f_i.

Lemma 4.5. *Let P be a closed convex cone satisfying (6) (in particular* $\mathrm{int}\, P \neq \emptyset$*). Assume that F is P-lsc and that h_q is quasiconvex for all $q \in B^*$. Then*

$$(E_{\mathrm{pr}})^\infty \subseteq E^\infty \subseteq (E_W)^\infty \subseteq \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcup_{q\in S_0} \{v \in K^\infty : \langle q, F(y+\lambda v) - F(y)\rangle \leq 0\} = R_W.$$

Proof. We only need to check the third inclusion. In case E_W is empty or bounded, such an inclusion follows. Let $v \in (E_W)^\infty$, $\|v\| = 1$. Then there exists $\{x_k\}_{k\in\mathbb{N}} \subseteq E_W$ such that $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to v$.
Take any $y \in K$ and $\lambda > 0$. Then, for some $k_0 \in \mathbb{N}$,

$$(1 - \frac{\lambda}{\|x_k\|})y + \lambda\frac{x_k}{\|x_k\|} \in K, \ \frac{\lambda}{\|x_k\|} < 1, \quad \forall\ k > k_0.$$

Since $x_k \in E_W$, there existe $\{q_k\}_{k\in\mathbb{N}} \subseteq S_0$, such that

$$h_{q_k}(x_k) \leq h_{q_n}(y) \ \text{ and } \ q_k \to q \in S_0.$$

By quasiconvexity of h_{q_k} we obtain,

$$\langle q_k, F((1 - \frac{\lambda}{\|x_k\|})y + \lambda\frac{x_k}{\|x_k\|})\rangle \leq \langle q_k, F(y)\rangle, \quad \forall\ k > k_0.$$

Let $\varepsilon > 0$, by [17, Theorem 5.5],

$$\langle q_k, F(y)\rangle \geq \langle q_k, F((1 - \frac{\lambda}{\|x_k\|})y + \lambda\frac{x_k}{\|x_k\|})\rangle \geq \langle q_k, F(y + \lambda v)\rangle - \varepsilon, \quad \forall\ k \geq k_0.$$

Letting $k \to +\infty$, and since $\varepsilon > 0$ was arbitrary, we get

$$\langle q, F(y + \lambda v)\rangle \leq \langle q, F(y)\rangle,$$

proving the desired result. □

Next lemma provides an interesting inner estimate for $(E_W)^\infty$.

Lemma 4.6. *Let P be a closed convex cone such that* $\operatorname{int} P \neq \emptyset$*, and let $J \subseteq P^* \setminus \{0\}$. If for all $q \in J$, h_q is quasiconvex, lower semicontinuous and* $\operatorname{argmin}_K h_q \neq \emptyset$*. Then,*

$$\bigcup_{q\in J}\bigcap_{y\in K}\{v \in K^\infty : \ h_q(y + \lambda v) \leq h_q(y), \quad \forall\ \lambda > 0\} \subseteq (E_W)^\infty.$$

Proof. Since $\operatorname{argmin}_K h_q \subseteq E_W$ for all $q \in J$, we get

$$\bigcup_{q\in J}\operatorname{argmin}_K h_q \subseteq E_W.$$

Thus,
$$(E_W)^\infty \supseteq \bigcup_{q\in J}(\operatorname{argmin}_K h_q)^\infty.$$

The quasiconvexity and lower semicontinuity of h_q, imply that

$$(\operatorname{argmin}_K h_q)^\infty = \bigcap_{y\in K}\{v \in K^\infty : h_q(y + \lambda v) \leq h_q(y), \ \forall\ \lambda > 0\},$$

from which the result follows. □

Lemma 4.7. *Let P be closed convex cone. If $E \neq \emptyset$, then*

$$R_P \doteq \bigcap_{y \in K} \{v \in K^{\infty} : \; F(y + \lambda v) - F(y) \in -P, \; \forall \; \lambda > 0\} \subseteq E^{\infty}.$$

Proof. Let $v \in R_P$ and $\overline{x} \in E$. We claim that $\overline{x} + tv \in E$, for all $t > 0$. Take any $y \in K$, then

$$F(y) - F(\overline{x} + tv) = F(y) - F(\overline{x}) + F(\overline{x}) - F(\overline{x} + tv).$$

Thus, $F(y) - F(\overline{x}) + F(\overline{x}) - F(\overline{x} + tv) \in Y \setminus (-P \setminus l(P)) + P \subseteq Y \setminus (-P \setminus l(P))$, which proves the claim, and therefore $v \in E^{\infty}$. □

Remark 4.8. *If h_q is convex for all $q \in J$, then the previous lemma reduces to*

$$\bigcap_{q \in J} \{v \in K^{\infty} : \; h_q^{\infty}(v) \leq 0\} \subseteq E^{\infty}.$$

Next lemma extends and generalizes Lemma 3.2 in [6] which is valid only for polyhedral cones.

Lemma 4.9. *Assume that P is a closed convex cone. The following assertions hold.*

(*a*) *Assume that $E_{\text{pr}} \neq \emptyset$. If there exists $\emptyset \neq J \subseteq P^* \setminus \{0\}$ such that*

$$\sup_{x \in E_{\text{pr}}} \sup_{q \in J} \; h_q(x) < +\infty, \tag{12}$$

then

$$(E_{pr})^{\infty} \subseteq \bigcap_{q \in J} \{v \in K^{\infty} : \; h_q^{\infty}(v) \leq 0\}.$$

(*b*) *Assume that $E \neq \emptyset$. If there exists $\emptyset \neq J \subseteq P^* \setminus \{0\}$ such that*

$$\sup_{x \in E} \sup_{q \in J} h_q(x) < +\infty, \tag{13}$$

then

$$E^{\infty} \subseteq \bigcap_{q \in J} \{v \in K^{\infty} : \; h_q^{\infty}(v) \leq 0\}.$$

(*c*) *Assume that* int $P \neq \emptyset$ *and $E_W \neq \emptyset$. If there exists $\emptyset \neq J \subseteq P^* \setminus \{0\}$ such that*

$$\sup_{x \in E_W} \sup_{q \in J} h_q(x) < +\infty, \tag{14}$$

then

$$(E_W)^{\infty} \subseteq \bigcap_{q \in J} \{v \in K^{\infty} : \; h_q^{\infty}(v) \leq 0\}.$$

Proof. Let $A \in \{E, E_{\rm pr}, E_W\}$. If either A is empty or bounded there nothing to prove. Let $v \in A^\infty, \|v\| = 1$. Then, there exists $\{x_k\}_{k\in\mathbb{N}} \subseteq A$, con $\|x_k\| \to +\infty$, such that $\dfrac{x_k}{\|x_k\|} \to v$. By assumption on A, that is, (12), (13), (14), we have, for some $M > 0$, $h_q(x_k) \le M$, for all $k \in \mathbb{N}$, for all $q \in J$, i.e., $(x_k, M) \in$ epi h_q. Since $\dfrac{1}{\|x_k\|}(x_k, M) \to (v, 0)$, we obtain $(v, 0) \in (\text{epi } h_q)^\infty = \text{epi } h_q^\infty$, for all $q \in J$. Hence $h_q^\infty(v) \le 0$, and therefore

$$A^\infty \subseteq \bigcap_{q\in J} \{v \in K^\infty : \ \ h_q^\infty(v) \le 0\}.$$

□

Remark 4.10. *By Lemma 4.7 and under assumptions of Lemma 4.9, we have*

$$\bigcap_{y\in K}\bigcap_{\lambda>0}\bigcap_{q\in P^*}\{v \in K^\infty : \ \ h_q(y+\lambda v) \le h_q(y)\} \subseteq E^\infty \subseteq \bigcap_{q\in J}\{v \in K^\infty : \ h_q^\infty(v) \le 0\}.$$

$$\bigcap_{y\in K}\bigcap_{\lambda>0}\bigcap_{q\in P^*}\{v \in K^\infty : \ \ h_q(y+\lambda v) \le h_q(y)\} \subseteq E_W^\infty \subseteq \bigcap_{q\in J}\{v \in K^\infty : \ \ h_q^\infty(v) \le 0\}.$$

5 Characterizing the nonemptiness and boundedness of E and E_W

We are now devoted to find a formula for E^∞ and $(E_W)^\infty$ under the standard convexity assumption. It extends partially Lema 3.2 in [6].
We continue to consider throughout this section that $Y = \mathbb{R}^m$, $X = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a closed convex set.

Theorem 5.1. *Let P be a closed convex cone such that $P^* \setminus \{0\} = \text{co}(\text{extrd } P^*)$. Assume that h_q is convex for all $q \in \text{extrd } P^*$.*

(*a*) *If assumption* (*13*) *holds for $J = \text{extrd } P^*$, then*

$$E^\infty = \bigcap_{q\in\text{extrd } P^*} \{v \in K^\infty : h_q^\infty(v) \le 0\}.$$

(*b*) *If* int $P \neq \emptyset$ *and assumption* (*14*) *holds for $J = \text{extrd } P^*$, then*

$$(E_W)^\infty = \bigcap_{q\in\text{extrd } P^*} \{v \in K^\infty : \ h_q^\infty(v) \le 0\}.$$

Proof. It is a consequence of Lemma 4.9 and Remark 4.10. □

Having found a formula for E^∞ and $(E_W)^\infty$, we proceed to characterize the nonemptiness and boundedness of E and E_W.

Theorem 5.2. *Let P be a closed convex cone such that $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ with* extrd P^* *being a finite set, that is, P is a polyhedral cone. Assume that h_q is convex and continuous for all $q \in \mathrm{extrd}\ P^*$.*

(*a*) *E is nonempty and bounded if, and only if*

$$\bigcap_{q\in \mathrm{extrd}\ P^*} \{v \in K^\infty : h_q^\infty(v) \leq 0\} = \{0\},$$

and assumption (13) holds for $J = \mathrm{extrd}\ P^*$.

(*b*) *If* int $P \neq \emptyset$ *then, E_W is nonempty and compact if, and only if*

$$\bigcap_{q\in \mathrm{extrd}\ P^*} \{v \in K^\infty : h_q^\infty(v) \leq 0\} = \{0\},$$

and assumption (14) holds for $J = \mathrm{extrd}\ P^*$.

Proof. We only check (a), the other being similar.
($\Leftarrow$) We apply Theorem 2.1 of [5] to conclude that $E \neq \emptyset$. Since assumption (13) holds, the previous theorem implies that

$$E^\infty = \bigcap_{q\in \mathrm{extrd}\ P^*} \{v \in K^\infty :\ h_q^\infty(v) \leq 0\}.$$

Thus, E is bounded.
($\Rightarrow$) Since E is nonempty and bounded and each h_q is continuous, assumption (13) holds, and therefore,

$$\{0\} = E^\infty = \bigcap_{q\in \mathrm{extrd}\ P^*} \{v \in K^\infty : h_q^\infty(v) \leq 0\}$$

by the previous theorem. □

Remark 5.3. *Trying to extend the previous result to a non-convex setting, one can think in semistrictly quasiconvex functions. Unfortunately, the following example shows that in such a case, Theorem 5.2 may be false. Indeed, take $K = \mathbb{R}$, $P = \mathbb{R}^2_+$, and consider $F = (f_1, f_2)$, with $f_1(x) = \dfrac{x}{1+|x|}$, $f_2(x) = -\dfrac{x}{1+|x|}$.*

It is easy to check that $E = \mathbb{R}$ and so $E^\infty = \mathbb{R}$; whereas assumption (13) holds and $R_P = \{0\}$.

We now present some characterizations of the nonemptiness of E and E_W by means of R_P, R_W and assumptions (13) and (14).

Theorem 5.4. *Let P be a closed convex cone such that $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ with* extrd P^* *being a finite set, that is, P is polyhedra. Assume that h_q is convex and continuous for all $q \in \mathrm{extrd}\ P^*$. Let consider the following statements:*

(*a*) *$E \neq \emptyset$ and bounded*;

(*b*) *$R_P = \{0\}$ and assumption (13) holds for $J = \text{extrd } P^*$.*

In case $\text{int } P \neq \emptyset$, *consider also*

(*c*) *$R_P = \{0\}$ and assumption (14) holds for $J = \text{extrd } P^*$*;

(*d*) *$E_W \neq \emptyset$ and compact*;

(*e*) $\text{argmin}_K h_q \neq \emptyset$ *and compact for all* $q \in \text{extrd } P^*$;

(*f*) $R_W = \{0\}$.

The following hold: $(a) \Leftrightarrow (b) \Leftarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$.

Proof. $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ are Theorem 5.2. That $(c) \Rightarrow (b)$ is obvious. The equivalences $(d) \Leftrightarrow (e) \Leftrightarrow (f)$ follows from Theorem 5.1 in [12]. □

6 Necessary conditions: the general convex and the non-convex quadratic cases

This section developes necessary conditions for existence of efficient and weak efficient solutions. We first consider the convex case and afterwards, we particularize a nonconvex situation when each component is quadratic and nonhomogeneous.

Next lemma generalizes [10, Proposition 8.3] and [6, Proposition 4.1(b)].

Lemma 6.1. *Let X, Y be normed vector spaces, $P \subsetneq Y$ be convex closed cone and $K \subseteq X$ be closed and convex.*

(*a*) *Assume that P^* is the weak-star closed convex hull of* extrd P^* *and that h_q is convex and lower semicontinuous for all $q \in \text{extrd } P^*$. If $E \neq \emptyset$ then,*

$$v \in K^\infty : \; h_q^\infty(v) \leq 0, \forall\, q \in \text{extrd } P^* \Rightarrow h_q^\infty(v) = 0, \;\forall\, q \in \text{extrd } P^*.$$

(*b*) *Assume that $P^* \setminus \{0\}$ is the convex hull of* extrd P^*, *that h_q is convex and lower semicontinuous for all $q \in \text{extrd } P^*$, and that* $\text{int } P \neq \emptyset$. *If $E_W \neq \emptyset$ then,*

$$v \in K^\infty : \quad h_q^\infty(v) \leq 0, \;\forall\, q \in \text{extrd } P^* \Rightarrow \exists\, q_0 \in \text{extrd } P^*, h_{q_0}^\infty(v) = 0.$$

Proof. (a): If on the contrary, there exists $\overline{q} \in \text{extrd } P^*$ such that $h_{\overline{q}}^\infty(v) < 0$, then

$$\frac{h_{\overline{q}}(y + \lambda v) - h_{\overline{q}}(y)}{\lambda} \leq \sup_{\lambda > 0} \frac{h_{\overline{q}}(y + \lambda v) - h_{\overline{q}}(y)}{\lambda} < 0, \;\forall\, y \in K.$$

Thus,

$$h_q(y+\lambda v) \le h_q(y),\ \forall\ q \in \text{extrd } P^* \text{ and } h_{\overline{q}}(y+\lambda v) < h_{\overline{q}}(y),$$

that is,

$$F(y+\lambda v) - F(y) \in -P \setminus l(P),\ \ \forall\ y \in K,$$

contradicting the fact that E is nonempty. Hence, $h_q^\infty(v) = 0$, for all $q \in \text{extrd } P^*$.
(b): Suppose, on the contrary, that for all $q \in \text{extrd } P^*$, $h_q^\infty(v) < 0$. Then,

$$\frac{h_q(y+\lambda v) - h_q(y)}{\lambda} \le \sup_{\lambda>0} \frac{h_q(y+\lambda v) - h_q(y)}{\lambda} < 0,\ \forall\ y \in K,$$

from which $h_q(y+\lambda v) < h_q(y),\ \forall\ y \in K, \forall\ q \in \text{extrd } P^*$. That is, $h_q(y+\lambda v) < h_q(y),\ \forall\ y \in K, \forall\ q \in P^* \setminus \{0\}$. This cannot happen if $E_W \neq \emptyset$, proving the result. ☐

We now consider $Y = \mathbb{R}^m$, $P = \mathbb{R}^m_+$ and $F = (f_1, f_2, \ldots, f_m)$ with $f_i(x) = x^\top A_i x + b_i^\top x + \alpha_i$ with $A_i \in M(n \times n)$ being symmetric, $b_i \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$. In addition we assume that K is closed and convex.

Lemma 6.2. *Let $K \subseteq \mathbb{R}^n$ be closed and convex, let f_i be a (not necessarily convex) queadratic function as above.*

(a) *If $E \neq \emptyset$ then,*

$$v \in K^\infty \bigcap_{i=1}^m \ker(A_i),\ b_i^\top v \le 0, \forall\ i = 1, 2, \ldots, m \Rightarrow b_i^\top v = 0,\ \forall\ i = 1, 2, \ldots, m.$$

(b) *If $E_W \neq \emptyset$ then,*

$$v \in K^\infty \bigcap_{i=1}^m \ker(A_i),\ b_i^\top v \le 0,\ \forall\ i = 1, 2, \ldots, m \Rightarrow \exists\ i_0 \in \{1, 2, \ldots, m\},\ b_{i_0}^\top v = 0.$$

Proof. The proofs being similar, we only check (a).
(a): Let $v \in K^\infty \bigcap_{i=1}^m \ker(A_i)$ such that $b_i^\top v \le 0$ for all $i = 1, 2, \ldots, m$. Then

$$f_i(y+\lambda v) \le f_i(y),\ \forall\ i = 1, 2, \ldots, m,\ \ \forall\ y \in K. \tag{15}$$

Suppose, on the contrary, that there exits $i_0 \in \{1, 2, \ldots, m\}$ such that $b_{i_0}^\top v < 0$. Then

$$f_{i_0}(y+\lambda v) < f_{i_0}(y),\ \ \forall\ y \in K. \tag{16}$$

Both inequalities imply $F(y+\lambda v) - F(y) \in -\mathbb{R}^m_+ \setminus \{0\}, \forall\ y \in K$, which is a contradiction since E is nonempty. ☐

Corollary 6.3. *Let $K \subseteq \mathbb{R}^n$ be closed and convex, let f_i be a (not necessarily convex) quadratic function as above.*

(a) If $E \neq \emptyset$ and bounded, then

$$K^\infty \bigcap_{i=1}^m \ker(A_i)(\bigcap_{i=1}^m \{-b_i\}^*) = \{0\}. \tag{17}$$

(b) If $E_W \neq \emptyset$ and bounded, then

$$K^\infty \bigcap_{i=1}^m \ker(A_i)(\bigcap_{i=1}^m \{-b_i\}^*) = \{0\}. \tag{18}$$

Proof. We only prove (a). Let $v \in K^\infty \bigcap_{i=1}^m \ker(A_i)(\bigcap_{i=1}^m \{-b_i\}^*)$, $v \neq 0$. By the previous lemma, $b_i^\top v = 0$ for all $i = 1, 2, \ldots, m$. This means that

$$f_i(y + \lambda v) = f_i(y), \ \forall\, y \in K, \ \forall\, \lambda > 0, \ \forall\, i = 1, 2, \ldots, m.$$

If $y \in E$, the previous equality implies that $y + \lambda v \in E$ for all $\lambda > 0$, which cannot happen if E is bounded. Hence, $v = 0$. □

Remark 6.4. *When each A_i is positive semidefinite, i.e., each f_i is convex, the necessary conditions in Lemma 6.2 and Corollary 6.3 become also sufficient as Example 8.4 in [10] shows.*

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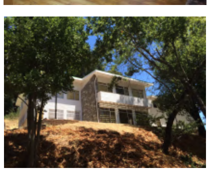

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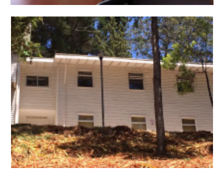

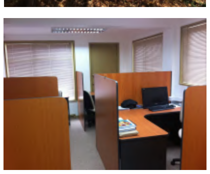

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