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Tomás P. Barrios† and Rommel Bustinza‡

Abstract

In this paper we present an augmented mixed discontinuous formulation for the stationary Stokes problem. More precisely, we derive a new stabilized discontinuous formulation by adding appropriate Galerkin least squares terms to the velocity-pseudostress formulation associated to Stokes problem. Then, using the Lax-Milgram theorem, we prove the well-posedness of the resulting discrete scheme, and under suitable regularity assumptions, we obtain the optimal rate of convergence of the method, with respect to the $h-$version. Finally, several numerical experiments confirming the theoretical properties of the augmented discontinuous scheme are also reported.

Key words: Discontinuous Galerkin, augmented formulation, Stokes problem.

Mathematics subject classifications (1991): 65N30; 65N12; 65N15

1 Introduction

In the last years, there have been done much efforts concerning the stabilization of the discontinuous mixed finite element methods. We refer to [8] (and the references therein) for an overview on stabilized DG methods for elliptic problems. In particular, stabilized discontinuous Galerkin (DG) methods are introduced in [20] and [17] for Stokes system and linear elasticity problems in the incompressible and nearly incompressible cases, respectively. Concerning Darcy flow, we can refer to [10] and [18], where the stabilized DG formulation is derived by adding a suitable Galerkin least square element term instead of the usual jump penalty terms. On the other hand, different alternatives for the Bassi-Rebay DG method can be found in [21] and [19].

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In this framework, we recently developed an augmented discontinuous formulation (see, for e.g., [5] and [6]), whose approach is based on the introduction of suitable Galerkin least squares terms, which usually arise from constitutive and equilibrium equations, to the discontinuous mixed formulation. In the present case, we look for the tensor-valued unknown such that locally belongs to $H(\mathbf{div})$, which motivates the employment of local row-wise Raviart-Thomas matrix space to approximate it. We note that this choice reduces the degrees of freedom than the needed in the standard form (polynomial-wise matrix space). In addition, we can establish existence and uniqueness of the discrete scheme thanks to Lax-Milgram theorem. This way we circumvent the use of any lifting operators (mild condition), and thus allow us to use any pair of discrete spaces to approximate the unknowns.

Now, in order to describe the model of interest, we let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ. Then, given the source terms $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity (vector field) $\boldsymbol{u}$ and the pressure (scalar field) p such that

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \boldsymbol{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{1.1}$$

where $\nu > 0$ is the viscosity of the flow and the datum $\mathbf{g}$ satisfies the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0$, with $\boldsymbol{\nu}$ stands for the unit outward normal at Γ. In addition, for uniqueness purposes, we seek $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q = 0\}$.

We remark that problem (1.1) has already been analyzed in previous works, such as [14] and [20], applying different strategies. In [14] the authors use the local discontinuous Galerkin method, introducing the gradient of the velocity as an additional unknown. In this case, they prove the optimal rates of convergence of the method in agreement with the $h-$version. On the other hand, in [20] it is developed two stabilized DG formulations, which are obtained in two steps. First, they deduce a nonconforming formulation, introducing suitable Lagrange multipliers instead of the standard numerical fluxes (acting as mortar unknowns). Then, three suitable interior stabilized terms are added to the formulation, which make it possible to eliminate the mortar unknowns, circumventing therefore the validation of the corresponding discrete inf-sup conditions. In the present work, we proceed as in [12], and introduce the corresponding pseudostress as an additional unknown, which let us to eliminate (in the continuous context) the pressure p from the variational formulation, deriving the *so-called* nonstandard velocity-pseudostress formulation for the Stokes system. Next, we partially follow [6] and [7], and develop an augmented discrete discontinuous mixed formulation, which is well-posed, and have the optimal rates of convergence under suitable regularity assumption. In addition, and as we remark in the above paragraph, this new discrete scheme also works for any combination of the approximation spaces for the velocity and the pseudostress. Moreover, we present a cheap postprocess to recover approximations for the pressure as well as for the velocity gradient, showing that their respective rates of convergence behave as expected. This way, we observe a significant decrease in the degrees of freedom, since we have to solve a linear system of equatuions involving only the pseudostress and the velocity as unknowns.

The rest of the paper is organized as follows. For the sake of completeness, in Section 2 we present a review of the nonstandard velocity-pseudostress formulation for Stokes equations. In Section 3, we proceed as in [6] and [7] to derive an augmented discontinuous

Galerkin scheme, we prove its unique solvability and the optimal rates of convergence. Finally, in Section 4 we give some numerical examples confirming our theoretical results.

We end this section with some notations to be used throughout the paper. Given any Hilbert space H, we denote by H^2 the space of vectors of order 2 with entries in H, and by $H^{2\times 2}$ the space of square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write, as usual, $\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})$, $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$, we denote by $\boldsymbol{v} \otimes \boldsymbol{w}$ the matrix whose ij-th entry is $v_i\, w_j$. We also use the standard notations for Sobolev spaces and norms. We denote by $H = H(\mathrm{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}$, and by $H_0 := \{\boldsymbol{\tau} \in H : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\}$. Note that $H = H_0 \oplus \mathbb{R}\,\mathbf{I}$, that is, for any $\boldsymbol{\tau} \in H$ there exist unique $\boldsymbol{\tau}_0 \in H_0$ and $d \in \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$. In addition, we define the deviator of the matrix $\boldsymbol{\tau} \in H$ by $\boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$. We remark that $\mathrm{tr}(\boldsymbol{\tau}^d) = 0$ in Ω, then for any $\boldsymbol{\tau} \in H$, $\boldsymbol{\tau}^d \in H_0$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameters, which may take different values at different occurrences.

2 The velocity-pseudostress formulation

We begin this work reviewing the nonstandard velocity-pseudostress continuous formulation for the Stokes system, described in detail in [12]. We point out also that this formulation can be derived following the ideas given in an earlier work [3] (see Section 8), where the authors propose a displacement-pressure formulation for an anisotropic elasticity problem. To this end, we first introduce the pseudostress $\boldsymbol{\sigma} := \nu\nabla\boldsymbol{u} - p\boldsymbol{I}$ in Ω as additional unknown. Next, thanks to the incompressibility condition $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω, it is not difficult to check that $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$ in Ω, which implies that $\boldsymbol{\sigma} \in H_0$. This relation allows us to rewrite the problem (1.1) as the following linear first order system: *Find* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [H^1(\Omega)]^2$ *such that*

$$\boldsymbol{\sigma}^d = \nu\nabla\boldsymbol{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\boldsymbol{f} \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{2.1}$$

Now, proceeding in the usual way, we deduce the variational formulation based on velocity-pseudostress, which reads: *Find* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [L^2(\Omega)]^2$ *such that*

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}) &= \ G(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in H_0\,, \\ b(\boldsymbol{\sigma}, \mathbf{v}) &= \ F(\mathbf{v}) \quad \forall\, \mathbf{v} \in [L^2(\Omega)]^2\,, \end{aligned} \tag{2.2}$$

where the bilinear forms $a : H \times H \to \mathbb{R}$ and $b : H \times [L^2(\Omega)]^2 \to \mathbb{R}$ are defined by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d \qquad \text{and} \qquad b(\boldsymbol{\tau}, \mathbf{v}) := \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau})\,,$$

and the linear funtionals $G : H \to \mathbb{R}$ and $F : [L^2(\Omega)]^2 \to \mathbb{R}$ are given by

$$G(\boldsymbol{\tau}) := \langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle_\Gamma \qquad \text{and} \qquad F(\mathbf{v}) := -\int_\Omega \boldsymbol{f} \cdot \mathbf{v}\,.$$

Hereafter, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing of $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2-$ inner product. Existence and uniqueness are established in the next result.

Theorem 2.1 *Problem (2.2) has a unique solution* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [L^2(\Omega)]^2$*. Moreover, there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{H \times [L^2(\Omega)]^2} \leq C(\|\boldsymbol{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2}). \tag{2.3}$$

Proof. Clearly, the bilinear forms a and b, as well as the linear functionals F and G, are bounded. In addition, we note that

$$V := \{\boldsymbol{\tau} \in H_0 : \quad b(\boldsymbol{\tau}, \mathbf{v}) = 0\,, \ \ \forall \mathbf{v} \in [L^2(\Omega)]^2\} = \{\boldsymbol{\tau} \in H_0 : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \ \text{ in } \ \Omega\}\,.$$

Then, applying Proposition IV.3.1 in [9], we deduce that there exists $c > 0$ such that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq c\|\boldsymbol{\tau}\|_H^2 \quad \forall \boldsymbol{\tau} \in V\,.$$

On the other hand, the continuous inf-sup condition satisfied by b is proved in Lemma 4.3 in [4]. Then the proof follows from the classical Babuška-Brezzi theory. □

3 The augmented DG formulation

In this section, we partially follow [18] (see also [5] and [6]) to derive a discrete formulation for the linear model (2.1), applying a consistent and conservative discontinuous Galerkin method in divergence form, adding then suitable stabilization terms to obtain a well-posed formulation.

3.1 Meshes, averages and jumps

We let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular triangulations of $\bar{\Omega}$ (with possible hanging nodes) made up of straight-side triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max_{T \in \mathcal{T}_h} h_T$. Then, given $\mathcal{T}_h$, its edges are defined as follows. An *interior edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial \Omega$, where T is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ the list of all interior edges of (counted only once) on Ω, and by $\mathcal{E}_\Gamma$ the lists of all boundary edges, and put $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_\Gamma$ the interior grid generated by the triangulation $\mathcal{T}_h$. Further, for each $e \in \mathcal{E}$, h_e represents its length. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, which means that there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \leq \frac{h_T}{h_{T'}} \leq l$ for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

Next, to define average and jump operators, we let T and T' be two adjacent elements of $\mathcal{T}_h$ and $\boldsymbol{x}$ be an arbitrary point on the interior edge $e = \partial T \cap \partial T' \in \mathcal{E}_I$. In addition, let q , $\boldsymbol{v}$ and $\boldsymbol{\tau}$ be scalar-, vector-, and tensor-valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(\boldsymbol{v}_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(\boldsymbol{v}_T, \boldsymbol{\tau}_T)$ to e. Then, we define the averages at $\boldsymbol{x} \in e$ by:

$$\{\boldsymbol{v}\} := \frac{1}{2}\big(\boldsymbol{v}_{T,e} + \boldsymbol{v}_{T',e}\big)\,, \quad \{\boldsymbol{\tau}\} := \frac{1}{2}\big(\boldsymbol{\tau}_{T,e} + \boldsymbol{\tau}_{T',e}\big)\,.$$

Similarly, the jumps at $\boldsymbol{x} \in e$ are given by

$$[\![\boldsymbol{v}]\!] := \boldsymbol{v}_{T,e} \cdot \boldsymbol{\nu}_T + \boldsymbol{v}_{T',e} \cdot \boldsymbol{\nu}_{T'}\,, \quad \underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}_{T,e} \otimes \boldsymbol{\nu}_T + \boldsymbol{v}_{T',e} \otimes \boldsymbol{\nu}_{T'}\,, \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{T,e}\,\boldsymbol{\nu}_T + \boldsymbol{\tau}_{T',e}\,\boldsymbol{\nu}_{T'}\,.$$

On boundary edges e, we set $\{\boldsymbol{v}\} := \boldsymbol{v}$, $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$, as well as $[\![q]\!] := q\,\boldsymbol{\nu}$, $[\![\boldsymbol{v}]\!] := \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v} \otimes \boldsymbol{\nu}$ and $[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\,\boldsymbol{\nu}$. Hereafter, as usual, $\mathbf{div}_h$ and ∇_h denote the piecewise divergence and gradient operators, respectively.

3.2 The augmented discrete formulation

Given a mesh $\mathcal{T}_h$, we proceed as in [22] (or [11]) and multiply each one of the equations of (2.1) by suitable test functions. Our purpose is to approximate the exact solution $(\boldsymbol{\sigma}, \boldsymbol{u})$ of (2.1) by discrete functions $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ in appropriate finite element space $\boldsymbol{\Sigma}_{h,0} \times V_h$ such that for all $T \in \mathcal{T}_h$ we have

$$\begin{aligned} &\frac{1}{\nu}\int_T \boldsymbol{\sigma}_h^d : \boldsymbol{\tau}^d + \int_T \boldsymbol{u}_h \cdot \mathbf{div}\boldsymbol{\tau} - \int_{\partial T} \widehat{\boldsymbol{u}} \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T = 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}\,, \\ &\int_T \nabla \boldsymbol{v} : \boldsymbol{\sigma}_h - \int_{\partial T} \boldsymbol{v} \cdot \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T = \int_T \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h\,, \end{aligned} \tag{3.1}$$

where the *numerical fluxes* $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on $\boldsymbol{u}_h$, $\boldsymbol{\sigma}_h$, and the boundary data, are set so that some compatibility conditions are satisfied (see [2]). Indeed, taking into account the approach from [22] and [13], we define the numerical fluxes $\widehat{\boldsymbol{u}} := \widehat{\boldsymbol{u}}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \mathbf{g})$ and $\widehat{\boldsymbol{\sigma}} := \widehat{\boldsymbol{\sigma}}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \mathbf{g})$ for each $T \in \mathcal{T}_h$ as follows:

$$\widehat{\boldsymbol{u}}_{T,e} := \begin{cases} \{\boldsymbol{u}_h\} + \underline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{\beta} - \gamma[\![\boldsymbol{\sigma}_h]\!] & \text{if } e \in \mathcal{E}_I, \\ \mathbf{g} & \text{if } e \in \mathcal{E}_\Gamma, \end{cases} \tag{3.2}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \alpha\underline{[\![\boldsymbol{u}_h]\!]} & \text{if } e \in \mathcal{E}_I, \\ \boldsymbol{\sigma}_h - \alpha(\boldsymbol{u}_h - \mathbf{g}) \otimes \boldsymbol{\nu} & \text{if } e \in \mathcal{E}_\Gamma, \end{cases} \tag{3.3}$$

where the auxiliary functions α, γ (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single valued on each edge $e \in \mathcal{E}$ and such that they allow us to prove the optimal rates of convergence of our approximation. To this aim, we set $\alpha := \frac{\widehat{\alpha}}{\mathtt{h}}$, and $\boldsymbol{\beta}$ as an arbitrary vector in $\mathbb{R}^2$. Hereafter, $\widehat{\alpha} > 0$ is arbitrary, while $\mathtt{h}$ is defined by

$$\mathtt{h} := \begin{cases} \max\{h_T, h_{T'}\} & \text{if } e \in \mathcal{E}_I\,, \\ h_T & \text{if } e \in \mathcal{E}_\Gamma\,. \end{cases}$$

Then, integrating by parts in the second equation in (3.1), summing up over all $T \in \mathcal{T}_h$, we arrive to: *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *such that*

$$\begin{aligned} &\frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}_h^d : \boldsymbol{\tau}^d + \int_{\mathcal{E}_I} \gamma[\![\boldsymbol{\sigma}_h]\!] \cdot [\![\boldsymbol{\tau}]\!] + \int_\Omega \boldsymbol{u}_h \cdot \mathbf{div}_h\boldsymbol{\tau} - \int_{\mathcal{E}_I} \big(\{\boldsymbol{u}_h\} + \underline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{\beta}\big) \cdot [\![\boldsymbol{\tau}]\!] = \int_{\mathcal{E}_\Gamma} \mathbf{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu}\,, \\ &-\int_\Omega \boldsymbol{v} \cdot \mathbf{div}_h\boldsymbol{\sigma}_h + \int_{\mathcal{E}_I} \big(\{\boldsymbol{v}\} + \underline{[\![\boldsymbol{v}]\!]}\boldsymbol{\beta}\big) \cdot [\![\boldsymbol{\sigma}]\!] + \boldsymbol{\alpha}(\boldsymbol{u}_h, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\mathcal{E}_\Gamma} \alpha\big(\mathbf{g} \otimes \boldsymbol{\nu}\big) : \big(\boldsymbol{v} \otimes \boldsymbol{\nu}\big) \end{aligned} \tag{3.4}$$

for all $(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$, with $\boldsymbol{\alpha} : [H^1(\mathcal{T}_h)]^2 \times [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ being the bilinear form defined by:

$$\boldsymbol{\alpha}(\boldsymbol{w},\boldsymbol{v}) := \int_{\mathcal{E}_I} \alpha\, \underline{[\![\boldsymbol{v}]\!]} : \underline{[\![\boldsymbol{w}]\!]} + \int_{\mathcal{E}_\Gamma} \alpha(\boldsymbol{v}\otimes\boldsymbol{\nu}) : (\boldsymbol{w}\otimes\boldsymbol{\nu}) \,, \quad \forall\, \boldsymbol{v},\boldsymbol{w} \in [H^1(\mathcal{T}_h)]^2 \,.$$

Now, we proceed as in [6] (see also [5], [7] and [18]) and we include the Galerkin-least squares terms given by

$$\delta_1 \int_\Omega (\nu\nabla_h \boldsymbol{u}_h - \boldsymbol{\sigma}_h^d) : (\nu\nabla_h \boldsymbol{v}_h + \boldsymbol{\tau}^d) = 0 \,, \tag{3.5}$$

for all $(\boldsymbol{\tau},\boldsymbol{v}) \in H(\mathrm{div};\mathcal{T}_h) \times H^1(\mathcal{T}_h)$, and

$$\delta_2 \int_\Omega \mathbf{div}_h \boldsymbol{\sigma}_h \cdot \mathbf{div}_h \boldsymbol{\tau} \;=\; -\delta_2 \int_\Omega \boldsymbol{f} \cdot \mathbf{div}_h \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};\mathcal{T}_h) \,. \tag{3.6}$$

where δ_1 and δ_2 are real parameters to be determined in a suitable way, and will be describe soon.

Hence, adding (3.5), (3.6) and (3.4), we obtain the following discrete augmented discontinuous Galerkin formulation: *Find* $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *such that*

$$A_{DG}^{stab}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\tau},\boldsymbol{v})) = F_{DG}^{stab}(\boldsymbol{\tau},\boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h \,, \tag{3.7}$$

where the bilinear form $A_{DG}^{stab} : ((H(\mathrm{div};\mathcal{T}_h) \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h)) \times ((H(\mathrm{div};\mathcal{T}_h) \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h)) \to \mathbb{R}$ and the linear functional $F_{DG}^{stab} : (H(\mathrm{div};\mathcal{T}_h) \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ are defined by

$$\begin{aligned} A_{DG}^{stab}((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\tau},\boldsymbol{v})) &:= \frac{1}{\nu}\int_\Omega \boldsymbol{\zeta}^d : \boldsymbol{\tau}^d + \int_\Omega \boldsymbol{w}\cdot\mathbf{div}_h\boldsymbol{\tau} - \int_{\mathcal{E}_I} \big(\{\boldsymbol{w}\} + \underline{[\![w]\!]}\boldsymbol{\beta}\big)\cdot[\![\boldsymbol{\tau}]\!] \\ &- \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h\boldsymbol{\zeta} + \int_{\mathcal{E}_I} \big(\{\boldsymbol{v}\} + \underline{[\![\boldsymbol{v}]\!]}\boldsymbol{\beta}\big)\cdot[\![\boldsymbol{\zeta}]\!] + \int_{\mathcal{E}_I} \gamma[\![\boldsymbol{\zeta}]\!]\cdot[\![\boldsymbol{\tau}]\!] + \boldsymbol{\alpha}(\boldsymbol{w},\boldsymbol{v}) \\ &+ \delta_1 \int_\Omega (\nu\nabla_h\boldsymbol{w} - \boldsymbol{\zeta}^d) : (\nu\nabla_h\boldsymbol{v} + \boldsymbol{\tau}^d) + \delta_2 \int_\Omega \mathbf{div}_h\boldsymbol{\zeta}\cdot\mathbf{div}_h\boldsymbol{\tau} \,, \end{aligned}$$

and

$$F_{DG}^{stab}(\boldsymbol{\tau},\boldsymbol{v}) := \int_{\mathcal{E}_\Gamma} \mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu} + \int_{\mathcal{E}_\Gamma} \alpha(\mathbf{g}\otimes\boldsymbol{\nu}) : (\boldsymbol{v}\otimes\boldsymbol{\nu}) - \delta_2 \int_\Omega \boldsymbol{f}\cdot\mathbf{div}_h\boldsymbol{\tau} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \,,$$

for all $(\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\tau},\boldsymbol{v}) \in (H(\mathrm{div};\mathcal{T}_h) \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h)$, with an appropriate $\varepsilon > 1/2$. At this point we define the discrete spaces $\boldsymbol{\Sigma}_h$, $\boldsymbol{\Sigma}_{h,0}$, and $\boldsymbol{V}_h$:

$$\boldsymbol{\Sigma}_h := \Big\{ \boldsymbol{\tau}_h \in [L^2(\mathcal{T}_h)]^{2\times 2} : \quad \boldsymbol{\tau}_h\big|_T \in [\mathbf{RT}_r(T)^{\mathrm{t}}]^2 \quad \forall T \in \mathcal{T}_h \Big\},$$

$$\boldsymbol{\Sigma}_{h,0} := \Big\{ \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) = 0 \Big\},$$

$$\boldsymbol{V}_h := \Big\{ \boldsymbol{v}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{v}_h\big|_T \in [\mathbf{P}_k(T)]^2 \quad \forall T \in \mathcal{T}_h \Big\},$$

with $k \geq 1$ and $r \geq 0$. Hereafter, given an integer $\kappa \geq 0$ we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of degree at most κ on T. In addition, for each $T \in \mathcal{T}_h$, we introduce the local Raviart-Thomas space of order κ (cf. [23]), $\mathbf{RT}_\kappa(T) := [\mathbf{P}_\kappa(T)]^2 \oplus \boldsymbol{x}\mathbf{P}_\kappa(T) \subseteq [\mathbf{P}_{\kappa+1}(T)]^2$.

The space $\boldsymbol{\Sigma}_h$ ($\boldsymbol{\Sigma}_{h,0}$) is provided with the norm of $\boldsymbol{\Sigma} := H(\mathrm{div}; \mathcal{T}_h)$, which is defined by

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}^2 := ||\boldsymbol{\tau}||^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}_h \boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2} \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}\,,$$

while for $\boldsymbol{V}_h$ we introduce its seminorms $|\cdot|_h : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ and its norm $\|\!|\cdot|\!\|_h : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ as

$$|\boldsymbol{v}|_h^2 := ||\alpha^{1/2}\underline{[\![\boldsymbol{v}]\!]}||^2_{[L^2(\mathcal{E}_I)]^{2\times 2}} + ||\alpha^{1/2}\boldsymbol{v}\otimes\boldsymbol{\nu}||^2_{[L^2(\mathcal{E}_\Gamma)]^{2\times 2}} \quad \forall\, \boldsymbol{v} \in [H^1(\mathcal{T}_h)]^2\,,$$

and

$$\|\!|\boldsymbol{v}|\!\|_h^2 := ||\nu \nabla_h \boldsymbol{v}||^2_{[L^2(\Omega)]^{2\times 2}} + |\boldsymbol{v}|_h^2 \quad \forall\, \boldsymbol{v} \in [H^1(\mathcal{T}_h)]^2\,,$$

respectively. In addition, we define the norm $||(\cdot,\cdot)||_{DG} : \boldsymbol{\Sigma} \times [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ by

$$||(\boldsymbol{\tau},\boldsymbol{v})||^2_{DG} \;:=\; ||\boldsymbol{\tau}||^2_{\boldsymbol{\Sigma}} \,+\, \|\!|\boldsymbol{v}|\!\|_h^2 \quad \forall\, (\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{\Sigma} \times [H^1(\mathcal{T}_h)]^2\,.$$

Our next aim is to guarantee that Problem (3.7) is well-posed, by checking that we can apply the Lax-Milgram theorem. In this direction, the next lemma will be useful to prove the coerciveness of A^{stab}_{DG}.

Lemma 3.1 , *There exists a constant $c_1 > 0$, independent of the meshsize, such that*

$$c_1\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} \;\leq\; \|\boldsymbol{\tau}^d\|^2_{[L^2(\Omega)]^{2\times 2}} \,+\, \|\mathbf{div}_h\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2} \,+\, \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2}\,, \tag{3.8}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_0 := \{\boldsymbol{\zeta} \in \boldsymbol{\Sigma} : \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0\}$.

Proof. We adapt the proof of Proposition IV.3.1 in [9]. First, we pick $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_0$, and since $\boldsymbol{\tau} = \boldsymbol{\tau}^d + \dfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}$, it is enough to prove that

$$\|\,\mathrm{tr}(\boldsymbol{\tau})\|^2_{L^2(\Omega)} \;\leq\; C\,\left(\|\boldsymbol{\tau}^d\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}_h(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \,+\, \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2}\right)\,.$$

To this end, using that $\displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0$, we deduce that there exists $\boldsymbol{v} \in [H^1_0(\Omega)]^2$ such that

$$\mathrm{div}(\boldsymbol{v}) = \mathrm{tr}(\boldsymbol{\tau}) \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{v}\|_{[H^1(\Omega)]^2} \;\leq\; c\,\|\,\mathrm{tr}(\boldsymbol{\tau})\|_{L^2(\Omega)}\,,$$

where $c > 0$ is a constant independent of the meshsize. Then, noting that $\nabla\boldsymbol{v} = (\nabla\boldsymbol{v})^d - \frac{1}{2}\,\mathrm{div}(\boldsymbol{v})\,\mathbf{I}$, and

$$\int_\Omega \boldsymbol{\tau}^d : \nabla\boldsymbol{v} \;=\; \int_\Omega \boldsymbol{\tau}^d : (\nabla\boldsymbol{v})^d \;=\; \int_\Omega \boldsymbol{\tau} : (\nabla\boldsymbol{v})^d\,,$$

we deduce

$$\|\,\mathrm{tr}(\boldsymbol{\tau})\|^2_{L^2(\Omega)} \;=\; \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\;\mathrm{div}(\boldsymbol{v}) \;=\; \int_\Omega \boldsymbol{\tau} : (\mathrm{div}(\boldsymbol{v})\,\mathbf{I})$$

$$=\; 2\int_\Omega \boldsymbol{\tau} : \left(\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^d\right) \;=\; 2\left\{\int_\Omega \boldsymbol{\tau} : \nabla\boldsymbol{v} - \int_\Omega \boldsymbol{\tau}^d : \nabla\boldsymbol{v}\right\}\,.$$

Now, since

$$\int_\Omega \boldsymbol{\tau} : \nabla \boldsymbol{v} \,=\, \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\tau} : \nabla \boldsymbol{v} = \sum_{K\in\mathcal{T}_h} \left\{ - \int_K \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\tau} + \int_{\partial K} \boldsymbol{v} \cdot \boldsymbol{\tau}\boldsymbol{\nu} \right\}$$

$$= - \int_\Omega \boldsymbol{v} \, \operatorname{div}_h \boldsymbol{\tau} \,+\, \int_{\mathcal{E}_I} \left(\gamma^{-1/2} \{ \boldsymbol{v} \} \right) \cdot \left(\gamma^{1/2} [\![\boldsymbol{\tau}]\!] \right),$$

the proof follows from Cauchy-Schwarz inequality, and a very well-known local trace inequality. □

At this point, we are in position to establish the ellipticity of A_{DG}^{stab}.

Lemma 3.2 *Let $(\delta_1, \delta_2) \in \mathbb{R}^2$ such that $0 < \delta_1 < \frac{1}{\nu}$ and $\delta_2 > 0$, with $\epsilon_1 := \min\{\frac{1}{\nu} - \delta_1, \frac{\delta_2}{2}, 1\}$ and $c_1 > 0$ being the constant in (3.8). Then, there exists a constant $C > 0$, independent of the meshsize, such that*

$$A_{DG}^{stab}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) \;\; \geq \;\; C \, \|(\boldsymbol{\tau}, \boldsymbol{v})\|_{DG}^2 \,, \tag{3.9}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in (\boldsymbol{\Sigma}_0 \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times [H^1(\mathcal{T}_h)]^2$.

Proof. According to the definition of A_{DG}^{stab}, we obtain, for each $(\boldsymbol{\tau}, \boldsymbol{v}) \in (\boldsymbol{\Sigma}_0 \cap [H^\varepsilon(\Omega)]^{2\times 2}) \times [H^1(\mathcal{T}_h)]^2$

$$A_{DG}^{stab}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) = \frac{1}{\nu} \|\boldsymbol{\tau}^d\|_{[L^2(\Omega)]^{2\times 2}}^2 \,+\, \|\gamma^{1/2} [\![\boldsymbol{\tau}]\!] \|_{[L^2(\mathcal{E}_I)]^2}^2 + \delta_2 \, \|\mathbf{div}_h \boldsymbol{\tau}\|_{[L^2(\Omega)]^2}^2$$

$$+ |\boldsymbol{v}|_h^2 \,+\, \delta_1 \nu^2 \|\nabla_h \boldsymbol{v}\|_{[L^2(\Omega)]^{2\times 2}}^2 \,-\, \delta_1 \|\boldsymbol{\tau}^d\|_{[L^2(\Omega)]^{2\times 2}}^2 \,.$$

Now, Lemma 3.1 implies that

$$A_{DG}^{stab}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) \geq c_1 \epsilon_1 \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}^2 \,+\, \frac{\delta_2}{2} \|\mathbf{div}_h \boldsymbol{\tau}\|_{[L^2(\Omega)]^2}^2$$

$$+ |\boldsymbol{v}|_h^2 \,+\, \delta_1 \nu^2 \|\nabla_h \boldsymbol{v}\|_{[L^2(\Omega)]^{2\times 2}}^2 \,.$$

Then choosing $C := \min\{c_1 \epsilon_1, \frac{\delta_2}{2}, \delta_1, 1\}$, we complete the proof. □

Theorem 3.1 *Assume the same hypotheses of Lemma 3.2. Then, Problem (3.7) is uniquely solvable, and there exists a positive constant $C_{\mathtt{F}}$, independent of the meshsize, such that there holds*

$$||(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)||_{DG} \;\leq\; C_{\mathtt{F}} \, \mathcal{B}(\boldsymbol{f}, \mathbf{g}) \,, \tag{3.10}$$

with $\mathcal{B}(\boldsymbol{f}, \mathbf{g}) := \left(\|\boldsymbol{f}\|_{[L^2(\Omega)]^2}^2 + \|\alpha^{1/2} \mathbf{g}\|_{[L^2(\mathcal{E}_\Gamma)]^2}^2 \right)^{1/2}$. In addition, assuming that the exact solution $(\boldsymbol{\sigma} := \nu \nabla \boldsymbol{u} - p\mathbf{I}, \boldsymbol{u})$ of (2.1) is such that $\boldsymbol{u} \in [H^{1+t}(\Omega)]^2$ and $p \in H^t(\Omega)$, with $t > 1/2$, then there is $C_{\mathtt{A}} > 0$, independent of the meshsize, such that

$$\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)\|_{DG} \leq \, C_{\mathtt{A}} \inf_{(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \|(\boldsymbol{\sigma} - \boldsymbol{\tau}, \boldsymbol{u} - \boldsymbol{v})\|_{DG} \,, \tag{3.11}$$

Proof. It is not difficult to check that the bilinear form A_{DG}^{stab} and the linear functional F_{DG}^{stab} are bounded. Then the unique solvability of (3.7) and the continuous dependence (3.10) follow straightforwardly from Lemma 3.2 and the Lax-Milgram theorem. Finally, the derivation of the estimate (3.11) follows from Second Strang Lemma, and take into account the fact that the formulation (3.7) is consistent with the exact solution $(\boldsymbol{\sigma}, \boldsymbol{u})$, that is

$$A_{DG}^{stab}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = F_{DG}^{stab}(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h\,.$$

□

In order to obtain the a priori error estimates for the scheme (3.7) we need the following lemmas, which establish local approximation properties of piecewise polynomials approximations of $H^1(T)$ and $H(\mathrm{div}; T)$.

Lemma 3.3 *Let $\mathcal{T}_h$ be an element of a shaped-regular triangulation family $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a nonnegative integer m, let $\Pi_T^m : L^2(T) \to \mathbf{P}_m(T)$ be the linear and bounded operator given by the $L^2(T)-$orthogonal projection, which satisfies $\Pi_T^m(p) = p$ for all $p \in \mathbf{P}_m(T)$. Then there exists $C > 0$, independent of the meshsize, such that for each s, t satisfying $0 \le s \le m+1$ and $0 \le s < t$, there holds*

$$|(I - \Pi_T^m)(w)|_{H^s(T)} \le C\, h_T^{\min\{t,m+1\}-s} ||w||_{H^t(T)} \quad \forall\, w \in H^t(T)\,, \tag{3.12}$$

and for each $t > 1/2$ there holds

$$|(I - \Pi_T^m)(w)|_{L^2(\partial T)} \le C\, h_T^{\min\{t,m+1\}-1/2} ||w||_{H^t(T)} \quad \forall\, w \in H^t(T)\,, \tag{3.13}$$

Proof. We refer to [15], [16]. □

Lemma 3.4 *Let $\mathcal{T}_h$ be an element of a shaped-regular triangulation family $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a positive integer k, let $\mathcal{E}_T^k : [H^1(T)]^2 \to \mathbf{RT}_{k-1}(T)$ be the local interpolation operator, which verifies $\mathrm{div}(\mathcal{E}_T^k(\boldsymbol{\tau})) = \Pi_T^{k-1}(\mathrm{div}(\boldsymbol{\tau}))$ for all $\boldsymbol{\tau} \in [H^1(T)]^2$. Then, for any $\boldsymbol{\tau} \in [H^l(T)]^2$ with $\mathrm{div}(\boldsymbol{\tau}) \in H^s(T)$, where l and s are positive and nonnegative integers, respectively, there exists $C > 0$, independent of the meshsize, such that*

$$||\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau})||_{[L^2(T)]^2} \le C\, h_T^l\, |\boldsymbol{\tau}|_{[H^l(T)]^2} \quad 1 \le l \le k\,, \tag{3.14}$$

and

$$||\,\mathrm{div}(\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau}))||_{L^2(T)} \le C\, h_T^s\, |\boldsymbol{\tau}|_{[H^s(T)]^2} \quad 0 \le s \le k\,. \tag{3.15}$$

Proof. We refer to [1]. □

The a priori error estimate is described next.

Theorem 3.2 *Let $(\boldsymbol{\sigma}, \boldsymbol{u})$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ be the unique solutions of (2.1) and (3.7), respectively. Then, assuming that $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}|_T) \in [H^t(T)]^2$ and $\boldsymbol{u}|_T \in [H^{1+t}(T)]^2$ with $t > 1/2$, for all $T \in \mathcal{T}_h$, there exists $C_{\mathtt{err}} > 0$, independent of the meshsize, such that*

$$\begin{aligned} &\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)\|_{DG}^2 \\ &\le C_{\mathtt{err}} \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k,r+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{[H^t(T)]^{2\times 2}}^2 + \|\mathbf{div}\boldsymbol{\sigma}\|_{[H^t(T)]^2}^2 + \|\boldsymbol{u}\|_{[H^{t+1}(T)]^2}^2 \Big\}\,. \end{aligned} \tag{3.16}$$

Proof. It is consequence of Theorem 3.1, and Lemmas 3.3 and 3.4. We omit further details. □

4 Numerical examples

This section is dedicated to show and discuss some numerical results, which illustrate the performance of the proposed augmented finite element scheme (3.7). For implementation purposes, we notice that it is hard to find a basis of $\mathbf{\Sigma}_{h,0}$, due to the null media condition of the trace of any element belonging to this space. In order to overcome this computational difficulty, we impose this condition in the discrete formulation by using a Lagrange multiplier. Therefore, we consider the following mixed discrete scheme: *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *such that*

$$\begin{aligned} A_{DG}^{stab}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v})) + \varphi_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau})\, dx &= F_{DG}^{stab}(\boldsymbol{\tau}, \boldsymbol{v})\,, \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h\,, \\ \psi \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h)\, dx &= 0\,, \forall\, \psi \in \mathbb{R}\,. \end{aligned} \tag{4.1}$$

The next theorem establishes the equivalence between the variational problems (3.7) and (4.1).

Theorem 4.1

i) Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *be a solution of* (3.7). *Then* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0)$ *is a solution of* (4.1).

ii) Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *be a solution of* (4.1). *Then* $\varphi_h = 0$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ *is a solution of* (3.7).

Proof. We first observe, according to the definition of A_{DG}^{stab}, that for each $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h$ there holds

$$A_{DG}^{stab}((\boldsymbol{\tau}, \mathbf{v}), (\mathbf{I}, \mathbf{0})) = 0 \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h\,. \tag{4.2}$$

Now, let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$ be a solution of (3.7), and let $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h$. We write $\boldsymbol{\tau} = \boldsymbol{\tau}_{0,h} + d_h\mathbf{I}$, with $\boldsymbol{\tau}_{0,h} \in \mathbf{\Sigma}_{h,0}$ and $d_h \in \mathbb{R}$, and observe that $(\boldsymbol{\tau}_{0,h}, \mathbf{v}) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$, whence the definition of F_{DG}^{stab}, (3.7) and (4.2) yield

$$F_{DG}^{stab}(\boldsymbol{\tau}, \mathbf{v}) = F_{DG}^{stab}(\boldsymbol{\tau}_{0,h}, \mathbf{v}) = A_{DG}^{stab}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_{0,h}, \mathbf{v})) = A_{DG}^{stab}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}, \mathbf{v}))$$

This identity and the fact that $\boldsymbol{\sigma}_h$ clearly satisfies the second equation of (4.1), show that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0)$ is indeed a solution of (4.1).

Conversely, let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ be a solution of (4.1). Then taking $(\boldsymbol{\tau}, \mathbf{v}) = (\mathbf{I}, \mathbf{0})$ in the first equation of (4.1) and using the definition of F_{DG}^{stab} and (4.2), we find that $\varphi_h = 0$, whence $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ becomes a solution of (3.7). □

It is important to remark here that we are also able to give a reasonable approximation of the pressure p as well as of the deviator $\boldsymbol{\sigma}^d$, which is related to the flux $\nabla \boldsymbol{u}$. These would be given by $p_h := -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ and $\boldsymbol{\sigma}_h^d := \boldsymbol{\sigma}_h - \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)\mathbf{I}$. This way, it is easy to check that

$$||p - p_h||_{L^2(\Omega)} = \frac{1}{2}||\operatorname{tr}(\boldsymbol{\sigma}) - \operatorname{tr}(\boldsymbol{\sigma}_h)|| \lesssim ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}}\,,$$

and

$$||\boldsymbol{\sigma}^d - \boldsymbol{\sigma}_h^d||_{[L^2(\Omega)]^{2\times 2}} \leq ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}} + \frac{\sqrt{2}}{2}||\,\mathrm{tr}(\boldsymbol{\sigma}) - \mathrm{tr}(\boldsymbol{\sigma}_h)|| \lesssim ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}}\,,$$

and therefore, we would expect (at least) that the rates of convergence of the corresponding error norms of p and $\boldsymbol{\sigma}^d$ behave like the $H(\mathbf{div};\mathcal{T}_h)-$ error of $\boldsymbol{\sigma}$.

Next, we provide several numerical examples illustrating the performance of the equivalent augmented DG method (4.1), instead of (3.7). Hereafter, N is the number of degrees of freedom defining the subspaces $\boldsymbol{\Sigma}_h$, $\boldsymbol{V}_h$ and $\mathbb{R}$, that is $N := 12\times$(number of triangles of $\mathcal{T}_h) + 1$ for the $\mathbf{RT}_0 - \mathbf{P}_1$ approximation and $N := 18\times$(number of triangles of $\mathcal{T}_h) + 1$ for the $\mathbf{RT}_0 - \mathbf{P}_2$ one. Moreover, the individual and total errors are defined as follows

$$\boldsymbol{e}_h(\boldsymbol{u}) := |\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_h\,, \quad \boldsymbol{e}_{\boldsymbol{\sigma}} := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{\Sigma}}\,, \quad \boldsymbol{e} := \Big([\boldsymbol{e}_h(\boldsymbol{u})]^2 + [\boldsymbol{e}_{\boldsymbol{\sigma}}]^2 \Big)^{1/2}\,,$$

$$\boldsymbol{e}_0(p) := \left\| p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h) \right\|_{L^2(\Omega)}\,, \quad \boldsymbol{e}_0(\boldsymbol{\sigma}^d) := \|\boldsymbol{\sigma}^d - \boldsymbol{\sigma}_h^d\|_{[L^2(\Omega)]^{2\times 2}}$$
$$\text{and} \quad \boldsymbol{e}_0(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{[L^2(\Omega)]^2}\,,$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [H^1(\Omega)]^2$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ are the unique solutions of the continuous and discrete formulations, respectively. In addition, if $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ stand for the errors at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, then the experimental rate of convergence is given by $\boldsymbol{r} := -2\dfrac{\log(\boldsymbol{e}/\tilde{\boldsymbol{e}})}{\log(N/\tilde{N})}$. The definitions of $\boldsymbol{r}_h(\boldsymbol{u})$, $\boldsymbol{r}(\boldsymbol{\sigma})$, $\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$, $\boldsymbol{r}_0(\boldsymbol{u})$ and $r_0(p)$ are given in analogous way.

We present three examples, and by simplicity we assume the viscosity $\nu = 1$ in all of them. In Example 1 we consider the unit square $\Omega := (0,1)^2$, in Example 2 we take the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]\times[-1,0]$, while in Example 3 we use the circular section $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 :\, x_1^2 + x_2^2 < 1\} \setminus [0,1]\times[-1,0]$. The data $\boldsymbol{f}$ and $\mathbf{g}$ are chosen so that the exact solutions $\boldsymbol{u}$ and p are shown in Table 4.1, where $s = \sqrt{(x_1-2)^2 + (x_2-2)^2}$ (examples 1 and 3) and p_0 represents the mean value of the function $\tilde{p}(x_1,x_2) := \dfrac{x_1 - 2}{4\pi s^2}$ in Ω (example 1). We remind that in all cases, $\mathbf{div}(\boldsymbol{u}) = 0$ in Ω and $\boldsymbol{\sigma} = \nu\nabla\boldsymbol{u} - p\mathbf{I}$ in Ω. In addition, we emphasize that the solution of Example 1 is a smooth function, while the solutions of Examples 2 and 3 have singularities. Indeed, the singularity of p (and thus of $\boldsymbol{\sigma}$) in Example 2 is localized in an exterior neighborhood of the segment $\{1\}\times[-1,0]$, while in Example 3 , the gradient of the solution p (and therefore $\mathbf{div}(\boldsymbol{\sigma})$) which is given in polar coordinates, has a singularity at $(0,0)$. The numerical results presented below were obtained using a MATLAB code, and consider the parameters $\gamma = \alpha$, $\widehat{\alpha} = 1$, $\boldsymbol{\beta} = (1,1)^t$, $\delta_1 = \frac{1}{2}$ and $\delta_2 = 1$ to define the bilinear form A_{DG}^{stab} as well as the linear functional F_{DG}^{stab}.

In Tables 4.2-4.4 we give the individual and global errors and the corresponding experimental rates of convergence for the uniform refinements as applied to Examples 1-3, considering the approximation spaces $\mathbf{RT}_0 - \mathbf{P}_1$. Hereafter, uniform refinement means that, given a uniform initial triangulation, each subsequent mesh is obtained from the previous one by dividing each triangle into the four ones arising when connecting the midpoints of its sides. We remark that the errors are computed on each triangle using a 7 point-Gaussian quadrature rule.

We notice that the orders of convergence predicted by the theory are achieved for all examples. Indeed, for Example 1, since we have a smooth solution in a convex region, the global orders behaves as $\mathcal{O}(h)$ with $\mathbf{RT}_0 - \mathbf{P}_1$ and $\mathbf{RT}_0 - \mathbf{P}_2$ approximations. In Example 2 the singularity is out of the domain Ω, then we observe the global orders behaviour is asymptotically as $\mathcal{O}(h)$, while for Example 3, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^{2/3}(\Omega)]^2$, which in accordance with Theorem 3.2, allow us to expect $\mathcal{O}(h^{2/3})$ for its velocity of convergence, and thus for the total error. This fact motivates us to develop a reliable and efficient aposteriori error estimate, in order to improve the quality of the approximation, by adapting such parts of the domain where the error is dominant. This will be the scope of a future work.

On the other hand, we also note a quadratic convergence for the error $\boldsymbol{e}_0(\boldsymbol{u})$, whose theoretical proof should be deduced from the usual duality arguments. It is worth pointing out that our augmented DG scheme works for any combination of the finite element subspaces $\boldsymbol{\Sigma}_h \times V_h$, not necessarily satisfying the well known mild condition $\nabla_h V_h \leq \boldsymbol{\Sigma}_h$ needed in the standard DG context (see, for e.g., [8, 11, 22]). This is the case, for example, when considering the $\mathbf{RT}_0 - \mathbf{P}_2$ approximation in Examples 1, 2 and 3, which results are shown on Tables 4.5-4.7, and are in agreement with the developed analysis.

Table 4.1. Summary of data for the three examples.

Ex.	Solution $\boldsymbol{u}$	Solution p
1	$\frac{1}{8\pi}\left\{-\ln(s)\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{s^2}\begin{pmatrix}(x_1-2)^2\\(x_1-2)(x_2-1)\end{pmatrix}\right\}$	$\frac{x_1-2}{4\pi s^2}-p_0$
2	$\frac{1}{\sqrt{(x_1-0.1)^2+(x_2-0.1)^2}}\begin{pmatrix}x_2-0.1\\0.1-x_1\end{pmatrix}$	$\frac{1}{1.1-x_1}-\frac{1}{3}\ln\left(\frac{441}{11}\right)$
3	$\frac{1}{8\pi}\left\{-\ln(s)\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{s^2}\begin{pmatrix}(x_1-2)^2\\(x_1-2)(x_2-1)\end{pmatrix}\right\}$	$r^{2/3}\sin\left(\frac{2}{3}\theta\right)-\frac{3}{2\pi}$

Table 4.2. Example 1 with $\mathbf{RT}_0 - \mathbf{P}_1$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
49	6.037e-03	——	9.748e-03	——	5.257e-03	——
193	3.086e-03	0.9792	5.071e-03	0.9534	2.498e-03	1.0858
769	1.534e-03	1.0110	2.545e-03	0.9976	1.155e-03	1.1158
3073	7.618e-04	1.0107	1.254e-03	1.0217	5.297e-04	1.1255
12289	3.791e-04	1.0068	6.203e-04	1.0156	2.512e-04	1.0764
49153	1.891e-04	1.0037	3.086e-04	1.0072	1.227e-04	1.0336
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
49	6.304e-03	——	1.147e-02	——	1.016e-03	——
193	3.638e-03	0.8019	5.936e-03	0.9605	3.058e-04	1.7517
769	1.951e-03	0.9015	2.971e-03	1.0012	8.411e-05	1.8675
3073	1.006e-03	0.9569	1.467e-03	1.0187	2.207e-05	1.9316
12289	5.085e-04	0.9840	7.270e-04	1.0133	5.618e-06	1.9743
49153	2.552e-04	0.9946	3.619e-04	1.0062	1.412e-06	1.9921

Table 4.3. Example 2 with $\mathbf{RT}_0 - \mathbf{P}_1$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
73	3.708e+00	——	1.396e+01	——	1.510e+00	——
289	2.235e+00	0.7360	1.372e+01	0.0258	9.754e-01	0.6348
1153	1.344e+00	0.7347	1.134e+01	0.2756	6.999e-01	0.4797
4609	7.392e-01	0.8634	7.672e+00	0.5636	4.552e-01	0.6212
18433	3.888e-01	0.9268	4.499e+00	0.7701	2.500e-01	0.8643
73729	1.974e-01	0.9782	2.394e+00	0.9100	1.287e-01	0.9584
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
73	2.854e+00	——	1.445e+01	——	7.126e-01	——
289	2.165e+00	0.4015	1.390e+01	0.0563	2.199e-01	1.7093
1153	1.519e+00	0.5126	1.142e+01	0.2845	6.750e-02	1.7069
4609	8.810e-01	0.7858	7.707e+00	0.5670	1.910e-02	1.8224
18433	4.826e-01	0.8683	4.516e+00	0.7714	5.415e-03	1.8185
73729	2.503e-01	0.9474	2.402e+00	0.9105	1.450e-03	1.9004

Table 4.4. Example 3 with $\mathbf{RT}_0 - \mathbf{P}_1$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
73	1.629e-01	——	3.971e-01	——	1.608e-01	——
289	7.614e-02	1.1054	2.452e-01	0.7007	8.547e-02	0.9190
1153	3.666e-02	1.0564	1.443e-01	0.7664	4.236e-02	1.0145
4609	1.788e-02	1.0366	8.542e-02	0.7567	2.085e-02	1.0231
18433	8.817e-03	1.0200	5.139e-02	0.7333	1.034e-02	1.0119
73729	4.379e-03	1.0098	3.136e-02	0.7126	5.154e-03	1.0045
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
73	2.179e-01	——	4.292e-01	——	4.049e-02	——
289	1.319e-01	0.7301	2.568e-01	0.7469	1.104e-02	1.8884
1153	7.070e-02	0.9010	1.489e-01	0.7877	2.877e-03	1.9444
4609	3.627e-02	0.9634	8.727e-02	0.7709	7.345e-04	1.9705
18433	1.830e-02	0.9871	5.214e-02	0.7433	1.854e-04	1.9864
73729	9.177e-03	0.9957	3.166e-02	0.7196	4.649e-05	1.9956

Table 4.5. Example 1 with $\mathbf{RT}_0 - \mathbf{P}_2$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
73	4.954e-03	——	9.801e-03	——	5.307e-03	——
289	2.541e-03	0.9706	5.136e-03	0.9392	2.568e-03	1.0548
1153	1.269e-03	1.0037	2.569e-03	1.0014	1.184e-03	1.1196
4609	6.310e-04	1.0081	1.260e-03	1.0278	5.379e-04	1.1386
18433	3.141e-04	1.0065	6.216e-04	1.0199	2.530e-04	1.0884
73729	1.566e-04	1.0041	3.089e-04	1.0091	1.230e-04	1.0399
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
73	6.303e-03	——	1.098e-02	——	9.468e-04	——
289	3.631e-03	0.8016	5.730e-03	0.9455	3.045e-04	1.6490
1153	1.949e-03	0.8998	2.865e-03	1.0018	8.662e-05	1.8170
4609	1.005e-03	0.9557	1.410e-03	1.0239	2.309e-05	1.9081
18433	5.083e-04	0.9834	6.965e-04	1.0172	5.918e-06	1.9645
73729	2.552e-04	0.9944	3.463e-04	1.0081	1.492e-06	1.9883

Table 4.6. Example 2 with $\mathbf{RT}_0 - \mathbf{P}_2$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
109	3.695e+00	——	1.396e+01	——	1.500e+00	——
433	2.062e+00	0.8458	1.372e+01	0.0252	1.010e+00	0.5728
1729	1.148e+00	0.8455	1.134e+01	0.2760	7.003e-01	0.5296
6913	6.153e-01	0.9006	7.672e+00	0.5635	4.548e-01	0.6229
27649	3.215e-01	0.9367	4.499e+00	0.7701	2.507e-01	0.8593
110593	1.625e-01	0.9840	2.394e+00	0.9100	1.292e-01	0.9570
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
109	2.868e+00	——	1.445e+01	——	7.789e-01	——
433	2.169e+00	0.4053	1.388e+01	0.0581	2.290e-01	1.7745
1729	1.520e+00	0.5133	1.139e+01	0.2848	6.698e-02	1.7761
6913	8.814e-01	0.7866	7.697e+00	0.5662	1.899e-02	1.8188
27649	4.825e-01	0.8695	4.510e+00	0.7710	5.306e-03	1.8398
110593	2.502e-01	0.9475	2.400e+00	0.9104	1.409e-03	1.9135

Table 4.7. Example 3 with $\mathbf{RT}_0 - \mathbf{P}_2$ approximation: uniform refinement.

N	$\boldsymbol{e}_h(\boldsymbol{u})$	$\boldsymbol{r}_h(\boldsymbol{u})$	$\boldsymbol{e}_{\boldsymbol{\sigma}}$	$\boldsymbol{r}_{\boldsymbol{\sigma}}$	$\boldsymbol{e}_0(p)$	$r_0(p)$
109	2.031e-01	——	3.969e-01	——	1.594e-01	——
433	1.014e-01	1.0069	2.450e-01	0.6998	8.484e-02	0.9142
1729	5.030e-02	1.0127	1.443e-01	0.7647	4.226e-02	1.0066
6913	2.497e-02	1.0106	8.542e-02	0.7563	2.084e-02	1.0203
27649	1.243e-02	1.0067	5.139e-02	0.7332	1.034e-02	1.0113
110593	6.199e-03	1.0036	3.136e-02	0.7126	5.154e-03	1.0044
N	$\boldsymbol{e}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{r}_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{e}$	$\boldsymbol{r}$	$\boldsymbol{e}_0(\boldsymbol{u})$	$\boldsymbol{r}_0(\boldsymbol{u})$
109	2.199e-01	——	4.458e-01	——	5.408e-02	——
433	1.322e-01	0.7371	2.651e-01	0.7537	1.521e-02	1.8393
1729	7.076e-02	0.9034	1.528e-01	0.7961	4.007e-03	1.9269
6913	3.628e-02	0.9641	8.900e-02	0.7799	1.027e-03	1.9651
27649	1.830e-02	0.9873	5.287e-02	0.7514	2.592e-04	1.9859
110593	9.177e-03	0.9957	3.196e-02	0.7260	6.499e-05	1.9959

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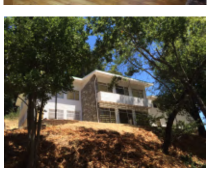

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