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AN A POSTERIORI ERROR ESTIMATOR FOR AN UNSTEADY ADVECTION–DIFFUSION PROBLEM

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Abstract. In this work we introduce an a posteriori error estimator, of the residual type, for the unsteady advection–diffusion–reaction problem. For the discretization in time we use an implicit Euler scheme and a continuous, piecewise linear triangular finite elements for the space together with a stabilized scheme. We prove that the approximation error is bounded, by above and below, by the error estimator. Using that, an adaptive algorithm is proposed, analyzed and tested numerically to prove the efficiency of our approach.

1. Introduction

In this work we deal with the unsteady advection–diffusion–reaction equation. This kind of problems arises in many applications, for instance the transport of pollutant in a river. We are interested in the convection–dominate regime where a characteristic feature of the solutions is the presence of sharp layers (see [17]), for that reason we extended a stabilized method presented in [11], for the stationary case, to the parabolic framework. The idea is to introduce an a posteriori error estimator, extending the techniques developed in [21] and [3] (for elliptic problems) and in [15] (for the linear parabolic problem). The main result of this paper consist of exhibiting a local error indicator which can be computed explicitly as a function of the discrete solution and the data.

There are several works devoted to the development of a posteriori error estimates applied to linear and non linear parabolic problems. In particular, we can cite the works of Bieterman and Babuška [5, 6] for problems in dimension one; Eriksson and Johnson [9], Picasso [15] and Verfürth [22, 23], for problems in higher dimensions. Also we can mention [13] for error estimators based in recovered gradients; [1] and [16] where estimators are developed in junction with Crank–Nicolson time scheme both for linear and nonlinear parabolic problems, or [14] and [18] where the discontinuous Galerkin method is used to approximate the solution of the parabolic equation.

This work is organized as follows. In Section 2 we introduce the unsteady advection–diffusion–reaction problem and some standard results concerning the solvability of the variational formulation associate with it. Also we introduce a fully discrete formulation using a stabilized scheme introduced in [11] for the stationary case, which will be used in the development of our a posteriori error estimator and in the numerical experiments. In Section 3 we present some standard auxiliary results that will be used in the sequel. In Section 4 we present our a posteriori error estimate and the main result of this work: the equivalence between the error estimate and the true error. Finally, in Section 5 we present several numerical experiments showing the quality of the adaptive scheme based in our a posteriori error estimate introduced in Section 4.

2. Model problem

Consider the scalar advection–diffusion–reaction equation given by

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$$
\text{(P)}\quad\left\{\begin{array}{rcll}
\partial_t u - \varepsilon\Delta u + \boldsymbol{a}(x)\cdot\nabla u + b(x)u &=& f(t,x) & \text{in}\quad]0,T[\times\Omega,\\
u &=& 0 & \text{on}\quad]0,T[\times\Gamma_D,\\
\varepsilon\dfrac{\partial u}{\partial \boldsymbol{n}} &=& g(t,x) & \text{on}\quad]0,T[\times\Gamma_N,\\
u(0,\cdot) &=& u_0 & \text{in}\quad \Omega,
\end{array}\right.
$$

where $\Omega\subset\mathbb{R}^2$ is a bounded polygonal domain with boundary $\partial\Omega=\Gamma_N\cup\Gamma_D$, $\Gamma_N\cap\Gamma_D=\emptyset$, $|\Gamma_D|>0$, and $T>0$. We denote by $\boldsymbol{n}$ the outer unit normal vector to $\partial\Omega$.

We assume that:

$(\mathcal{A}_1)$ $\quad\varepsilon\in\mathbb{R},\ 0<\varepsilon\ll 1,$

$(\mathcal{A}_2)$ $\quad f\in C^0\left(]0,T[;L^2(\Omega)\right)$, $g\in C^0(]0,T[;L^2(\Gamma_N))$, $\boldsymbol{a}\in W^{1,\infty}(\Omega)^2,\ b\in L^\infty(\Omega)$,

$(\mathcal{A}_3)$ $\quad -\dfrac{1}{2}\nabla\cdot\boldsymbol{a}\geq 0,\quad b\geq 1,$

$(\mathcal{A}_4)$ $\quad \Gamma_-:=\{x\in\partial\Omega:\boldsymbol{a}(x)\cdot\boldsymbol{n}(x)<0\}\subset\Gamma_D,$

where $L^2(]0,T[;V)$ and $C^0(]0,T[;V)$, are the spaces of square–integrable, respectively of continuous, functions with values in a Hilbert space V. We use standard notation for Sobolev and Lebesgue spaces and norms, for instance $(\cdot,\cdot)_S$ denotes the usual inner product in $L^2(S)$ with $S\subseteq\Omega$ and $(\cdot,\cdot)$ if $S=\Omega$. Also, we introduce the following Hilbert space:

$$H^1_D(\Omega):=\{v\in H^1(\Omega):\ v=0\quad\text{on}\quad\Gamma_D\}.$$

The usual variational formulation of (P) is: *Find $u\in\mathcal{W}$ such that, for all $t\in]0,T[$*

$$
\text{(VP)}\quad\left\{\begin{array}{rcl}
\langle\partial_t u,v\rangle+\varepsilon(\nabla u,\nabla v)+(\boldsymbol{a}\cdot\nabla u,v)+(b\,u\,,v) &=& (f,v)+(g,v)_{\Gamma_N},\\
u(0,\cdot) &=& u_0,
\end{array}\right.
$$

for all $v\in H^1_D(\Omega)$, here $\langle\cdot,\cdot\rangle$ is the duality product between $H^1_D(\Omega)$ and $H^1_D(\Omega)'$, $u(0,\cdot)=u_0\in L^2(\Omega)$ and

$$\mathcal{W}:=\{v\in L^2(]0,T[;H^1_D(\Omega))\ :\ \partial_t v\in L^2(]0,T[;H^1_D(\Omega)')\}.$$

Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ and integration by parts imply that, for all v, $w\in H^1_D(\Omega)$,

$$\varepsilon(\nabla v,\nabla v)+(\boldsymbol{a}\cdot\nabla v,v)+(b\,v\,,v)\geq\varepsilon\|\nabla v\|^2_{0,\Omega}+\|v\|^2_{0,\Omega}\tag{2.1}$$

and

$$\varepsilon(\nabla v,\nabla w)+(\boldsymbol{a}\cdot\nabla v,w)+(b\,v\,,w)\leq(\varepsilon+\|b\|_{\infty,\Omega}+\|\boldsymbol{a}\|_{\infty,\Omega})\|v\|_{1,\Omega}\|w\|_{1,\Omega}.\tag{2.2}$$

Remark 2.1. *Using* (2.1), (2.2) *and Lions Theorem (see Theorem* 6.6 *in* [10]*), we get that there exists a unique solution of problem* (VP). □

2.1. **Space–time discretization.** The implicit Euler method is used for time discretization, with a partition of $[0,T]$, not necessarily uniform, given by

$$0=t^0<t^1<\ldots<t^N=T,$$

with $N\geq 1$. We denote by τ_n the size of the interval $[t^{n-1},t^n]$, i.e.,

$$\tau_n:=t^n-t^{n-1},\quad 1\leq n\leq N.$$

For every time t^n, $0\leq n\leq N$, we associate a family of partitions $\{\mathcal{T}^n_h\}_h$ $(h>0)$ of Ω into triangles, which satisfy the following two properties:

(1) Any two triangles in $\mathcal{T}^n_h$ are either disjoint, share a complete edge or have a common vertex,

(2) $\sup\limits_{h>0}\ \sup\limits_{K\in\mathcal{T}^n_h}\dfrac{h_K}{\rho_K}\leq\kappa$, with κ a constant independent of n and the mesh–size h,

here h_K is the diameter of K and ρ_K is the diameter of the largest ball inscribed into K.

We consider the usual Lagrange finite element space

$$H_h^n := \{\varphi \in H_D^1(\Omega) : \varphi\,|_K \in \mathbb{P}_k(K),\, \forall\, K \in \mathcal{T}_h^n\}, \qquad k \geq 1, \qquad 0 \leq n \leq N,$$

where $\mathbb{P}_k$ denotes the set of polynomials of degree at most k. Note that $H_h^n \subset H_D^1(\Omega)$.

Using the above definitions, and applying an implicit Euler scheme in time, we obtain the following fully discrete scheme of problem (P): *Find* $u_h^n \in H_h^n$*, an approximation of* $u(t^n,\cdot)$*, such that, for* $n, 1 \leq n \leq N$,

$$\text{(D)} \left\{ \begin{array}{rcl} (\dfrac{u_h^n - u_h^{n-1}}{\tau_n}, v_h) + B(u_h^n, v_h) &=& (f^n, v_h) + (g^n, v_h)_{\Gamma_N} \quad \forall\, v_h \in H_h^n, \\ \\ u_h^0 &=& u_{0h}, \end{array} \right.$$

where $f^n := f(t^n,\cdot)$, $g^n := g(t^n,\cdot)$, $u_{0h} \in H_h^0$ is an approximation of $u^0 := u(0,\cdot)$, and for all $w_h, v_h \in H_h^n$

$$B(w_h, v_h) := \int_\Omega \varepsilon \nabla w_h \cdot \nabla v_h + \int_\Omega (\boldsymbol{a} \cdot \nabla w_h) v_h + \int_\Omega b\, w_h\, v_h. \tag{2.3}$$

We can rewrite (D) as follows: *Find* $u_h^n \in H_h^n$ *such that, for* $n,\ 1 \leq n \leq N$,

$$B_n(u_h^n, v_h) = F_n(v_h) \quad \forall v_h \in H_h^n(\Omega), \tag{2.4}$$

where, for $v_h, w_h \in H_h^n$, we define

$$B_n(w_h, v_h) := \int_\Omega \varepsilon \nabla w_h \cdot \nabla v_h + \int_\Omega (\boldsymbol{a} \cdot \nabla w_h) v_h + \int_\Omega b_n\, w_h\, v_h, \tag{2.5}$$

with $b_n := \left(b + \dfrac{1}{\tau_n}\right)$, and

$$F_n(v_h) := \int_\Omega \left(f^n + \frac{u_h^{n-1}}{\tau_n}\right) v_h + \int_{\Gamma_N} g^n\, v_h\, ds.$$

Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ imply that, for all $v_h, w_h \in H_h^n$, there hold

$$B_n(v_h, v_h) \geq \varepsilon \|\nabla v_h\|_{0,\Omega}^2 + \|v_h\|_{0,\Omega}^2, \tag{2.6}$$

$$B_n(v_h, w_h) \leq (\varepsilon + \|b_n\|_{\infty,\Omega} + \|\boldsymbol{a}\|_{\infty,\Omega}) \|v_h\|_{1,\Omega} \|w_h\|_{1,\Omega}. \tag{2.7}$$

Remark 2.2. *Using* (2.6), (2.7) *and Lax–Milgram's Lemma we have that the problem* (D) *has a unique solution.* □

It is well known that the standard finite element method yields to poor approximation when $\varepsilon \ll |\boldsymbol{a}|$ or $\varepsilon \ll b$ (see, for instance [17]). For this reason we consider the following stabilized formulation introduced in [11]: *Find* $u_h^n \in H_h^n$ *such that, for* $n,\ 1 \leq n \leq N$,

$$\text{(SP)} \left\{ \begin{array}{rcl} B_S(u_h^n, v_h) &=& F_S(v_h) \quad \forall v_h \in H_h^n, \\ \\ u_h^0 &=& u_{0h}, \end{array} \right.$$

where for all v_h, $w_h \in H_h^n$

$$B_S(w_h, v_h) := B_n(w_h, v_h) + \sum_{K \in \mathcal{T}_h^n} \int_K \delta_K (\varepsilon \Delta w_h - \boldsymbol{a} \cdot \nabla w_h - b_n w_h)(-\varepsilon \Delta v_h - \boldsymbol{a} \cdot \nabla v_h + b_n v_h), \tag{2.8}$$

and

$$F_S(v_h) := F_n(v_h) - \sum_{K \in \mathcal{T}_h^n} \int_K \delta_K (f^n + \frac{u_h^{n-1}}{\tau_n})(-\varepsilon \Delta v_h - \boldsymbol{a} \cdot \nabla v_h + b_n v_h). \tag{2.9}$$

The stabilization parameter δ_K is given by

$$\delta_K := \frac{h_K^2}{b_n h_K^2 \max\{1, Pe_K^R(x)\} + (2\varepsilon/m_k) \max\{1, Pe_K^A(x)\}}, \tag{2.10}$$

with

$$Pe_K^R(x) := \frac{2\varepsilon}{m_k b_n h_K^2} \qquad \text{and} \qquad Pe_K^A(x) := \frac{m_k|\boldsymbol{a}(x)|h_K}{\varepsilon}, \tag{2.11}$$

being the local Peclet numbers. Here $|\cdot|$ is the standard euclidean norm in $\mathbb{R}^2$ and $m_k := \min\{1/3, C_k\}$, with C_k being a positive constant, which depends on the polynomial degree k, appearing in the following inverse inequality

$$C_k \sum_{K\in\mathcal{T}_h^n} h_K^2 \|\Delta v_h\|_{0,K}^2 \le \|\nabla v_h\|_{0,\Omega}^2, \qquad \forall\, v_h \in H_h^n. \tag{2.12}$$

Remark 2.3. *From the definition of C_k we obtain $m_k = 1/3$ when $k = 1$. For higher order continuous interpolation spaces, the inverse inequality constant must be computed. This may be achieved in an elegant way by computing the biggest non–zero generalized eigenvalue of the problem associated with* (2.12) *in each element (for details, see* [12]*).* □

Lemma 2.4. *The stabilized problem* (SP) *admit a unique solution.*

Proof. The result is a direct consequence of the inequality (see [2])

$$B_S(v_h, v_h) \ge \sum_{K\in\mathcal{T}_h^n} \left[\int_K \varepsilon\beta_K |\nabla v_h|^2 + \int_K \delta_K (\boldsymbol{a}\cdot\nabla v_h)^2 + \int_K \beta_K b_n |v_h|^2 \right],$$

for all $v_h \in H_h^n$, where

$$\beta_K := \frac{2\varepsilon}{m_k b_n h_K^2 + 2\varepsilon} > 0.$$

□

We define the *energy norm* $|\!|\!|\cdot|\!|\!|_S$ as follows

$$|\!|\!|v|\!|\!|_S := \left\{\varepsilon\|\nabla v\|_{0,S}^2 + \|v\|_{0,S}^2\right\}^{1/2} \qquad \forall\, v \in H^1(S), \tag{2.13}$$

with $S \subset \Omega$. If $S = \Omega$ we write $|\!|\!|\cdot|\!|\!|$ instead of $|\!|\!|\cdot|\!|\!|_\Omega$.

We finish this section recalling the following two technical lemmas proved in [2] (see lemmas 2 and 3).

Lemma 2.5. *Given $K \in \mathcal{T}_h^n$ and δ_K defined by* (2.10)*, the following bounds hold for all $x \in K$ and n, $1 \le n \le N$.*

$$\begin{aligned} \varepsilon\delta_K(x) &\le \frac{h_K^2}{6}, \\ |\boldsymbol{a}(x)|\delta_K(x) &\le \frac{h_K}{2}, \\ b_n(x)\delta_K(x) &\le 1. \end{aligned}$$

Furthermore,

$$b_n(x)\delta_K(x) \le C\alpha_K,$$

with $C := \max\{1, \left(\|b_n\|_{\infty,\Omega}/6\right)^{1/2}\}$ *and*

$$\alpha_K := \min\{h_K\varepsilon^{-1/2}, 1\}, \qquad \forall K \in \mathcal{T}_h^n. \tag{2.14}$$

Lemma 2.6. *There exists a positive constant C, independent of h, such that the following bounds hold for all $v_h \in H_h^n$ and $K \in \mathcal{T}_h^n$*

$$\|\nabla v_h\|_{0,K} \le C\, h_K^{-1}\alpha_K |\!|\!|v_h|\!|\!|_K \qquad \text{and} \qquad \|\Delta v_h\|_{0,K} \le C\, h_K^{-2}\alpha_K |\!|\!|v_h|\!|\!|_K.$$

3. Auxiliary results

Let $\mathcal{E}_h^n$ be the set of all edges in $\mathcal{T}_h^n$, which can be split as follows

$$\mathcal{E}_h^n = \mathcal{E}_{h,\Omega}^n \cup \mathcal{E}_{h,N}^n \cup \mathcal{E}_{h,D}^n$$

where $\mathcal{E}_{h,\Omega}^n$, $\mathcal{E}_{h,N}^n$ and $\mathcal{E}_{h,D}^n$ denote the edges in the interior, on the Neumann boundary Γ_N and on the Dirichlet boundary Γ_D, respectively. For $E \in \mathcal{E}_h^n$, let h_E be the diameter of E.

For any $K \in \mathcal{T}_h^n$ we denote by $\mathcal{E}_K$ and $\mathcal{V}_K$ the set of edges and vertices of K, respectively. For $K \in \mathcal{T}_h^n$ and $E \in \mathcal{E}_h^n$ we can define the following neighborhoods

$$\widetilde{\omega}_K := \bigcup_{\mathcal{V}_K \cap \mathcal{V}_{K'} \neq \emptyset} K', \qquad \omega_E := \bigcup_{E \,\in\, \mathcal{E}_{K'}} K'.$$

Let $\varphi \in L^2(\omega_E)$, with $E \in \mathcal{E}_{h,\Omega}^n$, such that $\varphi|_K \in C(K)$, $\forall\, K \subset \omega_E$, we define the jump of φ on E in the direction of the vector $\boldsymbol{n}_E$ by

$$[\![\varphi]\!]_E(x) := \lim_{t\to 0^+} \varphi(x + t\boldsymbol{n}_E) - \lim_{t\to 0^-} \varphi(x - t\boldsymbol{n}_E).$$

Finally, in the sequel we will use the following notation:

$$\begin{aligned} a \preceq b \quad &\Longleftrightarrow \quad a \le cb \\ a \simeq b \quad &\Longleftrightarrow \quad a \preceq b \quad \text{and} \quad b \preceq a, \end{aligned}$$

where the constant $c > 0$, is independent of ε, the time step τ_n and the mesh size h.

3.1. **Bubble functions.** We denote by ψ_K the usual bubble function of the triangle $K \in \mathcal{T}_h^n$, and by $\psi_{E,\theta}$, with $0 < \theta \le 1$, the perturbed bubble function of the edge $E \in \mathcal{E}_h^n$, defined in [21]. Let $E \in \mathcal{E}_{h,\Omega}^n$ be an inner edge and denote by K_1, K_2 the two triangles sharing E. Let $G_{E,i}$, $i = 1, 2$, be the orientation preserving affine transformations which maps $\hat{K}$ onto K_i and $\hat{E}$ onto E (see Figure 1).

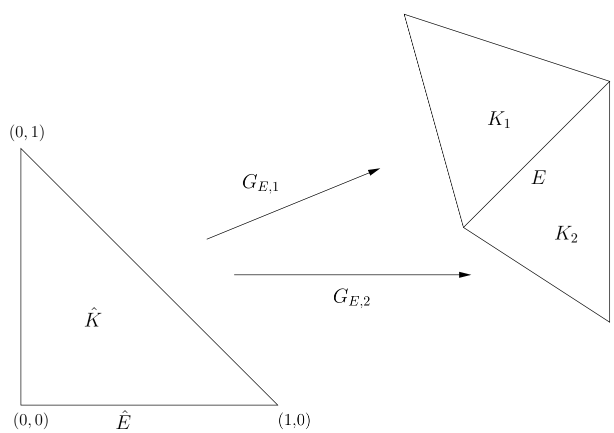

Figure 1. Affine transformation $G_{E,i}$, $i = 1, 2$.

Let $\hat{\Pi}$ the hyperplane defined by $\hat{\Pi} := \{(x, 0) : x \in \mathbb{R}\}$ and let $\hat{Q} : \mathbb{R}^2 \to \hat{\Pi}$ be the orthogonal projection from $\mathbb{R}^2$ onto $\hat{\Pi}$. We introduce the lifting operator $\hat{P}_{\hat{E}} : \mathbb{P}_k(\hat{E}) \to \mathbb{P}_k(\hat{K})$ by

$$\hat{P}_{\hat{E}}(\hat{\sigma}) := \hat{\sigma} \circ \hat{Q}.$$

Let $P_{E,K_i} : \mathbb{P}_k(E) \to \mathbb{P}_k(K_i)$ given by

$$P_{E,K_i} := \hat{P}_{\hat{E}}(\sigma \circ G_{E,i}) \circ G_{E,i}^{-1}, \quad i = 1, 2.$$

We define a lifting operator for $\sigma \in \mathbb{P}_k(E)$ by

$$P_E(\sigma) := \begin{cases} P_{E,K_1}(\sigma) & \text{in} \quad K_1, \\ \\ P_{E,K_2}(\sigma) & \text{in} \quad K_2. \end{cases}$$

For $E \in \mathcal{E}_{h,N}^n$ the lifting operator P_E is defined similarly, with obvious modifications.

Finally, we introduce the time bubble function ψ_n, given by

$$\psi_n(t) := \begin{cases} -\dfrac{4}{\tau_n^2}(t-t^{n-1})(t-t^n) & \text{if} \quad t \in]t^{n-1}, t^n[, \\ \\ 0 & \text{if} \quad t \in]-\infty, t^{n-1}] \cup [t^n, +\infty[. \end{cases} \tag{3.1}$$

Concerning the space bubble functions ψ_K and ψ_{E,θ_E} and the time bubble function ψ_n we have the following estimates:

Lemma 3.1. *For all $K \in \mathcal{T}_h^n$ and $v \in \mathbb{P}_k(K)$ we have*

(i) $\|v\|_{0,K}^2 \preceq (v, \psi_K v)_K$,
(ii) $\|v\psi_K\|_{0,K} \le \|v\|_{0,K}$,
(iii) $\|v\psi_K\| \preceq \min\{h_K\varepsilon^{-1/2}, 1\}^{-1}\|v\|_{0,K}$.

For $E \in \mathcal{E}_h^n$, we consider $\theta_E := \min\{\varepsilon^{1/2}h_E^{-1}, 1\}$. Then for all $\sigma \in \mathbb{P}_k(E)$ it holds

(iv) $\|\sigma\|_{0,E}^2 \preceq (\sigma, \psi_{E,\theta_E} P_E\sigma)_E$,
(v) $\|\psi_{E,\theta_E} P_E\sigma\|_{0,\omega_E} \preceq \varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2}, 1\}^{1/2}\|\sigma\|_{0,E}$,
(vi) $\|\psi_{E,\theta_E} P_E\sigma\|_{\omega_E} \preceq \varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2}, 1\}^{-1/2}\|\sigma\|_{0,E}$.

Proof. See Lemma 3.4 in [21]. □

Lemma 3.2. *For all $v \in \mathbb{P}_k(]t^{n-1}, t^n[)$ we have*

$$\begin{aligned} \|v\|_{0,]t^{n-1},t^n[}^2 &\preceq \int_{t^{n-1}}^{t^n} v^2\psi_n dt, \\ \|v\psi_n\|_{0,]t^{n-1},t^n[} &\le \|v\|_{0,]t^{n-1},t^n[}, \\ \|\frac{\partial}{\partial t}(\psi_n v)\|_{0,]t^{n-1},t^n[} &\preceq \frac{1}{\tau_n}\|v\|_{0,]t^{n-1},t^n[}. \end{aligned}$$

Proof. The first two inequalities follows from the definition of ψ_n, using the fact that $0 \le \psi_n \le 1$. For the last inequality we use standard scaling arguments. In fact, a change of variables give us

$$\int_{t^{n-1}}^{t^n} \left[\frac{\partial}{\partial t}(\psi_n v)\right]^2 dt = \int_0^1 \left[\frac{\partial}{\partial \widehat{t}}(\widehat{\psi}_n \widehat{v})(t^n - t^{n-1})^{-1}\right]^2 |J_t|\, d\hat{t},$$

where $|J_t|$ is the jacobian of the change of variable. Using that $0 \le \widehat{\psi}_n \le 1$ and the equivalence of norms on finite dimensional spaces, we obtain

$$\begin{aligned} \int_{t^{n-1}}^{t^n} \left[\frac{\partial}{\partial t}(\psi_n v)\right]^2 dt &\preceq \frac{1}{(t^n - t^{n-1})^2}\int_0^1 \widehat{v}^2 |J_t|\, d\widehat{t} \\ &\preceq \frac{1}{(t^n - t^{n-1})^2}\int_{t^{n-1}}^{t^n} v^2\, dt, \end{aligned}$$

and the result follows. □

We denote by $I_h^n : L^2(\Omega) \longrightarrow H_h^n$ the usual Clément's interpolator (see [8]). Now we establish the local approximation properties of I_h^n.

Lemma 3.3. *The following results hold for all $K \in \mathcal{T}_h^n$, $E \subset \partial K$ and $v \in H^1(\widetilde{\omega}_K)$*

$$\begin{aligned} \|v - I_h v\|_{0,K} &\preceq \min\{h_K \varepsilon^{-1/2}, 1\} \|\!|v|\!\|_{\widetilde{\omega}_K}, \\ \|v - I_h v\|_{0,E} &\preceq \varepsilon^{-1/4} \min\{h_K \varepsilon^{-1/2}, 1\}^{1/2} \|\!|v|\!\|_{\widetilde{\omega}_K}, \\ \|\!|I_h v|\!\|_K &\preceq \|\!|v|\!\|_{\widetilde{\omega}_K}. \end{aligned}$$

Proof. See Lemma 3.2 in [21]. □

4. Residual a posteriori error estimator

In this section our aim is to build a *residual a posteriori error estimator* for the stabilized scheme defined in (SP). To this end, we consider the function $u_{h\tau}$, defined in $[0, T]$, such that, restricted to $[t^{n-1}, t^n]$ is given by

$$u_{h\tau} := \frac{t - t^{n-1}}{t^n - t^{n-1}} u_h^n + \frac{t^n - t}{t^n - t^{n-1}} u_h^{n-1}. \tag{4.1}$$

Note that the function $t \longrightarrow u_{h\tau}(t, \cdot)$ is continuous and piecewise affine with values in $H_D^1(\Omega)$, therefore differentiable in the classical sense in $[t^{n-1}, t^n], 1 \le n \le N$. Let u be the solution of (VP), we define the error in $[0, T]$ as

$$e := u - u_{h\tau}$$

and we measured it, in $[t^{n-1}, t^n]$, using the following norm

$$\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2.$$

Remark 4.1. *Using integration by parts we obtain, for all $v \in H_D^1(\Omega)$ and $t \in [t^{n-1}, t^n]$, that*

$$\langle \partial_t e, v\rangle + \varepsilon(\nabla e, v) + (\boldsymbol{a} \cdot \nabla e, \nabla v) + (be, v) = \sum_{K \in \mathcal{T}_h^n} (f - \partial_t u_{h\tau} + \varepsilon \Delta u_{h\tau} - \boldsymbol{a} \cdot \nabla u_{h\tau} - b u_{h\tau}, v)_K$$

$$+ \sum_{E \in \mathcal{E}_{h,N}^n} (g - \varepsilon \partial_{\boldsymbol{n}_E} u_{h\tau}, v)_E + \sum_{E \in \mathcal{E}_{h,\Omega}^n} (-\varepsilon [\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E, v)_E.$$

From the stabilized problem (SP)*, we have that for all $w_h \in H_h^n$*

$$\langle \partial_t u_{h\tau}, w_h\rangle + \varepsilon(\nabla u_{h\tau}, w_h) + (\boldsymbol{a} \cdot \nabla u_{h\tau}, \nabla w_h) + (b u_{h\tau}, w_h) = (f^n, w_h) + (g^n, w_h)_{\Gamma_N} + \frac{t - t^{n-1}}{\tau_n} B(u_h^n - u_h^{n-1}, w_h)$$

$$+ \sum_{K \in \mathcal{T}_h^n} \int_K \delta_K (f^n - \partial_t u_{h\tau} + \varepsilon \Delta u_h^n - \boldsymbol{a} \cdot \nabla u_h^n - b u_h^n)(-\varepsilon \Delta w_h - \boldsymbol{a} \cdot \nabla w_h + b_n w_h).$$

Thus, using problem (VP) *and the above equation, we obtain*

$$\begin{aligned} &\langle \partial_t e, w_h\rangle + \varepsilon(\nabla e, w_h) + (\boldsymbol{a} \cdot \nabla e, \nabla w_h) + (be, w_h) \\ = \; &(f - f^n, w_h) + (g - g^n, w_h)_{\Gamma_N} - \frac{t - t^{n-1}}{\tau_n} B(u_h^n - u_h^{n-1}, w_h) \\ &- \sum_{K \in \mathcal{T}_h^n} \int_K \delta_K (f^n - \partial_t u_{h\tau} + \varepsilon \Delta u_h^n - \boldsymbol{a} \cdot \nabla u_h^n - b u_h^n)(-\varepsilon \Delta w_h - \boldsymbol{a} \cdot \nabla w_h + b_n w_h). \end{aligned}$$

□

For $t \in [t^{n-1}, t^n]$, let R_K^n and R_E^n be the volumetric and edge residuals, respectively, defined by

$$R_K^n := \Pi_K^n f - \partial_t u_{h\tau} + \varepsilon \Delta u_{h\tau} - \boldsymbol{a} \cdot \nabla u_{h\tau} - b u_{h\tau} \qquad \forall\, K \in \mathcal{T}_h^n,$$

and

$$R_E^n := \begin{cases} -\dfrac{\varepsilon}{2}\Big[\!\Big[\dfrac{\partial u_{h\tau}}{\partial \boldsymbol{n}_E}\Big]\!\Big]_E & \text{if} \quad E \in \mathcal{E}_{h,\Omega}^n, \\ \Pi_E^n g - \varepsilon \dfrac{\partial u_{h\tau}}{\partial \boldsymbol{n}_E} & \text{if} \quad E \in \mathcal{E}_{h,N}^n, \\ 0 & \text{if} \quad E \in \mathcal{E}_{h,D}^n, \end{cases}$$

where Π_K^n is the $L^2(]t^{n-1}, t^n[\times K)$ projection onto the constants (see [3] and [15]), i.e.,

$$\Pi_K^n f := \frac{1}{(t^n - t^{n-1})|K|} \int_{t^{n-1}}^{t^n} \int_K f$$

and Π_E^n is the $L^2(]t^{n-1}, t^n[\times E)$ projection onto the constants given by

$$\Pi_E^n g := \frac{1}{(t^n - t^{n-1})|E|} \int_{t^{n-1}}^{t^n} \int_E g\, ds.$$

We use the residuals R_K^n and R_E^n, to define an *a posteriori error estimator*, in the interval $[t^{n-1}, t^n]$, $1 \le n \le N$, as follows:

$$(\eta_K^n)^2 := \int_{t^{n-1}}^{t^n} \Big\{ \alpha_K^2 \|R_K^n\|_{0,K}^2 + \sum_{E \in \mathcal{E}_K} \varepsilon^{-1/2} \alpha_E \|R_E^n\|_{0,E}^2 \Big\}, \tag{4.2}$$

$$(\eta_\Omega^n)^2 := \sum_{K \in \mathcal{T}_h^n} (\eta_K^n)^2 .$$

with α_K defined as in (2.14) and $\alpha_E := \min\{h_E \varepsilon^{-1/2}, 1\}$.

The main result of this work is stated in the following theorem.

Theorem 4.2. *Let u and u_h^n be the solutions of* (VP) *and* (SP)*, respectively. Consider $e = u - u_{h\tau}$ for all $t \in [t^{n-1}, t^n]$, with $u_{h\tau}$ defined as in* (4.1)*. Let η_K^n be the a posteriori error estimator defined in* (4.2)*, then we have that*

$$\begin{aligned} &\|e(t^n,\cdot)\|_{0,\Omega}^2 + \int_{t^{n-1}}^{t^n} |\!|\!| e |\!|\!|^2 \preceq \|e(t^{n-1},\cdot)\|_{0,\Omega}^2 + \sum_{K \in \mathcal{T}_h^n} (\eta_K^n)^2 + \sum_{K \in \mathcal{T}_h^n} \int_{t^{n-1}}^{t^n} \alpha_K^2 \|f - \Pi_K^n f\|_{0,K}^2 \\ &+ \sum_{E \in \mathcal{E}_{h,N}^n} \int_{t^{n-1}}^{t^n} \varepsilon^{-1/2} \alpha_E \|g - \Pi_E^n g\|_{0,E}^2 + \int_{t^{n-1}}^{t^n} \|f - f^n\|_{0,\Omega}^2 + \int_{t^{n-1}}^{t^n} \|g - g^n\|_{0,\Gamma_N}^2 \\ &+ \int_{t^{n-1}}^{t^n} \sum_{K \in \mathcal{T}_h^n} \alpha_K^2 \|f^n - \Pi_K^n f\|_{0,K}^2 + \sum_{K \in \mathcal{T}_h^n} \frac{\tau_n}{3} |\!|\!| u_h^n - u_h^{n-1} |\!|\!|_K^2 \Big\{ \alpha_K^2 (\varepsilon^{-1/2} \|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2 \\ &+ \big(1 + \|b\|_{\infty,\Omega} + \varepsilon^{-1/2} \|\boldsymbol{a}\|_{\infty,\Omega}\big)^2 \Big\}. \end{aligned} \tag{4.3}$$

Moreover,

$$\begin{aligned} \eta_K^n \quad \preceq \quad & \alpha_K \left(\int_{t^{n-1}}^{t^n} \|f - \Pi_K^n f\|_{0,\omega_K}^2 \right)^{1/2} + \left(\int_{t^{n-1}}^{t^n} |\!|\!| e |\!|\!|_{\omega_K}^2 \right)^{1/2} \left\{ 1 + \|b\|_{\infty,\omega_K} + \varepsilon^{-1/2} \|\boldsymbol{a}\|_{\infty,\omega_K} \alpha_K + \frac{\alpha_K}{\tau_n} \right\} \\ & + \left(\sum_{E \in \mathcal{E}_{K,N}^n} \varepsilon^{-1/4} \alpha_E^{1/2} \int_{t^{n-1}}^{t^n} \|g - \Pi_E^n g\|_{0,E}^2 \right)^{1/2} . \end{aligned} \tag{4.4}$$

Proof. Upper bound

From (2.1) and the definition of the error e, we have that

$$\varepsilon(\nabla e,\nabla e)+(\boldsymbol{a}\cdot\nabla e,e)+(b\,e,e)\geq |||e|||^2,\quad \forall\, t\in[t^{n-1},t^n].$$

Thus, using the above inequality, adding in both sides $\langle\partial_t e,e\rangle$ and integrating between t^{n-1} y t^n, we get

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\langle\partial_t e,e\rangle+\int_{t^{n-1}}^{t^n}|||e|||^2 \quad\leq\quad & \int_{t^{n-1}}^{t^n}\langle\partial_t e,e-I_h^n e\rangle+\int_{t^{n-1}}^{t^n}B(e,e-I_h^n e)\\ &+\int_{t^{n-1}}^{t^n}\langle\partial_t e,I_h^n e\rangle+\int_{t^{n-1}}^{t^n}B(e,I_h^n e).\end{aligned}\tag{4.5}$$

Taking $v=e(t,\cdot)-I_h^n e(t,\cdot)$, for $t\in[t^{n-1},t^n]$, in (4.2), and integrating between t^{n-1} and t^n, we obtain

$$\begin{aligned}&\int_{t^{n-1}}^{t^n}\langle\partial_t e,e-I_h^n e\rangle+\int_{t^{n-1}}^{t^n}\{\varepsilon(\nabla e,\nabla(e-I_h^n e))+(\boldsymbol{a}\cdot\nabla e,e-I_h^n e)+(b\,e,e-I_h^n e)\}\\ &=\int_{t^{n-1}}^{t^n}\sum_{K\in\mathcal{T}_h^n}\Bigg\{(R_K^n,e-I_h^n e)_K+(f-\Pi_K^n f,e-I_h^n e)_K+\sum_{E\in\partial K}(R_E^n,e-I_h^n e)_E\\ &+\sum_{E\in\,\mathcal{E}_{K,N}^n}(g-\Pi_E^n g,e-I_h^n e)_E\Bigg\},\end{aligned}\tag{4.6}$$

Thus, using (4.6), Cauchy–Schwarz's inequality and Lemma 3.3, we arrive at

$$\begin{aligned}&\int_{t^{n-1}}^{t^n}\langle\partial_t e,e-I_h^n e\rangle+\int_{t^{n-1}}^{t^n}\{\varepsilon(\nabla e,\nabla(e-I_h^n e))+(\boldsymbol{a}\cdot\nabla e,e-I_h^n e)+(b\,e,e-I_h^n e)\}\\ \preceq\quad&\int_{t^{n-1}}^{t^n}\sum_{K\in\mathcal{T}_h^n}\Bigg\{\alpha_K\|R_K^n\|_{0,K}+\alpha_K\|f-\Pi_K^n f\|_{0,K}+\sum_{E\in\,\partial K}\varepsilon^{-1/4}\alpha_E^{1/2}\|R_E^n\|_{0,E}\\ &+\sum_{E\in\,\mathcal{E}_{K,N}^n}\varepsilon^{-1/4}\alpha_E^{1/2}\|g-\Pi_E^n g\|_{0,E}\Bigg\}|||e|||_{\widetilde{\omega}_K}\\ \preceq\quad&\int_{t^{n-1}}^{t^n}\Bigg(\sum_{K\in\mathcal{T}_h^n}\Bigg\{\alpha_K^2\|R_K^n\|_{0,K}^2+\sum_{E\in\,\partial K}\varepsilon^{-1/2}\alpha_E\|R_E^n\|_{0,E}^2\Bigg\}+\sum_{K\in\mathcal{T}_h^n}\alpha_K^2\|f-\Pi_K^n f\|_{0,K}^2\\ &+\sum_{E\in\,\mathcal{E}_{h,N}^n}\varepsilon^{-1/2}\alpha_E\|g-\Pi_E^n g\|_{0,E}^2\Bigg)^{1/2}\Bigg(\sum_{K\in\mathcal{T}_h^n}|||e|||_{\widetilde{\omega}_K}^2\Bigg)^{1/2}\\ \preceq\quad&\Bigg(\sum_{K\in\mathcal{T}_h^n}\int_{t^{n-1}}^{t^n}\Bigg\{\alpha_K^2\|R_K^n\|_{0,K}^2+\sum_{E\in\,\partial K}\varepsilon^{-1/2}\alpha_E\|R_E^n\|_{0,E}^2\Bigg\}+\sum_{K\in\,\mathcal{T}_h^n}\int_{t^{n-1}}^{t^n}\alpha_K^2\|f-\Pi_K^n f\|_{0,K}^2\\ &+\sum_{E\in\,\mathcal{E}_{h,N}^n}\int_{t^{n-1}}^{t^n}\varepsilon^{-1/2}\alpha_E\|g-\Pi_E^n g\|_{0,E}^2\Bigg)^{1/2}\Bigg(\int_{t^{n-1}}^{t^n}|||e|||^2\Bigg)^{1/2}.\end{aligned}\tag{4.7}$$

On the other hand, taking $w_h=I_h^n e(t,\cdot)$, $t\in[t^{n-1},t^n]$ in (4.2) and integrating between t^{n-1} and t^n, we get

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \langle \partial_t e, I_h^n e\rangle + \int_{t^{n-1}}^{t^n} \{\varepsilon(\nabla e, \nabla(I_h^n e)) + (\boldsymbol{a}\cdot\nabla e, I_h^n e) + (b\, e, I_h^n e)\}\\
&= \int_{t^{n-1}}^{t^n} (f - f^n, I_h^n e) + \int_{t^{n-1}}^{t^n} (g - g^n, I_h^n e)_{\Gamma_N} + \int_{t^{n-1}}^{t^n} \frac{t - t^{n-1}}{\tau_n} B(u_h^n - u_h^{n-1}, I_h^n e)\\
&+ \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \int_K \delta_K (f^n - \partial_t u_{h\tau} - \varepsilon\Delta u_h^n - \boldsymbol{a}\cdot\nabla u_h^n + b u_h^n)(-\varepsilon\Delta I_h^n e - \boldsymbol{a}\cdot\nabla I_h^n e + b_n I_h^n e).
\end{aligned}
\tag{4.8}
$$

Now, we will bound the last two terms on the right-hand side of (4.8). First, by a classical result (equation 2.4 in [2] and [21]), we obtain that for all $t \in [t^{n-1}, t^n]$

$$
B(u_h^n - u_h^{n-1}, I_h^n e(t,\cdot)) \le |||e(t,\cdot)|||\, |||u_h^n - u_h^{n-1}|||\,\big(1 + \|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big).
$$

Thus, using the above inequality, Cauchy–Schwarz's inequality and integrating between t^{n-1} and t^n, we arrive at

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \frac{t - t^{n-1}}{\tau_n} B(u_h^n - u_h^{n-1}, I_h^n e)\\
\le\ & \int_{t^{n-1}}^{t^n} \frac{t - t^{n-1}}{\tau_n} \Big\{ |||e|||\, |||u_h^n - u_h^{n-1}|||\,\big(1 + \|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)\Big\}\\
\le\ & \left(\int_{t^{n-1}}^{t^n} \left(\frac{t - t^{n-1}}{\tau_n}\right)^2 |||u_h^n - u_h^{n-1}|||^2 \big(1 + \|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)^2\right)^{1/2}\\
& \times \left(\int_{t^{n-1}}^{t^n} |||e|||^2\right)^{1/2}\\
=\ & \left(\frac{\tau_n}{3} |||u_h^n - u_h^{n-1}|||^2 \big(1 + \|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} |||e|||^2\right)^{1/2}.
\end{aligned}
\tag{4.9}
$$

The last term on the right-hand side of (4.8) can be bounded using lemmas 2.5 and 2.6, and Cauchy–Schwarz's inequality, as follows

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \int_K \delta_K(f^n - \partial_t u_{h\tau} - \varepsilon\Delta u_h^n - \boldsymbol{a}\cdot\nabla u_h^n + bu_h^n)(-\varepsilon\Delta I_h^n e - \boldsymbol{a}\cdot\nabla I_h^n e + b_n I_h^n e) \\
\preceq\ & \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K \|f^n - \partial_t u_{h\tau} - \varepsilon\Delta u_h^n - \boldsymbol{a}\cdot\nabla u_h^n + bu_h^n\|_{0,K} |\!|\!| e |\!|\!|_{\widetilde{\omega}_K} \\
\preceq\ & \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \Big\{ \alpha_K \|R_K^n\|_{0,K} + \alpha_K\|f^n - \Pi_h^n f\|_{0,K} + \frac{t^n - t}{\tau_n}\alpha_K \|\varepsilon\Delta(u_h^n - u_h^{n-1}) \\
& -\boldsymbol{a}\cdot\nabla(u_h^n - u_h^{n-1}) - b(u_h^n - u_h^{n-1})\|_{0,K} \Big\} |\!|\!| e |\!|\!|_{\widetilde{\omega}_K} \\
\preceq\ & \int_{t^{n-1}}^{t^n} \left(\sum_{K\in\mathcal{T}_h^n} \Big\{ \alpha_K^2 \|R_K^n\|_{0,K}^2 + \alpha_K^2 \|f^n - \Pi_h^n f\|_{0,K}^2 \right. \\
& \left. + \frac{(t^n - t)^2}{\tau_n^2} \alpha_K^2 |\!|\!| u_h^n - u_h^{n-1} |\!|\!|_K^2 (\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2 \Big\} \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h^n} |\!|\!| e |\!|\!|_{\widetilde{\omega}_K}^2 \right)^{1/2} \\
\preceq\ & \left(\int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|R_K^n\|_{0,K}^2 + \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|f^n - \Pi_h^n f\|_{0,K}^2 \right. \\
& \left. + \sum_{K\in\mathcal{T}_h^n} \frac{\tau_n}{3} \alpha_K^2 |\!|\!| u_h^n - u_h^{n-1} |\!|\!|_K^2 (\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2 \right)^{1/2} \left(\int_{t^{n-1}}^{t^n} |\!|\!| e |\!|\!|^2 \right)^{1/2}. \qquad (4.10)
\end{aligned}
$$

Finally, using Cauchy–Schwarz's inequality, (4.9), (4.10) and Lemma 3.3, we bound (4.8) as follows

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \langle \partial_t e, I_h^n e\rangle + \int_{t^{n-1}}^{t^n} \left\{\varepsilon(\nabla e, \nabla(I_h^n e)) + (\boldsymbol{a}\cdot\nabla e, I_h^n e) + (b\, e, I_h^n e)\right\} \\
&\leq \left(\int_{t^{n-1}}^{t^n} \|f-f^n\|_{0,\Omega}^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} + \left(\int_{t^{n-1}}^{t^n} \|g-g^n\|_{0,\Gamma_N}^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} \\
&+ \left(\int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|R_K^n\|_{0,K}^2 + \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|f^n - \Pi_h^n f\|_{0,K}^2 \right. \\
&\left. + \sum_{K\in\mathcal{T}_h^n} \frac{\tau_n}{3}\alpha_K^2 \|\!|u_h^n - u_h^{n-1}|\!\|_K^2 (\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} \\
&+ \left(\frac{\tau_n}{3}\|\!|u_h^n - u_h^{n-1}|\!\|^2 \big(1+\|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} \\
\preceq\ & \left(\int_{t^{n-1}}^{t^n} \|f-f^n\|_{0,\Omega}^2 + \int_{t^{n-1}}^{t^n} \|g-g^n\|_{0,\Gamma_N}^2 + \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|R_K^n\|_{0,K}^2 \right. \\
&+ \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|f^n - \Pi_h^n f\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h^n} \frac{\tau_n}{3}\alpha_K^2 \|\!|u_h^n - u_h^{n-1}|\!\|_K^2 (\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2 \\
&\left. + \frac{\tau_n}{3}\|\!|u_h^n - u_h^{n-1}|\!\|^2 \big(1+\|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} \\
\preceq\ & \left(\int_{t^{n-1}}^{t^n} \|f-f^n\|_{0,\Omega}^2 + \int_{t^{n-1}}^{t^n} \|g-g^n\|_{0,\Gamma_N}^2 + \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|R_K^n\|_{0,K}^2 \right. \\
&+ \int_{t^{n-1}}^{t^n} \sum_{K\in\mathcal{T}_h^n} \alpha_K^2 \|f^n - \Pi_h^n f\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h^n} \frac{\tau_n}{3} \|\!|u_h^n - u_h^{n-1}|\!\|_K^2 \Big(\alpha_K^2 (\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K} + \|b\|_{\infty,K})^2 \\
&\left. + \big(1+\|b\|_{\infty,\Omega} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\Omega}\big)^2\Big)\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2}. \qquad (4.11)
\end{aligned}
$$

Thus the upper bound (4.3) is a consequence of (4.5), (4.7) and (4.11), Cauchy–Schwarz's and Young's inequalities and the identity

$$
\int_{t^{n-1}}^{t^n} \langle \partial_t e, e\rangle = \frac{1}{2}\left(\|e(t^n,\cdot)\|_{0,\Omega}^2 - \|e(t^{n-1},\cdot)\|_{0,\Omega}^2\right).
$$

Lower bound

To prove the lower bound (4.4), we consider the bubble functions defined in Section 3. Integration by part implies that for all $w \in H^1_D(\Omega)$ and $t \in [t^{n-1}, t^n]$

$$\begin{aligned}(R^n_K, w)_K &= \int_K (fw - \partial_t u_{h\tau} w - \varepsilon \nabla u_{h\tau} \nabla w - \boldsymbol{a} \cdot \nabla u_{h\tau} w - b u_{h\tau} w - (f - \Pi^n_K f) w) \\ &\quad + \sum_{E \in \mathcal{E}_K} \int_E \nabla u_{h\tau} \cdot \boldsymbol{n}_E \, w \, ds \\ &= \langle \partial_t e \,, w\rangle + \varepsilon(\nabla e, \nabla w)_K + (\boldsymbol{a} \cdot \nabla e, w)_K + (b\, e, w)_K \\ &\quad + \sum_{E \in \mathcal{E}_K} \int_E \nabla u_{h\tau} \cdot \boldsymbol{n}_E \, w \, ds - \sum_{E \in \mathcal{E}_K} \int_E \nabla u \cdot \boldsymbol{n}_E \, w \, ds - \int_K (f - \Pi^n_K f) w.\end{aligned} \tag{4.12}$$

If we consider $w = w^n_K(t, \cdot)$ in (4.12), with $w^n_K(t, \cdot) := \psi_K(\cdot)\psi_n(t)R^n_K(t, \cdot)$ for $t \in [t^{n-1}, t^n]$, we have

$$\begin{aligned}(R^n_K, w^n_K(t, \cdot))_K &= \langle \partial_t e, w^n_K(t, \cdot)\rangle + \varepsilon(\nabla e, \nabla w^n_K(t, \cdot))_K + (\boldsymbol{a} \cdot \nabla e, w^n_K(t, \cdot))_K + (b\, e, w^n_K(t, \cdot))_K \\ &\quad - \int_K (f - \Pi^n_K f) w^n_K(t, \cdot).\end{aligned}$$

Integrating between t^{n-1} and t^n in the above equation and using integration by part in time, we obtain

$$\begin{aligned}\int_{t^{n-1}}^{t^n} (R^n_K, w^n_K)_K &= -\int_{t^{n-1}}^{t^n} \int_K \partial_t w^n_K \, e + \int_{t^{n-1}}^{t^n} \{\varepsilon(\nabla e, \nabla w^n_K)_K + (\boldsymbol{a} \cdot \nabla e, w^n_K)_K + (b\, e, w^n_K)_K\} \\ &\quad - \int_{t^{n-1}}^{t^n} \int_K (f - \Pi^n_K f) w^n_K.\end{aligned} \tag{4.13}$$

On the other hand, from lemmas 3.1 and 3.2 we have

$$\begin{aligned}\int_{t^{n-1}}^{t^n} (R^n_K, w^n_K)_K &= \int_{t^{n-1}}^{t^n} \psi_n (R^n_K, \psi_K R^n_K)_K \\ &\succeq \int_{t^{n-1}}^{t^n} \|R^n_K\|^2_{0,K} \psi_n \\ &\succeq \int_{t^{n-1}}^{t^n} \|R^n_K\|^2_{0,K}.\end{aligned} \tag{4.14}$$

Thus, using Cauchy–Schwarz's inequality, lemmas 3.1 and 3.2, (4.13) and multiplying (4.14) by α_K, we obtain

$$\begin{aligned}\alpha_K \left(\int_{t^{n-1}}^{t^n} \|R^n_K\|^2_{0,K}\right)^{1/2} &\preceq \left(\int_{t^{n-1}}^{t^n} \|\!|e|\!\|^2\right)^{1/2} \left(1 + \|b\|_{\infty,K} + \alpha_K \varepsilon^{-1/2} \|\boldsymbol{a}\|_{\infty,K} + \frac{\alpha_K}{\tau_n}\right) \\ &\quad + \alpha_K \left(\int_{t^{n-1}}^{t^n} \|f - \Pi^n_K f\|^2_{0,K}\right)^{1/2}.\end{aligned} \tag{4.15}$$

Now, for $E \in \mathcal{E}^n_{h,\Omega}$ we define $w^n_E(t, \cdot) := \psi_{E,\theta_E}(\cdot)\psi_n(t)(-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}(t, \cdot)]\!]_E)$, for $t \in [t^{n-1}, t^n]$. Taking $v = w^n_E(t, \cdot)$ in (4.2), we obtain

$$\begin{aligned}&\int_E -\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E w^n_E(t, \cdot)\, ds = \langle \partial_t e, w^n_E(t, \cdot)\rangle + B(e, w^n_E(t, \cdot)) \\ &- \sum_{K \subset \omega_E} \int_K R^n_K w^n_E(t, \cdot) - \sum_{K \subset \omega_E} \int_K (f - \Pi^n_K f) w^n_E(t, \cdot).\end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n}\int_E -\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E w_E^n\,ds \\
= &- \sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K \partial_t w_E^n\, e + \int_{t^{n-1}}^{t^n}\{\varepsilon(\nabla e,\nabla w_E^n) + (\boldsymbol{a}\cdot\nabla e, w_E^n) + (b\,e, w_E^n)\} \\
&- \sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K R_K^n w_E^n - \sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K (f-\Pi_K^n f) w_E^n. \qquad (4.17)
\end{aligned}$$

We bound each term on the right side of (4.17). By Cauchy–Schwarz's inequality, lemmas 3.1 and 3.2 we have

$$\sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K \partial_t w_E^n\, e \preceq \left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}^2\right)^{1/2}\left(\int_{t^{n-1}}^{t^n}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E\|_{0,E}^2\right)^{1/2}\frac{\varepsilon^{1/4}\alpha_E^{1/2}}{\tau_n}, \qquad (4.18)$$

also

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n}\{\varepsilon(\nabla e,\nabla w_E^n) + (\boldsymbol{a}\cdot\nabla e, w_E^n) + (b\,e, w_E^n)\} \\
&\preceq \int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}\left\{(1+\|b\|_{\infty,\omega_E})|\!|\!|w_E^n|\!|\!|_{\omega_E} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\omega_E}\|w_E^n\|_{0,\omega_E}\right\} \\
&\preceq \int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}\left\{(1+\|b\|_{\infty,\omega_E})\varepsilon^{1/4}\alpha_E^{-1/2} + \varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\omega_E}\varepsilon^{1/4}\alpha_E^{1/2}\right\}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E\|_{0,E} \\
&\preceq \left(\int_{t^{n-1}}^{t^n}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E\|_{0,E}^2\right)^{1/2}\left\{(1+\|b\|_{\infty,\omega_E)})\varepsilon^{1/4}\alpha_E^{-1/2} + \varepsilon^{-1/4}\|\boldsymbol{a}\|_{\infty,\omega_E}\alpha_E^{1/2}\right\} \\
&\times\left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}^2\right)^{1/2}. \qquad (4.19)
\end{aligned}$$

By Cauchy–Schwarz's inequality, (4.15), lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
&\sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K R_K^n w_E^n \preceq \left(\int_{t^{n-1}}^{t^n}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E\|_{0,E}^2\right)^{1/2}\left\{\varepsilon^{1/4}\alpha_E^{-1/2}\left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}^2\right)^{1/2}\right. \\
&\left.\times\left\{(1+\|b\|_{\infty,\omega_E}) + \alpha_E\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\omega_E} + \frac{\alpha_E}{\tau_n}\right\} + \varepsilon^{1/4}\alpha_E^{1/2}\left(\int_{t^{n-1}}^{t^n}\|f-\Pi_K^n f\|_{0,\omega_E}^2\right)^{1/2}\right\}, \qquad (4.20)
\end{aligned}$$

also

$$\begin{aligned}
\sum_{K\subset\omega_E}\int_{t^{n-1}}^{t^n}\int_K (f-\Pi_K^n f) w_E^n \quad &\preceq \quad \varepsilon^{1/4}\alpha_E^{1/2}\left(\int_{t^{n-1}}^{t^n}\|f-\Pi_K^n f\|_{0,\omega_E}^2\right)^{1/2} \\
&\times\left(\int_{t^{n-1}}^{t^n}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E} u_{h\tau}]\!]_E\|_{0,E}^2\right)^{1/2}. \qquad (4.21)
\end{aligned}$$

Thus, using (4.17)–(4.21) and lemmas 3.1 and 3.2, we arrive at

$$\varepsilon^{-1/4}\alpha_E^{1/2}\left(\int_{t^{n-1}}^{t^n}\|-\varepsilon[\![\partial_{\boldsymbol{n}_E}u_{h\tau}]\!]_E\|_{0,E}^2\right)^{1/2} \preceq \alpha_E\left(\int_{t^{n-1}}^{t^n}\|f-\Pi_K^n f\|_{0,\omega_E}^2\right)^{1/2}$$
$$+\left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{\omega_E}\right)^{1/2}\left\{1+\|b\|_{\infty,\omega_E}+\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,\omega_E}\alpha_E+\frac{\alpha_E}{\tau_n}\right\}. \tag{4.22}$$

Similarly, if $E\in\mathcal{E}_{h,N}^n$ and $w_E^n:=\psi_{E,\theta_E}\psi_n(\Pi_E^n g-\varepsilon\partial_{\boldsymbol{n}_E}u_{h\tau})$, it follows that

$$\varepsilon^{-1/4}\alpha_E^{1/2}\left(\int_{t^{n-1}}^{t^n}\|\Pi_E^n g-\varepsilon\partial_{\boldsymbol{n}_E}u_{h\tau}\|_{0,E}^2\right)^{1/2} \preceq \alpha_E\left(\int_{t^{n-1}}^{t^n}\|f-\Pi_K^n f\|_{0,K}^2\right)^{1/2}$$
$$+\left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|_{K}\right)^{1/2}\left\{1+\|b\|_{\infty,K}^2+\varepsilon^{-1/2}\|\boldsymbol{a}\|_{\infty,K}\alpha_E+\frac{\alpha_E}{\tau_n}\right\}$$
$$+\varepsilon^{-1/4}\alpha_E^{1/2}\left(\int_{t^{n-1}}^{t^n}\|g-\Pi_E^n g\|_{0,E}^2\right)^{1/2}. \tag{4.23}$$

The lower bound (4.4) is a direct consequence of (4.15), (4.22) and (4.23). □

5. Numerical experiments

In order to develop an adaptive algorithm, we introduce the time error estimator (see [4] and [15]) defined by

$$\left(\eta_{t,K}^n\right)^2 := \frac{1}{3}\tau_n|\!|\!|u_h^n-u_h^{n-1}|\!|\!|_K^2 \qquad \forall t\in[t^{n-1},t^n], \tag{5.1}$$
$$\eta_t^n := \left(\sum_{K\in\mathcal{T}_h^n}\left(\eta_{t,K}^n\right)^2\right)^{1/2}.$$

From Theorem 4.2 we note that $((\eta_\Omega^n)^2+(\eta_t^n)^2)^{1/2}$ is a good approximation of

$$|\!|\!|e|\!|\!|_{[t^{n-1},t^n]\times\Omega} := \left(\int_{t^{n-1}}^{t^n}|\!|\!|e|\!|\!|^2\right)^{1/2}.$$

The adaptive procedure in space is such that for each n, $1\le n\le N$, we compute the space–time local error estimator $\eta_{h,K}^n:=\eta_K^n+\eta_{t,K}^n$ for K in $\mathcal{T}_h^n$ and $t\in[t^{n-1},t^n]$, and refine those elements $K\in\mathcal{T}_h^n$ such that

$$\eta_{h,K}^n\ge\theta_h\max\left\{\eta_{h,K}^n:\ K\in\mathcal{T}_h^n\right\}, \tag{5.2}$$

where $\theta_h\in]0,1[$ is a prescribed parameter. This procedure is repeated a fixed number of iterations or until a stop criteria is fulfilled. The aim of the time adaptive procedure is to develop a partition $0=t^0<t^1<\ldots<t^N=T$, such that in each interval $[t^{n-1},t^n]$ the following condition is fulfilled (see [15])

$$\frac{\left(\sum_{n=1}^N\left(\eta_t^n\right)^2\right)^{1/2}}{\left(\sum_{n=1}^N|\!|\!|u_h^n|\!|\!|^2\right)^{1/2}}\le\theta_N, \tag{5.3}$$

where $\theta_N\in]0,1[$ is a prescribed parameter, which means that the relative error is close to a preset tolerance. In our computations, to satisfy (5.3), we consider the following necessary condition

$$\left(\eta_t^n\right)^2\le\theta_N^2|\!|\!|u_h^n|\!|\!|^2,$$

for each n, $1\le n\le N$.

Remark 5.1. *To solve Problem* (SP) *we use piecewise linear functions which correspond to take $k = 1$ in the definition of the Lagrange finite element space. On the other hand, for the adaptive procedure we use the mesh generator* `Triangle` *which allows to create successively refined meshes (for details, see* [20]*). Other mesh adapted approach is obtained using the* `BL2D` *mesh generator (see* [7]*). The difference between* `Triangle` *and* `BL2D` *is the refinement algorithm.* □

Remark 5.2. *Note that the solution u_h^{n-1}, at time t^{n-1}, is defined on the mesh $\mathcal{T}_h^{n-1}$, thus to compute the solution u_h^n, at time t^n, using our adapted scheme, we need to interpolate u_h^{n-1} onto the new mesh $\mathcal{T}_h^n$ (see step* 13 *in the Adaptive algorithm below). In our case, we use the Shepard's algorithm to compute this interpolation (for details, see* [19]*).* □

The adaptive algorithm that we use is introduced in Algorithm 1, this procedure is similar to the one introduced in [15].

Algorithm 1 : Adaptive algorithm

1: **Input:** $\mathcal{T}_h^0,\ u_h^0,\ n = 1,\ \tau_n$
2: **while** $t \leq T$ **do**
3:   Compute $u_h^n,\ \eta_{t,K}^n,\ \eta_K^n$
4:   **if** $\eta_t^n < \theta_N \|u_h^n\|_\Omega$ **then**
5:     **while** stop criteria **do**
6:       **for all** $K \in \mathcal{T}_h^n$ **do**
7:         **if** $\eta_{h,K}^n \geq \theta_h \max\left\{\eta_{h,K}^n : K \in \mathcal{T}_h^n\right\}$ **then**
8:           mark the element K for refinement
9:         **end if**
10:       **end for**
11:       create a refined mesh $\mathcal{T}$
12:       $\mathcal{T}_h^n \leftarrow \mathcal{T}$
13:       interpolate u_h^{n-1} on $\mathcal{T}_h^n$ (Shepard's algorithm)
14:       Compute $u_h^n,\ \eta_{t,K}^n,\ \eta_K^n$
15:     **end while**
16:     $t \leftarrow t + \tau_n$
17:     $n \leftarrow n + 1$
18:   **else**
19:     $t \leftarrow t - \tau_n$
20:     $\tau_n \leftarrow \tau_n/2$
21:     $t \leftarrow t + \tau_n$
22:   **end if**
23: **end while**

In the following, we present three numerical test to show the performance of the stabilized scheme, introduced in problem (SP), and the adapted scheme based in the a posteriori error estimator introduced in (4.2) and (5.1).

The first test concern the numerical validation of our stabilized scheme trough a convergence analysis, in this case we see a perfect agreement between the theoretical order of convergence and the numerical ones. The other two tests are introduced to show the effectiveness of our adapted scheme, to produce adapted meshes that can capture the physical phenomena with precision.

5.1. **An analytical solution.** We consider the time-space domain $]0,1[\times\Omega$ where $\Omega :=]0,1[^2$, $\varepsilon = 1$, $a = (1, 1)$, $b = 1$, $u(0,\cdot) = 0$ and f such that the solution of (P) is given by

$$u(t,x,y) = 16txy(1-x)(1-y).$$

We consider a sequence of uniform meshes and compute the norms: $\|\!|u-u_{h\tau}\|\!|_{[0,1]\times\Omega}$, $|\,u-u_{h\tau}|_{[0,1]\times H^1(\Omega)}$ and $\|u-u_{h\tau}\|_{[0,1]\times L^2(\Omega)}$ defined by

$$\begin{aligned}
\|u-u_{h\tau}\|^2_{[0,1]\times L^2(\Omega)} &:= \int_0^1 \|u-u_{h\tau}\|^2_{0,\Omega},\\
|u-u_{h\tau}|^2_{[0,1]\times H^1(\Omega)} &:= \int_0^1 \|\nabla(u-u_{h\tau})\|^2_{0,\Omega},\\
\|\!|u-u_{h\tau}\|\!|^2_{[0,1]\times\Omega} &:= \int_0^1 \|\!|u-u_{h\tau}\|\!|^2,
\end{aligned}$$

where $u_{h\tau}$ is defined as in (4.1).

The results show, as expected, an $O(h^2)$ order of convergence for $\|u-u_{h\tau}\|_{[0,1]\times L^2(\Omega)}$ and $O(h)$ for $\|\!|u-u_{h\tau}\|\!|_{[0,1]\times\Omega}$ and $|\,u-u_{h\tau}|_{[0,1]\times H^1(\Omega)}$.

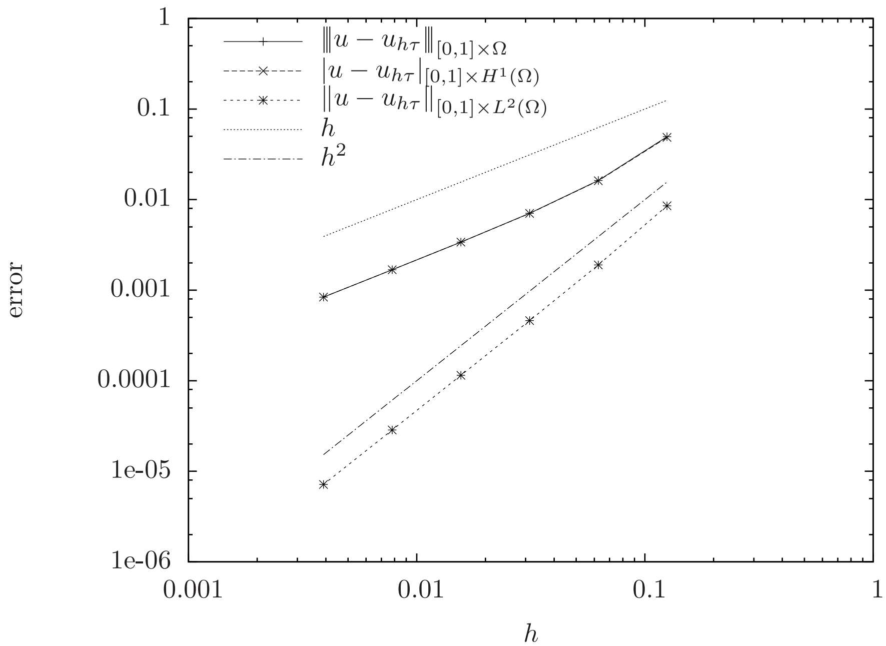

FIGURE 2. Convergence history for $\varepsilon = 1$.

5.2. **A diffusion problem.** In this case we consider $\Omega :=]-0.25, 1.25[^2$, $t \in]0,1[$, $\varepsilon = 1$, $\boldsymbol{a} = (0,0)$, $b = 1$, f is obtained assuming that the solution of (P) is given by

$$u(t,x,y) := \beta(t)e^{-50r^2(t,x,y)},$$

with

$$\begin{aligned}
r^2(t,x,y) &:= (x-0.4t-0.3)^2 + (y-0.4t-0.3)^2,\\
\beta(t) &:= 1 - e^{-50(0.98t+0.01)^2}.
\end{aligned}$$

Note that u is a Gaussian function, which center moves from (0.3,0.3), at $t = 0$, to (0.7,0.7), at $t = 1$ (see [15]).

We consider time and space adaptation with $\theta_N = 9\times 10^{-2}$, $\tau_1 = 0.33$ and $\theta_h = 0.03$. The initial mesh is presented in the following figure.

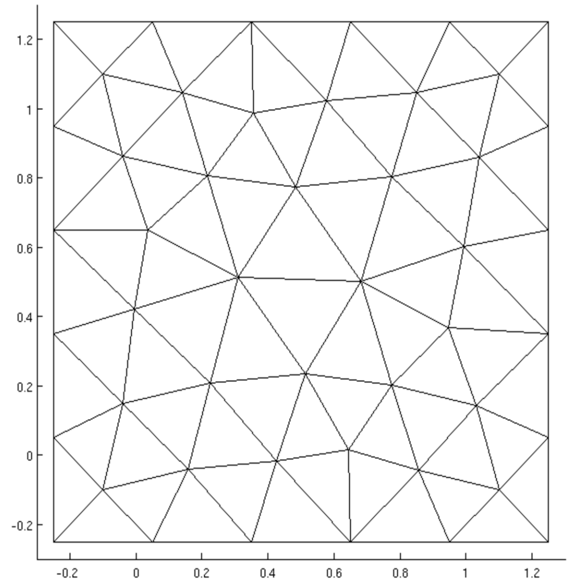

FIGURE 3. Initial mesh with 74 elements.

Below we show the adapted meshes and solutions obtained using our adaptive scheme at different times.

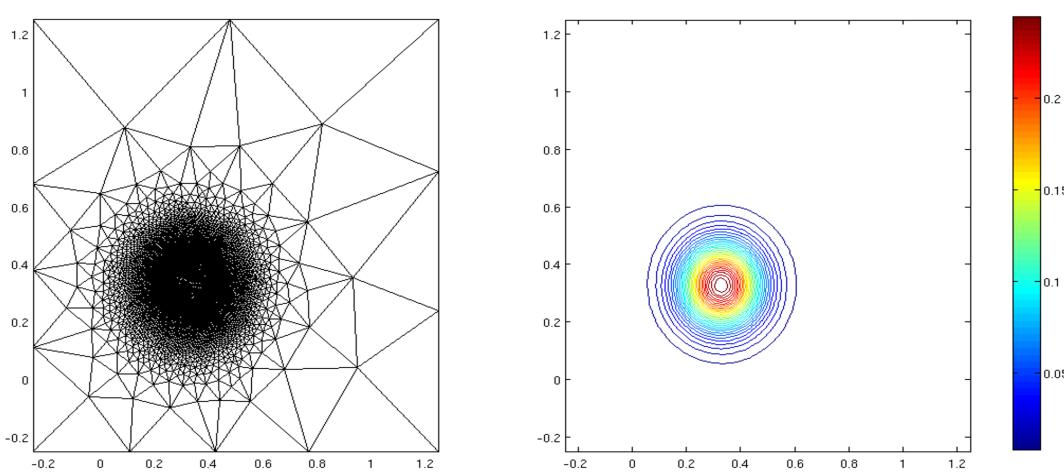

FIGURE 4. Adapted mesh and solution at time $t = 0.0667$ ($\tau_n = 0.0167$).

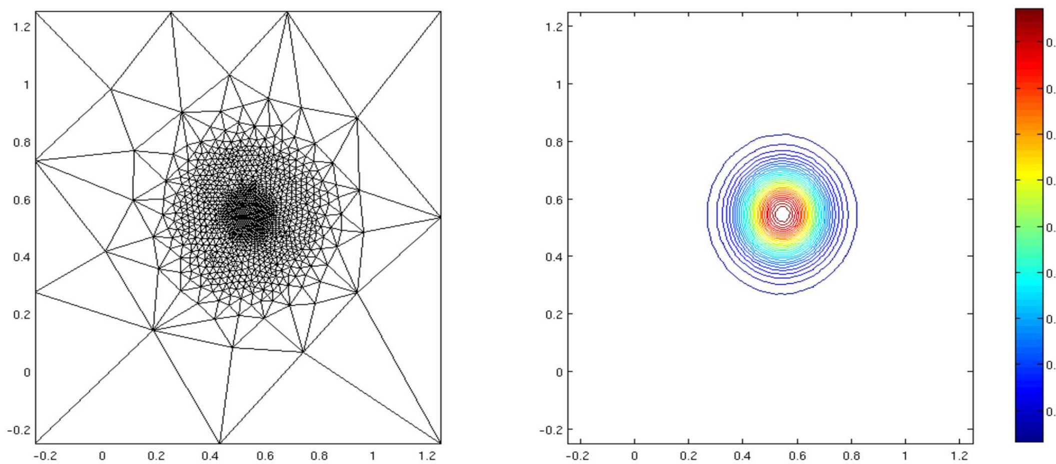

FIGURE 5. Adapted mesh and solution at time $t = 0.6167$ ($\tau_n = 0.0167$).

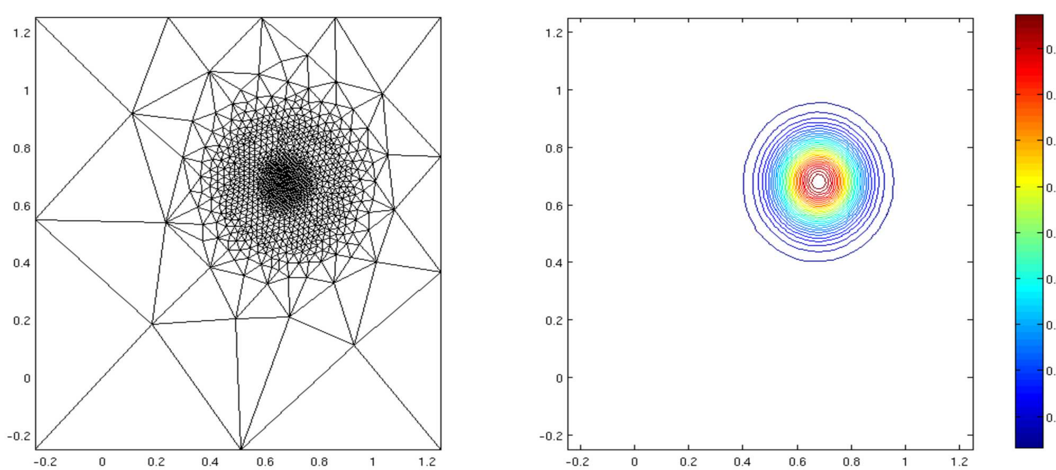

FIGURE 6. Adapted mesh and solution at time $t = 0.9500$ ($\tau_n = 0.0167$).

As we show in Figures 4–6, the evolution of the solution is captured by the adapted meshes generated with the use of our error estimator.

5.3. **An advection–diffusion problem.** This example consists of solving (P) with $\Omega :=]0, 200[\times]0, 50[$, $t \in]0, 150[$, $\boldsymbol{a} = (1, 0)$, $b = 0$ and $f = 1$ in the circle of center (25,25) and ratio 0.5. In the first case we consider $\varepsilon = 1$ and in the second one $\varepsilon = 10^{-5}$, which allows us to test the performance of our error estimator in an extreme case. We choose the boundary conditions as shown in Figure 7, with $u(0, \cdot) = 0$. The initial mesh is presented in Figure 8. Note that our adapted scheme produces, see Figures 9–11 and 12–14, meshes and solutions that are in perfect agreement with the physical phenomena even when the diffusion parameter is very small.

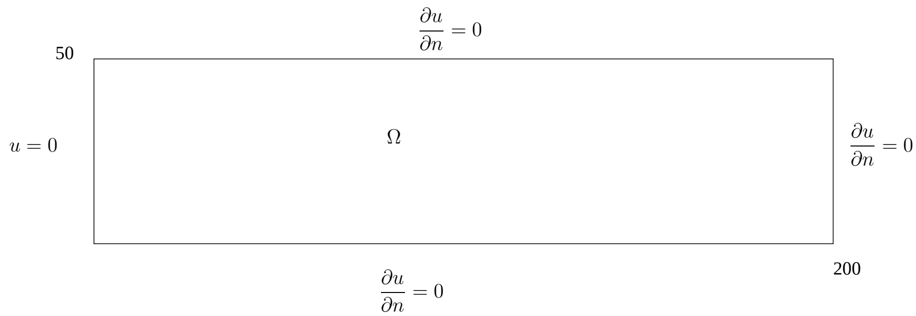

FIGURE 7. Boundary conditions.

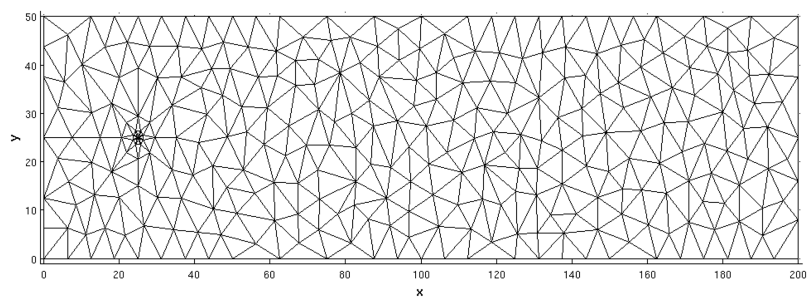

FIGURE 8. Initial mesh. 828 elements.

To solve the stabilized problem (SP), we consider $\tau_n = 3$ for $1 \leq n \leq N$. We show the solution at three different times for each value of ε.

Case $\varepsilon = 1$

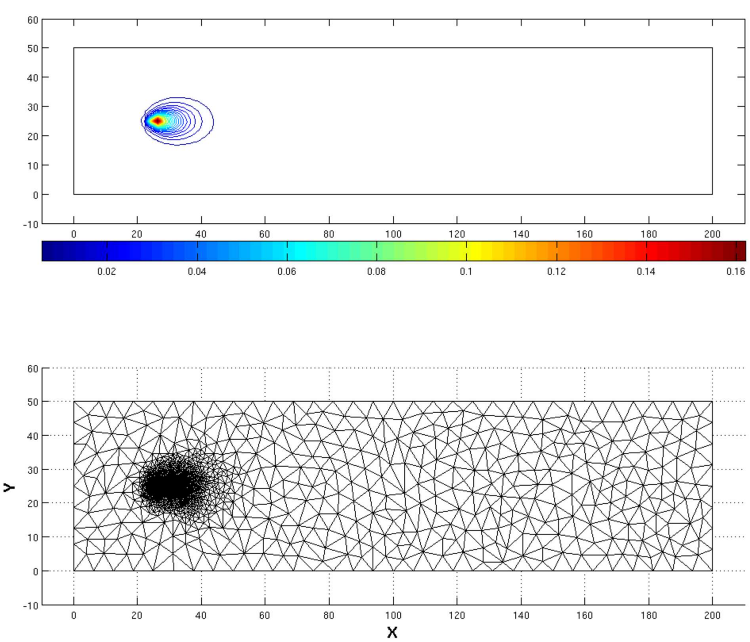

FIGURE 9. Solution and adapted mesh obtained by the adaptive process at time $t = 9$ ($\varepsilon = 1$).

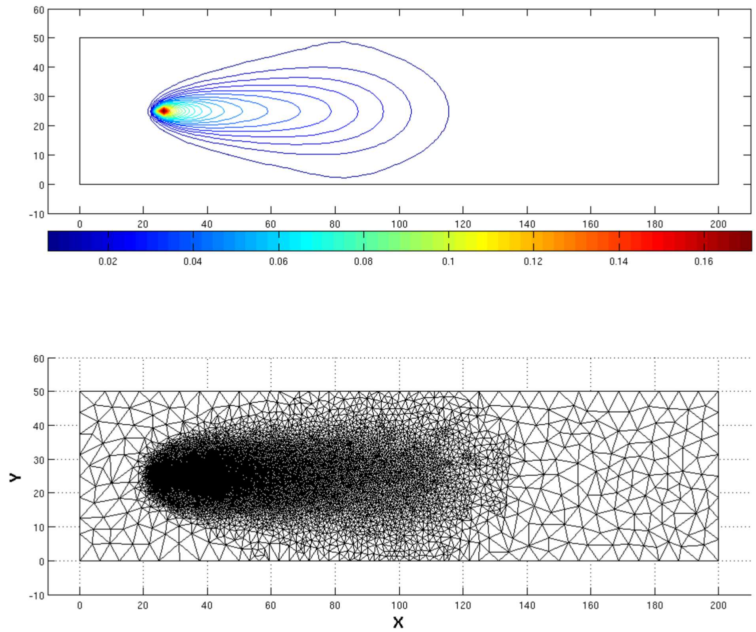

FIGURE 10. Solution and adapted mesh obtained by the adaptive process at time $t = 75$ ($\varepsilon = 1$).

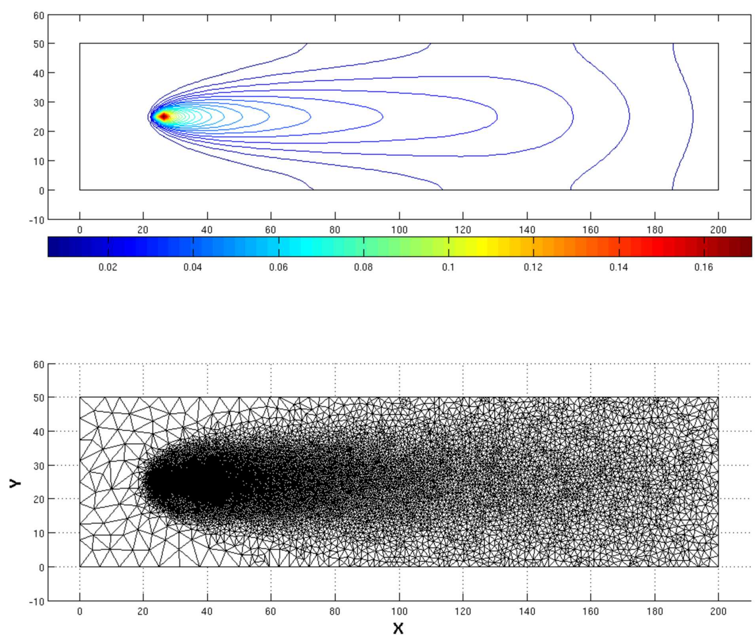

FIGURE 11. Solution and adapted mesh obtained by the adaptive process at time $t = 150$ ($\varepsilon = 1$).

Case $\varepsilon = 10^{-5}$

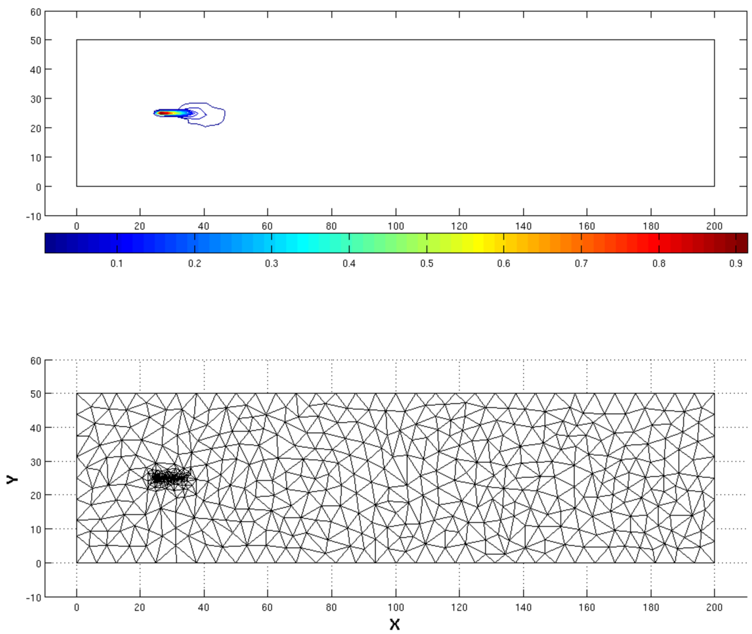

FIGURE 12. Solution and adapted mesh obtained by the adaptive process at time $t = 9$ ($\varepsilon = 10^{-5}$).

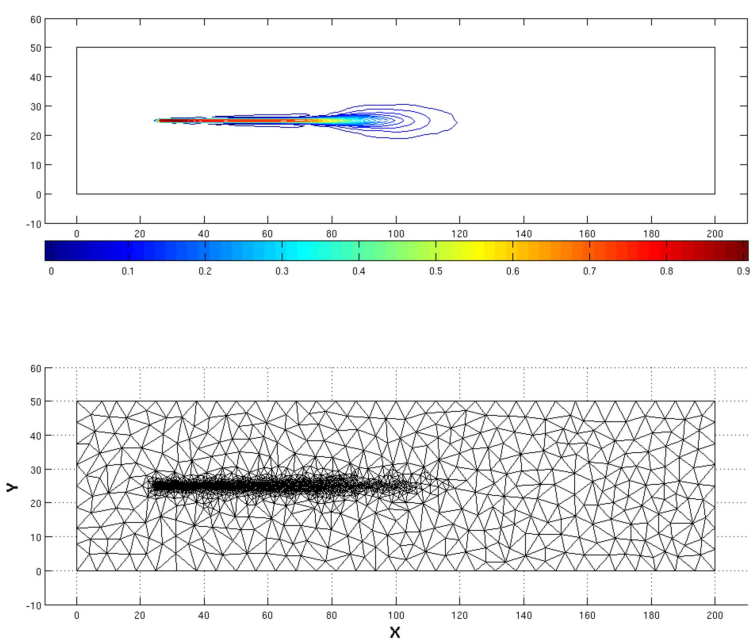

FIGURE 13. Solution and adapted mesh obtained by the adaptive process at time $t = 75$ ($\varepsilon = 10^{-5}$).

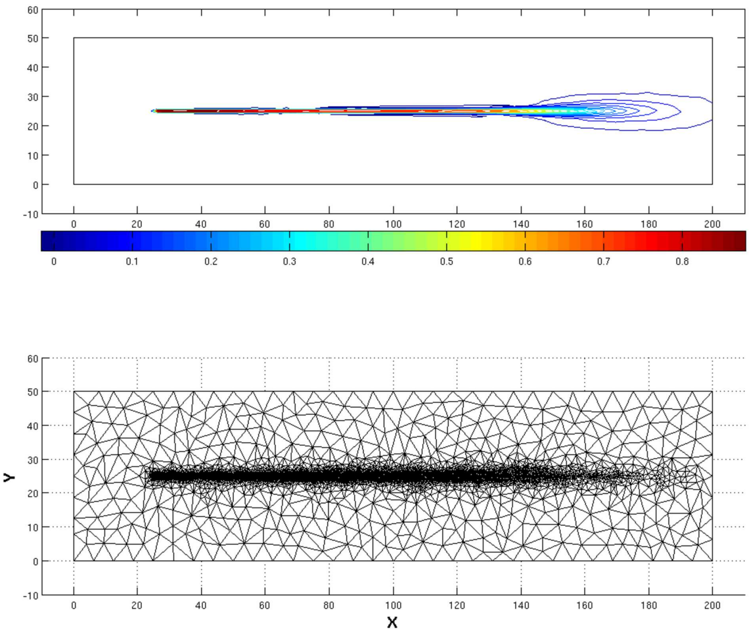

FIGURE 14. Solution and adapted mesh obtained by the adaptive process at time $t = 150$ ($\varepsilon = 10^{-5}$).

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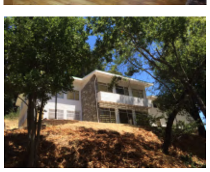

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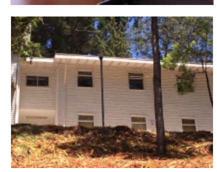

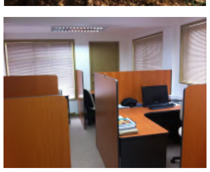

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