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# MATHEMATICAL AND NUMERICAL ANALYSIS OF A TRANSIENT EDDY CURRENT PROBLEM ARISING FROM ELECTROMAGNETIC FORMING

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**Abstract.** Electromagnetic forming is a type of high velocity cold forming process for electrically conductive metals. The aim of this paper is to introduce and analyze a weak formulation of the underlying transient axisymmetric eddy current problem governing such phenomenon. The resulting problem is degenerate parabolic with the time derivative acting on a moving subdomain. Because of this, we have to resort to regularization arguments in order to prove its well-posedness. We propose a finite element method in space combined with a backward Euler time scheme for its numerical solution. We obtain error estimates and report numerical results which allow us to assess the performance of the proposed method.

**Key words.** Electromagnetic forming, finite elements, eddy current, transient electromagnetics, degenerate parabolic problems, moving domains

**AMS subject classifications.** 78M10, 65N30

## 1. Introduction.

The Electromagnetic Forming (EMF) is a metal working process that relies on the use of electromagnetic forces to deform metallic workpieces at high speeds. A transient electric current is induced in a coil using a capacitor bank and high-speed switches. This current creates a magnetic field that penetrates the nearby conductive workpiece where an eddy current is generated. The magnetic field, together with the eddy current, produces Lorentz forces that drive the deformation of the workpiece [9, 12, 15]. The workpiece can be reshaped without any contact from a tool, although in some instances the piece is pressed against a die or former. The technique is sometimes called high velocity forming. The process works better with good electrical conductors such as copper or aluminum but it can be also adapted to work with poorer conductors such as steel.

The motion of the workpiece introduces two difficulties to the problem. First, the conducting domain changes along time, because the workpiece changes its position. Also the velocity in the workpiece produces currents that in principle should be added in the Ohm's law. The difficulties arising from this additional term have been studied in [3] on a fixed domain. However, in EMF, the current density induced from the velocity terms is not significant, so that it is typically neglected.

In [6], authors analyze a sliding mesh-mortar method for a two-dimensional model

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of electric engines. This model takes into account the motion of the rotor but the domain occupied by the moving part is always the same, which is not the case in the problem we are dealing with.

In this paper we focus on problems with cylindrical symmetry, which allows stating the eddy current equations in terms of the azimuthal component of a magnetic vector potential defined in a meridian section of the domain (see, for instance, [2]). In the thorough problem, the eddy current equations must be coupled with an adequate mechanical model for the deformation of the workpiece. In this paper we restrict our attention to the underlying electromagnetic model and take the motion of the workpiece as a data. This leads to considering a transient problem where the term involving the time derivative appears only in a part of the domain which changes with time. Because of this, neither the classical theory for abstract parabolic problems (see, for instance, [10]) nor results for degenerate parabolic problems on fixed domains (as those in [16]) can be used for the mathematical analysis. This is the reason why we resort to a regularization argument to prove the well-posedness of the continuous problem.

For the numerical solution, we discretize in space by standard finite elements. This leads to a singular differential algebraic system (see, for instance, [7]) which is proved to be well-posed using the same arguments as for the continuous problem. We prove error estimates for this semidiscrete approximation by adapting the classical theory (cf. [10]) to the degenerate character of the problem.

Next, we combine finite elements in space with a backward Euler time discretization. The resulting scheme avoids dealing with the additional terms arising from the Reynolds transport theorem. On the other hand, the spatial mesh does not need to be fitted to the workpiece, which allows using a fixed mesh for the whole process. All these features lead to a numerical scheme easy to implement computationally. We prove error estimates for the fully discretized scheme by adapting once more the classical theory to the degenerate character of the problem. These error estimates are valid provided some additional regularity holds for the source current density and the initial data, as well as for the solution.

The outline of this paper is as follows. In section 2, we describe the transient eddy current model and introduce a magnetic vector potential formulation under axisymmetric assumptions. In section 3, we state the weak formulation and prove its well-posedness. In section 4, we introduce the finite element space discretization and prove error estimates. In section 5, we propose a backward Euler scheme for time discretization and prove error estimates for the fully discrete problem. In section 6, we report some numerical tests which allow us to asses the performance of the proposed method. Finally, in an appendix, we give a sketch of the proof of a trace result on weighted Sobolev spaces that we have used to prove the well-posedness of the problem.

**2. Statement of the problem.** We are interested in computing the electromagnetic field produced by a coil in a cylindrical workpiece (see Figure 2.1 for a couple of examples). To ensure the cylindrical symmetry, we model the coil by several concentric rings with toroidal geometry, all carrying the same current intensity. On the other hand, to solve the electromagnetic model in a bounded domain, we introduce a three dimensional cylinder $\widetilde{\Omega}$ containing the coil and the workpiece with its boundary $\partial\widetilde{\Omega}$ sufficiently far from them. Then, because of the cylindrical symmetry, we are allowed to state the problem in a meridian section of $\widetilde{\Omega}$ which we denote by $\Omega$. We denote by $\Omega_t$ the meridian section of the workpiece at time $t$ and $\Omega_{\mathrm{S}} := \Omega_1 \cup \cdots \cup \Omega_m$, where $\Omega_k$ $(k = 1, \ldots, m)$ are the meridian sections of the turns of the coil. We assume that

$\Omega_t$ and the sets $\Omega_k$ are open and that $\overline{\Omega}_t \cap \overline{\Omega}_S = \emptyset$ for all $t$. Finally, we denote by $\Omega_t^A := \Omega \setminus (\overline{\Omega}_S \cup \overline{\Omega}_t)$ the section of the domain occupied by air, $\Gamma_0$ the intersection between $\partial\Omega$ and the symmetry axis ($r = 0$), and $\Gamma_D := \partial\Omega \setminus \Gamma_0$ (see Figure 2.2).

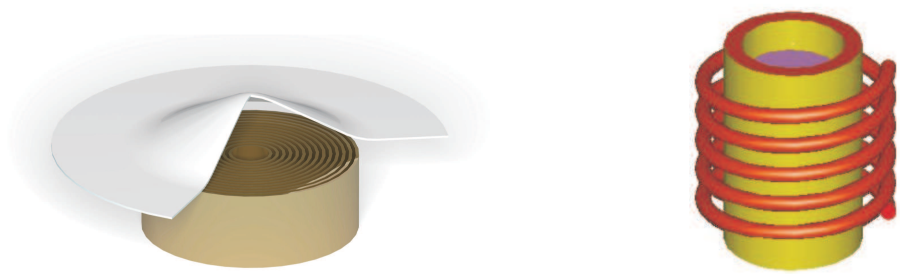

FIG. 2.1. *Sketch of 3D-domains of EMF systems.*

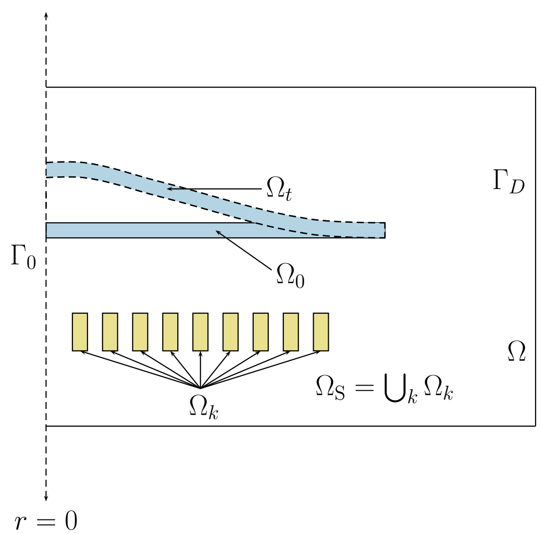


FIG. 2.2. *Sketch of the meridian section of the EMF system from Figure 2.1 (left).*

We will use standard notation in electromagnetism:

- $\boldsymbol{E}$ is the electric field,
- $\boldsymbol{B}$ is the magnetic induction,
- $\boldsymbol{H}$ is the magnetic field,
- $\boldsymbol{J}$ is the current density,
- $\mu$ is the magnetic permeability,
- $\sigma$ is the electric conductivity.

The magnetic permeability $\mu$ is taken as a positive constant in the whole domain. The conductivity $\sigma$ vanishes outside the workpiece. This piece can be made of different materials, each with a different conductivity. We will make this assumption more precise below; by the moment we just assume

$$\begin{aligned} 0 < \underline{\sigma} \leq \sigma \leq \overline{\sigma}, &\qquad \text{in the workpiece,} \\ \sigma = 0, &\qquad \text{outside the workpiece.} \end{aligned}$$

In this kind of problem, the electric displacement can be neglected in Ampère's law, leading to the so called eddy current model:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J}, \tag{2.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0}, \tag{2.2}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.3}$$

This system must be completed with the relations

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{2.4}$$

and

$$\boldsymbol{J} = \begin{cases} \sigma \boldsymbol{E} & \text{in the workpiece (unknown)}, \\ \boldsymbol{J}_{\mathrm{S}} & \text{in the coil (data)}, \\ \mathbf{0} & \text{in the air.} \end{cases} \tag{2.5}$$

Notice that since the source current density $\boldsymbol{J}_{\mathrm{S}}$ is taken as a given data, the conductivity $\sigma$ is taken as vanishing in the coil. The relation above can be written in a single equation as follows:

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}}.$$

We assume that all the physical quantities are independent of the azimuthal coordinate $\theta$ and that the source current density field has only azimuthal non-zero component, i.e.,

$$\boldsymbol{J}_{\mathrm{S}}(t, r, \theta, z) = J_{\mathrm{S}}(t, r, z)\boldsymbol{e}_{\theta}. \tag{2.6}$$

Then, proceeding as in [2] and [3], it can be shown that

$$\begin{aligned} \boldsymbol{H}(t, r, \theta, z) &= H_r(t, r, z)\boldsymbol{e}_r + H_z(t, r, z)\boldsymbol{e}_z, \\ \boldsymbol{B}(t, r, \theta, z) &= B_r(t, r, z)\boldsymbol{e}_r + B_z(t, r, z)\boldsymbol{e}_z, \\ \boldsymbol{E}(t, r, \theta, z) &= E(t, r, z)\boldsymbol{e}_{\theta}, \\ \boldsymbol{J}(t, r, \theta, z) &= J(t, r, z)\boldsymbol{e}_{\theta}. \end{aligned}$$

On the other hand, because of (2.3), we can introduce a magnetic vector potential $\boldsymbol{A}$ for $\boldsymbol{B}$, so that

$$\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A}. \tag{2.7}$$

According to [3], this vector potential can be chosen of the form

$$\boldsymbol{A}(t, r, \theta, z) = A(t, r, z)\boldsymbol{e}_{\theta} \tag{2.8}$$

and such that (cf. (2.2))

$$-\boldsymbol{E} = \frac{\partial \boldsymbol{A}}{\partial t}.$$

Therefore, the eddy current equations (2.1)–(2.4) can be rewritten in terms of this vector potential as follows:

$$\mathbf{curl}\left(\frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{A}\right) = \boldsymbol{J} = J\boldsymbol{e}_{\theta},$$

where

$$J = \begin{cases} -\sigma(t)\dfrac{\partial A}{\partial t} & \text{in } \Omega_t, \\ J_{\mathrm{S}} & \text{in } \Omega_{\mathrm{S}} \text{ (data)}, \\ 0 & \text{in } \Omega_t^{\mathrm{A}}. \end{cases} \tag{2.9}$$

Thus, we are led to the following parabolic-elliptic problem:

$$\sigma(t)\frac{\partial A}{\partial t}\boldsymbol{e}_\theta + \mathbf{curl}\left[\frac{1}{\mu}\,\mathbf{curl}\,(A\boldsymbol{e}_\theta)\right] = J_{\mathrm{S}}\boldsymbol{e}_\theta \quad \text{in } \Omega. \tag{2.10}$$

Finally we impose homogeneous Dirichlet boundary conditions for $A$ on $\Gamma_{\mathrm{D}}$, which makes sense provided the boundary $\partial\widetilde{\Omega}$ is sufficiently far from from $\Omega_t$ and $\Omega_{\mathrm{S}}$.

**3. Weak formulation.** The aim of this section is to introduce a weak formulation of the degenerate parabolic problem (2.10) and to prove that it has a unique solution. With this end, first we introduce the functional framework we will use.

Let $L^2_r(\Omega)$ be the weighted Lebesgue space of all measurable functions $Z$ defined in $\Omega$ such that

$$\|Z\|^2_{L^2_r(\Omega)} := \int_\Omega |Z|^2 \, r\,dr\,dz < \infty.$$

The weighted Sobolev space $H^k_r(\Omega)$ consists of all functions in $L^2_r(\Omega)$ whose derivatives up to the order $k$ are also in $L^2_r(\Omega)$. We define the norms and semi-norms of these spaces in the standard way. Let $L^2_{1/r}(\Omega)$ be the space of all measurable functions $Z$ defined in $\Omega$ such that

$$\|Z\|^2_{L^2_{1/r}(\Omega)} := \int_\Omega \frac{|Z|^2}{r}\,dr\,dz < \infty$$

and let

$$\widetilde{H}^1_r(\Omega) := \left\{ Z \in H^1_r(\Omega) :\ Z \in L^2_{1/r}(\Omega) \right\},$$

endowed with the norm defined by $\|Z\|^2_{\widetilde{H}^1_r(\Omega)} := \|Z\|^2_{H^1_r(\Omega)} + \|Z\|^2_{L^2_{1/r}(\Omega)}$. It is well-known (see, for instance, [5, 13]) that $Z\boldsymbol{e}_\theta \in [H^1(\widetilde{\Omega})]^3$ if and only if $Z \in \widetilde{H}^1_r(\Omega)$. Finally, let

$$\mathcal{V} := \left\{ Z \in \widetilde{H}^1_r(\Omega) :\ Z = 0 \text{ on } \Gamma_{\mathrm{D}} \right\}.$$

Since the domain $\Omega_t$ changes with time, we define a reference domain $\widehat{\Omega} \subset (0,\infty)\times\mathbb{R}$ and an application $\boldsymbol{X} \in \mathcal{C}([0,T]\times\overline{\widehat{\Omega}};[0,\infty)\times\mathbb{R})$ such that, for all $t\in[0,T]$,

$$\begin{aligned} \boldsymbol{X}_t :\ \overline{\widehat{\Omega}} &\longrightarrow \overline{\Omega}_t, \\ \widehat{\boldsymbol{x}} &\longmapsto \boldsymbol{X}(t,\widehat{\boldsymbol{x}}) \end{aligned} \tag{3.1}$$

is a one-to-one correspondence such that $\boldsymbol{X}_t(\widehat{\Omega}) = \Omega_t$ (see Figure 3.1). Moreover, we assume that $\boldsymbol{X}$ is sufficiently smooth with respect to space and time and that $\boldsymbol{X}_0$ is the identity, so that $\Omega_0 = \widehat{\Omega}$. Additional assumptions on $\boldsymbol{X}$ are given in Theorem 3.1 below.

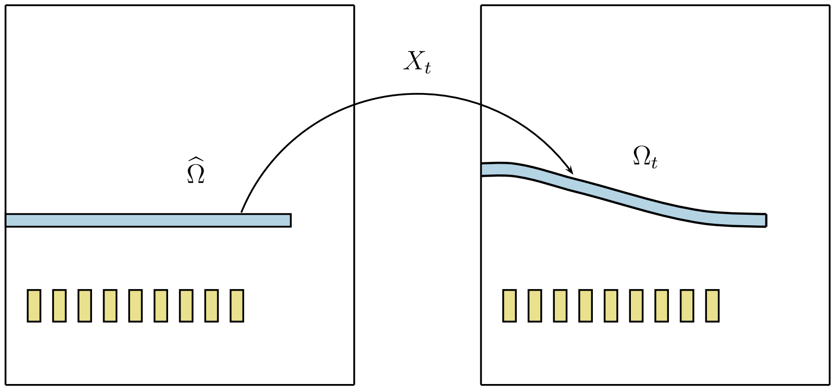


FIG. 3.1. *Reference Domain.*

A usual way to define $\boldsymbol{X}$ is through a vector field $\boldsymbol{v}$ which represents the velocity of the workpiece. In such a case, we define $t \mapsto \boldsymbol{X}_t$ as the solution to the following problem:

$$\frac{\partial \boldsymbol{X}_t}{\partial t}(\widehat{\boldsymbol{x}}) = \boldsymbol{v}(t, \boldsymbol{X}_t(\widehat{\boldsymbol{x}})),$$
$$\boldsymbol{X}_0(\widehat{\boldsymbol{x}}) = \widehat{\boldsymbol{x}}.$$

We also impose some geometrical constraints on $\widehat{\Omega}$. First, we assume that it is a Lipschitz bounded domain. Secondly, we consider that either $\overline{\widehat{\Omega}}$ does not intersect the axis $r = 0$ or there exists $a > 0$ such that the set $\{(r, z) \in \widehat{\Omega}\ \ 0 < r < a\}$ is a trapezoid with parallel sides aligned with the axis $r = 0$ (see [13] for further details).

On the other hand, the conductivity $\sigma$ is taken such that

$$\sigma(t, \boldsymbol{x}) = \widehat{\sigma}(\widehat{\boldsymbol{x}}), \tag{3.2}$$

where $\boldsymbol{x} = \boldsymbol{X}_t(\widehat{\boldsymbol{x}})$ and $\widehat{\sigma}$ is the conductivity in the reference domain $\widehat{\Omega}$, which is a given measurable function satisfying

$$0 < \underline{\sigma} \leq \widehat{\sigma}(\widehat{\boldsymbol{x}}) \leq \overline{\sigma}, \qquad \widehat{\boldsymbol{x}} \in \widehat{\Omega}.$$

From a physical point of view, this means that the conductivity of each material point remains constant along the process.

Next, let us introduce the non-cylindrical open subset of $(0, T) \times \Omega$,

$$Q := \{(t, \boldsymbol{x}) : \ \boldsymbol{x} \in \Omega_t,\ t \in (0, T)\}$$

and the following Banach spaces of functions defined in $Q$:

$$L^p_r(Q) := \left\{ \varphi : \ Q \to \mathbb{R} \text{ measurable with } \int_0^T \int_{\Omega_t} |\varphi|^p \ r\, dr\, dz\, dt < \infty \right\},$$
$$W^{1,p}_r(Q) := \left\{ \varphi \in L^p_r(Q) : \ \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial r}, \frac{\partial \varphi}{\partial z} \in L^p_r(Q) \right\}$$

$(1 \leq p < \infty)$, respectively endowed with the norms defined by

$$\|\varphi\|^p_{L^p_r(Q)} := \int_0^T \int_{\Omega_t} |\varphi|^p \; r\,dr\,dz\,dt,$$
$$\|\varphi\|^p_{W^{1,p}_r(Q)} := \|\varphi\|^p_{L^p_r(Q)} + \left\|\frac{\partial \varphi}{\partial t}\right\|^p_{L^p_r(Q)} + \left\|\frac{\partial \varphi}{\partial r}\right\|^p_{L^p_r(Q)} + \left\|\frac{\partial \varphi}{\partial z}\right\|^p_{L^p_r(Q)}.$$

Moreover we denote $H^1_r(Q) := W^{1,2}_r(Q)$.

Finally, for any bounded open set $G \subset \mathbb{R}^n$, we denote by $\mathcal{C}^1(\overline{G})$ the set of functions in $\mathcal{C}(\overline{G}) \cap \mathcal{C}^1(G)$ such that all its first-order partial derivatives have continuous extensions to all of $\overline{G}$.

Now, we are in a position to write a weak formulation of (2.10) and to prove that it is well-posed. For this purpose, let us multiply (2.10) by a test vector field $Z\boldsymbol{e}_\theta$ with $Z \in \mathcal{V}$, integrate over $\Omega$, and use a Green's formula, to obtain

$$\int_{\Omega_t} \sigma \frac{\partial A}{\partial t} Z\, r\,dr\,dz + a(A,Z) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z\, r\,dr\,dz \qquad \forall Z \in \mathcal{V},$$

where

$$a(A,Z) := \int_\Omega \frac{1}{\mu}\, \mathbf{curl}\,(A\boldsymbol{e}_\theta) \cdot \mathbf{curl}\,(Z\boldsymbol{e}_\theta)\; r\,dr\,dz.$$

It is shown in [11, Prop. 2.1 & 3.1] that $a$ is a $\mathcal{V}$-elliptic bilinear form; namely, there exists $\alpha > 0$ such that

$$a(Z,Z) \geq \alpha\, \|Z\|^2_{\widetilde{H}^1_r(\Omega)} \qquad \forall Z \in \mathcal{V}.$$

Thus we are led to the following weak problem. (For the sake of notational compactness, here and thereafter $\partial_t$ will be often used to denote derivative with respect to time.)

Problem 1. *Given $J_{\mathrm{S}} \in L^2(0,T; L^2_r(\Omega_{\mathrm{S}}))$ and $A^0 \in L^2_r(\Omega_0)$, find $A \in L^2(0,T;\mathcal{V})$ with $\partial_t A \in L^2_r(Q)$, such that*

$$\int_{\Omega_t} \sigma\,(\partial_t A)\, Z\, r\,dr\,dz + a(A,Z) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z\, r\,dr\,dz \quad \forall Z \in \mathcal{V}, \quad a.e.\ t \in [0,T],$$
$$A(0) = A^0 \qquad in\ \Omega_0.$$

*Remark* 3.1. Since $A \in L^2(0,T;\mathcal{V})$ and $\partial_t A \in L^2_r(Q)$, it follows that $A \in H^1_r(Q)$. From a trace result (see Lemma A.1 in the appendix), this implies that $A|_{\{0\}\times\Omega_0} \in L^2_r(\{0\}\times\Omega_0) \simeq L^2_r(\Omega_0)$. Thus the initial condition in Problem 1 makes sense.

Notice that Problem 1 is a degenerate parabolic problem, because the term including the time derivative of $A$ is only defined in the moving domain $\Omega_t$. Our next goal is to show that this degenerate problem has a unique solution. With this aim, we will use the following form of the Reynolds transport theorem. The proof of a closely related version of this theorem can be found in [4].

Theorem 3.1. *Let $\boldsymbol{X} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; [0,\infty) \times \mathbb{R})$ and $\boldsymbol{X}_t$ be defined as in (3.1). Let us assume that these mappings satisfy the following hypotheses:*

*i) $\forall t \in [0,T]$, $\boldsymbol{X}_t : \overline{\widehat{\Omega}} \to \overline{\Omega}_t$ is a one-to-one correspondence and $\boldsymbol{X}_t(\widehat{\Omega}) = \Omega_t$;*

*ii)* $\forall t \in [0,T]$, $\boldsymbol{X}(t,\widehat{\boldsymbol{x}})$ *lies on the axis* $r=0$ *if only if* $\widehat{\boldsymbol{x}}$ *lies on the same axis;*

*iii)* $\boldsymbol{X} \in \mathcal{C}^1([0,T];[\mathcal{C}^1(\overline{\widehat{\Omega}})]^2)$*;*

*iv)* $\det(D_{\widehat{\boldsymbol{x}}}\boldsymbol{X})(t,\widehat{\boldsymbol{x}}) > 0 \quad \forall (t,\widehat{\boldsymbol{x}}) \in [0,T]\times\overline{\widehat{\Omega}}$*;*

*Let* $\boldsymbol{v} = v_r\boldsymbol{e}_r + v_z\boldsymbol{e}_z$ *be defined for all* $t\in[0,T]$ *and* $\boldsymbol{x}\in\overline{\Omega}_t$ *by*

$$\boldsymbol{v}(t,\boldsymbol{x}) := \frac{\partial \boldsymbol{X}}{\partial t}(t,\widehat{\boldsymbol{x}}), \qquad \text{with } \widehat{\boldsymbol{x}} \in \overline{\widehat{\Omega}}: \quad \boldsymbol{x} = \boldsymbol{X}_t(\widehat{\boldsymbol{x}}). \tag{3.3}$$

*Let* $\sigma$ *be given by (3.2). Then, for all* $\varphi \in W^{1,1}_r(Q)$*, there holds*

$$\frac{d}{dt}\int_{\Omega_t}\sigma\varphi\, r\,dr\,dz = \int_{\Omega_t}\sigma\frac{\partial\varphi}{\partial t}\, r\,dr\,dz + \int_{\Omega_t}\sigma\varphi\operatorname{div}\boldsymbol{v}\, r\,dr\,dz + \int_{\Omega_t}\sigma\,\mathbf{grad}\,\varphi\cdot\boldsymbol{v}\, r\,dr\,dz$$

*in* $\mathcal{D}'((0,T))$ *and a.e. in* $[0,T]$.

*Remark* 3.2. Assumptions (i)–(iv) together with our hypothesis on $\widehat{\Omega}$ imply that

$$\operatorname{div}\boldsymbol{v} := \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} \tag{3.4}$$

is bounded in $Q$ (see [4, Remark 4]). Therefore, the above theorem is an immediate consequence of the Reynolds transport theorem proved in [4].

*Remark* 3.3. Assumptions (i) and (ii) are natural when $\boldsymbol{X}$ is the description in a meridian plane of a three-dimensional axisymmetric motion keeping invariant the azimuthal coordinate. Moreover, in such a case, assumptions (iii)–(iv) are satisfied provided the three-dimensional motion is sufficiently smooth (see [4, Remark 5]). In this case, $\operatorname{div}\boldsymbol{v}$ actually corresponds to the divergence of the velocity associated to the three-dimensional motion, written in polar coordinates. Hence the boundedness of $\operatorname{div}\boldsymbol{v}$ is an immediate consequence of the smoothness of the three-dimensional motion.

From now on, we assume that $\boldsymbol{X}$ satisfies assumptions (i)–(iv) from Theorem 3.1 and $\boldsymbol{v}$ is defined by (3.3). In the sequel, we will also use the $t$-dependent bilinear form

$$c(t,Y,Z) := -\int_{\Omega_t}\sigma YZ\operatorname{div}\boldsymbol{v}\, r\,dr\,dz - \int_{\Omega_t}\sigma\,\mathbf{grad}(YZ)\cdot\boldsymbol{v}\, r\,dr\,dz, \quad Y,Z\in\widetilde{H}^1_r(\Omega),$$

which is related with the last two terms from the Reynolds transport formula. The following Gårding-like inequality holds true.

Lemma 3.2. *There exist* $\lambda > 0$ *such that*

$$a(Z,Z) \pm \frac{1}{2}c(t,Z,Z) + \lambda\int_{\Omega_t}\sigma\,|Z|^2\, r\,dr\,dz \geq \frac{\alpha}{2}\,\|Z\|^2_{\widetilde{H}^1_r(\Omega)} \quad \forall Z\in\widetilde{H}^1_r(\Omega) \quad \forall t\in(0,T).$$

*Proof.* By using Young's inequality, we can write for any $\delta > 0$

$$\begin{aligned}|c(t,Z,Z)| &= \left|\int_{\Omega_t}\sigma Z^2\operatorname{div}\boldsymbol{v}\, r\,dr\,dz + 2\int_{\Omega_t}\sigma Z\,\mathbf{grad}\,Z\cdot\boldsymbol{v}\, r\,dr\,dz\right| \\ &\leq \left(\|\operatorname{div}\boldsymbol{v}\|_\infty + \frac{\overline{\sigma}\,\|\boldsymbol{v}\|^2_\infty}{\delta}\right)\int_{\Omega_t}\sigma\,|Z|^2\, r\,dr\,dz + \delta\,\|Z\|^2_{H^1_r(\Omega_t)}.\end{aligned}$$

Thus, we conclude the proof by using the ellipticity of $a$ and taking $\delta = \alpha$. ∎

Next result is the first step to show that Problem 1 is well-posed. Here and thereafter, $C$ will denote a constant not necessarily the same at each occurrence, but always independent of the data $J_{\rm S}$ and $A^0$ of Problem 1. Moreover, in the forthcoming sections, $C$ will also be always independent of the discretization parameters $h$ and $\Delta t$.

THEOREM 3.3. *If* $J_{\rm S} \in H^1(0,T;L^2_r(\Omega_{\rm S}))$ *and* $A^0 \in \widetilde{H}^1_r(\Omega_0)$*, then there exists a solution to Problem 1 which satisfies* $A \in L^\infty(0,T;\mathcal{V})$ *and*

$$\|\partial_t A\|_{L^2_r(Q)} + \|A\|_{L^\infty(0,T;\mathcal{V})} \le C\left[\|A^0\|_{\widetilde{H}^1_r(\Omega_0)} + \|J_{\rm S}\|_{H^1(0,T;L^2_r(\Omega_{\rm S}))}\right]. \tag{3.5}$$

*Proof.* We proceed by space discretization and passing to the limit. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of the space $\mathcal{V}$. Consider the family of finite-dimensional subspaces $\mathcal{V}_N := \langle \phi_1, \cdots, \phi_N\rangle$. The first step of the proof is to look for a function of the form $A_N(t,r,z) = \sum_{j=1}^N A_{jN}(t)\phi_j(r,z)$ satisfying

$$\int_{\Omega_t} \sigma\,(\partial_t A_N)\,\phi_i\, r\,dr\,dz + a(A_N,\phi_i) = \int_{\Omega_{\rm S}} J_{\rm S}\phi_i\, r\,dr\,dz, \qquad 1\le i\le N, \quad \text{a.e. } t\in[0,T], \tag{3.6}$$

$$A_N(0)|_{\Omega_0} = A^0_N|_{\Omega_0}. \tag{3.7}$$

The initial condition $A^0_N \in \mathcal{V}_N$ must be chosen so that

$$A^0_N|_{\Omega_0} \xrightarrow{N} A^0 \qquad \text{in } L^2_r(\Omega_0), \tag{3.8}$$

and

$$\|A^0_N\|_{\widetilde{H}^1_r(\Omega)} \le C\,\|A^0\|_{\widetilde{H}^1_r(\Omega_0)}. \tag{3.9}$$

To obtain $A^0_N$ we may proceed as follows. Let $\bar{A}^0 \in \mathcal{V}$ be an extension of $A^0$ to $\Omega$ satisfying

$$\|\bar{A}^0\|_{\widetilde{H}^1_r(\Omega)} \le C\,\|A^0\|_{\widetilde{H}^1_r(\Omega_0)}. \tag{3.10}$$

Such an $\bar{A}^0$ can be obtained for instance by means of a Nikolskii extension operator as in [13, Lemma 4.1]. (Here we make use of the geometric assumptions on the domain $\widehat{\Omega}$.) We write $\bar{A}^0$ in the Hilbert basis, $\bar{A}^0 = \sum_{j=1}^\infty b_j\phi_j$, and define

$$A^0_N := \sum_{j=1}^N b_j\phi_j. \tag{3.11}$$

Hence, $\|A^0_N - \bar{A}^0\|_{\widetilde{H}^1_r(\Omega)} \xrightarrow{N} 0$, which implies (3.8) and together with (3.10) lead to (3.9).

Since problem (3.6)–(3.7) is degenerate, to prove the existence of a solution we introduce a parabolic regularization as follows: For small $\varepsilon > 0$, we replace (3.6)–(3.7) by the "approximate" parabolic problem

$$\int_{\Omega_t} \sigma\,(\partial_t A^\varepsilon_N)\,\phi_i\, r\,dr\,dz + \varepsilon\int_{\Omega} (\partial_t A^\varepsilon_N)\,\phi_i\, r\,dr\,dz + a(A^\varepsilon_N,\phi_i) = \int_{\Omega_{\rm S}} J_{\rm S}\phi_i\, r\,dr\,dz, \quad i=1,\dots,N, \quad \text{a.e. } t\in[0,T], \tag{3.12}$$

$$A^\varepsilon_N(0) = A^0_N \quad \text{in } \Omega. \tag{3.13}$$

To obtain the matrix form of this problem, we write $A_N^\varepsilon(t) = \sum_{j=1}^N A_{jN}^\varepsilon(t)\phi_j$ and define $\boldsymbol{A}_N^\varepsilon(t) := (A_{jN}^\varepsilon(t))_{1\le j\le N}$,

$$\begin{aligned}
\boldsymbol{F}_N(t) &:= (F_{iN}(t))_{1\le i\le N}\,, && F_{iN}(t) := \int_{\Omega_\mathrm{S}} J_\mathrm{S}(t)\phi_i\, r\, dr\, dz, \quad 1\le i\le N,\\
\boldsymbol{\mathcal{K}} &:= (\mathcal{K}_{ij})_{1\le i,j\le N}\,, && \mathcal{K}_{i,j} := a(\phi_i,\phi_j), \quad 1\le i,j\le N,\\
\boldsymbol{\mathcal{M}}(t) &:= (\mathcal{M}_{ij}(t))_{1\le i,j\le N}\,, && \mathcal{M}_{i,j}(t) := \int_{\Omega_t} \sigma\phi_i\phi_j\, r\, dr\, dz, \quad 1\le i,j\le N,\\
\boldsymbol{\mathcal{N}} &:= (\mathcal{N}_{ij})_{1\le i,j\le N}\,, && \mathcal{N}_{i,j} := \int_{\Omega} \phi_i\phi_j\, r\, dr\, dz, \quad 1\le i,j\le N.
\end{aligned}$$

Finally, let $\boldsymbol{b}_N := (b_i)_{1\le i\le N}$, with $b_i$ as in (3.11). Then, problem (3.12)–(3.13) reads as follows: Find $\boldsymbol{A}_N^\varepsilon : [0,T] \longrightarrow \mathbb{R}^N$ such that

$$[\boldsymbol{\mathcal{M}}(t) + \varepsilon\boldsymbol{\mathcal{N}}]\frac{d}{dt}\boldsymbol{A}_N^\varepsilon(t) + \boldsymbol{\mathcal{K}}\boldsymbol{A}_N^\varepsilon(t) = \boldsymbol{F}_N(t), \tag{3.14}$$

$$\boldsymbol{A}_N^\varepsilon(0) = \boldsymbol{b}_N. \tag{3.15}$$

Since $\boldsymbol{\mathcal{M}}(t)$ is symmetric positive semidefinite and $\boldsymbol{\mathcal{N}}$ is symmetric positive definite, we have that $\boldsymbol{\mathcal{M}}(t) + \varepsilon\boldsymbol{\mathcal{N}}$ is invertible and this problem has a unique solution in $W^{1,1}(0,T;\mathbb{R}^N)$. Furthermore, $\boldsymbol{A}_N^\varepsilon \in H^1(0,T;\mathbb{R}^N)$, because

$$\begin{aligned}
\|\partial_t\boldsymbol{A}_N^\varepsilon\|_{L^2(0,T;\mathbb{R}^N)} &\le \operatorname*{ess\,sup}_{0\le t\le T}\left|[\boldsymbol{\mathcal{M}}(t)+\varepsilon\boldsymbol{\mathcal{N}}]^{-1}\right|\left[\|\boldsymbol{F}_N\|_{L^2(0,T;\mathbb{R}^N)} + \|\boldsymbol{\mathcal{K}}\boldsymbol{A}_N^\varepsilon\|_{L^2(0,T;\mathbb{R}^N)}\right]\\
&\le \frac{1}{\varepsilon}\left|\boldsymbol{\mathcal{N}}^{-1}\right|\left[\|\boldsymbol{F}_N\|_{L^2(0,T;\mathbb{R}^N)} + |\boldsymbol{\mathcal{K}}|\,\|\boldsymbol{A}_N^\varepsilon\|_{L^2(0,T;\mathbb{R}^N)}\right] < \infty,
\end{aligned}$$

where $|\cdot|$ denotes the matrix norm induced by the Euclidean norm in $\mathbb{R}^N$.

In order to pass to the limit as $\varepsilon$ goes to 0, we need a priori estimates. With this end, we multiply (3.12) by $\partial_t A_{iN}^\varepsilon$ and sum up from $i = 1$ to $N$ to obtain

$$\int_{\Omega_t}\sigma\left|\partial_t A_N^\varepsilon\right|^2\, r\, dr\, dz + \varepsilon\int_\Omega \left|\partial_t A_N^\varepsilon\right|^2\, r\, dr\, dz + \frac{1}{2}\frac{d}{dt}a(A_N^\varepsilon, A_N^\varepsilon) = \int_{\Omega_\mathrm{S}} J_\mathrm{S}\left(\partial_t A_N^\varepsilon\right)\, r\, dr\, dz.$$

Integrating in time from 0 to $\tau$ ($0 < \tau \le T$) and using an integration by parts formula on the right-hand side, we deduce

$$\begin{aligned}
&\int_0^\tau\int_{\Omega_t}\sigma\left|\partial_t A_N^\varepsilon(t)\right|^2\, r\, dr\, dz\, dt + \varepsilon\int_0^\tau\int_\Omega\left|\partial_t A_N^\varepsilon(t)\right|^2\, r\, dr\, dz\, dt + \frac{1}{2}a(A_N^\varepsilon(\tau), A_N^\varepsilon(\tau))\\
&\quad = \frac{1}{2}a(A_N^0, A_N^0) + \int_{\Omega_\mathrm{S}} J_\mathrm{S}(\tau)A_N^\varepsilon(\tau)\, r\, dr\, dz - \int_{\Omega_\mathrm{S}} J_\mathrm{S}(0)A_N^0\, r\, dr\, dz\\
&\quad\quad - \int_0^\tau\int_{\Omega_\mathrm{S}}\left[\partial_t J_\mathrm{S}(t)\right]A_N^\varepsilon(t)\, r\, dr\, dz\, dt.
\end{aligned}$$

Hence, the ellipticity of $a$, (3.9), and a Young's inequality lead to

$$\begin{aligned}
&\int_0^\tau\int_{\Omega_t}\sigma\left|\partial_t A_N^\varepsilon(t)\right|^2\, r\, dr\, dz\, dt + \varepsilon\int_0^\tau\int_\Omega\left|\partial_t A_N^\varepsilon(t)\right|^2\, r\, dr\, dz\, dt + \frac{\alpha}{4}\left\|A_N^\varepsilon(\tau)\right\|^2_{\widetilde{H}_r^1(\Omega)}\\
&\qquad \le C\left[\left\|A^0\right\|^2_{\widetilde{H}_r^1(\Omega_0)} + \|J_\mathrm{S}\|^2_{H^1(0,T;L_r^2(\Omega_\mathrm{S}))} + \int_0^\tau \left\|A_N^\varepsilon(t)\right\|^2_{\widetilde{H}_r^1(\Omega)}\, dt\right],
\end{aligned}$$

whence, by applying the Gronwall's lemma (see, for instance, [14, Lemma 1.4.1]), it follows that

$$\int_0^\tau \int_{\Omega_t} \sigma \left|\partial_t A_N^\varepsilon(t)\right|^2 \, r\,dr\,dz\,dt + \varepsilon \int_0^\tau \int_{\Omega} \left|\partial_t A_N^\varepsilon(t)\right|^2 \, r\,dr\,dz\,dt + \frac{\alpha}{4}\left\|A_N^\varepsilon(\tau)\right\|^2_{\widetilde{H}^1_r(\Omega)} \leq C\left[\left\|A^0\right\|^2_{\widetilde{H}^1_r(\Omega_0)} + \left\|J_{\mathrm{S}}\right\|^2_{H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))}\right]. \tag{3.16}$$

Thus, we have proved the following a priori estimates:

- $\partial_t A_N^\varepsilon$ is bounded in $L^2_r(Q)$,
- $\sqrt{\varepsilon}\partial_t A_N^\varepsilon$ is bounded in $L^2(0,T;L^2_r(\Omega))$,
- $A_N^\varepsilon$ is bounded in $L^\infty(0,T;\mathcal{V})$.

Therefore, for fixed $N$, there exists $A_N \in L^\infty(0,T;\mathcal{V})$ with $\partial_t A_N \in L^2_r(Q)$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to 0 such that

- $\partial_t A_N^{\varepsilon_n} \rightharpoonup \partial_t A_N$ weakly in $L^2_r(Q)$,
- $\sqrt{\varepsilon_n}\partial_t A_N^{\varepsilon_n} \rightharpoonup 0$ weakly in $L^2(0,T;L^2_r(\Omega))$,
- $A_N^{\varepsilon_n} \rightharpoonup A_N$ weakly-star in $L^\infty(0,T;\mathcal{V})$.

In particular, this implies that $A_N(0)|_{\Omega_0} = \lim_{n\to\infty}\left[A_N^{\varepsilon_n}(0)|_{\Omega_0}\right] = A_N^0|_{\Omega_0}$ weakly in $L^2_r(\Omega_0)$, so that $A_N$ satisfies the initial condition (3.7). Moreover, one can pass to the limit in (3.12) for $\varepsilon = \varepsilon_n$ as $n \to \infty$ and show that $A_N$ satisfies (3.6), too.

Furthermore, it is also possible to pass to the limit in estimate (3.16) (see, for instance, [8, Prop. III.5]) to obtain

$$\int_0^\tau \int_{\Omega_t} \sigma \left|\partial_t A_N(t)\right|^2 \, r\,dr\,dz\,dt + \frac{\alpha}{4}\left\|A_N(\tau)\right\|^2_{\widetilde{H}^1_r(\Omega)} \leq C\left[\left\|A^0\right\|^2_{\widetilde{H}^1_r(\Omega_0)} + \left\|J_{\mathrm{S}}\right\|^2_{H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))}\right] \qquad \text{a.e. } \tau\in[0,T]. \tag{3.17}$$

The above estimate allows us to conclude that there exists $A \in L^\infty(0,T;\mathcal{V})$ with $\partial_t A \in L^2_r(Q)$ and a subsequence of $\{A_N\}$ still denoted in the same way such that

- $A_N \rightharpoonup A$ weakly-star in $L^\infty(0,T;\mathcal{V})$,
- $\partial_t A_N \rightharpoonup \partial_t A$ weakly in $L^2_r(Q)$.

In particular, this and (3.8) imply that $A(0)|_{\Omega_0} = \lim_{N\to\infty}\left[A_N(0)|_{\Omega_0}\right] = A^0$, where the limit is weak in $L^2_r(\Omega_0)$, so that $A$ satisfies the initial condition from Problem 1.

Next, take any fixed $i \in \mathbb{N}$. Then, for $N \geq i$, $\phi_i \in \mathcal{V}_N$ and we can pass to the limit in (3.6) as $N \to \infty$ to obtain

$$\int_{\Omega_t} \sigma\left(\partial_t A\right)\phi_i \, r\,dr\,dz + a(A,\phi_i) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}\phi_i \, r\,dr\,dz \quad \forall i\in\mathbb{N}, \quad \text{a.e. } t\in[0,T].$$

Hence, since the linear combinations of functions $\phi_i$ are dense in $\mathcal{V}$, we deduce the first equation in Problem 1. Finally, passing to the limit as $N \to \infty$ in estimate (3.17), we obtain (3.5). □

In principle, estimate (3.5) only holds for a solution obtained by the regularization procedure used in this theorem. Thus, we cannot derive uniqueness of solution from this inequality. In what follows we prove a stability estimate valid for any solution to Problem 1, which allows us to conclude that this is a well-posed problem.

THEOREM 3.4. *Problem 1 has at most one solution $A$ and the following a priori* *estimate holds:*

$$\sup_{0\le t\le T}\left[\int_{\Omega_t}\sigma\left|A(t)\right|^2\, r\,dr\,dz\right]^{1/2}+\|A\|_{L^2(0,T;\mathcal{V})} \le C\left[\left\|A^0\right\|_{L^2_r(\Omega_0)}+\|J_{\mathrm{S}}\|_{L^2(0,T;L^2_r(\Omega_{\mathrm{S}}))}\right]. \tag{3.18}$$

*Proof.* The uniqueness follows immediately from (3.18). Thus, we only have to prove this estimate.

Let $A$ be a solution to Problem 1. Taking $Z=A(t)$ in the first equation of this problem, we obtain

$$\int_{\Omega_t}\sigma\left[\partial_t A(t)\right]A(t)\,r\,dr\,dz+a(A(t),A(t))=\int_{\Omega_{\mathrm{S}}}J_{\mathrm{S}}(t)A(t)\,r\,dr\,dz \quad \text{a.e. } t\in[0,T].$$

Hence, from Theorem 3.1,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\sigma\left|A(t)\right|^2\, r\,dr\,dz+\frac{1}{2}c(t,A(t),A(t))+a(A(t),A(t))=\int_{\Omega_{\mathrm{S}}}J_{\mathrm{S}}(t)A(t)\,r\,dr\,dz.$$

Next, we use Lemma 3.2 and a Young's inequality to write

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\sigma\left|A(t)\right|^2\, r\,dr\,dz+\frac{\alpha}{4}\left\|A(t)\right\|^2_{\widetilde{H}^1_r(\Omega)} \le\frac{1}{\alpha}\left\|J_{\mathrm{S}}(t)\right\|^2_{L^2_r(\Omega_{\mathrm{S}})}+\lambda\int_{\Omega_t}\sigma\left|A(t)\right|^2\, r\,dr\,dz. \tag{3.19}$$

Therefore, by applying the Gronwall's lemma we conclude that

$$\int_{\Omega_t}\sigma\left|A(t)\right|^2\, r\,dr\,dz\le C\left[\int_{\Omega_0}\sigma\left|A^0\right|^2\, r\,dr\,dz+\int_0^t\left\|J_{\mathrm{S}}(s)\right\|^2_{L^2_r(\Omega_{\mathrm{S}})}\,ds\right],$$

where we have also used the initial condition of Problem 1. Finally, we integrate (3.19) in time from 0 to $\tau$ ($0\le\tau\le T$) and use the above estimate to derive

$$\frac{1}{2}\int_{\Omega_\tau}\sigma\left|A(\tau)\right|^2\, r\,dr\,dz+\frac{\alpha}{4}\int_0^\tau\left\|A(t)\right\|^2_{\widetilde{H}^1_r(\Omega)}\,dt \le C\left[\int_{\Omega_0}\sigma\left|A^0\right|^2\, r\,dr\,dz+\int_0^T\left\|J_{\mathrm{S}}(t)\right\|^2_{L^2_r(\Omega_{\mathrm{S}})}\,dt\right], \tag{3.20}$$

which clearly leads to (3.18). Thus, we conclude the proof. ☐

The following result is a direct consequence of the two previous theorems.

COROLLARY 3.5. *Under the assumptions of Theorem 3.3, Problem 1 has a unique solution and it satisfies the a priori estimates (3.18) and (3.5).*

*Remark* 3.4. Estimate (3.18) shows that the linear mapping giving the solution to Problem 1 from the data $A^0\in\widetilde{H}^1_r(\Omega_0)$ and $J_{\mathrm{S}}\in H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))$ is continuous from $L^2_r(\Omega_0)\times L^2(0,T;L^2_r(\Omega_{\mathrm{S}}))$ into $L^2(0,T;\mathcal{V})$. Hence it can be uniquely extended by continuity and density for data $A^0\in L^2_r(\Omega_0)$ and $J_{\mathrm{S}}\in L^2(0,T;L^2_r(\Omega_{\mathrm{S}}))$. The extended solution belongs to the space $L^2(0,T;\mathcal{V})$.

*Remark* 3.5. Under the assumptions of Theorem 3.3, the unique solution to Problem 1 satisfies additional space regularity. In fact, for almost all $t \in [0,T]$, the solution $A(t)$ of this problem can be seen as the solution of an elliptic problem in the space $\mathcal{V}$, with bilinear form $a(\cdot,\cdot)$ and right-hand side $\widetilde{f}(t) := \chi_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}(t) - \chi_Q(t)\sigma(t)\partial_t A(t)$, where $\chi_{\Omega_{\mathrm{S}}}$ (resp. $\chi_Q$) stands for the characteristic function of $\Omega_{\mathrm{S}}$ (resp. $Q$). By virtue of Theorem 3.3, $\widetilde{f}(t) \in L^2(0,T;L^2_r(\Omega))$; hence, we can apply [11, Theorem 4.1] to conclude that $A \in L^2(0,T;H^2_r(\Omega))$.

Notice that in principle the time derivative of the solution to Problem 1 is only defined in $Q$. However, the following theorem shows additional regularity of this derivative, which in particular implies that it is well defined in the whole $(0,T)\times\Omega$.

THEOREM 3.6. *Under the assumptions of Theorem 3.3, the solution to Problem 1 satisfies* $\sqrt{t}\partial_t A \in L^2(0,T;\mathcal{V})$.

*Proof.* We keep on using notation and partial results from the proof of Theorem 3.3. Recalling the definition of $\mathcal{M}_{i,j}(t)$ and using Theorem 3.1, we obtain

$$\frac{d}{dt}\mathcal{M}_{i,j}(t) = \int_{\Omega_t} \sigma \operatorname{div} \boldsymbol{v}\phi_j\phi_i \, r\,dr\,dz + \int_{\Omega_t} \sigma\boldsymbol{v}\cdot\mathbf{grad}(\phi_j\phi_i)\, r\,dr\,dz.$$

Since $\sigma$, $\boldsymbol{v}$, and $\operatorname{div}\boldsymbol{v}$ are essentially bounded, functions $\mathcal{M}_{i,j} \in W^{1,\infty}(0,T)$. This, together with (3.14) and the assumption that $J_{\mathrm{S}} \in H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))$ imply that $\boldsymbol{A}^\varepsilon_N \in H^2(0,T;\mathbb{R}^N)$. Thus, we are allowed to derive (3.14) with respect to time, which leads to

$$\sum_{j=1}^{N}\left[\mathcal{M}_{i,j}(t)+\varepsilon\mathcal{N}_{i,j}\right]\frac{d^2}{dt^2}A^\varepsilon_{jN}(t) + \sum_{j=1}^{N}\frac{d}{dt}\mathcal{M}_{i,j}(t)\frac{d}{dt}A^\varepsilon_{jN}(t) + \sum_{j=1}^{N}\mathcal{K}_{i,j}\frac{d}{dt}A^\varepsilon_{jN}(t) = \frac{d}{dt}F_{iN}(t).$$

Next, we multiply the above equation by $t\frac{d}{dt}A^\varepsilon_{iN}(t)$ and sum up from $i=1$ to $N$. Thus, we obtain

$$\begin{aligned}
&\int_{\Omega_t} t\sigma\left(\partial_{tt}A^\varepsilon_N\right)\left(\partial_t A^\varepsilon_N\right) r\,dr\,dz + \varepsilon\int_{\Omega} t\left(\partial_{tt}A^\varepsilon_N\right)\left(\partial_t A^\varepsilon_N\right) r\,dr\,dz + ta(\partial_t A^\varepsilon_N,\partial_t A^\varepsilon_N)\\
&+\int_{\Omega_t} t\sigma\operatorname{div}\boldsymbol{v}\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz + \int_{\Omega_t} t\sigma\boldsymbol{v}\cdot\mathbf{grad}\left(\left|\partial_t A^\varepsilon_N\right|^2\right) r\,dr\,dz\\
&\qquad = \int_{\Omega_{\mathrm{S}}} t\left(\partial_t J_{\mathrm{S}}\right)\left(\partial_t A^\varepsilon_N\right) r\,dr\,dz.
\end{aligned}$$

From Theorem 3.1 we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t} t\sigma\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz\\
&\quad = 2\int_{\Omega_t} t\sigma\left(\partial_{tt}A^\varepsilon_N\right)\left(\partial_t A^\varepsilon_N\right) r\,dr\,dz + \int_{\Omega_t}\sigma\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz - tc(t,\partial_t A^\varepsilon_N,\partial_t A^\varepsilon_N).
\end{aligned}$$

This, together with the previous equation, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} t\sigma\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz + \frac{\varepsilon}{2}\frac{d}{dt}\int_{\Omega} t\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz\\
&+ ta(\partial_t A^\varepsilon_N,\partial_t A^\varepsilon_N) - \frac{t}{2}c(t,\partial_t A^\varepsilon_N,\partial_t A^\varepsilon_N)\\
&\qquad = \frac{1}{2}\int_{\Omega_t}\sigma\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz + \frac{\varepsilon}{2}\int_{\Omega}\left|\partial_t A^\varepsilon_N\right|^2 r\,dr\,dz + \int_{\Omega_{\mathrm{S}}} t\left(\partial_t J_{\mathrm{S}}\right)\left(\partial_t A^\varepsilon_N\right) r\,dr\,dz.
\end{aligned}$$

Now, we repeat the arguments used in the proof of Theorem 3.4 to obtain

$$\int_0^T t\,\|\partial_t A_N^\varepsilon(t)\|^2_{\widetilde{H}^1_r(\Omega)}\,dt \le C\left[\int_0^T\int_{\Omega_t}\sigma\,|\partial_t A_N^\varepsilon(t)|^2\,r\,dr\,dz\,dt + \varepsilon\int_0^T\int_{\Omega}|\partial_t A_N^\varepsilon(t)|^2\,r\,dr\,dz\,dt + \int_0^T t\,\|\partial_t J_{\mathrm{S}}(t)\|^2_{L^2_r(\Omega_{\mathrm{S}})}\,dt\right].$$

This estimate together with (3.16) imply that $\sqrt{t}\partial_t A_N^\varepsilon$ is bounded in $L^2(0,T;\mathcal{V})$. Thus, working along the same lines as in the proof of Theorem 3.3, we first deduce, for fixed $N$, that $\sqrt{t}\partial_t A_N^\varepsilon$ converges weakly in $L^2(0,T;\mathcal{V})$ as $\varepsilon \to 0+$ to a function that turns out to be $\sqrt{t}\partial_t A_N$. Next, owing to [Brezis, Prop. III.5], $\sqrt{t}\partial_t A_N$ is also bounded in $L^2(0,T;\mathcal{V})$. From this we conclude analogously that $\sqrt{t}\partial_t A \in L^2(0,T;\mathcal{V})$ (and also the weak convergence $\sqrt{t}\partial_t A_N \rightharpoonup \sqrt{t}\partial_t A$ in $L^2(0,T;\mathcal{V})$). Thus we conclude the proof. □

*Remark* 3.6. As a by-product of the preceding proof, we obtain that the solution to the semi-discrete problem (3.6)–(3.7), which in principle is only known to satisfy $A_N \in L^2(0,T;\mathcal{V}_N)$ and $\partial_t A_N \in L^2_r(Q)$, has the following additional time regularity: $\sqrt{t}\partial_t A_N \in L^2(0,T;\mathcal{V}_N)$. In particular, $\partial_t A_N(t) \in \mathcal{V}_N$ a.e. $t \in [0,T]$ and $A_N \in \mathcal{C}((0,T];\mathcal{V}_N)$. This will be used in the following section to analyze a space discretization of Problem 1.

**4. Finite element space discretization.** Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\Omega$, with $h$ being the mesh-size. Let us emphasize that the meshes do not need to be fitted to $\Omega_0$. Let

$$\mathcal{V}_h := \{A_h \in \mathcal{V}:\ A_h|_T \in \mathbb{P}_1\ \forall T \in \mathcal{T}_h\}.$$

We introduce the following semi-discrete problem.

Problem 2. *Given $J_{\mathrm{S}} \in L^2(0,T;L^2_r(\Omega_{\mathrm{S}}))$ and $A_h^0 \in \mathcal{V}_h$, find $A_h \in L^2(0,T;\mathcal{V}_h)$ with $\partial_t A_h \in L^2_r(Q)$, such that*

$$\int_{\Omega_t}\sigma\,(\partial_t A_h)\,Z_h\,r\,dr\,dz + a(A_h,Z_h) = \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}} Z_h\,r\,dr\,dz \quad \forall Z_h \in \mathcal{V}, \quad a.e.\ t \in [0,T],$$

$$A_h(0)|_{\Omega_0} = A_h^0|_{\Omega_0}.$$

Notice that the initial data $A_h^0$ is assumed to be defined in the whole domain $\Omega$, although only its values in $\Omega_0$ are actually needed. The reason is that, in general, this term also depends on degrees of freedom lying outside $\Omega_0$ (unless the meshes were fitted to $\Omega_0$; see Remark 4.1 for a convenient choice of $A_h^0$ in such a case).

If $J_{\mathrm{S}} \in H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))$, then we can apply to Problem 2 the parabolic regularization used in the proof of Theorem 3.3 to show that (3.6)–(3.7) has a solution. By doing so, we conclude that Problem 2 has a solution, too. Moreover, the arguments used in the proof of Theorem 3.4 allow us to obtain the following estimate (cf. (3.20))

$$\sup_{t\in[0,T]} \int_{\Omega_t} \sigma \left|A_h(t)\right|^2 r\,dr\,dz + \int_0^T \|A_h(t)\|^2_{\widetilde{H}^1_r(\Omega)}\,dt \leq C \left[ \int_{\Omega_0} \sigma \left|A^0_h\right|^2 r\,dr\,dz + \int_0^T \|J_{\mathrm{S}}(t)\|^2_{L^2_r(\Omega_{\mathrm{S}})}\,dt \right]$$

and whence the uniqueness of solution to Problem 2.

In the remainder of this section, we assume that $J_{\mathrm{S}} \in H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))$ and $A^0 \in \widetilde{H}^1_r(\Omega_0)$, so that Problems 1 and 2 have unique solutions.

In what follows we will prove error estimates for this semi-discrete problem. With this end, we introduce the elliptic projector $P_h \in \mathcal{L}(\mathcal{V},\mathcal{V}_h)$ associated to $a$:

$$Y \in \mathcal{V} \longmapsto P_h Y \in \mathcal{V}_h : \qquad a(P_h Y, Z_h) = a(Y, Z_h) \qquad \forall Z_h \in \mathcal{V}_h.$$

Next result follows from Cea's lemma and a duality argument (see [3, section 4]).

Lemma 4.1. *For all $Z \in H^2_r(\Omega) \cap \mathcal{V}$,*

$$h\,\|Z - P_h Z\|_{\widetilde{H}^1_r(\Omega)} + \|Z - P_h Z\|_{L^2_r(\Omega)} \leq Ch^2\,\|Z\|_{H^2_r(\Omega)}.$$

Let $A$ and $A_h$ be the solutions to Problems 1 and 2, respectively. According to Theorem 3.6 and Remark 3.6, $A$ and $A_h$ belong to $\mathcal{C}((0,T];\mathcal{V})$. Then, for all $t \in (0,T]$ we write

$$A(t) - A_h(t) = \delta_h(t) + \rho_h(t), \tag{4.1}$$

where

$$\delta_h(t) := P_h A(t) - A_h(t) \qquad \text{and} \qquad \rho_h(t) := A(t) - P_h A(t).$$

Provided $A$ is smooth enough (for instance $A \in H^1(0,T;H^2_r(\Omega)\cap\mathcal{V})$, as in Theorem 4.3 below), $\rho_h$ is well defined in the whole domain $\Omega$ at $t=0$, too. Instead, $\delta_h(0)$ is only defined in $\Omega_0$:

$$\delta_h(0) := P_h A(0) - A_h(0) = P_h A(0) - A^0_h \qquad \text{in } \Omega_0.$$

In fact, our results only imply that $A_h$ has a well defined trace on $\{0\}\times\Omega_0$.

It is easy to show that $\partial_t(P_h A) = P_h(\partial_t A)$ a.e. $t \in [0,T]$. Hence, assuming again $A \in H^1(0,T;H^2_r(\Omega)\cap\mathcal{V})$, we have from Lemma 4.1

$$h\,\|\partial_t\rho_h(t)\|_{\widetilde{H}^1_r(\Omega)} + \|\partial_t\rho_h(t)\|_{L^2_r(\Omega)} \leq Ch^2\,\|\partial_t A(t)\|_{H^2_r(\Omega)} \qquad \text{a.e. } t \in [0,T]. \tag{4.2}$$

Furthermore, $\rho_h \in \mathcal{C}([0,T];\widetilde{H}^1_r(\Omega))$ and

$$h \sup_{t\in[0,T]} \|\rho_h(t)\|_{\widetilde{H}^1_r(\Omega)} + \sup_{t\in[0,T]} \|\rho_h(t)\|_{L^2_r(\Omega)} \leq Ch^2\,\|A\|_{H^1(0,T;H^2_r(\Omega))}. \tag{4.3}$$

Regarding the other term in decomposition (4.1), we have the following estimates.

Lemma 4.2. *Let $A$ and $A_h$ be the solutions to Problems 1 and 2, respectively. If $A \in H^1(0,T;\mathcal{V})$, then*

$$\sup_{t\in[0,T]} \|\delta_h(t)\|^2_{L^2_r(\Omega_t)} + \int_0^T \|\delta_h(t)\|^2_{\widetilde{H}^1_r(\Omega)}\,dt \leq C\left[\|\delta_h(0)\|^2_{L^2_r(\Omega_0)} + \int_0^T \|\partial_t\rho_h(t)\|^2_{L^2_r(\Omega)}\,dt\right], \tag{4.4}$$

$$\sup_{t\in[\varepsilon,T]} \|\delta_h(t)\|^2_{\widetilde{H}^1_r(\Omega)} + \int_\varepsilon^T \|\partial_t\delta_h(t)\|^2_{L^2_r(\Omega_t)}\,dt \leq C\left[\|\delta_h(\varepsilon)\|^2_{\widetilde{H}^1_r(\Omega)} + \int_\varepsilon^T \|\partial_t\rho_h(t)\|^2_{L^2_r(\Omega)}\,dt\right] \tag{4.5}$$

*for all $\varepsilon \in (0,T]$, with a constant $C$ independent of $\varepsilon$.*

*Proof.* Testing Problem 1 with $Z = Z_h \in \mathcal{V}_h$ and subtracting from Problem 2, we obtain

$$\int_{\Omega_t} \sigma\partial_t[A(t) - A_h(t)]\,Z_h\,r\,dr\,dz + a(A(t) - A_h(t), Z_h) = 0.$$

Using the definitions of $\delta_h$, $\rho_h$, and $P_h$, this equation becomes

$$\int_{\Omega_t} \sigma\,[\partial_t\delta_h(t)]\,Z_h\,r\,dr\,dz + a(\delta_h(t), Z_h) = -\int_{\Omega_t} \sigma\,[\partial_t\rho_h(t)]\,Z_h\,r\,dr\,dz. \tag{4.6}$$

Choosing in this equation $Z_h = \delta_h(t)$, we have

$$\int_{\Omega_t} \sigma\,[\partial_t\delta_h(t)]\,\delta_h(t)\,r\,dr\,dz + a(\delta_h(t), \delta_h(t)) = -\int_{\Omega_t} \sigma\,[\partial_t\rho_h(t)]\,\delta_h(t)\,r\,dr\,dz.$$

Hence, by using Theorem 3.1 and Lemma 3.2, we write

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \sigma\,|\delta_h(t)|^2\,r\,dr\,dz + \frac{\alpha}{2}\|\delta_h(t)\|^2_{\widetilde{H}^1_r(\Omega)} \leq \left|\int_{\Omega_t} \sigma\,[\partial_t\rho_h(t)]\,\delta_h(t)\,r\,dr\,dz\right| + \lambda\int_{\Omega_t} \sigma\,|\delta_h(t)|^2\,r\,dr\,dz,$$

which together with a Young's inequality yield

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \sigma\,|\delta_h(t)|^2\,r\,dr\,dz + \frac{\alpha}{2}\|\delta_h(t)\|^2_{\widetilde{H}^1_r(\Omega)} \leq C\left[\int_{\Omega} |\partial_t\rho_h(t)|^2\,r\,dr\,dz + \int_{\Omega_t} \sigma\,|\delta_h(t)|^2\,r\,dr\,dz\right]. \tag{4.7}$$

Next, we use the Gronwall's lemma in this equation to obtain for all $\tau \in [0,T]$,

$$\int_{\Omega_\tau} \sigma\,|\delta_h(\tau)|^2\,r\,dr\,dz \leq C\left[\int_{\Omega_0} \sigma\,|\delta_h(0)|^2\,r\,dr\,dz + \int_0^T\int_{\Omega} |\partial_t\rho_h(t)|^2\,r\,dr\,dz\,dt\right].$$

Then, we integrate (4.7) with respect to time and use the above estimate to write

$$\int_0^T \|\delta_h(t)\|^2_{\widetilde{H}^1_r(\Omega)}\, dt \leq C \left[ \int_{\Omega_0} \sigma \left|\delta_h(0)\right|^2\, r\, dr\, dz + \int_0^T \int_\Omega \left|\partial_t \rho_h(t)\right|^2\, r\, dr\, dz\, dt \right].$$

Thus, (4.4) follows from the last two inequalities.

On the other hand, due to the regularity of $A$ and $A_h$ (see Theorem 3.6 and Remark 3.6), $\partial_t \delta_h(t) \in \mathcal{V}_h$ a.e. $t \in [0, T]$. By testing (4.6) with $Z_h = \partial_t \delta_h(t)$ and using once more a Young's inequality, we obtain

$$\begin{aligned} &\int_{\Omega_t} \sigma \left|\partial_t \delta_h(t)\right|^2\, r\, dr\, dz + a(\delta_h(t), \partial_t \delta_h(t)) \\ &\qquad \leq \frac{1}{2} \int_{\Omega_t} \sigma \left|\partial_t \rho_h(t)\right|^2\, r\, dr\, dz + \frac{1}{2} \int_{\Omega_t} \sigma \left|\partial_t \delta_h(t)\right|^2\, r\, dr\, dz \qquad \text{a.e. } t \in [0, T]. \end{aligned}$$

Hence, integrating with respect to time, we have for $\varepsilon < \tau \leq T$

$$\begin{aligned} &\int_\varepsilon^\tau \int_{\Omega_t} \sigma \left|\partial_t \delta_h(t)\right|^2\, r\, dr\, dz\, dt + a(\delta_h(\tau), \delta_h(\tau)) \\ &\qquad \leq a(\delta_h(\varepsilon), \delta_h(\varepsilon)) + \int_\varepsilon^\tau \int_\Omega \overline{\sigma} \left|\partial_t \rho_h(t)\right|^2\, r\, dr\, dz\, dt. \end{aligned}$$

Finally, (4.5) follows from this inequality and the continuity and ellipticity of $a$. □

Now we are in a position to prove error estimates for the discrete vector potential $A_h$ as well as for the physical quantities of interest that can be derived from it, namely, the magnetic induction and the current density. According to (2.7) and (2.8), we define

$$\boldsymbol{B}_h := \mathbf{curl}(A_h \boldsymbol{e}_\theta),$$

whereas, according to (2.6) and (2.9),

$$\boldsymbol{J}_h := -\sigma \frac{\partial A_h}{\partial t} \boldsymbol{e}_\theta \qquad \text{in } \Omega_t.$$

The following error estimates hold true.

Theorem 4.3. *Let $A$ and $A_h$ be solutions to Problems 1 and 2, respectively. Let $\boldsymbol{B} := \mathbf{curl}(A\boldsymbol{e}_\theta)$ and $\boldsymbol{J} := -\sigma\, (\partial_t A)\,|_{\Omega_t} \boldsymbol{e}_\theta$. Let $\boldsymbol{B}_h$ and $\boldsymbol{J}_h$ be defined as above. If $A \in H^1(0, T; H^2_r(\Omega) \cap \mathcal{V})$, then*

$$\begin{aligned} \|A - A_h\|_{L^\infty(0,T;L^2_r(\Omega_t))} &\leq C \left[ \left\|A^0 - A^0_h\right\|_{L^2_r(\Omega_0)} + h^2 \|A\|_{H^1(0,T;H^2_r(\Omega))} \right], \\ \|\boldsymbol{B} - \boldsymbol{B}_h\|_{L^2(0,T;L^2_r(\Omega))} &\leq C \left[ \left\|A^0 - A^0_h\right\|_{L^2_r(\Omega_0)} + h \|A\|_{H^1(0,T;H^2_r(\Omega))} \right], \\ \|A - A_h\|_{L^\infty(0,T;\widetilde{H}^1_r(\Omega))} &\leq C \left[ \liminf_{\varepsilon \to 0+} \|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + h \|A\|_{H^1(0,T;H^2_r(\Omega))} \right], \\ \|\boldsymbol{J} - \boldsymbol{J}_h\|_{L^2(0,T;L^2_r(\Omega_t))} &\leq C \left[ \liminf_{\varepsilon \to 0+} \|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + h \|A\|_{H^1(0,T;H^2_r(\Omega))} \right]. \end{aligned}$$

*Proof.* The first inequality is a consequence of (4.1), (4.4), (4.2), and (4.3):

$$\begin{aligned}
\|A - A_h\|_{L^\infty(0,T;L^2_r(\Omega_t))} &\le \operatorname*{ess\,sup}_{t\in[0,T]} \|\delta_h(t)\|_{L^2_r(\Omega_t)} + \operatorname*{ess\,sup}_{t\in[0,T]} \|\rho_h(t)\|_{L^2_r(\Omega_t)} \\
&\le C\left[\|\delta_h(0)\|_{L^2_r(\Omega_0)} + \|\partial_t\rho_h\|_{L^2(0,T;L^2_r(\Omega))}\right] \\
&\quad + \sup_{t\in[0,T]} \|\rho_h(t)\|_{L^2_r(\Omega_t)} \\
&\le C\left[\left\|A^0 - A^0_h\right\|_{L^2_r(\Omega_0)} + h^2\|A\|_{H^1(0,T;H^2_r(\Omega))}\right],
\end{aligned}$$

where we have also used the initial conditions of Problems 1 and 2.

The second inequality is proved by following exactly the same steps; however, since (4.3) is now also used to derive $\|\rho_h\|_{L^2(0,T;\widetilde{H}^1_r(\Omega))} \le Ch\|A\|_{H^1(0,T;H^2_r(\Omega))}$, this term appears in the inequality instead of $h^2\|A\|_{H^1(0,T;H^2_r(\Omega))}$.

For the third inequality we essentially repeat the same arguments again, but using now (4.5) instead of (4.4):

$$\begin{aligned}
\|A - A_h\|_{L^\infty(\varepsilon,T;\widetilde{H}^1_r(\Omega))} &\le C\left[\|\delta_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + \|\partial_t\rho_h\|_{L^2(0,T;L^2_r(\Omega))}\right] \\
&\quad + \operatorname*{ess\,sup}_{t\in[\varepsilon,T]} \|\rho_h(t)\|_{\widetilde{H}^1_r(\Omega)} \\
&\le C\Big[\|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + \|\rho_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} \\
&\qquad + h\|A\|_{H^1(0,T;H^2_r(\Omega))}\Big] \\
&\le C\left[\|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + h\|A\|_{H^1(0,T;H^2_r(\Omega))}\right]
\end{aligned}$$

for all $\varepsilon \in (0,T]$. Thus, the third inequality follows by taking $\liminf$ as $\varepsilon \to 0+$.

Finally, the last inequality is a consequence of the following estimate:

$$\|\partial_t A - \partial_t A_h\|_{L^2(\varepsilon,T;L^2_r(\Omega_t))} \le C\left[\|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)} + h\|A\|_{H^1(0,T;H^2_r(\Omega))}\right],$$

which, in its turn, follows once more by repeating similar steps. Thus, we conclude the proof. ∎

*Remark* 4.1. The first two estimates can be improved when $\Omega_0$ is polygonal and the meshes are fitted to this subdomain. In fact, if $A^0 \in H^2_r(\Omega_0) \cap \widetilde{H}^1_r(\Omega_0)$ (what necessarily happens under the assumptions of the previous theorem), then the initial condition for Problem 2 can be taken as its Lagrange interpolant $A^0_h|_{\Omega_0} := \mathcal{I}_h A^0$. In such a case, there holds (cf. [1, Prop. 3])

$$\|A(0) - A_h(0)\|_{L^2_r(\Omega_0)} = \left\|A^0 - \mathcal{I}_h A^0\right\|_{L^2_r(\Omega_0)} \le Ch^2\left\|A^0\right\|_{H^2_r(\Omega_0)},$$

which allows us to improve the first estimates in the above theorem as follows:

$$\begin{aligned}
\|A - A_h\|_{L^\infty(0,T;L^2_r(\Omega_t))} &\le Ch^2\|A\|_{H^1(0,T;H^2_r(\Omega))}, \\
\|\boldsymbol{B} - \boldsymbol{B}_h\|_{L^2(0,T;L^2_r(\Omega))} &\le Ch\|A\|_{H^1(0,T;H^2_r(\Omega))}.
\end{aligned}$$

The last two inequalities in the previous theorem are not actual a priori error estimates, since they involve $\liminf_{\varepsilon\to 0+}\|A(\varepsilon) - A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)}$. This term is finite. In fact, $A$ and $A_h$ both belong to $L^\infty(0,T;\mathcal{V})\cap\mathcal{C}((0,T],\mathcal{V})$ (cf. Theorems 3.3 and 3.6). Thus, $A$ and $A_h$ are actually bounded for all $t\in(0,T]$. If $A_h$ were continuous at $t=0$, then we would have $\liminf_{\varepsilon\to 0+}\|A(\varepsilon)-A_h(\varepsilon)\|_{\widetilde{H}^1_r(\Omega)}=\|A(0)-A_h(0)\|_{\widetilde{H}^1_r(\Omega)}$. However, we do not have estimates allowing us to prove such continuity. Consequently, we do not have an a priori error estimate for the approximation of the current density $\boldsymbol{J}$, which is typically a quantity of interest. In the following section, we will introduce a fully discrete scheme which does suffer of this drawback.

**5. Fully discrete problem.** We consider a uniform partition of the time interval $[0,T]$ with time step $\Delta t:=\frac{T}{N}$: $\{t_k:=k\Delta t,\ k=1,\dots,N\}$. We use the backward Euler approximation for the time discretization. Thus, the fully discrete approximation of Problem 1 reads as follows.

PROBLEM 3. *Given* $J_{\mathrm{S}}\in\mathcal{C}([0,T];L^2_r(\Omega_{\mathrm{S}}))$ *and* $A_h^0\in\mathcal{V}_h$, *for* $k=1,\dots,N$, *find* $A_h^k\in\mathcal{V}_h$ *such that*

$$\int_{\Omega_{t_k}}\sigma(t_k)\frac{A_h^k-A_h^{k-1}}{\Delta t}Z_h\,r\,dr\,dz+a(A_h^k,Z_h)=\int_{\Omega_{\mathrm{S}}}J_{\mathrm{S}}(t_k)Z_h\,r\,dr\,dz\qquad\forall Z_h\in\mathcal{V}_h.$$

Notice that this scheme needs the initial data $A_h^0$ in $\Omega_{t_1}$ and not in $\Omega_0$. As in the previous section, we assume that $A(0)$ is known in the whole domain $\Omega$ and take $A_h^0$ as an appropriate approximation of $A(0)$.

Moreover, since the domain where the derivative of $A$ is approximated changes with time, terms of the form $\int_{\Omega_{t_k}}\sigma(t_k)A_h^{k-1}$ appear in the numerical scheme. This is the reason why we cannot follow a more standard approach (as that used for the continuous and the semi-discrete problems) to prove its stability. Anyway the fully discrete scheme can be proved to be stable by assuming further regularity for $J_{\mathrm{S}}$.

THEOREM 5.1. *If* $J_{\mathrm{S}}\in H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))$, *then Problem 3 has a unique solution and it satisfies*

$$\max_{1\le k\le N}\left\|A_h^k\right\|_{\widetilde{H}^1_r(\Omega)}\le C\left[\left\|A_h^0\right\|_{\widetilde{H}^1_r(\Omega)}+\|J_{\mathrm{S}}\|_{H^1(0,T;L^2_r(\Omega_{\mathrm{S}}))}\right].$$

*Proof.* The well-posedness is an immediate consequence of the following inequality, which in its turn results from the ellipticity of $a$ and the positiveness of $\sigma$:

$$\frac{1}{\Delta t}\int_{\Omega_{t_k}}\sigma(t_k)Z_h^2\,r\,dr\,dz+a(Z_h,Z_h)\ge C\,\|Z_h\|^2_{\widetilde{H}^1_r(\Omega)}.$$

To prove the stability estimate, we test Problem 3 with $Z_h=A_h^k-A_h^{k-1}$:

$$\int_{\Omega_{t_k}}\sigma(t_k)\frac{\left(A_h^k-A_h^{k-1}\right)^2}{\Delta t}\,r\,dr\,dz+a(A_h^k,A_h^k-A_h^{k-1})$$
$$=\int_{\Omega_{\mathrm{S}}}J_{\mathrm{S}}(t_k)(A_h^k-A_h^{k-1})\,r\,dr\,dz,\quad k=1,\dots,N.$$

Hence, by using the algebraic identity $2(p-q)p=p^2+(p-q)^2-q^2$ and again the ellipticity of $a$ and the positiveness of $\sigma$, we write

$$a(A_h^k,A_h^k)-a(A_h^{k-1},A_h^{k-1})\le 2\int_{\Omega_{\mathrm{S}}}J_{\mathrm{S}}(t_k)(A_h^k-A_h^{k-1})\,r\,dr\,dz,\quad k=1,\dots,N.$$

Summing up the above equations from $k=1$ to $n$ $(1 \leq n \leq N)$, we obtain

$$
\begin{aligned}
a(A_h^n, A_h^n) &\leq a(A_h^0, A_h^0) + 2 \sum_{k=1}^{n} \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}(t_k)(A_h^k - A_h^{k-1})\, r\, dr\, dz \\
&= a(A_h^0, A_h^0) + 2 \left[ \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}(t_n) A_h^n\, r\, dr\, dz - \int_{\Omega_{\mathrm{S}}} J_{\mathrm{S}}(t_0) A_h^0\, r\, dr\, dz \right. \\
&\qquad \left. - \Delta t \sum_{k=1}^{n} \int_{\Omega_{\mathrm{S}}} \frac{J_{\mathrm{S}}(t_k) - J_{\mathrm{S}}(t_{k-1})}{\Delta t}\, A_h^{k-1}\, r\, dr\, dz \right],
\end{aligned}
$$

where we have used summation by parts. Next, we use once more the ellipticity of $a$ and a Young's inequality to write

$$
\begin{aligned}
\alpha \left\| A_h^n \right\|_{\widetilde{H}_r^1(\Omega)}^2 &\leq a(A_h^0, A_h^0) + \frac{2}{\alpha} \left\| J_{\mathrm{S}}(t_n) \right\|_{L_r^2(\Omega_{\mathrm{S}})}^2 + \frac{\alpha}{2} \left\| A_h^n \right\|_{L_r^2(\Omega_{\mathrm{S}})}^2 \\
&\quad + \left\| J_{\mathrm{S}}(t_0) \right\|_{L_r^2(\Omega_{\mathrm{S}})}^2 + \left\| A_h^0 \right\|_{L_r^2(\Omega_{\mathrm{S}})}^2 \\
&\quad + \Delta t \sum_{k=1}^{n} \int_{\Omega_{\mathrm{S}}} \left| \frac{J_{\mathrm{S}}(t_k) - J_{\mathrm{S}}(t_{k-1})}{\Delta t} \right|^2 r\, dr\, dz + \Delta t \sum_{k=0}^{n-1} \left\| A_h^k \right\|_{L_r^2(\Omega_{\mathrm{S}})}^2 .
\end{aligned}
$$

Hence, using the relationship $J_{\mathrm{S}}(t_k) - J_{\mathrm{S}}(t_{k-1}) = \int_{t_{k-1}}^{t_k} J_{\mathrm{S}}'(t)\, dt$ and a Cauchy-Schwartz inequality, straightforward computations lead to

$$
\left\| A_h^n \right\|_{\widetilde{H}_r^1(\Omega)}^2 \leq C \left[ \left\| A_h^0 \right\|_{\widetilde{H}_r^1(\Omega)}^2 + \left\| J_{\mathrm{S}} \right\|_{H^1(0,T;L_r^2(\Omega_{\mathrm{S}}))}^2 \right] + \frac{2\Delta t}{\alpha} \sum_{k=0}^{n-1} \left\| A_h^k \right\|_{\widetilde{H}_r^1(\Omega)}^2 .
$$

Finally, by using the discrete Gronwall's lemma (see, for instance, [14, Lemma 1.4.2]), we obtain

$$
\left\| A_h^n \right\|_{\widetilde{H}_r^1(\Omega)}^2 \leq C \left[ \left\| A_h^0 \right\|_{\widetilde{H}_r^1(\Omega)}^2 + C \left\| J_{\mathrm{S}} \right\|_{H^1(0,T;L_r^2(\Omega_{\mathrm{S}}))}^2 \right].
$$

Since this holds for all $n = 1, \ldots, N$, we conclude the proof. □

Our next goal is to prove error estimates for the solution to Problem 3. To do this we introduce some notation. Given $(\phi^0, \ldots, \phi^N) \in \mathbb{R}^{N+1}$, we define the backward difference quotient

$$
\bar{\partial} \phi^k := \frac{\phi^k - \phi^{k-1}}{\Delta t}, \qquad k = 1, \ldots, N.
$$

For $A$ being the solution to Problem 1 and $A_h^k$ that to Problem 3, we write

$$
A(t_k) - A_h^k = \delta_h^k + \rho_h^k, \qquad k = 0, \ldots, N, \tag{5.1}
$$

with

$$
\delta_h^k := P_h A(t_k) - A_h^k \qquad \text{and} \qquad \rho_h^k := A(t_k) - P_h A(t_k) = \rho_h(t_k).
$$

Let us remark that the definition of $\delta_h^0$ involves the chosen approximation $A_h^0$ of the initial data $A(0)$. Finally, we define the truncation errors

$$
\tau^k := \bar{\partial} A(t_k) - \partial_t A(t_k), \qquad k = 1, \ldots, N.
$$

Notice that all these quantities are related as follows:

$$\partial_t A(t_k) - \bar{\partial} A_h^k = -\tau^k + \bar{\partial}\rho_h^k + \bar{\partial}\delta_h^k, \qquad k = 1, \dots, N. \tag{5.2}$$

The first step is to estimate $\delta_h^k$ in terms of $\rho_h^k$ and $\tau^k$.

LEMMA 5.2. *Let $A$ and $A_h^k$ be solutions to Problems 1 and 3, respectively. If $A \in \mathcal{C}([0,T];\mathcal{V}) \cap \mathcal{C}^1([0,T];L_r^2(\Omega))$, then*

$$\Delta t \sum_{k=1}^{N} \int_{\Omega_{t_k}} \sigma(t_k) \left|\bar{\partial}\delta_h^k\right|^2 r\,dr\,dz + \max_{1\le k\le N} \left\|\delta_h^k\right\|_{\widetilde{H}_r^1(\Omega)}^2$$
$$\le C \left\{ \left\|\delta_h^0\right\|_{\widetilde{H}_r^1(\Omega)}^2 + \Delta t \sum_{k=1}^{N} \left[ \left\|\bar{\partial}\rho_h^k\right\|_{L_r^2(\Omega_{t_k})}^2 + \left\|\tau^k\right\|_{L_r^2(\Omega_{t_k})}^2 \right] \right\}.$$

*Proof.* Testing Problems 1 and 3 with $Z_h \in \mathcal{V}_h \subset \mathcal{V}$ and subtracting, we obtain

$$\int_{\Omega_{t_k}} \sigma(t_k) \left[\partial_t A(t_k) - \bar{\partial} A_h^k\right] Z_h\, r\,dr\,dz + a(A(t_k) - A_h^k, Z_h) = 0.$$

Hence, by using (5.2), straightforward calculations yield

$$\int_{\Omega_{t_k}} \sigma(t_k) \left(\bar{\partial}\delta_h^k\right) Z_h\, r\,dr\,dz + a(\delta_h^k, Z_h)$$
$$= -\int_{\Omega_{t_k}} \sigma(t_k) \left(\bar{\partial}\rho_h^k\right) Z_h\, r\,dr\,dz + \int_{\Omega_{t_k}} \sigma(t_k) \tau^k Z_h\, r\,dr\,dz.$$

Now, taking $Z_h = \Delta t \bar{\partial}\delta_h^k$, we write

$$\Delta t \int_{\Omega_{t_k}} \sigma(t_k) \left|\bar{\partial}\delta_h^k\right|^2 r\,dr\,dz + \Delta t\, a(\delta_h^k, \bar{\partial}\delta_h^k)$$
$$= -\Delta t \int_{\Omega_{t_k}} \sigma(t_k) \left(\bar{\partial}\rho_h^k\right)\left(\bar{\partial}\delta_h^k\right) r\,dr\,dz + \Delta t \int_{\Omega_{t_k}} \sigma(t_k)\tau^k \left(\bar{\partial}\delta_h^k\right) r\,dr\,dz.$$

On the other hand, from the algebraic identity $2(p-q)p = p^2 + (p-q)^2 - q^2$, we have

$$2\Delta t\, a(\delta_h^k, \bar{\partial}\delta_h^k) \ge a(\delta_h^k, \delta_h^k) - a(\delta_h^{k-1}, \delta_h^{k-1}).$$

Using this and a Young's inequality in the equation above, we obtain

$$\Delta t \int_{\Omega_{t_k}} \sigma(t_k) \left|\bar{\partial}\delta_h^k\right|^2 r\,dr\,dz + \frac{1}{2} a(\delta_h^k, \delta_h^k) - \frac{1}{2} a(\delta_h^{k-1}, \delta_h^{k-1})$$
$$\le \Delta t \left[ \left\|\bar{\partial}\rho_h^k\right\|_{L_r^2(\Omega_{t_k})}^2 + \left\|\tau^k\right\|_{L_r^2(\Omega_{t_k})}^2 \right] + \frac{\Delta t}{2} \int_{\Omega_{t_k}} \sigma(t_k) \left|\bar{\partial}\delta_h^k\right|^2 r\,dr\,dz.$$

Therefore, summing up from $k = 1$ to $n$ ($1 \le n \le N$), we have

$$\Delta t \sum_{k=1}^{n} \int_{\Omega_{t_k}} \sigma(t_k) \left|\bar{\partial}\delta_h^k\right|^2 r\,dr\,dz + a(\delta_h^n, \delta_h^n)$$
$$\le a(\delta_h^0, \delta_h^0) + 2\Delta t \sum_{k=1}^{n} \left[ \left\|\bar{\partial}\rho_h^k\right\|_{L_r^2(\Omega_{t_k})}^2 + \left\|\tau^k\right\|_{L_r^2(\Omega_{t_k})}^2 \right].$$

Hence the result follows from the continuity and the ellipticity of $a$. ∎

Next step is to obtain appropriate estimates for $\bar{\partial}\rho_h^k$ and $\tau^k$.

LEMMA 5.3. *If $A \in H^1(0,T;H_r^2(\Omega))$, then*

$$\left[\Delta t \sum_{k=1}^{N} \left\|\bar{\partial}\rho_h^k\right\|_{L_r^2(\Omega_{t_k})}^2\right]^{1/2} \leq Ch^2 \left\|A\right\|_{H^1(0,T;H_r^2(\Omega))}.$$

*Moreover, if $A \in H^2(0,T;L_r^2(\Omega))$, then*

$$\left[\Delta t \sum_{k=1}^{N} \left\|\tau^k\right\|_{L_r^2(\Omega_{t_k})}^2\right]^{1/2} \leq C\Delta t \left\|A\right\|_{H^2(0,T;L_r^2(\Omega))}.$$

*Proof.* Equality $\bar{\partial}\rho_h^k = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} \partial_t \rho_h(t)\,dt$ and (4.2) lead to

$$\Delta t \sum_{k=1}^{N} \left\|\bar{\partial}\rho_h^k\right\|_{L_r^2(\Omega_{t_k})}^2 \leq \int_0^T \left\|\partial_t\rho_h(t)\right\|_{L_r^2(\Omega)}^2 \,dt \leq Ch^4 \left\|A\right\|_{H^1(0,T;H_r^2(\Omega))}^2,$$

whence we conclude the first estimate of the lemma.

On the other hand, from Taylor's formula,

$$\tau^k := \bar{\partial}A(t_k) - \partial_t A(t_k) = -\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} (t - t_{k-1})\,\partial_{tt}A(t)\,dt.$$

Therefore, using Cauchy-Schwartz inequality, it is easy to obtain

$$\Delta t \sum_{k=1}^{N} \left\|\tau^k\right\|_{L_r^2(\Omega_{t_k})}^2 \leq \Delta t^2 \left\|A\right\|_{H^2(0,T;L_r^2(\Omega))}^2.$$

Thus we conclude the second estimate of the lemma. ∎

We end this section by proving error estimates for the computed vector potential $A_h^k$ as well as for the physical quantities of interest that can be derived from it. We define approximations $\boldsymbol{B}_h^k$ of the magnetic induction and $\boldsymbol{J}_h^k$ of the current density as follows (cf. (2.5)–(2.9)):

$$\boldsymbol{B}_h^k := \mathbf{curl}(A_h^k \boldsymbol{e}_\theta), \qquad \boldsymbol{J}_h^k := -\sigma(t_k)\bar{\partial}A_h^k \boldsymbol{e}_\theta \quad \text{in } \Omega_{t_k}, \qquad k = 1,\dots,N.$$

The error estimates for this quantities will be a consequence of Lemmas 5.2 and 5.3. The former depends on $\delta_h^0$, which in its turn depends on the particular approximation $A_h^0$ of $A(0)$ used as initial condition in Problem 3. If the solution to Problem 1 is sufficiently smooth at time $t = 0$, namely $A(0) \in H_r^2(\Omega) \cap \mathcal{V}$, then we can take for instance the Lagrange interpolant of $A(0)$, $A_h^0 := \mathcal{I}_h A(0)$. In such a case we have the following result.

THEOREM 5.4. *Let $A$ and $A_h^k$ be solutions to Problems 1 and 3, respectively. Assume that $A \in H^1(0,T;H_r^2(\Omega) \cap \mathcal{V}) \cap H^2(0,T;L_r^2(\Omega))$ and $A_h^0 := \mathcal{I}_h A(0)$ is taken as initial condition for Problem 3. Let $\boldsymbol{B} := \mathbf{curl}(A\boldsymbol{e}_\theta)$ and $\boldsymbol{J} := -\sigma\,(\partial_t A)\,|_{\Omega_t}\boldsymbol{e}_\theta$. Let $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$ be defined as above. Then,*

$$\max_{1\leq k\leq N} \left\|\boldsymbol{B}(t_k) - \boldsymbol{B}_h^k\right\|_{L_r^2(\Omega)} \leq C\left[h\left\|A\right\|_{H^1(0,T;H_r^2(\Omega))} + \Delta t\left\|A\right\|_{H^2(0,T;L_r^2(\Omega))}\right],$$

$$\left[\Delta t \sum_{k=1}^{N} \left\|\boldsymbol{J}(t_k) - \boldsymbol{J}_h^k\right\|_{L_r^2(\Omega_{t_k})}^2\right]^{1/2} \leq C\left[h\left\|A\right\|_{H^1(0,T;H_r^2(\Omega))} + \Delta t\left\|A\right\|_{H^2(0,T;L_r^2(\Omega))}\right].$$

*Proof.* Recalling (5.1), from Lemmas 5.2 and 5.3 and estimate (4.3), we obtain

$$\max_{1\le k\le N} \left\|\boldsymbol{B}(t_k) - \boldsymbol{B}_h^k\right\|_{L_r^2(\Omega)} \le C\left[\left\|\delta_h^0\right\|_{\widetilde{H}_r^1(\Omega)} + h\left\|A\right\|_{H^1(0,T;H_r^2(\Omega))} + \Delta t\left\|A\right\|_{H^2(0,T;L_r^2(\Omega))}\right].$$

The first term in the right-hand side is bounded as follows:

$$\begin{aligned}\left\|\delta_h^0\right\|_{\widetilde{H}_r^1(\Omega)} &\le \left\|P_h A(0) - A(0)\right\|_{\widetilde{H}_r^1(\Omega)} + \left\|A(0) - \mathcal{I}_h A(0)\right\|_{\widetilde{H}_r^1(\Omega)}\\ &\le Ch\left\|A(0)\right\|_{H^1(0,T;H_r^2(\Omega))},\end{aligned}$$

where we have used (4.3) again and an error estimate for the Lagrange interpolant (cf. [1, Prop. 3]). Thus, the first estimate of this theorem follows from these inequalities. The proof of the second one makes use of (5.2) and is essentially identical. ◻

**6. Numerical tests.** We have developed a FORTRAN code which implements the fully discrete numerical scheme analyzed in the previous section (cf. Problem 3). Let us recall that the method does not need a mesh fitted to the workpiece, which allows us to avoid remeshing at each time step. In fact, we have used in all cases a fixed mesh for the whole process. In what follows we report the results of two tests, one with a known analytical solution and another one corresponding to an EMF process.

**6.1. Test 1: An example with analytical solution.** First, the code has been validated by solving a problem with known analytical solution. We have used this test to confirm the theoretical order of convergence. In this case, $\Omega := [0,1]\times[0,3]$ and the workpiece moves as a rigid body, with velocity $\boldsymbol{v} = \boldsymbol{e}_z$ and initial position $\Omega_0 := [0,1]\times[1,2]$ (see Figure 6.1 (left)).

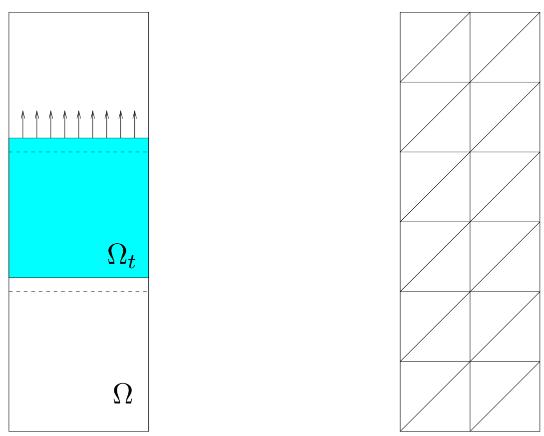


Fig. 6.1. *Test 1. Sketch of the domain (left) and coarse initial mesh (right).*

We have solved the following problem:

$$\sigma\frac{\partial A}{\partial t}\boldsymbol{e}_\theta + \mathbf{curl}\left[\frac{1}{\mu}\,\mathbf{curl}\,(A\boldsymbol{e}_\theta)\right] = f\boldsymbol{e}_\theta,$$

with $\mu = 1$ and

$$\sigma = \begin{cases} 1 & \text{in } \Omega_t,\\ 0 & \text{in } \Omega\setminus\Omega_t.\end{cases}$$

Function $f$ in the right-hand side has been chosen so that the solution be

$$A(t,r,z) = \sin(2\pi t)(r^3 + rz^2).$$

We have used uniform meshes obtained by successively refining the coarse one shown in Figure 6.1 (right). Notice that although the meshes are fitted to the initial position $\Omega_0$ of the workpiece, this does not happen at the subsequent time steps $t_k$. Therefore, the computation of terms $\int_{\Omega_{t_k}} \sigma(t_k)(A_h^k - A_h^{k-1})$ in Problem 3 involve integrals on triangles of piecewise smooth discontinuous functions. These integrals were computed by using low order quadrature rules with a large number of integrations points; this number was determined in advance so that the results be essentially indifferent to it.

The method has been used on several successively refined meshes by reducing the time step in a convenient way to analyze the convergence with respect to both, the mesh-size and the time step. With this aim, the numerical approximations have been compared with the analytical solution. As a first step, for each quantity $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$, the dependence of the error on $h$ and $\Delta t$ was studied separately. To do this, first we fixed the time step to a sufficiently small value, so that the error practically depends only on the mesh-size. In this case we observed that the error of $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$ reduces linearly with respect to $h$. Then, we fixed the mesh-size to a sufficiently small value for the time discretization error to prevail. In such a case we observed a linear dependence on $\Delta t$ for both quantities. We illustrate in Figure 6.2 the convergence behavior of the method for each of these quantities. These figure shows log-log plots of the errors of $\boldsymbol{B}_h^k$ and $\boldsymbol{J}_h^k$ in the discrete norms considered in Theorem 5.4 versus the number of degrees of freedom (d.o.f.). To report in one only figure the simultaneous dependence on $h$ and $\Delta t$, we proceeded in the following way: first, we chose initial values of $h$ and $\Delta t$, so that the time and the space discretization errors were both of approximately the same size; secondly, for each of the successively refined meshes, we have taken values of $\Delta t$ proportional to $h$, according to the previously observed dependence of the errors on the mesh-size.

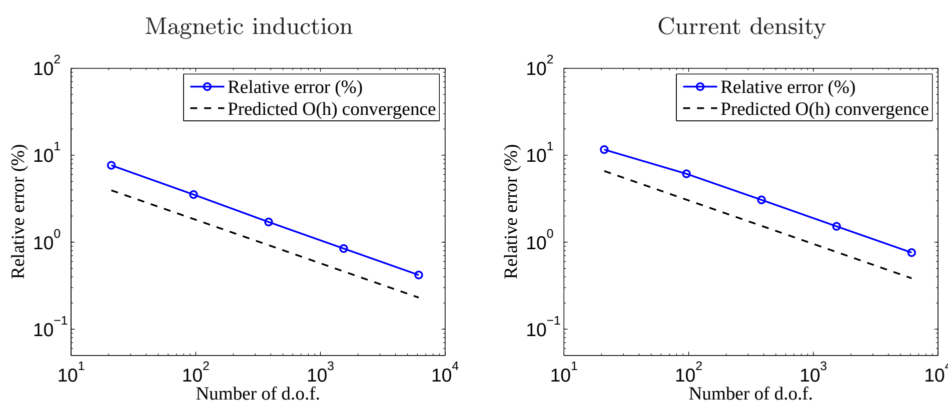


FIG. 6.2. *Test 1. Relative errors for the magnetic induction field* $\max_{1\le k\le N} \left\|\boldsymbol{B}(t_k) - \boldsymbol{B}_h^k\right\|_{L_r^2(\Omega)}$ *(left) and the current density* $\left[\Delta t \sum_{k=1}^N \left\|\boldsymbol{J}(t_k) - \boldsymbol{J}_h^k\right\|^2_{L_r^2(\Omega_{t_k})}\right]^{1/2}$ *(right) versus number of degrees of freedom (d.o.f.) (log-log scale) with* $\Delta t = Ch$ *in both cases.*

**6.2. Test 2: Numerical solution of an EMF process.** We have used the same code as above to compute the current density and the Lorentz force in an example taken from an EMF process. We consider the geometry and physical data of the axisymmetric test described in [12] (see Figure 6.3), which is a classical benchmark (see [12, 15] for more details). The geometrical and physical data are given in Table 6.1.

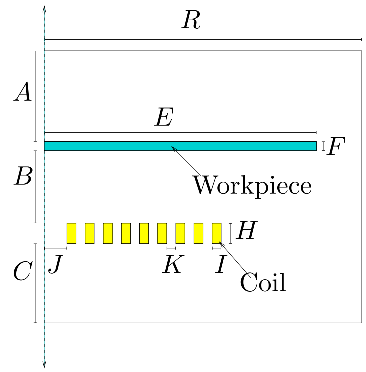


FIG. 6.3. *EMF. Geometry of the benchmark problem.*

TABLE 6.1
*Test 2. Geometrical data and physical parameters.*

| | |
|---|---|
| Thickness of the workpiece ($F$): | 0.0012 m |
| Height of each coil turn ($H$): | 0.0115 m |
| Width of each coil turn ($I$): | 0.0025 m |
| Distance from the symmetry axis to the inner turn coil ($J$): | 0.009 m |
| Distance between coil turns ($K$): | 0.0003 m |
| Initial distance coil-workpiece ($B$): | 0.002 m |
| Vertical distance from coil to bottom ($C$): | 0.05 m |
| Vertical distance from workpiece to the top ($A$): | 0.05 m |
| Width of the workpiece ($E$): | 0.115 m |
| Width of the rectangular box ($R$): | 0.2 m |
| Number of coil turns: | 9 |
| Electrical conductivity of metal ($\sigma$): | 25900 (Ohm m)$^{-1}$ |
| Magnetic permeability of all materials ($\mu$): | $4\pi 10^{-7}$ Hm$^{-1}$ |
| Density of the workpiece: | 2700 kg/m$^3$ |
| Final time ($T$): | 90 $\mu$s |

At each time $t \in [0, T]$, the source current density $J_{\mathrm{S}}(t)$ is assumed to be constant in each turn of the coil and the same for all turns. The value of $J_{\mathrm{S}}(t)$ has been obtained from [12], where the corresponding intensity was reported as a function of time. Figure 6.4 shows this intensity during the whole process.

We have used a mesh significantly more refined in a subdomain covering the coil and $\bigcup_{t\in[0,T]} \Omega_t$. Figure 6.5 shows a zoom of the mesh on this zone. A low order integration rule has been used as in the previous test, to compute the integrals on triangles of piecewise smooth discontinuous functions.

We denote by $\boldsymbol{f}$ the Lorentz force density which is given by $\boldsymbol{f}(r, z, t) := \boldsymbol{J} \times \boldsymbol{B} = f_r(r, z, t)\boldsymbol{e}_r + f_z(r, z, t)\boldsymbol{e}_z$. To determine the position of $\Omega_t$ along the time, we assume that the workpiece moves as a rigid body under the action of the (3D) resultant of

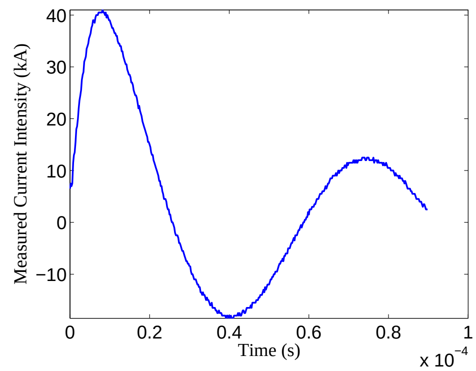


FIG. 6.4. *Test 2. Current intensity vs. time.*

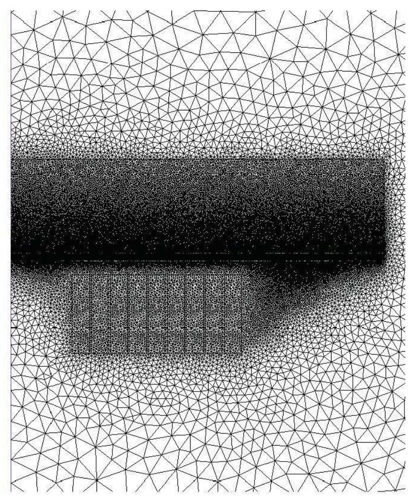

FIG. 6.5. *Test 2. Zoom of the mesh around the coil and* $\bigcup_{t\in[0,T]}\Omega_t$.

the axial component of $\boldsymbol{f}$, which is given by

$$2\pi \int_{\Omega_t} f_z(r,z,t)\, r\, dr\, dz. \tag{6.1}$$

(Notice that the 3D-resultant of the radial component has to vanish, because of the axisymmetric assumption.) More precisely, we have coupled the electromagnetic problem with the motion equation

$$mv_z'(t) = 2\pi \int_{\Omega_t} f_z(r,z,t)\, r\, dr\, dz, \tag{6.2}$$

$$v_z(0) = 0, \tag{6.3}$$

where $v_z$ and $m$ denotes the axial velocity and the mass of the rigid body, respectively. Notice that $m$ is computed from the volume of the workpiece and its density, which is given in Table 6.1.

The coupling between the electromagnetic and the mechanical problem has been done by using the following iterative scheme where the notation $(\cdot)^s$ refers to the quantities computed at iteration $s$:

1. Take $(v_z(t))^0 = 0$ and, consequently, $(\Omega_t)^0 = \Omega_0$ for all $t \in [0,T]$.
2. For $s = 0, 1, 2, \ldots$
   (a) compute $(\boldsymbol{J}(t))^s$ and $(\boldsymbol{B}(t))^s$, $t \in [0,T]$, by using $(\Omega_t)^s$ as the domain occupied by the workpiece at each time $t \in [0,T]$;
   (b) compute $(f_z(t))^s$ by using $(\boldsymbol{J}(t))^s$ and $(\boldsymbol{B}(t))^s$, $t \in [0,T]$;
   (c) compute $(v_z(t))^{s+1}$ by solving problem (6.2)–(6.3) with forcing data $(f_z(t))^s$, $t \in [0,T]$;
   (d) compute the location of the domain $(\Omega_t)^{s+1}$ for all $t \in [0,T]$, by taking $(v_z(t))^{s+1}$, $t \in [0,T]$, as the velocity of the workpiece, which is taken as a rigid body occupying the domain $\Omega_0$ at time $t = 0$;
   (e) if $(v_z(t))^s$ and $(v_z(t))^{s+1}$ differ in $\|\cdot\|_\infty$ less than a prescribe tolerance, then stop; else, next $s$.

Figure 6.6 (left) shows a plot of the velocity $v_z$ versus time corresponding to several iterations. The convergence of the iterative scheme can be clearly seen from this figure. We remark that convergence has been achieved in seven iterations by using a relative tolerance of 0.1%. Figure 6.6 (right) shows the resultant of the axial force (cf. (6.1)) versus time corresponding to the last iteration of the scheme.

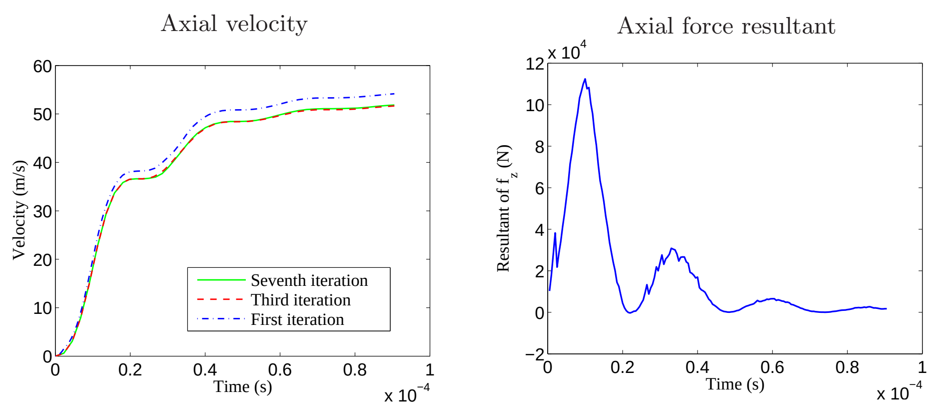


FIG. 6.6. *Test 2. Axial velocity vs. time (left) at three iterations and axial force resultant vs. time (right) at the last iteration.*

Finally, Figure 6.7 shows the computed current densities at $10\,\mu$s and $90\,\mu$s, respectively.

**Appendix. A trace result in $W_r^{1,p}(Q)$.**

As stated in Remark 3.1, the initial condition in Problem 1 makes sense because we are searching a solution of this problem in $H_r^1(Q)$ and a trace result holds in this space. Notice that the additional regularity proved in Theorem 3.6 does not imply that $A(t)$ has a limit as $t$ goes to zero, because of the term $\sqrt{t}$ in this theorem. Thus, we have to prove that a function in $H_r^1(Q)$ has a well defined trace at $t = 0$.

The aim of this appendix is to prove such a result in $W_r^{1,p}(Q)$ for $p \in [1, \infty)$. We keep here the notation for functional spaces on $Q$ introduced in section 3. An analogous notation will be used for functional spaces on the cylinder $(0,T) \times \widehat{\Omega}$ (with $\Omega_t$ replaced with $\widehat{\Omega}$). Moreover, we denote by $\widehat{\boldsymbol{x}} = (\widehat{r}, \widehat{z})$ a generic point in $\overline{\widehat{\Omega}}$ and by

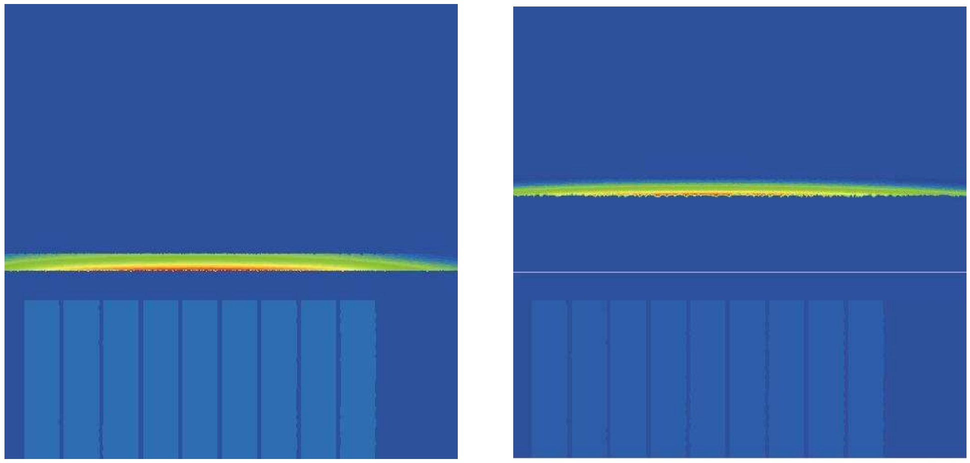

FIG. 6.7. *Test 2. Current Density at* $10\,\mu$s *(left) and* $90\,\mu$s *(right).*

$\boldsymbol{x} = (r, z)$ a generic point in $\overline{\Omega}_t$.

LEMMA A.1. *Let $Q$ and $\Omega_t$, $t \in [0, T]$, be as defined in section 3 by means of a mapping $\boldsymbol{X} \in \mathcal{C}([0, T] \times \overline{\widehat{\Omega}}; [0, \infty) \times \mathbb{R})$ satisfying hypotheses (i)–(iv) from Theorem 3.1. Let $p \in [1, \infty)$ be fixed. For all $t \in [0, T]$, there exists a unique linear continuous operator $\gamma_t : W_r^{1,p}(Q) \to L_r^p(\Omega_t)$ such that, for all $\varphi \in \mathcal{C}^1(\overline{Q})$,*

$$(\gamma_t \varphi)(\boldsymbol{x}) = \varphi(t, \boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega_t.$$

*Moreover, the norm of this operator is bounded independently of $t$.*

*Proof.* We will give a sketch of the proof, which we decompose into six steps.

- *Step 1: $\mathcal{C}^1([0, T] \times \overline{\widehat{\Omega}})$ is dense in $W_r^{1,p}((0, T) \times \widehat{\Omega})$.*

Let $u \in W_r^{1,p}((0, T) \times \widehat{\Omega})$. Using standard arguments, $u$ can be approximated by finite sums of the form $\sum \psi_i(t) v_i(\boldsymbol{x})$, where $\psi_i \in \mathcal{C}^1([0, T])$ and $v_i \in W_r^{1,p}(\widehat{\Omega})$. Then, thanks to Theorem 4.3 from [13], each $v_i$ can be approximated, in the sense of $W_r^{1,p}(\widehat{\Omega})$, by functions in $\mathcal{C}^1(\overline{\widehat{\Omega}})$. This yields the result.

- *Step 2: Let $\boldsymbol{\Psi} : [0, T] \times \overline{\widehat{\Omega}} \to \mathbb{R}^3$ be the mapping defined by $\boldsymbol{\Psi}(t, \widehat{\boldsymbol{x}}) := (t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}}))$. Then, $\boldsymbol{\Psi} : [0, T] \times \overline{\widehat{\Omega}} \to \overline{Q}$ is a homeomorphism, $\boldsymbol{\Psi} \in \mathcal{C}^1([0, T] \times \overline{\widehat{\Omega}})$ and $\boldsymbol{\Psi}^{-1} \in \mathcal{C}^1(\overline{Q})$.*

The proof that $\boldsymbol{\Psi}$ is a homeomorphism onto $\overline{Q}$ follows from assumption (i) and the continuity of $\boldsymbol{X}$, by noticing that $Q = \boldsymbol{\Psi}((0, T) \times \widehat{\Omega})$. The regularity of $\boldsymbol{\Psi}$ is clear from assumption (iii), whereas that of $\boldsymbol{\Psi}^{-1}$ follows from assumptions (iii) and (iv) by using the inverse function theorem.

- *Step 3: Let $X_1$ and $X_2$ be the components of $\boldsymbol{X}$. There exist $c_1, c_2 > 0$ such that*

$$c_1 \widehat{r} \le X_1(t, \widehat{r}, \widehat{z}) \le c_2 \widehat{r} \qquad \forall (t, \widehat{r}, \widehat{z}) \in [0, T] \times \overline{\widehat{\Omega}}. \tag{A.1}$$

First, we consider the case that $\overline{\widehat{\Omega}}$ does not intersect the axis $\widehat{r} = 0$. Then, assumption (ii) and the fact that $\boldsymbol{X} \in \mathcal{C}([0, T] \times \overline{\widehat{\Omega}}; [0, \infty) \times \mathbb{R})$ imply that, for all points $(t, \widehat{r}, \widehat{z})$ in the compact set $[0, T] \times \overline{\widehat{\Omega}}$, $X_1(t, \widehat{r}, \widehat{z})$ is bounded above and below by strictly positive constants. Since the same happens to $\widehat{r}$ for all points $(\widehat{r}, \widehat{z})$ in the compact set $\overline{\widehat{\Omega}}$, the property holds in this case.

Next, we consider the case that $\overline{\widehat{\Omega}}$ intersects the axis $\widehat{r}=0$. Then, according to the geometrical assumptions on $\widehat{\Omega}$, there exists $a>0$ such that, for all $\delta\in(0,a)$, $G_\delta:=\{(\widehat{r},\widehat{z})\in\overline{\widehat{\Omega}}:\ \widehat{r}\le\delta\}$ is a trapezoid with parallel sides aligned with the axis $\widehat{r}=0$. We write $\overline{\widehat{\Omega}}=G_\delta\cup F_\delta$, with $F_\delta:=\{(\widehat{r},\widehat{z})\in\overline{\widehat{\Omega}}:\ \widehat{r}\ge\delta\}$. Since $F_\delta$ does not intersect the axis $\widehat{r}=0$, we have just proved that (A.1) holds in $[0,T]\times F_\delta$. Thus, we only need to find $\delta\in(0,a)$ such that (A.1) holds in $[0,T]\times G_\delta$, too.

The proof of the latter follows by means of standard arguments based on the mean value theorem and the facts that $X_1(t,0,\widehat{z})$ vanishes (cf. assumption (ii)), $\frac{\partial X_1}{\partial\widehat{r}}$ and $\frac{\partial X_1}{\partial\widehat{z}}$ are bounded above (cf. assumption (iii)), and that, for a sufficiently small $\delta>0$, $\left|\frac{\partial X_1}{\partial\widehat{z}}\right|$ is bounded in $G_\delta$ by a suitable small constant and $\frac{\partial X_1}{\partial\widehat{r}}$ is bounded below by a strictly positive constant. In its turn, this lower bound can be proved by using the uniform continuity of $\frac{\partial X_1}{\partial\widehat{r}}$ and the following facts:

a) $\exists\alpha>0:\ \det(D_{\widehat{\boldsymbol{x}}}\boldsymbol{X})(t,0,\widehat{z})\ge\alpha$ (because of assumptions (iii) and (iv));
b) $\det(D_{\widehat{\boldsymbol{x}}}\boldsymbol{X})(t,0,\widehat{z})=\frac{\partial X_1}{\partial\widehat{r}}(t,0,\widehat{z})\,\frac{\partial X_2}{\partial\widehat{z}}(t,0,\widehat{z})$ (because $\frac{\partial X_1}{\partial\widehat{z}}(t,0,\widehat{z})=0$);
c) $\frac{\partial X_1}{\partial\widehat{r}}(t,0,\widehat{z})\ge0$ (because $X_1(t,\widehat{r},\widehat{z})\ge0$ and $X_1(t,0,\widehat{z})=0$);
d) $\frac{\partial X_2}{\partial\widehat{z}}$ is bounded above (because of assumption (iii)).

• *Step 4: For any measurable function $u:Q\to\mathbb{R}$, let $\widehat{u}:=u\circ\boldsymbol{\Psi}:(0,T)\times\widehat{\Omega}\to\mathbb{R}$. Let $L$ be the linear operator defined by $Lu:=\widehat{u}$. For all $p\in[1,\infty)$, $L$ is an isomorphism between $L^p_r(Q)$ and $L^p_r((0,T)\times\widehat{\Omega})$ and also an isomorphism between $W^{1,p}_r(Q)$ and $W^{1,p}_r((0,T)\times\widehat{\Omega})$.*

Using (A.1), the change of variables $(r,z)=\boldsymbol{X}(t,\widehat{r},\widehat{z})$, and step 2, it is easy to prove that $u\in L^p_r(Q)$ if and only if $\widehat{u}\in L^p_r((0,T)\times\widehat{\Omega})$ and that there exist $c_3,c_4>0$ such that

$$c_3\,\|u\|_{L^p_r(Q)}\le\|\widehat{u}\|_{L^p_r((0,T)\times\widehat{\Omega})}\le c_4\,\|u\|_{L^p_r(Q)}\,. \tag{A.2}$$

Hence $L:L^p_r(Q)\to L^p_r((0,T)\times\widehat{\Omega})$ is an isomorphism.

Next we prove that $L$ maps $W^{1,p}_r(Q)$ into $W^{1,p}_r((0,T)\times\widehat{\Omega})$. For $\varepsilon>0$, let $\widehat{\Omega}_\varepsilon:=\{(\widehat{r},\widehat{z})\in\widehat{\Omega}:\ \widehat{r}>\varepsilon\}$ and $Q_\varepsilon:=\boldsymbol{\Psi}((0,T)\times\widehat{\Omega}_\varepsilon)$. Since $u\in W^{1,p}_r(Q_\varepsilon)\simeq W^{1,p}(Q_\varepsilon)$, by applying the chain rule (see [8, Prop. IX.6], for instance), we have that $\widehat{u}\in W^{1,p}((0,T)\times\widehat{\Omega}_\varepsilon)$. Since $\varepsilon>0$ can be taken arbitrarily small, $\widehat{u}\in W^{1,p}_{\mathrm{loc}}((0,T)\times\widehat{\Omega})$ and the chain rule is valid, in fact, a.e. in $(0,T)\times\widehat{\Omega}$. Using this, assumption (iii), and (A.2), we obtain that $\widehat{u}\in W^{1,p}_r((0,T)\times\widehat{\Omega})$ and $\|\widehat{u}\|_{W^{1,p}_r((0,T)\times\widehat{\Omega})}\le C\|u\|_{W^{1,p}_r(Q)}$. Therefore, $L:W^{1,p}_r(Q)\to W^{1,p}_r((0,T)\times\widehat{\Omega})$ is a bounded operator. A similar argument, using now that $\boldsymbol{\Psi}^{-1}\in\mathcal{C}^1(\overline{Q})$ (cf. step 2), allows us to prove that $L^{-1}:W^{1,p}_r((0,T)\times\widehat{\Omega})\to W^{1,p}_r(Q)$ is also bounded.

• *Step 5: For all $p\in[1,\infty)$, the space $\mathcal{C}^1(\overline{Q})$ is dense in $W^{1,p}_r(Q)$.* It follows from steps 4 and 1 and the fact that $\widehat{u}\in\mathcal{C}^1([0,T]\times\overline{\widehat{\Omega}})$ if and only if $u\in\mathcal{C}^1(\overline{Q})$, which in its turn is a consequence of step 2.

• *Step 6: Conclusion of the proof.*

Let $u \in \mathcal{C}^1(\overline{Q})$. It is easy to show that, for all $t \in [0,T]$,

$$\int_{\widehat{\Omega}} |\widehat{u}(t,\widehat{r},\widehat{z})|^p \ \widehat{r}\, d\widehat{r}\, d\widehat{z}$$
$$\leq \frac{2^{p-1}}{T}\left[\int_0^T \int_{\widehat{\Omega}} |\widehat{u}(\tau,\widehat{r},\widehat{z})|^p \ \widehat{r}\, d\widehat{r}\, d\widehat{z}\, d\tau + T^p \int_0^T \int_{\widehat{\Omega}} \left|\frac{\partial \widehat{u}}{\partial t}(\tau,\widehat{r},\widehat{z})\right|^p \ \widehat{r}\, d\widehat{r}\, d\widehat{z}\, d\tau\right].$$

Straightforward calculations using again the change of variables $(r,z) = \boldsymbol{X}(t,\widehat{r},\widehat{z})$, equation (A.1), the boundedness of $\det(D_{\widehat{\boldsymbol{x}}}\boldsymbol{X})(t,\widehat{\boldsymbol{x}})$, the inequality above, and step 4 lead to

$$\int_{\Omega_t} |u(t,r,z)|^p \ r\, dr\, dz \leq C(T)\, \|u\|^p_{W^{1,p}_r(Q)}.$$

This estimate together with step 5 allow us to conclude the proof by means of a density argument. □

*Remark* A.1. The above lemma still holds true under the weaker regularity assumption $\boldsymbol{X} \in \mathcal{C}^1([0,T] \times \overline{\widehat{\Omega}}, \mathbb{R}^2)$ instead of assumption (iii).

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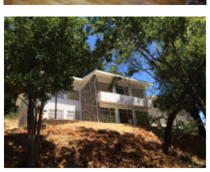

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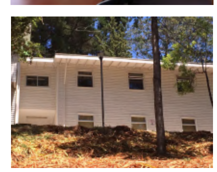
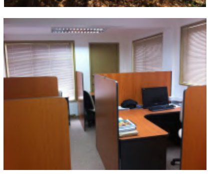

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