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A unified vector optimization problem: complete scalarizations and applications

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Abstract

The aim of this paper is to introduce and analyse a general vector optimization problem in a unified framework. Using a well-known nonlinear scalarizing function defined by a solid set, we present complete scalarizations of the solution set to the vector problem without any convexity assumptions. As applications of our results we obtain new optimality conditions for several classical optimization problems by characterizing their solution set.

Key words. vector optimization, efficiency, approximate efficiency, scalarization, weak efficiency, strict efficiency, optimality conditions;

Mathematics subject classification 2000. 90C26, 90C29, 90C30, 90C46

1 Introduction

In most real-life problems, optimization problems concern the minimization of several criterion functions simultaneously. Very often, no single point minimizing all criteria at once may be found, and therefore others concepts of optimality arise. Among them, the so-called weak efficient, efficient, strict efficient, or proper efficient solution are discussed in the literature.

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By algorithmic and theoretical purposes, one needs to describe the whole solution set to a vector problem via scalarization. For instance, given a function $f : K \to \mathbb{R}^m$, if we are interested in its weakly efficient minima with respect to the nonnegative orthant in $\mathbb{R}^m$, E_W, one desires to know under which conditions, the equivalence

$$\bar{x} \in E_W \Longleftrightarrow \bar{x} \in \bigcup_{p^* \in \mathbb{R}^m_+, p^* \neq 0} \operatorname{argmin}_K \langle p^*, f(\cdot)\rangle \tag{1}$$

holds. It is well-known that such an equivalence (1) is true whenever each component of f is convex, but it is still true under weaker assumption as shown in [19, 9]. The authors in [9] established a necessary and sufficient condition in order the equivalence (1) is satisfied: it requires the convexity of the set $\operatorname{cone}(f(K) - f(\bar{x}) + \mathbb{R}^2_+)$, where $\operatorname{cone}(A)$ means the smallest cone containing the set A. In spite of this fact, solving the vector problem via the equivalence (1) (giving rise to the weighting method), requires to know p^* in advance. This is the main drawback of the procedure. In fact, by taking the function $f(x) = x = (x_1, x_2)$, and $K = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 + x_2 \geq 1\}$, we get, for given $p^* = (p_1^*, p_2^*) \in \mathbb{R}^2_+$, $p^* \neq 0$,

$$\inf_{x \in K} \langle p^*, f(x)\rangle \in \mathbb{R} \Longleftrightarrow \ p_1^* = p_2^*.$$

Here $E_W = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 + x_2 = 1\}$. A way to choose the parameter p^* is discussed in [5] under the boundedness from below of $\langle p^*, f(\cdot)\rangle$ on K.

The outline of the paper is as follows. In Section 2 we formulate the unified vector optimization problem and discuss its generality. Section 3 introduces the scalarizing function and recalls its useful properties under very mild conditions. Section 4, is devoted to describe the scalarization procedure for (approximate) efficiency by establishing complete scalarizations under a more general assumption, termed (B). Section 5 provides optimality conditions for specific optimization problems and gives several examples. In Section 6, the main conclusions are presented.

2 A unified vector optimization problem

Let Y be a real topological vector space and let X be a Banach space.

Given a nonempty set $S \subsetneq Y$, a nonempty set $K \subseteq X$ and a function $f : K \to Y$, we are interested in the problem

$$(\mathcal{P}) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in S \quad \forall\, x \in K,\ x \neq \bar{x}.$$

We emphasize the generality of problem $(\mathcal{P})$ from the economic point of view since the preference order can be given on X by a function f or/and on Y by a set S which is not necessarily a cone (see Figure 1).

The set of such vectors $\bar{x} \in K$ is denoted by $E_S = E_S(K)$, and each one of its elements is called a (global) S-minimizer of f on K.

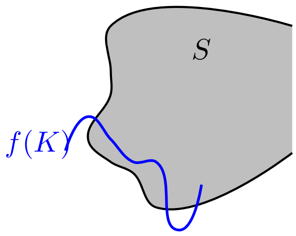

Figure 1: Illustration of problem $(\mathcal{P})$.

Since most resolution methods, like the iterative and heuristic ones, yield feasible points near to the exact solution, we also are interested in the main notion of approximate solutions.

Given $\varepsilon \geq 0$ and $0 \neq q \in Y$, we consider the following approximated problem associated to $(\mathcal{P})$:

$$(\mathcal{P}(\varepsilon q)) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in -\varepsilon q + S \quad \forall\, x \in K,\ x \neq \bar{x},$$

where S is any set satisfying $S + \mathbb{R}_{++}q \subseteq S$, where $\mathbb{R}_{++} \doteq\ ]0, +\infty[$. We denote by $E_S(\varepsilon q)$ the solution set to $(\mathcal{P}(\varepsilon q))$. Thus, the previous inclusion implies that:

$$0 \leq \varepsilon_1 < \varepsilon_2 \Longrightarrow E_S(\varepsilon_1 q) \subseteq E_S(\varepsilon_2 q); \qquad E_S = E_S(0) \subseteq E_S(\varepsilon q) \ \ \forall\, \varepsilon > 0.$$

Consequently,

$$E_S \subseteq \bigcap_{\varepsilon > 0} E_S(\varepsilon q) \subseteq E_{\mathrm{cl}\, S}.$$

When f is a real function we denote by $E(f, K, \varepsilon)$ the set of ε-solutions, that is, $\bar{x} \in E(f, K, \varepsilon)$ if and only if $f(x) - f(\bar{x}) \geq -\varepsilon$ for all $x \in K$. The above approximate problem is defined in the Kutateladze's sense [20]. In White [29] several notions of approximate solutions are discussed.

If we consider the classical framework of vector spaces ordered by a proper convex cone $\{0\} \neq P \subseteq Y$ we obtain the following well-known notions of optimality.

In what follows, given any $\emptyset \neq A \subseteq Y$, we denote by $\mathcal{C}(A)$, $\mathrm{int}\, A$, $\mathrm{cl}\, A$ and ∂A the complement, the topological interior, the topological closure and the boundary of A respectively. A convex cone P is called pointed if $l(P) \doteq P \cap (-P) = \{0\}$. If

- $S = P$, the solutions are termed "*ideal*" or "*strong*" minima of f and the solution set is denoted by E_P;

- $(\mathrm{int}\, P \neq \emptyset)$ $S = \mathcal{C}(-\,\mathrm{int}\, P)$, the solutions are called "*weakly efficient*" minima of f and for the set of solutions we use E_W;

- $S = \mathcal{C}(-P) \cup l(P)$, the solutions are said to be "*efficient*" minima of f and we set $E \doteq E_S$;

- when P is not pointed and $S = \mathcal{C}(-P) \cup \{0\}$, such solutions are named "*weakly strict efficient*" minima of f, in this case $E_{W1} \doteq E_S$;

- $S = \mathcal{C}(-P)$, such solutions are named "*strict efficient*" minima of f and the solution set is denoted by E_1;

- $S = \mathcal{C}(-D) \cup l(D)$ for some proper convex cone $D \subseteq Y$ with nonempty interior satisfying $P \setminus l(P) \subseteq \operatorname{int} D$, the solutions are called "*Henig proper efficient*" minima of f, and the solution set is denoted by E_2;

- $S = \mathcal{C}(-\operatorname{int} A)$ for some closed convex set $A \subsetneq Y$ with nonempty interior satisfying $0 \in \partial A$ and $A + (P \setminus \{0\}) \subseteq \operatorname{int} A$, the solutions are called "*proper efficient*" minima and the solution set is denoted by E_3. Note that $A + (P \setminus \{0\}) \subseteq \operatorname{int} A$ implies the pointedness of P.

Note that if $\operatorname{int} P = \emptyset$ then we can consider P with algebraic interior or relative algebraic interior nonempty.

Since $l(P) \cup \mathcal{C}(-P) \subseteq \mathcal{C}(-\operatorname{int} P)$ and $l(D) \cup \mathcal{C}(-D) \subseteq l(P) \cup \mathcal{C}(-P)$ we have $E_2 \subseteq E \subseteq E_W$. In this case, if $E_P \neq \emptyset$ then $E_P = E$. On the other hand: $E_1 \subseteq E_{W1} \subseteq E$; $E = E_{W1}$ whenever P is pointed; $E_W = E_P$ provided P is a closed halfspace (see Lemma 2.5 in [8]); $E_1 = E_{W1}$ whenever f is injective. Moreover, when P is pointed every Henig proper efficient solution is also proper efficient.

The set E_3 has been studied in [28], see the references therein for more details; whereas the notion of strict efficiency is further developed in [10].

As we shall see in Section 5, problem $(\mathcal{P})$ categorizes optimization problems given by a not necessarily pre-order relation. Such non-transitive preferences are very interesting in mathematical economics see [23] and references therein. Moreover, $(\mathcal{P}(\varepsilon q))$ encompasses several notions of ε-efficiency since as a particular case we obtain the $Q(\varepsilon)$-efficiency concept introduced by Gutiérrez et al. in [15] to unified the approximated solutions in vector optimization.

3 The nonlinear scalarizing function

A nonlinear scalarizing function that nowadays is having a great impact in the development of a theoretical and algorithmic treatment of vector optimization problems, is a particular Minkowski-type function which is known under different names in several areas of applied mathematics. Specially, such a scalarizing function has been considered in optimization theory by several authors. For instance, in mathematical economics we refer to Bonnisseau and Cornet [1], [22]; for vector solutions, to Pascoletti and Serafini [24], Jahn [18], Luc [21], Gerth and Weidner [11], Rubinov and Gasimov [27]; for set solutions, to Hamel and Löhne [16], Hernández and Rodríguez-Marín [17], for approximate solutions, Gutiérrez, Jiménez and Novo [13] and [14]; for fuzzy optimality conditions to Durea and Tammer [6]; and for computing ε-efficient solutions to Engau and Wiecek [7].

Let $q \in Y$, $q \neq 0$, be fixed. Set $\mathbb{R}_+ \doteq [0, +\infty[$, $\mathbb{R}_{++} \doteq \,]0, +\infty[$. Let $\emptyset \neq A \subseteq Y$.

Definition 3.1. *Let $\xi_{q,A}\colon Y \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$\xi_{q,A}(y) \doteq \inf\{t \in \mathbb{R} \colon y \in tq - A\} \quad (y \in Y).$$

We use the convention $\inf \emptyset = +\infty$. It is well-known that such a function has many useful properties of separation and monotonicity which play a central rol in the proofs of the main previously mentioned results in a nonconvex setting. We emphasize that a good account of its properties is given in Göpfert, Riahi, Tammer and Zălinescu [12] and Tammer and Zălinescu [28].

Now we recall some of its properties which are used to establish complete scalarizations of problem $(\mathcal{P})$ in the following section. Some of them are given in [12, Theorem 2.3.1] for closed sets, A.

Firstly, we need directions of the recession cone of a nonempty set $A \subseteq Y$, $A_\infty = \{y \in Y \colon a + \mathbb{R}_{++} y \in A, \ \forall\, a \in A\}$, then $A + A_\infty = A$.

Proposition 3.2. *Let $\lambda \in \mathbb{R}$, $q \neq 0$, The following assertions hold:*

(*a*) $\{y \in Y : \xi_{q,A}(y) < \lambda\} = \,]-\infty, \lambda[\, q - A$; *thus*

$$\{y \in Y : \xi_{q,A}(y) < 0\} = -\mathbb{R}_{++} q - A; \quad \{y \in Y : \xi_{q,A}(y) < +\infty\} = \mathbb{R} q - A.$$

If, in addition, $A + \mathbb{R}_{++} q \subseteq A \subsetneq Y$.

(*b*) $\lambda q - \operatorname{int} A \subseteq \{y \in Y : \xi_{q,A}(y) < \lambda\} \subseteq \lambda q - A \subseteq \{y \in Y : \xi_{q,A}(y) \leq \lambda\} \subseteq \lambda q - \operatorname{cl} A$;

(*c*) $\{y \in Y : \xi_{q,A}(y) = \lambda\} \subseteq \lambda q - \partial A$;

From the preceding result we obtain the next corollary.

Corollary 3.3. *Let $q \in Y$, $q \neq 0$, $A \subsetneq Y$.*

(*a*) *Assume that* $\operatorname{cl} A + \mathbb{R}_{++}q \subseteq A$, *then*

$$\{y \in Y: \quad \xi_{q,A}(y) \leq \lambda\} = \lambda q - \operatorname{cl} A, \quad \forall\, \lambda \in \mathbb{R} \quad \textit{and} \quad \xi_{q,A}(y) = \xi_{q,\operatorname{cl} A}(y).$$

(*b*) *Assume that* $\operatorname{int} A \neq \emptyset$ *and* $A + \mathbb{R}_{++}q \subseteq \operatorname{int} A$, *then*

$$\{y \in Y: \quad \xi_{q,A}(y) < \lambda\} = \lambda q - \operatorname{int} A, \quad \forall\, \lambda \in \mathbb{R} \quad \textit{and} \quad \xi_{q,A}(y) = \xi_{q,\operatorname{int} A}(y).$$

Taking into account that $[\operatorname{cl} A + \mathbb{R}_{++}q \subseteq A$ and $A + \mathbb{R}_{++}q \subseteq \operatorname{int} A] \Longleftrightarrow \operatorname{cl} A + \mathbb{R}_{++}q \subseteq \operatorname{int} A$ we have the following result.

Corollary 3.4. *Let* $q \in Y$, $q \neq 0$, $A \subsetneq Y$.

(*a*) *If* $\operatorname{cl} A + \mathbb{R}_{++}q \subseteq \operatorname{int} A$, *then for all* $\lambda \in \xi_{q,A}(Y)$,

$$\{y \in Y: \quad \xi_{q,A}(y) = \lambda\} = \lambda q - \partial A.$$

(*b*) *If* $\operatorname{cl} A + \mathbb{R}_{++}q \subseteq A$ *and* $\operatorname{cl} A$ *is convex, then* $\xi_{q,A}$ *is convex.*

Remark 3.5. *Note that* $\operatorname{cl} A + \mathbb{R}_{++}q \subseteq \operatorname{int} A$ *implies that* $q \in (\operatorname{cl} A)_\infty$. *On the other hand, the condition* $A + \operatorname{int} P \subseteq A \neq Y$, *with* P *being any convex cone with nonempty interior, implies* $\operatorname{cl} A + \mathbb{R}_{++}q \subseteq \operatorname{int} A \ \ \forall\, q \in \operatorname{int} P$. *Indeed, by* [3, *Lemma* 2.5], *we obtain*

$$\operatorname{cl} A + \mathbb{R}_{++}q \subseteq \operatorname{cl} A + \operatorname{int} P = \operatorname{int}(\operatorname{cl} A + P) = \operatorname{int}(\operatorname{cl}(A + P)) = \operatorname{int}(A + P) = \operatorname{int} A.$$

Lemma 3.6. *Let* $\emptyset \neq A \subseteq Y$ *and* $P \subseteq Y$ *be any proper convex cone with nonempty interior. Then, for all* $q \in \operatorname{int} P$,

$$\xi_{q,A+P} = \xi_{q,A+(P\setminus l(P))} = \xi_{q,A+(P\setminus\{0\})} = \xi_{q,\operatorname{cl}(A+P)} = \xi_{q,A+\operatorname{cl} P} = \xi_{q,A+\operatorname{int} P}.$$

Proof. It is a consequence of Corollary 3.3 and [3, Lemma 2.5]. □

Lemma 3.7. *Suppose that* A, $B \subseteq Y$, *such that* $B + B \subseteq B$. *Let* y, $y' \in Y$ *and* $q \in Y$.

(*a*) *If* $y - y' \in -B$, *then* $\xi_{q,B-A}(y) \leq \xi_{q,B-A}(y')$;

(*b*) *if* $y - y' \in -\operatorname{int} B$, *then* $\xi_{q,B-A}(y) < \xi_{q,B-A}(y')$.

Proof. By definition $\xi_{q,B-A}(y) = \inf\{t \in \mathbb{R}\colon y \in tq + A - B\}$ for every $y \in Y$. We only prove (*b*) since (*a*) is entirely similar. Since $y - y' \in -\operatorname{int} B$ there exists $\varepsilon < 0$ such that $y - y' - \varepsilon q \in -B$. Thus, if $t \in \mathbb{R}$ is such that $y' \in tq - (B - A)$, then

$$y \in \varepsilon q - B + tq - (B - A) \subseteq (\varepsilon + t)q - (B - A)$$

since $B + B \subseteq B$. It follows that $\xi_{q,B-A}(y) \leq \varepsilon + t$ and hence $\xi_{q,B-A}(y) < \xi_{q,B-A}(y')$. □

4 Scalarizations for a unified vector optimization problem

Following the notations introduced in the previous sections, we establish scalarizing conditions for the problems

$$(\mathcal{P}) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in S \quad \forall\, x \in K,\ x \neq \bar{x},$$

and

$$(\mathcal{P}(\varepsilon q)) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in -\varepsilon q + S \quad \forall\, x \in K,\ x \neq \bar{x},$$

where $\varepsilon \geq 0$, $q \in Y$, $q \neq 0$, $\emptyset \neq K \subseteq X$ and $f : K \to Y$ by introducing families of scalar optimization problems which will describe the solution set to $(\mathcal{P})$ and $(\mathcal{P}(\varepsilon q))$, denoted by E_S and $E_S(\varepsilon q)$ respectively. This will be carried out through the scalarizing function discussed in the previous section.

According to [21, Definition 3.1, pag. 95], given a family G of functions $g : Y \to \mathbb{R}$, we say that G is a complete scalarization for $(\mathcal{P})$ if for every $x \in E_S$ there exists $g \in G$ such that $x \in E(g \circ f, K)$ and $E(g \circ f, K) \subseteq E_S$, where $E(g \circ f, K)$ denotes the solution set to $(\mathcal{SP})$:

$$(\mathcal{SP}) \qquad \min\{(g \circ f)(x) \colon x \in K\}.$$

In other words, G is a complete scalarization for $(\mathcal{P})$ if and only if there exists $G' \subseteq G$ such that

$$E_S = \bigcup_{g \in G'} E(g \circ f, K).$$

Similar representations will be established for $(\mathcal{P}(\varepsilon q))$.

In this section we consider sets satisfying the so-called free-disposal assumption $(S + P = S)$ introduced by Debreu [4] in the setting of mathematical economics.

Assumption (A): $P \subseteq Y$ is a proper (not necessarily closed or pointed) convex cone with nonempty interior, and $S \subsetneq Y$ is such that $0 \in \partial S$ and $S + \operatorname{int} P = \operatorname{int} S$, or equivalently, $S + \operatorname{int} P \subseteq S$.

Since then, several conditions related to Assumption (A) have been considered in economic theory and optimization. More precisely, given a closed convex cone P, we say that a closed set $S \subsetneq Y$ satisfies the free-disposal Assumption (P) [2, 28] if $S + P = S$; whereas S satisfies the strong free disposal Assumption (P_S) [28]: if $S + (P \setminus \{0\}) = \operatorname{int} S$, or equivalently, $S + (P \setminus \{0\}) \subseteq \operatorname{int} S$.

Obviously, when $0 \in \partial S$, $\operatorname{int} S \neq \emptyset$ and $\operatorname{int} P \neq \emptyset$, we get $(\mathrm{P}_S) \Longrightarrow (\mathrm{P}) \Longrightarrow (\mathrm{A})$. Certainly, the set $S = \mathcal{C}(-P) \cup l(P)$ satisfies $S + P = S$ without being closed; whereas $S = \mathcal{C}(-P) \cup \{0\}$ satisfies (A), but it is non-closed and $S + P \neq S$ whenever P is non-pointed.

Throughout this section we impose the following assumption on S which is more general that Assumption (A) since no convex cone P is involved.

Assumption (B): $0 \neq q \in Y$, and $S \subsetneq Y$ is a (not necessarily closed) set such that $0 \in \partial S$ and $\operatorname{cl}(\mathcal{C}(-S)) + \mathbb{R}_{++}q \subseteq \operatorname{int} \mathcal{C}(-S)$.

It is clear that, the condition $0 \in \partial S$ can be assumed, after a translation, whenever $S \neq Y$. Moreover, under this assumption, $\operatorname{int} \mathcal{C}(-S) \neq \emptyset \neq \operatorname{int} S$, because of $S \neq Y$ and we have the equivalence

$$\operatorname{cl}(\mathcal{C}(-S)) + \mathbb{R}_{++}q \subseteq \operatorname{int} \mathcal{C}(-S) \iff \operatorname{cl} S + \mathbb{R}_{++}q \subseteq \operatorname{int} S.$$

By virtue of Remark 3.5, if P and S satisfy Assumption (A) then S fulfills Assumption (B) for every $q \in \operatorname{int} P$ since $S + \operatorname{int} P \subseteq S \iff \mathcal{C}(-S) + \operatorname{int} P \subseteq \mathcal{C}(-S)$, as one can check easily. We recall that Assumption (A) holds for a wide class of (not necessarily closed) sets including those classical models:

$$S = P,\ S = \mathcal{C}(-\operatorname{int} P),\ S = \mathcal{C}(-P) \cup l(P),\ S = \mathcal{C}(-P) \cup \{0\},\ S = \mathcal{C}(-P).$$

Notice that any set S satisfying $0 \in \partial S$ and $S + P = S$ fulfills Assumption (A) provided int $P \neq emptyset$, but such an equality is not verified by $S = \mathcal{C}(-P) \cup \{0\}$ when P is not pointed.

If, instead, S is closed and satisfies Assumption (P_S), then S fulfills Assumption (B) for every $q \in P \setminus \{0\}$. This allows us to deal with proper efficiency (E_3), where $S = \mathcal{C}(-\operatorname{int} A)$ for some closed convex set A such that $0 \in \partial A$ and $A + (P \setminus \{0\}) \subseteq \operatorname{int} A$. Here, P may have empty interior.

The previous remarks point out the generality of our problem ($\mathcal{P}$) due to geometric structure of S under Assumption (B) since S could not be a cone or a convex set as Figure 2 shows.

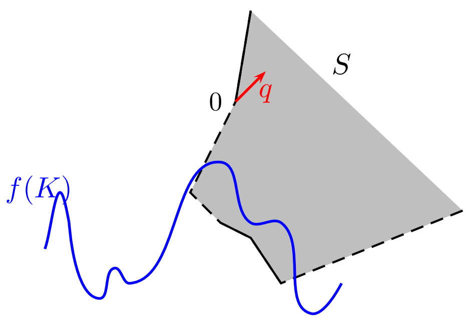

Figure 2: Illustration of problem ($\mathcal{P}$) satisfying Assumption (B).

Remark 4.1. *If* $\operatorname{cl} S + \mathbb{R}_{++}q \subseteq \operatorname{int} S$, *then* $S + \mathbb{R}_{++}q = \operatorname{cl} S + \mathbb{R}_{++}q = \operatorname{int} S$, *and so* $\operatorname{int}(\operatorname{cl} S) = \operatorname{int} S$, $\operatorname{cl}(\operatorname{int} S) = \operatorname{cl} S$. *Similar expressions hold for* $\mathcal{C}(-S)$.

The next two main theorems characterize when a point $\bar{x} \in K$ belongs to E_S (resp. $E_S(\varepsilon q)$) in terms of $E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ (resp. $E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$) under Assumption (B).

Theorem 4.2. *Suppose that q and S satisfy Assumption (B). Let $\bar{x} \in K$, the following assertions are equivalent:*

(a) $\bar{x} \in E_S$;

(b) $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ *and*

$$E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K) \setminus \{\bar{x}\} = \{x \in K : \ x \neq \bar{x}, \ f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S))\}$$

$$= \{x \in K : \ x \neq \bar{x}, \ f(x) - f(\bar{x}) \in S \setminus \operatorname{int} S\}.$$

Proof. $(a) \Longrightarrow (b)$: It is clear that $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ since $0 \in \partial(\mathcal{C}(S))$. From $\bar{x} \in E_S$, we have $f(x) - f(\bar{x}) \notin -\mathcal{C}(-S)$ for all $x \in K$, $x \neq \bar{x}$. Thus $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \geq 0$ by Proposition 3.2, which turns out $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$. On the other hand, take any $x \in K$, $x \neq \bar{x}$, such that

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) = (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0.$$

Then $f(x) - f(\bar{x}) \in \partial(-\mathcal{C}(-S))$ by Proposition 3.2. We also have $f(x) - f(\bar{x}) \in S$. From both relations, we obtain $f(x) - f(\bar{x}) \in [\partial(-\mathcal{C}(-S))] \cap S$. By simplifying, we get

$$f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S))$$

which proves one inclusion in (b).
For the other inclusion simply observe that if $x \in K \setminus \{\bar{x}\}$ is such that $f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S))$, then $f(x) - f(\bar{x}) \in -\partial(\mathcal{C}(-S))$. Hence, $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) = 0$, implying that $x \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$.
The remaining equality follows from the fact that $\mathrm{cl}(\mathcal{C}(A)) = \mathcal{C}(\operatorname{int} A)$ for every A.
$(b) \Longrightarrow (a)$: Let $x \in K$, $x \neq \bar{x}$. We distinguish two cases. If x is such that

$$f(x) - f(\bar{x}) \in S \setminus \operatorname{int} S \subseteq S,$$

we are done. If, on the contrary, $f(x) - f(\bar{x}) \notin S \setminus \operatorname{int} S$, then $x \notin E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ by assumption. Thus $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) > 0$ since $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ and $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$. Again, by Proposition 3.2, $f(x) - f(\bar{x}) \notin -\mathcal{C}(-S)$, which implies that $f(x) - f(\bar{x}) \in S$. Hence $\bar{x} \in E_S$. $\square$

In particular, we deduce that Lemma 5.2 in [28] follows from $(a) \Longrightarrow (b)$ in the previous theorem by taking S such that $E_S = E_3$.

Before continuing, some remarks are in order.

Remark 4.3. (*i*) *It may happen that the set of the right-hand side in* (*b*) *be empty* (*this occurs for instance when P is closed and $S = \mathcal{C}(-P)$): in such a situation Theorem 4.2 reduces*

$$\bar{x} \in E_1 \iff (\xi_{q,-f(\bar{x})+P} \circ f)(x) > 0 \quad \forall\, x \in K,\ x \neq \bar{x}.$$

We will discuss related points later on.
(*ii*) *When $0 \in S$ (some classical models have been described before),* (*b*) *of the previous theorem admits the following formulation:*

$$E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K) = \{x \in K :\ f(x) - f(\bar{x}) \in S \setminus \operatorname{int} S\}.$$

Now, we establish a similar characterization for the problem $(\mathcal{P}(\varepsilon q))$. Notice that it also provides another characterization for $\varepsilon = 0$.

Theorem 4.4. *Suppose that q and S satisfy Assumption (B). Let us consider problem $(\mathcal{P}(\varepsilon q))$ with $\varepsilon \geq 0$, and $\bar{x} \in K$. The following assertions are equivalent:*

(*a*) $\bar{x} \in E_S(\varepsilon q)$;

(*b*) $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$ *and*

$$E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon) \setminus \{\bar{x}\} \subseteq \{x \in K :\ x \neq \bar{x},\ f(x) - f(\bar{x}) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S)))\}.$$

Proof. (*a*) $\Longrightarrow$ (*b*): Obviously $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ since $0 \in \partial S$. From $\bar{x} \in E_S(\varepsilon q)$, we have $f(x) - f(\bar{x}) \notin -\varepsilon q - \mathcal{C}(-S)$ for all $x \in K$, $x \neq \bar{x}$. By Proposition 3.2, $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \geq -\varepsilon$, which turns out $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon$. Thus $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$.
Let us prove the inclusion in (*b*). If $x' \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$, $x' \neq \bar{x}$, then

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x') \geq -\varepsilon \quad \forall\, x \in K.$$

Since $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ we have $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x') \leq \varepsilon$. Therefore, $f(x') \in f(\bar{x}) + \varepsilon q - \operatorname{cl}(\mathcal{C}(-S))$ by Proposition 3.2. On the other hand, by hypothesis, we have $f(x') - f(\bar{x}) \in -\varepsilon q + S$. Thus, $f(x') - f(\bar{x}) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S))$.
(*b*) $\Longrightarrow$ (*a*): Let $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$. Then,

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall\, x \in K.$$

Since $\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ we have $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \geq -\varepsilon$ for all $x \in K$.
If on the contrary $\bar{x} \notin E_S(\varepsilon q)$, there exists $x' \in K$, $x' \neq \bar{x}$, such that $f(x') - f(\bar{x}) \in -\varepsilon q - \mathcal{C}(-S)$. Then $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x') \leq -\varepsilon$. From the above inequality we obtain

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x') = -\varepsilon.$$

Thus, $x' \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon) \setminus \{\bar{x}\}$, which by (b) implies that $f(x') - f(\bar{x}) \in -\varepsilon q + S$, contradicting a previous relation. Hence $\bar{x} \in E_S(\varepsilon q)$. $\square$

Remark 4.5. *We point out that taking into account that $\bar{x} \in E_S$ if and only if $[f(K \setminus \{\bar{x}\}) - f(\bar{x})] \cap \mathcal{C}(S) = \emptyset$, the implication $\bar{x} \in E_S \Rightarrow \bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ of Theorem 4.2 may be obtained by using a similar reasoning to that employed in the proof of Theorem 2.3.6 in [12] about nonconvex separation (where Assumption (B) substitutes (i) of [12, Theorem 2.3.6]). A similar argument can be used to prove the "if part" of Theorem 4.4.*

Next example shows the inclusion in Theorem 4.4(b) for $\varepsilon > 0$ may be strict.

Example 4.6. *Take $K = [-\frac{5}{2}, 2]$ and $f : K \to \mathbb{R}^2$, $f(x) = (x, x+2)$ if $-\frac{5}{2} \leq x < 0$ and $f(x) = (x, 0)$ if $0 \leq x \leq 2$. Let $S = \mathcal{C}(-\operatorname{int} \mathbb{R}^2_+)$, $q = (\frac{1}{2}, \frac{1}{2})$ and $\varepsilon = 2$. It is clear that $0 \in E_S(\varepsilon q)$, in addition,*

$$-1, -\frac{6}{5} \in \{x \in K : x \neq 0,\ f(x) - f(0) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S)))\} =$$

$$\{x \in K : x \neq 0,\ f(x) \in ((-1,-1) + S) \cap ((1,1) - \operatorname{cl}(\mathcal{C}(-S)))\}.$$

However

$$-1, -\frac{6}{5} \notin E(\xi_{q,-f(0)+\mathcal{C}(-S)} \circ f, K, \varepsilon) = E(\xi_{q,\mathcal{C}(-S)} \circ f, K, 2)$$

since

$$(\xi_{q,\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) - (\xi_{q,\mathcal{C}(-S)} \circ f)(-1) \not\geq -2$$

and

$$(\xi_{q,\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) - (\xi_{q,\mathcal{C}(-S)} \circ f)(-\frac{6}{5}) \not\geq -2$$

taking into account that $(\xi_{q,\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) = -1$, $(\xi_{q,\mathcal{C}(-S)} \circ f)(-1) = 2$ and $1 < (\xi_{q,\mathcal{C}(-S)} \circ f)(-\frac{6}{5}) < 2$.

A simpler equivalence than those in Theorems 4.2 and 4.4 can be obtained under an additional assumption on S.

Theorem 4.7. *Consider problem $(\mathcal{P}(\varepsilon q))$ and suppose that q and S satisfy Assumption (B). Let $\bar{x} \in K$. Then,*

$$\bar{x} \in E_S(\varepsilon q) \Longrightarrow \bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon) \Longrightarrow \bar{x} \in E_S(\delta q)\ \ \forall\, \delta > \varepsilon \Longrightarrow \bar{x} \in E_{\operatorname{cl} S}(\varepsilon q).$$

Consequently if, in addition, S is closed then

(a) $\bar{x} \in E_S(\varepsilon q) \Longleftrightarrow \bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$;

(b) $E_S(\varepsilon q) = \bigcap_{\delta > \varepsilon} E_S(\delta q)$.

Proof. The first implication is in Theorem 4.4.
For the second we proceed as follows. If on the contrary $\bar{x} \notin E_S(\delta q)$, then $f(x) - f(\bar{x}) \notin -\delta q + S$ for some $x \in K$, $x \neq \bar{x}$. Then, $f(x) - f(\bar{x}) \in -\delta q - \mathcal{C}(-S)$. Thus, $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \leq -\delta$. By assumption,

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x') - (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall\, x' \in K.$$

Hence, if $\delta > \varepsilon$ then $-\varepsilon \leq (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \leq -\delta < -\varepsilon$, a contradiction.
The third implication is obtained by taking the limit as δ goes to ε in $(\mathcal{P}(\delta q))$. ☐

By considering $(\mathcal{P}(\varepsilon q))$, we deduce that Theorem 4.4 extends and refines Theorems 4.5, 5.1(a) in [14]. In addition, the first part of Theorem 4.7 extends Theorem 5.1 of [14]; whereas the second part can be applied when P is any (not necessarily closed or pointed) convex cone and $S = \mathcal{C}(-\operatorname{int} P)$; or when P is a closed halfspace, to $S = P$, and when $P = Q \cup \{0\}$ with Q being open and convex satisfying $tQ \subseteq Q$ for all $t > 0$, to $S = \mathcal{C}(-P \setminus \{0\}) = \mathcal{C}(-P) \cup \{0\}$. This particular case extends Theorems 4.5 and 5.2 in [14].

The examples below show that under the assumptions given in Theorem 4.7 the implication $\bar{x} \in E_S(\delta q)\ \ \forall\, \delta > \varepsilon \Longrightarrow \bar{x} \in E_S(\varepsilon q)$ may be false when S is not closed.

Example 4.8. *Here consider $S = \mathcal{C}(-P) \cup \{(0,0)\}$ where $P = \{(x,y) \in \mathbb{R}^2 \colon x \leq 0,\ y < 0\}$. Let f be a function from $K = \mathbb{R}$ to $Y = \mathbb{R}^2$ defined by*

$$f(x) = \begin{cases} (0,-x) & if \quad x < 0; \\ (0,1) & if \quad x = 0; \\ (x,x) & if \quad 0 < x < 1; \\ (x, 2x-1) & if \quad x \geq 1, \end{cases}$$

and take $q = (1,1)$, $\varepsilon = 1$. Then, it is easy to check that $1 = \bar{x} \notin E_S(\varepsilon q)$ since $f(0) - f(\bar{x}) \notin -\varepsilon q + S$. On the other hand, taking into account that $\mathcal{C}(-S) = P$ and $\xi_{q,-f(\bar{x})+\mathcal{C}(-S)}(f(x)) = \xi_{q,\mathcal{C}(-S)}(f(x) - f(\bar{x}))$ we easily compute

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) = \begin{cases} -x-1 & if \quad x < 0; \\ -1 & if \quad x = 0; \\ x-1 & if \quad 0 < x < 1; \\ x-1 & if \quad x \geq 1. \end{cases}$$

Thus,

$$(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall x \in \mathbb{R},$$

that is, $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$, and therefore $1 = \bar{x} \in E_S(\delta q)\ \forall\, \delta > \varepsilon = 1$. Note that $E_S(0) = E_S = \emptyset$.

Example 4.9. *Consider $S = P$ with $P = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y \geq 0\} \cup \{(0,0)\}$. Let f, q and ε be as in the previous example. Then, we see that $1 = \bar{x} \notin E_S(\varepsilon q)$ since $f(0) - f(\bar{x}) \notin -q + P$. However, we can also check that $\bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K, \varepsilon)$ and so $1 = \bar{x} \in E_S(\delta q)\ \forall\ \delta > \varepsilon = 1$. Note that $E_S(0) = E_S = \emptyset$.*

In order to obtain complete scalarizations for E_S, we need the next theorem.

Theorem 4.10. *Suppose that q and S satisfy Assumption (B).*

(a) If $\emptyset \neq A \subseteq E_S$, then $A \subseteq E(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f, K) \subseteq E(\xi_{q,-f(E_S)+\mathcal{C}(-S)} \circ f, K)$ and

$$\min\{(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f)(x) :\ x \in K\} = 0.$$

(b) If $0 \in S$ and $S + [\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S$ then,

$$\bar{x} \in E_S \Longleftrightarrow \bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K) \subseteq E_S.$$

Proof. (a): Since for each $\bar{x} \in A$, $f(x) \notin f(\bar{x}) - \mathcal{C}(-S)$ for all $x \in K \setminus \{\bar{x}\}$, we obtain by Proposition 3.2, $(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(x) \geq 0$ for all $x \in K$. Thus, $(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f)(x) \geq 0$ for all $x \in K$. Since $(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f)(\bar{x}) \leq (\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$, we get

$$\bar{x} \in E(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f, K) \quad \text{and} \quad \min\{(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f)(x) :\ x \in K\} = 0.$$

The same reasoning also proves

$$\min\{(\xi_{q,-f(E_S)+\mathcal{C}(-S)} \circ f)(x) :\ x \in K\} = 0.$$

(b): It only remains to prove the inclusion. Let $\bar{x} \in E_S$ and $x' \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-S)} \circ f, K)$ with $x' \neq \bar{x}$. By Theorem 4.2, $f(\bar{x}) - f(x') \in \mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)$. Hence, for every $x \in K$ with $x \neq x'$, $x \neq \bar{x}$,

$$f(x) - f(x') = f(x) - f(\bar{x}) + f(\bar{x}) - f(x') \in S + [\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S,$$

and so $x' \in E_S$ since $0 \in S$. □

Remark 4.11. *Taking into account the previos results, we point out that Theorem 4.10(a) applies when P is any (not necessarily closed or pointed) convex cone with nonempty interior, to $S = P$; $S = \mathcal{C}(-\operatorname{int} P)$; $\mathcal{C}(-P) \cup l(P)$; $\mathcal{C}(-P) \cup \{0\}$; $\mathcal{C}(-P)$; whereas (b) applies when P is any (not necessarily pointed) closed convex cone to $S = \mathcal{C}(-\operatorname{int} P)$; $\mathcal{C}(-P) \cup l(P)$; $\mathcal{C}(-P) \cup \{0\}$. Notice that $0 \in S \cap \partial S$ implies that $\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S) \neq \emptyset$.*

Theorem 4.12. *Suppose that q and S satisfy Assumption (B) and consider $(\mathcal{P}(\varepsilon q))$, $\varepsilon \geq 0$. Assume that $\mathcal{C}(-S) + \mathcal{C}(-S) \subseteq \mathcal{C}(-S)$ and S is closed. If $\emptyset \neq A \subseteq K$ then*

$$E(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f, K, \varepsilon) \subseteq E_S(\varepsilon q).$$

Proof. Let $\bar{x} \in E(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f, K, \varepsilon)$ and $\bar{x} \notin E_S(\varepsilon q)$. Then, there exists $x \in K$, $x \neq \bar{x}$, such that $f(x) - f(\bar{x}) \in -\varepsilon q - \mathcal{C}(-S)$. By the closedness of S, Lemma 3.7(b) implies that

$$(\xi_{q,-f(A)+\mathcal{C}(-S)})(f(x)) + \varepsilon = (\xi_{q,-f(A)+\mathcal{C}(-S)})(f(x) + \varepsilon q) < (\xi_{q,-f(A)+\mathcal{C}(-S)})(f(\bar{x})).$$

It follows that $\bar{x} \notin E(\xi_{q,-f(A)+\mathcal{C}(-S)} \circ f, K, \varepsilon)$, which cannot happen. □

Remark 4.13. *When P is any (not necessarily closed or pointed) convex cone, the previous theorem can be applied to $S = \mathcal{C}(-\operatorname{int} P)$, and to $S = P$ provided P is a closed halfspace. In addition, it also applies when $S = \mathcal{C}(-P \setminus \{0\}) = \mathcal{C}(-P) \cup \{0\}$ where $P = Q \cup \{0\}$ is pointed with Q being open and convex set satisfying $tQ \subseteq Q$ for all $t > 0$.*

We are ready to state our main result of complete scalarization for $(\mathcal{P})$ which is a consequence of Theorems 4.10 and 4.12.

Theorem 4.14. *Suppose that q and S satisfy Assumption (B). Assume that $E_S \neq \emptyset$.*

(a) *If $0 \in S$ and $S + [\operatorname{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S$, then*

$$E_S = \bigcup_{x \in E_S} E(\xi_{q,-f(x)+\mathcal{C}(-S)} \circ f, K) \subseteq E(\xi_{q,-f(E_S)+\mathcal{C}(-S)} \circ f, K).$$

(b) *If S is closed and $\mathcal{C}(-S) + \mathcal{C}(-S) \subseteq \mathcal{C}(-S)$, then*

$$E_S = E(\xi_{q,-f(E_S)+\mathcal{C}(-S)} \circ f, K) = \bigcup_{x \in E_S} E(\xi_{q,-f(x)+\mathcal{C}(-S)} \circ f, K) = \bigcup_{x \in K} E(\xi_{q,-f(x)+\mathcal{C}(-S)} \circ f, K),$$

$$E_S(\varepsilon q) = \bigcup_{x \in K} E(\xi_{q,-f(x)+\mathcal{C}(-S)} \circ f, K, \varepsilon) \quad \forall\, \varepsilon > 0.$$

5 Some applications

In this section we present three important applications of our results: the first one develops complete characterizations of the well-known notions of solutions to vector optimization as described at the introduction, say weakly efficient, efficient, proper efficient, strict efficient, etc. The second application shows that our Problem $(\mathcal{P})$ also subsumes the notion of ε-efficiency recently introduced in [15]. Finally, as a third application we deal with a nontransitive relation since the set S, which models the vector optimization problem, is neither convex nor a cone; thus, the results established in [27] are improved.

5.1 Complete scalarizations for the classical problems

In this section we apply the previous results to establish new optimality conditions and characterize different types of well-known efficient solutions.

¿From now on, we consider vector optimization problems defined by a solid convex cone P as described at the introduction.

Let $q \in \operatorname{int} P$. From Theorem 4.2, Remark 4.3 and Lemma 3.6, we can prove the following results.

Corollary 5.1. *Let $\bar{x} \in K$. Then,*

(a) $\bar{x} \in E_P \Longleftrightarrow E(\xi_{q,-f(\bar{x})+\mathcal{C}(-P))} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in P \setminus \operatorname{int} P\}$;

(b) $\bar{x} \in E_W \Longleftrightarrow E(\xi_{q,-f(\bar{x})+P} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -\partial P\}$;

(c) $\bar{x} \in E \Longleftrightarrow E(\xi_{q,-f(\bar{x})+P} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -(\operatorname{cl} P \setminus P) \cup l(P)\}$;

(d) $\bar{x} \in E_{W1} \Longleftrightarrow E(\xi_{q,-f(\bar{x})+P} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -(\operatorname{cl} P \setminus P) \cup \{0\}\}$;

(e) $\bar{x} \in E_1 \Longleftrightarrow E(\xi_{q,-f(\bar{x})+P} \circ f, K) \setminus \{\bar{x}\} = \{x \in K : f(x) - f(\bar{x}) \in -(\operatorname{cl} P \setminus P)\}$.

(f) *Denoting by $\mathcal{H}(P)$ to be the family of all proper convex cones D with nonempty interior satisfying $P \setminus l(P) \subseteq \operatorname{int} D$, we get $(q \in P \setminus l(P))$,*

$$\bar{x} \in E_2 \Longleftrightarrow \exists\, D \in \mathcal{H}(P)\colon E(\xi_{q,-f(\bar{x})+D}\circ f, K) = \{x \in K : f(x)-f(\bar{x}) \in -(\operatorname{cl} D\setminus D)\cup l(D)\}.$$

(g) *Denoting by $\mathcal{D}(P)$ to be the family of all closed convex sets A satisfying $0 \in \partial A$ and $A + (P \setminus \{0\}) \subseteq \operatorname{int} A$, we get $(q \in P \setminus \{0\})$,*

$$\bar{x} \in E_3 \iff \exists\ A \in \mathcal{D}(P) :\ E(\xi_{q,-f(\bar{x})+\operatorname{int}\ A} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -\partial A\}.$$

When P is closed and pointed, Part (c) was earlier proved in [17, Corollary 4.9]. Compare also the above results with those given in [27, Section 6].

Next result, whose proof follows from Theorem 4.7 and Corollary 5.1, provides some characterizations for a point to be in E_S when $S = \mathcal{C}(-\operatorname{int} P)$, $S = P$, $S = \mathcal{C}(-P)\cup\{0\}$ or $S = \mathcal{C}(-P)$. In particular, we recover Corollary 5.5 in [14] and extends Lemma 5.2(ii) of [28]. The last equality of Part $(a1)$ is really interesting since it says that approximate weakly efficient solutions can be approximated by efficient solutions.

Corollary 5.2. *The following assertions hold.*

(a) *Let $\varepsilon \geq 0$. Then,*

$(a1)$ $\bar{x} \in E_W(\varepsilon q) \Longleftrightarrow \bar{x} \in E(\xi_{q,-f(\bar{x})+P} \circ f, K, \varepsilon)$; $E_W(\varepsilon q) = \bigcap_{\delta>\varepsilon} E(\delta q) = \bigcap_{\delta>\varepsilon} E_W(\delta q)$.

$(a2)$ $E(\xi_{q,-f(K)+P} \circ f, K, \varepsilon) \subseteq E_W(\varepsilon q)$.

(b) *if, in addition, P is closed, then*

$(b1)$ $\bar{x} \in E \iff [x \in K,\ (\xi_{q,-f(x)+P} \circ f)(\bar{x}) > 0 \Longrightarrow (\xi_{q,-f(\bar{x})+P} \circ f)(x) > 0]$;

$(b2)$ $\bar{x} \in E_{W1} \iff (\xi_{q,-f(\bar{x})+P} \circ f)(x) > 0 \ \ \forall\, x \in K$ *such that* $f(x) \neq f(\bar{x})$;

$(b3)$ $\bar{x} \in E_1 \iff (\xi_{q,-f(\bar{x})+P} \circ f)(x) > 0 \ \ \forall\, x \in K,\ x \neq \bar{x}$;

$(b4)$ $\bar{x} \in E_P \iff \bar{x} \in E(\xi_{q,-f(\bar{x})+\mathcal{C}(-P)} \circ f, K)$.

Proof. $(a1)$ follows from Theorem 4.7 and $(a2)$ results by particularizing $S = \mathcal{C}(-\operatorname{int} P)$ in Theorem 4.12.
$(b1)$ is a consequence of the following equivalence:

$$\bar{x} \in E \Longleftrightarrow [x \in K,\ f(x) - f(\bar{x}) \in -P \Longrightarrow f(\bar{x}) - f(x) \in -P],$$

and the closedness of P, along with Corollary 3.3; $(b2)$ results from (d) of Corollary 5.1; $(b3)$ is Remark 4.3(i).
To prove $(b4)$, we write

$$\bar{x} \in E_P \Longleftrightarrow f(x) - f(\bar{x}) \in P \ \ \forall\, x \in K \Longleftrightarrow f(x) - f(\bar{x}) \notin -\mathcal{C}(-P) \ \ \forall\, x \in K.$$

We use the closedness of P and Corollary 3.3 to conclude with the desired result. □

The next example shows that the closedness of P is necessary in $(b1)$, $(b2)$, $(b3)$ and $(b4)$.

Example 5.3. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}^2$ defined by*

$$f(x) = \begin{cases} (-1, -x-1) & if \quad x \leq -1; \\ (x, 0) & if \quad -1 < x < 0; \\ (x, x) & if \quad x \geq 0. \end{cases}$$

Let $P = \{(x, y) \in \mathbb{R}^2 \colon x,\ y > 0\} \cup \{(0, 0)\}$ and $q = (1, 1)$. It is clear that $E_P = \emptyset$ and $E = E_1 = E_{W1} = (-\infty, 0]$. However $(b4)$ is false because $-1 \in E(\xi_{f(-1)-\mathcal{C}(-P)} \circ f, K)$ since $(\xi_{q,-f(-1)+\mathcal{C}(-P)} \circ f)(x) = 0$ if $x \leq 0$ and $(\xi_{q,-f(-1)+\mathcal{C}(-P)} \circ f)(x) > 0$ if $x > 0$. In addition, $(b1)$, $(b2)$ and $(b3)$ do not hold since $(\xi_{q,-f(0)+P} \circ f)(-1) = 0$, $(\xi_{q,-f(-1)+P} \circ f)(0) > 0$ and $f(0) \neq f(-1)$.

From Corollary 5.2 we deduce Corollary 4.8(a) in [10].

By particularizing Theorem 4.14 to our classical models, we obtain complete scalarization for E_W, E, E_1, E_{W1}, E_2 and E_3.

Corollary 5.4. *Let $P \subseteq Y$ be a (not necessarily pointed) convex cone with nonempty interior and $q \in \operatorname{int} P$.*

(a) If $E_W \neq \emptyset$ then

$$E_W = E(\xi_{q,-f(E_W)+P} \circ f, K) = \bigcup_{x \in E_W} E(\xi_{q,-f(x)+P} \circ f, K) = \bigcup_{x \in K} E(\xi_{q,-f(x)+P} \circ f, K),$$

$$E_W(\varepsilon q) = \bigcup_{x \in K} E(\xi_{q,-f(x)+P} \circ f, K, \varepsilon) \quad \forall\, \varepsilon > 0;$$

(b) If P is closed and $E \neq \emptyset$ then

$$E = \bigcup_{x \in E} E(\xi_{q,-f(x)+P} \circ f, K) \subseteq E(\xi_{q,-f(E)+P} \circ f, K);$$

(c) If P is closed and $E_{W1} \neq \emptyset$ then

$$E_{W1} = \bigcup_{x \in E_{W1}} E(\xi_{q,-f(x)+P} \circ f, K) \subseteq E(\xi_{q,-f(E_{W1}+P)} \circ f, K);$$

(d) If P is closed and $E_1 \neq \emptyset$ then

$$E_1 = \bigcup_{x \in E_1} E(\xi_{q,-f(x)+P} \circ f, K) \subseteq E(\xi_{q,-f(E_1)+P} \circ f, K);$$

(e) Let $\mathcal{H}(P)$ be as in Corollary 5.1, we get

$$E_2 = \bigcup_{D \in \mathcal{H}(P)} \bigcup_{x \in E(D)} E(\xi_{q,-f(x)+D} \circ f, K),$$

where $E(D)$ corresponds to the efficient solution set for D instead of P.

(f) Let $\mathcal{D}(P)$ be as in Corollary 5.1, we get

$$E_3 = \bigcup_{A \in \mathcal{D}(P)} \bigcup_{x \in K} E(\xi_{q,-f(x)+\operatorname{int} A} \circ f, K).$$

Proof. By taking into account Remark 4.11, the corollary is a consequence of Theorem 4.14. Notice the equality of (c) may be also obtained from Corollary 5.1(d) since $\bar{x} \in E_1$ if and only if $E(\xi_{q,-f(\bar{x})+P} \circ f, K) = \{x \in K : f(x) = f(\bar{x})\}$. Part (d) trivially holds by Remark 4.3(i) since $\bar{x} \in E_1$ if and only if $E(\xi_{q,-f(\bar{x})+P} \circ f, K) = \{\bar{x}\}$.
The remaining part follows from Corollary 5.1. $\square$

The first part in (a) of the above result was established in the proof of [21, Theorem 3.4, pag. 96]; whereas the second part was proved in [14, Theorem 5.11] for P pointed. Observe also that (a) is sharper than Theorem 3.1 in [25] when restricted to single-valued functions.

We cannot expect an equality in Corollary 5.4 for E_P even when P is closed. Indeed, take $P = \mathbb{R}^2_+$, $q = (1,1)$ and $f\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $f(x) = (-1, -x-1)$ if $x \leq -1$, $f(x) = (x, 0)$ if $-1 < x < 0$ and $f(x) = (x, x)$ if $x \geq 0$. We have $E_P = \{-1\}$ and $E(\xi_{q,-f(-1)+\mathcal{C}(-P)} \circ f, K) = \ ]-\infty, 0]$.

Remark 5.5. *In* $[10, Corollary\ 4.14]$ *a free boundary Stefan problem is discussed taking into account the definitions introduced in* [*18*]. *Exactly, the scalarizing function* $\xi_{q,f(\bar{x})}$ *is computed. We point out that according to previous results* (*see, for instance, Theorem 4.7 and Corollary 5.2*) *we may obtain optimality conditions for the* (*approximate*) *free boundary Stefan problem.*

5.2 About $Q(\varepsilon)$-efficiency

Recently, a new ε-efficiency notion in vector optimization was introduced in [15] which is more general than that (C, ε)-efficiency discussed by the same authors in [13] (see more details in [15, Remark 2.1]).

We show that our problem $(\mathcal{P})$ also subsumes such approximated solutions, and so new optimality conditions will be obtained as applications of our results established in Section 4.

Throughout this subsection, we consider Y to be a real topological vector space (note that additionally local convexity of Y was needed in [15]), $P \subseteq Y$ to be a proper pointed convex cone.

Following the notations in [15]), let $Q_P\colon \mathbb{R}_+ \to 2^P$ be defined by $Q_P(\varepsilon) = Q(\varepsilon) + P \setminus \{0\}$ for all $\varepsilon \in \mathbb{R}_+$ where $Q\colon \mathbb{R}_+ \to 2^P$ is a proper set-valued map ($Q(\varepsilon) \neq \emptyset$ for all $\varepsilon \in \mathbb{R}_+$). From the pointedness of P, we obtain that $0 \notin Q_P(\mathbb{R}_+)$ and $Q_P(\varepsilon) + \mathbb{R}_+ p \subseteq Q_P(\varepsilon)$ for all $p \in P$.

Definition 5.6. [*15*] *Let* $M \subseteq Y$ *and* $\varepsilon \in \mathbb{R}_+$. $y \in M$ *is a* $Q(\varepsilon)$*-efficient point of* M, $y \in E(M, Q(\varepsilon))$, *if* $(M - y) \cap (-Q_P(\varepsilon)) = \emptyset$

As stated in Remark 2.1 of [15], the above approximated notion of efficiency can be considered as an extension of several approximated solutions, and when applied to the vector optimization problem (V-P)

$$\mathrm{Min}\{f(x)\colon x \in K\},$$

we obtain the notion of $Q(\varepsilon)$-efficient solution of (V-P) as follows: $\bar{x} \in K$ is $Q(\varepsilon)$-efficient solution of (V-P), denoted by $\bar{x} \in E(f, K, Q(\varepsilon))$, if $f(\bar{x}) \in E(f(K), Q(\varepsilon))$.

Lemma 5.7. $\bar{x} \in E(f, K, Q(\varepsilon))$ *if and only if* $\bar{x} \in E_S$ *being* $S = \mathcal{C}(-Q_P(\varepsilon))$.

From now on, we denote by $S_1 = \mathcal{C}(-Q_P(\varepsilon))$. Note that $0 \in S_1$ and we have two cases:

Case I: $0 \in \partial S_1$.

It is not much interesting from the point of view of approximate solutions since if $0 \in \partial S_1$, then $\mathrm{cl}(Q_P(\varepsilon)) = \mathrm{cl}(P)$. Indeed, it is clear that $\mathrm{cl}(Q_P(\varepsilon)) \subseteq \mathrm{cl}(P)$ (note that $Q_P(\varepsilon) \subseteq P$). Conversely, if $p \in \mathrm{cl}(P)$ then $p = p + 0 \in \mathrm{cl}(Q_P(\varepsilon))$ since $0 \in \mathrm{cl}(Q_P(\varepsilon))$.

Case II: $0 \in \mathrm{int}\, S_1$.

Since $Q_P(\varepsilon) \subseteq P$ we can chose $q \in P \setminus \{0\}$ such that $0 \in \partial(q + S_1)$ (note q depends on $Q(\varepsilon)$, see Figure 3).

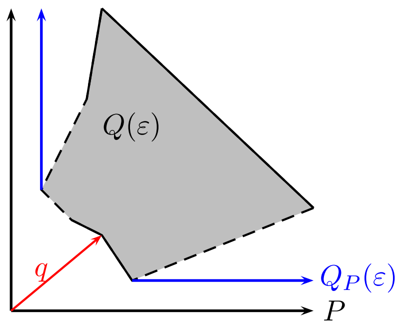

Figure 3: Illustration of $Q_P(\varepsilon)$.

Thus, if $S_2 \doteq q + S_1$ then Problem $(\mathcal{P})$ for $S = S_1$ can be rewritten as follows:

$$(\mathcal{P}_1) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in -q + S_2 \quad \forall\, x \in K.$$

By Lemma 5.7, $E_{S_1} = E_{S_2}(q) = E(f, K, Q(\varepsilon))$ and $(\mathcal{P}_1)$ is a particular case of $(\mathcal{P}(\varepsilon q))$ for $\varepsilon = 1$. Moreover, $S_2 + \mathbb{R}_{++}q \subseteq S_2$, $q \in \mathrm{int}\, S_2$ and $0 \in \partial S_2$. From $\mathcal{C}(-S_2) = -q + Q_P(\varepsilon)$, we can easily deduce that Assumption (B) for problem $(\mathcal{P}_1)$ holds if, and only if

$$\mathrm{cl}(Q(\varepsilon) + P \setminus \{0\}) + \mathbb{R}_{++}q \subseteq \mathrm{int}(Q(\varepsilon) + P \setminus \{0\}). \tag{2}$$

In particular, if P is solid and $q \in \mathrm{int}\, P$, by Remark 3.5, the inclusion (2) always is satisfied and, by Lemma 3.6 the scalarizing function for problem $(\mathcal{P}_1)$ is:

$$\xi_{q,-f(\bar{x})+\mathcal{C}(-S_2)} = \xi_{q,-f(\bar{x})-q+Q(\varepsilon)+P\setminus\{0\}} = \xi_{q,-f(\bar{x})-q+Q(\varepsilon)+P} = \xi_{q,-f(\bar{x})-q+Q(\varepsilon)+\mathrm{int}\, P} = \ldots$$

(note that $\xi_{q,A-q}(\cdot) = \xi_{q,A}(\cdot) - 1$).

As particular cases of Theorems 4.4 and 4.7 we obtain new characterizations for $Q(\varepsilon)$-efficiency under the solidness of P.

Corollary 5.8. *Let us consider problem $(\mathcal{P}_1)$, $q \in \mathrm{int}\, P$ and $\bar{x} \in K$. The following assertions are equivalent:*

(a) $\bar{x} \in E(f, K, Q(\varepsilon))$;

(b) $\bar{x} \in E((\xi_{q,-f(\bar{x})+Q_P(\varepsilon)} - 1) \circ f, K, 1)$ *and*

$$E((\xi_{q,-f(\bar{x})+Q_P(\varepsilon)} - 1) \circ f, K, 1) \setminus \{\bar{x}\} \subseteq \{x \in K : \ x \neq \bar{x}, \ f(x) - f(\bar{x}) \in (-q + \mathcal{C}(-Q_P(\varepsilon))) \cap (q - \operatorname{cl} Q_P(\varepsilon))\}.$$

Corollary 5.9. *Consider problem* $(\mathcal{P}_1)$ *and* $q \in \operatorname{int} P$. *Let* $\bar{x} \in K$. *Then,*

$$\bar{x} \in E(f, K, Q(\varepsilon)) \Longrightarrow \bar{x} \in E((\xi_{q,-f(\bar{x})+Q_P(\varepsilon)} - 1) \circ f, K, \varepsilon) \Longrightarrow \bar{x} \in E_S(\delta q) \ \ \forall\, \delta > 1 \Longrightarrow \bar{x} \in E_{\operatorname{cl} S_2}(q).$$

Consequently if, in addition, $Q_P(\varepsilon)$ *is open then*

(a) $\bar{x} \in E(f, K, Q(\varepsilon)) \Longleftrightarrow \bar{x} \in E((\xi_{q,-f(\bar{x})+Q_P(\varepsilon)} - 1) \circ f, K, 1)$;

(b) $\bar{x} \in E(f, K, Q(\varepsilon)) = \bigcap\limits_{\delta > 1} E_{S_2(\delta q)}$.

Corollaries 5.8 and 5.9 can be considered more interesting than Corollaries 5.1 and 5.2 of [15] since from the practical point of view the scalarizing function could be easily computed and the assumptions on $Q_P(\varepsilon)$ could be relaxed.

On the other hand, the authors in [15] establish necessary or sufficient conditions for $Q(\varepsilon)$-efficiency where P is pointed and not necessarily solid; instead, they consider scalarizing function satisfying suitable separation properties. See Definitions 3.1, 4.1 in [15] and compare with Proposition 3.2(*a*) and (*b*) respectively.

Remark 5.10. *Note that it is possible to assume* P *not necessarily pointed by replacing in the above results* $P \setminus \{0\}$ *by* $P \setminus l(P)$.

5.3 About a non-transitive preference relation

We now provide a geometric condition on q and S satisfying Assumption (B) by considering preferences that are not necessarily transitive (examples are given in [27]), it means that non-convex cones are involved giving rise such preferences.

Following notations used in [27], we consider Y a Banach space (although Y to be a real topological vector space suffices) and say that a set $S \subseteq Y$ is strongly star-shaped if there exists $u \in \operatorname{int} S$ such that $u + \mathbb{R}_+ y$ does not intersect the boundary ∂S of the set A more than once for each $y \in Y$. The set of points u, which enjoy this property is denoted by $\operatorname{Ker}_* S$. That is, $\operatorname{Ker}_* S \doteq \{u \in \operatorname{int} S \colon (u + \mathbb{R}_+ y) \cap \partial S = \{z\} \text{ or } \emptyset \text{ for each } y \in Y\} = \{u \in \operatorname{int} S \colon (u + \mathbb{R}_+ y) \cap \partial S = \{z\} \text{ or } u + \mathbb{R}_+ y \subseteq \operatorname{int} S \text{ for each } y \in Y\}$.

A set $S \subseteq Y$ is star-shaped if there exists $u \in S$ such that $\alpha u + (1 - \alpha)x \in S$ for all $x \in S$ and $\alpha \in [0, 1]$. We denote by
$\operatorname{Ker} S = \{u \in S \colon \alpha u + (1 - \alpha)a \in S, \ \ \forall\, \alpha \in [0, 1], \forall a \in S\} = \{u \in S \colon u + \alpha(a - u) \in S, \ \ \forall \alpha \in [0, 1], \forall\, a \in S\}$. Note that, in general, $\operatorname{Ker}_* S \not\subseteq \operatorname{Ker} S$.

Lemma 5.11. [*26, Proposition* 5.18] *Let* $S \subseteq Y$. *If* S *is closed, then* $\operatorname{Ker}_* S \subseteq \operatorname{Ker} S$.

It is easy to check that if S is a conic set, then $\alpha S = S$ for any $\alpha > 0$, or equivalently $\mathbb{R}_{++}S = S$, and $\mathcal{C}(S)$, $\operatorname{int} S$, ∂S and $\operatorname{Ker}_* S$ are also conic sets.

Proposition 5.12. *Let $S \subseteq Y$ be a conic set. Then the following assertions hold.*

(a) $S + \mathbb{R}_{++}q \subseteq S \Longleftrightarrow S + \mathbb{R}_{+}q \subseteq S \Longleftrightarrow S + q \subseteq S$;

(b) $\operatorname{Ker} S = S_\infty$;

(c) *If* $\operatorname{Ker}_* S \subseteq \operatorname{Ker} S$, *then u and S satisfy Assumption* (B) *for all* $u \in \operatorname{Ker}_* S$.

Proof. (a): The equivalences are straightforward.
(b): $\operatorname{Ker} S = \{u \in S \colon \alpha u + (1-\alpha)S \subseteq S, \ \ \forall\, \alpha \in [0,1]\}$. Since S is conic, $\operatorname{Ker} S = \{u \in S \colon \alpha u + S \subseteq S, \ \ \forall\, \alpha \in [0,1]\} = \{u \in S \colon u + S \subseteq S\}$ and from (a), the conclusion follows.
(c): Let $u \in \operatorname{Ker}_* S$. By assumption, $u \in \operatorname{Ker} S$. From ($a$) it follows that $\operatorname{int} S + u \subseteq \operatorname{int} S$. Since $\operatorname{cl} S = \operatorname{int} S \sqcup \partial S$ we need to check that $\partial S + u \subseteq \operatorname{int} S$. Let $a \in \partial S$. Since $u \in \operatorname{Ker}_* S$, we have $[u + \mathbb{R}_+(a-u)] \cap \partial S = \{a\}$. On the other hand, $u + \alpha(a-u) \in S$ for all $\alpha \in (0,1)$ since $u \in \operatorname{Ker} S$. Thus, $u + \alpha(a-u) \in \operatorname{int} S$ for $\alpha \in (0,1)$. In particular, $\frac{1}{2}(a+u) = u + \frac{1}{2}(a-u) \in \operatorname{int} A$ and taking account that A is conic the proof is concluded. $\square$

The previous proposition allows us to state the following result which ensures the applicability of our approach developed in Section 4.

Proposition 5.13. *Suppose that P is a closed conic set and strongly star-shaped. Then* $P + \operatorname{Ker}_* P \subseteq \operatorname{int} P$. *In particular, if* $u \in \operatorname{Ker}_* P$, *$P$ and u satisfy Assumption* (B).

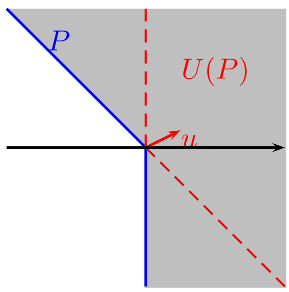

Figure 4: Illustration of $u \in U(P)$.

Consequently, we obtain a class of sets which satisfy Assumption (B) and point out that the definition of weakly minimal, minimal and proper minimal considered in [27] can be rewritten as minimal solutions of problem $(\mathcal{P})$ where f is the identity function on (a suitable) S (for instance, $S = \mathcal{C}(-\operatorname{int} P)$ for weakly minimal points). Thus, according Proposition 5.13, it is easy to check that the results given in Section 5 of [27] could be improved and extended by those results established in Section 4 under Assumption (B) and P not necessarily closed and/or conic (note that the scalarizing function considered in [27] $p_{u,P}$ where $u \in U(P) \doteq \{u \in \operatorname{Ker}_* P\colon \text{ for each } y \in Y,\ y + \mathbb{R}u \not\subseteq P\}$, see Figure 4, coincides with $\xi_{u,P}$). Moreover, optimality conditions for approximate solutions in the framework of [27] can be also given.

6 Conclusions

We have provided an alternative approach to study several efficient notions via an abstract optimization problem. By using a well-known nonlinear function and considering the standard procedure of scalarization we obtain new optimality conditions for several classical efficient notions without any convexity assumption. This has been carried out by considering a solid set which satisfies an assumption of free-disposal type.

The unified optimization problem $(\mathcal{P})$ subsumes several vector optimization problems and sheds a new light since offers an alternative to study more general problems, for instance, when f is a economic function and S is a production set.

The main result lies in establishing complete characterizations of the solution sets to the (approximate) vector optimization problems. Moreover, many instances satisfying our assumptions are exhibited, showing the wide applicability of our results.

It would be interesting to obtain new characterizations without solidness on S in order to include ordering cones like L^p_+, l^p_+, $1 \leq p < \infty$, for instance. On the other hand, applications of our complete scalarizations to derive convergence results and analyzing $Q_P(\varepsilon)$-efficiency for set-valued maps Q and Q_P not necessarily with values into the power set of P would be welcome.

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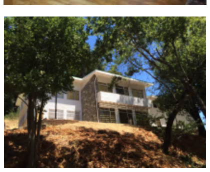

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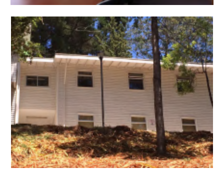
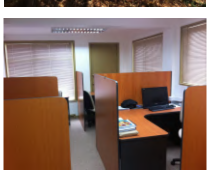

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