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CONVERGENCE ANALYSIS OF A RESIDUAL LOCAL PROJECTION FINITE ELEMENT METHOD FOR THE NAVIER–STOKES EQUATIONS

RODOLFO ARAYA, GABRIEL R. BARRENECHEA, ABNER H. POZA, AND FRÉDÉRIC VALENTIN

Abstract. This work presents and analyzes a new Residual Local Projection stabilized finite element method (RELP) for the non-linear incompressible Navier–Stokes equations. Stokes problems defined element–wisely drive the construction of the residual-based terms which make the present method stable for the finite element pairs $\mathbb{P}_1/\mathbb{P}_l$, $l = 0, 1$. Numerical upwinding is incorporated through an extra control on the advective derivative and on the residual of the divergence equation. Existence of the discrete solution and uniqueness of a non–singular branch of solutions, as well as optimal error estimates in natural norms are proved under standard assumptions. Next, a divergence-free velocity field is provided by a simple post-processing of the computed velocity and pressure using the lowest order Raviart–Thomas basis functions. This updated velocity is proved to converge optimally to the exact solution. Numerics asses the theoretical results and validate the RELP method.

1. Introduction

The numerical solution of the incompressible Navier–Stokes equations by standard finite element methods demands the selection of inf–sup stable pairs of interpolation spaces for the velocity and the pressure [17]. This condition prevents the most desirable choices of spaces to be adopted, such as the simplest and lowest equal order elements [12]. Also, numerical methods should include upwinding strategies to avoid spurious oscillations when the exact solution develops boundary layers [20], behavior that appears for high Reynold numbers flows.

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In the quest of bypassing these issues, a branch of new stabilized finite element methods, called Local Projection Stabilized (LPS) methods, have been recently introduced for mixed problems (see [4, 11]). Like classical stabilized methods, the main idea lies on the addition of extra terms to the Galerkin method but with the difference that they are no longer dependent on residuals, but rather constructed using the fluctuation between the variables (or their derivatives) and their projection onto a given finite dimensional space (see [15] for an interesting overview). LPS methods have been extended to the Oseen model in [6, 7], although their capacity to handle singularly perturbed models still deserves more investigations [18].

Also recently, a relationship between enriching polynomial spaces with the solution of local problems and LPS methods has been established (see [14, 1]). The resulting methods, called Residual Local Projection (RELP) stabilized methods, reintroduced residuals as the main ingredient in their construction but now they are included through fluctuation operators. Thereby, some of the standard LPS extra terms can be seen as a consequence of an enriching space procedure. Next, a new RELP method [2] has been proposed for the Oseen equation and validated for advection dominated flows. Also, a simplified version of the method, specially suited for lower order methods, was presented in [3]. Well-posedness and optimal error estimates were obtained for both methods which were extensively validated through singularly perturbed benchmarks.

In this work, we extend the RELP method to the non-linear incompressible Navier–Stokes equations. We seek the method bearing in mind a certain set of desired characteristics. Among these we can quote:

- Be stable and achieve optimal convergent in natural norms for $\mathbb{P}_1/\mathbb{P}_l$, $l = 0, 1$;
- Bring balanced numerical diffusion;
- Be easily post-processed such as the discrete solution is divergence-free.

To face these requirements, the method is developed within the framework proposed in [1] in which boundary value problems account for residuals at the element level, which ultimately, are responsible for stabilizing the Galerkin method. In particular, we choose to set up a Stokes model element wisely. This accounts for diffusive processes that dominate flows at small scales and might be modeled through the residuals at large scales. Now, since no analytical solution is available for this local problem, a two-level numerical strategy is needed to implement the method which makes the approach more involved. In view of making the present method attractive for practitioners, we project the residual onto the space of piecewise constant functions before solving the local problem. This simplification makes the local problem analytically solvable and provides an alternative RELP method that does not undermine the convergence of the method. In the process, a way to construct an exactly divergence-free velocity field is also set up.

The stability and convergence analysis of the RELP method is based on the fixed point theory presented in [8, 17], and used in [23] to analyze the SDFEM method originally proposed in [13] in the case diffusion dominates, and in [10] for a pressure-stabilized finite element method. Here, we extend that idea and prove that the original RELP method is well-posed and achieve optimal convergence in natural norms. Due to the particular structure of the method, the proof requests the construction of a new stabilized finite element method for the Stokes equation, related to the one given in [1], which is also analyzed. Finally, we establish that the post-processed divergence-free velocity field is also optimally convergent.

The paper is outlined as follows: we end this section with some notations and definitions to be used throughout this manuscript. Next section is devoted to the presentation of the RELP methods. Section 3 includes a well-posedness result and error estimates, and the post-processing to get a divergence-free velocity field is described in Section 4. Numerical validations are in Section 5 and some conclusions are drawn in Section 6. In Appendix A we investigate numerical aspects of a new stabilized method for the Stokes model.

1.1. **Notations and preliminaries.** Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal open domain. The steady incompressible Navier–Stokes equations consists of finding the velocity and pressure $(\boldsymbol{u}, \tilde{p})$ as the solution of

$$
\begin{gathered}
-\nu\,\Delta \boldsymbol{u} + (\nabla \boldsymbol{u})\,\boldsymbol{u} + \nabla \tilde{p} = \tilde{\boldsymbol{f}}, \quad \nabla\cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega\,, \\
\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega\,,
\end{gathered} \tag{1}
$$

where $\nu \in \mathbb{R}^+$ is the fluid viscosity and $\tilde{\boldsymbol{f}} \in L^2(\Omega)^2$. Adopting standard notations for Sobolev spaces, the weak form associated to (1) reads: *Find* $(\boldsymbol{u}, \tilde{p}) \in \mathbf{V} \times Q := H_0^1(\Omega)^2 \times L_0^2(\Omega)$ *such that:*

$$
\nu\,(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\nabla \boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v}) - (\tilde{p}, \nabla\cdot \boldsymbol{v}) + (q, \nabla\cdot \boldsymbol{u}) = (\tilde{\boldsymbol{f}}, \boldsymbol{v}) \quad \text{for all } (\boldsymbol{v}, q) \in \mathbf{V} \times Q, \tag{2}
$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product (or $L^2(\Omega)^2$ if necessary). We place the continuous problem (2) in the case in which the hypothesis of Theorem 2.4, Chap. 4 from [17] are satisfied such that (2) has a unique solution.

Let D be an open subset of Ω, we denote by $\|\cdot\|_{m,D}$ the norm in $H^m(D)$, and by $\|\cdot\|_{m,q,D}$ the norm in $W^{m,q}(D)$ with $m \geq 0$ and $1 \leq q \leq \infty$. We denote, as usual, $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ equipped with the dual norm $\|\cdot\|_{-1,\Omega}$ and the duality product $\langle\cdot,\cdot\rangle$, and

$H^0(\Omega) = L^2(\Omega)$ and $W^{0,q}(\Omega) = L^q(\Omega)$. Also, we define the norm $\|\!|\cdot|\!\|$, in $\mathbf{V}\times Q$, by

$$\|\!|(\boldsymbol{v},q)|\!\| := \left\{|\boldsymbol{v}|^2_{1,\Omega} + \|q\|^2_{0,\Omega}\right\}^{1/2}, \tag{3}$$

and the norm $\|\cdot\|_{(\mathbf{V}\times Q)'}$ in $(\mathbf{V}\times Q)'$, the dual space of $\mathbf{V}\times Q$, by

$$\|(\boldsymbol{v},q)\|_{(\mathbf{V}\times Q)'} := \sup_{\|\!|(\boldsymbol{w},r)|\!\|\le 1} \{\langle \boldsymbol{v},\boldsymbol{w}\rangle + (q,r)\}\,. \tag{4}$$

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular, quasi-uniform triangulations of Ω, built up using triangles K with boundary ∂K and characteristic length $h_K := diam(K)$, and $h := \max\{h_K : K \in \mathcal{T}_h\}$. The set of internal edges F of the triangulation is denoted $\mathcal{E}_h$ with $h_F = |F|$. We denote by $\boldsymbol{n}$ the normal outward vector on ∂K; also, $[\![v]\!]$ stands for the jump of v across F. In addition, for $K\in\mathcal{T}_h$ and $F\in\mathcal{E}_h$, we define the neighborhoods $\omega_K = \{K'\in\mathcal{T}_h : K'\cap K \neq \emptyset\}$ and $\omega_F = \{K\in\mathcal{T}_h : F\cap K\neq\emptyset\}$. Finally, we denote by Π_S, where $S\subset\mathbb{R}^2$, the orthogonal projection onto the constant space, i.e., $\Pi_S(q) := \frac{(q,1)_S}{|S|}$, and by $H^m(\mathcal{T}_h)$, $m\geq 1$ we denote the space of functions whose restriction to $K\in\mathcal{T}_h$ belongs to $H^m(K)$.

Associated to the triangulation $\mathcal{T}_h$, the discrete space for the velocity $\mathbf{V}_h$ is the usual space of vector-valued piecewise linear continuous functions with zero trace on $\partial\Omega$. To approximate the pressure we use Q_h, the space of piecewise polynomial functions of degree $l, (l = 0,1)$ with zero mean value on Ω. If $l = 1$, the space of pressures may contain continuous or discontinuous functions. Analogous to (4) we introduce the following norm in the dual of the discrete space

$$\|(\boldsymbol{v},q)\|_{(\mathbf{V}_h\times Q_h)'} := \sup_{\|\!|(\boldsymbol{w}_h,r_h)|\!\|\le 1} \{\langle \boldsymbol{v},\boldsymbol{w}_h\rangle + (q,r_h)\}\,. \tag{5}$$

In what follows, we will employ the differential of a mapping $F : \mathbf{V}\times Q \to \mathbf{V}\times Q$ with respect to $(\boldsymbol{u},q)$ at $(\boldsymbol{v},q)\in\mathbf{V}\times Q$ denoted by $D_{\boldsymbol{u},p}F(\boldsymbol{v},q)\in\mathcal{L}(\mathbf{V}\times Q)$, where $\mathcal{L}(\mathbf{V}\times Q)$ represents the space of linear mappings acting on elements of $\mathbf{V}\times Q$ with values in $\mathbf{V}\times Q$ and equipped with the usual norm $\|\cdot\|_{\mathcal{L}(\mathbf{V}\times Q)}$.

Finally, in the forthcoming analysis we will use the following classical result.

Lemma 1. *For all $\boldsymbol{v},\boldsymbol{w}\in\mathbf{V}$ and $q\in Q$, we have*

$$\sup_{\boldsymbol{v}\in\mathbf{V}} \frac{(q,\nabla\cdot\boldsymbol{v})}{|\boldsymbol{v}|_{1,\Omega}} \;\geq\; \beta\,\|q\|_{0,\Omega}, \tag{6}$$

$$((\nabla\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}) \;\leq\; \alpha\,|\boldsymbol{u}|_{1,\Omega}|\boldsymbol{w}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega}, \tag{7}$$

where α and β are positive constants depending only on Ω. Moreover, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbf{V}$, it holds

$$((\nabla\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}) = -((\nabla\boldsymbol{v})\boldsymbol{w},\boldsymbol{u}) - (\nabla\cdot\boldsymbol{w},\boldsymbol{u}\cdot\boldsymbol{v}), \tag{8}$$

$$((\nabla\boldsymbol{v})\boldsymbol{w},\boldsymbol{v}) = -\frac{1}{2}(\nabla\cdot\boldsymbol{w},\boldsymbol{v}\cdot\boldsymbol{v}). \tag{9}$$

Proof. See [17] and [21]. □

2. The residual local projection method

The finite element method that we analyze in this work reads: *Find $(\boldsymbol{u}_h,\tilde{p}_h)\in\mathbf{V}_h\times Q_h$ such that*

$$\begin{aligned}&\nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h)+((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h,\boldsymbol{v}_h)-(\tilde{p}_h,\nabla\cdot\boldsymbol{v}_h)+(\tilde{q}_h,\nabla\cdot\boldsymbol{u}_h)\\&+\sum_{K\in\mathcal{T}_h}\Big[\frac{\alpha_K}{\nu}\Big(p_e^M(-\tilde{\boldsymbol{f}}-\Delta\boldsymbol{u}_h+(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h+\nabla\tilde{p}_h),p_e^M((\nabla\boldsymbol{v}_h)\boldsymbol{u}_h+\nabla\tilde{q}_h)\Big)_K\\&+\frac{\gamma_K}{\nu}(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h),\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h))_K\Big]+\sum_{F\in\mathcal{E}_h}\tau_F\,([\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h+\tilde{p}_h\,\boldsymbol{n}]\!],[\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{v}_h+\tilde{q}_h\,\boldsymbol{n}]\!])_F=(\tilde{\boldsymbol{f}},\boldsymbol{v}_h),\end{aligned} \tag{10}$$

for all $(\boldsymbol{v}_h,\tilde{q}_h)\in\mathbf{V}_h\times Q_h$, where $\chi_h:=I-\Pi_K$ is the fluctuation operator.
Here, for a function $\boldsymbol{v}\in L^2(K)^2$, $(\boldsymbol{u}_e^M(\boldsymbol{v}),p_e^M(\boldsymbol{v}))$ stands for the solution of the local Stokes problem

$$\begin{aligned}-\nu\,\Delta\boldsymbol{u}_e^M(\boldsymbol{v})+\nabla p_e^M(\boldsymbol{v})=\boldsymbol{v},\quad \nabla\cdot\boldsymbol{u}_e^M(\boldsymbol{v})&=0\quad\text{in } K,\\ \boldsymbol{u}_e^M(\boldsymbol{v})&=\boldsymbol{0}\quad\text{on }\partial K.\end{aligned} \tag{11}$$

Also, the stabilization parameters are given by

$$\alpha_K:=\frac{1}{\max\{1,Pe_K\}}\quad\text{and}\quad\gamma_K:=\frac{1}{\max\left\{1,\frac{Pe_K}{24}\right\}}, \tag{12}$$

where

$$Pe_K:=\frac{|\boldsymbol{u}_h|_K h_K}{18\,\nu}\quad\text{with}\quad|\boldsymbol{u}_h|_K:=\frac{\|\boldsymbol{u}_h\|_{0,K}}{|K|^{\frac{1}{2}}},$$

and

$$\tau_F:=\begin{cases}\dfrac{h_F}{12\,\nu} & \text{if } |\boldsymbol{u}_h|_F=0,\\[2ex] \dfrac{1}{2\,|\boldsymbol{u}_h|_F}-\dfrac{1}{|\boldsymbol{u}_h|_F\,(1-\exp(Pe_F))}\Big(1+\dfrac{1}{Pe_F}(1-\exp(Pe_F))\Big) & \text{otherwise.}\end{cases} \tag{13}$$

Here

$$Pe_F:=\frac{|\boldsymbol{u}_h|_F\,h_F}{\nu}\quad\text{with}\quad|\boldsymbol{u}_h|_F:=\frac{\|\boldsymbol{u}_h\|_{0,F}}{h_F^{1/2}}.$$

In general (10) requests a two-level discretization since problem (11) can not be exactly solved. On the other hand, a closer inspection of problem (11) reveals that some terms in (10) can indeed be exactly computed, thus simplifying the implementation. In particular, we realize that, for all $q \in H^1(K)$ it holds

$$\boldsymbol{u}_e^M(\nabla q) = \mathbf{0} \quad \text{and} \quad p_e^M(\nabla q) = \chi_h(q) \,. \tag{14}$$

As for the remaining terms, we replace $p_e^M(-\tilde{\boldsymbol{f}}+(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h+\nabla\tilde{p}_h)$ by $p_e^M(\Pi_K(-\tilde{\boldsymbol{f}}+(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h+\nabla\tilde{p}_h))$, and $p_e^M((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h+\nabla q_h)$ by $p_e^M(\Pi_K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h+\nabla q_h))$ in (10) to get the following simplified method: *Find* $(\boldsymbol{u}_h, \tilde{p}_h) \in \mathbf{V}_h \times Q_h$ *such that*

$$\begin{aligned}
&\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (\tilde{p}_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h)\\
&+ \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu}\Big(p_e^M(\Pi_K(-\tilde{\boldsymbol{f}} + (\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\tilde{p}_h)), p_e^M(\Pi_K((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h + \nabla q_h))\Big)_K\\
&+ \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu}\left(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h), \chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h)\right)_K\\
&+ \sum_{F\in\mathcal{E}_h} \tau_F\ \left([\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \tilde{p}_h\,\boldsymbol{n}]\!], [\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\,\boldsymbol{n}]\!]\right)_F = (\tilde{\boldsymbol{f}}, \boldsymbol{v}_h)\,.
\end{aligned}$$

Finally, noting that every constant is a gradient, we apply (14) to recast the final form of the simplified RELP method: *Find* $(\boldsymbol{u}_h, \tilde{p}_h) \in \mathbf{V}_h \times Q_h$ *such that*

$$\begin{aligned}
&\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (\tilde{p}_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h)\\
&+ \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu}\left(\chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h + \tilde{p}_h), \chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h + q_h)\right)_K\\
&+ \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu}\left(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h), \chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h)\right)_K + \sum_{F\in\mathcal{E}_h} \tau_F\ \left([\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \tilde{p}_h\,\boldsymbol{n}]\!], [\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\,\boldsymbol{n}]\!]\right)_F\\
&= (\tilde{\boldsymbol{f}}, \boldsymbol{v}_h) + \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu}\Big(\chi_h(\boldsymbol{x}\cdot\Pi_K\tilde{\boldsymbol{f}}), \chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h + q_h)\Big)_K\,.
\end{aligned} \tag{15}$$

Remark. The method (15) is the one we will implement since it is not a two-level method. The next section deals with the error analysis for the original RELP method (10), as the analysis can be extended, with minor differences, to the simplified RELP method (15). The reason to analyze method (10) is its generality, which opens the door to different solution strategies for the local problem (11), and it can be seen as the first steps toward the analysis of general two-level finite element methods for the Navier–Stokes equations.

3. Error analysis

We consider a scaled form of (1) by setting $\tilde{p} = \nu\, p$, $\tilde{\boldsymbol{f}} = \nu\, \boldsymbol{f}$, and $\lambda = \nu^{-1}$

$$-\Delta \boldsymbol{u}_\lambda + \lambda\,(\nabla \boldsymbol{u}_\lambda)\,\boldsymbol{u}_\lambda + \nabla p_\lambda = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u}_\lambda = 0 \qquad \text{in } \Omega\,, \tag{16}$$
$$\boldsymbol{u}_\lambda = \mathbf{0} \quad \text{on } \partial\Omega\,.$$

The standard weak formulation of problem (16) is given by the following: *Find* $(\boldsymbol{u}_\lambda, p_\lambda) \in \mathbf{V} \times Q$ *such that*

$$(\nabla \boldsymbol{u}_\lambda, \nabla \boldsymbol{v}) + \lambda\,((\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda, \boldsymbol{v}) - (p_\lambda, \nabla\cdot \boldsymbol{v}) + (q, \nabla\cdot \boldsymbol{u}_\lambda) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall (\boldsymbol{v}, q) \in \mathbf{V} \times Q\,. \tag{17}$$

We assume in this work that problem (17) admits at least one solution, which is unique provided λ is sufficiently small. Also, (17) can be written in the operator form as follows

$$F(\lambda, \boldsymbol{u}_\lambda, p_\lambda) := (\boldsymbol{u}_\lambda, p_\lambda) + TG(\lambda, \boldsymbol{u}_\lambda, p_\lambda) = \mathbf{0}, \tag{18}$$

where $G(\lambda, \boldsymbol{u}_\lambda, p_\lambda) \in \mathbf{V}' \times Q$ is given by

$$\langle G(\lambda, \boldsymbol{u}_\lambda, p_\lambda), (\boldsymbol{v}, q)\rangle := \lambda\,((\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \mathbf{V} \times Q, \tag{19}$$

and $T : \mathbf{V}' \times Q \longrightarrow \mathbf{V} \times Q$ denotes the Stokes operator, which associates for each $(\boldsymbol{w}, r) \in \mathbf{V}' \times Q$ the unique solution $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ of

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla\cdot \boldsymbol{v}) + (q, \nabla\cdot \boldsymbol{u}) = \langle \boldsymbol{w}, \boldsymbol{v}\rangle + (r, q), \tag{20}$$

for all $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$.

The stabilized method for problem (16) reads as follows: *Find* $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \mathbf{V}_h \times Q_h$ *such that for all* $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$\begin{aligned}
&(\nabla \boldsymbol{u}_{h,\lambda}, \nabla \boldsymbol{v}_h) + \lambda\,((\nabla \boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}, \boldsymbol{v}_h) - (p_{h,\lambda}, \nabla\cdot \boldsymbol{v}_h) + (q_h, \nabla\cdot \boldsymbol{u}_{h,\lambda}) \\
&+ \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(-\Delta \boldsymbol{u}_{h,\lambda} + \lambda(\nabla \boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda} + \nabla p_{h,\lambda}), p_e^M(\lambda(\nabla \boldsymbol{v}_h)\boldsymbol{u}_{h,\lambda} + \nabla q_h)\right)_K \\
&+ \sum_{K\in\mathcal{T}_h} \gamma_K\,(\lambda\,\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_{h,\lambda}), \lambda\,\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{v}_h))_K + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F\,\left([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_{h,\lambda} + p_{h,\lambda}\,\boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h\,\boldsymbol{n}]\!]\right)_F \\
&= (\boldsymbol{f}, \boldsymbol{v}_h) + \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\boldsymbol{f}), p_e^M(\lambda(\nabla \boldsymbol{v}_h)\boldsymbol{u}_{h,\lambda} + \nabla q_h)\right)_K\,,
\end{aligned} \tag{21}$$

where $\tilde{\tau}_F = \dfrac{\tau_F}{\lambda}$ and p_e^M solves the Stokes local problem (11). Multiplying (21) by ν, and substituting $\tilde{p}_h = \nu\, p_h$, $\tilde{\boldsymbol{f}} = \nu\,\boldsymbol{f}$, and $\tilde{q}_h = \nu\, q_h$, we recover the RELP method (10) for the original, unscaled Navier–Stokes equations.

3.1. **Technical Preliminaries.**

We first state the following local trace theorem [22].

Lemma 2. *There exists a constant $C > 0$, independent of h, such that for all $v \in H^1(K)$*

$$\|v\|_{0,\partial K}^2 \leq C\left\{h_K^{-1}\,\|v\|_{0,K}^2 + h_K\,|v|_{1,K}^2\right\}. \tag{22}$$

The following inverse estimates are satisfied for $\mathbf{V}_h$ and Q_h.

Lemma 3. *There exists a constant $C > 0$, independent of h, such for all $\boldsymbol{v}_h \in \mathbf{V}_h$ and all $q_h \in Q_h$ we have*

$$\|\boldsymbol{v}_h\|_{\infty,\Omega} \leq C\,h^{-\kappa}|\boldsymbol{v}_h|_{1,\Omega}, \tag{23}$$

$$\|\boldsymbol{v}_h\|_{\infty,K} \leq C\,h_K^{-1}\,\|\boldsymbol{v}_h\|_{0,K}, \tag{24}$$

$$\|[\![q_h]\!]\|_{0,F} \leq C\,h_F^{-1/2}\|q_h\|_{0,\omega_F}, \tag{25}$$

$$h_K\,|\boldsymbol{v}_h|_{1,K} \leq C\,\|\boldsymbol{v}_h\|_{0,K}, \tag{26}$$

for all κ with $0 < \kappa \leq \frac{1}{2}$.

Proof. The first estimate is a direct consequence of global inverse inequalities and Sobolev's embedding Theorem (cf. [12]). For the remaining estimates, see [12]. □

The properties of the orthogonal projection Π_K onto the constant space are summarized in the following result.

Lemma 4. *There exists a constant $C > 0$, independent of h, such that*

$$\|v - \Pi_K v\|_{0,K} \leq C\,h_K\,|v|_{1,K} \qquad \forall v \in H^1(K)\,, \tag{27}$$

$$\|\Pi_K v\|_{0,K} \leq \|v\|_{0,K} \qquad \forall v \in L^2(K), \tag{28}$$

$$\|\Pi_K v\|_{\infty,K} \leq C\,h_K^{-1}\|v\|_{0,K} \qquad \forall v \in L^2(K). \tag{29}$$

Proof. For estimates (27) and (28), see [12]. Finally, (29) follows from (24) and (28). □

We introduce the Lagrange interpolation operator $\mathcal{I}_h : \mathbf{V} \cap H^2(\Omega)^2 \longrightarrow \mathbf{V}_h$ and the operator $\mathcal{J}_h : Q \longrightarrow Q_h$, for the velocity and pressure, respectively, where $\mathcal{J}_h$ is a modified Clément operator for continuous pressures ($l = 1$) or the orthogonal projection onto Q_h for discontinuous pressures. These interpolation operators satisfy (see [9, 12])

$$
\begin{aligned}
|\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v}|_{m,K} &\leq C\,h_K^{2-m}\,|\boldsymbol{v}|_{2,K} \qquad \forall \boldsymbol{v}\in H^2(K)^2, && (30)\\
|\mathcal{I}_h\boldsymbol{v}|_{1,K} &\leq C\,\|\boldsymbol{v}\|_{2,K} \qquad \forall \boldsymbol{v}\in H^2(K)^2, && (31)\\
\|\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v}\|_{i,F} &\leq C\,h_F^{3/2-i}\,\|\boldsymbol{v}\|_{2,\omega_F} \qquad \forall \boldsymbol{v}\in H^2(\omega_F)^2, && (32)\\
\|\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v}\|_{1,r,K} &\leq C\,\|\boldsymbol{v}\|_{1,r,K} \qquad \forall \boldsymbol{v}\in W^{1,r}(K)^2\ ,\ \forall r\in(2,\infty]\,, && (33)\\
\|\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v}\|_{\infty,K} &\leq C\,\|\boldsymbol{v}\|_{\infty,K} \qquad \forall \boldsymbol{v}\in \mathcal{C}^0(K)^2\,, && (34)\\
|q-\mathcal{J}_h q|_{i,K} &\leq C\,h_K^{j-i}\,|q|_{j,\omega_K} \qquad \forall q\in H^j(\omega_K), && (35)\\
\|q-\mathcal{J}_h q\|_{0,F} &\leq C\,h_F^{j-1/2}\,\|q\|_{j,\omega_F} \qquad \forall q\in H^j(\omega_F), && (36)
\end{aligned}
$$

where $0\leq m\leq 2$, and $0\leq i\leq 1$, $1\leq j\leq l+1$ and $C>0$ independent of h.
Before heading to well-posedness results, we give the following technical result whose proof can be found in [1] (see Lemma 3.2 for (37) and (38), and equation (3.18) for (39)).

Lemma 5. *Let $\boldsymbol{v}\in L^2(K)^2$ and let $(\boldsymbol{u}_e^M(\boldsymbol{v}),p_e^M(\boldsymbol{v}))$ be the solution of problem* (11). *Then, there exists $C>0$, independent of h_K, such that*

$$
\begin{aligned}
|\boldsymbol{u}_e^M(\boldsymbol{v})|_{1,K} &\leq \lambda\,h_K\,\|\boldsymbol{v}\|_{0,K}, && (37)\\
\|p_e^M(\boldsymbol{v})\|_{0,K} &\leq C\,h_K\,\|\boldsymbol{v}\|_{0,K}. && (38)
\end{aligned}
$$

Moreover, there exists a constant $C>0$, independent of h, such that for all $q\in\mathbb{P}_1(K)$ we have

$$
C h_K\,\|\nabla q\|_{0,K}\leq \|p_e^M(\nabla q)\|_{0,K}\leq h_K\,\|\nabla q\|_{0,K}. \tag{39}
$$

3.2. **Existence of a discrete solution.** We start defining the operator $\mathcal{P}:\mathbf{V}_h\longrightarrow Q_h$ by

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\alpha_K\big(p_e^M(\nabla\mathcal{P}(\boldsymbol{u}_h)),p_e^M(\nabla q_h)\big)_K+\sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\left([\![\mathcal{P}(\boldsymbol{u}_h)]\!],[\![q_h]\!]\right)_F \qquad (40)\\
=\ & -(q_h,\nabla\cdot\boldsymbol{u}_h)-\sum_{K\in\mathcal{T}_h}\alpha_K\left(p_e^M(\lambda(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h-\boldsymbol{f}),p_e^M(\nabla q_h)\right)_K-\sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\left([\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h]\!],[\![q_h\,\boldsymbol{n}]\!]\right)_F\,,
\end{aligned}
$$

for all $\boldsymbol{u}_h\in\mathbf{V}_h$, $q_h\in Q_h$. The operator $\mathcal{P}$ is well defined due to Lax Milgram's Theorem with the norm

$$
\|q_h\|_*:=\left\{\sum_{K\in\mathcal{T}_h}\alpha_K\|p_e^M(\nabla q_h)\|_{0,K}^2+\sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\,\|[\![q_h]\!]\|_{0,F}^2\right\}^{1/2}.
$$

Also, we define the mapping $\mathcal{N} : \mathbf{V}_h \longrightarrow \mathbf{V}_h$ by

$$\begin{aligned}(\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{v}_h) &= (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + (\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (\mathcal{P}(\boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h) - (\boldsymbol{f}, \boldsymbol{v}_h) \\ &\quad + \sum_{K\in\mathcal{T}_h} \gamma_K \left(\lambda\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h), \lambda\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h)\right)_K \\ &\quad + \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - \boldsymbol{f} + \nabla\mathcal{P}(\boldsymbol{u}_h)), p_e^M(\lambda(\nabla \boldsymbol{v}_h)\boldsymbol{u}_h)\right)_K \\ &\quad + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \left([\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}}\boldsymbol{v}_h]\!]\right)_F ,\end{aligned} \tag{41}$$

for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h$.

The following result provides a characterization of the solution of (21) in terms of the mappings $\mathcal{P}$ and $\mathcal{N}$.

Lemma 6. *The pair $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \mathbf{V}_h \times Q_h$ is a solution of problem (21) if and only if $\mathcal{N}(\boldsymbol{u}_{h,\lambda}) = \mathbf{0}$ and $p_{h,\lambda} = \mathcal{P}(\boldsymbol{u}_{h,\lambda})$.*

Proof. If $\mathcal{N}(\boldsymbol{u}_{h,\lambda}) = \mathbf{0}$ and $p_{h,\lambda} = \mathcal{P}(\boldsymbol{u}_{h,\lambda})$, then adding (40) and (41) we see that $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \mathbf{V}_h \times Q_h$ is a solution of problem (21). Moreover, let $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \mathbf{V}_h \times Q_h$ be a solution of (21). If $\boldsymbol{v}_h = \mathbf{0}$ in (21), we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\nabla p_{h,\lambda}), p_e^M(\nabla q_h)\right)_K + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \left([\![p_{h,\lambda}]\!], [\![q_h]\!]\right)_F = \\ &-(q_h, \nabla\cdot\boldsymbol{u}_{h,\lambda}) - \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(-\boldsymbol{f} + \lambda(\nabla\boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}), p_e^M(\nabla q_h)\right)_K - \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \left([\![\partial_{\boldsymbol{n}}\boldsymbol{u}_{h,\lambda}]\!], [\![q_h\boldsymbol{n}]\!]\right)_F\end{aligned}$$

and hence, since $\mathcal{P}$ is well defined, $p_{h,\lambda} = \mathcal{P}(\boldsymbol{u}_{h,\lambda})$. Finally, if $q_h = 0$ in (21), we have $\mathcal{N}(\boldsymbol{u}_{h,\lambda}) = \mathbf{0}$ and the result follows. ∎

We are now in position of proving the well-posedeness of the discrete problem (21).

Theorem 7. *There is a positive constant C, which is independent of h and λ, such that problem (21) admits at least one solution $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda})$ provided*

$$\lambda\, h^{1-\kappa} \left\{ \|\boldsymbol{f}\|_{-1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\boldsymbol{f})\|_{0,K}^2 \right\}^{1/2} \leq C.$$

Proof. Let $R > 0$ and $\boldsymbol{u}_h \in \mathbf{V}_h$, with $|\boldsymbol{u}_h|_{1,\Omega} = R$, be arbitrary and for abbreviation let

$$x := \left\{ \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \right\}^{1/2}, \; y := \left\{ \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{u}_{h,\lambda} + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \right\}^{1/2},$$

$$z := \left\{ \|\boldsymbol{f}\|_{-1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\boldsymbol{f})\|_{0,K}^2 \right\}^{1/2}, \quad w := \left\{ \sum_{K\in\mathcal{T}_h} \gamma_K \|\lambda \chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h)\|_{0,K}^2 \right\}^{1/2}.$$

Taking $q_h = \mathcal{P}(\boldsymbol{u}_h)$ in (40) gives

$$\begin{aligned} -(\mathcal{P}(\boldsymbol{u}_h), \nabla\cdot \boldsymbol{u}_h) \;&=\; \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h) - \boldsymbol{f}), p_e^M(\nabla\mathcal{P}(\boldsymbol{u}_h)) \right)_K \\ &\quad + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F ([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!], [\![\mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!])_F. \end{aligned}$$

Then, using Cauchy-Schwarz's inequality and (9) we have

$$\begin{aligned} & (\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{u}_h) \\ = \;& |\boldsymbol{u}_h|_{1,\Omega}^2 + \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - (\boldsymbol{f}, \boldsymbol{u}_h) \\ & + \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h) - \boldsymbol{f}), p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h)) \right)_K \\ & + \sum_{K\in\mathcal{T}_h} \gamma_K \|\lambda\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\ \geq \;& |\boldsymbol{u}_h|_{1,\Omega}^2 + \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - \|\boldsymbol{f}\|_{-1,\Omega} |\boldsymbol{u}_h|_{1,\Omega} + \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \\ & - \sum_{K\in\mathcal{T}_h} \alpha_K \left(p_e^M(\boldsymbol{f}), p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h)) \right)_K \\ & + \sum_{K\in\mathcal{T}_h} \gamma_K \|\lambda\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\ \geq \;& \frac{1}{2}|\boldsymbol{u}_h|_{1,\Omega}^2 + \lambda((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) + \frac{1}{2} \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\lambda(\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \\ & + \sum_{K\in\mathcal{T}_h} \gamma_K \|\lambda\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tilde{\tau}_F \, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\ & -\frac{1}{2} \sum_{K\in\mathcal{T}_h} \alpha_K \|p_e^M(\boldsymbol{f})\|_{0,K}^2 - \frac{1}{2}\|\boldsymbol{f}\|_{-1,\Omega}^2 \\ \geq \;& \frac{1}{2}R^2 + \frac{1}{2}x^2 + y^2 + w^2 - \frac{1}{2}z^2 - \frac{\lambda}{2}(\nabla\cdot \boldsymbol{u}_h, \boldsymbol{u}_h\cdot \boldsymbol{u}_h). \end{aligned} \tag{42}$$

Now, if we use the definition of the method with a test function given by $(\mathbf{0}, \mathcal{J}_h(\boldsymbol{u}_h \cdot \boldsymbol{u}_h))$, Cauchy–Schwarz's inequality, Lemma 5, the fact that $\alpha_K \leq 1$ and (35), we get

$$
\begin{aligned}
& |(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h)| \leq |(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h - J_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))| + |(\nabla\cdot\boldsymbol{u}_h, \mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))| \\
\leq\ & \sqrt{2}|\boldsymbol{u}_h|_{1,\Omega}\,\|\boldsymbol{u}_h\cdot\boldsymbol{u}_h - \mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\|_{0,\Omega} \\
& + \left|\sum_{K\in\mathcal{T}_h} \alpha_K(p_e^M(\lambda(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h) - \boldsymbol{f}), p_e^M(\nabla\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)))_K\right| \\
& + \left|\sum_{F\in\mathcal{E}_h} \tilde{\tau}_F\,\|[\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\ \|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F}\right| \\
\leq\ & CRh\,|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\Omega} + \sum_{K\in\mathcal{T}_h}\alpha_K\|p_e^M(\lambda(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h) - \boldsymbol{f})\|_{0,K}\ \|p_e^M(\nabla\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))\|_{0,K} \\
& + \sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\,\|[\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\ \|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F} \\
\leq\ & CRh\,|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\Omega} + C\sum_{K\in\mathcal{T}_h}\alpha_K\|p_e^M(\lambda(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h + \nabla\mathcal{P}(\boldsymbol{u}_h))\|_{0,K}\ h_K|\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|_{1,K} \\
+\ & C\sum_{K\in\mathcal{T}_h}\alpha_K\|p_e^M(\boldsymbol{f})\|_{0,K}\ h_K|J_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|_{1,K} + \sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\,\|[\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\ \|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F} \\
\leq\ & CRh\,|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\Omega} + Cx\left\{\sum_{K\in\mathcal{T}_h}\alpha_K h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\omega_K}^2\right\}^{1/2} \\
& + C\left\{\sum_{K\in\mathcal{T}_h}\alpha_K\|p_e^M(\boldsymbol{f})\|_{0,K}^2\right\}^{1/2}\left\{\sum_{K\in\mathcal{T}_h}\alpha_K h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\omega_K}^2\right\}^{1/2} \\
& + \left\{\sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\|[\![\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2\right\}^{1/2}\left\{\sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)]\!]\|_{0,F}^2\right\}^{1/2} \\
\leq\ & C\{R + x + y + z\}\left\{h^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\Omega}^2 + \sum_{F\in\mathcal{E}_h}\tilde{\tau}_F\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h) - \boldsymbol{u}_h\cdot\boldsymbol{u}_h]\!]\|_{0,F}^2\right\}^{1/2}.
\end{aligned} \tag{43}
$$

But, in [2], Lemma 2, it is proved that

$$\tilde{\tau}_F = \frac{1}{\lambda}\tau_F \leq C\nu\frac{h_F}{\nu} = Ch_F,$$

and then, applying (36) and the mesh regularity, we arrive at

$$|(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h)| \leq C\{R + x + y + z\}\,h\,|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\Omega}.$$

Moreover, using (23) we get

$$\begin{aligned} |\boldsymbol{u}_h \cdot \boldsymbol{u}_h|_{1,\Omega} &= \|\nabla(\boldsymbol{u}_h \cdot \boldsymbol{u}_h)\|_{0,\Omega} \\ &= 2\|\nabla(\boldsymbol{u}_h)\boldsymbol{u}_h\|_{0,\Omega} \\ &\leq C|\boldsymbol{u}_h|_{1,\Omega}\,\|\boldsymbol{u}_h\|_{\infty,\Omega} \\ &\leq Ch^{-\kappa}|\boldsymbol{u}_h|^2_{1,\Omega}, \end{aligned}$$

and then from (42) and (43) we arrive at

$$\begin{aligned} &(\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{u}_h) \\ \geq\ & \frac{1}{2}R^2 + \frac{1}{2}x^2 + w^2 + y^2 - \frac{1}{2}z^2 - \frac{\lambda}{2}(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h) \\ \geq\ & \frac{1}{2}R^2 + \frac{1}{2}x^2 + w^2 + y^2 - \frac{1}{2}z^2 - Ch^{1-\kappa}\lambda\{R+x+y+z\}\,R^2 \\ \geq\ & \frac{1}{2}R^2 + \frac{1}{2}x^2 + w^2 + y^2 - \frac{1}{2}z^2 - Ch^{1-\kappa}\lambda R^3 - Ch^{1-\kappa}\lambda\{x+y+z\}\,R^2 \\ \geq\ & \frac{1}{2}R^2 + \frac{1}{2}x^2 + w^2 + y^2 - \frac{1}{2}z^2 - Ch^{1-\kappa}\lambda R^3 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 - \frac{3}{2}C^2h^{2(1-\kappa)}\lambda^2R^4 \\ \geq\ & \frac{1}{2}R^2 + w^2 + \frac{1}{2}y^2 - z^2 - Ch^{1-\kappa}\lambda R^3 - \frac{3}{2}C^2h^{2(1-\kappa)}\lambda^2R^4. \end{aligned}$$

Let us define

$$R := \frac{1}{MC\lambda h^{1-\kappa}},$$

with $M \in \mathbb{N}$ sufficiently large such that

$$\frac{1}{2}R^2 - \frac{1}{M}R^2 - \frac{3}{2M^2}R^2 \geq \frac{1}{4}R^2.$$

Next, imposing the following hypothesis on h:

$$2MC\lambda h^{1-\kappa}z \leq 1,$$

we conclude that

$$\begin{aligned} (\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{u}_h) &\geq \frac{1}{2}R^2 - z^2 - \frac{1}{M}R^2 - \frac{3}{2M^2}R^2 + \frac{1}{2}y^2 + w^2 \\ &\geq \frac{1}{4}R^2 - z^2 + \frac{1}{2}y^2 + w^2 \\ &\geq \frac{1}{2}y^2 + w^2 > 0\,. \end{aligned}$$

Hence, Brouwer's fixed point theorem implies that there is a $\boldsymbol{u}_h \in \mathbf{V}_h$ with $|\boldsymbol{u}_h|_{1,\Omega} \leq R$ and $\mathcal{N}(\boldsymbol{u}_h) = \mathbf{0}$. □

3.3. **A priori error analysis.** We place the method in the case existence is assured, i.e., at the diffusion dominated regime. To this end, we suppose that $\alpha_K = \gamma_K = 1$ on every element and $\tau_F = \dfrac{h_F}{12}$ on each edge. Then, method (21) simplifies to: *Find* $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in \mathbf{V}_h \times Q_h$ *such that*

$$
\begin{aligned}
&(\nabla \boldsymbol{u}_{h,\lambda}, \nabla \boldsymbol{v}_h) + \lambda\,((\nabla \boldsymbol{u}_{h,\lambda})\boldsymbol{u}_{h,\lambda}, \boldsymbol{v}_h) - (p_{h,\lambda}, \nabla\cdot \boldsymbol{v}_h) + (q_h, \nabla\cdot \boldsymbol{u}_{h,\lambda}) \\
&- \sum_{K\in\mathcal{T}_h} \big(p_e^M(\boldsymbol{f} + \Delta \boldsymbol{u}_{h,\lambda} - \lambda\,(\nabla \boldsymbol{u}_{h,\lambda})\,\boldsymbol{u}_{h,\lambda} - \nabla p_{h,\lambda}), p_e^M(\nabla q_h + \lambda\,(\nabla \boldsymbol{v}_h)\,\boldsymbol{u}_{h,\lambda})\big)_K \\
&+ \sum_{K\in\mathcal{T}_h} (\lambda\,\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_{h,\lambda}), \lambda\,\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{v}_h))_K + \sum_{F\in\mathcal{E}_h} \frac{h_F}{12}\,([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_{h,\lambda} + p_{h,\lambda}\boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!])_F \\
&= (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall (\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h.
\end{aligned} \tag{44}
$$

Next, we denote by $T_h : \mathbf{V}' \times Q \longrightarrow \mathbf{V}_h \times Q_h$ the discrete Stokes operator, which associates with each $(\boldsymbol{w}, r) \in \mathbf{V}' \times Q$ the unique solution $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of

$$
(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla\cdot \boldsymbol{v}_h) + (q_h, \nabla\cdot \boldsymbol{u}_h) + \sum_{K\in\mathcal{T}_h} \big(p_e^M(\nabla p_h), p_e^M(\nabla q_h)\big)_K \tag{45}
$$

$$
+(\lambda\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h), \lambda\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{v}_h))_K + \sum_{F\in\mathcal{E}_h} \frac{h_F}{12}\,([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h\boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!])_F = \langle \boldsymbol{w}, \boldsymbol{v}_h\rangle + (r, q_h),
$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. This discrete Stokes operator, to the best of our knowledge, does not coincide with any stabilized method available in the literature, although it might be seen as a variation of the PPS method (cf. [11]) if the pressure space is the continuous piecewise linear functions, or the RELP method (cf. [1]), if the pressures are discontinuous. Adapting the analysis presented in [5, 1] we obtain the following result (see Appendix A for a sketch of the analysis).

Lemma 8. *There exist constants $C, C' > 0$, independent of h and λ, such that it holds*

$$
|\!|\!|(T - T_h)(\boldsymbol{w}, 0)|\!|\!| \le C\,h\,(1 + \lambda\,h)^2\,\|\boldsymbol{w}\|_{0,\Omega} \quad \forall \boldsymbol{w} \in L^2(\Omega)\,, \tag{46}
$$

$$
|\!|\!|T_h(\boldsymbol{w}, q)|\!|\!| \le C'\,(1 + \lambda\,h)^2\,\|(\boldsymbol{w}, q)\|_{(\mathbf{V}_h\times Q_h)'} \quad \forall (\boldsymbol{w}, q) \in (\mathbf{V} \times Q)'\,. \tag{47}
$$

To write the discrete method as a fixed point equation we also introduce the mapping $G_h : \Lambda \times H^2(\mathcal{T}_h)^2 \times H^1(\mathcal{T}_h) \longrightarrow \mathbf{V}_h \times Q_h$ by $G_h(\lambda, \boldsymbol{z}, t) = (\boldsymbol{w}_h, r_h)$ where

$$
\begin{aligned}
&(\boldsymbol{w}_h, \boldsymbol{v}_h) + (r_h, q_h) = -(\boldsymbol{f} - \lambda(\nabla \boldsymbol{z})\boldsymbol{z}, \boldsymbol{v}_h) \\
&- \sum_{K\in\mathcal{T}_h} \big(p_e^M(\boldsymbol{f} + \Delta \boldsymbol{z} - \lambda\,(\nabla \boldsymbol{z})\boldsymbol{z} - \nabla t), p_e^M(\lambda\,(\nabla \boldsymbol{v}_h)\boldsymbol{z})\big)_K - \sum_{K\in\mathcal{T}_h} \big(p_e^M(\boldsymbol{f} - \lambda\,(\nabla \boldsymbol{z})\boldsymbol{z}), p_e^M(\nabla q_h)\big)_K\,,
\end{aligned} \tag{48}
$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$.

Using these operators, problem (44) can be written in a form analogous to (18):

$$F_h(\lambda, \boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) := (\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) + T_h G_h(\lambda, \boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) = \mathbf{0}. \tag{49}$$

The next result states some properties of F_h and its derivatives, and will be fundamental for the error analysis carried out below. The first one proves that the differential operator $D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)$ is an isomorphism under appropriate conditions on h and λ.

Lemma 9. *Assume that on a given compact interval $\Lambda \subset \mathbb{R}$ a regular branch $\lambda \to (\boldsymbol{u}_\lambda, p_\lambda)$ of solutions of problem (18) exists and that $(\boldsymbol{u}_\lambda, p_\lambda)$ belongs to the space $H^2(\Omega)^2 \times H^1(\Omega)$. Therefore, there exists a constant $h_0 > 0$ such that, for all $h \le h_0$, the mapping $D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)$ is an isomorphism on $\mathbf{V}_h \times Q_h$.*

Proof. We start noting that, since T is a linear and continuous operator, and using (7) and (30), we arrive at

$$\begin{aligned}
&\|D_{\boldsymbol{u},p}F(\lambda, \boldsymbol{u}_\lambda, p_\lambda) - D_{\boldsymbol{u},p}F(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)\|_{\mathcal{L}(\mathbf{V}\times Q)} \\
&= \sup_{\|(\boldsymbol{v},q)\|\le 1} \|T\big((D_{\boldsymbol{u},p}G(\lambda, \boldsymbol{u}_\lambda, p_\lambda) - D_{\boldsymbol{u},p}G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda))[\boldsymbol{v}, q]\big)\| \\
&\le C \sup_{\|(\boldsymbol{v},q)\|\le 1} \|(D_{\boldsymbol{u},p}G(\lambda, \boldsymbol{u}_\lambda, p_\lambda) - D_{\boldsymbol{u},p}G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda))[\boldsymbol{v}, q]\|_{(\mathbf{V}\times Q)'} \\
&\le C\lambda \sup_{\|(\boldsymbol{w},t)\|\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} ((\nabla\boldsymbol{u}_\lambda)\boldsymbol{v} + (\nabla\boldsymbol{v})\boldsymbol{u}_\lambda - (\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v} - (\nabla\boldsymbol{v})\mathcal{I}_h\boldsymbol{u}_\lambda, \boldsymbol{w}) \\
&= C\lambda \sup_{\|(\boldsymbol{w},t)\|\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} ((\nabla(\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda))\boldsymbol{v} + (\nabla\boldsymbol{v})(\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda), \boldsymbol{w}) \\
&\le C\lambda \sup_{\|(\boldsymbol{w},t)\|\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} 2|\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda|_{1,\Omega}\, |\boldsymbol{v}|_{1,\Omega}\, |\boldsymbol{w}|_{1,\Omega} \\
&\le C\lambda\, h\, L,
\end{aligned} \tag{50}$$

where $L := sup_{\lambda\in\Lambda} \max\{\|\boldsymbol{f}\|_{0,\Omega}, \|\boldsymbol{u}_\lambda\|_{2,\Omega}, \|p_\lambda\|_{1,\Omega}\}$. Then, since the set of isomorphisms on $\mathbf{V} \times Q$ is open, there exists $h_2 > 0$ such that $D_{\boldsymbol{u},p}F(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)$ is an isomorphism on $\mathbf{V} \times Q$ for all $h \le h_2$.

Next, we define the mapping $A_1 := I + T_h D_{\boldsymbol{u},p}G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)$, which belongs to $\mathcal{L}(\mathbf{V} \times Q)$, but also to $\mathcal{L}(\mathbf{V}_h \times Q_h)$. Then, we use Lemma 8, Hölder's inequality, (33), (34), and the inclusions $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, to obtain

$$\begin{aligned}
&\|A_1 - D_{\boldsymbol{u},p}F(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)\|_{\mathcal{L}(\mathbf{V}\times Q)} = \sup_{\|(\boldsymbol{v},q)\|\le 1} \|(T_h - T) D_{\boldsymbol{u},p} G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)[\boldsymbol{v}, q]\| \\
&\le C\,(1+\lambda h)^2 \lambda\, h \sup_{\|\boldsymbol{w}\|_{0,\Omega}\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} ((\nabla \mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v} + (\nabla\boldsymbol{v})\mathcal{I}_h\boldsymbol{u}_\lambda, \boldsymbol{w}) \\
&\le C\,(1+\lambda h)^2 \lambda\, h \sup_{\|\boldsymbol{w}\|_{0,\Omega}\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} \{\|\nabla \mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,4,\Omega}\|\boldsymbol{v}\|_{0,4,\Omega}\|\boldsymbol{w}\|_{0,\Omega} + \|\nabla\boldsymbol{v}\|_{0,\Omega}\|\mathcal{I}_h\boldsymbol{u}_\lambda\|_{\infty,\Omega}\|\boldsymbol{w}\|_{0,\Omega}\} \\
&\le C\,(1+\lambda h)^2 \lambda\, h \sup_{\|\boldsymbol{w}\|_{0,\Omega}\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} \{\|\nabla \boldsymbol{u}_\lambda\|_{0,4,\Omega}\|\boldsymbol{v}\|_{0,4,\Omega}\|\boldsymbol{w}\|_{0,\Omega} + \|\boldsymbol{u}_\lambda\|_{\infty,\Omega}\|\nabla\boldsymbol{v}\|_{0,\Omega}\|\boldsymbol{w}\|_{0,\Omega}\} \\
&\le C\,(1+\lambda h)^2 \lambda\, h \sup_{\|\boldsymbol{w}\|_{0,\Omega}\le 1} \sup_{\|(\boldsymbol{v},q)\|\le 1} \{\|\boldsymbol{u}_\lambda\|_{2,\Omega}|\boldsymbol{v}|_{1,\Omega}\|\boldsymbol{w}\|_{0,\Omega} + \|\boldsymbol{u}_\lambda\|_{2,\Omega}|\boldsymbol{v}|_{1,\Omega}\|\boldsymbol{w}\|_{0,\Omega}\} \\
&\le C\,(1+\lambda h)^2\, \lambda\, h\, L\,, \qquad (51)
\end{aligned}$$

and then there exists $h_1 \le h_2$ such that for all $h \le h_1$ the mapping A_1 is an isomorphism in $\mathbf{V}\times Q$. Also, since A_1 also maps $\mathbf{V}_h \times Q_h$ onto itself, and is injective, is also an isomorphism on $\mathbf{V}_h \times Q_h$.

Finally, using Lemma 8, it holds

$$\begin{aligned}
&\|A_1 - D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)\|_{\mathcal{L}(\mathbf{V}_h\times Q_h)} \\
= &\sup_{\|(\boldsymbol{v}_h,q_h)\|\le 1} \|T_h(D_{\boldsymbol{u},p}G_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda) - D_{\boldsymbol{u},p}G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda))[\boldsymbol{v}_h, q_h]\| \\
\le\ & C\,(1+\lambda\, h)^2 \sup_{\|(\boldsymbol{v}_h,q_h)\|\le 1} \|(D_{\boldsymbol{u},p}G_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda) - D_{\boldsymbol{u},p}G(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda))[\boldsymbol{v}_h, q_h]\|_{(\mathbf{V}_h\times Q_h)'} \\
\le\ & C\,(1+\lambda\, h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\le 1} \sup_{\|(\boldsymbol{v}_h,q_h)\|\le 1} \left\{ \sum_{K\in\mathcal{T}_h} \big(p_e^M(\boldsymbol{f} - \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - \nabla\mathcal{J}_h p_\lambda), p_e^M(\lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h)\big)_K \right. \\
&- \sum_{K\in\mathcal{T}_h} \big(p_e^M(\lambda(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda + \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h + \nabla q_h), p_e^M(\lambda(\nabla\boldsymbol{w}_h)\mathcal{I}_h\boldsymbol{u}_\lambda)\big)_K \\
&\left. - \sum_{K\in\mathcal{T}_h} \big(p_e^M(\lambda(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda + \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h), p_e^M(\nabla t_h)\big)_K \right\} \\
=\ & C\,(1+\lambda\, h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\le 1} \sup_{\|(\boldsymbol{v}_h,q_h)\|\le 1} \{\mathrm{I} + \mathrm{II} + \mathrm{III}\}\,.
\end{aligned}$$

Now, using (24), (30), (31) and the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain

$$\begin{aligned}
\|(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda - (\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} &\le \|(\nabla(\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda))\boldsymbol{u}_\lambda\|_{0,K} + \|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)(\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda)\|_{0,K} \\
&\le \|\boldsymbol{u}_\lambda\|_{\infty,K}\,|\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda|_{1,K} + \|\nabla\mathcal{I}_h\boldsymbol{u}_\lambda\|_{\infty,K}\,\|\boldsymbol{u}_\lambda - \mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\
&\le \|\boldsymbol{u}_\lambda\|_{\infty,\Omega} C h_K \|\boldsymbol{u}_\lambda\|_{2,K} + C h_K^{-1}\|\nabla\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} h_K^2 \|\boldsymbol{u}_\lambda\|_{2,K} \\
&\le C h_K L \|\boldsymbol{u}_\lambda\|_{2,K}, \qquad (52)
\end{aligned}$$

and then, using (35), we get

$$
\begin{aligned}
&\| -\boldsymbol{f} + \lambda(\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\mathcal{I}_h \boldsymbol{u}_\lambda + \nabla \mathcal{J}_h p_\lambda \|_{0,K} \\
&= \|\Delta \boldsymbol{u}_\lambda - \lambda((\nabla \boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda - (\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\mathcal{I}_h \boldsymbol{u}_\lambda) - \nabla(p_\lambda - \mathcal{J}_h p_\lambda)\|_{0,K} \\
&\leq \|\Delta \boldsymbol{u}_\lambda\|_{0,K} + C\lambda h_K L \|\boldsymbol{u}_\lambda\|_{2,K} + C\|p_\lambda\|_{1,\omega_K} \\
&\leq C\{(1 + \lambda\, h\, L)\|\boldsymbol{u}_\lambda\|_{2,K} + \|p_\lambda\|_{1,\omega_K}\}. \qquad (53)
\end{aligned}
$$

The item I is addressed using Lemma 5, (53) and (24), and Poincaré's inequality as follows

$$
\begin{aligned}
\mathrm{I} &= \sum_{K\in\mathcal{T}_h} \left(p_e^M(\boldsymbol{f} - \lambda(\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\mathcal{I}_h \boldsymbol{u}_\lambda - \nabla \mathcal{J}_h p_\lambda), p_e^M(\lambda(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h)\right)_K \\
&\leq C \sum_{K\in\mathcal{T}_h} \lambda\, h_K^2 \|\boldsymbol{f} - \lambda(\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\mathcal{I}_h \boldsymbol{u}_\lambda - \nabla \mathcal{J}_h p_\lambda\|_{0,K} \|(\nabla \boldsymbol{w}_h)\boldsymbol{v}_h\|_{0,K} \\
&\leq C \sum_{K\in\mathcal{T}_h} \lambda\, h_K\, L(1 + \lambda\, h_K\, L)\|\nabla \boldsymbol{w}_h\|_{0,K} h_K \|\boldsymbol{v}_h\|_{\infty,K} \\
&\leq C\,\lambda\, h\, L\,(1 + \lambda\, h\, L) \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{w}_h\|_{0,K} \|\boldsymbol{v}_h\|_{0,K} \\
&\leq C\,\lambda\, h\, L\,(1 + \lambda\, h\, L) |\!|\!|(\boldsymbol{w}_h, t_h)|\!|\!| \, |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|. \qquad (54)
\end{aligned}
$$

Also, using again Lemma 5, the embedding of $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, (26), (33), (34) and the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the item II is bounded as follows

$$
\begin{aligned}
\mathrm{II} &= -\sum_{K\in\mathcal{T}_h} \left(p_e^M(\lambda(\nabla \boldsymbol{v}_h)\mathcal{I}_h \boldsymbol{u}_\lambda + \lambda(\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\boldsymbol{v}_h + \nabla q_h), p_e^M(\lambda(\nabla \boldsymbol{w}_h)\mathcal{I}_h \boldsymbol{u}_\lambda)\right)_K \\
&\leq C\,\lambda \sum_{K\in\mathcal{T}_h} \left\{\lambda \|p_e^M((\nabla \boldsymbol{v}_h)\mathcal{I}_h \boldsymbol{u}_\lambda + (\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\boldsymbol{v}_h)\|_{0,K} + \|p_e^M(\nabla q_h)\|_{0,K}\right\} \|p_e^M((\nabla \boldsymbol{w}_h)\mathcal{I}_h \boldsymbol{u}_\lambda)\|_{0,K} \\
&\leq C\,\lambda \sum_{K\in\mathcal{T}_h} \{\lambda\, h_K(\|(\nabla \boldsymbol{v}_h)\mathcal{I}_h \boldsymbol{u}_\lambda\|_{0,K} + \|(\nabla \mathcal{I}_h \boldsymbol{u}_\lambda)\boldsymbol{v}_h\|_{0,K}) + h_K \|\nabla q_h\|_{0,K}\}\, h_K \|(\nabla \boldsymbol{w}_h)\mathcal{I}_h \boldsymbol{u}_\lambda\|_{0,K} \\
&\leq C\,\lambda\, h \sum_{K\in\mathcal{T}_h} \{\lambda\, h_K \|\nabla \boldsymbol{v}_h\|_{0,K} \|\mathcal{I}_h \boldsymbol{u}_\lambda\|_{\infty,K} + \lambda\, h_K \|\nabla \mathcal{I}_h \boldsymbol{u}_\lambda\|_{0,4,K} \|\boldsymbol{v}_h\|_{0,4,K} \\
&\quad + h_K \|\nabla q_h\|_{0,K}\}\, \|\nabla \boldsymbol{w}_h\|_{0,K} \|\mathcal{I}_h \boldsymbol{u}_\lambda\|_{\infty,K} \\
&\leq C\,\lambda\, h\, L \left\{\lambda\, h \|\nabla \boldsymbol{v}_h\|_{0,\Omega} \|\boldsymbol{u}_\lambda\|_{\infty,\Omega} + \lambda\, h \|\nabla \boldsymbol{u}_\lambda\|_{0,4,\Omega} \|\boldsymbol{v}_h\|_{0,4,\Omega} + \|q_h\|_{0,\Omega}\right\} \|\nabla \boldsymbol{w}_h\|_{0,\Omega} \\
&\leq C\,\lambda\, h\, L\, \{1 + \lambda\, h\, L\}\, |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!| \, |\!|\!|(\boldsymbol{w}_h, t_h)|\!|\!|, \qquad (55)
\end{aligned}
$$

and the item III as

$$\begin{aligned}
\mathrm{III} &= -\sum_{K\in\mathcal{T}_h}\left(p_e^M(\lambda(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda+\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{v}_h),p_e^M(\nabla t_h)\right)_K\\
&\le C\lambda\,h\sum_{K\in\mathcal{T}_h}\left(\|\nabla\boldsymbol{v}_h\|_{0,K}\|\mathcal{I}_h\boldsymbol{u}_\lambda\|_{\infty,K}+\|\nabla\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,4,K}\|\boldsymbol{v}_h\|_{0,4,K}\right)\|p_e^M(\nabla t_h)\|_{0,K}\\
&\le C\lambda\,h\left\{|\boldsymbol{v}_h|_{1,\Omega}\|\boldsymbol{u}_\lambda\|_{\infty,\Omega}+\|\mathcal{I}_h\boldsymbol{u}_\lambda\|_{1,4,\Omega}\|\boldsymbol{v}_h\|_{0,4,\Omega}\right\}\left(\sum_{K\in\mathcal{T}_h}h_K^2\|\nabla t_h\|_{0,K}^2\right)^{1/2}\\
&\le C\lambda\,h\,\|\boldsymbol{u}_\lambda\|_{2,\Omega}|\boldsymbol{v}_h|_{1,\Omega}\|t_h\|_{0,\Omega}\\
&\le C\lambda\,h\,L\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|\;|\!|\!|(\boldsymbol{w}_h,t_h)|\!|\!|. \qquad (56)
\end{aligned}$$

Therefore, from (54), (55) and (56) it follows that

$$\|A_1-D_{\boldsymbol{u},p}F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)\|_{\mathcal{L}(\mathbf{V}_h\times Q_h)}\le C\,\lambda\,h\,L\,(1+\lambda\,h)^2\,(1+\lambda\,h\,L)\,, \qquad (57)$$

and the result follows for some $h_0\le h_1$ using analogous arguments. □

Along with the previous Lemma, the next result states further properties of the mapping F_h and its derivative.

Lemma 10. *Assume the hypothesis of Lemma 9 hold. Therefore, there exists a constant C, which does not depend on h or λ, such that*

$$|\!|\!|F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)|\!|\!|\le C\,h\left\{\lambda L^2(1+\lambda h)^2\,(1+h+\lambda h^2L)+L\right\}. \qquad (58)$$

Furthermore, for each $\rho>0$ and for all $(\boldsymbol{v}_h,q_h)\in\mathbf{V}_h\times Q_h$ such that $(\boldsymbol{v}_h,q_h)$ belongs to the ball centered at $(\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)$ with radius ρ, there exists a constant $C>0$, independent of h and λ but depending on ρ, such that

$$\begin{aligned}
&\|D_{\boldsymbol{u},p}F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)-D_{\boldsymbol{u},p}F_h(\lambda,\boldsymbol{v}_h,q_h)\|_{\mathcal{L}(\mathbf{V}_h\times Q_h)}\\
&\qquad\le C\,\lambda\,\left\{(1+\lambda\,h)^2\,(1+\lambda+\lambda\,L)\right\}\,|\!|\!|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathcal{J}_hp_\lambda-q_h)|\!|\!|\,. \qquad (59)
\end{aligned}$$

Proof. We first note that $F_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)=\mathbf{0}$, and then using the linearity of T_h we obtain

$$\begin{aligned}
|\!|\!|F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)|\!|\!| &= |\!|\!|F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)-F_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)|\!|\!|\\
&\le |\!|\!|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda-p_\lambda)|\!|\!|+|\!|\!|T_h(G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda)-G_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda))|\!|\!|\\
&= S_1+S_2. \qquad (60)
\end{aligned}$$

To estimate S_1 we use (30) and (35) and easily obtain

$$S_1\le C\,h\,L\,. \qquad (61)$$

Next, using the continuity of T_h (cf. Lemma 8) and the dual norm (5), it holds

$$
\begin{aligned}
S_2 &= \|\!|T_h(G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda) - G_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_h p_\lambda))\|\!| \\
&\le C\,(1+\lambda\,h)^2 \sup_{\|\!|(\boldsymbol{v}_h,q_h)\|\!|\le 1} (G_h(\lambda,\boldsymbol{u}_\lambda,p_\lambda) - G_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_h p_\lambda),(\boldsymbol{v}_h,q_h)) \\
&\le C\,(1+\lambda\,h)^2 \sup_{\|\!|(\boldsymbol{v}_h,q_h)\|\!|\le 1} \Bigg\{ \lambda((\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda - (\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda,\boldsymbol{v}_h)_\Omega \\
&\quad - \sum_{K\in\mathcal{T}_h} \big(p_e^M(\boldsymbol{f} + \Delta\boldsymbol{u}_\lambda - \lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda - \nabla p_\lambda), p_e^M(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{u}_\lambda)\big)_K \\
&\quad - \sum_{K\in\mathcal{T}_h} \big(p_e^M(-\boldsymbol{f} + \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda + \nabla\mathcal{J}_h p_\lambda), p_e^M(\lambda(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda)\big)_K \\
&\quad - \sum_{K\in\mathcal{T}_h} \big(p_e^M(\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - \lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda), p_e^M(\nabla q_h)\big)_K \Bigg\} \\
&\le C\,(1+\lambda\,h)^2 \sup_{\|\!|(\boldsymbol{v}_h,q_h)\|\!|\le 1} \{\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\}\,.
\end{aligned}
$$

As for the first term, using (52) and Cauchy–Schwarz's and Poincaré's inequalities we have

$$
\begin{aligned}
\mathrm{I} &= \lambda((\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - (\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda,\boldsymbol{v}_h)_\Omega \\
&\le \lambda\|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - (\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda\|_{0,\Omega}\,\|\boldsymbol{v}_h\|_{0,\Omega} \\
&\le C\,\lambda\,h\,L^2\,\|\!|(\boldsymbol{v}_h,q_h)\|\!|. \qquad (62)
\end{aligned}
$$

Since $(\boldsymbol{u}_\lambda,p_\lambda)$ is the solution of problem (16) then $\mathrm{II}=0$. We are left with bounding III and IV, for which we use Lemma 5, (34) and the continuous embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, and (53) as follows

$$
\begin{aligned}
\mathrm{III} &= -\sum_{K\in\mathcal{T}_h} \big(p_e^M(-\boldsymbol{f} + \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda + \nabla\mathcal{J}_h p_\lambda), p_e^M(\lambda(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda)\big)_K \\
&\le C\sum_{K\in\mathcal{T}_h} \lambda h_K^2\,\|-\boldsymbol{f} + \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda + \nabla\mathcal{J}_h p_\lambda\|_{0,K}\|(\nabla\boldsymbol{v}_h)\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\
&\le C\sum_{K\in\mathcal{T}_h} \lambda h_K^2\,\{(1+\lambda\,h\,L)\|\boldsymbol{u}_\lambda\|_{2,K} + \|p_\lambda\|_{1,\omega_K}\}\;|\boldsymbol{v}_h|_{1,K}\,L \\
&\le C\,\lambda\,h^2\,L^2\,\{1+\lambda\,h\,L\}\,\|\!|(\boldsymbol{v}_h,q_h)\|\!|. \qquad (63)
\end{aligned}
$$

As for item IV, we further use (52) and (26) to get

$$\begin{aligned}\mathrm{IV} &= \sum_{K\in\mathcal{T}_h} \left(p_e^M(\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - \lambda(\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda), p_e^M(\nabla q_h)\right)_K \\ &\leq C\sum_{K\in\mathcal{T}_h} \lambda\, h_K \,\|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - (\nabla\boldsymbol{u}_\lambda)\boldsymbol{u}_\lambda\|_{0,K}\,\|p_e^M(\nabla q_h)\|_{0,K} \\ &\leq C\sum_{K\in\mathcal{T}_h} \lambda\, L\, h_K^2\|\boldsymbol{u}_\lambda\|_{2,K}\, h_K\|\nabla q_h\|_{0,K} \\ &\leq C\,\lambda\, h^2\, L^2\, |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|. \end{aligned} \tag{64}$$

Gathering (62)–(64), we get

$$S_2 \leq C\,\lambda h L^2\,(1+\lambda h)^2\,(1+h+\lambda h^2 L) \tag{65}$$

and thus, using (60) and the bounds given in (61) and (65), we establish (58).

Estimate (59) is addressed next. Let $(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h), (\boldsymbol{w}_h, r_h) \in \mathbf{V}_h \times Q_h$ with $|\!|\!|(\boldsymbol{w}_h, r_h)|\!|\!| = 1$. From the stability of the discrete Stokes operator in Lemma 8 we get

$$\begin{aligned} &|\!|\!|D_{\boldsymbol{u},p}F_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h) - D_{\boldsymbol{u},p}F_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)(\boldsymbol{w}_h,r_h)|\!|\!| \\ =\ & |\!|\!|T_h(D_{\boldsymbol{u},p}G_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h) - D_{\boldsymbol{u},p}G_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)(\boldsymbol{w}_h,r_h))|\!|\!| \\ \leq\ & C\,(1+\lambda\,h)^2 \sup_{|\!|\!|(\boldsymbol{z}_h,s_h)|\!|\!|\leq 1} (D_{\boldsymbol{u},p}G_h(\lambda,\boldsymbol{v}_h,q_h)(\boldsymbol{w}_h,r_h) - D_{\boldsymbol{u},p}G_h(\lambda,\mathcal{I}_h\boldsymbol{u}_\lambda,\mathcal{J}_hp_\lambda)(\boldsymbol{w}_h,r_h),(\boldsymbol{z}_h,s_h)) \\ \leq\ & C\,(1+\lambda\,h)^2 \sup_{|\!|\!|(\boldsymbol{z}_h,s_h)|\!|\!|\leq 1} \Big\{ -(\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\mathcal{I}_h\boldsymbol{u}_\lambda, \boldsymbol{z}_h) \\ & + (\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h, \boldsymbol{z}_h) + \sum_{K\in\mathcal{T}_h}\left(p_e^M(\boldsymbol{f} - \lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda - \nabla\mathcal{J}_hp_\lambda), p_e^M(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\right)_K \\ & - \sum_{K\in\mathcal{T}_h}\left(p_e^M(\boldsymbol{f} - \lambda(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h - \nabla q_h), p_e^M(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\right)_K \\ & - \sum_{K\in\mathcal{T}_h}\left(p_e^M(\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\mathcal{I}_h\boldsymbol{u}_\lambda + \nabla r_h), p_e^M(\lambda(\nabla\boldsymbol{z}_h)\mathcal{I}_h\boldsymbol{u}_\lambda)\right)_K \\ & + \sum_{K\in\mathcal{T}_h}\left(p_e^M(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h + \nabla r_h), p_e^M(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{v}_h)\right)_K \\ & - \sum_{K\in\mathcal{T}_h}\left(p_e^M(\lambda(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\mathcal{I}_h\boldsymbol{u}_\lambda), p_e^M(\nabla s_h)\right)_K \\ & + \sum_{K\in\mathcal{T}_h}\left(p_e^M(\lambda(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h + \lambda(\nabla\boldsymbol{w}_h)\boldsymbol{v}_h), p_e^M(\nabla s_h)\right)_K \Big\} \\ =\ & C\,(1+\lambda\,h)^2 \sup_{|\!|\!|(\boldsymbol{z}_h,s_h)|\!|\!|\leq 1} \Big\{\mathrm{V} + \mathrm{VI} + \mathrm{VII} + \mathrm{VIII} + \mathrm{IX} + \mathrm{X}\Big\}. \end{aligned}$$

We bound the item V by using (7) as follows

$$
\begin{aligned}
\mathrm{V} &= -\lambda\left((\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\boldsymbol{w}_h+(\nabla\boldsymbol{w}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h),\boldsymbol{z}_h\right)\\
&\le 2\alpha\,\lambda\,|\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega}|\boldsymbol{z}_h|_{1,\Omega}\\
&\le 2\alpha\,\lambda\,\|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathcal{J}_hp_\lambda-q_h)\|\;\|(\boldsymbol{w}_h,r_h)\|\;\|(\boldsymbol{z}_h,s_h)\|, \qquad (66)
\end{aligned}
$$

and for the item VI we use Cauchy-Schwarz's inequality, Lemma 5, (24) and (26) to obtain

$$
\begin{aligned}
\mathrm{VI} &= -\sum_{K\in\mathcal{T}_h}\left(p_e^M(\nabla(\mathcal{J}_hp_\lambda-q_h)),p_e^M(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\right)_K\\
&\le C\,\lambda\sum_{K\in\mathcal{T}_h}h_K\,\|\nabla(\mathcal{J}_hp_\lambda-q_h)\|_{0,K}h_K\|\nabla\boldsymbol{z}_h\|_{0,K}\|\boldsymbol{w}_h\|_{\infty,K}\\
&\le C\,\lambda\sum_{K\in\mathcal{T}_h}\|\mathcal{J}_hp_\lambda-q_h\|_{0,K}|\boldsymbol{z}_h|_{1,K}\|\boldsymbol{w}_h\|_{0,K}\\
&\le C\,\lambda\,\|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathcal{J}_hp_\lambda-q_h)\|\;\|(\boldsymbol{z}_h,s_h)\|\;\|(\boldsymbol{w}_h,r_h)\|\,. \qquad (67)
\end{aligned}
$$

Following analogous steps, and using Poincaré's inequality, we can establish the following estimates for items VII – X:

$$
\begin{aligned}
\mathrm{VII} &= -\sum_{K\in\mathcal{T}_h}\lambda\left(p_e^M((\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h),p_e^M(\lambda(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h)\right)_K\\
&\le C\lambda^2\sum_{K\in\mathcal{T}_h}h_K^2\|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h\|_{0,K}\|(\nabla\boldsymbol{z}_h)\boldsymbol{w}_h\|_{0,K}\\
&\le C\lambda^2\sum_{K\in\mathcal{T}_h}h_K\|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)\mathcal{I}_h\boldsymbol{u}_\lambda-(\nabla\boldsymbol{v}_h)\boldsymbol{v}_h\|_{0,K}|\boldsymbol{z}_h|_{1,K}\|\boldsymbol{w}_h\|_{0,K}\\
&\le C\lambda^2\left\{\sum_{K\in\mathcal{T}_h}h_K\|(\nabla\mathcal{I}_h\boldsymbol{u}_\lambda)(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,K}+\sum_{K\in\mathcal{T}_h}h_K\|(\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h))\boldsymbol{v}_h\|_{0,K}\right\}|\boldsymbol{z}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega}\\
&\le C\lambda^2\left\{\sum_{K\in\mathcal{T}_h}h_K\|\nabla\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K}\|\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{\infty,K}+\sum_{K\in\mathcal{T}_h}h_K\|\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h)\|_{0,K}\|\boldsymbol{v}_h\|_{\infty,K}\right\}|\boldsymbol{z}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega}\\
&\le C\lambda^2\left\{\left(\sum_{K\in\mathcal{T}_h}\|\nabla\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\|\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h\|_{0,K}^2\right)^{1/2}\right.\\
&\qquad\left.+\left(\sum_{K\in\mathcal{T}_h}|\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h|_{1,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{v}_h\|_{0,K}^2\right)^{1/2}\right\}|\boldsymbol{z}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega}\\
&\le C\lambda^2\left\{\|\boldsymbol{u}_\lambda\|_{2,\Omega}+|\boldsymbol{v}_h|_{1,\Omega}\right\}\|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathcal{J}_hp_\lambda-q_h)\|\;\|(\boldsymbol{w}_h,r_h)\|\;\|(\boldsymbol{z}_h,s_h)\|,\\
&\le C\lambda^2\left\{L+\rho\right\}\|(\mathcal{I}_h\boldsymbol{u}_\lambda-\boldsymbol{v}_h,\mathcal{J}_hp_\lambda-q_h)\|\;\|(\boldsymbol{w}_h,r_h)\|\;\|(\boldsymbol{z}_h,s_h)\|, \qquad (68)
\end{aligned}
$$

and

$$\begin{aligned}
\mathrm{VIII} = &- \sum_{K\in\mathcal{T}_h} \lambda \left(p_e^M((\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h))\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h)), p_e^M(\lambda(\nabla\boldsymbol{z}_h)\mathcal{I}_h\boldsymbol{u}_\lambda)\right)_K \\
&\le C\,\lambda^2 \sum_{K\in\mathcal{T}_h} h_K^2 \|(\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h))\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h)\|_{0,K}\|(\nabla\boldsymbol{z}_h)\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\
&\le C\,\lambda^2 \sum_{K\in\mathcal{T}_h} \left\{|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h|_{1,K}\|\boldsymbol{w}_h\|_{0,K} + |\boldsymbol{w}_h|_{1,K}\|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h\|_{0,K}\right\} |\boldsymbol{z}_h|_{1,K}\|\mathcal{I}_h\boldsymbol{u}_\lambda\|_{0,K} \\
&\le C\,\lambda^2 \left\{ \left(\sum_{K\in\mathcal{T}_h} |\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h|_{1,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{w}_h\|_{0,K}^2\right)^{1/2} \right. \\
&\quad \left. + \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{w}_h|_{1,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h\|_{0,K}^2\right)^{1/2} \right\} |\boldsymbol{z}_h|_{1,\Omega}\|\boldsymbol{u}_\lambda\|_{2,\Omega} \\
&\le C\,\lambda^2 L\, |||(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h, \mathcal{J}_h p_\lambda - q_h)|||\; |||(\boldsymbol{w}_h, r_h)|||\; |||(\boldsymbol{z}_h, s_h)|||\,, \qquad (69)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{IX} = &- \sum_{K\in\mathcal{T}_h} \lambda \left(p_e^M((\nabla\boldsymbol{v}_h)\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)\boldsymbol{v}_h + \nabla r_h), p_e^M(\lambda(\nabla\boldsymbol{z}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h))\right)_K \\
&\le C\,\lambda^2 \sum_{K\in\mathcal{T}_h} h_K^2 \|(\nabla\boldsymbol{v}_h)\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)\boldsymbol{v}_h + \nabla r_h\|_{0,K}\|(\nabla\boldsymbol{z}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h)\|_{0,K} \\
&\le C\,\lambda^2 \sum_{K\in\mathcal{T}_h} \left\{|\boldsymbol{v}_h|_{1,K}\|\boldsymbol{w}_h\|_{0,K} + |\boldsymbol{w}_h|_{1,K}\|\boldsymbol{v}_h\|_{0,K} + \|r_h\|_{0,K}\right\} |\boldsymbol{z}_h|_{1,K}\|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h\|_{0,K} \\
&\le C\,\lambda^2 |\boldsymbol{v}_h|_{1,\Omega}\, |||(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h, \mathcal{J}_h p_\lambda - q_h)|||\; |||(\boldsymbol{w}_h, r_h)|||\; |||(\boldsymbol{z}_h, s_h)||| \\
&\le C\,\lambda^2 \{L + \rho\}\, |||(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h, \mathcal{J}_h p_\lambda - q_h)|||\; |||(\boldsymbol{w}_h, r_h)|||\; |||(\boldsymbol{z}_h, s_h)|||\,, \qquad (70)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{X} = &- \sum_{K\in\mathcal{T}_h} \lambda \left(p_e^M((\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h))\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h)), p_e^M(\nabla s_h)\right)_K \\
&\le C\,\lambda \sum_{K\in\mathcal{T}_h} h_K\, \|(\nabla(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h))\boldsymbol{w}_h + (\nabla\boldsymbol{w}_h)(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h)\|_{0,K}\, h_K\, \|\nabla s_h\|_{0,K} \\
&\le C\,\lambda \sum_{K\in\mathcal{T}_h} \left\{|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h|_{1,K}\, h_K\, \|\boldsymbol{w}_h\|_{\infty,K} + |\boldsymbol{w}_h|_{1,K} h_K\, \|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h\|_{\infty,K}\right\} \|s_h\|_{0,K} \\
&\le C\,\lambda \left(|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h|_{1,\Omega}\, \|\boldsymbol{w}_h\|_{0,\Omega} + |\boldsymbol{w}_h|_{1,\Omega}\|\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h\|_{0,\Omega}\right)\|s_h\|_{0,\Omega} \\
&\le C\,\lambda\, |||(\mathcal{I}_h\boldsymbol{u}_\lambda - \boldsymbol{v}_h, \mathcal{J}_h p_\lambda - q_h)|||\; |||(\boldsymbol{z}_h, s_h)|||\; |||(\boldsymbol{w}_h, r_h)|||\,. \qquad (71)
\end{aligned}$$

Finally, gathering (66)–(71) the estimate (59) follows. □

We are now ready to prove the existence and uniqueness of a local discrete solution, and to present an error estimate.

Theorem 11. *Assume the hypothesis of Lemma 9 hold. Therefore, there is a positive constant $h_0(\Lambda)$ such that for all h with $0 < h \leq h_0$ a unique branch $\lambda \to (\boldsymbol{u}_{h,\lambda}, p_{h,\lambda})$ of solutions of problem* (21) *exists in a neighborhood of $(\boldsymbol{u}_\lambda, p_\lambda)$. Moreover, the following estimate holds*

$$\sup_{\lambda\in\Lambda} \left\{ |\boldsymbol{u}_\lambda - \boldsymbol{u}_{h,\lambda}|_{1,\Omega}^2 + \|p_\lambda - p_{h,\lambda}\|_{0,\Omega}^2 \right\}^{1/2} \;\leq\; C\, h\,, \tag{72}$$

where $C = C(L,\Lambda) > 0$ does not depend on h.

Proof. From Lemma 9 we have that $D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)$ is an isomorphism of $\mathbf{V}_h \times Q_h$ onto itself for each $\lambda \in \Lambda$, provided that $h \sup_{\lambda\in\Lambda} \lambda$ is sufficiently small. In addition, from (59) (cf. Lemma 10) we obtain that

$$\begin{aligned} &|\!|\!| F_h(\lambda, \boldsymbol{v}_h, q_h) - F_h(\lambda, \boldsymbol{w}_h, t_h) - D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)[(\boldsymbol{v}_h, q_h) - (\boldsymbol{w}_h, t_h)] |\!|\!| \\ &\qquad\qquad\qquad \leq\; C(\lambda, L)\rho\, |\!|\!| (\boldsymbol{v}_h, q_h) - (\boldsymbol{w}_h, t_h) |\!|\!|\,, \end{aligned} \tag{73}$$

for all $(\boldsymbol{v}_h, q_h), (\boldsymbol{w}_h, t_h) \in B((\mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda); \rho)$. The two above facts constitute the hypothesis of Theorem IV.3.6 in [17]. Hence, supposing that h is small enough such as

$$4\, C(\lambda, L)\, \|\{D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)\}^{-1}\|^2_{\mathcal{L}(\mathbf{V}_h\times Q_h)}\, |\!|\!| F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda) |\!|\!| \;<\; 1\,, \tag{74}$$

and applying (58) (cf. Lemma 10) and Theorem IV.3.6 in [17] we conclude that problem (49) has a unique solution $(\boldsymbol{u}_{h,\lambda}, p_{h,\lambda}) \in B((\mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda); \rho)$, where ρ is given by

$$\rho := 2\, \|\{D_{\boldsymbol{u},p}F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda)\}^{-1}\|_{\mathcal{L}(\mathbf{V}_h\times Q_h)}\, |\!|\!| F_h(\lambda, \mathcal{I}_h\boldsymbol{u}_\lambda, \mathcal{J}_h p_\lambda) |\!|\!| \;\leq\; C(L,\Lambda)h\,,$$

and the result follows using (30), (35) and the triangular inequality. □

4. A divergence-free discrete velocity.

Since continuous piecewise linear interpolations for the velocity can not be divergence-free, some additional work is needed in terms of a post-processing. Whenever discontinuous pressure interpolations are used, this can be accomplished by either solving local problems (as in [1]), or by adding a particular Raviart-Thomas vector field to the discrete velocity, which is easy to compute. Here we follow the latter option, slightly modifying the approach presented in [1, 2] and defining in each $K \in \mathcal{T}_h$

$$\boldsymbol{u}_{nc,\lambda} = \sum_{F\subset\partial K\cap\Omega} \tilde{\tau}_F \Pi_F([\![\partial_{\boldsymbol{n}}\boldsymbol{u}_{h,\lambda} + p_{h,\lambda}\boldsymbol{n}]\!]) \cdot \boldsymbol{n}\, \varphi_F, \tag{75}$$

where $\varphi_F := \pm \dfrac{h_F}{2\,|K|}(\boldsymbol{x} - \boldsymbol{x}_F)$, with $\boldsymbol{x}_F$ the opposite node to the edge F, stands for the lowest–order Raviart–Thomas basis function. Using (75), we build the non–conforming velocity field $\hat{\boldsymbol{u}}_{h,\lambda} := \boldsymbol{u}_{h,\lambda} + \boldsymbol{u}_{nc,\lambda}$ which is point-wise divergence-free and shares the same convergence properties as $\boldsymbol{u}_{h,\lambda}$. This is stated in the following result.

Lemma 12. *Let $\boldsymbol{u}_{h,\lambda}$ be the solution of* (21) *and $\boldsymbol{u}_{nc,\lambda}$ given in* (75)*, respectively. Then, the velocity field $\hat{\boldsymbol{u}}_{h,\lambda} := \boldsymbol{u}_{h,\lambda} + \boldsymbol{u}_{nc,\lambda}$ satisfies*

$$\nabla \cdot \hat{\boldsymbol{u}}_{h,\lambda} = 0 \qquad \forall K \in \mathcal{T}_h. \tag{76}$$

Moreover, under the hypothesis of Theorem 11, there exists a constant $C = C(L, \Lambda) > 0$, independent of h, such that

$$\sup_{\lambda\in\Lambda} \left\{ \sum_{K\in\mathcal{T}_h} |\boldsymbol{u}_\lambda - \hat{\boldsymbol{u}}_{h,\lambda}|^2_{1,K} \right\}^{\frac{1}{2}} \leq C\,h.$$

Proof. The proof of the first part is a slight variation of Lemma 3.8 in [1], and hence we omit it. The error estimate reduces to prove a bound on $|\boldsymbol{u}_{nc,\lambda}|_{1,K}$ which, after using the fact that $\tilde{\tau}_F \leq C h_F$ (see [2]), follows the same steps as in [1], Lemma 3.9. □

5. Numerical validations

This section is devoted to testing the new RELP method (15) having as discrete spaces the pairs $\mathbb{P}_1^2/\mathbb{P}_0$ and $\mathbb{P}_1^2/\mathbb{P}_1$ (with continuos pressures). The domain is set to be the unit square $\Omega = (0,1) \times (0,1)$. In what follows, we first validate theoretical convergence rates through a benchmark with an analytical solution. Next, we address the standard lid-driven cavity problem in the high Reynolds number setting.

5.1. A study of convergence.

We set $\nu = 1$ and $\nu = 10^{-2}$, and $\boldsymbol{f}$ is such as the exact solution $\boldsymbol{u}(x,y) := (u_1(x,y), u_2(x,y))$ and $p(x,y)$ of the Navier–Stokes equations is given by

$$u_1(x,y) := e^x \sin(y), \quad u_2(x,y) := e^x \cos(y), \quad p(x,y) := -\frac{1}{2}\,e^{2x} + \frac{1}{4}(e^2 - 1)\,.$$

We remark that the velocity is a harmonic function and then the RELP method is fully consistent for the element $\mathbb{P}_1^2/\mathbb{P}_1$. In Figures 1-2 we provide the convergence history using the norms $\|p - p_h\|_{0,\Omega}$, $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$ and $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega}$, where $(\boldsymbol{u}_h, p_h)$ is the solution of (15).

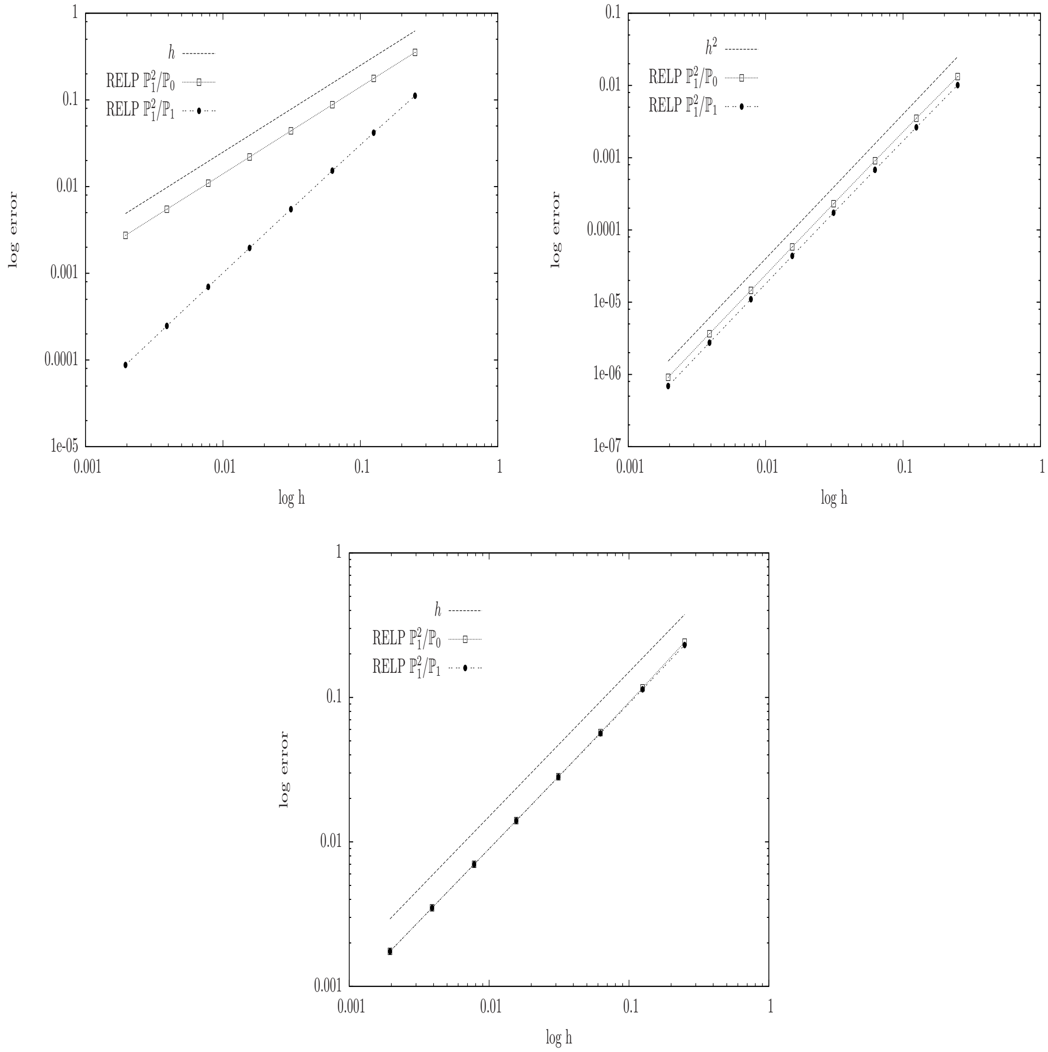

FIGURE 1. Convergence history of $\|p - p_h\|_{0,\Omega}$ (top left), $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$ (top right) and $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega}$ (bottom). Here $\nu = 1$.

We observe that the numerical orders show a perfect agreement with the theoretical ones, plus a second order convergence for the pressure for the $\mathbb{P}_1^2/\mathbb{P}_1$ case. Next, for the element $\mathbb{P}_1^2/\mathbb{P}_0$ we update the solution $\boldsymbol{u}_h$ with (75) and produce the divergence-free velocity field $\hat{\boldsymbol{u}}_h = \boldsymbol{u}_h + \boldsymbol{u}_{nc}$. The results are given in Table 1 assuming different values for ν. The procedure preserves the optimality of the error as pointed out in Figure 3, in agreement with the theory.

h	0.25	0.125	0.0625	0.03125	0.015625
$\max_{K\in\mathcal{T}_h} \lvert\nabla\cdot \hat{\boldsymbol{u}}_h\lvert_K\rvert$ $(\nu = 1)$	8×10^{-15}	4.9×10^{-14}	2.4×10^{-13}	1.4×10^{-12}	5.7×10^{-12}
$\max_{K\in\mathcal{T}_h} \lvert\nabla\cdot \hat{\boldsymbol{u}}_h\lvert_K\rvert$ $(\nu = 10^{-2})$	9.1×10^{-12}	2.6×10^{-11}	2.4×10^{-11}	8×10^{-11}	2.6×10^{-13}

TABLE 1. Error in the divergence for $\mathbb{P}_1/\mathbb{P}_0$.

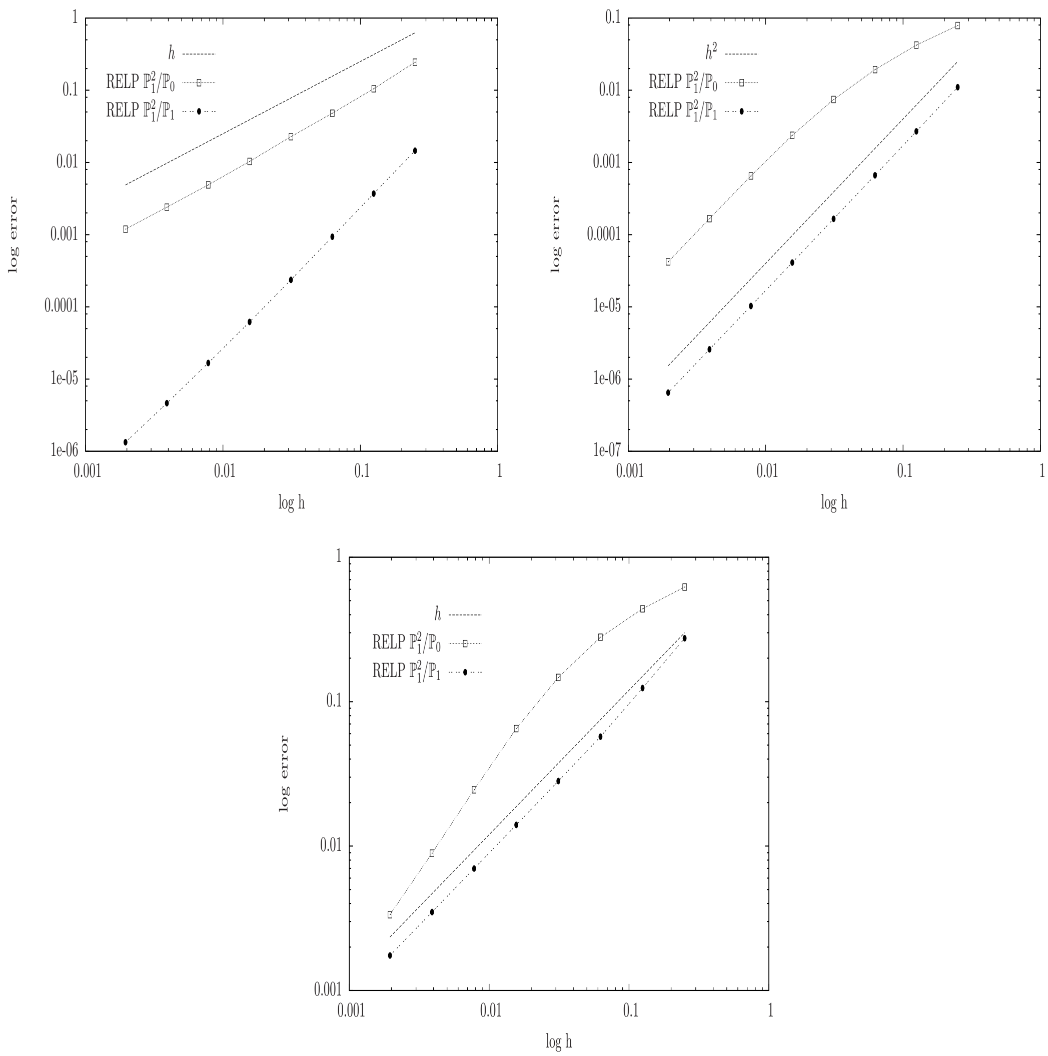

FIGURE 2. Convergence history of $\|p - p_h\|_{0,\Omega}$ (top left), $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$ (top right) and $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega}$ (bottom). Here $\nu = 10^{-2}$.

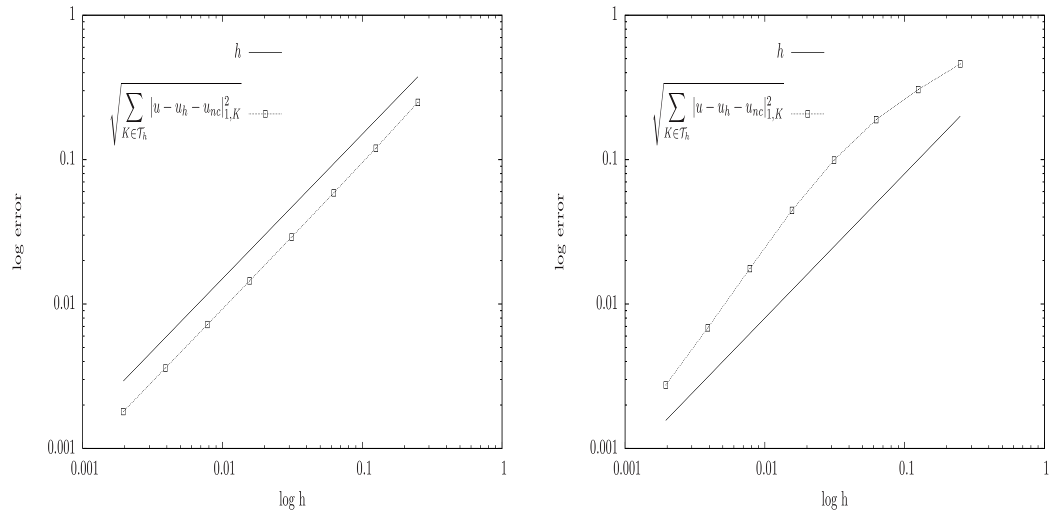

FIGURE 3. Convergence history for the updated velocity $\hat{\boldsymbol{u}}_h$ with the element $\mathbb{P}_1^2/\mathbb{P}_0$. Here $\nu = 1$ (left) and $\nu = 10^{-2}$ (right).

5.2. **The lid-driven cavity flow.** Next, a challenging test involving a high Reynolds number flow is addressed for which no exact solution is available. We attempt to solve the lid-driven cavity problem with $\boldsymbol{f} = \boldsymbol{0}$ and $\nu = \frac{1}{5000}$, and consider a structured mesh of around 65,000 elements. We depict in Figure 4 the streamlines obtained using both pairs of interpolation spaces.

We can see that the method provides a well-balanced dose of numerical diffusion as the secondary vortices are recovered. Also, the precision of the RELP method is validated comparing the numerical solutions provided by the present method with previously available reference solutions. We can see in Table 2 that such results are in accordance.

Re	Ghia et al. [16]	NSIKE [19]	RELP $\mathbb{P}_1^2/\mathbb{P}_0$	RELP $\mathbb{P}_1^2/\mathbb{P}_1$
5000	$x = 0.5117$	$x = 0.53$	$x = 0.5285$	$x = 0.5298$
	$y = 0.5352$	$y = 0.53$	$y = 0.521$	$y = 0.5370$

TABLE 2. Primary vortex center position.

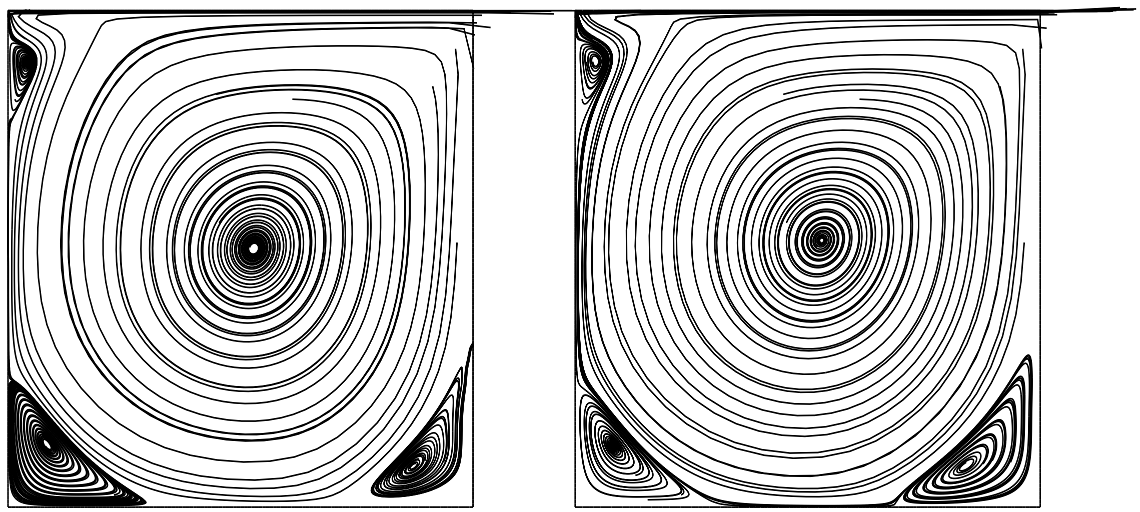

FIGURE 4. Streamlines with elements $\mathbb{P}_1^2 \times \mathbb{P}_0$ (left) and $\mathbb{P}_1^2 \times \mathbb{P}_1$ (right). Here the Reynolds number is 5000.

6. Conclusion

New RELP methods made stable the simplest and lowest equal order pairs of interpolation spaces for the fully non-linear Navier–Stokes equations, introducing the right dose of numerical diffusion. In the process of proving well-posedness and optimal convergence in the natural norms, a new stabilized method for the Stokes model was also introduced and analyzed. In addition, a simplified version of the RELP method, which shares the same desired properties of the original method, avoided the use of two level approaches and became computationally competitive. Next, this method was combined with a simple post-processing procedure to produce a locally conservative solution which is optimally convergent in the discontinuous pressure case. As such, the methods in this work may be seen as an appealing alternative to simulate complex flows using the cheapest and simplest elements in a precise way while respecting the divergence-free constraint exactly.

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Appendix A. The discrete Stokes operator

The discrete Stokes operator includes a formally new stabilized finite element method, given by: *Find* $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *such that :*

$$\boldsymbol{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \langle \boldsymbol{w}, \boldsymbol{v}_h \rangle + (r, q_h)\,, \tag{77}$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where $(\boldsymbol{w}, r) \in \mathbf{V}' \times Q$ is given, and $\boldsymbol{B}(.,.)$ reads as follows

$$\begin{aligned} \boldsymbol{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) &:= (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) \\ &+ \sum_{K \in \mathcal{T}_h} \left(p_e^M(\nabla p_h), p_e^M(\nabla q_h)\right)_K + (\lambda \chi_h(\boldsymbol{x}\, \nabla \cdot \boldsymbol{u}_h), \lambda \chi_h(\boldsymbol{x}\, \nabla \cdot \boldsymbol{v}_h))_K \\ &+ \sum_{F \in \mathcal{E}_h} \frac{h_F}{12}\, ([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!])_F\,. \end{aligned} \tag{78}$$

The next result establishes the existence and unicity of solution for (77).

Lemma 13. *The mapping* T_h *is well-defined.*

Proof. Defining the mesh-dependent norm

$$\|(\boldsymbol{v}_h, q_h)\|_h := \left\{ |\boldsymbol{v}_h|_{1,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \|\chi_h(q_h)\|_{0,K}^2 + \|\lambda \chi_h(\boldsymbol{x}\, \nabla \cdot \boldsymbol{v}_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{h_F}{12}\, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}}, \tag{79}$$

and using (14) it is easy to realize that, for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$

$$\boldsymbol{B}((\boldsymbol{v}_h, q_h), (\boldsymbol{v}_h, q_h)) = \|(\boldsymbol{v}_h, q_h)\|_h^2\,, \tag{80}$$

and, thus, the problem (77) is well-posed and the operator T_h is well-defined. □

Lemma 14. *The operator* T_h *is continuous. More precisely, there exists* $C > 0$*, independent of* h *and* λ*, such that*

$$\|T_h(\boldsymbol{w}, r)\| \leq C\, (1 + \lambda\, h)^2\, \|(\boldsymbol{w}, r)\|_{(\mathbf{V}_h \times Q_h)'}\,,$$

for all $(\boldsymbol{w}, r) \in (\mathbf{V} \times Q)'$.

Proof. The proof follows standard arguments, but we present it here for completeness. Let $(\boldsymbol{u}_h, p_h) = T_h(\boldsymbol{w}, r)$. From (80) we see that

$$\|(\boldsymbol{u}_h, p_h)\|_h^2 = \boldsymbol{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) = \langle \boldsymbol{w}, \boldsymbol{u}_h \rangle + (r, p_h) \leq \|\boldsymbol{w}\|_{\mathbf{V}_h'} |\boldsymbol{u}_h|_{1,\Omega} + \|r\|_{Q_h'} \|p_h\|_{0,\Omega}\,. \tag{81}$$

To bound the $L^2(\Omega)$-norm of p_h, let $\boldsymbol{z} \in H_0^1(\Omega)^2$ be such that

$$\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} \leq (p_h, \nabla \cdot \boldsymbol{z}) \,, \tag{82}$$

and let $\boldsymbol{z}_h$ be the Clément interpolate of $\boldsymbol{z}$. Then, integrating by parts, using that $(\boldsymbol{u}_h, p_h)$ is the solution of (77), (35) and (36) we arrive at

$$\begin{aligned}
&\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} \leq (p_h, \nabla \cdot (\boldsymbol{z} - \boldsymbol{z}_h)) + (p_h, \nabla \cdot \boldsymbol{z}_h) \\
&= - \sum_{K \in \mathcal{T}_h} (\nabla p_h, \boldsymbol{z} - \boldsymbol{z}_h)_K + \sum_{F \in \mathcal{E}_h} ([\![p_h \boldsymbol{n}]\!], \boldsymbol{z} - \boldsymbol{z}_h)_F + (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{z}_h) \\
&+ \sum_{K \in \mathcal{T}_h} (\lambda \chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{u}_h), \lambda \chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h))_K + \sum_{F \in \mathcal{E}_h} \frac{h_F}{12} ([\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \boldsymbol{n}]\!], [\![\partial_{\boldsymbol{n}} \boldsymbol{z}_h]\!])_F - \langle \boldsymbol{w}, \boldsymbol{z}_h \rangle \\
&\leq C \sum_{K \in \mathcal{T}_h} h_K \|\nabla p_h\|_{0,K} |\boldsymbol{z}|_{1,\omega_K} + C \sum_{F \in \mathcal{E}_h} h_F^{\frac{1}{2}} \| [\![p_h \boldsymbol{n}]\!] \|_{0,F} |\boldsymbol{z}|_{1,\omega_F} + |\boldsymbol{u}_h|_{1,\Omega} |\boldsymbol{z}_h|_{1,\Omega} + \|\boldsymbol{w}\|_{\mathbf{V}_h'} |\boldsymbol{z}_h|_{1,\Omega} \\
&\quad + C \sum_{K \in \mathcal{T}_h} \lambda \|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{u}_h)\|_{0,K} \lambda \|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h)\|_{0,K} + \sum_{F \in \mathcal{E}_h} \frac{h_F}{12} \| [\![\partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \boldsymbol{n}]\!] \|_{0,F} \| [\![\partial_{\boldsymbol{n}} \boldsymbol{z}_h]\!] \|_{0,F} \,.
\end{aligned} \tag{83}$$

Next, using the generalized Poincaré's inequality and the fact that $|\boldsymbol{x}|_{1,K} \leq C h_K$ and (35) we obtain

$$\|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h)\|_{0,K} = \frac{\|\chi_h(\boldsymbol{x})\|_{0,K}}{|K|^{\frac{1}{2}}} \|\nabla \cdot \boldsymbol{z}_h\|_{0,K} \leq C h_K \, \|\boldsymbol{z}\|_{1,\omega_K} \,, \tag{84}$$

and then (25), (26), (83) and the mesh regularity lead to

$$\begin{aligned}
\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} &\leq C \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F \| [\![p_h \boldsymbol{n}]\!] \|_{0,F}^2 + \|(\boldsymbol{u}_h, p_h)\|_h^2 + \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 \right\}^{\frac{1}{2}} \\
&\quad \left\{ |\boldsymbol{z}|_{1,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \lambda^2 h_K^2 \|\boldsymbol{z}\|_{1,\omega_K}^2 \right\}^{\frac{1}{2}} \\
&\leq \; C \; \left\{ \|(\boldsymbol{u}_h, p_h)\|_h^2 + \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 \right\}^{\frac{1}{2}} (1 + \lambda h) \, |\boldsymbol{z}|_{1,\Omega} \,,
\end{aligned} \tag{85}$$

and dividing by $|\boldsymbol{z}|_{1,\Omega}$ we arrive at

$$\|p_h\|_{0,\Omega} \leq C \, (1 + \lambda h) \, \left\{ \|(\boldsymbol{u}_h, p_h)\|_h^2 + \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 \right\}^{\frac{1}{2}} \,. \tag{86}$$

Then, using (86) in (81), and $a \, b \leq a^2 + \frac{1}{4} \, b^2$ with $a, \, b \in \mathbb{R}^+$, we arrive at

$$\begin{aligned}
\|(\boldsymbol{u}_h, p_h)\|_h^2 &\leq C \, (\, \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 + (1 + \lambda \, h)^2 \|r\|_{Q_h'}^2) \\
&\leq C \, (1 + \lambda \, h)^2 \, (\, \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 + \|r\|_{Q_h'}^2) \,,
\end{aligned} \tag{87}$$

and then replacing this in (86) it holds

$$\|p_h\|_{0,\Omega} \leq C\,(1+\lambda h)^2\,(\|\boldsymbol{w}\|_{\mathbf{V}_h'} + \|r\|_{Q_h'})\,.$$

Finally, the proof ends remarking that $\|\!|(\boldsymbol{u}_h,p_h)\|\!| \leq \left(\|(\boldsymbol{u}_h,p_h)\|_h^2 + \|p_h\|_{0,\Omega}^2\right)^{1/2}$ and using $\|\boldsymbol{w}\|_{\mathbf{V}_h'} + \|r\|_{Q_h'} \leq C\,\|(\boldsymbol{w},r)\|_{(\mathbf{V}_h\times Q_h)'}$. □

Lemma 15. *There exists a constant $C>0$, independent of h and λ such that, for all $\boldsymbol{w}\in L^2(\Omega)^2$, it holds*

$$\|\!|(T-T_h)(\boldsymbol{w},0)\|\!| \leq C\,(1+\lambda\,h)^2 h\;\|\boldsymbol{w}\|_{0,\Omega}\,. \tag{88}$$

Proof. Let $(\boldsymbol{u},p)=T(\boldsymbol{w},0)$ and $(\boldsymbol{u}_h,p_h)=T_h(\boldsymbol{w},0)$. The proof consists of proving the error estimates

$$\|(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)\|_h \leq C\,(1+\lambda\,h)\,h\,(\|\boldsymbol{u}\|_{2,\Omega}+\|p\|_{1,\Omega})\,, \tag{89}$$

$$\|p-p_h\|_{0,\Omega} \leq C\,(1+\lambda\,h)^2\,h\,(\|\boldsymbol{u}\|_{2,\Omega}+\|p\|_{1,\Omega})\,, \tag{90}$$

and then applying classical regularity results for the Stokes problem (cf. [17]).
To prove (88) we split the error into interpolation error $(\eta^{\boldsymbol{u}},\eta^p) := (\boldsymbol{u}-\mathcal{I}_h(\boldsymbol{u}), p-\mathcal{J}_h(p))$ and discrete error $(e_h^{\boldsymbol{u}},e_h^p) := (\mathcal{I}_h(\boldsymbol{u})-\boldsymbol{u}_h, \mathcal{J}_h(p)-p_h)$. Then, using the stability and approximation properties of χ_h, the fact that $\boldsymbol{u}$ is solenoidal, (30)-(36), and the mesh regularity we obtain

$$\begin{aligned}
\|(\eta^{\boldsymbol{u}},\eta^p)\|_h^2 &= |\eta^{\boldsymbol{u}}|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\|\chi_h(\eta^p)\|_{0,K}^2 + \|\lambda\chi_h(\boldsymbol{x}\nabla\cdot\eta^{\boldsymbol{u}})\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h}\frac{h_F}{12}\,\|[\![\partial_{\boldsymbol{n}}\eta^{\boldsymbol{u}}+\eta^p\boldsymbol{n}]\!]\|_{0,F}^2\\
&\leq Ch^2\,|\boldsymbol{u}|_{2,\Omega}^2 + \|\eta^p\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{\lambda^2\|\chi_h(\boldsymbol{x})\|_{0,K}^2}{|K|}\,\|\nabla\cdot\mathcal{I}_h(\boldsymbol{u})\|_{0,K}^2 + C\sum_{F\in\mathcal{E}_h}h_F^2\,(|\boldsymbol{u}|_{2,\omega_F}^2+|p|_{1,\omega_F}^2)\\
&\leq Ch^2(|\boldsymbol{u}|_{2,\Omega}^2+|p|_{1,\Omega}^2) + C\sum_{K\in\mathcal{T}_h}\lambda^2h_K^2\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,K}^2\\
&\leq Ch^2\big((1+\lambda^2h^2)\,|\boldsymbol{u}|_{2,\Omega}^2+|p|_{1,\Omega}^2\big)\,.
\end{aligned} \tag{91}$$

Next, using arguments very close to the analysis from [1] we can prove that

$$\|(e_h^{\boldsymbol{u}},e_h^p)\|_h \leq C(1+\lambda\,h)\,h\,(|\boldsymbol{u}|_{2,\Omega}+|p|_{1,\Omega})\,, \tag{92}$$

and then (89) follows using the triangle inequality. To prove (90) we use the continuous inf-sup condition and the definition of the method. We omit further details, and refer to [1] for an analysis that can be easily adapted to the present case. □

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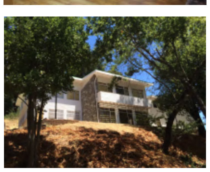

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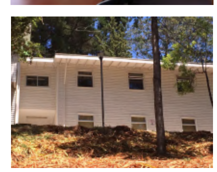

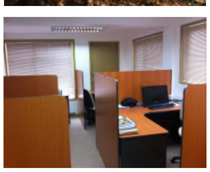

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