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NUMERICAL ANALYSIS FOR HP FOOD CHAIN SYSTEM WITH NONLOCAL AND CROSS DIFFUSION

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Abstract. In this paper, we consider a reaction-diffusion system describing three interacting species in the HP food chain structure with nonlocal and cross diffusion. We construct a finite volume scheme for this system, we establish existence and uniqueness of the discrete solution, and it is also showed that the scheme converges to the corresponding weak solution for the model studied. The convergence proof is based on the use of the discrete Sobolev embedding inequalities with general boundary conditions and a space-time L^1 compactness argument that mimics the compactness lemma due to S. N. Kruzhkov. Finally we give some numerical examples.

1. Introduction

It is clear that species do not exist alone in nature; therefore, to study the persistence and extinction of each population in systems of two or more interacting species have more biological significance. The classical ecological models of interacting populations typically have focussed on two species. Two species systems have long played dominating roles in ecology, systems such as predator-prey, plant-herbivore or plant-pest, etc. However, it has been recognized that this kind of ecological systems by two interacting species can account for only a small number of the phenomena that are commonly exhibited in nature. This is particularly true in community studies where the essence of the behaviour of a complex system may only be understood when the interactions among a large number of species are incorporated. Of course, the increasing number of differential equations and the increasing dimensionality raise considerable additional problems both for the experimenter and the theoretician. Nonetheless, such models need to be analyzed because certain three-species communities have become the focus of considerable attention.

Mathematical developments also suggest that models which involve only two species as the basic buildings blocks may miss important ecological behavior. Results that are much more complicated than those seen in two-species models appeared in early theoretical studies of three-species (e. g. [17]), models based on local stability analyses. Rosenzweig began the exploitation of three species systems by adding a third species and the trophic level [17]. Hastings and Powell [11] studied the HP food chain, and they found that there is a "tea-cup" attractor in the system. Yodzis and Innes [21] showed that changes in dynamics are associated with increased resource carrying capacity and gave estimates for resource-consumer body mass ratios that permit robust limit cycles. Klebanoff and Hastings [12] studied the dynamics of the model given by Hastings and Powell in [11] using the co-dimension two bifurcation theory to show the existence of chaotic dynamics. McCann and Yodzis [15] performed numerical simulation for the same system and showed that chaos ocurred in some region of the parameter space.

By the way, first-order differential equations like the system studied by Hastings and Powell in [11] reflects only population changes because of the predation in a situation where predator and prey are not spatially dependent. It does not take into account either the fact that predators and preys naturally develop strategies to survive, nor the fact that population is usually not homegeneously distributed. The two aspects mentioned before involve process of diffusion which can be a little

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complicated since different concentration levels of preys and predators cause different population movements. These movements can be determine by the concentration of the same species or the other species (diffusion and cross-diffusion, respectively). With this in mind, Shigesada, Kawasaki and Teramoto in [18] proposed a strongly coupled reaction-diffusion model (SKT model) with reaction terms of Lotka-Volterra type to describe spatial segregation of interacting population species in one-dimensional space. Since that time the two species SKT competing system continue being of great interest in mathematical analysis as well as in real-life modelling. Furthermore, three or multi-species system and the SKT model in any space dimension has recently focused a lot of attention due to their more complicated patterns, besides the SKT models with other kind of reaction terms are also proposed and investigated. Considering the above two aspects, Wen and Zhong investigated a strongly coupled reaction-diffusion system of the HP model given by Hastings and Powell in [11], in which the population is not homogeneously distributed caused by the consideration of diffusions. The authors established the existence of non-constant positive steady states of their system through using the Leray-Schauder degree theory [22].

The model that we considered in this paper is based on the HP food chain model given in [11], besides we consider a diffusion terms as Wen and Zhong in their model studied in [22]. We have a reaction-diffusion system in which the population is not homogeneously distributed due to the consideration of nonlocal and cross diffusion terms. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. The dynamics of interacting population with cross-diffusion are investigated by several researchers. Beginning with Turing [**?**] in 1952, diffusion and cross-diffusion have been observed as causes of the spontaneous emergence of ordered structures, namely stationary patterns. For the ecological systems with cross-diffusion and Lotka-Volterra type reaction terms in [14] is studied the effect of diffusion, self-diffusion and cross-diffusion of the two species SKT competition model. Moreover, in [16] the authors investigated a three species predator-prey model with cross-diffusion and found that the stationary patterns do not emerge from the diffusion of individual species but only appear with the introduction of cross-diffusion. The concept of this phenomena was also studied by Galiano et al [9, 10], Bendahmane et al [1, 6], and many other authors. In this kind of models were noticed that when the cross-diffusion is nonlinear, difficulties increased in the mathematical analysis. Furthermore, there is not general theory available that covers all possible cross-diffusion models. Tian, Lin and Pedersen in [19] studied the reaction-diffusion systems with nonlinear cross-diffusion, the aim of the authors is to study what role the cross-diffusion plays in the process of pattern formation.

Let be $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ a physical domain with smooth boundary $\partial\Omega$, over a time span $(0, T)$, $T > 0$ and $\Omega_T := \Omega \times (0, T)$, we have the following model:

(1.1)

$$\begin{cases} \partial_t u_1 - \operatorname{div}\left(d_1\left(\int_\Omega u_1\,dx\right)\nabla u_1\right) - \operatorname{div}\left[(\alpha_1 u_1 + u_2)\nabla u_1 + u_1 \nabla u_2\right] = F(u_1, u_2, u_3), & \text{in } \Omega_T, \\ \partial_t u_2 - \operatorname{div}\left(d_2\left(\int_\Omega u_2\,dx\right)\nabla u_2\right) & \\ \qquad -\operatorname{div}\Big[u_2\nabla u_1 + (u_1 + \alpha_2 u_2 + u_3)\nabla u_2 + u_2 \nabla u_3\Big] = G(u_1, u_2, u_3), & \text{in } \Omega_T, \\ \partial_t u_3 - \operatorname{div}\left(d_3\left(\int_\Omega u_3\,dx\right)\nabla u_3\right) - \operatorname{div}\left[u_3\nabla u_2 + (u_2 + \alpha_3 u_3)\nabla u_3\right] = H(u_1, u_2, u_3), & \text{in } \Omega_T, \\ u_i(\cdot, 0) = u_{i0}(\cdot) > 0 \qquad \text{in } \Omega, \text{ for } \quad i = 1, 2, 3, & \end{cases}$$

We complete the system (1.1) with Neumann boundary conditions:

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0 \text{ on } \quad \Sigma_T := \partial\Omega \times (0, T), \tag{1.2}$$

where ν is the unit outward normal to $\partial\Omega$. Herein $\alpha_i > 0$ is known as self-diffusion rate for $i = 1, 2, 3$. The cross-diffusion rate is assumed to be equal 1. The nonlinearities F, G, and H take the form:

$$
\begin{aligned}
F(u_1, u_2, u_3) &= \left(1 - \frac{u_1}{k}\right) u_1 - \frac{L_2 M_2 u_1 u_2}{R_0 + u_1}, \\
G(u_1, u_2, u_3) &= -L_2 u_2 + \frac{L_2 M_2 u_1 u_2}{R_0 + u_1} - \frac{L_3 M_{32} u_2 u_3}{C_0 + u_2}, \\
H(u_1, u_2, u_3) &= -L_3 u_3 + \frac{L_3 M_{32} u_2 u_3}{C_0 + u_2}.
\end{aligned}
\tag{1.3}
$$

In our model, $u_i(x, t)$ for $i = 1, 2, 3$, represent the density of the population of the ith species at time t. The constant k is the carrying capacity of u_1 species. R_0 and C_0 are the half saturation densities of u_1 and u_2, respectively. Moreover L_2 and L_3 are the mass-specific metabolic rates of u_2 and u_3, respectively. M_2 is a measure of ingestion rate per unit metabolic rate of u_2, and M_{32} denotes the ingestion rate for u_3 on prey term u_2.

In this work, the diffusion rates $d_1 > 0$, $d_2 > 0$ and $d_3 > 0$ are supposed to depend to the whole of each populations in the domain rather than on the local density, i. e. moves are guided by considering the global state of the medium, we assume that

$d_i : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying: there exist constants $a_i, C > 0$ such that

$$
a_i \leq d_i \quad \text{and} \quad |d_i(I_1) - d_i(I_2)| \leq C\,|I_1 - I_2| \quad \text{for all} \quad I_1, I_2 \in \mathbb{R}, \quad \text{for} \quad i = 1, 2, 3.
\tag{1.4}
$$

In this work, we shall assume that the coefficients α_i, satisfy

$$
\alpha_1 > \frac{1}{2},\ \alpha_2 > 1 \text{ and } \alpha_3 > \frac{1}{2}.
\tag{1.5}
$$

Observe that one can replace (1.1) by

$$
\begin{cases}
\partial_t u_1 - \operatorname{div}\left(d_1\left(\displaystyle\int_\Omega u_1\,dx\right)\nabla u_1\right) - \operatorname{div}\left(\mathcal{A}_{11}\nabla u_1 + \mathcal{A}_{12}\nabla u_2\right) = F(u_1, u_2, u_3), \\
\partial_t u_2 - \operatorname{div}\left(d_2\left(\displaystyle\int_\Omega u_2\,dx\right)\nabla u_2\right) - \operatorname{div}\left(\mathcal{A}_{21}\nabla u_1 + \mathcal{A}_{22}\nabla u_2 + \mathcal{A}_{23}\nabla u_3\right) = G(u_1, u_2, u_3), \\
\partial_t u_3 - \operatorname{div}\left(d_3\left(\displaystyle\int_\Omega u_3\,dx\right)\nabla u_3\right) - \operatorname{div}\left(\mathcal{A}_{32}\nabla u_2 + \mathcal{A}_{33}\nabla u_3\right) = G(u_1, u_2, u_3),
\end{cases}
\tag{1.6}
$$

where the diffusion matrix $\mathcal{A} = (\mathcal{A}_{ij})_{1\leq i,j\leq 3}$ is defined by

$$
\mathcal{A} = \begin{pmatrix} \alpha_1 u_1 + u_2 & u_1 & 0 \\ u_2 & u_1 + \alpha_2 u_2 + u_3 & u_2 \\ 0 & u_3 & u_2 + \alpha_3 u_3 \end{pmatrix},
$$

which is uniformly nonnegative: using condition (1.5) and the inequality $ab \geq -\frac{a^2}{2} - \frac{b^2}{2}$ for all $a, b \in \mathbb{R}$ one gets: for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$
\begin{aligned}
\xi^T \mathcal{A} \xi \geq& \left(\Big(\alpha_1 - \frac{1}{2}\Big)u_1 + \frac{u_2}{2}\right)\xi_1^2 + \left(\frac{u_1}{2} + \Big(\alpha_2 - 1\Big)u_2 + \frac{u_3}{2}\right)\xi_2^2 \\
&+ \left(\frac{u_2}{2} + \Big(\alpha 3 - \frac{1}{2}\Big)u_3\right)\xi_3^2 \\
\geq& c\Big((u_1 + u_2)\xi_1^2 + (u_1 + u_2 + u_3)\xi_2^2 + (u_2 + u_3)\xi_3^2\Big),
\end{aligned}
\tag{1.7}
$$

for some constant $c > 0$.

The next goal is to discretize our model. There are many finite volume schemes to tackle numerically a nonlinear reaction-diffusion system. One of them is the well-known method introduced by Gallouët [8]. In [2, 3, 4, 7] was used this idea by doing a convergence analysis of the method.

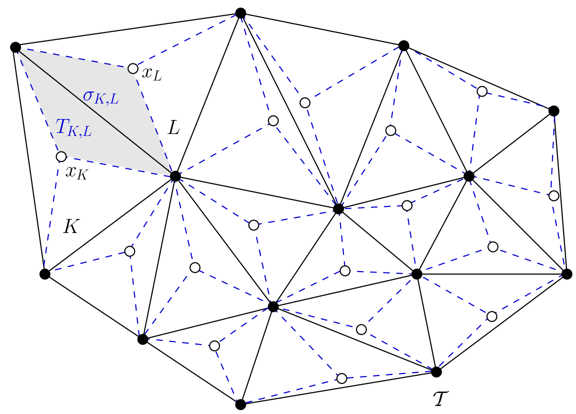

FIGURE 1. Control volumes, centers and diamonds (in dashed lines).

The plan of this paper is as follows: In Section 2 we define weak solution to system (1.1)-(1.2). We introduce some notations for the finite volume method, we present our scheme and the main theorem of convergence. The proof of this convergence result is divided into Section 3 a priori estimates and existence of solution, Section 4 compactness for discrete solution and Section 5 convergence to a weak solution. Finally, in section 6 we give some numerical examples to our model.

2. Finite Volume Approximation

2.1. **Admissible mesh.** In this work we assume that $\Omega \subset \mathbb{R}^d$, $d = 2$ (respectively, $d = 3$) is an open bounded polygonal (resp., polyhedral) connected domain with boundary $\partial\Omega$. We consider a family $\mathcal{T}_h$ of admissible meshes of the domain Ω consisting of disjoint open and convex polygons (resp., polyhedra) called control volumes. The parameter h has the sense of an upper bound for the maximum diameter of the control volumes in $\mathcal{T}_h$. Whenever $\mathcal{T}_h$ is fixed, we will drop the subscript h in the notation. Of course, the mesh should be admissible in the sense of [8].
A generic volume in $\mathcal{T}_h$ is denoted by K. For all $K \in \mathcal{T}_h$, we denote by $|K|$ the d-dimensional Lebesgue measure of K. For a given finite volume K, we denote by $N(K)$ the set of neighbors of K which have a common interface with K; a generic neighbor of K is often denoted by L. For all $L \in N(K)$, we denote by $\sigma_{K,L}$ the interface between K and L; we denote by $\eta_{K,L}$ the unit normal vector to $\sigma_{K,L}$ outward to K. We have $\eta_{L,K} = -\eta_{K,L}$. For an interface $\sigma_{K,L}$, $|\sigma_{K,L}|$ will denote its $(d-1)$-dimensional measure.
By saying that $\mathcal{T}_h$ is admissible, we mean that there exists a family $(x_K)_{K\in\mathcal{T}_h}$ such that the straight line $\overline{x_K x_L}$ is orthogonal to the interface $\sigma_{K,L}$. The point x_K is referred to as the center of K. In the case where $\mathcal{T}_h$ is a simplicial mesh of Ω (a triangulation, in dimension $d = 2$), one takes for x_K the center of the circumscribed ball of K. We also require that $\eta_{K,L} \cdot (x_L - x_K) > 0$ (in the case of simplicial meshes, this restriction amounts to the Delaunay condition, see e.g. Ref. [8]). The "diamond" constructed from the neighbor centers x_K, x_L and the interface $\sigma_{K,L}$ is denoted by $T_{K,L}$; e.g. in the case $x_K \in K$, $x_L \in L$, $T_{K,L}$ is the convex hull of x_K,x_L and $\sigma_{K,L}$) (see Figure 1). We have $\Omega = \cup_{K\in\mathcal{T}_h}\Big(\cup_{L\in N(K)}\overline{T}_{K,L}\Big)$.
We require local regularity restrictions on the family of meshes $\mathcal{T}_h$; namely,

$$\exists\gamma > 0 \qquad \forall h \ \ \forall K \in \mathcal{T}_h \ \forall L \in N(K) \quad \mathrm{diam}\,(K) + \mathrm{diam}\,(L) \le \gamma d_{K,L}, \tag{2.1}$$

$$\exists\gamma > 0 \qquad \forall h \ \ \forall K \in \mathcal{T}_h \ \forall L \in N(K) \quad |\sigma_{K,L}| d_{K,L} \ \le \ \gamma\, |K|. \tag{2.2}$$

Herein, $d_{K,L}$ is the distance between x_K and x_L.

Remark 2.1. *Note that* (2.1) *is used in* [5] *to prove the discrete* L^1 *compactness. Moreover* (2.2) *is used in the proof of the discrete Sobolev embeddings (see Proposition* B.1 *in* [5]*).*

A discrete function on the mesh $\mathcal{T}_h$ is a set $(w_K)_{K\in\mathcal{T}_h}$. Whenever convenient, we identify it with the piecewise constant function w_h on Ω such that $w_h|_K = w_K$. Finally, the discrete gradient $\nabla_h w_h$ of a constant per control volume function w_h is defined as the constant per diamond $T_{K,L}$ function, $\mathbb{R}^d$-valued, with the values

$$\Big(\nabla_h w_h\Big)\Big|_{T_{K,L}} = \nabla_{K,L} w_h := d\,\frac{w_L - w_K}{d_{K,L}}\eta_{K,L}. \tag{2.3}$$

Remark 2.2. *Because we consider the zero-flux boundary condition, we do not need to distinguish between interior and exterior control volumes; only inner interfaces between volumes are needed in order to formulate the scheme.*

2.2. **Approximation of the nonlocal cross-diffusion model and the main result.** To discretize (1.1), we choose an admissible discretization of Ω_T consisting of an admissible mesh $\mathcal{T}_h$ of Ω and of a time step size $\Delta t_h > 0$; both Δt_h and the size $\max_{K\in\mathcal{T}_h}\operatorname{diam}(K)$ tend to zero as $h \to 0$. We define $N_h > 0$ as the smallest integer such that $(N_h+1)\Delta t_h \geq T$, and set $t^n := n\Delta t_h$ for $n \in \{0,\dots,N_h\}$. Whenever Δt_h is fixed, we will drop the subscript h in the notation.

Furthermore, we denote

$$\begin{aligned} F_K^{n+1} &= F\big(u_{1,K}^{n+1^+}, u_{2,K}^{n+1^+}, u_{3,K}^{n+1^+}\big),\\ G_K^{n+1} &= G\big(u_{1,K}^{n+1^+}, u_{2,K}^{n+1^+}, u_{3,K}^{n+1^+}\big),\\ H_K^{n+1} &= H\big(u_{1,K}^{n+1^+}, u_{2,K}^{n+1^+}, u_{3,K}^{n+1^+}\big). \end{aligned} \tag{2.4}$$

To approximate the diffusive terms, we introduce the terms $\mathcal{A}_{ij,K}^{n+1}$ for $i,j = 1,2,3$. Herein, we make the choice

$$\mathcal{A}_{ij,K,L}^{n+1} := \mathcal{A}_{ij}\Big(\min\{u_{1,K}^{n+1^+}, u_{1,L}^{n+1^+}\}, \min\{u_{2,K}^{n+1^+}, u_{2,L}^{n+1^+}\}, \min\{u_{3,K}^{n+1^+}, u_{3,L}^{n+1^+}\}\Big) \tag{2.5}$$

Remark 2.3. *Note that the choice of the minimum in the discretization of $\mathcal{A}_{ij,K,L}^{n+1}$ for $i \neq j$ and $i,j = 1,2,3$, is imposed to justify the non-negativity of our discrete solution. Moreover, the choice of the diagonal terms $\mathcal{A}_{ii,K,L}^{n+1}$ for $i = 1,2,3$, is made in order to preserve, at the discrete level, the structure of the cross-diffusion matrix $\mathcal{A}$.*

The computation starts from the initial cell averages

$$u_{i,K}^0 = \frac{1}{|K|}\int_K u_{i,0}(x)\,dx \text{ for } i = 1,2,3. \tag{2.6}$$

To advance the numerical solution from t^n to $t^{n+1} = t^n + \Delta t$, we use the following implicit finite volume scheme: Determine $(u_{i,K}^{n+1})_{K\in\mathcal{T}_h}$ for $i = 1,2,3$ such that

$$\begin{aligned} &|K|\frac{u_{1,K}^{n+1} - u_{1,K}^n}{\Delta t} - d_1\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{1,K_0}^n\right)\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(u_{1,L}^{n+1} - u_{1,K}^{n+1})\\ &\qquad - \sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\Big[\mathcal{A}_{11,K,L}^{n+1}(u_{1,L}^{n+1} - u_{1,K}^{n+1}) + \mathcal{A}_{12,K,L}^{n+1}(u_{2,L}^{n+1} - u_{2,K}^{n+1})\Big]\\ &\qquad = |K|F_K^{n+1}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &|K|\frac{u_{2,K}^{n+1} - u_{2,K}^n}{\Delta t} - d_2\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{2,K_0}^n\right)\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(u_{2,L}^{n+1} - u_{2,K}^{n+1})\\ &- \sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\Big[\mathcal{A}_{21,K,L}^{n+1}(u_{1,L}^{n+1} - u_{1,K}^{n+1}) + \mathcal{A}_{22,K,L}^{n+1}(u_{2,L}^{n+1} - u_{2,K}^{n+1}) + \mathcal{A}_{23,K,L}^{n+1}(u_{3,L}^{n+1} - u_{3,K}^{n+1})\Big]\\ &\qquad = |K|G_K^{n+1}, \end{aligned} \tag{2.8}$$

(2.9)
$$
\begin{aligned}
|K|\frac{u_{3,K}^{n+1}-u_{3,K}^{n}}{\Delta t} - d_3\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{3,K_0}^n\right)\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(u_{3,L}^{n+1}-u_{3,K}^{n+1})\\
-\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left[\mathcal{A}_{32,K,L}^{n+1}(u_{2,L}^{n+1}-u_{2,K}^{n+1})+\mathcal{A}_{33,K,L}^{n+1}(u_{3,L}^{n+1}-u_{3,K}^{n+1})\right]\\
=|K|H_K^{n+1},
\end{aligned}
$$

for all $K\in\mathcal{T}_h$ and $n\in[0,N_h]$. Herein

(2.10)
$$
\begin{aligned}
&\mathcal{A}_{11,K,L}^{n+1} := \alpha_1\min\{u_{1,K}^{n+1^+},u_{1,L}^{n+1^+}\}+\min\{u_{2,K}^{n+1^+},u_{2,L}^{n+1^+}\},\quad \mathcal{A}_{12,K,L}^{n+1}:=\min\{u_{1,K}^{n+1^+},u_{1,L}^{n+1^+}\},\\
&\mathcal{A}_{22,K,L}^{n+1} := \min\{u_{1,K}^{n+1^+},u_{1,L}^{n+1^+}\}+\alpha_2\min\{u_{2,K}^{n+1^+},u_{2,L}^{n+1^+}\}+\min\{u_{3,K}^{n+1^+},u_{3,L}^{n+1^+}\},\\
&\mathcal{A}_{21,K,L}^{n+1}=\mathcal{A}_{23,K,L}^{n+1}:=\min\{u_{2,K}^{n+1^+},u_{2,L}^{n+1^+}\},\\
&\mathcal{A}_{33,K,L}^{n+1} := \min\{u_{1,K}^{n+1^+},u_{1,L}^{n+1^+}\}+\alpha_3\min\{u_{3,K}^{n+1^+},u_{3,L}^{n+1^+}\},\quad \mathcal{A}_{32,K,L}^{n+1}:=\min\{u_{3,K}^{n+1^+},u_{3,L}^{n+1^+}\}.
\end{aligned}
$$

Note that the homogeneous Neumann boundary condition is taken into account implicitly. Indeed, the parts of ∂K that lie in $\partial\Omega$ do not contribute to the $\sum_{L\in N(K)}$ terms, which means that the flux zero is imposed on the external edges of the mesh.

The set of values $(u_{1,K}^{n+1},u_{2,K}^{n+1},u_{3,K}^{n+1})_{K\in\mathcal{T}_h,n\in[0,N_h]}$ satisfying (2.6), (2.7), (2.8) and (2.9) will be called a discrete solution. Whenever convenient, we will assimilate a discrete solution of the scheme at the time step n to the triple $\mathbf{u}_h^{n+1}=(u_{1,h}^{n+1},u_{2,h}^{n+1},u_{3,h}^{n+1})$ of piecewise constant on Ω functions given by

$$\forall K\in\mathcal{T}_h\ \forall n\in[0,N_h]\qquad u_{i,h}^{n+1}|_K=u_{i,K}^{n+1},\ \text{for } i=1,2,3.$$

We will write $\mathbf{u}_h$ for the discrete solution on Ω_T, assimilated to the piecewise constant function

$$\Bigg(\sum_{\substack{K\in\mathcal{T}_h,\\ n\in[0,N_h]}}u_{1,K}^{n+1}\mathbb{1}_{(t^n,t^{n+1}]\times K}\ ,\ \sum_{\substack{K\in\mathcal{T}_h,\\ n\in[0,N_h]}}u_{2,K}^{n+1}\mathbb{1}_{(t^n,t^{n+1}]\times K}\ ,\ \sum_{\substack{K\in\mathcal{T}_h,\\ n\in[0,N_h]}}u_{3,K}^{n+1}\mathbb{1}_{(t^n,t^{n+1}]\times K}\Bigg).$$

We will assume that the following mild time step condition is satisfied

$$\Delta t<\sup\left\{\frac{1}{2},\frac{1}{2L_2M_2},\frac{1}{2L_3M_{32}}\right\}, \tag{2.11}$$

which will be used to prove the existence of solutions to the scheme.

Before stating our main results, we give the definition of a weak solution.

Definition 2.1. *A triple* $\mathbf{u}=(u_1,u_2,u_3)$ *of nonnegative functions is a* weak solution *of* (1.1)-(1.2) *if* $u_1,u_2,u_3\in L^2(0,T;H^1(\Omega))$ *and for all test functions* $\varphi,\psi,\xi\in\mathcal{D}([0,T)\times\bar\Omega)$*:*

$$
\begin{aligned}
-\iint_{\Omega_T}u_1\partial_t\varphi\,dx\,dt+\int_0^T d_1\left(\int_\Omega u_1\,dx\right)\int_\Omega\nabla u_1\cdot\nabla\varphi\,dx\,dt\\
+\iint_{\Omega_T}[\mathcal{A}_{11}\nabla u_1+\mathcal{A}_{12}\nabla u_2]\cdot\nabla\varphi\,dx\,dt\\
=\iint_{\Omega_T}F(u_1,u_2,u_3)\varphi\,dx\,dt+\int_\Omega u_{1,0}(x)\varphi(0,x)\,dx,\\
-\iint_{\Omega_T}u_2\partial_t\psi\,dx\,dt+\int_0^T d_2\left(\int_\Omega u_2\,dx\right)\int_\Omega\nabla u_2\cdot\nabla\psi\,dx\,dt\\
+\iint_{\Omega_T}[\mathcal{A}_{21}\nabla u_1+\mathcal{A}_{22}\nabla u_2+\mathcal{A}_{23}\nabla u_3]\cdot\nabla\psi\,dx\,dt\\
=\iint_{\Omega_T}G(u_1,u_2,u_3)\xi\,dx\,dt+\int_\Omega u_{2,0}(x)\psi(0,x)\,dx,
\end{aligned}
$$

$$-\iint_{\Omega_T} u_3\partial_t\xi \, dx\,dt + \int_0^T d_3\left(\int_\Omega u_3\,dx\right)\int_\Omega \nabla u_3\cdot\nabla\xi\,dx\,dt + \iint_{\Omega_T}[\mathcal{A}_{32}\nabla u_2 + \mathcal{A}_{33}\nabla u_3]\cdot\nabla\xi\,dx\,dt = \iint_{\Omega_T} H(u_1,u_2,u_3)\xi\,dx\,dt + \int_\Omega u_{3,0}(x)\xi(0,x)\,dx.$$

The existence of a weak non-negative solution for system (1.1) will be shown by proving convergence of our numerical scheme (2.7), (2.8) and (2.9).

Our main result is

Theorem 2.1. *Assume that $u_{i,0}\in(L^2(\Omega))^+$ for $i=1,2,3$. Let $\mathbf{u}_h$ be the discrete solution generated by the finite volume scheme (2.6), (2.7), (2.8) and (2.9) on a family of meshes satisfying (2.1),(2.2). Then, as $h\to 0$, $\mathbf{u}_h$ converges (along a subsequence) a.e. on Ω_T to a limit $\mathbf{u}=(u_1,u_2,u_3)$ that is a weak solution $\mathbf{u}$ of (1.1) - (1.2).*

3. A priori estimates and existence

3.1. **Nonnegativity.** We have the following lemma.

Lemma 3.1. *Let $(u_{1,K}^{n+1},u_{2,K}^{n+1},u_{3,K}^{n+1})_{K\in\mathcal{T}_h,n\in[0,N_h]}$ be a solution of the finite volume scheme (2.6), (2.7), (2.8) and (2.9). Then, $(u_{1,K}^{n+1},u_{2,K}^{n+1},u_{3,K}^{n+1})_{K\in\mathcal{T}_h,n\in[0,N_h]}$ is nonnegative.*

Proof. We prove the nonnegativity by induction, that for all $n\in[0,N_h]$, $\min\{u_{1,K}^{n+1}\}_{K\in\mathcal{T}_h}\geq 0$.

For $n\geq 0$, we fix K such that $u_{1,K}^{n+1}=\min\{u_{1,L}^{n+1}\}_{L\in\mathcal{T}_h}$. We multiply equation (2.7) by $-\vartriangle t u_{1,K}^{n+1^-}$ to deduce

(3.1)
$$-|K|\,u_{1,K}^{n+1^-}(u_{1,K}^{n+1}-u_{1,K}^n) = -\,d_1\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{1,K_0}^n\right)\vartriangle t\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(u_{1,L}^{n+1}-u_{1,K}^{n+1})\,u_{1,K}^{n+1^-} - \vartriangle t\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\Bigg[\mathcal{A}_{11,K,L}^{n+1}(u_{1,L}^{n+1}-u_{1,K}^{n+1}) + \mathcal{A}_{12,K,L}^{n+1}(u_{2,L}^{n+1}-u_{2,K}^{n+1})\Bigg]u_K^{n+1^-} - \vartriangle t|K|\,F_K^{n+1}\,u_K^{n+1^-}.$$

By the choice of K and the non-negativity of $\mathcal{A}_{11,K,L}^{n+1}$, we get

$$\vartriangle t\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left(d_1\left(\sum_{K_0\in\mathcal{T}_h}m(K_0)u_{1,K_0}^n\right)+\mathcal{A}_{11,K,L}^{n+1}\right)(u_{1,L}^{n+1}-u_{1,K}^{n+1})\,u_{1,K}^{n+1^-}\geq 0.$$

Moreover, by the choice (2.5) of $\mathcal{A}_{12,K,L}^{n+1}$, we obtain

$$\vartriangle t\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left[\mathcal{A}_{12,K,L}^{n+1}(u_{2,L}^{n+1}-u_{2,K}^{n+1})\right]\,u_K^{n+1^-} \;=\; 0,$$

Similarly, by the definition of F_K^{n+1} we have

$$F_K^{n+1}\,u_K^{n+1^-} = \left(\left(1-\frac{u_{1,K}^{n+1^+}}{k_p}\right)u_{1,K}^{n+1^+} - \frac{L_2M_2u_{1,K}^{n+1^+}u_{2,K}^{n+1^+}}{R_0+u_{1,K}^{n+1^+}}\right)u_K^{n+1^-}=0. \tag{3.2}$$

Finally we use the identity $u_{1,K}^{n+1}=(u_{1,K}^{n+1})^+-(u_{1,K}^{n+1})^-$ and the nonnegativity of $u_{1,K}^0$ to deduce from (3.1) and (3.2) that $(u_{1,K}^{n+1})^-=0$. By induction in n, we infer that

$$u_{1,L}^{n+1}\geq 0 \text{ for all } n\in[0,N_h] \text{ and } L\in\mathcal{T}_h.$$

Along the same lines as $u_{1,K}^{n+1}$, we obtain the nonnegativity of the discrete solution $u_{i,K}^{n+1}$ for all $K \in \mathcal{T}_h$ and $n \in [0, N_h]$ for $i = 2, 3$. $\square$

3.2. **A priori estimates.** The goal now is to establish several a priori (discrete energy) estimates for the finite volume scheme, which eventually will imply the desired convergence results.

Proposition 3.2. *Let* $(u_{1,K}^{n+1}, u_{2,K}^{n+1}, u_{3,K}^{n+1})_{K\in\mathcal{T}_h, n\in[0,N_h]}$*, be a solution of the finite volume scheme* (2.6)*,* (2.7)*,* (2.8) *and* (2.9)*. Then there exist a constant* $C > 0$*, depending on* Ω*,* T*,* $\|u_{1,0}\|_2$*,* $\|u_{2,0}\|_2$*,* $\|u_{3,0}\|_2$*,* $L_2, L_3, M_2, M_{32}, C_0, R_0$*,* a_1, a_2, a_3 *such that for* $i, j = 1, 2, 3$

$$\max_{[0,N_h]} \sum_{K\in\mathcal{T}_h} |K| \left|u_{i,K}^{n+1}\right|^2 \leq C, \tag{3.3}$$

$$\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2 \leq C, \tag{3.4}$$

and

$$\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \mathcal{A}_{ij,K,L}^{n+1} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2 \leq C. \tag{3.5}$$

Proof. We multiply (2.7), (2.8) and (2.9) by $\Delta t u_{1,K}^{n+1}$, $\Delta t u_{2,K}^{n+1}$ and $\Delta t u_{3,K}^{n+1}$, respectively, and add together the outcomes. Summing the resulting equation over K and n yields

$$S_1 + S_2 + S_3 + S_4 = 0,$$

where

$$\begin{aligned}
S_1 &= \sum_{n=0}^{N_h} \sum_{K\in\mathcal{T}_h} |K| \Big((u_{1,K}^{n+1} - u_{1,K}^n) u_{1,K}^{n+1} + (u_{2,K}^{n+1} - u_{2,K}^n) u_{2,K}^{n+1} + (u_{3,K}^{n+1} - u_{3,K}^n) u_{3,K}^{n+1} \Big), \\
S_2 &= -\sum_{i=1}^{3} \sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left(d_i \left(\sum_{K_0\in\mathcal{T}_h} m(K_0) u_{i,K_0}^n \right) (u_{i,L}^{n+1} - u_{i,K}^{n+1}) u_{i,K}^{n+1} \right), \\
S_3 &= -\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \\
&\quad \times \Bigg(\left[\mathcal{A}_{11,K,L}^{n+1} (u_{1,L}^{n+1} - u_{1,K}^{n+1}) + \mathcal{A}_{12,K,L}^{n+1} (u_{2,L}^{n+1} - u_{2,K}^{n+1}) \right] u_{1,K}^{n+1} \\
&\qquad + \left[\mathcal{A}_{21,K,L}^{n+1} (u_{1,L}^{n+1} - u_{1,K}^{n+1}) + \mathcal{A}_{22,K,L}^{n+1} (u_{2,L}^{n+1} - u_{2,K}^{n+1}) + \mathcal{A}_{23,K,L}^{n+1} (u_{3,L}^{n+1} - u_{3,K}^{n+1}) \right] u_{2,K}^{n+1} \\
&\qquad + \left[+ \mathcal{A}_{32,K,L}^{n+1} (u_{2,L}^{n+1} - u_{2,K}^{n+1}) + \mathcal{A}_{33,K,L}^{n+1} (u_{3,L}^{n+1} - u_{3,K}^{n+1}) \right] u_{3,K}^{n+1} \Bigg), \\
S_4 &= -\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} |K| \Big(F_K^{n+1} u_{1,K}^{n+1} + G_K^{n+1} u_{2,K}^{n+1} + H_K^{n+1} u_{3,K}^{n+1} \Big).
\end{aligned}$$

Observe that

$$\begin{aligned}
S_1 &= \sum_{n=0}^{N_h} \sum_{K\in\mathcal{T}_h} |K| \Big((u_{1,K}^{n+1} - u_{1,K}^n) u_{1,K}^{n+1} + (u_{2,K}^{n+1} - u_{2,K}^n) u_{2,K}^{n+1} + (u_{3,K}^{n+1} - u_{3,K}^n) u_{3,K}^{n+1} \Big) \\
&\geq \frac{1}{2} \sum_{n=0}^{N_h} \sum_{K\in\mathcal{T}_h} |K| \left(\left|u_{1,K}^{n+1}\right|^2 - \left|u_{1,K}^n\right|^2 + \left|u_{2,K}^{n+1}\right|^2 - \left|u_{2,K}^n\right|^2 + \left|u_{3,K}^{n+1}\right|^2 - \left|u_{3,K}^n\right|^2 \right) \\
&= \frac{1}{2} \sum_{K\in\mathcal{T}_h} |K| \left(\left|u_{1,K}^{N_h+1}\right|^2 - \left|u_{1,K}^0\right|^2 + \left|u_{2,K}^{N_h+1}\right|^2 - \left|u_{2,K}^0\right|^2 + \left|u_{3,K}^{N_h+1}\right|^2 - \left|u_{3,K}^0\right|^2 \right),
\end{aligned}$$

where we have used the inequality "$a(a-b) \geq \frac{1}{2}(a^2-b^2)$". Gathering by edges, we obtain

$$S_2 = \sum_{i=1}^{3}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \frac{d_i\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{i,K_0}^n\right)}{2} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2 .$$

Next, using (1.7) where u_i is replaced by $\min\{u_{i,K}^{n+1^+}, u_{i,L}^{n+1^+}\}$ for $i=1,2,3$, we deduce

$$S_3 \geq c\sum_{i=1}^{3}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \mathcal{A}_{ij,K,L}^{n+1} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2,$$

for some constant $c>0$. Now we use the nonnegativity of $u_{i,K}^{n+1}$ for $i=1,2,3$, and the discrete expressions of F,G,H to deduce

$$S_4 \geq -\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} |K| \Big(\left|u_{1,K}^{n+1}\right|^2 + L_2M_2\left|u_{2,K}^{n+1}\right|^2 + L_3M_{32}\left|u_{3,K}^{n+1}\right|^2 \Big).$$

Collecting the previous inequalities we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{K\in\mathcal{T}_h}|K|\left(\left|u_{1,K}^{N_h+1}\right|^2 - \left|u_{1,K}^0\right|^2 + \left|u_{2,K}^{N_h+1}\right|^2 - \left|u_{2,K}^0\right|^2 + \left|u_{3,K}^{N_h+1}\right|^2 - \left|u_{3,K}^0\right|^2\right)\\
&\quad + \sum_{i=1}^{3}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \frac{d_i\left(\sum_{K_0\in\mathcal{T}_h} m(K_0)u_{i,K_0}^n\right)}{2} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2\\
&\quad + c\sum_{i=1}^{3}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \mathcal{A}_{ij,K,L}^{n+1} \left|u_{i,K}^{n+1} - u_{i,L}^{n+1}\right|^2\\
&\leq \sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_h} |K| \Big(\left|u_{1,K}^{n+1}\right|^2 + L_2M_2\left|u_{2,K}^{n+1}\right|^2 + L_3M_{32}\left|u_{3,K}^{n+1}\right|^2 \Big).
\end{aligned} \tag{3.6}$$

An application of the discrete Gronwall inequality, (3.3) follows from (3.6). Consequently, (3.6) entails the estimates (3.4)–(3.5). This concludes the proof of Proposition 3.2. □

3.3. **Existence of a solution for the finite volume scheme.** The existence of a solution for the finite volume scheme is given in the following proposition.

Proposition 3.3. *Let $\mathcal{T}_h$ be an admissible discretization of Ω_T and assume that* (2.11) *holds. Then the discrete problem* (2.6), (2.7) *and* (2.8) *admits at least one solution* $(u_{1,K}^{n+1}, u_{2,K}^{n+1}, u_{3,K}^{n+1})$ *where* $(K,n)\in\mathcal{T}_h\times[0,N_h]$.

Proof. First we introduce the Hilbert space

$$E_h = H_h(\Omega)\times H_h(\Omega)\times H_h(\Omega),$$

of triples $\mathbf{u}_h = (u_{1,h},u_{2,h},u_{3,h})$ of discrete functions on Ω under the norm

$$\|\mathbf{u}_h\|_{E_h}^2 := \sum_{i=1}^{3} |u_{i,h}|_{H_h(\Omega)}^2 + \|u_{i,h}\|_{L^2(\Omega)}^2,$$

where the "discrete H_0^1 seminorm" $|\cdot|_{H_h(\Omega)}^2$ is given by

$$|w_h|_{H_h(\Omega)}^2 := \frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} |T_{K,L}| \left|\frac{w_L - w_K}{d_{K,L}}\right|^2,$$

and the $L^2(\Omega)$ norm of w_h is given by

$$\|w_h\|_{L^2(\Omega)}^2 := \sum_{K\in\mathcal{T}_h} |K|\ |w_K|^2.$$

Let $\Phi_h = (\varphi_{1,h}, \varphi_{2,h}, \varphi_{3,h}) \in E_h$ and define the discrete bilinear forms

$$T_h(\mathbf{u}_h, \Phi_h) = \sum_{i=1}^{3} \sum_{K\in\mathcal{T}_h} |K| \; u_{i,K}\varphi_{i,K},$$

$$a_{1,h}(\mathbf{u}_h^{n+1}, \Phi_h) = \frac{1}{2} \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} d_i \left(\sum_{K_0\in\mathcal{T}_h} m(K_0) u_{i,K_0}^n \right) \Big((u_{i,L}^{n+1} - u_{i,K}^{n+1})(\varphi_{i,L}^{n+1} - \varphi_{i,K}^{n+1}) \Big).$$

Similarly, for given matrices $\mathcal{A}_h^{n+1} := \left(\left(\mathcal{A}_{ij,K,L}^{n+1} \right)_{1\le i,j\le 3} \right)_{K\in\mathcal{T}_h, L\in N(K)}$, define the bilinear form

$$\begin{aligned} a_{2,h}(\mathcal{A}_h^{n+1}; \mathbf{u}_h^{n+1}, \Phi_h) = \frac{1}{2} \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \Bigg[& \mathcal{A}_{11,K,L}^{n+1}(u_{1,L}^{n+1} - u_{1,K}^{n+1})(\varphi_{1,L}^{n+1} - \varphi_{1,K}^{n+1}) \\ & + \mathcal{A}_{12,K,L}^{n+1}(u_{2,L}^{n+1} - u_{2,K}^{n+1})(\varphi_{1,L}^{n+1} - \varphi_{1,K}^{n+1}) + \mathcal{A}_{21,K,L}^{n+1}(u_{1,L}^{n+1} - u_{1,K}^{n+1})(\varphi_{2,L}^{n+1} - \varphi_{2,K}^{n+1}) \\ & + \mathcal{A}_{22,K,L}^{n+1}(u_{2,L}^{n+1} - u_{2,K}^{n+1})(\varphi_{2,L}^{n+1} - \varphi_{2,K}^{n+1}) + \mathcal{A}_{21,K,L}^{n+1}(u_{3,L}^{n+1} - u_{3,K}^{n+1})(\varphi_{2,L}^{n+1} - \varphi_{2,K}^{n+1}) \\ & + \mathcal{A}_{32,K,L}^{n+1}(u_{2,L}^{n+1} - u_{2,K}^{n+1})(\varphi_{3,L}^{n+1} - \varphi_{3,K}^{n+1}) + \mathcal{A}_{33,K,L}^{n+1}(u_{3,L}^{n+1} - u_{3,K}^{n+1})(\varphi_{3,L}^{n+1} - \varphi_{3,K}^{n+1}) \Bigg]. \end{aligned}$$

Multiplying (2.7), (2.8) and (2.9) by $\varphi_{1,K}$, $\varphi_{2,K}$ and $\varphi_{3,K}$, respectively, summing in $K \in \mathcal{T}_h$, we get the equation

$$\begin{aligned} \frac{1}{\Delta t}\Big(T_h(\mathbf{u}_h^{n+1}, \Phi_h) - T_h(\mathbf{u}_h^n, \Phi_h) \Big) + a_{1,h}(\mathbf{u}_h^{n+1}, \Phi_h) &+ a_{2,h}(\mathcal{A}_h^{n+1}(\mathbf{u}_h^{n+1}); \mathbf{u}_h^{n+1}, \Phi_h) \\ &+ T_h(R^h(\mathbf{u}_h^{n+1}), \Phi_h) = 0; \end{aligned}$$

here the entries $\mathcal{A}_{ij,K,L}^{n+1}$ of $\mathcal{A}_h^{n+1}$ are defined from $\mathbf{u}_h^{n+1}$ with the help of formulas (2.5), furthermore, $R^h(\mathbf{u}_h^{n+1}) := (F_h^{n+1}, G_h^{n+1}, H_h^{n+1})$ with the discrete functions $F_h^{n+1}, G_h^{n+1}, H_h^{n+1}$ defined from $\mathbf{u}_h^{n+1}$ by formulas (2.4). It is clear that, $\mathbf{u}_h^n$ being given, there exists a solution $\mathbf{u}_h^{n+1}$ of the above equation if and only if there exists a discrete solution of (2.7) - (2.9) at the time step $(n+1)$. Now we define, by duality, the mapping $\mathcal{P}$ from E_h into itself:

$$\begin{aligned} \forall \Phi_h \in E_h \quad [\mathcal{P}(\mathbf{u}_h^{n+1}), \Phi_h] = \frac{1}{\Delta t}(T_h(\mathbf{u}_h^{n+1}, \Phi_h) - T_h(\mathbf{u}_h^n, \Phi_h)) + a_{1,h}(\mathbf{u}_h^{n+1}, \Phi_h) \\ + a_{2,h}(\mathcal{A}_h^{n+1}(\mathbf{u}_h^{n+1}); \mathbf{u}_h^{n+1}, \Phi_h) + T_h(R^h(\mathbf{u}_h^{n+1}), \Phi_h). \end{aligned}$$

The continuity of the mapping $\mathcal{P}$ follows from the continuity of the nonlinearities F, G, H, $\left(\mathcal{A}_{ij}\right)_{1\le i,j\le 3}$ and from the continuity of $a_{h,1}(\cdot,\cdot)$, $a_{h,2}(\mathcal{A}_h^{n+1};\cdot,\cdot)$ and $T_h(\cdot,\cdot)$. Now we are looking for $\mathbf{u}_h^{n+1} \in E_h$ such that $\mathcal{P}(\mathbf{u}_h^{n+1}) = 0$. According to [13], in order to prove the existence of $\mathbf{u}_h^{n+1}$ it remains to show that

$$[\mathcal{P}(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}] > 0 \quad \text{whenever } \left\|\mathbf{u}_h^{n+1}\right\|_{E_h} = r > 0, \tag{3.7}$$

for a sufficiently large r. We observe that

$$\begin{aligned} [\mathcal{P}(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}] = & \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| \left| u_{1,K}^{n+1} \right|^2 + \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| \left| u_{2,K}^{n+1} \right|^2 + \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| \left| u_{3,K}^{n+1} \right|^2 \\ & + a_{1,h}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + a_{2,h}(\mathcal{A}_h^{n+1}(\mathbf{u}_h^{n+1}); \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + T_h(R^h(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}) \\ & - \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| u_{1,K}^n u_{1,K}^{n+1} - \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| u_{2,K}^n u_{2,K}^{n+1} - \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} |K| u_{3,K}^n u_{3,K}^{n+1}. \end{aligned} \tag{3.8}$$

Using the definition of F_K^{n+1}, G_K^{n+1} and H_K^{n+1}, estimate (2.11) and Young's inequality we deduce from (3.8)

$$
\begin{aligned}
&[\mathcal{P}(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}] \\
&\geq \frac{1}{\Delta t}\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h}|K|\left|u_{i,K}^{n+1}\right|^2 + \sum_{i=1}^{3} a_i \left\|u_{i,h}^{n+1}\right\|^2_{H_h(\Omega)} \\
&\quad - \frac{1}{2\Delta t}\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h}|K|\left|u_{i,K}^{n+1}\right|^2 - \frac{1}{2\Delta t}\sum_{i=1}^{3}\sum_{K\in\mathcal{T}_h}|K|\left|u_{i,K}^{n}\right|^2 \\
&\quad - \sum_{K\in\mathcal{T}_h}|K|\left|u_{1,K}^{n+1}\right|^2 - L_2M_2\sum_{K\in\mathcal{T}_h}|K|\left|u_{2,K}^{n+1}\right|^2 - L_3M_{32}\sum_{K\in\mathcal{T}_h}|K|\left|u_{3,K}^{n+1}\right|^2 \\
&\geq \sum_{i=1}^{3} a_i \left\|u_{i,h}^{n+1}\right\|^2_{H_h(\Omega)} \\
&\quad + \Big(\frac{1}{2\Delta t} - 1\Big)\sum_{K\in\mathcal{T}_h}|K|\left|u_{1,K}^{n+1}\right|^2 + \Big(\frac{1}{2\Delta t} - L_2M_2\Big)\sum_{K\in\mathcal{T}_h}|K|\left|u_{2,K}^{n+1}\right|^2 \\
&\quad + \Big(\frac{1}{2\Delta t} - L_3M_{32}\Big)\sum_{K\in\mathcal{T}_h}|K|\left|u_{3,K}^{n+1}\right|^2 - const(\Delta t, \mathbf{u}_h^n) \\
&\geq const(a_1,a_2,a_3,L_2,L_3,M_2,M_{32},\Delta t)\Big(\sum_{i=1}^{3}\left|u_{i,h}^{n+1}\right|^2_{H_h(\Omega)} + \sum_{i=1}^{3}\|u_{i,h}\|^2_{L^2(\Omega)}\Big) - const(\Delta t, \mathbf{u}_h^n).
\end{aligned}
$$

The constant $const(a_1,a_2,a_3,L_2,L_3,M_2,M_{32},\Delta t)$ in the above expression is greater than zero, provided (2.11) holds. This implies that (3.7) holds for r large enough (recall that $\left\|\mathbf{u}_h^{n+1}\right\|_{E_h} = r$). By induction in n, we deduce the existence of at least one solution to the scheme (2.7) - (2.9). □

4. Compactness arguments

In this section, we prove that the family $\mathbf{u_h}$ of discrete solutions constructed in Proposition 3.3 is relatively compact in $L^1(\Omega_T)$. We first apply the following lemma (see the proof of this lemma in Appendix A in [5]):

Lemma 4.1. *Let Ω be an open domain in $\mathbb{R}^d$, $T>0$, $\Omega_T=(0,T)\times\Omega$. Let $(\mathcal{T}^h)_h$ be an admissible family of meshes of Ω satisfying the restriction (2.1); let $(\Delta t^h)_h$ be the associated time steps.*

For all $h>0$, assume that discrete functions $\left(u_h^{n+1}\right)_{n\in[0,N_h]}$, $\left(f_h^{n+1}\right)_{n\in[0,N_h]}$ and discrete fields $\left(\vec{\mathcal{F}}_h^{n+1}\right)_{n\in[0,N_h]}$ satisfy the discrete evolution equations

$$
\text{(4.1)}\qquad \text{for } n\in[0,N_h],\quad \frac{u_h^{n+1}-u_h^n}{\Delta t} = \operatorname{div}_h[\vec{\mathcal{F}}_h^{n+1}] + f_h^{n+1}
$$

with a family $(u_h^0)_h$ of initial data. Assume that for all $\Omega' \Subset \Omega$, there exists a constant $M(\Omega')$ such that

$$
\text{(4.2)}\qquad \sum\nolimits_{n=0}^{N_h}\Delta t\left\| u_h^{n+1}\right\|_{L^1(\Omega')} + \sum\nolimits_{n=0}^{N_h}\Delta t\left\| f_h^{n+1}\right\|_{L^1(\Omega')} + \sum\nolimits_{n=0}^{N_h}\Delta t\left\| \vec{\mathcal{F}}_h^{n+1}\right\|_{L^1(\Omega')} \leq M(\Omega').
$$

and, moreover,

$$
\text{(4.3)}\qquad \sum\nolimits_{n=0}^{N_h}\Delta t\left\| \nabla_h u_h^{n+1}\right\|_{L^1(\Omega')} \leq M(\Omega').
$$

Assume that the family $(u_0^h)_h$ is bounded in $L^1_{loc}(\Omega)$. Then there exists a measurable function u on Ω_T such that, along a subsequence,

$$
\sum\nolimits_{n=0}^{N_h}\sum\nolimits_{K\in\mathcal{T}_h} u_K^{n+1}\mathbb{1}_{(t^n,t^{n+1}]\times K} \longrightarrow u \quad \text{in } L^1_{loc}([0,T]\times\Omega) \ \text{ as } h\to 0.
$$

Denote by $\mathcal{A}^h$ the 3×3 matrix on Ω_T with the entries $\mathcal{A}^h_{ij}$ given by

$$\mathcal{A}^h_{ij} := \frac{1}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} \mathcal{A}^{n+1}_{ij,K,L}\ \mathbb{1}_{(t^n,t^{n+1}]\times T_{K,L}}.$$

We have the following convergence results along a subsequence:

Proposition 4.2. *There exists a triple* $\mathbf{u}\in(L^{10/3}(\Omega_T))^3\cap L^2(0,T;H^1(\Omega))^3$ *and a subsequence of* $\mathbf{u}_h=(u_{1,h},u_{2,h},u_{3,h})$*, not labelled, such that, as* $h\to 0$,

(i) $\mathbf{u}_h\to\mathbf{u}$ *strongly in* $(L^1(\Omega_T))^3$ *and a.e. in* Ω_T,

(ii) $\nabla_h\mathbf{u}_h\longrightarrow\nabla\mathbf{u}$ *weakly in* $(L^2(\Omega_T))^{3\times 3}$,

(iii) $\mathcal{A}^h\nabla_h\mathbf{u}_h\longrightarrow\mathcal{A}(\mathbf{u})\nabla\mathbf{u}$ *weakly in* $(L^1(\Omega_T))^{3\times 3}$,

(iv) $(F(\mathbf{u}_h),G(\mathbf{u}_h),H(\mathbf{u}_h))\longrightarrow(F(\mathbf{u}),G(\mathbf{u}),H(\mathbf{u}))$ *strongly in* $(L^1(\Omega_T))^3$.

Proof. In this proof we apply Lemma 4.1 using the estimates shown in Proposition 3.2.
Observe that we may consider that the evolution of the first component $(u^{n+1}_{1,h})_{n\in[0,N_h]}$ the solution (2.7) is governed by the system of discrete equations

$$\frac{u^{n+1}_{1,K}-u^n_{1,K}}{\Delta t}=\frac{1}{|K|}\sum_{L\in N(K)}|\sigma_{K,L}|\,\vec{\mathcal{F}}^{n+1}_{K,L}\cdot\eta_{K,L}+f^{n+1}_K. \tag{4.4}$$

Herein,

$$\begin{aligned}
f^{n+1}_K :=&F(u^{n+1}_{1,K},u^{n+1}_{2,K},u^{n+1}_{3,K}),\\
\vec{\mathcal{F}}^{n+1}_{K,L} :=&d_1\left(\sum_{K_0\in\mathcal{T}_h}m(K_0)u^n_{1,K_0}\right)\frac{u^{n+1}_{1,L}-u^{n+1}_{1,K}}{d_{K,L}}\eta_{K,L}+\mathcal{A}^{n+1}_{11,K,L}\frac{u^{n+1}_{1,L}-u^{n+1}_{1,K}}{d_{K,L}}\eta_{K,L}\\
&+\mathcal{A}^{n+1}_{12,K,L}\frac{u^{n+1}_{2,L}-u^{n+1}_{2,K}}{d_{K,L}}\eta_{K,L}\\
\equiv&\frac{1}{d}\left[d_1\left(\sum_{K_0\in\mathcal{T}_h}m(K_0)u^n_{1,K_0}\right)\nabla_{K,L}u^{n+1}_{1,h}+\mathcal{A}^{n+1}_{11,K,L}\nabla_{K,L}u^{n+1}_{1,h}+\mathcal{A}^{n+1}_{12,K,L}\nabla_{K,L}u^{n+1}_{1,h}\right].
\end{aligned}$$

It is easy to see that equations (4.4) have the form (4.1) required in Lemma 4.1.

The next step is to check that the local L^1 bounds (4.2),(4.3) are verified. Using the $L^\infty(0,T;L^2(\Omega))$ estimate (3.3), the discrete $L^2(0,T;H^1(\Omega))$ estimate (3.4) and the estimate (3.5) (recall that (3.5) is exactly the $L^2(\Omega_T)$ estimate of the product $\sqrt{|\mathcal{A}^h|}\nabla_h\mathbf{u}_h$), we get the global $L^1(\Omega_T)$ uniform estimates on the families

$$\begin{aligned}
u^h &:= \textstyle\sum_{n=0}^{N_h}u^{n+1}_{1,h}\mathbb{1}_{(t^n,t^{n+1}]}, &\vec{\mathcal{F}}_h &:= \frac{1}{2}\textstyle\sum_{n=0}^{N_h}\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}\ \vec{\mathcal{F}}^{n+1}_{K,L}\ \ \mathbb{1}_{(t^n,t^{n+1}]\times T_{K,L}}\\
f^h &:= \textstyle\sum_{n=0}^{N_h}f^{n+1}_h\mathbb{1}_{(t^n,t^{n+1}]}, &\nabla_h u_h &:= \frac{1}{2}\textstyle\sum_{n=0}^{N_h}\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}\nabla_{K,L}u^{n+1}_h\ \mathbb{1}_{(t^n,t^{n+1}]\times T_{K,L}}.
\end{aligned}$$

Because $\vec{\mathcal{F}}^h$ is precisely the first component of the vector $\frac{1}{d}\Big[d_1\left(\sum_{K_0\in\mathcal{T}_h}m(K_0)u^n_{1,K_0}\right)\nabla_h\mathbf{u}_h+\mathcal{A}^h\nabla_h\mathbf{u}_h\Big]$, and $\sqrt{d_1\left(\sum_{K_0\in\mathcal{T}_h}m(K_0)u^n_{1,K_0}\right)}\nabla_h\mathbf{u}_h$ is $L^2(\Omega_T)$ bounded by estimate (3.4), by the Cauchy-Schwarz inequality we infer a uniform $L^1(\Omega_T)$ estimate of $\vec{\mathcal{F}}^h$ and also the one of $\nabla_h u^h$. Thus (4.2),(4.3) are verified; the uniform $L^1(\Omega)$ bound on the initial data $u^0_{1,h}$ is also clear from (2.6), and Lemma 4.1 can be applied to derive the $L^1(\Omega_T)$ compactness of $(u_{1,h})_h$.
Along the same lines as $u_{1,h}$, we obtain the compactness of $u_{2,h}$ and $u_{3,h}$. Consequently, we can define the limit $\mathbf{u}=(u_1,u_2,u_3)$ of (a subsequence of) $\mathbf{u}_h$ and obtain the claim (i).
Further, it is easy to see that the claim (ii) is the consequence of the estimate (3.4).

Finally, the claims $(iii),(iv)$ follow because the uniform $L^2(\Omega_T)$ estimates of $\mathbf{u}_h$ and of $\sqrt{|\mathcal{A}^h_{ij}|}$.

Using in addition the quadratic growth of F and the a.e. convergence of $\mathbf{u}_h$ to $\mathbf{u}$, by the Vitali theorem we get (iv). Similarly, we get the strong $L^2(\Omega_T)$ convergence of $\sqrt{|\mathcal{A}_{ij}^h|}$ to $\sqrt{|\mathcal{A}_{ij}(\mathbf{u})|}$. Then, we pass to the limit first in $\sqrt{|\mathcal{A}_{ij}^h|}\nabla_h \mathbf{u}_h$ and then in $\mathcal{A}^h \nabla_h \mathbf{u}_h$; hence we get (iii). Finally, the $L^{10/3}$ bound of $\mathbf{u}$ is a consequence of the interpolation between $L^\infty(0,T;L^2(\Omega))$ estimate (3.3) and the discrete $L^2(0,T;H^1(\Omega))$ estimate (3.4). □

5. Convergence Analysis

Our final goal is to show that the limit functions $\mathbf{u} = (u_1, u_2, u_3)$ constructed in Proposition 4.2 constitute a weak solution of system (1.1). We start by passing to the limit in (2.7) to get the first equality in Definition 2.1.

Let $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$. Set $\varphi_K^n := \varphi(t^n, x_K)$ for all $K \in \mathcal{T}_h$ and $n \in [0, N_h + 1]$. We multiply the discrete equation (2.7) by $\Delta t \varphi_K^{n+1}$. Summing the result over $K \in \mathcal{T}_h$ and $n \in [0, N_h]$, yields

$$S_1^h + S_2^h + S_3^h = S_4^h,$$

where

$$S_1^h = \sum_{n=0}^{N_h} \sum_{K \in \mathcal{T}_h} |K| \, (u_{1,K}^{n+1} - u_{1,K}^n) \, \varphi_K^{n+1},$$

$$S_2^h = -\sum_{n=0}^{N_h} \Delta t \, d_1 \left(\sum_{K_0 \in \mathcal{T}_h} m(K_0) u_{1,K_0}^n \right) \sum_{K \in \mathcal{T}_h} \sum_{L \in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (u_{1,L}^{n+1} - u_{1,K}^{n+1}) \, \varphi_K^{n+1},$$

$$S_3^h = -\sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathcal{T}_h} \sum_{L \in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left[\mathcal{A}_{11,K,L}^{n+1} (u_{1,L}^{n+1} - u_{1,K}^{n+1}) + \mathcal{A}_{12,K,L}^{n+1} (u_{2,L}^{n+1} - u_{2,K}^{n+1}) \right] \varphi_K^{n+1},$$

$$S_4^h = \sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathcal{T}_h} |K| F_K^{n+1} \, \varphi_K^{n+1}.$$

Performing integration-by-parts and keeping in mind that $\varphi_K^{N_h+1} = 0$ for all $K \in \mathcal{T}_h$, we get from Proposition 4.2 (i) the convergence (along a subsequence)

$$\lim_{h \to 0} S_1^h = -\int_0^T \int_\Omega u_1 \partial_t \varphi - \int_\Omega u_{1,0} \varphi(0, \cdot).$$

Gathering by edges and using the definition (2.3) of ∇_h, we have

$$\begin{aligned} S_2^h &= \frac{1}{2} \sum_{n=0}^{N_h} \Delta t d_1 \left(\sum_{K_0 \in \mathcal{T}_h} m(K_0) u_{1,K_0}^n \right) \sum_{K \in \mathcal{T}_h} \sum_{L \in N(K)} \frac{1}{d} |\sigma_{K,L}| \, d_{K,L} \; d \, \frac{u_{1,L}^{n+1} - u_{1,K}^{n+1}}{d_{K,L}} \; \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{d_{K,L}} \\ &= \frac{1}{2} \sum_{n=0}^{N_h} \Delta t d_1 \left(\sum_{K_0 \in \mathcal{T}_h} m(K_0) u_{1,K_0}^n \right) \sum_{K \in \mathcal{T}_h} \sum_{L \in N(K)} |T_{K,L}| \left(\nabla_{K,L} u_{1,h}^{n+1} \cdot \eta_{K,L} \right) \left(\nabla \varphi(t^{n+1}, \overline{x_{K,L}}) \cdot \eta_{K,L} \right), \end{aligned}$$

where $\overline{x_{K,L}}$ is some point on the segment with the endpoints x_K, x_L. Since the values of $\nabla_{K,L}$ are directed by $\eta_{K,L}$, we have

$$\left(\nabla_{K,L} u_{1,h}^{n+1} \cdot \eta_{K,L} \right) \left(\nabla \varphi(t^{n+1}, \overline{x_{K,L}}) \cdot \eta_{K,L} \right) \equiv \nabla_{K,L} u_{1,h}^{n+1} \cdot \nabla \varphi(t^{n+1}, \overline{x_{K,L}}).$$

Moreover, each term corresponding to $T_{K,L}$ appears twice in the above formula,

$$S_2^h = \int_0^T d_1 \Big(\int_\Omega u_{1,h}(t,x) \, dx \Big) \int_\Omega \nabla_h u_{1,h} \cdot (\nabla \varphi)_h,$$

where

$$(\nabla \varphi)_h |_{(t^n, t^{n+1}] \times T_{K,L}} := \nabla \varphi(t^{n+1}, \overline{x_{K,L}}).$$

Observe that from the continuity of $\nabla\varphi$ we get $(\nabla\varphi)_h \to \nabla\varphi$ in $L^\infty(\Omega_T)$. Hence using (1.4), the strong L^p convergence of $u_{1,h}$ to u for $p < 10/3$, and the weak L^2 convergence of $\nabla_h u_{1,h}$ to ∇u_1, we pass to the limit in S_2^h and S_3^h, as $h \to 0$. Then, again along a subsequence, we have

$$\lim_{h\to 0} S_2^h = \int_0^T d_1\Big(\int_\Omega u_1(t,x)\,dx\Big)\int_\Omega \nabla u_1 \cdot \nabla\varphi,$$
$$\lim_{h\to 0} S_3^h = \iint_{\Omega_T}\Big(\mathcal{A}_{11}(u_1,u_2,u_3)\nabla u_1 + \mathcal{A}_{12}(u_1,u_2,u_3)\nabla u_2\Big)\cdot\nabla\varphi.$$

Note that our proof is slightly different from the classical one (cf. Ref. [8]), adapted to the definition (2.3) of the discrete gradient and to the associated weak convergence statements of Proposition 4.2 items (*ii*) and (*iii*). Let us put forward the arguments for the term S_2^h. In the proof of the convergence claim for S_3^h, we use Proposition 4.2(*iii*) in the place of Proposition 4.2(*ii*).

Finally, using Proposition 4.2(*iv*), we deduce that S_4^h converges to $\iint_{\Omega_T} F(u_1,u_2,u_3)\,\varphi$ as $h \to 0$. Gathering the obtained results, we justify the first equality in Definition 2.1 and . Reasoning along the same lines as above, we conclude that also the second and the third equality in Definition 2.1 hold. This concludes the proof of Theorem 2.1

6. Numerical Results

Numerical results presented in this section refer to the system given before (1.1) - (1.3). We now show some numerical experiments in two dimension, where the spatial domain corresponds to a simple square $\Omega =]-1,1[\times]-1,1[$. We used uniform meshes.

6.1. **Example 1.** We take the following ecological parameters:

$$K = 1.0, \quad L_2 = 1/6, \quad M_2 = 6.0, \quad R_0 = 1.0.$$
$$C_0 = 3/4, \quad L_3 = 1.0, \quad M_{32} = 2.0.$$

Let us precise the initial conditions for the first numerical test

The initial conditions showed in Figure 2, which we use for the simulations, are the following:

$$u_1(x,y,0) = u_1^* \qquad \forall (x,y) \in \Omega.$$

$$u_2(x,y,0) = \begin{cases} u_2^*, & \text{if } -0.2 \le x \le 0.2 \text{ and } \forall y,\\ 0, & \text{in other cases.}\end{cases}$$

$$u_3(x,y,0) = \begin{cases} u_3^*, & \text{if } -0.2 \le y \le 0.2 \text{ and } \forall x,\\ 0, & \text{in other cases.}\end{cases}$$

where $(u_1^*, u_2^*, u_3^*) = (0.5, 0.75, 0.125)$ a positive stable state of the system.

In Figure 3, we can observe the behaviour of u_1, u_2 and u_3 for different times when we do not have nonlocal diffusion neither cross diffusion, i.e. in our system (1.1) - (1.3) the second and third terms of the left hand side of each equation are vanished. In other words, we only have an ordinary differential equation system. At the beginning, the biggest change that we can noticed is that preys u_1 decreased in the region where predators u_2 are located ($-0.2 \le x \le 0.2$, $-1 \le y \le 1$), with u_2 and u_3 the changes are minor. As the time passes, we can observe the interaction between preys, predators and superpredators u_3, preys tried to evade the region occupied by predator and for this reason the population of them increased out of this area and continue decreasing where predators are located. Something similar happens with predators and superpredators, the predadors go out of the region where superpredators are located ($-1 \le x \le 1$, $-0.2 \le y \le 0.2$), as we can noticed, the superpredators increased their population in the region where is common for u_2 and u_3 ($-0.2 \le x \le 0.2$, $-0.2 \le y \le 0.2$).

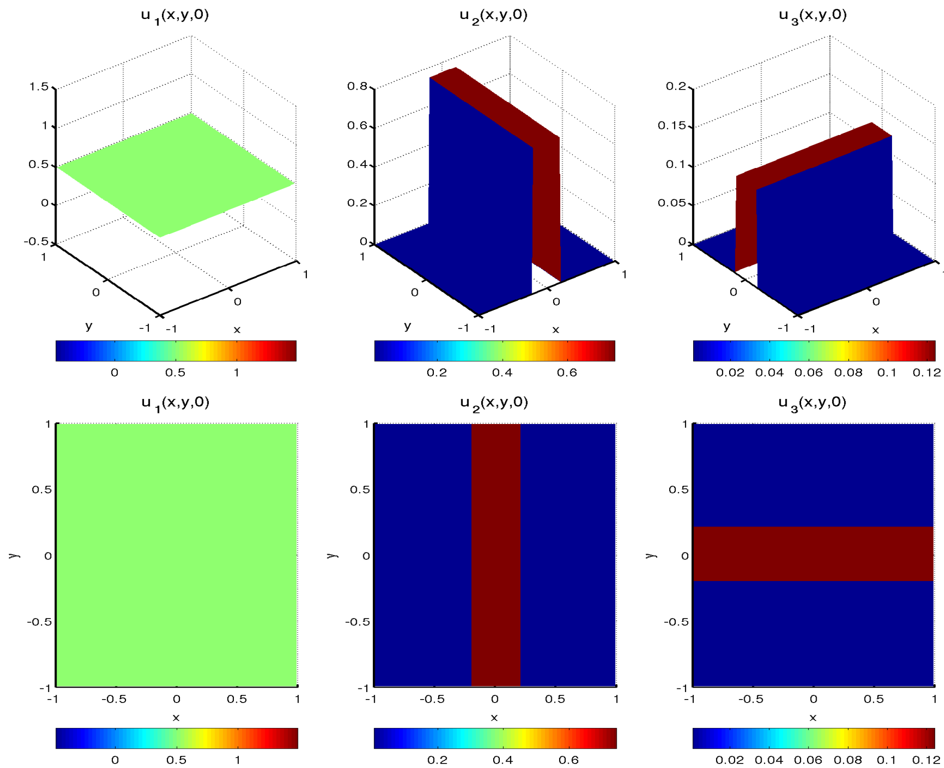

FIGURE 2. Initial data

In Figure 4, we can observe the behaviour of u_1, u_2 and u_3 for different times when we only have nonlocal diffusion, without cross diffusion, i.e. in our system (1.1) - (1.3) the third terms of the left hand side of each equation are vanished. We can clearly observe the effect of diffusion over the three populations (preys, predators and superpredators). At the beginning, we see that preys escape from the area where predators are located (central vertical band). The preys are directed to the boundary of the domain escaping to the central region. At the same time, we note that predators and superpredators remain in the initial regions (vertical and horizontal band). We also observed a large decrease in both populations (predators and superpredators), the superpredators almost dissapear. The prey population increase due to the decline of the other species (u_2 and u_3). As the time passes, because of the large decrease in the population of superpredators (u_3), the population of predators (u_2) is increasing so we can see a decline in the prey population (u_1). In this example we can clearly see the interaction between the three species and the effect of diffusion.

In Figure 5, we can observe the behaviour of u_1, u_2 and u_3 for different times when we have nonlocal and cross diffusion. Initially, we can observe the effect of the nonlocal and cross diffusion over the three populations. We notice the rapid movement of superpredators (u_3) towards the regions occupied by predators (u_2) and at the same time predators spread out to the areas where preys (u_1) are located. As the time passes, we can observe the dynamics between the three species.

Aditionally, in Figure 6 we observe the graphics about the behaviour of the diffusion. The picture on the left side is only nonlocal diffusion which is proportional to the total population and the other on the right side is the graphic of the nonlocal and cross diffusion. We can observe that the behaviour in both cases are almost similar. The diffusion of the preys (blue), at the beggining tends to increase for a while and then it decreases. The diffusion of predators (red) increases until to remain constant. Finally, the diffusion of the superpredators (green) decreases reaching almost void values.

6.2. **Example 2.** As a second numerical example, we consider the system (1.1) - (1.3), where we have chosen the same biological parameters as in Example 1. For this example, the initial

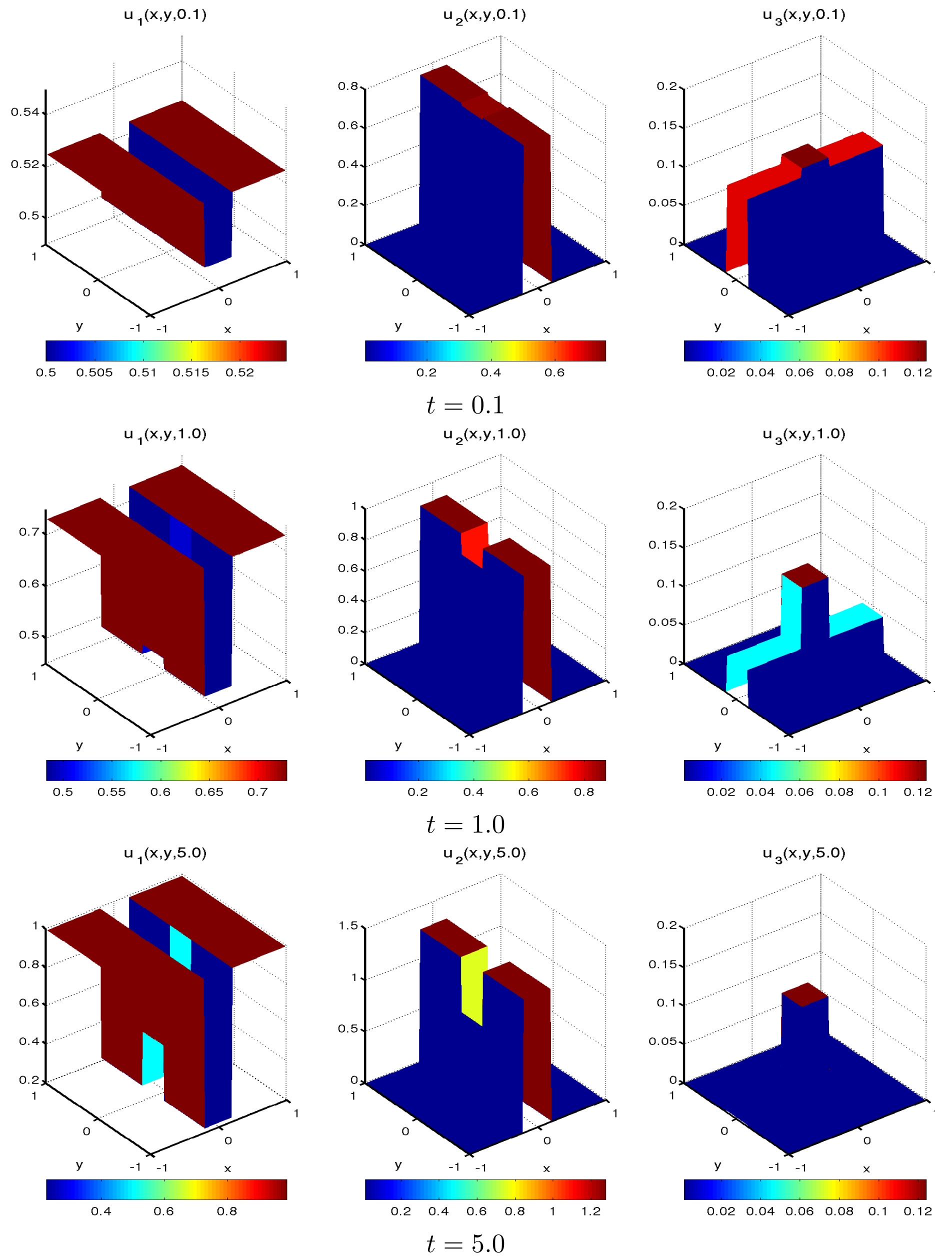

FIGURE 3. Interaction of the three species without nonlocal diffusion and cross diffusion, at time $t = 0.1, 1.0, 5.0$.

distribution for the species is chosen to be a normally distributed random perturbation around the equilibrium state $u_1^* = 0.5$, $u_2^* = 0.75$ and $u_3^* = 0.125$ on the entire domain. We consider the initial condition as follows:

$$\begin{aligned} u_1(x,y,0) &= u_1^* + \epsilon * \omega_{u_1}, & (x,y) \in \Omega \\ u_2(x,y,0) &= u_2^* + \epsilon * \omega_{u_2}, & (x,y) \in \Omega \\ u_3(x,y,0) &= u_3^* + \epsilon * \omega_{u_3}, & (x,y) \in \Omega \end{aligned}$$

with $\epsilon = 0.001$, and $\omega_{u_i} \in [0,1]$, $i = 1,2,3$. are random variables.

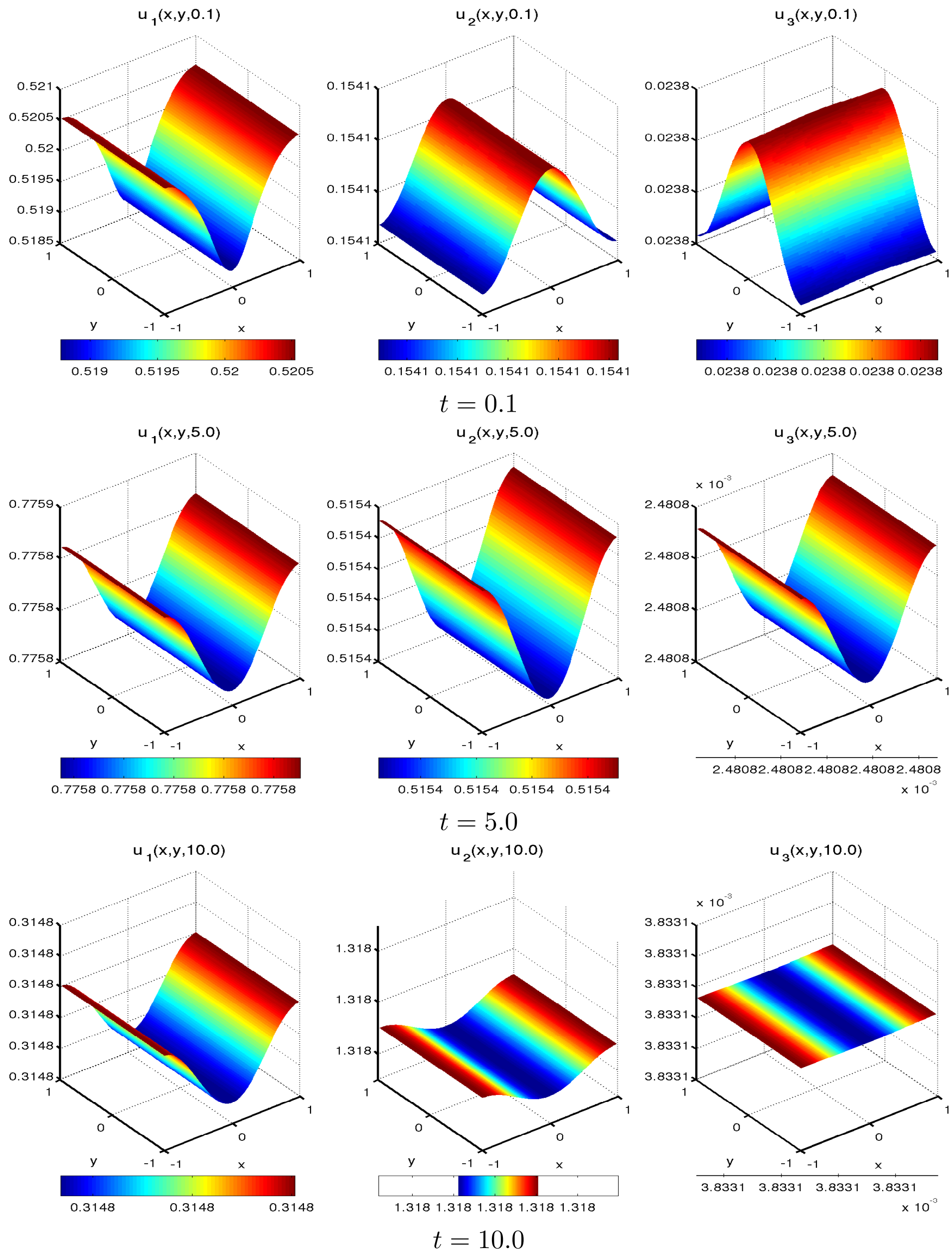

FIGURE 4. Interaction of the three species with nonlocal diffusion, at time $0.1, 5.0, 10.0$.

In Figure 7, it is clearly seen that the system evolves from noise to spatial patterns. It is observed that even when solution seems to fall in the constant equilibrium state, instabilities lead the solution to a stationary nonuniform spatial pattern. At the beggining, we can see that the preys are small groups scattered throughout the region, groups in which the number of individuals is variable. Similarly, we can see that predators and superpredators are distributed in three major groups throughout the region. As the time passes, we can clearly see the interaction between the three species.

In Figure 8, it is clearly seen that the system evolves from noise to spatial patterns. It is observed that even when solution seems to fall in the constant equilibrium state, instabilities lead the solution to a stationary nonuniform spatial pattern. We can see that the preys quickly form

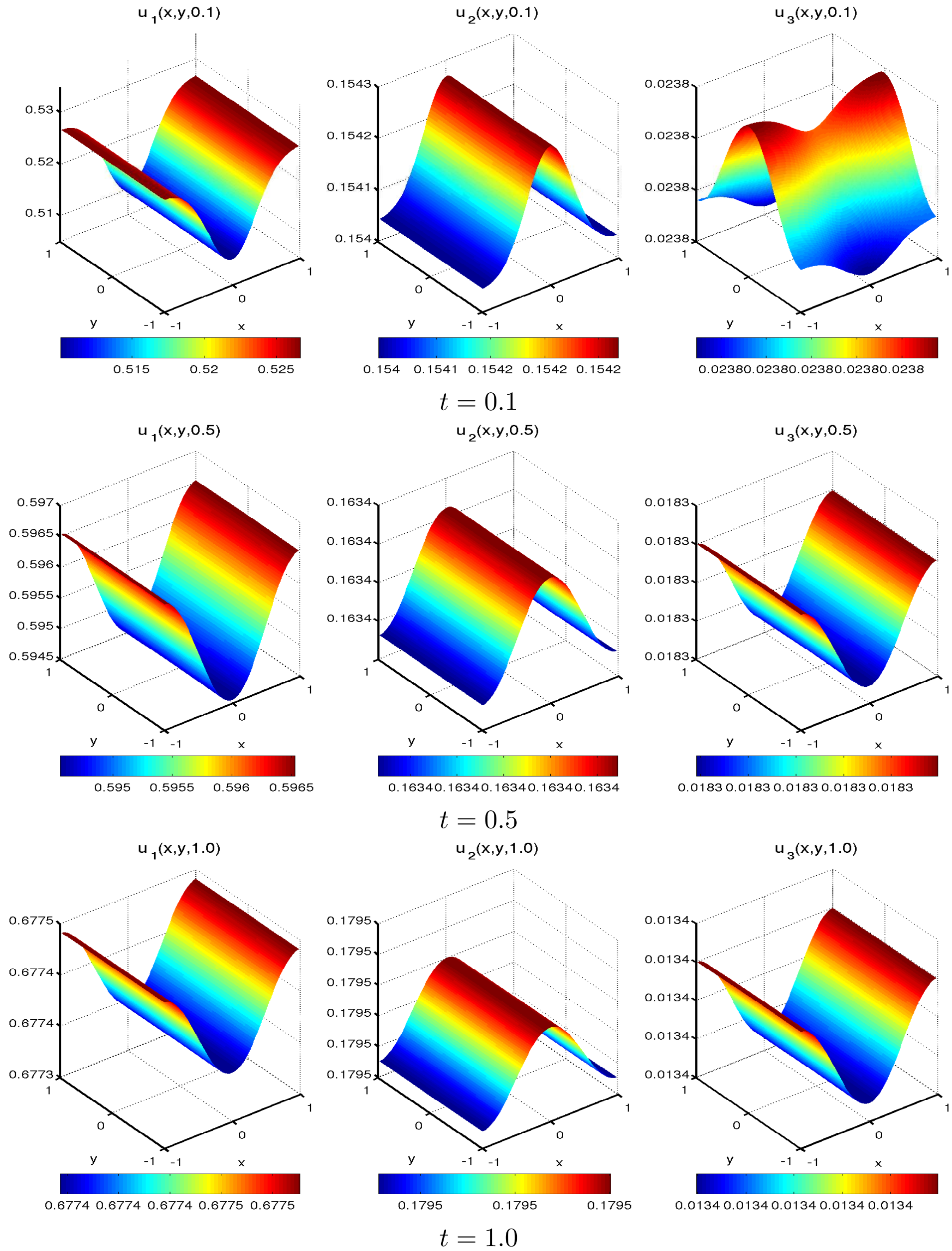

Figure 5. Interaction of the three species with nonlocal and cross diffusion, at time $t = 0.1, 0.5, 1.0$.

small groups distributed throughout the region, it is observed that these groups have the number of individuals variable. Similarly, we can see that predators and superpredators are distributed in only three big groups throughout the region. As the time passes, we can clearly see the interaction between the three species.

It is observed in Figure 9 the graphics about the behaviour of the diffusion. The picture on the left side is only nonlocal diffusion which is proportional to the total population and the other on the right side is the graphic of the nonlocal and cross diffusion. We can observe that the behaviour in both cases are almost similar. The diffusion of the preys (blue), at the beggining tends to decrease for a while and then it increases until to remain constant. The diffusion of

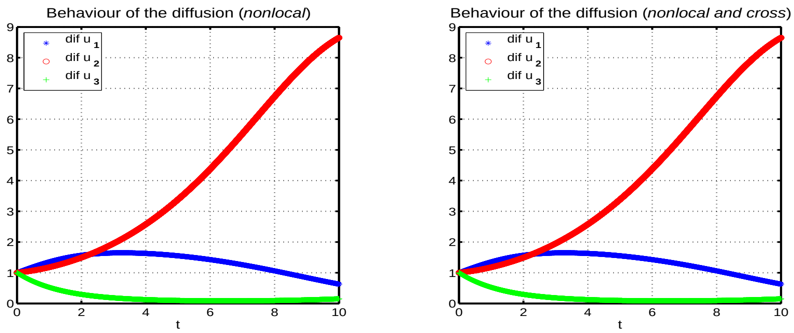

FIGURE 6. The behaviour of the diffusion

predators (red) tends to decrease. Finally, the diffusion of the superpredators (green) tends to increase for a moment and then it decreases.

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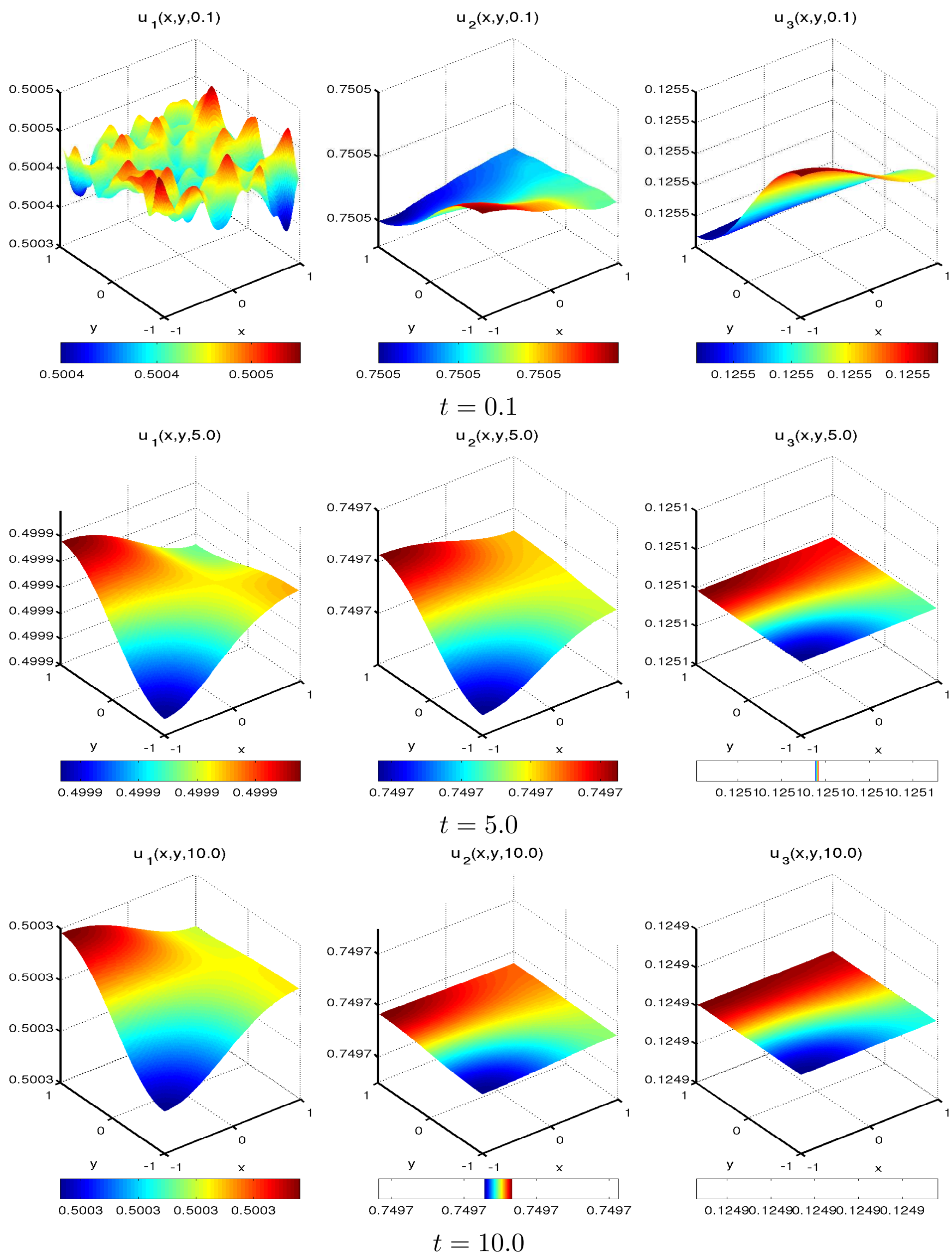

FIGURE 7. Interaction of the three species with nonlocal diffusion, at time $0.1, 5.0, 10.0$.

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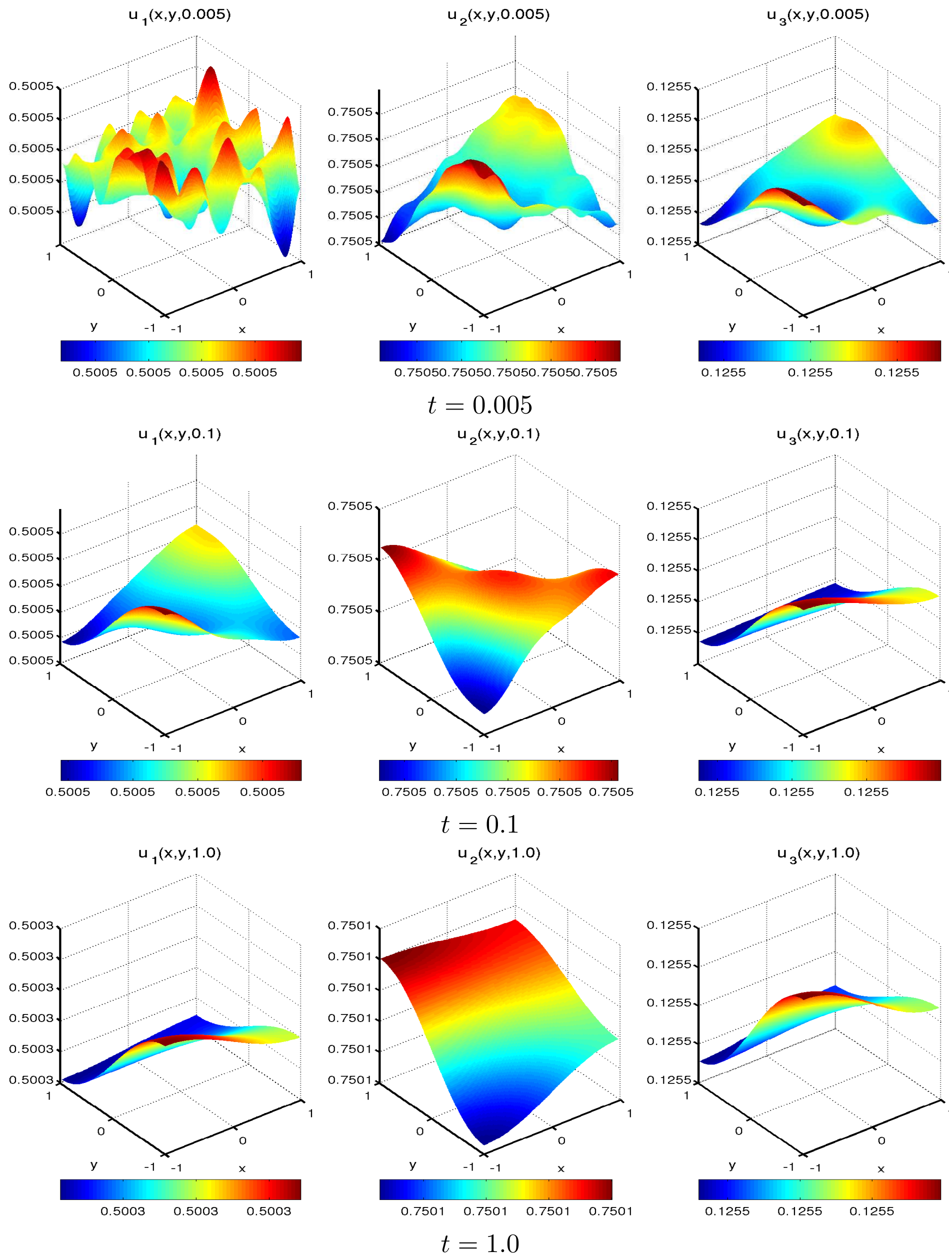

FIGURE 8. Interaction of the three species with nonlocal and cross diffusion, at time $t = 0.005, 0.1, 1.0$.

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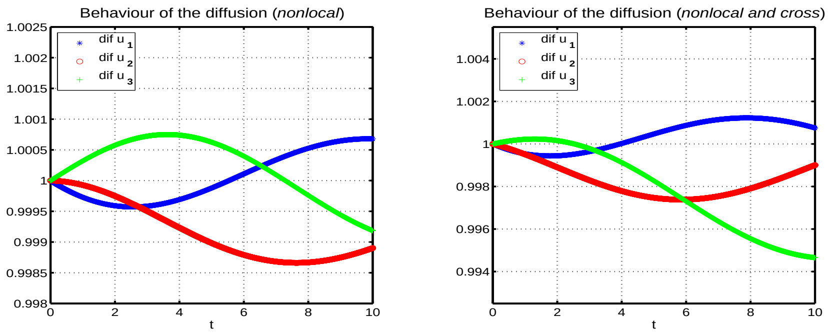

FIGURE 9. The behaviour of the diffusion

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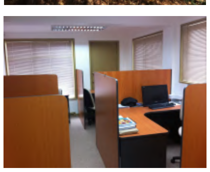

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