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# Analysis of a FEM-BEM model posed on the conducting domain for the time-dependent eddy current problem

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**Abstract**

The three-dimensional eddy current time-dependent problem is considered. We formulate it in terms of new variables lying only on the conducting domain and on its boundary. We combine finite elements (FEM) and boundary elements (BEM), to obtain a FEM-BEM coupled variational formulation. We prove existence and uniqueness of the solution in the continuous and the fully discrete case. Finally, we investigate the convergence order of the fully discrete scheme.

*Keywords:* Boundary elements; eddy current problem; finite elements; time-dependent electromagnetic problem.

## 1. Introduction

The eddy current model is commonly used in many problems in sciences and industry, for example, in induction heating, electromagnetic braking, electric generation, etc. An overview of the mathematical analysis of the eddy current model and its numerical solution in harmonic regime can be found in the recent book by Alonso and Valli [3], which provides a large list of references on this subject.

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In this paper, we deal with the numerical solution of the time-dependent eddy current problem, which is naturally formulated in the whole space, with adequate decay conditions at infinity. The literature on the numerical analysis of this kind of problems is more scarce. Among the few papers devoted to this subject, both in bounded and unbounded domains, by using finite element (FEM), boundary element methods (BEM) and coupled FEM-BEM methods, we can mention [1, 2, 8, 9, 10, 11, 16]. These articles differ from each other by the physical quantities chosen for the formulation (magnetic field, electric field or different kind of potentials) and by the way of treating the decay condition to reduce the problem to a bounded domain.

We consider a FEM-BEM method to compute the eddy currents generated in a three-dimensional conductor $\Omega_C$ by a time-dependent source current. The problem is reformulated by expressing the magnetic and the electric fields in terms of convenient new variables. We use FEM only on the conducting domain $\Omega_C$, the integral conditions being imposed on its boundary $\partial\Omega_C$. Therefore, the domain where FEM is used results as small as possible, leading to a more efficient method as compared, for instance, with [1] and [2], where similar formulations are considered. Another important feature of this approach is that it preserves the coercivity of the original problem. The purpose of the paper is to analyze the convergence of a fully discrete finite element scheme for this formulation and to investigate the convergence order.

The paper is organized as follows. In Section 2 we give some basic definitions. In Section 3 we present the model problem with the necessary assumptions over the data and introduce a new variable, the time-primitive of the electric field, which plays the role of a vector potential for the magnetic field. In Section 4 we introduce the integral operators and recall their properties. Then, we derive the FEM-BEM formulation and show existence and uniqueness of the solution to the problem. In Section 5, we introduce a space-discretization of the problem based on Nédélec edge elements in $\Omega_C$ and piecewise linear continuous elements for the variable on $\partial\Omega_C$ arising from the integral equations. Then, a backward Euler method is employed to obtain a time discretization. Finally, the results presented in Section 6 prove that the proposed fully discrete scheme is convergent with optimal order.

## 2. Preliminaries

In the sequel we deal with real valued functions. Boldface letters will denote vectors (in $\mathbb{R}^n$) or vector-valued functions. The symbol $|\cdot|$ will represent the 2-norm for vectors:

$$|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} := \sum_{i=1}^{n} v_i^2.$$

In all the paper the conductor $\Omega_C \subset \mathbb{R}^3$ is a bounded connected polyhedron, with a connected boundary $\Gamma := \partial\Omega_C$ such that the insulator $\Omega_I := \mathbb{R}^3\backslash\overline{\Omega}_C$ is connected.

We remark that, under the above conditions, $\Omega_C$ and $\Omega_I$ have the same number of non-bounding cycles $L$. There exists $L$ disjoint connected open surfaces $\Sigma_j^{int} \subset \Omega_C$ (respectively $\Sigma_j^{ext} \subset \Omega_I$), $j = 1, ..., L$, such that $\widetilde{\Omega}_C := \Omega_C\backslash\bigcup_{j=1}^{L}\Sigma_j^{int}$ (respectively $\widetilde{\Omega}_I := \Omega_I\backslash\bigcup_{j=1}^{L}\Sigma_j^{ext}$) is simply connected. The boundary curves $\partial\Sigma_j^{int}$ and $\partial\Sigma_j^{ext}$ lie on $\Gamma$.

We denote by

$$(f, g)_{0,\Omega_*} := \int_{\Omega_*} f g \,\mathrm{d}\mathbf{x}$$

the inner product in $L^2(\Omega_*)$ and $\|\cdot\|_{0,\Omega_*}$ the corresponding norm with $* \in \{C, I\}$. As usual, $\|\cdot\|_{s,\Omega_C}$ stands for the norm of the Hilbertian Sobolev space $H^s(\Omega_C)$ for all $s \in \mathbb{R}$. We also recall that, for any the space $H^t(\Gamma)$ has an intrinsic definition (by localization) on the Lipschitz surface $\Gamma$ due to their invariance under Lipschitz coordinate transformations. We denote by $\|\cdot\|_{t,\Gamma}$ the norm in $\mathrm{H}^t(\Gamma)$.

In this paper, the spaces that are product forms of the previous function spaces are endowed with the natural product norm and duality pairing without changing the notations since it will be clear from the context when scalar or vector functions are used.

We introduce the functional space

$$\mathbf{H}(\mathbf{curl}\,; \Omega_C) := \{\mathbf{v} \in (L^2(\Omega_C))^3 \,:\, \mathbf{curl}\,\mathbf{v} \in (L^2(\Omega_C))^3\},$$

endowed with the natural norm: $\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} := \|\mathbf{v}\|^2_{0,\Omega_C} + \|\mathbf{curl}\,\mathbf{v}\|^2_{0,\Omega_C}$.

We will also need to define $\mathbf{H}(\mathrm{div}\,; \Omega_C) := \{\mathbf{v} \in (L^2(\Omega_C))^3 \,:\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega_C)\}$, endowed with the norm $\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div}\,;\Omega_C)} := \|\mathbf{v}\|^2_{0,\Omega_C} + \|\mathrm{div}\,\mathbf{v}\|^2_{0,\Omega_C}$.

*2.1. Basic spaces for time dependent problems*

As we shall deal with an evolutionary problem, we need to introduce spaces of functions defined on a bounded interval $[0,T]$ and with values in a separable Hilbert space $V$ whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0,T];V)$ for the Banach space consisting of all continuous functions $f : [0,T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0,T];V)$ denotes the subspace of $\mathcal{C}^0([0,T];V)$ of all functions $f$ with (strong) derivatives $d^j f/dt^j$ in $\mathcal{C}^0([0,T];V)$ for all $1 \leq j \leq k$. In the following, we will use indistinctly the notations

$$\frac{d}{dt} f = \partial_t f$$

to express the derivative with respect to the variable $t$.

We also consider the space $L^2(0,T;V)$ of classes of functions $f : (0,T) \to V$ that are Böchner-measurable and such that

$$\|f\|^2_{L^2(0,T;V)} := \int_0^T \|f(t)\|^2_V dt < +\infty.$$

Furthermore, we will work with

$$H^1(0,T;V) := \left\{ f \in L^2(0,T;V) \, : \, \frac{d}{dt} f \in L^2(0,T;V) \right\}.$$

Analogously, we define $H^k(0,T;V)$, for all $k \in \mathbf{N}$.

## 3. The model problem

The unit normal vector on $\Gamma$ that points from $\Omega_C$ to $\Omega_I$ (respectively from $\Omega_I$ to $\Omega_C$) is denoted by $\mathbf{n}_C$ (respectively $\mathbf{n}_I = -\mathbf{n}_C$).

Let $\mathbf{E}(\mathbf{x},t)$ the electric field and $\mathbf{H}(\mathbf{x},t)$ the magnetic field. Given a time-dependent compactly supported current density $\mathbf{J}$, our aim is to furnish an approximate solution to the problem below:

$$\begin{array}{lll}
\partial_t (\boldsymbol{\mu}\mathbf{H}) + \mathbf{curl}\,\mathbf{E} = \mathbf{0} & \text{in } \mathbb{R}^3 \times (0,T), & \\
\mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\,\mathbf{E} = \mathbf{J} & \text{in } \mathbb{R}^3 \times [0,T], & \\
\operatorname{div}(\boldsymbol{\varepsilon}\mathbf{E}) = 0 & \text{in } \Omega_I \times [0,T], & \\
\displaystyle\int_\Gamma \boldsymbol{\varepsilon}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}\zeta = 0, & \text{in } [0,T], & \\
\mathbf{H}(\mathbf{x},t) = O(|\mathbf{x}|^{-1}) & \text{and} \quad \mathbf{E}(\mathbf{x},t) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \\
\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) & \text{in } \mathbb{R}^3, &
\end{array} \tag{1}$$

where the asymptotic behavior $(1)_5$ holds uniformly in $[0, T]$. The initial data $\mathbf{H}_0$ in $(1)_6$ has to satisfy $\operatorname{div}(\boldsymbol{\mu}\mathbf{H}_0) = 0$ in $\mathbb{R}^3$. The electric permittivity $\boldsymbol{\varepsilon}$, the electric conductivity $\boldsymbol{\sigma}$, and the magnetic permeability $\boldsymbol{\mu}$ are symmetric and uniformly positive definite matrices in $\Omega_C$, with entries belonging to $L^\infty(\Omega_C)$. Moreover, $\boldsymbol{\varepsilon} = \varepsilon_0 I$, $\boldsymbol{\mu} = \mu_0 I$ and $\boldsymbol{\sigma} = \mathbf{0}$, bf. in $\Omega_I$ ($I$ is the identity matrix).

It is important to notice that, since $\boldsymbol{\sigma} = \mathbf{0}$ in $\Omega_I$, $(1)_2$ implies that the data $\mathbf{J}$ satisfies the compatibility conditions

$$\operatorname{div} \mathbf{J} = 0 \text{ in } \Omega_I \qquad \text{and} \qquad \int_\Gamma \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}\zeta = 0, \tag{2}$$

for all $t \in [0, T]$.

For the sake of simplicity, we consider that $\operatorname{supp}(\mathbf{J}) \subset \overline{\Omega}_C$, namely, $\mathbf{J} = 0$ in $\Omega_I$. Moreover we consider $\mathbf{J} \in L^2(0, T; (L^2(\Omega_C))^3)$.

We define $\mathbf{H}_C := \mathbf{H}|_{\Omega_C}$ and $\mathbf{H}_I := \mathbf{H}|_{\Omega_I}$. Analogously, $\mathbf{H}_{C,0} := \mathbf{H}_0|_{\Omega_C}$, $\mathbf{H}_{I,0} := \mathbf{H}_0|_{\Omega_I}$, $\mathbf{E}_C := \mathbf{E}|_{\Omega_C}$ and $\mathbf{E}_I := \mathbf{E}|_{\Omega_I}$. Moreover, we consider the space $\mathbb{H}(\Omega_C)$, defined as

$$\mathbb{H}(\Omega_C) \;:=\; \{\mathbf{v} \in (L^2(\Omega_C))^3 \;:\; \mathbf{curl}\, \mathbf{v} = \mathbf{0},\; \operatorname{div}(\boldsymbol{\sigma}\,\mathbf{v}) = 0,\; \boldsymbol{\sigma}\,\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

We recall that each "cutting" surface $\Sigma_j^{int}$, $j = 1, ..., L$, "cuts" an independent non-bounding cycle in $\Omega_C$. They are connected orientable Lipschitz surfaces with $\partial\Sigma_j^{int} \subset \Gamma$, such that every curl-free vector field in $\Omega_C$ has a global potential in $\widetilde{\Omega}_C$. The basis functions $\boldsymbol{\omega}_j$ are the $(L^2(\Omega_C))^3$-extension of $\nabla p_j$, where $p_j \in H^1(\Omega_C \backslash \Sigma_j^{int})$ is the solution of the problem

$$\begin{aligned}
&\operatorname{div}(\boldsymbol{\sigma}\,\nabla p_j) \;=\; 0 && \text{in } \Omega_C \backslash \Sigma_j^{int},\\
&\boldsymbol{\sigma}\,\nabla p_j \cdot \mathbf{n}_C \;=\; 0 && \text{on } \Gamma \backslash \partial\Sigma_j^{int},\\
&[\![\boldsymbol{\sigma}\,\nabla p_j \cdot \mathbf{n}_j]\!]_{\Sigma_j^{int}} = 0, && j = 1, ..., L,\\
&[\![p_j]\!]_{\Sigma_j^{int}} = 1, && j = 1, ..., L,
\end{aligned}$$

having denoted by $[\![\,\cdot\,]\!]_{\Sigma_j^{int}}$ the jump across the surface $\Sigma_j^{int}$ and by $\mathbf{n}_j$ the unit normal vector on $\Sigma_j^{int}$.

In order to obtain a suitable formulation for the problem (1), we introduce the variable

$$\mathbf{A}_C(\mathbf{x},t) \;:=\; -\int_0^t \mathbf{E}_C(\mathbf{x},s)\,\mathrm{d}s \;+\; \mathbf{A}_{C,0}(\mathbf{x}) \tag{3}$$

where $\mathbf{A}_{C,0}$ is the solution of this problem:

$$\begin{aligned}
&\mathbf{curl}\,\mathbf{A}_{C,0} \;=\; \boldsymbol{\mu}_C\mathbf{H}_{C,0} && \text{in } \Omega_C,\\
&\mathrm{div}\,(\boldsymbol{\sigma}\mathbf{A}_{C,0}) \;=\; 0 && \text{in } \Omega_C,\\
&\boldsymbol{\sigma}\mathbf{A}_{C,0}\cdot\mathbf{n}_C \;=\; 0 && \text{on } \Gamma.\\
&\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{A}_{C,0}\cdot\boldsymbol{\omega}_j\,\mathrm{d}\mathbf{x} = \;0, && j=1,...,L.
\end{aligned} \tag{4}$$

This problem has a unique solution, because $\mathrm{div}\,(\boldsymbol{\mu}_C\mathbf{H}_{C,0}) \;=\; 0$ in $\Omega_C$. (See [3]).

We obtain directly from (3) that $\mathbf{E}_C = -\partial_t\mathbf{A}_C$ in $\Omega_C\times(0,T)$. Moreover, if we apply **curl** to (3) and use $(1)_1$ and $(4)_1$ we also deduce that $\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{curl}\,\mathbf{A}_C$ in $\Omega_C\times[0,T]$ and, replacing the new equalities in $(1)_2$, we obtain

$$\boldsymbol{\sigma}\,\partial_t\,\mathbf{A}_C \;+\; \mathbf{curl}\,(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C) \;=\; \mathbf{J} \qquad \text{in } \Omega_C\times(0,T).$$

We need some other tools to continue.

We introduce the Beppo-Levi space:

$$W^1(\Omega_I) \;:=\; \Big\{\varphi\in\mathfrak{D}'(\Omega_I)\,;\,\frac{\varphi}{\sqrt{1+|\mathbf{x}|^2}}\in L^2(\Omega_I)\,,\,\nabla\varphi\in(L^2(\Omega_I))^3\Big\},$$

and recall that the seminorm $\|\nabla(\cdot)\|_{0,\Omega_I}$ is a norm in $W^1(\Omega_I)$ equivalent to the natural norm; i.e., there exists a constant $C>0$ such that (see [14]) :

$$\left\|\frac{\varphi}{\sqrt{1+|\mathbf{x}|^2}}\right\|_{0,\Omega_I}^2 \;\leq\; C\|\nabla\varphi\|_{0,\Omega_I}^2 \qquad \forall\varphi\in W^1(\Omega_I). \tag{5}$$

Moreover we define the harmonic Neumann vector-fields associated with $\Omega_I$ by

$$\mathbb{H}(\Omega_I) \;:=\; \{\mathbf{v}\in(L^2(\Omega_I))^3 \;:\; \mathbf{curl}\,\mathbf{v}=\mathbf{0},\ \mathrm{div}\,\mathbf{v}=0,\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma\}.$$

We will need a basis of the finite dimensional space $\mathbb{H}(\Omega_I)$. To this end, we consider the set $\{\Sigma_j^{ext}\;:\;j=1,...,L\}$ of orientable cutting surfaces in $\Omega_I$ introduced above. We fix a unit normal $\mathbf{n}_j$ on each $\Sigma_j^{ext}$.

**Theorem 3.1.** *For any $j = 1, ..., L$, the following problems admit unique solutions: Find $z_j \in W^1(\Omega_I \backslash \Sigma_j^{ext})$ such that*

$$\begin{aligned} \triangle z_j &= 0 && \text{in } \Omega_I \backslash \Sigma_j^{ext}, \\ \nabla z_j \cdot \mathbf{n}_I &= 0 && \text{on } \Gamma \backslash \partial \Sigma_j^{ext}, \\ [\![\nabla z_j \cdot \mathbf{n}_j]\!]_{\Sigma_j^{ext}} &= 0, && , \\ [\![z_j]\!]_{\Sigma_j^{ext}} &= 1. \end{aligned} \tag{6}$$

*Moreover the set $\{\widetilde{\nabla}\, z_j \,:\, j = 1, ..., L\}$, where $\widetilde{\nabla}\, z_j$ are the $(L^2(\Omega_I))^3$-extension of $\nabla z_j$, is a basis of $\mathbb{H}(\Omega_I)$.*

We have the following representation of rotational free vector-fields in $\Omega_I$, see Remark 7 in [6].

**Lemma 3.1.** *There holds*

$$\{\mathbf{u} \in (L^2(\Omega_I))^3 \,:\, \mathbf{curl}\,\mathbf{u} = \mathbf{0} \text{ in } \Omega_I\} = \nabla(W^1(\Omega_I)) \oplus \mathbb{H}(\Omega_I).$$

*Moreover this is an $L^2(\Omega_I)$-orthogonal decomposition.*

We know of $(1)_2$ that $\mathbf{curl}\,\mathbf{H}_I = \mathbf{0}$ in $\Omega_I$ at each time $t \in [0, T]$. Then, the previous lemma ensures the existence at each time $t \in [0, T]$, of a function $\psi_I(t)$ in $W^1(\Omega_I)$ and real constants $\{\alpha_j(t)\}_{j=1}^L$ such that

$$\mathbf{H}_I(\mathbf{x}, t) = \nabla \psi_I(\mathbf{x}, t) + \sum_{j=1}^{L} \alpha_j(t) \widetilde{\nabla}\, z_j(\mathbf{x}) \qquad \text{in } \Omega_I \times [0, T]. \tag{7}$$

On the other hand, taking divergence in the equation $(1)_1$ and using that $\boldsymbol{\mu} = \mu_0 I$ in $\Omega_I$, we obtain that $\partial_t(\operatorname{div} \mathbf{H}_I) = 0$ in $\Omega_I \times (0, T)$. Hence, as we know that $\operatorname{div} \mathbf{H}_I(\mathbf{x}, 0) = \operatorname{div} \mathbf{H}_{I,0} = 0$ in $\Omega_I$, we conclude that $\operatorname{div} \mathbf{H}_I = 0$ in $\Omega_I \times [0, T]$. Then, using (7), the last equality and $(6)_1$, we obtain that

$$\triangle \psi_I = 0 \qquad \text{in } \Omega_I \times [0, T].$$

On the other hand, multiplying $(1)_1$ by $\widetilde{\nabla}\, z_i$, using a Green's formula and the fact that $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$, we obtain

$$\int_{\Omega_I} \partial_t(\mu_0 \mathbf{H}_I) \cdot \widetilde{\nabla}\, z_i \,\mathrm{d}\mathbf{x} = -\int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \widetilde{\nabla}\, z_i \,\mathrm{d}\zeta \qquad i = 1, ..., L.$$

Replacing $\mathbf{H}_I$ by $\nabla\psi_I + \sum_{j=1}^{L}\alpha_j\widetilde{\nabla}\, z_j$ and $\mathbf{E}_C$ by $-\partial_t\mathbf{A}_C$, using the orthogonality between $\nabla W^1(\Omega_I)$ and $\mathbb{H}(\Omega_I)$ and integrating by parts in $\Omega_I$, we obtain

$$\mu_0\sum_{j=1}^{L}\alpha_j'(t)\int_{\Omega_I}\widetilde{\nabla}\, z_j(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \;=\; \int_\Gamma \partial_t\mathbf{A}_C(\mathbf{x},t)\times\mathbf{n}_C(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\zeta,$$

$i=1,...,L$. Next, integrating in time between 0 and $s$, with $s\in(0,T)$ and recalling that $\mathbf{A}_C(\mathbf{x},0) \;=\; \mathbf{A}_{C,0}(\mathbf{x})$, we obtain

$$\begin{aligned}&\mu_0\sum_{j=1}^{L}\alpha_j(s)\int_{\Omega_I}\widetilde{\nabla}\, z_j(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \;-\; \int_\Gamma \mathbf{A}_C(\mathbf{x},s)\times\mathbf{n}_C(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\zeta\\ &=\mu_0\sum_{j=1}^{L}\alpha_j(0)\int_{\Omega_I}\widetilde{\nabla}\, z_j(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \;-\; \int_\Gamma \mathbf{A}_{C,0}(\mathbf{x})\times\mathbf{n}_C(\mathbf{x})\cdot\widetilde{\nabla}\, z_i(\mathbf{x})\,\mathrm{d}\zeta,\end{aligned} \tag{8}$$

with $i=1,...,L$. From (6), Green's formula yields

$$\int_{\Omega_I}\widetilde{\nabla}\, z_j\cdot\widetilde{\nabla}\, z_i\,\mathrm{d}\mathbf{x} \;=\; \int_{\Sigma_j^{ext}}\frac{\partial z_i}{\partial\mathbf{n}_j}\,\mathrm{d}\zeta,$$

for all $1\le i,j\le L$. Then, we introduce the matrix

$$\mathbf{N} := \left(\int_{\Sigma_j^{ext}}\frac{\partial z_k}{\partial\mathbf{n}_j}\,\mathrm{d}\zeta\right)_{1\le k,j\le L}. \tag{9}$$

It is clear that $\mathbf{N}$ is symmetric and positive definite. We also define the matrix $\mathbf{Z}$ and the vector $\boldsymbol{\alpha}$ by

$$\mathbf{Z} \;:=\; [\widetilde{\nabla}\, z_1\ \cdots\ \widetilde{\nabla}\, z_L]^t, \qquad \boldsymbol{\alpha}^t \;:=\; [\alpha_1\ \cdots\ \alpha_L]. \tag{10}$$

Thus, we can write equation (8) as follows:

$$\mu_0\,\mathbf{N}\,\boldsymbol{\alpha} \;-\; \int_\Gamma \mathbf{Z}\,(\mathbf{A}_C\times\mathbf{n}_C)\,\mathrm{d}\zeta \;=\; \mu_0\,\mathbf{N}\,\boldsymbol{\alpha}_0 \;-\; \int_\Gamma \mathbf{Z}\,(\mathbf{A}_{C,0}\times\mathbf{n}_C)\,\mathrm{d}\zeta, \tag{11}$$

where $\boldsymbol{\alpha}_0 := \boldsymbol{\alpha}(0)$ is known.

Finally, we also impose that

$$|\psi_I(\mathbf{x},t)| \,=\, O(|\mathbf{x}|^{-1}) \qquad \text{as } \; |\mathbf{x}| \to \infty.$$

In conclusion, we are led to the following problem in terms of the variables $(\mathbf{A}_C, \psi_I, \boldsymbol{\alpha})$:

$$\begin{aligned}
&\boldsymbol{\sigma}\partial_t \mathbf{A}_C \,+\, \mathbf{curl}\,(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C) \,=\, \mathbf{J} && \text{in } \Omega_C \times (0,T),\\
&\mu_0\,\mathbf{N}\,\boldsymbol{\alpha} \,-\, \int_\Gamma \mathbf{Z}\,(\mathbf{A}_C \times \mathbf{n}_C)\,\mathrm{d}\zeta && \\
&\qquad\quad = \mu_0\,\mathbf{N}\,\boldsymbol{\alpha}_0 - \textstyle\int_\Gamma \mathbf{Z}\,(\mathbf{A}_{C,0} \times \mathbf{n}_C)\,\mathrm{d}\zeta, && \\
&\Delta\psi_I \,=\, 0 && \text{in } \Omega_I \times [0,T],\\
&(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C) \times \mathbf{n}_C + (\nabla\psi_I + \mathbf{Z}^t\boldsymbol{\alpha}) \times \mathbf{n}_I \,=\, \mathbf{0} && \text{on } \Gamma \times [0,T],\\
&\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mu_0 \nabla\psi_I \cdot \mathbf{n}_I \,=\, 0 && \text{on } \Gamma \times [0,T],\\
&|\psi_I(\mathbf{x})| \,+\, |\nabla\psi_I(\mathbf{x})| \,=\, O(|\mathbf{x}|^{-1}) && \text{as } |\mathbf{x}| \to \infty.\\
&\mathbf{A}_C(\mathbf{x},0) \,=\, \mathbf{A}_{C,0} && \text{in } \Omega_C,
\end{aligned} \tag{12}$$

Notice that equations $(12)_4$ and $(12)_5$ come from the fact $\mathbf{H}_C \times \mathbf{n}_C = -\mathbf{H}_I \times \mathbf{n}_I$ and $\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{n}_C = -\mu_0 \mathbf{H}_I \cdot \mathbf{n}_I$ on $\Gamma$, respectively.

## 4. A FEM-BEM coupling variational formulation

It is well-known from potential theory (see, e.g. McLean [13], Nédélec [14]) that we can introduce on $\Gamma$ the single layer and double layer potentials, which satisfy

$$\mathcal{S} \,:\, H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \;\; \mathcal{S}(\xi)(\mathbf{x}) \,:=\, \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\xi(\mathbf{y})\,\mathrm{d}\zeta_y,$$

$$\mathcal{D} \,:\, H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \quad \mathcal{D}(\eta)(\mathbf{x}) \,:=\, \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y})\,\mathbf{n}_C(\mathbf{y})\,\mathrm{d}\zeta_y,$$

and the hypersingular operator $\mathcal{H} \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, which is defined as follows:

$$\mathcal{H}(\eta)(\mathbf{x}) \,:=\, -\nabla\Big(\int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y})\,\mathbf{n}_C(\mathbf{y})\,\mathrm{d}\zeta_y\Big) \cdot \mathbf{n}_C(\mathbf{x}).$$

The three operators are linear and bounded. We also recall that the adjoint operator $\mathcal{D}' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ reads

$$\mathcal{D}'(\xi)(\mathbf{x}) := \Big( \int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \xi(\mathbf{y})\, \mathbf{n}_C(\mathbf{y})\, \mathrm{d}\zeta_y \Big) \cdot \mathbf{n}_C(\mathbf{x}).$$

In what follows we recall some basics properties of these operators. See, e.g. McLean [13], Nédélec [14] for the corresponding proofs.

**Theorem 4.1.** *Let $\varphi \in W^1(\Omega_I)$ be a harmonic function. Then*

$$\left(\frac{1}{2}\mathcal{I} - \mathcal{D}\right)(\varphi) \,+\, \mathcal{S}\left(\frac{\partial\varphi}{\partial \mathbf{n}_C}\right) \,=\, 0 \quad \textit{and} \quad \left(\frac{1}{2}\mathcal{I} + \mathcal{D}'\right)\left(\frac{\partial\varphi}{\partial \mathbf{n}_C}\right) \,+\, \mathcal{H}(\varphi) \,=\, 0$$

*on* $\Gamma$.

**Lemma 4.1.** *(i) There exists $k_1$ >0 such that*

$$\int_\Gamma \mathcal{S}(\eta)\, \eta\, d\zeta \ge k_1 \|\eta\|^2_{-1/2,\Gamma} \qquad \forall \eta \in H^{-1/2}(\Gamma).$$

*(ii) There exists a constant $k_2 > 0$ such that*

$$\int_\Gamma \mathcal{H}(\varphi)\, \varphi\, d\zeta \ge k_2 \|\varphi\|^2_{1/2,\Gamma}$$

*for all $\varphi \in H_0^{1/2}(\Gamma)$ where,*

$$H_0^{1/2}(\Gamma) \;:=\; \left\{ \varphi \in H^{1/2}(\Gamma) \;:\; \quad \int_\Gamma \varphi = 0 \right\}.$$

**Lemma 4.2.** *$\mathcal{H}(1) = 0$, $\mathcal{D}(1) = -1/2$ and $\int_\Gamma \mathcal{H}(\eta) = 0$, for all $\eta \in H^{1/2}(\Gamma)$.*

**Theorem 4.2.** *The linear operator $\mathcal{H} : H^{1/2}(\Gamma)/\mathbb{R} \to H_0^{-1/2}(\Gamma)$ defines an isomorphism.*

Let $(\mathbf{A}_C, \psi_I, \boldsymbol{\alpha})$ satisfying (12). Let $\psi(t) := \psi_I(t) - c(t)$, where $c : [0,T] \to \mathbb{R}$ such that $\psi(t) \in H_0^{1/2}(\Gamma)$. By using $(12)_3$ and $(12)_5$, according to Theorem 4.1 and Lemma 4.2, for all $t \in [0,T]$, we have

$$-\frac{1}{2}\psi \,-\, \mathcal{D}(\psi) \,+\, \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C) \,=\, -\psi_I \qquad \text{on } \Gamma, \tag{13}$$

$$\frac{1}{2\mu_0}\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C) + \mathcal{H}(\psi) = 0 \qquad \text{on } \Gamma. \tag{14}$$

The following is a variational formulation of problem (12), where $\mathcal{V} := \mathbf{H}(\mathbf{curl}\,;\Omega_C)$. For the ease of notation, we write the integration symbol on $\Gamma$ instead of the pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, as usual:
*Problem P.* Find $\mathbf{A}_C \in L^2(0,T;\mathcal{V})\cap H^1(0,T;(L^2(\Omega_C))^3)$, $\psi \in L^2(0,T;H_0^{1/2}(\Gamma))$ and $\boldsymbol{\alpha} \in \mathcal{C}^0(0,T;\mathbb{R}^L)$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{A}_C\cdot\mathbf{w}_C\,\mathrm{d}\mathbf{x} + \int_{\Omega_C} \boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{curl}\,\mathbf{w}_C\,\mathrm{d}\mathbf{x}\\
&+\int_\Gamma \Big[-\frac{1}{2}\psi \,-\, \mathcal{D}(\psi) \,+\, \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C)\Big]\mathbf{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C\,\mathrm{d}\zeta\\
&+\int_\Gamma (\mathbf{w}_C\times\mathbf{n}_C)\cdot(\mathbf{Z}^t\boldsymbol{\alpha})\,\mathrm{d}\zeta \qquad\qquad = \int_{\Omega_C}\mathbf{J}\cdot\mathbf{w}_C\,\mathrm{d}\mathbf{x},\\
&\int_\Gamma \Big[\frac{1}{2}\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C \,+\, \mathcal{D}'(\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C) \,+\, \mu_0\,\mathcal{H}(\psi)\Big]\eta\,\mathrm{d}\zeta \,=\, 0,\\
&\mu_0\,\boldsymbol{\beta}^t\,\mathbf{N}\,\boldsymbol{\alpha} - \boldsymbol{\beta}^t\int_\Gamma \mathbf{Z}\,(\mathbf{A}_C\times\mathbf{n}_C)\,\mathrm{d}\zeta\\
&\qquad\qquad = \mu_0\,\boldsymbol{\beta}^t\,\mathbf{N}\,\boldsymbol{\alpha}_0 \,-\, \boldsymbol{\beta}^t\textstyle\int_\Gamma \mathbf{Z}\,(\mathbf{A}_{C,0}\times\mathbf{n}_C)\,\mathrm{d}\zeta,
\end{aligned} \tag{15}$$

for all $\mathbf{w}_C \in \mathcal{V}$, $\eta \in H_0^{1/2}(\Gamma)$, $\boldsymbol{\beta}\in\mathbb{R}^L$ with

$$\mathbf{A}_C(0) = \mathbf{A}_{C,0} \quad \text{in} \quad \Omega_C.$$

In fact, to derive $(15)_1$, we have multiplied (12), by $\mathbf{w}_C$, integrated by parts in $\Omega_C$ and used that

$$\int_\Gamma \mathbf{n}_I\times\nabla\psi_I\cdot\mathbf{w}_C\,\mathrm{d}\zeta \,=\, \int_\Gamma \psi_I\;\mathbf{curl}\,\mathbf{w}_C\cdot\mathbf{n}_C\,\mathrm{d}\zeta, \tag{16}$$

which follows by integration by parts too. In its turn, equations $(15)_2$ and $(15)_3$ follows directly from (14) and $(12)_2$, respectively.

For the theoretical analysis it is convenient to eliminate $\boldsymbol{\alpha}$ and $\psi$ from the previous formulation. With this aim, we introduce the linear operator $\mathbb{T} : \mathcal{V} \to \mathbb{R}^L$ defined by

$$\mathbb{T}\left(\mathbf{w}_C\right) := \int_\Gamma \mathbf{Z}(\mathbf{w}_C \times \mathbf{n}_C)\,\mathrm{d}\zeta.$$

We can eliminate easily $\boldsymbol{\alpha}$ from $(15)_3$ and replace it in $(15)_1$. Then, the fourth term of this equation reads

$$\begin{aligned}\int_\Gamma (\mathbf{w}_C \times \mathbf{n}_C)\cdot(\mathbf{Z}^t\boldsymbol{\alpha})\,\mathrm{d}\zeta \;&=\; \big(\mathbb{T}\left(\mathbf{w}_C\right)\big)^t\,\boldsymbol{\alpha}\\ &=\; \mu_0\,\big(\mathbb{T}\left(\mathbf{w}_C\right)\big)^t\,\mathbf{N}^{-1}\,\mathbb{T}\left(\mathbf{A}_C\right) + \big(\mathbb{T}\left(\mathbf{w}_C\right)\big)^t\,\boldsymbol{\alpha}_0\\ &\quad -\mu_0\,\big(\mathbb{T}\left(\mathbf{w}_C\right)\big)^t\,\mathbf{N}^{-1}\,\mathbb{T}\left(\mathbf{A}_{C,0}\right).\end{aligned}$$

Moreover, we introduce the operator $\mathcal{R} : H_0^{-1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$ given by

$$\int_\Gamma \mathcal{H}(\mathcal{R}(\xi))\,\eta\,\mathrm{d}\zeta = \int_\Gamma \xi\,\eta\,\mathrm{d}\zeta \qquad \forall\,\eta \in H_0^{1/2}(\Gamma),\ \forall\,\xi \in H_0^{-1/2}(\Gamma), \tag{17}$$

where $H_0^{-1/2}(\Gamma) := \{\eta \in H^{-1/2}(\Gamma) : \int_\Gamma \eta\,\mathrm{d}\zeta = 0\}$. It is straightforward to show from Lemma 4.1 and the Lax-Milgram lemma that $\mathcal{R}$ is well defined and bounded. Therefore, the second equation of (15) may be equivalently written $\psi = -\mu_0^{-1}\mathcal{R}\Big(\frac{1}{2}\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C + \mathcal{D}'(\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C)\Big)$. Note that $\mathcal{R}$ is a self-adjoint and non-negative operator. Consequently, (15) admits the following equivalent reduced form:

Find $\mathbf{A}_C \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;(L^2(\Omega_C))^3)$ such that

$$\begin{aligned}\tfrac{d}{dt}(\mathbf{A}_C(t), \mathbf{w}_C)_{\boldsymbol{\sigma}} + \mathcal{A}(\mathbf{A}_C(t), \mathbf{w}_C) + \mathcal{B}(\mathbf{A}_C(t), \mathbf{w}_C)&\\ = (\mathbf{J}(t), \mathbf{w}_C)_{0,\Omega_C} + \boldsymbol{g}(\mathbf{w}_C),&\end{aligned} \tag{18}$$

for all $\mathbf{w}_C \in \mathcal{V}$, with

$$\mathbf{A}_C(0) = \mathbf{A}_{C,0} \quad \text{in} \quad \Omega_C,$$

where

$$(\mathbf{H}, \mathbf{G})_{\boldsymbol{\sigma}} := \int_{\Omega_C} \boldsymbol{\sigma}\,\mathbf{H}\cdot\mathbf{G}\,\mathrm{d}\mathbf{x}, \qquad \forall\ \mathbf{H}, \mathbf{G} \in (L^2(\Omega_C))^3,$$

$$
\begin{aligned}
&\mathcal{K}:\mathcal{V}\to H_0^{-1/2}(\Gamma), && \mathcal{K}(\mathbf{H}) := \frac{1}{2}\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}_C + \mathcal{D}'(\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}_C),\\
&\mathcal{A}:\mathcal{V}\times\mathcal{V}\to\mathbb{R}, && \mathcal{A}(\mathbf{H},\mathbf{G}) := \int_{\Omega_C}\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{H}\cdot\mathbf{curl}\,\mathbf{G}\,\mathrm{d}\mathbf{x}\\
& && \qquad + \mu_0^{-1}\int_\Gamma \mathcal{S}(\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}_C)\,\mathbf{curl}\,\mathbf{G}\cdot\mathbf{n}_C\,\mathrm{d}\zeta,\\
&\mathcal{B}:\mathcal{V}\times\mathcal{V}\to\mathbb{R}, && \mathcal{B}(\mathbf{H},\mathbf{G}) := \mu_0^{-1}\int_\Gamma \mathcal{K}(\mathbf{G})\mathcal{R}(\mathcal{K}(\mathbf{H}))\,\mathrm{d}\zeta\\
& && \qquad + \mu_0\left(\mathbb{T}(\mathbf{G})\right)^t\mathbf{N}^{-1}\,\mathbb{T}(\mathbf{H}),\\
&\boldsymbol{g}:\mathcal{V}\to\mathbb{R}, && \boldsymbol{g}(\mathbf{H}) := \mu_0^{-1}\left(\mathbb{T}(\mathbf{H})\right)^t\mathbf{N}^{-1}\,\mathbb{T}(\mathbf{A}_{C,0}) - \left(\mathbb{T}(\mathbf{H})\right)^t\boldsymbol{\alpha}_0.
\end{aligned}
$$

Note that $\mathcal{A}$ and $\mathcal{B}$ are bounded, symmetric and non-negative definite bilinear forms.

**Remark 4.1.** *Note that $\|\cdot\|_{0,\Omega_C}$ is equivalent to $\|\cdot\|_{\boldsymbol{\sigma}}$ in $\Omega_C$, and therefore, $\|\cdot\|_{\mathcal{V}}$ is equivalent to $\|\cdot\|_{\boldsymbol{\sigma}} + \|\mathbf{curl}\,(\cdot)\|_{0,\Omega_C}$ in $\Omega_C$.*

*4.1. Existence and Uniqueness.*

**Lemma 4.3.** *There exists a unique solution to* (18) *and*

$$
\sup_{t\in[0,T]}\|\mathbf{A}_C(t)\|_{\boldsymbol{\sigma}}^2 \le C\left\{\|\boldsymbol{J}\|^2_{L^2(0,T;\,(L^2(\Omega_C))^3)} + \|\mathbf{A}_{C,0}\|^2_{\mathcal{V}} + |\boldsymbol{\alpha}_0|^2\right\} \tag{19}
$$

*for some constant $C>0$.*

Proof. **Uniqueness.** First, we prove that any solution to (18) has to satisfy (19). To this end, we replace $\mathbf{w}_C$ by $\mathbf{A}_C$ in (18), recall that the bilinear forms $\mathcal{A}$ and $\mathcal{B}$ are non-negative definite and moreover,

$$
\mathcal{A}(\mathbf{A}_C,\mathbf{A}_C) \ge \mu_1^{-1}\|\mathbf{curl}\,\mathbf{A}_C\|^2_{0,\Omega_C}, \tag{20}
$$

where $\mu_1$ is a uniform upper bound in $\Omega_C$ for the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$.

Then we apply the Cauchy-Schwarz inequality and obtain

$$
\begin{aligned}
\frac{1}{2}\partial_t\|\mathbf{A}_C\|^2_{\boldsymbol{\sigma}} + \mu_1^{-1}\|\mathbf{curl}\,\mathbf{A}_C\|^2_{0,\Omega_C} \le \|\mathbf{J}\|_{0,\Omega_C}\|\mathbf{A}_C\|_{0,\Omega_C} + C_1\|\mathbf{A}_C\|_{\mathcal{V}}\|\mathbf{A}_{C,0}\|_{\mathcal{V}}\\
+C_2\|\mathbf{A}_C\|_{\mathcal{V}}|\boldsymbol{\alpha}_0|.
\end{aligned}
$$

Now, using Remark 4.1 and a Young's inequality, we obtain

$$\frac{1}{2}\partial_t\|\mathbf{A}_C\|^2_{\boldsymbol{\sigma}} \,\leq\, C\Big\{\|\mathbf{J}\|^2_{0,\Omega_C} \,+\, \|\mathbf{A}_{C,0}\|^2_{\mathcal{V}} \,+\, |\boldsymbol{\alpha}_0|^2\Big\} \,+\, \|\mathbf{A}_C\|^2_{\boldsymbol{\sigma}}.$$

Hence, integrating in time and using Gronwall's inequality, we obtain

$$\|\mathbf{A}_C(t)\|^2_{\boldsymbol{\sigma}} \,\leq\, C\left\{\int_0^T \|\mathbf{J}\|^2_{0,\Omega_C}\,\mathrm{d}s \,+\, \|\mathbf{A}_{C,0}\|^2_{\mathcal{V}} \,+\, |\boldsymbol{\alpha}_0|^2\right\} \qquad \forall t\in[0,T],$$

from which we conclude the uniqueness.

**Existence.** We fix $N\in\mathbb{N}$ and consider a uniform partition $\{t_n := n\triangle t\ :\ n=0,...,N\}$ of $[0,T]$ with step size $\triangle t := \frac{T}{N}$. For any finite sequence $\{\theta^n\,:\,n=0,...,N\}$, let

$$\overline{\partial}\theta^n \,:=\, \frac{\theta^n-\theta^{n-1}}{\triangle t}, \qquad n=1,...,N. \tag{21}$$

Moreover, we define

$$\mathbf{J}^n_N(\mathbf{x}) \,:=\, \frac{1}{\triangle t}\int_{(n-1)\,\triangle t}^{n\,\triangle t}\mathbf{J}(\mathbf{x},t)\,\mathrm{d}t \qquad \text{a.e. in } \Omega_C,\ \ n=1,...,N. \tag{22}$$

We approximate our problem by an implicit time discretization scheme. Find $\mathbf{A}^n_{CN}\in\mathcal{V}$ for $n=1,...,N$, such that

$$\begin{aligned}(\overline{\partial}\mathbf{A}^n_{CN},\mathbf{w}_C)_{\boldsymbol{\sigma}} \,+\, \mathcal{A}(\mathbf{A}^n_{CN},\mathbf{w}_C) \,+\, \mathcal{B}(\mathbf{A}^n_{CN},\mathbf{w}_C)&\\ =\, (\mathbf{J}^n_N,\mathbf{w}_C)_{0,\Omega_C} \,+\, \boldsymbol{g}(\mathbf{w}_C),&\end{aligned} \tag{23}$$

for all $\mathbf{w}_C\in\mathcal{V}$, with

$$\mathbf{A}^0_{CN} := \mathbf{A}_{C,0}.$$

For any $n\in\{1,...,N\}$, we assume that $\mathbf{A}^1_{CN},...,\mathbf{A}^{n-1}_{CN}\in\mathcal{V}$ are known, and consider the problem of determining $\mathbf{A}^n_{CN}$. It is clear that the Lax-Milgram theorem ensures existence and uniqueness of the solution for $n=1,...,N$.

Replacing $\mathbf{w}_C$ by $\mathbf{A}^n_{CN}$ in (23), using that $\mathcal{A}$ and $\mathcal{B}$ are non-negative definite, (20), a Young's inequality and the fact that

$$(\overline{\partial}\mathbf{A}^k_{CN},\mathbf{A}^k_{CN})_{\boldsymbol{\sigma}} \,\geq\, \frac{1}{2\triangle t}\left(\|\mathbf{A}^k_{CN}\|^2_{\boldsymbol{\sigma}} - \|\mathbf{A}^{k-1}_{CN}\|^2_{\boldsymbol{\sigma}}\right), \tag{24}$$

we obtain

$$\begin{aligned}\frac{1}{2\triangle t}\left(\|\mathbf{A}_{CN}^k\|_{\boldsymbol{\sigma}}^2\right. &- \left.\|\mathbf{A}_{CN}^{k-1}\|_{\boldsymbol{\sigma}}^2\right) + \tfrac{\mu_1^{-1}}{2}\|\mathbf{curl}\,\mathbf{A}_{CN}^k\|_{0,\Omega_C}^2 \\ &\le \frac{1}{4T}\|\mathbf{A}_{CN}^k\|_{\boldsymbol{\sigma}}^2 + C\Big\{\|\mathbf{J}_N^k\|_{0,\Omega_C}^2 + \|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\Big\}.\end{aligned}$$

In particular, we have

$$\|\mathbf{A}_{CN}^k\|_{\boldsymbol{\sigma}}^2 - \|\mathbf{A}_{CN}^{k-1}\|_{\boldsymbol{\sigma}}^2 \le \frac{\triangle t}{2T}\|\mathbf{A}_{CN}^k\|_{\boldsymbol{\sigma}}^2 + C\triangle t\Big\{\|\mathbf{J}_N^k\|_{0,\Omega_C}^2 + \|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\Big\}.$$

Then, summing over $k$, the discrete Gronwall's Lemma (see, for instance, Lemma 1.4.2 from Quarteroni & Valli, 1994) leads to

$$\|\mathbf{A}_{CN}^n\|_{\boldsymbol{\sigma}}^2 \le C\left\{\|\mathbf{A}_{CN}^0\|_{\boldsymbol{\sigma}}^2 + \triangle t\sum_{k=1}^n \|\mathbf{J}_N^k\|_{0,\Omega_C}^2 + \|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\right\}. \quad (25)$$

Thus, from (22) since

$$\triangle t\sum_{k=1}^n \|\mathbf{J}_N^k\|_{0,\Omega_C}^2 \le \int_0^T \|\mathbf{J}(t)\|_{0,\Omega_C}^2\,\mathrm{d}t,$$

we obtain that $\|\mathbf{A}_{CN}^n\|_{\boldsymbol{\sigma}}^2 \le C$, with $C$ independent of $N$.

Note that being $\mathcal{A}$ non-negative and symmetric, it is easy to check that

$$\mathcal{A}(\mathbf{A}_{CN}^k, \mathbf{A}_{CN}^k - \mathbf{A}_{CN}^{k-1}) \ge \frac{1}{2}\left(\mathcal{A}(\mathbf{A}_{CN}^k, \mathbf{A}_{CN}^k) - \mathcal{A}(\mathbf{A}_{CN}^{k-1}, \mathbf{A}_{CN}^{k-1})\right) \quad (26)$$

and similarly for $\mathcal{B}$. Let us now take $\mathbf{w}_C = \mathbf{A}_{CN}^k - \mathbf{A}_{CN}^{k-1}$ in (23). Summing for $k = 1, ..., n$, for any $n \in \{1, ..., N\}$, recalling that $\mathcal{B}$ is non-negative definite and $\mathcal{A}$ satisfies (20), we obtain

$$\begin{aligned}\triangle t\sum_{k=1}^n &\left\|\overline{\partial}\mathbf{A}_{CN}^k\right\|_{\boldsymbol{\sigma}}^2 + \frac{1}{2}\mu_1^{-1}\|\mathbf{curl}\,\mathbf{A}_{CN}^n,\|_{0,\Omega_C} \\ &\le \left(\triangle t\sum_{k=1}^n \|\mathbf{J}_N^k\|_{0,\Omega_C}^2\right)^{\frac{1}{2}}\left(\triangle t\sum_{k=1}^n \left\|\overline{\partial}\mathbf{A}_{CN}^k\right\|_{0,\Omega_C}^2\right)^{\frac{1}{2}} \\ &\qquad +\mu_0^{-1}\left(\mathbb{T}\left(\mathbf{A}_{CN}^n - \mathbf{A}_{CN}^0\right)\right)^t \mathbf{N}^{-1}\,\mathbb{T}\left(\mathbf{A}_{CN}^0\right) - \left(\mathbb{T}\left(\mathbf{A}_{CN}^n - \mathbf{A}_{CN}^0\right)\right)^t \boldsymbol{\alpha}_0 \\ &\qquad +\mathcal{A}(\mathbf{A}_{CN}^0, \mathbf{A}_{CN}^0) + \mathcal{B}(\mathbf{A}_{CN}^0, \mathbf{A}_{CN}^0).\end{aligned}$$

Applying Young's inequalities, recalling that $\mathbb{T}$ is continuous and Remark 4.1, we obtain that

$$\begin{aligned}
&\triangle t \sum_{k=1}^{n} \left\|\overline{\partial}\mathbf{A}_{CN}^{k}\right\|_{\boldsymbol{\sigma}}^{2} + \mu_1^{-1}\|\mathbf{curl}\,\mathbf{A}_{CN}^{n},\|_{0,\Omega_C}^{2} \\
&\qquad \leq C\left\{\triangle t \sum_{k=1}^{n} \|\mathbf{J}_N^k\|_{0,\Omega_C}^2 + \|\mathbf{A}_{CN}^0\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\right\} + \|\mathbf{A}_{CN}^n\|_{\boldsymbol{\sigma}}^2.
\end{aligned}$$

Hence, using (25), we conclude that

$$\triangle t \sum_{k=1}^{N} \left\|\overline{\partial}\mathbf{A}_{CN}^{k}\right\|_{\boldsymbol{\sigma}} + \max_{n=1,\dots,N} \|\mathbf{A}_{CN}^n\|_{\mathcal{V}} \leq C, \tag{27}$$

where $C$ is independent of $N$.

Let us introduce some further notation. Let $\overline{\mathbf{A}}_{CN}(\mathbf{x},t)$ be the piecewise constant in time function defined by

$$\overline{\mathbf{A}}_{CN}(\mathbf{x},t) := \mathbf{A}_{CN}^n(\mathbf{x}) \qquad \text{if } (n-1)\triangle t < t \leq n\triangle t, \text{ for } n = 1, ..., N.$$

We define $\overline{\mathbf{J}}_N$ similarly. Let $\widehat{A}_{CN}(\mathbf{x},t)$ be the piecewise linear and continuous time intepolant of the values $\mathbf{A}_{CN}^n(\mathbf{x})$, $n = 0, ..., N$ (namely, $\widehat{\mathbf{A}}_{CN}(\mathbf{x},t_n) = \mathbf{A}_{CN}^n(\mathbf{x})$, $n = 0, ..., N$). Thus, we can write (23) and (27) as follows:

$$\begin{aligned}
(\partial_t\widehat{\mathbf{A}}_{CN}, \mathbf{w}_C)_{\boldsymbol{\sigma}} + \mathcal{A}(\overline{\mathbf{A}}_{CN}, \mathbf{w}_C) + \mathcal{B}(\overline{\mathbf{A}}_{CN}, \mathbf{w}_C)& \\
= (\overline{\mathbf{J}}_N, \mathbf{w}_C)_{0,\Omega_C} + \boldsymbol{g}(\mathbf{w}_C)&
\end{aligned} \tag{28}$$

and

$$\|\widehat{\mathbf{A}}_{CN}\|_{H^1(0,T;\,(L^2(\Omega_C))^3)\cap L^\infty(0,T;\mathcal{V})} \leq C \quad \text{and} \quad \|\overline{\mathbf{A}}_{CN}\|_{L^\infty(0,T;\mathcal{V})} \leq C \tag{29}$$

a.e. in $[0,T]$. From these estimates, we conclude that there exists $\mathbf{A}_C$ such that, possibly taking $N \to \infty$ along a subsequence,

$$\begin{aligned}
&\widehat{\mathbf{A}}_{CN} \to \mathbf{A}_C \quad \text{weakly star in } H^1(0,T;\,(L^2(\Omega_C))^3) \cap L^\infty(0,T;\mathcal{V}), \\
&\overline{\mathbf{A}}_{CN} \to \mathbf{A}_C \quad \text{weakly star in } L^\infty(0,T;\mathcal{V}).
\end{aligned}$$

So, by taking $N \to \infty$ in (28), since $\overline{\mathbf{J}}_N \to \mathbf{J}$ in $(L^2(\Omega_C))^3$, we obtain (18) in the sense of $L^2(0,T;\,(L^2(\Omega_C))^3)$. □

**Remark 4.2.** *Problem* (15) *and* (23) *are actually equivalent. In fact, for* $\mathbf{A}_C$ *solution of* (23)*, if we define* $\psi = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\mathbf{A}_C))$ *and* $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \mu_0^{-1}\boldsymbol{N}^{-1}(\mathbb{T}(\mathbf{A}_C) - \mathbb{T}(\mathbf{A}_{C,0}))$*, then* $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ *is a solution of* (15)*. Moreover this problem has a unique solution, because* $\mathbf{A}_C$ *has to be the unique solution of* (18) *and* $\psi$ *and* $\boldsymbol{\alpha}$ *are determined via* $(15)_2$ *and* $(15)_3$*, respectively.*

**Theorem 4.3.** *Let* $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ *be the solution of problem* (15)*. Then there exists* $\psi_I \in L^2(0,T; W^1(\Omega_I))$ *and a function* $c : [0,T] \to \mathbb{R}$ *such that* $\psi = \psi_I|_\Gamma - c$ *and* $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ *satisfies* (12)*.*

Proof. Testing $(15)_1$ with $\mathbf{w}_C \in (\mathcal{C}_0^\infty(\Omega_C))^3$ we obtain

$$\boldsymbol{\sigma}\partial_t \mathbf{A}_C + \mathbf{curl}\,(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C) = \mathbf{J} \quad \text{in} \quad \Omega_C.$$

Testing $(15)_2$ with $\eta \in H^{1/2}(\Gamma)$ and using Lemma 4.1 we recover

$$\frac{1}{2}\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(\psi) = 0 \quad \text{on} \quad \Gamma. \tag{30}$$

Now, let $\psi_I \in W^1(\Omega_I)$ be the solution of this problem:

$$\begin{aligned} \triangle \psi_I &= 0 && \text{in} \quad \Omega_I, \\ \mu_0 \nabla \psi_I \cdot \mathbf{n}_I &= -\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C && \text{on} \quad \Gamma, \\ |\psi_I(\mathbf{x})| + |\nabla \psi_I(\mathbf{x})| &= \mathrm{O}(|\mathbf{x}|^{-1}) && \text{as} \quad |\mathbf{x}| \to \infty. \end{aligned} \tag{31}$$

Since $\psi_I \in W^1(\Omega_I)$ is a harmonic function, then Theorem 4.1 ensures that

$$\begin{aligned} &\tfrac{1}{2}\psi_I|_\Gamma - \mathcal{D}(\psi_I|_\Gamma) + \mathcal{S}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) = 0 && \text{and} \\ &\tfrac{1}{2}\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) + \mathcal{H}(\psi_I|_\Gamma) = 0 && \text{on} \quad \Gamma. \end{aligned} \tag{32}$$

Now, subtracting (30) to $(32)_2$ we obtain $\mathcal{H}(\psi - \psi_I) = 0$ on $\Gamma$. Therefore, we conclude from Theorem 4.2 that $\psi_I(t) = \psi(t) + c(t)$ on $\Gamma$, where $c(t)$ is a constant. As a consequence,

$$\begin{aligned} -\tfrac{1}{2}\psi \quad & - \mathcal{D}(\psi) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) \\ &= -\frac{1}{2}(\psi_I|_\Gamma - c) - \mathcal{D}(\psi_I|_\Gamma - c) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) \\ &= -\psi_I|_\Gamma. \end{aligned} \tag{33}$$

Now replacing this equality in $(15)_1$, using (16) and testing $(15)_1$ with $\mathbf{w}_C \in \mathbf{H}(\mathbf{curl}\,; \Omega_C)$, then we obtain

$$(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{A}_C) \times \mathbf{n}_C + (\nabla \psi_I + \mathbf{Z}^t \boldsymbol{\alpha}) \times \mathbf{n}_I = \mathbf{0} \quad \text{on} \quad \Gamma.$$

$\square$

## 5. Analysis of a fully-discrete scheme

Let $\{\mathcal{T}_h(\Omega_C)\}_h$ be a regular family of tetrahedral meshes of $\Omega_C$. As usual, $h$ stands for the largest diameter of the tetrahedra $K$ in $\mathcal{T}_h(\Omega_C)$. Furthermore, we consider the family of triangulations induced on $\Gamma$, $\{\mathcal{T}_h(\Gamma)\}_h$.

We define a fully-discrete version of (15) by means of Nédélec finite elements. The local representation on $K$ of the lowest order Nédélec finite element is given by

$$\mathcal{ND}(K) := \{\mathbf{a} \times \mathbf{x} + \mathbf{b} \,:\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

The corresponding global space $\mathcal{V}_h$ is the space of vector fields that are locally in $\mathcal{ND}(K)$ for all $K$ in $\Omega_C$, and globally in $\mathcal{V} = \mathbf{H}(\mathbf{curl}\,; \Omega_C)$. Moreover, we define

$$\mathcal{L}_h(\Gamma) := \{\eta \in H_0^{1/2}(\Gamma) \,:\, \eta|_T \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h(\Gamma)\}$$

which approximates the space $H_0^{1/2}(\Gamma)$, where $\mathbb{P}_1(K)$ is the set of polynomials of degree not greater than one.

Since $\Omega_C$ is not simply connected, problem (15) involve the matrices $\mathbf{N}$ and $\mathbf{Z}$ defined by (11) and (10), respectively. To compute these matrices we need to approximate numerically the basis $\{\widetilde{\nabla} z_k\}_{k=1}^L$ of the harmonic Neumann vector-fields $\mathbb{H}(\Omega_I)$. A similar need arose in [11], where the authors proposed a coupled BEM-FEM method to compute the entries of a matrix $\mathbf{N}_h$ approximating $\mathbf{N}$. For the sake of completeness, in what follows we describe briefly the method introduced in [11] to approximate $\mathbf{N}$ and the corresponding error estimates proved in this reference.

Consider a connected and simply connected polyhedron $\Omega$ with a connected boundary such that $\overline{\Omega}_C \cup \left(\bigcup_{k=1}^L \overline{\Sigma}_k^{ext}\right) \subset \Omega$. Set

$$\mathcal{Q}^0 := \Omega \setminus \left\{\overline{\Omega}_C \cup \left(\bigcup_{k=1}^{L} \overline{\Sigma_k^{ext}}\right)\right\}, \qquad \mathcal{Q} := \Omega \setminus \overline{\Omega}_C \quad \text{and} \quad \Lambda := \partial\Omega.$$

From (6), $\mathbf{p}_k := \widetilde{\nabla}\, z_k|_{\mathcal{Q}}$ belongs to the closed subspace of $\mathbf{H}(\text{div};\, \mathcal{Q})$

$$\mathcal{Y} := \{\mathbf{q} \in (L^2(\mathcal{Q}))^3 \,:\, \operatorname{div} \mathbf{q} = 0 \text{ in } \mathcal{Q} \text{ and } \mathbf{q}|_\Gamma \cdot \mathbf{n}_I = 0 \text{ in } H^{-1/2}(\Gamma)\}$$

and satisfies the variational equation

$$\int_{\mathcal{Q}} \mathbf{p}_k \cdot \mathbf{q} \,\mathrm{d}\mathbf{x} \;=\; \int_{\Sigma_k^{ext}} \mathbf{q} \cdot \mathbf{n}_k \,\mathrm{d}\zeta \;+\; \int_{\Lambda} \mathbf{q} \cdot \mathbf{n}\, z_k \,\mathrm{d}\zeta \qquad \forall \mathbf{q} \in \mathcal{Y},$$

where $\mathbf{n}$ correspond to the normal vector on $\Lambda$ outer to $\mathcal{Q}$. Furthermore, as $z_k$ is harmonic in $\mathbb{R}^3 \setminus \overline{\Omega}$, the last equation may be coupled with boundary integral equations relating $z_k$ and its normal derivative $\mathbf{p}_k \cdot \mathbf{n}$ on $\Lambda$. This leads to the following weak formulation (see [12] for more details)

Find $\mathbf{p}_k \in \mathcal{Y}$ and $\phi_k \in H^{1/2}(\Lambda)/\mathbb{R}$ such that

$$\begin{aligned} \int_{\mathcal{Q}} \mathbf{p}_k \cdot \mathbf{q} \,\mathrm{d}\mathbf{x} \;+\; \int_{\Lambda} \mathcal{S}(\mathbf{p}_k \cdot \mathbf{n}) \mathbf{q} \cdot \mathbf{n} \,\mathrm{d}\zeta \;-\; \int_{\Lambda} \Big[ \frac{1}{2}\phi_k + \mathcal{D}(\phi_k) \Big] \mathbf{q} \cdot \mathbf{n} \,\mathrm{d}\zeta & \\ = \int_{\Sigma_k^{ext}} \mathbf{q} \cdot \mathbf{n}_k \,\mathrm{d}\zeta, & \qquad (34) \\ \int_{\Lambda} \Big[ \frac{1}{2}\mathbf{p}_k \cdot \mathbf{n} + \mathcal{D}'(\mathbf{p}_k \cdot \mathbf{n}) \Big] \chi \,\mathrm{d}\zeta \;+\; \int_{\Lambda} \mathcal{H}(\phi_k) \chi \,\mathrm{d}\zeta = 0, & \end{aligned}$$

for all functions $\mathbf{q} \in \mathcal{Y}$ and $\chi \in H^{1/2}(\Lambda)/\mathbb{R}$. The variable $\phi_k$ represents (up to and additive constant) the trace of $z_k$ on $\Lambda$. Now, consider a regular family of triangulations $\{\mathcal{T}_h(\mathcal{Q})\}_h$ of $\mathcal{Q}$ by tetrahedra $K$ of diameter no greater than $h > 0$. Assume that, for any $h$, the set $\mathcal{T}_h(\Omega_C) \cup \mathcal{T}_h(\mathcal{Q})$ is a triangulation of $\Omega$. This implies that the triangulation induced by $\mathcal{T}_h(\mathcal{Q})$ on $\Gamma$ is identical to $\mathcal{T}_h(\Gamma)$. It can be assumed, without loss of generality, that the cutting surfaces $\Sigma_k^{ext}$ are union of faces of tetrahedra $T \in \mathcal{T}_h(\mathcal{Q})$ for each mesh $\mathcal{T}_h(\mathcal{Q})$. Finally, denote by $\mathcal{T}_h(\Lambda)$ the triangulation induced by $\mathcal{T}_h(\mathcal{Q})$ on $\Lambda$.

Consider a conforming discretization of $\mathbf{H}(\mathrm{div}\,;\Omega)$

$$\mathcal{RT}_h(\mathcal{Q}) \;:=\; \{\mathbf{q} \in \mathbf{H}(\mathrm{div}\,;\,\Omega) \;:\; \mathbf{q}|_T \in \mathcal{RT}(K) \;\; \forall K \in \mathcal{T}_h(\mathcal{Q})\}$$

where $\mathcal{RT}(K) \;:=\; \{a\mathbf{x} + \mathbf{b} :\;\; a \in \mathbb{R},\;\; \mathbf{b} \in \mathbb{R}^3\}$.

The following is a convenient way of discretizing problem (34) (for more details, see [11]):

Find $\mathbf{p}_{kh} \in \mathcal{RT}_h^0(\mathcal{Q})$, $\phi_{kh} \in \Phi_h/\mathbb{R}$ and $\beta_{kh} \in M_h$ such that

$$
\begin{aligned}
&\int_{\mathcal{Q}} \mathbf{p}_{kh} \cdot \mathbf{q} \,\mathrm{d}\mathbf{x} + \int_{\Lambda} \mathcal{S}(\mathbf{p}_{kh} \cdot \mathbf{n})\mathbf{q} \cdot \mathbf{n} \,\mathrm{d}\zeta - \int_{\Lambda} \Big[\frac{1}{2}\phi_{kh} + \mathcal{D}(\phi_{kh})\Big]\mathbf{q} \cdot \mathbf{n} \,\mathrm{d}\zeta \\
&+ \int_{\mathcal{Q}} \beta_{kh} \mathrm{div}\mathbf{q} \qquad\qquad\qquad\qquad\qquad\qquad = \int_{\Sigma_k^{ext}} \mathbf{q} \cdot \mathbf{n}_k \,\mathrm{d}\zeta, \\
&\int_{\Lambda} \Big[\frac{1}{2}\chi + \mathcal{D}(\chi)\Big]\mathbf{p}_{kh} \cdot \mathbf{n} \,\mathrm{d}\zeta + \int_{\Lambda} \mathcal{S}(\mathbf{curl}_\tau \phi_{kh})\, \mathbf{curl}_\tau \chi \,\mathrm{d}\zeta = 0, \\
&\int_{\mathcal{Q}} \mathrm{div}\, \mathbf{p}_{kh}\, v \,\mathrm{d}\mathbf{x} = 0,
\end{aligned}
\tag{35}
$$

for all functions $\mathbf{q} \in \mathcal{RT}_h^0(\mathcal{Q})$, $\chi \in \Phi_h/\mathbb{R}$, and $v \in M_h$, where

$$
\begin{aligned}
&\mathcal{RT}_h^0(\mathcal{Q}) := \{\mathbf{q} \in \mathcal{RT}_h(\mathcal{Q}) \,:\, \mathbf{q}|_\Gamma \cdot \mathbf{n} = 0\}, \\
&\Phi_h := \{\eta \in \mathcal{C}^0(\Lambda) \,:\, \eta|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h(\Lambda)\}, \\
&M_h := \{v \in L^2(\mathcal{Q}) \,:\, v|_{\mathcal{Q}} \in \mathbb{P}_0(\mathcal{Q})\ \forall K \in \mathcal{T}_h(\mathcal{Q})\}.
\end{aligned}
$$

We know from [12] that (35) is a well posed problem. Once the function $\mathbf{p}_{kh}$ computed for $1 \le k \le L$, the matrix $\mathbf{N}$ can be approximated by

$$
\mathbf{N}_h := \left( \int_{\Sigma_j^{ext}} \mathbf{p}_{kh} \cdot \mathbf{n}_j \,\mathrm{d}\zeta \right)_{1 \le k,j \le L}. \tag{36}
$$

Note that this matrix is symmetric and positive definite. Error estimates for the approximation $\mathbf{N}_h$ of $\mathbf{N}$ has been obtained in [11]. With this end, the following additional regularity result has been proved in [11]; here and thereafter.

In the sequel, we denote by $s_{\mathcal{Q}} \in (1/2, 1]$ the exponent of maximal regularity in $\mathcal{Q}$ of the solution of Laplace operators with $L^2(\mathcal{Q})$ right-hand side and homogeneous Neumann boundary datum.

**Theorem 5.1.** *Let* $(\boldsymbol{p}_k, \phi_k)$ *be the solution to problem* (34). *There exists* $1/2 < s < s_{\mathcal{Q}}$ *such that* $\boldsymbol{p}_k \in (H^s(\mathcal{Q}))^3$, $k = 1, ..., L$.

Proof. See [11, Theorem 7.1]. □

Finally we recall the error estimates obtained in [11]:

**Theorem 5.2.** *Problems* (34) *and* (35) *are well posed and there exists a constant* $C > 0$ *independent of* $h$ *such that*

$$\|\boldsymbol{p}_k - \boldsymbol{p}_{kh}\|_{0,\mathcal{Q}} + \|\phi_k - \phi_{kh}\|_{H^{1/2}(\Lambda)/\mathbb{R}} \leq C\, h^s \left\{ \|\boldsymbol{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda} \right\}$$

*for all* $1/2 < s < s_{\mathcal{Q}}$.

Proof. See [11, Theorem 7.2]. □

**Theorem 5.3.** *There exists* $h_0 > 0$ *such that* $\boldsymbol{N}_h$ *is invertible for all* $0 < h \leq h_0$. *Moreover, we have the error estimate*

$$\|\boldsymbol{N} - \boldsymbol{N}_h\| \;+\; \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\| \;\leq\; C\, h^s \max_{1\leq k\leq L} \left\{ \|\boldsymbol{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda} \right\},$$

*for some constant* $C$ *independent of* $h$.

Proof. See [11, Corollary 7.3]. □

Notice that $\|\phi_k\|_{s+1/2,\Lambda}$ is clearly bounded, since $\phi_k$ is the trace on $\Lambda$ of the harmonic function $z_k$. To compute an approximation of the entries of $\mathbf{Z}$, we need to resort to a different strategy. In fact, the previous methods yields good approximation of $\mathbf{p}_k|_\Gamma \cdot \mathbf{n}_I = \widetilde{\nabla} z_k|_\Gamma \cdot \mathbf{n}_I$, but not of $\widetilde{\nabla} z_k|_\Gamma \times \mathbf{n}_I$ (which are the terms defining the entries of $\mathbf{Z}$). A similar situation happened in [11], too. However, in this case, we follow an alternative approach that we think is simpler.

One can see that the solution of (6) satisfies the variational formulation: Find $z_k \in H^1(\mathcal{Q}\backslash\Sigma_k^{ext})/\mathbb{R}$ such that

$$[\![ z_k ]\!]_{\Sigma_k^{ext}} = 1 \qquad \text{and} \qquad \int_{\mathcal{Q}\backslash\Sigma_k^{ext}} \nabla z_k \cdot \nabla\varphi \,\mathrm{d}\mathbf{x} \;=\; \int_\Lambda \mathbf{p}_k \cdot \mathbf{n}\,\varphi \,\mathrm{d}\zeta, \tag{37}$$

for all $\varphi \in H^1(\mathcal{Q})/\mathbb{R}$.

We introduce

$$\mathcal{L}_h(\mathcal{Q}) := \{\theta \in H^1(\mathcal{Q}) \,:\, \theta|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_h(\mathcal{Q})\},$$
$$\mathcal{L}_h(\mathcal{Q}\backslash\Sigma_k^{ext}) := \{\theta \in H^1(\mathcal{Q}\backslash\Sigma_k^{ext}) \,:\, \theta|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_h(\mathcal{Q})\}.$$

Consider the following discrete version of problem (37):

Find $z_{kh} \in \mathcal{L}_h(\mathcal{Q}\backslash\Sigma_k^{ext})/\mathbb{R}$ such that

$$[\![ z_{kh} ]\!]_{\Sigma_k^{ext}} = 1 \qquad \text{and} \qquad \int_{\mathcal{Q}\backslash\Sigma_k^{ext}} \nabla z_{kh} \cdot \nabla\varphi \,\mathrm{d}\mathbf{x} \;=\; \int_\Lambda \mathbf{p}_{kh} \cdot \mathbf{n}\,\varphi \,\mathrm{d}\zeta, \tag{38}$$

for all $\varphi \in \mathcal{L}_h(\mathcal{Q})/\mathbb{R}$.

**Lemma 5.1.** *Let $z_k$ and $z_{kh}$ be the solutions to problems* (37) *and* (38), *respectively. Then*

$$\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{kh}\|_{0,\mathcal{Q}} \leq C h^s$$

*for all $k = 1, ..., L$, with $C$ independent of $h$ and $s$ as in Theorem* 5.1.

Proof. Let $\widehat{z}_k \in \mathcal{C}^\infty(\mathcal{Q}\backslash\Sigma_k^{ext})$ such that $[\![\widehat{z}_k]\!]_{\Sigma_k^{ext}} = 1$. Let $\widehat{z}_k^I$ be the Lagrange intepolant of $\widehat{z}_k$ in $\mathcal{Q}\backslash\Sigma_k^{ext}$. Notice that $[\![\widehat{z}_k^I]\!]_{\Sigma_k^{ext}} = 1$, too. We write

$$z_k = \widehat{z}_k + \overline{z}_k \quad \text{and} \quad z_{kh} = \widehat{z}_k^I + \overline{z}_{kh},$$

with $\overline{z}_k \in H^1(\mathcal{Q})/\mathbb{R}$ and $\overline{z}_{kh} \in \mathcal{L}_h(\mathcal{Q})/\mathbb{R}$. Substituting these expressions in (37) and (38), using the first Strang Lemma (see [5, Theorem 4.4.1] ) for these new problems, we have that we obtain new problems for $\overline{z}_k$ and $\overline{z}_{kh}$ respectively.

$$\begin{aligned}\|\nabla\overline{z}_k \,-\, \nabla\overline{z}_{kh}\|_{0,\mathcal{Q}} \,\leq\, C &\inf_{\varphi\,\in\,\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} \|\nabla\overline{z}_k - \nabla\varphi\|_{0,\mathcal{Q}} \\ +C \sup_{\varphi\,\in\,\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} &\frac{\left|\displaystyle\int_{\mathcal{Q}\backslash\Sigma_k^{ext}} \nabla(\widehat{z}_k - \widehat{z}_k^I)\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} \,+\, \int_\Lambda (\mathbf{p}_k - \mathbf{p}_{kh})\cdot\mathbf{n}\,\varphi\,\mathrm{d}\zeta\right|}{\|\nabla\varphi\|_{0,\mathcal{Q}}}.\end{aligned}$$

The second term on the right-hand side of the last inequality is bounded as follows:

$$\begin{aligned}\Big|\int_{\mathcal{Q}\backslash\Sigma_k^{ext}} &\nabla(\widehat{z}_k - \widehat{z}_k^I)\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} \,+\, \int_\Lambda (\mathbf{p}_k - \mathbf{p}_{kh})\cdot\mathbf{n}\,\varphi\,\mathrm{d}\zeta\Big| \\ &\leq\, \|\nabla\widehat{z}_k - \nabla\widehat{z}_k^I\|_{0,\mathcal{Q}\backslash\Sigma_j^{ext}}\|\nabla\varphi\|_{0,\mathcal{Q}} + C\|\mathbf{p}_k - \mathbf{p}_{kh}\|_{0,\mathcal{Q}}\|\nabla\varphi\|_{0,\mathcal{Q}},\end{aligned} \tag{39}$$

where we have used that $\operatorname{div}\mathbf{p}_k = \operatorname{div}\mathbf{p}_{kh} = 0$ in $\mathcal{Q}$ and the fact that $\|\nabla(\cdot)\|_{0,\mathcal{Q}}$ is equivalent to $\|\cdot\|_{1,\mathcal{Q}}$ on $H^1(\mathcal{Q})/\mathbb{R}$.

From Theorem 5.1, we know that $\nabla\overline{z}_k|_{\mathcal{Q}} \in (H^s(\mathcal{Q}))^3$. Hence

$$\inf_{\varphi\,\in\,\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} \|\nabla\overline{z}_k - \nabla\varphi\|_{0,\mathcal{Q}} \,\leq\, \|\nabla\overline{z}_k - \nabla\overline{z}_k^I\|_{0,\mathcal{Q}} \,\leq\, Ch^s\|\nabla\overline{z}_k\|_{s,\mathcal{Q}}.$$

Using the last two estimates and Theorem 5.2, we obtain

$$\|\nabla\overline{z}_k - \nabla\overline{z}_{kh}\|_{0,\mathcal{Q}} \,\leq\, Ch^s\{\|\nabla\widehat{z}_k\|_{s,\mathcal{Q}\backslash\Sigma_k^{ext}} + \|\mathbf{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda} + \|\nabla\overline{z}_k\|_{s,\mathcal{Q}}\}$$

for all $k = 1, ..., L$. Therefore, as a consequence of Theorem 5.1,

$$\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{kh}\|_{0,\mathcal{Q}} \leq C h^s$$

for all $k = 1, ..., L$.

□

Now, we are in a position to introduce the following fully-discretization of *Problem P*:

*Problem $P_h^n$*. Find $(\mathbf{A}_{Ch}^n, \psi_h^n, \boldsymbol{\alpha}_h^n) \in \mathcal{V}_h \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$, $n = 1, ..., N$, such that

$$\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma} \overline{\partial} \mathbf{A}_{Ch}^n \cdot \mathbf{w}_C \, \mathrm{d}\mathbf{x} + \int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \mathbf{A}_{Ch}^n \cdot \mathbf{curl}\, \mathbf{w}_C \, \mathrm{d}\mathbf{x} \\
&+ \int_\Gamma \Big[ -\frac{1}{2}\psi_h^n - \mathcal{D}(\psi_h^n) + \frac{1}{\mu_0} \mathcal{S}(\mathbf{curl}\, \mathbf{A}_{Ch}^n \cdot \mathbf{n}_C) \Big] \mathbf{curl}\, \mathbf{w}_C \cdot \mathbf{n}_C \, \mathrm{d}\zeta \\
&+ \int_\Gamma (\mathbf{w}_C \times \mathbf{n}_C) \cdot (\mathbf{Z}_h^t \boldsymbol{\alpha}_h^n) \, \mathrm{d}\zeta \qquad\qquad = \int_{\Omega_C} \mathbf{J}(t_n) \cdot \mathbf{w}_C \, \mathrm{d}\mathbf{x}, \qquad (40) \\
&\int_\Gamma \Big[ \frac{1}{2} \mathbf{curl}\, \mathbf{A}_{Ch}^n \cdot \mathbf{n}_C + \mathcal{D}'(\mathbf{curl}\, \mathbf{A}_{Ch}^n \cdot \mathbf{n}_C) + \mu_0 \, \mathcal{H}(\psi_h^n) \Big] \eta \, \mathrm{d}\zeta = 0, \\
&\mu_0 \, \boldsymbol{\beta}_h^t \, \mathbf{N}_h \, \boldsymbol{\alpha}_h^n - \boldsymbol{\beta}_h^t \, \mathbb{T}_h (\mathbf{A}_{Ch}^n) = \mu_0 \, \boldsymbol{\beta}_h^t \, \mathbf{N}_h \, \boldsymbol{\alpha}_0 - \boldsymbol{\beta}_h^t \, \mathbb{T}_h (\mathbf{A}_{C,0}),
\end{aligned}$$

for all $(\mathbf{w}_C, \eta, \boldsymbol{\beta}) \in \mathcal{V}_h \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$, with

$$\mathbf{A}_{Ch}^0 = \mathbf{A}_{Ch,0} \quad \text{in} \quad \Omega_C,$$

where $\mathbf{A}_{Ch,0}$ is an approximation of $\mathbf{A}_{C,0}$, $\overline{\partial} \mathbf{A}_{Ch}^n$ is defined in (21) and the linear and continuous operator $\mathbb{T}_h : \mathcal{V} \to \mathbb{R}^L$ is defined by $\mathbb{T}_h(\mathbf{w}) := \int_\Gamma \mathbf{Z}_h(\mathbf{w} \times \mathbf{n}_C) \, \mathrm{d}\zeta$, with $\mathbf{Z}_h := [\widetilde{\nabla} z_{1h} \cdots \widetilde{\nabla} z_{Lh}]^t$.

We proceed as in the continuous case to prove existence and uniqueness of solution to (40). Indeed, let $\mathcal{R}_h : H_0^{-1/2}(\Gamma) \to \mathcal{L}_h(\Gamma)$ be the operator defined by

$$\int_\Gamma \mathcal{H}(\mathcal{R}_h(\xi)) \, \eta \, \mathrm{d}\zeta = \int_\Gamma \xi \, \eta \, \mathrm{d}\zeta \qquad \forall \eta \in \mathcal{L}_h(\Gamma), \ \forall \xi \in H_0^{-1/2}(\Gamma). \qquad (41)$$

Note that (41) is a Galerkin discretization of the elliptic problem (17). Consequently, using the Galerkin orthogonality and the continuity and coercivity of $\mathcal{H}$ (cf. Lemma 4.1 $(ii)$), we have the following Céa estimate:

$$\|\mathcal{R}\xi - \mathcal{R}_h \xi\|_{1/2,\Gamma} \leq C \inf_{\eta \in \mathcal{L}_h(\Gamma)} \|\mathcal{R}\xi - \eta\|_{1/2,\Gamma} \qquad \forall \xi \in H_0^{-1/2}(\Gamma). \qquad (42)$$

It is important to note that $\mathcal{R}_h$ is a self-adjoint and non-negative operator. Here again using that $\psi_h^n := -\mu_0^{-1}\mathcal{R}_h\left(\mathcal{K}(\mathbf{A}_{Ch}^n)\right)$ (cf. $(40)_2$) we deduce the following equivalent formulation of (40):
Find $\mathbf{A}_{Ch}^n \in \mathcal{V}_h$ such that

$$\begin{aligned}(\bar{\partial}\mathbf{A}_{Ch}^n, \mathbf{w}_C)_{\boldsymbol{\sigma}} \,+\, \mathcal{A}(\mathbf{A}_{Ch}^n, \mathbf{w}_C) \,+\, \mathcal{B}_h(\mathbf{A}_{Ch}^n, \mathbf{w}_C)& \\ = \,(\mathbf{J}(t_n), \mathbf{w}_C)_{0,\Omega_C} \,+\, \boldsymbol{g}_h(\mathbf{w}_C)&\end{aligned} \tag{43}$$

for all $\mathbf{w}_C \in \mathcal{V}_h$, with

$$\mathbf{A}_{Ch}^0 = \mathbf{A}_{Ch,0} \quad \text{in} \quad \Omega_C,$$

where

$$\begin{aligned}\mathcal{B}_h\,:\mathcal{V}_h\times\mathcal{V}_h\to\mathbb{R},\quad \mathcal{B}_h\left(\mathbf{H},\mathbf{G}\right) &:= \mu_0^{-1}\int_\Gamma \mathcal{K}(\mathbf{G})\mathcal{R}_h(\mathcal{K}(\mathbf{H}))\,\mathrm{d}\zeta \\ &\quad + \mu_0\left(\mathbb{T}_h\left(\mathbf{G}\right)\right)^t\mathbf{N}_h^{-1}\,\mathbb{T}_h\left(\mathbf{H}\right),\\[1em] \boldsymbol{g}_h\,:\mathcal{V}_h\to\mathbb{R},\qquad\qquad \boldsymbol{g}_h\left(\mathbf{H}\right) &:= \mu_0^{-1}\left(\mathbb{T}_h\left(\mathbf{H}\right)\right)^t\mathbf{N}_h^{-1}\,\mathbb{T}_h\left(\mathbf{A}_{C,0}\right)\\ &\quad - \left(\mathbb{T}_h\left(\mathbf{H}\right)\right)^t\boldsymbol{\alpha}_0.\end{aligned}$$

Hence, at each iteration, we have to find $\mathbf{A}_{Ch}^n \in \mathcal{V}_h$ such that

$$\begin{aligned}(\mathbf{A}_{Ch}^n, \mathbf{w}_C)_{\boldsymbol{\sigma}} \,+\, \triangle t[\,\mathcal{A}(\mathbf{A}_{Ch}^n, \mathbf{w}_C) \,+\, \mathcal{B}_h(\mathbf{A}_{Ch}^n, \mathbf{w}_C)\,]& \\ = \,\triangle t[(\mathbf{J}(t_n), \mathbf{w}_C)_{0,\Omega_C} \,+\, \boldsymbol{g}_h(\mathbf{w}_C)] \,+\, (\mathbf{A}_{Ch}^{n-1}, \mathbf{w}_C)_{\boldsymbol{\sigma}}.&\end{aligned} \tag{44}$$

Since $\mathcal{B}_h$ and $\mathcal{A}$ are non-negative definite, the existence and uniqueness of $\mathbf{A}_{Ch}^n$, $n = 1, ..., N$, is immediate.

**Remark 5.1.** *It is easy to prove that if we define* $\psi_h^n := -\mu_0^{-1}\mathcal{R}_h(\mathcal{K}(\mathbf{A}_{Ch}^n))$ *and* $\boldsymbol{\alpha}_h^n := \boldsymbol{\alpha}_0 + \mu_0^{-1}\boldsymbol{N}_h^{-1}(\mathbb{T}_h(\mathbf{A}_{Ch}^n) - \mathbb{T}_h(\mathbf{A}_{C,0}))$*, then* $(\mathbf{A}_{Ch}^n, \psi_h^n, \boldsymbol{\alpha}_h^n)$ *is a solution of* (40)*. This solution is unique, because* $\mathcal{H}$ *is coercive in* $\mathcal{L}_h(\Gamma) \subset H_0^{1/2}(\Gamma)$ *and* $\boldsymbol{N}_h$ *is a symmetric and positive definite matrix.*

## 6. Error estimates

For any $s \geq 0$, we consider the Sobolev space

$$\mathbf{H}^s(\mathbf{curl}\,;\Omega_C) := \{\mathbf{v} \in (H^s(\Omega_C))^3 \,:\, \mathbf{curl}\,\mathbf{v} \in (H^s(\Omega_C))^3\}$$

endowed with the norm $\|\mathbf{v}\|^2_{\mathbf{H}^s(\mathbf{curl}\,;\,\Omega_C)} := \|\mathbf{v}\|^2_{s,\Omega_C} + \|\mathbf{curl}\,\mathbf{v}\|^2_{s,\Omega_C}$. It is well known that the Nédélec operator interpolation $\mathcal{I}_h^{\mathcal{N}}\mathbf{v} \in \mathcal{V}_h$ is well defined for any $\mathbf{v} \in \mathbf{H}^s(\mathbf{curl}\,;\,\Omega_C)$ with $s > 1/2$; see, for instance, Lemma 4.7 of [4]. Moreover, for $1/2 < s \leq 1$, the following interpolation error estimate holds true (see Proposition 5.6 of [**?** ]).

$$\|\mathbf{v} - \mathcal{I}_h^{\mathcal{N}}\mathbf{v}\|_{\mathcal{V}} \leq Ch^s\|\mathbf{v}\|_{\mathbf{H}^s(\mathbf{curl}\,;\,\Omega_C)}, \qquad \forall \mathbf{v} \in \mathbf{H}^s(\mathbf{curl}\,;\,\Omega_C). \tag{45}$$

To simplify the notation, we introduce

$$\mathcal{G}_h(\mathbf{w}) := \|(\mathcal{R} - \mathcal{R}_h)\mathcal{K}(\mathbf{w})\|_{1/2,\Gamma}.$$

**Lemma 6.1.** *Let $\mathbf{A}_C$ and $\mathbf{A}^n_{Ch}$ be solutions of problems* (15) *and* (40)*, respectively the latter with initial data $\mathbf{A}^0_{Ch} := \mathcal{I}_h^{\mathcal{N}}(\mathbf{A}_{C,0})$. Assume that $\mathbf{A}_C \in H^2(0,T;\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$, with $s > 1/2$. Moreover, let $\boldsymbol{\rho}^n := \mathbf{A}_C(t_n) - \mathcal{I}_h^{\mathcal{N}}\mathbf{A}_C(t_n)$, $\boldsymbol{\delta}^n := \mathcal{I}_h^{\mathcal{N}}\mathbf{A}_C(t_n) - \mathbf{A}^n_{Ch}$ and $\boldsymbol{\tau}^n := \overline{\partial}\mathbf{A}_C(t_n) - \partial_t\mathbf{A}_C(t_n)$. Then, there exists $C > 0$ independent of $h$ and $\triangle t$ such that*

$$\begin{aligned}
&\max_{1\leq k\leq n}\|\boldsymbol{\delta}^k\|^2_{\mathcal{V}} + \triangle t\sum_{k=1}^n\|\overline{\partial}\boldsymbol{\delta}^k\|^2_{\boldsymbol{\sigma}}\\
&\leq C\left\{\triangle t\sum_{k=1}^n\Big[\|\overline{\partial}\boldsymbol{\rho}^k\|^2_{\mathcal{V}} + \|\boldsymbol{\tau}^k\|^2_{\mathcal{V}} + \mathcal{G}_h(\partial_t\mathbf{A}_C(t_k))^2\right.\\
&\quad+\Big(\|\mathbf{A}_C(t_k)\|^2_{\mathcal{V}} + \|\partial_t\mathbf{A}_C(t_k)\|^2_{\mathcal{V}}\Big)\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|^2_{0,\mathcal{Q}} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\Big)\Big]\\
&\quad+\Big(\|\mathbf{A}_{C,0}\|^2_{\mathcal{V}} + |\boldsymbol{\alpha}_0\|^2_{\mathcal{V}}\Big)\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|^2_{0,\mathcal{Q}} + \|\boldsymbol{N}^{-1} - \boldsymbol{N}_h^{-1}\|^2\Big)\\
&\quad\left.+\max_{0\leq k\leq n}\|\boldsymbol{\rho}^k\|^2_{\mathcal{V}} + \max_{0\leq k\leq n}\mathcal{G}_h(\mathbf{A}_C(t_k))^2\right\}.
\end{aligned} \tag{46}$$

Proof. It is straightforward to show that

$$\begin{aligned}
&(\overline{\partial}\boldsymbol{\delta}^k,\mathbf{v})_{\boldsymbol{\sigma}} + \mathcal{A}(\boldsymbol{\delta}^k,\mathbf{v}) + \mathcal{B}_h(\boldsymbol{\delta}^k,\mathbf{v})\\
&\qquad= -(\overline{\partial}\boldsymbol{\rho}^k,\mathbf{v})_{\boldsymbol{\sigma}} + (\boldsymbol{\tau}^k,\mathbf{v})_{\boldsymbol{\sigma}} - \mathcal{A}(\boldsymbol{\rho}^k,\mathbf{v}) - \mathcal{B}_h(\boldsymbol{\rho}^k,\mathbf{v})\\
&\qquad\quad+\mathcal{B}_h(\mathbf{A}_C(t_k),\mathbf{v}) - \mathcal{B}(\mathbf{A}_C(t_k),\mathbf{v})\\
&\qquad\quad+\boldsymbol{g}(\mathbf{v}) - \boldsymbol{g}_h(\mathbf{v}), \quad \forall\mathbf{v} \in \mathcal{V}_h.
\end{aligned} \tag{47}$$

as well as the following inequalities:

$$
\begin{aligned}
&(\overline{\partial}\boldsymbol{\delta}^k,\boldsymbol{\delta}^k)_{\boldsymbol{\sigma}} \geq \frac{1}{2\triangle t}(\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 - \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2),\\
&\mathcal{A}(\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) \geq \mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_C}^2,\\
&\mathcal{B}(\mathbf{A}_C(t_k),\boldsymbol{\delta}^k) - \mathcal{B}_h(\mathbf{A}_C(t_k),\boldsymbol{\delta}^k)\\
&\quad \leq k_3\|\mathbf{A}_C(t_k)\|_{\mathcal{V}}\|\boldsymbol{\delta}^k\|_{\mathcal{V}}\left(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}} + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|\right)\\
&\qquad + k_4\|\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_C}\mathcal{G}_h(\mathbf{A}_C(t_k)),\\
&\boldsymbol{g}(\boldsymbol{\delta}^k) - \boldsymbol{g}_h(\boldsymbol{\delta}^k)\\
&\quad \leq\ k_5\Big(\|\mathbf{A}_{C,0}\|_{\mathcal{V}} + |\boldsymbol{\alpha}_0|\Big)\|\boldsymbol{\delta}^k\|_{\mathcal{V}}\left(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}} + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|\right).
\end{aligned}
$$

Hence, choose $\mathbf{v}=\boldsymbol{\delta}^k$ in (47), recall that $\mathcal{B}_h$ is non-negative, the Cauchy-Schwarz inequality, Remark 4.1 and Young inequality lead us to the following estimate:

$$
\begin{aligned}
&\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 - \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2 + \triangle t\mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_C}^2\\
&\quad \leq \frac{\triangle t}{2T}\|\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 + C\triangle t\Big[\|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 + \mathcal{G}_h(\mathbf{A}_C(t_k))^2\\
&\qquad + \|\mathbf{A}_C(t_k)\|_{\mathcal{V}}^2\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big)\\
&\qquad + (\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2)\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big)\Big].
\end{aligned}
\tag{48}
$$

Then, summing over $k$, using the discrete Gronwall's Lemma (see[15, Lemma 1.4.2]) and taking into account that $\boldsymbol{\delta}^0=\mathbf{0}$, we obtain

$$
\begin{aligned}
\|\boldsymbol{\delta}^n\|_{\sigma}^2 \quad &\leq C\left\{\triangle t\sum_{k=1}^{n}\Big[\|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 + \mathcal{G}_h(\mathbf{A}_C(t_k))^2\right.\\
&\qquad + \|\mathbf{A}_C(t_k)\|_{\mathcal{V}}^2\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big)\Big]\\
&\qquad \left. + (\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2)\Big(\max_{1\leq i\leq L}\|\widetilde{\nabla}z_i - \widetilde{\nabla}z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big)\right\}.
\end{aligned}
$$

for $n=1,...,N$. Inserting the last inequality in (48) and summing over $k$ we have the estimate

$$
\begin{aligned}
&\|\boldsymbol{\delta}^n\|_{\boldsymbol{\sigma}}^2 \;+\; \triangle t \sum_{k=1}^{n} \|\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega_C}^2 \\
&\quad \leq C \Bigg\{ \triangle t \sum_{k=1}^{n} \Big[ \|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 \;+\; \mathcal{G}_h(\mathbf{A}_C(t_k))^2 \\
&\qquad + \|\mathbf{A}_C(t_k)\|_{\mathcal{V}}^2 \Big( \max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\|^2 \Big) \Big] \\
&\qquad + (\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2) \Big( \max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\|^2 \Big) \Bigg\} .
\end{aligned}
\tag{49}
$$

Let us now take $\mathbf{v} = \overline{\partial}\boldsymbol{\delta}^k$ in (47):

$$
\begin{aligned}
&\|\overline{\partial}\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 \;+\; \mathcal{A}(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) + \mathcal{B}_h(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) \\
&\quad = -(\overline{\partial}\boldsymbol{\rho}^k, \overline{\partial}\boldsymbol{\delta}^k)_{\boldsymbol{\sigma}} \;+\; (\boldsymbol{\tau}^k, \overline{\partial}\boldsymbol{\delta}^k)_{\boldsymbol{\sigma}} \;+\; \mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) +\; \mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) \\
&\qquad + \mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) \;+\; \mathcal{B}(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1}) \\
&\qquad - \mathcal{B}_h(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1}) + \boldsymbol{g}(\overline{\partial}\boldsymbol{\delta}^k) - \boldsymbol{g}_h(\overline{\partial}\boldsymbol{\delta}^k) - \tfrac{1}{\triangle t}[\gamma_k - \gamma_{k-1}],
\end{aligned}
\tag{50}
$$

where $\gamma_k := \mathcal{A}(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) + \mathcal{B}_h(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) - \mathcal{B}_h(\mathbf{A}_C(t_k), \boldsymbol{\delta}^k) + \mathcal{B}(\mathbf{A}_C(t_k), \boldsymbol{\delta}^k)$.

On the other hand, as $\mathcal{A}$ is non-negative definite and symmetric, it is easy to check that

$$
\mathcal{A}(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) \geq \frac{1}{2\triangle t} \Big[ \mathcal{A}(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - \mathcal{A}(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \Big]
$$

and similarly for $\mathcal{B}_h$. Using these inequalities in (50) together with the Cauchy-Schwarz inequality, and then, summing over $k$ and recalling that $\mathcal{B}_h$ is non-negative, we deduce that

$$
\begin{aligned}
&\frac{1}{2} \sum_{k=1}^{n} \|\overline{\partial}\boldsymbol{\delta}^k\|_{\boldsymbol{\sigma}}^2 \;+\; \frac{1}{2\triangle t} \mu_1^{-1} \|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega_C}^2 \\
&\qquad \leq C \sum_{k=1}^{n} \Big[ \|\overline{\partial}\boldsymbol{\rho}^k\|_{\boldsymbol{\sigma}}^2 + \|\boldsymbol{\tau}^k\|_{\boldsymbol{\sigma}}^2 \Big] + \sum_{k=1}^{n} \Big[ \big| \mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) \big| \\
&\qquad\quad + \big| \mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) \big| + \big| \mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) \big| \\
&\qquad\quad + \big| \mathcal{B}(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1}) \big| \Big] \\
&\qquad\quad + \frac{1}{\triangle t} |\boldsymbol{g}(\boldsymbol{\delta}^n) - \boldsymbol{g}_h(\boldsymbol{\delta}^n)| - \frac{1}{\triangle t} |\gamma_n|.
\end{aligned}
\tag{51}
$$

It is easy to obtain from Young's inequality and Remark 4.1 the following bounds:

$$\begin{aligned}
&\sum_{k=1}^{n} |\mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})| \\
&\qquad \leq \sum_{k=1}^{n} \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega_C}^2 + C_1 \sum_{k=1}^{n} \|\mathbf{curl}\,\overline{\partial}\boldsymbol{\rho}^k\|_{0,\Omega_C}^2, \\
&\sum_{k=1}^{n} |\mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})| \\
&\qquad \leq \sum_{k=1}^{n} \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega_C}^2 + \sum_{k=1}^{n} \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2 + C_2 \sum_{k=1}^{n} \|\mathbf{curl}\,\overline{\partial}\boldsymbol{\rho}^k\|_{0,\Omega_C}^2, \\
&\sum_{k=1}^{n} |\mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1})| \\
&\qquad \leq \sum_{k=1}^{n} \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega_C}^2 + \sum_{k=1}^{n} \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2 + C_3 \sum_{k=1}^{n} \|\boldsymbol{\tau}^k\|_{\mathcal{V}}^2, \\
&\sum_{k=1}^{n} |\mathcal{B}(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\partial_t \mathbf{A}_C(t_k), \boldsymbol{\delta}^{k-1})| \\
&\qquad \leq \sum_{k=1}^{n} \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega_C}^2 + \sum_{k=1}^{n} \|\boldsymbol{\delta}^{k-1}\|_{\boldsymbol{\sigma}}^2 + C_4 \sum_{k=1}^{n} \mathcal{G}_h(\partial_t \mathbf{A}_C(t_k))^2 \\
&\qquad\quad + C_5 \sum_{k=1}^{n} \|\partial_t \mathbf{A}_C(t_k)\|^2 \Big( \max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\|^2 \Big),
\end{aligned}$$

and

$$\begin{aligned}
&|\boldsymbol{g}(\boldsymbol{\delta}^n) - \boldsymbol{g}_h(\boldsymbol{\delta}^n)| \\
&\qquad \leq C_6(\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2) \Big( \max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\|^2 \Big) \\
&\qquad\quad + \frac{1}{8}\mu_1^{-1} \|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega_C}^2 + \|\boldsymbol{\delta}^n\|_{\boldsymbol{\sigma}}^2, \\
&|\gamma_n| \leq \frac{1}{8}\mu_1^{-1} \|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega_C}^2 + \|\boldsymbol{\delta}^n\|_{\boldsymbol{\sigma}}^2 \\
&\qquad\quad + C_7 \Big[ \|\boldsymbol{\rho}^n\|_{\mathcal{V}}^2 + \|\mathbf{A}_C(t_n)\|_{\mathcal{V}}^2 \Big( \max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\|^2 \Big) \Big].
\end{aligned}$$

Substituting all these inequalities in (51), using (49) and Remark 4.1, we obtain

$$
\begin{aligned}
&\triangle t \sum_{k=1}^{n} \|\overline{\partial}\boldsymbol{\delta}_C^k\|_{\boldsymbol{\sigma}}^2 \;+\; \|\mathbf{curl}\,\boldsymbol{\delta}_C^n\|_{0,\Omega_C}^2 \\
&\quad\le C\,\Bigg\{\triangle t \sum_{k=1}^{n}\Big[\|\overline{\partial}\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 + \|\boldsymbol{\tau}^k\|_{\mathcal{V}}^2 + \mathcal{G}_h(\partial_t(\mathbf{A}_C(t_k)))^2 + \|\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 \\
&\qquad +\|\mathbf{A}_C(t_k)\|^2\Big(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big). \\
&\qquad +\|\partial_t\mathbf{A}_C(t_k)\|^2\Big(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big)\Big] \\
&\qquad +(\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2)\Big(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\|^2\Big) \\
&\qquad + \|\boldsymbol{\rho}^n\|_{\mathcal{V}}^2 + \mathcal{G}_h(\mathbf{A}_C(t_n))^2\Bigg\}
\end{aligned}
$$

Combining this last inequality with (49) and Remark 4.1 we conclude (46). □

**Lemma 6.2.** *Let* $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ *be the solution of* (15). *If we assume that* $\mathbf{A}_C \in H^1(0,T;\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$, $1/2 < s < s_{\mathcal{Q}}$, *then* $\psi \in H^1(0,T;H^{s+1/2}(\Gamma))$,

$$\inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\psi(t)-\eta\|_{1/2,\Gamma} \le Ch^s\|\mathbf{curl}\,\mathbf{A}_C(t)\|_{s,\Omega_C} \tag{52}$$

*and*

$$\inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\partial_t\psi(t)-\eta\|_{1/2,\Gamma}^2 \le Ch^s\|\partial_t(\mathbf{curl}\mathbf{A}_C(t))\|_{s,\Omega_C}. \tag{53}$$

Proof. Since $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ is the unique solution of (15), then by Theorem 4.3, we know that there exists $\psi_I \in W^1(\Omega_I)$ harmonic and for all $t\in[0,T]$, $\psi_I(t) = \psi(t) + c(t)$, with $c(t)\in\mathbb{R}$ such that $\int_\Gamma \psi(t)\,\mathrm{d}t = 0$; namely $c(t) = \frac{1}{|\Gamma|}\int_\Gamma \psi_I(t)\,\mathrm{d}t$. Therefore, $\psi_I|_{\mathcal{Q}}$ is the solution to

$$
\begin{aligned}
-\triangle\psi_I \;&=\; 0 && \text{in}\quad \mathcal{Q}, \\
\mu_0\tfrac{\partial\psi_I}{\partial\mathbf{n}_I} \;&=\; -\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C && \text{on}\quad \Gamma, \\
\psi_I|_\Lambda \in \mathcal{C}^\infty(\Lambda).
\end{aligned} \tag{54}
$$

with $1/2 < s < s_{\mathcal{Q}}$. Thus, applying classical results for the Laplace equation (see [7]), we have that $\psi_I \in H^{s+1}(\mathcal{Q})$ and

$$\|\psi_I\|_{s+1,\mathcal{Q}} \le C\|\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C\|_{s+1/2,\Gamma} \le C\|\mathbf{curl}\,\mathbf{A}_C\|_{s,\Omega_C}. \tag{55}$$

Since $s > 1/2$, the Lagrange interpolant $\psi_I^I$ of $\psi_I$ is well defined. Moreover,

$$(\psi_I - \psi_I^I)|_\Gamma = \psi - \psi^{I_\Gamma},$$

where $\psi^{I_\Gamma} \in \mathcal{L}_h(\Gamma)$ denotes the 2D Lagrange surface intepolant on $\Gamma$. Therefore, because of the trace theorem, standard estimates for the 3D Lagrange interpolant and (55)

$$\begin{aligned} \|\psi - \psi^{I_\Gamma}\|_{1/2,\Gamma} &\leq C\|\psi_I - \psi_I^I\|_{1,\mathcal{Q}} \\ &\leq Ch^s\|\psi_I\|_{s+1,\mathcal{Q}} \\ &\leq Ch^s\|\mathbf{curl}\,\mathbf{A}_C\|_{s,\Omega_C}. \end{aligned}$$

Thus, we conclude (52). To prove (53), first we differentiate each equation in (54). Since $\mathbf{A}_C \in H^1(0,T;\mathbf{H}^s(\mathbf{curl}\,;\,\Omega_C))$, we obtain an estimate analogous to (55) for $\frac{\partial \psi_I}{\partial t}$. Moreover

$$\partial_t\psi(t) = \partial_t\psi_I(t) - \frac{1}{|\Gamma|}\int_\Gamma \partial_t\psi_I(t)\,\mathrm{d}t.$$

Hence, the rest of the proof follows identically as above.

$\square$

Finally, with the aid of previous lemma, Corollary 5.3, Corollary 5.1 and the interpolation error estimate (45), we deduce the following asymptotic error estimate for our fully discrete scheme.

**Theorem 6.1.** *Let $(\mathbf{A}_C, \psi, \boldsymbol{\alpha})$ and $(\mathbf{A}_{Ch}^n, \psi_h^n, \boldsymbol{\alpha}_h^n)$, $n = 1, ..., N$, be the solutions to problem (15) and (40), respectively. Let us assume that $\mathbf{A}_C \in H^2(0,T;\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$ with $1/2 < s < s_{\mathcal{Q}}$. Then, there exists $h_0 > 0$ such that, for all $0 < h \leq h_0$, the following estimate holds:*

$$\begin{aligned} &\max_{1\leq n\leq N}\|\mathbf{A}_C(t_n) - \mathbf{A}_{Ch}^n\|_{\mathcal{V}}^2 + \triangle t\sum_{n=1}^N \|\overline{\partial}(\mathbf{A}_C(t_n) - \mathbf{A}_{Ch}^n)\|_{\boldsymbol{\sigma}}^2 \\ &\quad\leq Ch^{2s}\Bigg\{\int_0^T \|\partial_t\mathbf{A}_C(t)\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\,\Omega_C)}^2\,dt + \max_{1\leq n\leq N}\|\partial_t(\mathbf{curl}\,\mathbf{A}_C(t_n))\|_{s,\Omega_C}^2 \\ &\qquad + \max_{1\leq n\leq N}\Big(\|\mathbf{A}_C(t_n)\|_{\mathcal{V}}^2 + \|\partial_t\mathbf{A}_C(t_n)\|_{\mathcal{V}}^2\Big)\Big(\max_{1\leq k\leq L}\|\widetilde{\nabla} z_k\|_{s,\mathcal{Q}}^2 + \|z_k\|_{s+1/2,\Lambda}^2\Big) \\ &\qquad + \Big(\|\mathbf{A}_{C,0}\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|_{\mathcal{V}}^2\Big)\Big(\max_{1\leq k\leq L}\|\widetilde{\nabla} z_k\|_{s,\mathcal{Q}}^2 + \|z_k\|_{s+1/2,\Lambda}^2\Big) \\ &\qquad + \max_{1\leq n\leq N}\|\mathbf{A}_C(t_n)\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\,\Omega_C)}^2\Bigg\} + (\triangle t)^2\int_0^T \|\partial_{tt}\mathbf{A}_C(t)\|_{\mathcal{V}}^2\,dt \\ &\quad\leq C\big((\triangle t))^2 + h^{2s}\big)\|\mathbf{A}_C\|_{H^2(0,T;\boldsymbol{H}^s(\mathbf{curl}\,\Omega_C))}. \end{aligned} \tag{56}$$

*with $C > 0$ independent of $h$ and $\triangle t$, where $z_k$, $k = 1, .., L$ are the solutions of problem* (6).

Proof.

A Taylor expansion shows that

$$\overline{\partial}\mathbf{A}_C(t_k) = \partial_t \mathbf{A}_C(t_k) + \frac{1}{\triangle t}\int_{t_{k-1}}^{t_k} (t_{k-1} - t)\partial_{tt}\mathbf{A}_C(t)\,\mathrm{d}t.$$

Consequently,

$$\sum_{k=1}^{n} \|\boldsymbol{\tau}_C^k\|_{\mathcal{V}}^2 \leq \triangle t \int_0^T \|\partial_{tt}\mathbf{A}_C(t)\|_{\mathcal{V}}^2\,\mathrm{d}t.$$

Moreover, we have from (45)

$$\begin{aligned} \sum_{k=1}^{n} \|\overline{\partial}\boldsymbol{\rho}^k\|_{\mathcal{V}}^2 &\leq \frac{1}{\triangle t}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \|\partial_t(I - \mathcal{I}_h^{\mathcal{N}})\mathbf{A}_C(t)\|_{\mathcal{V}}^2\,\mathrm{d}t \\ &\leq \frac{C}{\triangle t}h^{2s}\int_0^T \|\partial_t\mathbf{A}_C(t)\|_{\mathbf{H}^s(\mathbf{curl}\,;\Omega_C)}^2\,\mathrm{d}t. \end{aligned}$$

We recall that $\psi(t) = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\mathbf{A}_C(t)))$ (cf. Remark 4.2). Hence, the regularity assumption on $\mathbf{A}_C$ implies that $\psi \in H^2(0, T; H_0^{s+1/2}(\Gamma))$ and $\partial_t\psi(t) = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\partial_t\mathbf{A}_C(t)))$. It follows from (42) that

$$\begin{aligned} \mathcal{G}_h(\mathbf{A}_C(t_n)) &\leq \inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\psi(t_n) - \eta\|_{1/2,\Gamma}^2, \\ \mathcal{G}_h(\partial_t\mathbf{A}_C(t_n)) &\leq \inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\partial_t\psi(t_n) - \eta\|_{1/2,\Gamma}^2. \end{aligned}$$

Thus, using the previous lemma, we obtain

$$\begin{aligned} \mathcal{G}_h(\mathbf{A}_C(t_n)) &\leq Ch^s\|\mathbf{curl}\,\mathbf{A}_C(t_n)\|_{s,\Omega_C}, \\ \mathcal{G}_h(\partial_t\mathbf{A}_C(t_n)) &\leq Ch^s\|\partial_t(\mathbf{curl}\,\mathbf{A}_C(t_n))\|_{s,\Omega_C}. \end{aligned} \tag{57}$$

Hence, the results follows by writing $\mathbf{A}_C(t_n) - \mathbf{A}_{Ch}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and using Lemma 6.1, Lemma 5.1, Theorem 5.3 and (45). □

**Remark 6.1.** *Let us recall that* $\psi(t_n) = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\mathbf{A}_C(t_n)))$ *and* $\psi_h^n = -\mu_0^{-1}\mathcal{R}_h(\mathcal{K}(\mathbf{A}_{Ch}^n))$*. Therefore, using* (57) *and the uniform boundedness of* $\mathcal{R}_h$*, we obtain*

$$\begin{aligned}\|\psi(t_n) - \psi_h^n\|_{1/2,\Gamma} &\leq \mathcal{G}_h(\mathbf{A}_C(t_n)) + \|\mathcal{R}_h(\mathcal{K}(\mathbf{A}_C(t_n) - \mathbf{A}_{Ch}^n))\|_{1/2,\Gamma} \\ &\leq C\left\{h^s\|\mathbf{curl}\,\mathbf{A}_C(t_n)\|_{s,\Omega_C} + \|\mathbf{A}_C(t_n) - \mathbf{A}_{Ch}^n\|_{\mathcal{V}}\right\}.\end{aligned}$$

*Then, using Lemma* 6.2 *and Theorem* 6.1 *we conclude*

$$\triangle t \sum_{n=1}^{N} \|\psi(t_n) - \psi_h^n\|_{1/2,\Gamma}^2 \leq C[h^{2s} + (\triangle t)^2].$$

*Moreover, since* $\boldsymbol{\alpha}(t_n) = \boldsymbol{\alpha}_0 - \mu_0^{-1}\boldsymbol{N}^{-1}(\mathbb{T}\,(\mathbf{A}_C(t_n) - \mathbf{A}_{C,0}))$ *and* $\boldsymbol{\alpha}_h^n = \boldsymbol{\alpha}_0 - \mu_0^{-1}\boldsymbol{N}_h^{-1}(\mathbb{T}_h\,(\mathbf{A}_{Ch}^n - \mathbf{A}_{C,0}))$*, then from Theorem* 5.3*, Lemma* 5.1 *and Theorem* 6.1 *we also conclude that*

$$\max_{1\leq n\leq N} |\boldsymbol{\alpha}(t_n) - \boldsymbol{\alpha}_h^n|^2 \leq C[h^{2s} + (\triangle t)^2].$$

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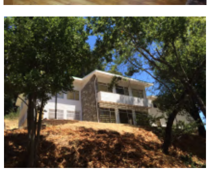


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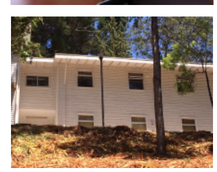
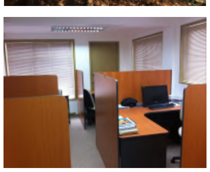


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