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On the number of update digraphs and its relation with the feedback arc sets and tournaments

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Abstract

An update digraph corresponds to a labeled digraph that indicates a relative order of its nodes introduced to define equivalence classes of deterministic update schedules yielding the same dynamical behavior of a Boolean network. In (Aracena et al., 2011), the authors exhibited relationships between update digraphs and the feedback arc sets of a given digraph G. In this paper, we delve into the study of these relations. Specifically, we show differences and similarities between both sets through of increasing and decreasing monotony properties in terms of its structural characteristics. Besides, we prove that these sets are equivalent if and only if all its circuits are cycles. On the other hand, we characterize the minimal feedback arc sets of a given digraph in terms of its update digraphs associated. In particular, for complete digraphs, this characterization shows a close relation with acyclic tournaments. For these latter, we show that the size of the associated equivalence classes is a power of two. Finally, we determine exactly the number of update digraphs associated to digraphs containing a tournament.

Key words: Update digraph, feedback arc set, tournament, update schedule.

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1 Introduction

Boolean networks (BN) are the simplest models for genetic regulatory networks, as well as for other simple distributed dynamical systems. Despite their simplicity, they provide a realistic model in which different phenomena can be reproduced and studied, and indeed, many regulatory models published in the biological literature fit within this framework (Kauffman, 1969; Thomas, 1973; Shmulevich et al., 2003).

A BN is defined by its connection digraph, its local activation functions, and the update schedule used (see (Aracena et al., 2009) for more details). In (Salinas, 2008) was defined equivalence classes of deterministic update schedules in BN's according to the associated connection digraph with labels on its arcs, named *update digraph*, which are of two types: an arc (a, b) has a *negative* label if the state of a is updated strictly before than b, and a *positive* label otherwise.

In (Salinas, 2008; Aracena et al., 2009), the authors showed that the elements of such equivalence classes yield exactly the same dynamical behavior of the network. This result was the starting point for the algorithmical and combinatorial study of the update digraphs done in (Aracena et al., 2011), where was showed, among other things, a close relationship between update digraphs and the feedback arc sets of a given digraph.

For a given digraph, a feedback arc set is a set of arcs which if removed leaves the resultant digraph free of cycles. They have been extensively studied, mainly concerning to the cardinality of a minimum feedback arc set (see (Bang-Jensen and Gutin, 2010) for a good survey). One of these works showed that such a minimum arcs set is determined by a sequential ordering of the nodes which minimizes the number of positive arcs (Younger, 1963).

In this paper, we delve into the study of the relationships between update digraphs and feedback arc sets showed in (Aracena et al., 2011) as well as their relations with some particular digraphs such as complete digraphs, acyclic digraphs and tournaments in order to determine exactly the number and size of the associated equivalence classes.

2 Preliminaries

We begin this section giving some basic definitions and introducing the necessary notations. Besides, we remember some known results that allow us to develop the following sections.

A **digraph** is an ordered pair of sets $G = (V, A)$ where $V = \{1, \ldots, n\}$ is a set of elements called **vertices** (or **nodes**) and A is a set of ordered pairs (called **arcs**) of vertices of V. The **vertex set** of G is referred to as $V(G)$, its **arc set** as $A(G)$. If more than one arc that connects a node i to a node j are admissible, then we will say that G is a **multidigraph**. A **subdigraph** of G is a digraph $G' = (V', A')$ where $V' \subseteq V$ and $A' \subseteq (V' \times V') \cap A$. We write $G' \subseteq G$. If $A' = (V' \times V') \cap A$, we say that G' is **induced** by V' and call G' an **induced subdigraph** of G which is denoted by $G' = G[V']$. If $V' \subsetneq V \vee A' \subsetneq A$, then we write $G' \subsetneq G$. We will often write $G - U$ whose meaning depends of the nature of U. Thus, if $U \subseteq V$, then $G - U$ is the subdigraph of G induced by $V - U$, i.e. $G - U = G[V - U]$. If $U \subseteq A$, then $G - U = (V, A - U)$.

G is a **complete** digraph if $A = \{(u, v) : \ u, v \in V \wedge u \neq v\}$. G is a **tournament** if $\forall x, y \in V$, $x \neq y$, $(x, y) \in A \veebar (y, x) \in A$.

A **walk** from a vertex v_1 to a vertex v_m in a digraph G is a sequence of vertices $v_1, v_2, \ldots, v_m$ of $V(G)$ such that $\forall k = 1, \ldots, m - 1$, $(v_k, v_{k+1}) \in A(G)$ or $(v_{k+1}, v_k) \in A(G)$. The vertices v_1 and v_m are the initial and terminal vertex of the walk. A walk is **elementary** if each vertex in the walk appears only once with the possible exception that the first and last vertex may coincide. A walk is closed if its initial and terminal vertices coincide. A **circuit** is a closed elementary walk. A walk $v_1, v_2 \ldots, v_m$ is a **path** if $(v_k, v_{k+1}) \in A(G)$ for all $k = 1, \ldots, m - 1$. A **cycle** is a directed circuit, that is a closed elementary path. G is an **acyclic digraph** if G has no cycle.

An arc set $U \subseteq A$ is a **feedback arc set** of G if $G - U$ is an acyclic digraph. U is said to be a **minimal** feedback arc set of G if U is a feedback arc set of G and there is no other feedback arc set $W \subseteq A$ of G such that $W \subsetneq U$. U is said to be a **minimum** feedback arc set of G if U is a feedback arc set of G and there is no other feedback arc set $W \subseteq A$ of G such that $|W| < |U|$. The sets of feedback arc sets, minimal feedback arc sets and non feedback arc sets of G are denoted by $FAS(G)$, $MFAS(G)$ and $NFAS(G)$ respectively.

A digraph G is said to be **connected** if there is a walk between every pair of its vertices, and **strongly connected** if there is a path between every pair of its vertices. $G' \subseteq G$ is a **strongly connected component** of G if G' is strongly connected and it is maximal for this property, i.e., there is no other strongly connected subdigraph G'' of G such that $G' \subsetneq G''$. If G' is a strongly connected component of G, composed by only one vertex, then G' is called a **trivial component**.

An **undirected graph** (or a **graph**) is an ordered pair $G = (V, E)$ where $V = \{1, \ldots, n\}$ is a non-empty finite set of elements called **vertices** (or **nodes**) and E is a finite set of unordered pairs of distinct vertices of V called **edges**. The

vertex set of G is referred to as $V(G)$, its **arc set** as $E(G)$. An **orientation** of G is a digraph $G' = (V', A)$ where $V' = V$, $|A| = |E|$ and $\forall \{x, y\} \in E$, either $(x, y) \in A$ or $(y, x) \in A$.

Also, in the sequel, for any integers a and b with $a \leq b$, we will write $[\![a, b]\!] = \{i \in \mathbb{Z} : a \leq i \leq b\}$.

An **update schedule** of the vertices of a digraph $G = (V, A)$, with $|V| = n$, is a function $s : V \rightarrow [\![1, n]\!]$ such that $s(V) = [\![1, m]\!]$ for some $m \leq n$. We denote S_n the **set of update schedules** over $[\![1, n]\!]$. A **block** of s is the set $B_i = \{v \in V : s(v) = i\}$, $1 \leq i \leq m$. The **number of blocks** of s is denoted by $nb(s) \equiv m$. If $nb(s) = 1$, then s is said to be a **parallel** update schedule. In this case, we will write $s = s_p$. If s is a permutation over the set $[\![1, n]\!]$, i.e. $nb(s) = n$, s is said to be a **sequential** update schedule. In all other cases, i.e. when $2 \leq nb(s) \leq n - 1$, s is said to be a **block sequential** update schedule. Frequently, s will be denoted by $s = (j \in B_1)(j \in B_2) \cdots (j \in B_{nb(s)})$ or more compactly $s = (B_i)_{i=1}^{nb(s)}$ and $P_s = \{B_1, ..., B_{nb(s)}\}$ the partition associated to s.

As mentioned in (Demongeot et al., 2008), the number of update schedules associated to a digraph of n vertices, i.e. $|S_n|$, is equal to the number of ordered partitions of a set of size n, that is

$$|S_n| = T_n \equiv \sum_{k=0}^{n-1} \binom{n}{k} T_k,$$

where $T_0 \equiv 1$.

Let G be a digraph. A function $lab : A(G) \rightarrow \{\ominus, \oplus\}$ is called a **label function** of G. An arc $a \in A(G)$ such that $lab(a) = \oplus$ is called a **positive arc** and an arc $a \in A(G)$ such that $lab(a) = \ominus$ is called a **negative arc**. A cycle C in G such that $\forall a \in A(C)$, $lab(a) = \oplus$ is called a **positive cycle** and a cycle C in G such that $\forall a \in A(C)$, $lab(a) = \ominus$ is called a **negative cycle**. (G, lab) is named a **labeled digraph**.

Let s be an update schedule of $V(G)$, we denote by lab_s the label function defined as follows (see Fig. 1):

$$\forall (j, i) \in A(G), lab_s(j, i) = \begin{cases} \oplus & \text{if } s(j) \geq s(i) \\ \ominus & \text{if } s(j) < s(i). \end{cases}$$

We define **equivalence classes** with respect to labeled digraphs: if s is an update schedule of the vertices of a digraph G, we write $[s]_G$ the set of update
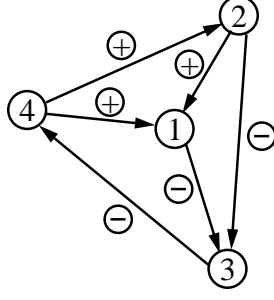
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Fig. 1. A digraph $G = (V, A)$ labeled by the function lab_s where $\forall i \in V = \{1, \ldots, 4\}$, $s(i) = i$.

schedules s' such that $s \overset{G}{\sim} s'$, that is

$$[s]_G = \{s' : (G, lab_s) = (G, lab_{s'})\}.$$

In simple words, an equivalence class, $[s]_G$, is a set of update schedules that all yield the same labeled digraph, and consequently the same dynamical behavior of the network (Aracena et al., 2009).

We will say that s has the **maximum number of blocks** if $\forall s' \in [s]_G$, $nb(s') \leq nb(s)$ and we will denote by S_n^* the set of update schedules having the maximum number of blocks.

A labeled digraph (G, lab) is said to be an **update digraph** if there exists an update schedule s such that $lab = lab_s$, that is $\forall a \in A(G)$, $lab(a) = lab_s(a)$. We denote by $U(G) = \{lab : A(G) \to \{\ominus, \oplus\}| (G, lab) \text{ is an update digraph}\}$ and $NU(G) = \{lab : A(G) \to \{\ominus, \oplus\}| (G, lab) \text{ is a non update digraph}\}$. An update digraph (G, lab) has a **maximal number of negative arcs** if there is no label function $lab^* \in U(G)$ with strictly more negative arcs that lab, and where $lab(u, v) = \ominus$ implies $lab^*(u, v) = \ominus$, for every $(u, v) \in A(G)$ (see example in Fig. 2).
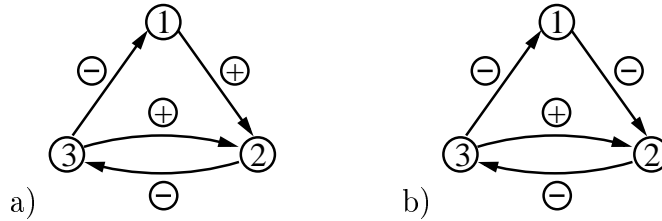

Fig. 2. a) A labeled digraph (G, lab) which is an update digraph with a maximal (but not maximum) number of negative arcs. b) A labeled digraph (G, lab') which is not an update digraph.

We define the **projection** of (G, lab) onto $G' \subseteq G$ as being the labeled digraph $(G', lab|_{G'})$, where $lab|_{G'}(a) = lab(a)$, $\forall a \in A(G')$.

Given an update digraph (G, lab), $G' \subseteq G$ is said to be a **positive strongly**

connected component of (G, lab) if G' is a strongly connected component of $G - \{a \in A(G) : lab(a) = \ominus\}$. In particular, if G' is an isolated vertex, G' is called a **trivial positive component**. Besides, any update digraph (G, lab) has a unique decomposition into positive strongly connected components.

Let (G, lab) be a labeled digraph. The **labeled reoriented multidigraph** associated to (G, lab), denoted by (G_R, lab_R), is the labeled multidigraph defined by construction as follows:

- $V(G_R) = V(G)$.
- $\forall (u, v) \in A(G)$ such that $lab(u, v) = \oplus$, $(u, v) \in A(G_R)$ and $lab_R(u, v) = \oplus$.
- $\forall (u, v) \in A(G)$ such that $lab(u, v) = \ominus$, $(v, u) \in A(G_R)$ and $lab_R(v, u) = \ominus$.

Observe that if (G, lab) has no cycle of length two with a positive arc and another negative, then (G_R, lab_R) is simply a digraph. An example of labeled reoriented multidigraph is shown in Fig. 3.
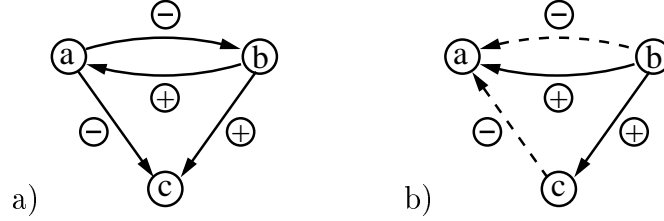

Fig. 3. a) A labeled digraph $G = (\{a, b, c\}, A)$. b) (G_R, lab_R) where the arcs drawn in dotted lines are the ones that have been inverted.

A **forbidden cycle** in (G_R, lab_R) is a cycle containing a negative arc.

In this context, the following characterization of update digraphs was proven in (Aracena et al., 2011).

Theorem 1 *A labeled digraph (G, lab) is an update digraph if and only if (G_R, lab_R) does not contain any forbidden cycle.*

Observe that if (G, lab) has a negative cycle, then (G, lab) is not an update digraph. Besides, if G is a digraph without circuits, then for every label function $lab : A(G) \to \{\ominus, \oplus\}$, (G, lab) is an update digraph.

On the other hand, in (Aracena et al., 2011) was also shown that the property of being update digraph can be extended or projected over update digraphs. Specifically, the authors showed the following Theorem.

Theorem 2 *Let G be a digraph and $G' \subseteq G$. If (G', lab') is an update digraph, then there exists a label function lab of $A(G)$ such that (G, lab) is an update digraph and $lab|_{G'} = lab'$.*

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The above Theorem will be fundamental in the following section, to be more explicit, in the relationships between the sizes of the sets $U(G)$, $MFAS(G)$ and $FAS(G)$ associated to a given digraph G.

3 Feedback arc sets and update digraphs

In this section we will investigate into the relationships that exist between the feedback arc sets and the update digraphs associated to a given digraph, work begun in (Aracena et al., 2011).

Next, we give a first result relating the sizes of the sets $U(G)$, $MFAS(G)$ and $FAS(G)$ that was shown in (Aracena et al., 2011).

Proposition 3 *Let G be a digraph. Then,*

$$|MFAS(G)| \ \leq \ |U(G)| \ \leq \ |FAS(G)|.$$

Basically, the proof of the Proposition 3 considers, on the one hand, an injective function $g : U(G) \rightarrow FAS(G)$ such that for every $lab \in U(G)$, $g(lab) = \{a \in A(G) : \ lab(a) = \oplus\}$ and on the other hand, an injective function $h : MFAS(G) \rightarrow U(G)$ such that for every $F \in MFAS(G)$, $h(F) = lab_F$, where $lab_F(a) = \oplus \Leftrightarrow a \in F$.

Now, being a little bit more specific, it is easy to see that the label function lab_p defined by $lab_p(a) = \oplus$, $\forall a \in A(G)$ is in $U(G)$ but there does not exist $F \in MFAS(G)$ such that $h(F) = lab_p$, because for each arc in F there exists a cycle in G which it does not contain another arc in F. Therefore, the function h cannot be surjective and hence, we have the following result.

Proposition 4 *Let G be a digraph. Then,*

$$|MFAS(G)| \ < \ |U(G)| \ \leq \ |FAS(G)|.$$

As shown in Proposition 4, there exist relationships between feedback arc sets and update digraphs. Next, we will study in greater depth this relation.

Proposition 5 *Let G be a digraph and $G' \subsetneq G$ where $V(G) = V(G')$. Then $|U(G')| < |U(G)|$ and $|FAS(G')| < |FAS(G)|$.*

PROOF. Let G be a digraph with $|V(G)| = n$ and $G' \subsetneq G$ where $V(G) = V(G')$. Due to Theorem 2, there is a function $f : U(G') \rightarrow U(G)$, $lab' \rightarrow$

$f(lab') = lab$, where $lab|_{G'} = lab'$. Thus, it is easy to see that f is injective. Hence, $|U(G')| \leq |U(G)|$.

Besides, since $G' \subsetneq G$ and $V(G) = V(G')$, then $\exists (a, b) \in A(G) - A(G')$. Let $s_1, s_2 \in S_n$ such that $s_1(a) = s_1(b) = s_2(a) = 1$, $s_2(b) = 2$, $s_1(c) = 2$ and $s_2(c) = 3$, $\forall c \in V(G) - \{a, b\}$. Then $lab_{s_1}|_{G'} = lab_{s_2}|_{G'}$, but $lab_{s_1} \neq lab_{s_2}$. Defining $lab' = lab_{s_1}|_{G'}$ and supposing w.l.o.g that $f(lab') = lab_{s_1}$, we deduce that f is not surjective because there is no $lab'' \neq lab'$, $lab'' \in U(G')$ such that $f(lab'') = lab_{s_2}$. Hence, $|U(G')| < |U(G)|$.

On the other hand, the function $g : FAS(G') \to FAS(G)$, $F' \to g(F') = F' \cup (A(G) - A(G'))$, is evidently injective and well-defined. Thus, $|FAS(G')| \leq |FAS(G)|$. But for a given $(a, b) \in A(G) - A(G')$, $F = A(G) - \{(a, b)\} \in FAS(G)$, however, there is no $F' \in FAS(G')$ such that $g(F') = F$. Hence, g is not surjective and in this way. $|FAS(G')| < |FAS(G)|$. □

Theorem 6 *Let G be an undirected graph and G_1 and G_2 two orientations of G such that every cycle of G_1 is also a cycle of G_2. Then $|U(G_1)| \leq |U(G_2)|$ and $|FAS(G_2)| \leq |FAS(G_1)|$.*

PROOF. Let G, G_1 and G_2 as in the hypothesis of the proposition. We define the function $f : NU(G_2) \longrightarrow NU(G_1)$, $lab \to f(lab) = lab'$ by:

$$lab'(u, v) = \begin{cases} lab(u, v), & \text{if } (u, v) \in A(G_1) \cap A(G_2), \\ \overline{lab}(u, v), & \text{in other case.} \end{cases} , \forall (u, v) \in A(G_1)$$

where $\overline{lab} : A(G_1) \longrightarrow \{\ominus, \oplus\}$ is defined by:

$$\overline{lab}(u, v) = \oplus \Leftrightarrow lab(v, u) = \ominus$$

f is well defined. First, note from definition of lab' that $(G_2)_R = (G_1)_R$, where $(G_1)_R$ and $(G_2)_R$ are the reoriented multidigraphs associated to (G_1, lab') and (G_2, lab) respectively. In this way, if C is a cycle in $((G_2)_R, lab_R)$ with some negative arc, then C is also a cycle in $((G_1)_R, lab'_R)$ with some negative arc (see Figure 4).

In fact, suppose on the contrary that C is a cycle in $((G_1)_R, lab'_R)$ with all the arcs positive. This is possible if and only if C is a cycle in (G_1, lab') with all the arcs positive and by hypothesis of the proposition, C is also a cycle in (G_2, lab) with all the arcs positive, which is a contradiction. Therefore, $lab \in NU(G_2) \Rightarrow f(lab) = lab' \in NU(G_1)$.

On the other hand, it is easy to see from the definition of lab' that f is injective. Therefore, $|NU(G_2)| \leq |NU(G_1)|$ and consequently $|U(G_1)| \leq |U(G_2)|$.

Finally, $|NFAS(G_1)| \leq |NFAS(G_2)|$ and consequently $|FAS(G_2)| \leq |FAS(G_1)|$. In fact, if $F \subseteq A(G_1)$ is a non-feedback arc set of G_1, then $F' = \{(u,v) \in G_2 : (u,v) \in F \vee (v,u) \in F\}$ is a non-feedback arc set of G_2. $\qquad\square$
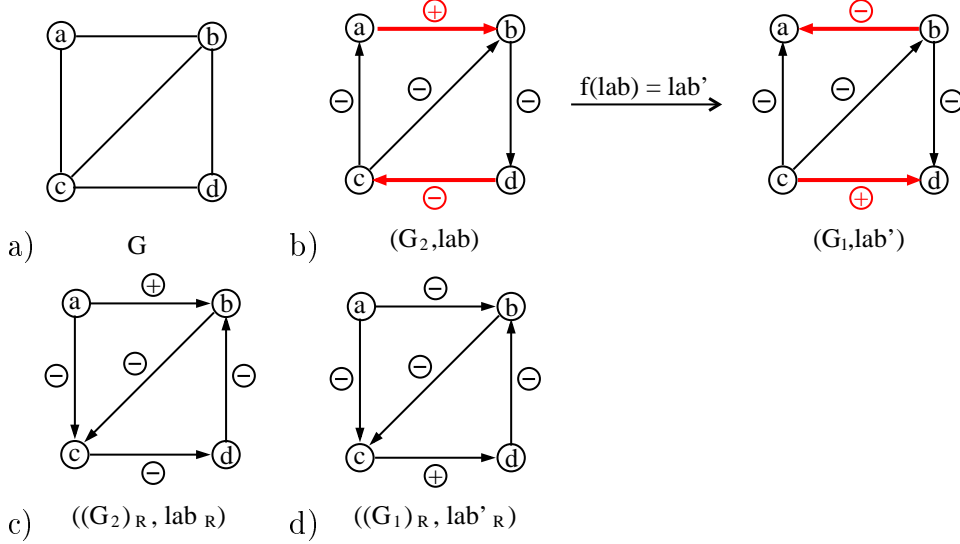

a)   G   b)   (G_2,lab)   f(lab) = lab'   (G_1,lab')

c)   ((G_2)$_R$, lab$_R$)   d)   ((G_1)$_R$, lab'$_R$)

Fig. 4. *a*) An undirected graph G. *b*) Two orientations G_1 and G_2 of G and the injective function f defined in the proof of Theorem 6. Observe that $|U(G_1)| = 18 < 20 = |U(G_2)|$ and $|FAS(G_2)| = 27 < 32 = |FAS(G_1)|$. *c*) The labeled reoriented multidigraph $((G_2)_R, lab_R)$ associated to (G_2, lab). *d*) The labeled reoriented multidigraph $((G_1)_R, lab'_R)$ associated to (G_1, lab').

Proposition 7 *Let G be a digraph. Then, $|U(G)| = |FAS(G)|$ if and only if all the circuits of G are cycles.*

PROOF. $\Rightarrow$) Let us suppose that G has a circuit C which is not a cycle. We will prove that $|U(G)| < |FAS(G)|$. Let (G, lab) be an update digraph, then $\{a \in A(G) : lab(a) = \oplus\}$ is a feedback arc set of G induced by the function lab, and thus $|U(G)| \leq |FAS(G)|$. We will show that there is another feedback arc set F' of G not induced by any element of $U(G)$. Indeed, there exists a label function lab' such that the circuit C is a forbidden circuit of (G, lab') with at least a positive arc and a negative arc in C. Let $F' = A(G) - \{a \in A(C) : lab'(a) = \ominus\}$. It is easy to check that F' is a feedback arc set of G which is not induced by any element in $U(G)$.

$\Leftarrow$) Let us suppose that all circuits of G are cycles. We want to prove that $|FAS(G)| \leq |U(G)|$.

We define the following function $T_G : FAS(G) \longrightarrow U(G)$ such that for each feedback arc set $F \in FAS(G)$, $T_G(F)$ is the label function:

$$T_G(F)(a) = \begin{cases} \oplus, & \text{if } a \in F, \\ \ominus, & \text{if } a \notin F \end{cases}$$

Then, $T_G(F) \in U(G)$. Indeed, if there exists a forbidden circuit C in $(G, T_G(F))$, there is at least an arc $a \in A(C)$ such that $T_G(F)(a) = \oplus$. Since, if for all $a \in C$, $T_G(F)(a) = \ominus$ then $A(C) \cap F = \emptyset$, which contradicts the fact: $F \in FAS(G)$. Besides, there exists $a' \in A(C)$, $T_G(F)(a') = \ominus$. Hence, C is a circuit in G which is not a cycle, which is a contradiction. Therefore, T_G is well-defined and obviously injective. □

Remark 8 *Observe that in a strongly connected digraph G, every circuit in G is a cycle if and only if all cycles are pairwise arc-disjoints.*

From the proof of Proposition 3, we know that to each $F \subseteq A(G)$, a minimal feedback arc set of G, can be associated a label function lab_F such that (G, lab_F) is an update digraph. However, the opposite is not always possible. Next, the following theorem shows a characterization of the update digraphs (G, lab) induced by a minimal feedback arc set.

Theorem 9 *Let G be a digraph with $|V(G)| = n$. Then, $F \subseteq A(G)$ is a minimal feedback arc set of G if and only if (G, lab_F) is an update digraph with a maximal number of negative arcs, where $lab_F(u, v) = \oplus \Leftrightarrow (u, v) \in F$.*

PROOF. $\Rightarrow$) Let G be a digraph with $|V(G)| = n$ and F a minimal feedback arc set of G. From the proof of Proposition 3, we know that (G, lab_F) is an update digraph.

Suppose on the contrary that (G, lab_F) has not a maximal number of negative arcs, i.e., there exists a label function lab with strictly more negative arcs than lab_F such that (G, lab) is an update digraph, and where $lab_F(u, v) = \ominus$ implies $lab(u, v) = \ominus$, for every $(u, v) \in A(G)$. Then $F' = \{(u, v) \in A(G) : lab(u, v) = \oplus\}$ is a feedback arc set of G and verifies that $F' \subsetneq F$, which contradicts the minimality of F. Hence, (G, lab_F) is an update digraph with maximal number of negative arcs.

$\Leftarrow$) Let lab_F be a label function for which (G, lab_F) is an update digraph with maximal number of negative arcs. We know that $F \subseteq A(G)$ is a feedback arc set of G because (G, lab_F) is an update digraph, which implies that $G - F$ is acyclic.

Now, suppose on the contrary that F is not minimal, i.e, there exists a minimal feedback arc set $F' \subseteq F$ of G and consequently a label function $lab_{F'}$ with which $(G, lab_{F'})$ is also an update digraph, but having the same negative arcs

as (G, lab_F) and more. This contradicts the maximality of (G, lab). Therefore, F is a minimal feedback arc set of G. □

Remark 10 *It is easy to see that a labeled digraph (G, lab) with all its arcs negative is an update digraph if and only if G is an acyclic digraph (see (Younger, 1963)). However, in the general case, we have the following result.*

Proposition 11 *An update digraph (G, lab) with maximum number of negative arcs has at least $\frac{|A(G)|}{2}$ negative arcs.*

PROOF. Let (G, lab) be an update digraph and $na(G, lab) = |\{(u, v) \in A(G) : lab(u, v) = \ominus\}|$ the number of negative arcs of (G, lab).

If G is an acyclic digraph, then (G, lab) with all its arcs negative is an update digraph, i.e. $na(G, lab) = |A(G)|$.

If G has a cycle, then let $s_{q_1} = (v_1)(v_2) \cdots (v_n)$ and $s_{q_2} = (v_n)(v_{n-1}) \cdots (v_1)$ two sequential update schedules with $v_i \in V(G)$, $\forall i \in [\![1, n]\!]$. Then, $\forall i, j \in [\![1, n]\!]$ such that $(v_i, v_j) \in A(G)$,

$$lab_{s_{q_1}}(v_i, v_j) = \ominus \Leftrightarrow lab_{s_{q_2}}(v_i, v_j) = \oplus$$

Thus, since (G, lab) has maximum number of negative arcs, must satisfy $na(G, lab) \geq max\{na(G, lab_{s_{q_1}}), na(G, lab_{s_{q_2}})\} \geq \frac{|A(G)|}{2}$. □

Proposition 12 *Let (G, lab) be an update digraph with a maximal number of negative arcs. Then, there is no positive cycle.*

PROOF. Let (G, lab) be an update digraph with a maximal number of negative arcs, i.e, there is no positive arc that can be changed by a negative arc holding the property of update digraph. Let s be an update schedule such that $(G, lab) = (G, lab_s)$ and suppose on the contrary that there is a positive cycle C, then there exists $(i, j) \in A(C)$ such that $s(i) = s(j)$. Defining s' by:

$s'(k) = s(k), \forall k, s(k) < s(i).$
$s'(i) = s(i)$
$s'(k) = s(k) + 1, \forall k \neq i, s(k) \geq s(i).$

we have that $lab_s(u, v) = \ominus \Rightarrow lab_{s'}(u, v) = \ominus$, but $lab_s(i, j) = \oplus \neq lab_{s'}(i, j)$ which contradicts the maximality of (G, lab). □

Proposition 13 *Let (G, lab) be an update digraph. Then, there is a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$ if and only if (G, lab) has no positive cycle.*

PROOF. $\Rightarrow$) Straightforward.

$\Leftarrow$) Let (G, lab) be an update digraph with $|V(G)| = n$ and without positive cycles, then (G_R, lab_R) has no cycles. Therefore, there exists a topological order $i_1, \ldots, i_n$ of the vertices, i.e., $(i_j, i_k) \in A(G_R) \Rightarrow k > j$, such that $s_q = (i_1)(i_2) \ldots (i_n)$ is a sequential schedule satisfying $(G, lab) = (G, lab_{s_q})$. $\square$

In general, the necessary condition of Proposition 12 is not sufficient. As a counter-example, we can consider a simple labeled digraph (G, lab) composed by only one positive arc (a, b) which is obviously an update digraph without positive cycles, but not maximal. Nevertheless, Theorem 17 of the next section shows that in update complete digraphs, the sufficiency holds.

3.1 Feedback arc sets in update complete digraphs

In the particular case of complete digraphs, we will see how many of the properties of update digraphs and feedback arc sets previously mentioned are equivalent between them, establishing an explicit form for determining the set $MFAS(G)$.

Remark 14 *Let G be a complete digraph with $|V(G)| = n$ and F a minimal feedback arc set of G. Then, $|F| = \binom{n}{2}$ and talk of minimum feedback arc set it is equivalent to talk of minimal feedback arc set. In fact, suppose that $|F| > \binom{n}{2}$, then necessarily there exist u, v in $V(G)$ such that $\{(u, v), (v, u)\} \subseteq F$. Since F is a feedback arc set, there exists path $P_1 = u_1, \ldots, u_k$ with $u_1 = v$, $u_k = u$ which does not contain any arc of F (in particular the arc (v, u)), for some $k \in [\![2, n]\!]$ and $u_i \in V(G)$, $\forall i \in [\![1, k]\!]$. Analogously, there exists path $P_2 = v_1, \ldots, v_j$ with $v_1 = u$, $v_j = v$ which does not contain any arc of F (in particular the arc (u, v)), for some $j \in [\![2, n]\!]$ and $v_i \in V(G)$, $\forall i \in [\![1, j]\!]$. Thus, joining the paths P_1 and P_2 we deduce the existence of a cycle which does not contain any arc of F, a contradiction because F is a feedback arc set. Hence $|F| \leq \binom{n}{2}$. On the other hand, $|F| \geq \binom{n}{2}$ because G has $\binom{n}{2}$ cycles of length two.*

The following Proposition is a simple characterization of update complete digraphs.

Proposition 15 *Let G be a complete digraph. Then, (G, lab) is a non update digraph if and only if there exists a forbidden cycle of length either two or three in G_R.*

PROOF. $\Leftarrow$) It is straightforward.

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$\Rightarrow$) Let (G, lab) be a non-update complete digraph. Then, there exists a forbidden cycle C_R of smallest length in (G_R, lab_R). Let suppose that the length of C_R is strictly greater than three.

Let C be the forbidden circuit associated to C_R in G. If C has a path of the form (a, b), (b, c) $\oplus$-labeled, then necessarily (a, c) is also positive in G because otherwise, there would exist a forbidden cycle C' of length three strictly smaller than C_R in G_R. Analogously, if C has a path of the form (a, b), (b, c) $\ominus$-labeled, then necessarily (a, c) is negative (see Fig. 5: a) and b)).

Therefore, we can suppose that C does not have two consecutive arcs with the same label. But when, for example, $lab(b, a) = \ominus$ and $lab(b, c) = \oplus$, necessarily the arc (c, a) is negative in G (i.e., (a, c) is negative in G_R), because on the contrary, there would exist a forbidden cycle C' of length three smaller than C_R in G_R (see Fig. 5: c) and d)).

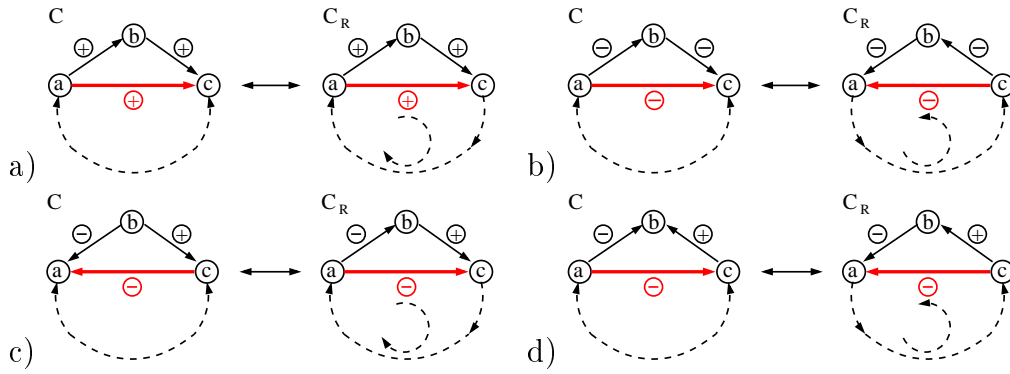In this way, it is always possible to reduce the length of the forbidden cycle C_R up to obtain a length of three. $\qquad\square$

Fig. 5. Proof idea of Proposition 15. a) When the arcs (a, b) and (b, c) are positive in C, then necessarily (a, c) is also positive in C. b) When the arcs (a, b) and (b, c) are negative in C, then necessarily (a, c) is also negative in C. c) If $lab(b, a) = \ominus$ and $lab(b, c) = \oplus$ in C, then necessarily $lab(c, a) = \ominus$. d) If $lab(a, b) = \ominus$ and $lab(c, b) = \oplus$ in C, then necessarily $lab(a, c) = \ominus$.

Proposition 16 *Let G be a complete digraph with $|V(G)| = n$. Then, F is a minimal feedback arc set of G if and only if $G' = G - F$ is an acyclic tournament.*

PROOF. $\Rightarrow$) Let F be a minimal feedback arc set of G, then $G' = G - F$ represents an acyclic digraph. Besides, G' is a tournament. In fact, suppose on the contrary that G' is not a tournament, i.e., there is a pair of vertices a, b in $V(G')$ such that $\{(a, b), (b, a)\} \cap A(G') = \emptyset$. That minds $\{(a, b), (b, a)\} \subseteq F$ and since F must have at least one arc for each pair of vertices of $V(G)$, we

have that $|F| \geq \binom{n}{2} + 1$, which is a contradiction because by Remark 14, $|F| = \binom{n}{2}$.

$\Leftarrow$) Let G' be an acyclic tournament, then $F = A(G) - A(G')$ is a minimal feedback arc set of G, because $|F| = |A(G) - A(G')| = \binom{n}{2}$. $\qquad\square$

Theorem 17 *Let (G, lab) be an update complete digraph with $|V(G)| = n$. The following statements are equivalent:*

(i) *(G, lab) has a maximal number of negative arcs.*
(ii) *The digraph induced by the negative arcs of (G, lab) is an acyclic tournament.*
(iii) *(G, lab) has no positive cycle.*

PROOF. Let (G, lab) be an update complete digraph with $|V(G)| = n$.

$(i) \Rightarrow (ii)$. If (G, lab) has a maximal number of negative arcs, then by Theorem 9, $F = \{(u, v) \in A(G) : lab(u, v) = \oplus\}$ is a minimal feedback arc set of G. Hence, by Proposition 16, $G' = G - F$ is an acyclic tournament, i.e., the digraph induced by the negative arcs of (G, lab) is an acyclic tournament.

$(ii) \Rightarrow (iii)$. If the digraph induced by the negative arcs of (G, lab) is an acyclic tournament, then the digraph induced by the positive arcs of (G, lab) is also an acyclic tournament because G is a complete digraph, i.e. (G, lab) has no positive cycle.

$(iii) \Rightarrow (i)$. If (G, lab) has no positive cycle, then all its cycles of length two have only one negative arc. Since a complete digraph has $\binom{n}{2}$ cycles of length two, (G, lab) has at least $\binom{n}{2}$ negative arcs, but this is the maximum number, and thus the maximal number of negative arcs that an update complete digraph can have. $\qquad\square$

Corollary 18 *Let G be a complete digraph with $|V(G)| = n$. Then, $|MFAS(G)| = |\{lab : (G, lab)$ is an update digraph with maximal number of negative arcs$\}| = n!$*

PROOF. It is easy to see that the number of acyclic tournaments of size n is $n!$. Then, the Corollary 18 follows from Theorem 9 and Proposition 16. $\qquad\square$

Note also due to Theorem 9, Proposition 13 and Theorem 17 that finding all possible minimal feedback arc sets of a complete digraph can be made considering the update digraphs associated with the $n!$ different sequential

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update schedules, where a minimal feedback arc set will consist of their positive arcs. In other words, if G is a complete digraph, then,

$$MFAS(G) = \{F \subseteq A(G) : \ lab_{s_q}(a) = \oplus \Leftrightarrow a \in F \ \wedge \ s_q \text{ sequential}\}$$

4 Tournaments and update digraphs

In the previous sections, we found bounds for the sizes of the schedule equivalence classes associated with a given update digraph G as well as relationships between the sets $U(G)$, $MFAS(G)$ and $FAS(G)$. In this section, we will restrict to classical families such as acyclic digraphs, complete digraphs and tournaments in order to determine exactly the number and size of their schedule equivalence classes. Also we will show how the number of negative arcs of a given update digraph (G, lab) is related to these families and the feedback arc sets.

Theorem 19 *Let G be an acyclic digraph with $|V(G)| = n$. Then,*

(a) $|U(G)| \leq n!$.
(b) $|U(G)| = n! \Leftrightarrow G$ *is a tournament.*
(c) *If G is a tournament, then for each update schedule s of G, $|[s]_G| = 2^k$, for some $k \in \mathbb{N} \cup \{0\}$.*

PROOF. (a) For every $lab \in U(G)$, (G, lab) has no positive cycle. Hence, by Proposition 13 there exists a sequential schedule s_q such that $(G, lab) = (G, lab_{s_q})$.

Therefore, since the number of different sequential update schedules is $n!$, we have that:

$$|U(G)| \leq n!$$

(b) $\Leftarrow$) Let G be a tournament with $|V(G)| = n$ and let s_1, s_2 be two different sequential update schedules, then by tournament definition there is $(i, j) \in A(G)$ such that $lab_{s_1}(i, j) \neq lab_{s_2}(i, j)$. Since every sequential update schedule s define a different update digraph (G, lab_s) and the number of different sequential update schedules is $n!$, we have that:

$$|U(G)| \geq n!$$

and due to (a), we conclude that $|U(G)| = n!$.

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$\Rightarrow$) Let G be an acyclic digraph with $|V(G)| = n$ and $|U(G)| = n!$. Suppose on the contrary that G is not a tournament, then necessarily there exist u, v in $V(G)$ without arcs between them (two arcs between them is not possible because G is acyclic) which implies that there are two different sequential update schedules with the same update digraph. This contradicts the fact that $|U(G)| = n!$. Therefore, G is also a tournament.

(c) Let T be an acyclic tournament. By induction on $m = |V(T)|$, we have:

Basis Step, $m{=}2$. There are two posibilities: the equivalence classes of the parallel update schedule s_p, where it is easy to see that $|[s_p]_T| = 2^1 = 2$, and of the sequential update schedule $s_q \notin [s_p]_T$, where clearly $|[s_q]_T| = 2^0 = 1$.

Induction Hypothesis. T an acyclic tournament with $m \leq n - 1$. Hence, for each update schedule s over $V(T)$, $|[s]_T| = 2^k$, for some $k \in \mathbb{N} \cup \{0\}$.

Let G be an acyclic tournament with $|V(G)| = n$. First, note that the proof of (b) implies the existence of a bijection $f : U(G) \rightarrow \{s \in S_n : s \text{ sequential update schedule on } V(G)\}$. Let (G, lab_s) be an update digraph with the sequential update schedule $s = (i_1)(i_2)\cdots(i_j)(n)(i_{j+1})\cdots(i_{n-1})$, where $\{i_1, ..., i_{n-1}\} = \{1, \ldots, n-1\}$.

Let $G' = G - \{n\}$. Clearly, G' is an acyclic tournament with $|V(G')| = n - 1$ and $s' = (i_1)\cdots(i_j)(i_{j+1})\cdots(i_{n-1})$ is the sequential update schedule such that $(G', lab_{s'})$ is the labeled subdigraph of (G, lab) induced by $\{i_1, \ldots, i_{n-1}\}$. Defining $S_{dif} = \{s^* \in [s']_{G'} : s^*(i_j) = s^*(i_{j+1}) - 1\} \neq \emptyset$ and $S_{eq} = \{s^* \in [s']_{G'} : s^*(i_j) = s^*(i_{j+1})\}$, we can see that if $S_{eq} \neq \emptyset$, then $|S_{dif}| = |S_{eq}| = 2^{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$.

There are the following $2^3 = 8$ cases:

Case 1. $lab_s(i_j, n) = \ominus$ and $lab_s(i_{j+1}, n) = lab_s(i_{j+1}, i_j) = \oplus$ (see Figure 6 a)).
To obtain all the elements of $[s]_G$, for every $s^* \in S_{dif}$, it is enough to generate s_1 and s_2 in $[s]_G$ where $s_1(p) = s_2(p) = s^*(p)$, $\forall p \in V(G')$ such that $s^*(p) \leq s^*(i_j)$; $s_1(n) = s_2(n) = s^*(i_j) + 1$ and $s_2(q) = s_1(q) - 1 = s^*(q)$, $\forall q \in V(G')$ satisfying $s^*(q) \geq s^*(i_{j+1})$. In this way, $|[s]_G| = 2 \cdot |S_{dif}| = 2^k$ for some $k \in \mathbb{N} \cup \{0\}$.

Case 2. $lab_s(n, i_j) = lab_s(i_{j+1}, n) = lab_s(i_{j+1}, i_j) = \oplus$ (see Figure 6 b)).
To obtain all the elements of $[s]_G$, for every $s_1^* \in S_{dif}$, we generate s_1, s_2 and s_3 in $[s]_G$ where $s_1(p) = s_2(p) = s_3(p) = s_1^*(p)$, $\forall p \in V(G')$ such that $s_1^*(p) \leq s_1^*(i_j)$; $s_1(n) = s_3(n) = s_2(n) + 1 = s_1^*(i_j) + 1$ and $s_2(q) = s_3(q) = s_1(q) - 1 = s_1^*(q)$, $\forall q \in V(G')$ satisfying $s_1^*(q) \geq s_1^*(i_{j+1})$. And for every $s_2^* \in S_{eq}$, we generate $s_4 \in [s]_G$ defined by $s_4(p) = s_2^*(p)$, $\forall p \in V(G')$, $p \neq n$ and $s_4(n) = s_2^*(i_j)$. Therefore, $|[s]_G| = 3 \cdot |S_{dif}| + |S_{eq}| = 3 \cdot 2^{k-1} + 2^{k-1} = 2^{k+1}$

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for some $k \in \mathbb{N} \cup \{0\}$.

Case 3. $lab_s(i_j, n) = lab_s(i_j, i_{j+1}) = lab_s(n, i_{j+1}) = \ominus$ (see Figure 6c)). Here, for each $s_1^* \in S_{dif}$, we generate $s_1 \in [s]_G$ where $s_1(p) = s^*(p)$, $\forall p \in V(G')$ such that $s^*(p) \leq s^*(i_j)$; $s_1(n) = s^*(i_j) + 1$ and $s_1(q) = s^*(q) + 1$, $\forall q \in V(G')$ satisfying $s^*(q) \geq s^*(i_{j+1})$. Therefore, $|[s]_G| = |S_{dif}| = 2^{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$.

Case 4. $lab_s(i_j, n) = lab_s(i_j, i_{j+1}) = \ominus$ and $lab_s(i_{j+1}, n) = \oplus$.

Case 5. $lab_s(n, i_j) = lab_s(i_{j+1}, i_j) = \oplus$ and $lab_s(n, i_{j+1}) = \ominus$.

Case 6. $lab_s(n, i_{j+1}) = lab_s(i_j, i_{j+1}) = \ominus$ and $lab_s(n, i_j) = \oplus$.

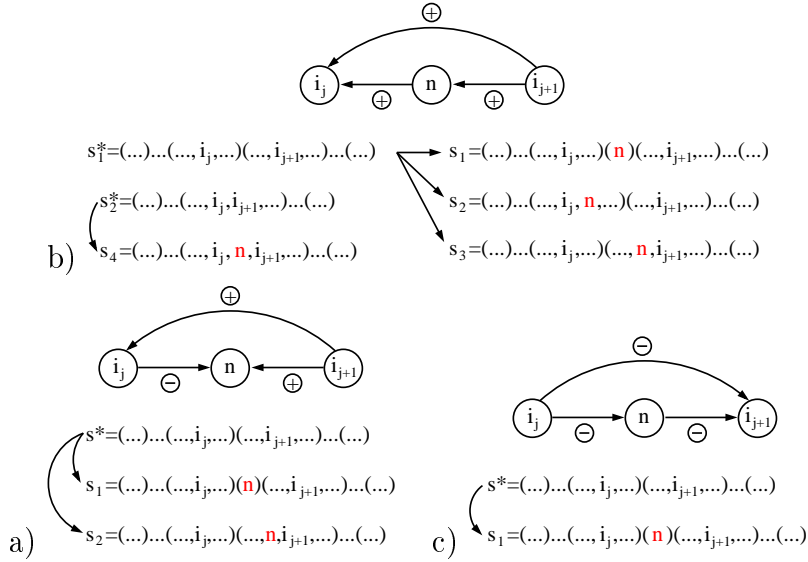Cases 4, 5 and 6 are similar to case 1. $\qquad\square$

Fig. 6. Proof idea of Theorem 19. a) Case 1.   b) Case 2.   c) Case 3.

Observe that the assertion (c) of Theorem 19 is not true for tournaments in general (see Fig. 7 as a counter-example).
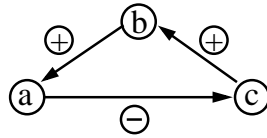

Fig. 7. An update digraph (G, lab_s) where $|[s]_G| = |\{(a)(b)(c), (a)(b, c), (a, b)(c)\}| = 3$.

Corollary 20 *Let (G, lab) be an update digraph where G is an acyclic tournament. Then, all arcs of (G, lab) are negative, if and only if there exists a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$ and $|[s_q]_G| = 1$.*

PROOF. Let G be an acyclic tournament. First, observe that every acyclic digraph (G, lab) with all its arcs negative is an update digraph.

$\Rightarrow$) Let (G, lab) with all its $|A(G)| = \binom{n}{2}$ arcs negative. Hence, by Proposition 13, there exists a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$. Now, suppose that there exists another update schedule $s \neq s_q$ such that $s \in [s_q]_G$. By Theorem 19, we know that s can not be a sequential schedule. If s is an update schedule but not sequential, then there is a block B_i of s, $1 \leq i \leq nb(s) < |V(G)|$, such that $|B_i| > 1$, i.e., there exists u, v in B_i such that the arc $(u, v) \in A(G)$ is positive in (G, lab_s), but the same arc is negative in (G, lab_{s_q}), which is a contradiction. Therefore, $|[s_q]_G| = 1$.

$\Leftarrow$) Let $s_q = (v_1)(v_2)\cdots(v_n)$ be a sequential update schedule such that $(G, lab) = (G, lab_{s_q})$, $|[s_q]_G| = 1$ and $V(G) = \{v_1, \ldots, v_n\}$. Since (G, lab) is an update tournament, for every pair of vertices v_i, v_j in $V(G)$, there is only one arc between them, either the negative arc (v_i, v_j) or the positive arc (v_j, v_i) with i, $j \in \{1, ..., n\}$. Observe that $\forall i = 1, \ldots, n-1$, $(v_i, v_{i+1}) \in A(G)$, since otherwise there would be another update schedule $s \neq s_q$, $s = (v_1)(v_2)\cdots(v_{i-1})(v_i, v_{i+1})(v_{i+2})\cdots(v_n)$ such that $(G, lab) = (G, lab_{s_q}) = (G, lab_s)$, which contradicts that $|[s_q]_G| = 1$ (see Fig. 8 a)). Besides, for every $i \neq j \in \{1, \ldots, n\}$ such that $j > i + 1$, $(v_i, v_j) \in A(G)$ because if there exists the positive arc (v_j, v_i), from the above mentioned, we would have the existence of the negative arcs (v_i, v_{i+1}), (v_{i+1}, v_{i+2}),...,(v_{j-1}, v_j) which implies the existence of a cycle (see Fig. 8 b)). It would be a contradiction, because G is acyclic. Therefore, all arcs of (G, lab) must be negative. $\square$
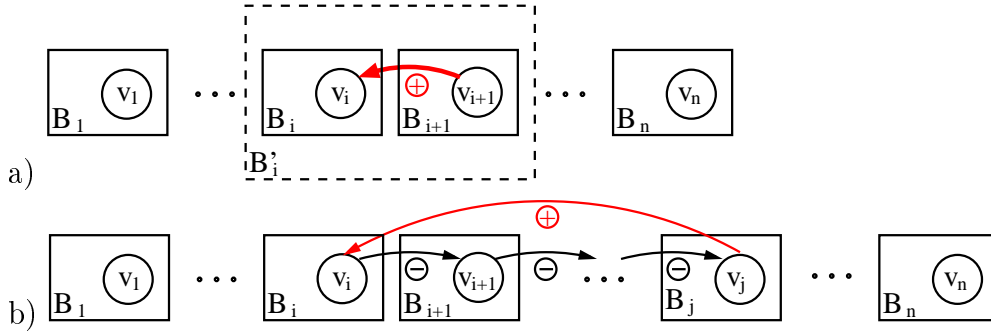

Fig. 8. Proof idea of Corollary 20. a) Case $j = i + 1$. If there is a positive arc from the B_{i+1} to B_i, it is possible to construct a new update schedule $s \neq s_q$ with block $B'_i = B_i \cup B_{i+1}$. b) Case $j > i + 1$. There is necessarily a cycle (v_i, v_{i+1}), (v_{i+1}, v_{i+2}),...,(v_{j-1}, v_j), (v_j, v_i).

Observe that from previous proof, if G is a tournament with at least a cycle, then there exists an update schedule s such that $[s]_G$ does not contain a sequential schedule. Hence, $|U(G)| > n!$.

Theorem 21 *Let G be a digraph with $|V(G)| = n$, $P = \{B_i\}_{i=1}^{m}$ a partition*

of $V(G)$, $m \leq n$, and $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n$, where π is a permutation over $[\![1, m]\!]$. Then, the following statements are equivalents:

(1) $s_\pi \in S_n^*$.
(2) $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}).
(3) $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a strongly connected subdigraph of G.

PROOF. Let G be a digraph with $|V(G)| = n$, $P = \{B_i\}_{i=1}^m$ a partition of $V(G)$, $m \leq n$, and $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n$, where π is a permutation over $[\![1, m]\!]$.

$(1) \Rightarrow (2)$ Suppose that $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n^*$, i.e., s_π has the maximum number of blocks. We note that P is the partition associated to s_π. Now, let us prove that $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}), i.e. $G[B_i]$ is a strongly connected component of $G' = G - \{a \in A(G) : lab_{s_\pi}(a) = \ominus\}$. If $|B_i| = 1$, then the proof is done. Hence, we can assume w.l.o.g. that $|B_i| \geq 2$. Suppose on the contrary that $G[B_i]$ is not a positive strongly connected component of (G, lab_{s_π}). Then, either $G[B_i]$ is not a subdigraph of G', which means that $\exists a \in A(G') \cap A(G)$, $lab_{s_\pi}(a) = \ominus$, but this is impossible because (G, lab_{s_π}) is an update digraph, or $G[B_i]$ is a strongly connected subdigraph of G', but not maximal for this property. i.e., $\exists u \notin B_i$ strongly connected with the nodes of B_i by positive arcs, this means that $s_\pi(u) = s_\pi(j), \forall j \in B_i$, which is a contradiction because $u \notin B_i$, or $G[B_i]$ is not strongly connected, then there exists a strongly connected component $U \subseteq B_i$ which is a source, i.e. without incoming arc to U, having initial vertex in $B_i - U$. Thus, since $G[B_i] = G[B_{\pi(j)}]$ for some $j \in [\![1, m]\!]$, there is another update schedule $s' = (k \in B_{\pi(1)}) \cdots (k \in B_{\pi(j)} - U)(k \in U)(k \in B_{\pi(j+1)}) \cdots (k \in B_{\pi(m)})$ on $V(G)$ such that $s' \in [s_\pi]_G$ and $nb(s') = nb(s_\pi) + 1 = m + 1$, which contradicts that s_π has the maximum number of blocks.

$(2) \Rightarrow (3)$ If $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}), then $G[B_i]$ is a strongly connected component of $G' = G - \{a \in A(G) : lab_{s_\pi}(a) = \ominus\} \subseteq G$, i.e., $G[B_i]$ is a strongly connected subdigraph of G.

$(3) \Rightarrow (1)$ Suppose that $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a strongly connected subdigraph of G. Let $s' \in [s_\pi]_G$, then $(G, lab_{s'}) = (G, lab_{s_\pi})$. Thus, s' must satisfy that for every $i \in [\![1, m]\!]$, $s'(j) = s'(k), \forall j, k \in B_i$, because s' is an update schedule on $V(G)$ and $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a strongly connected subdigraph of G with all its arcs positive in (G, lab_{s_π}). Hence, $P = \{B_i\}_{i=1}^m$ is also the partition of $V(G)$ associated to s', then necessarily $nb(s') = nb(s) = m$. Therefore, we have showed that, $\forall s' \in [s_\pi]_G$, $nb(s') \leq nb(s_\pi)$, i.e., $s_\pi \in S_n^*$. $\qquad \square$

Lemma 22 *Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph. Then, for every $s_1, s_2 \in S_n^*$ such that $s_1 \neq s_2$, we have: $(G, lab_{s_1}) \neq (G, lab_{s_2})$.*

PROOF. Let G be a digraph with $|V(G)| = n$ containing a tournament of n vertices as a subdigraph. Let $s_1, s_2 \in S_n^*$, $s_1 \neq s_2$ and suppose on the contrary that $(G, lab_{s_1}) = (G, lab_{s_2})$. Let $s_1 = (B_i^1)_{i=1}^{nb(s_1)}$, $s_2 = (B_i^2)_{i=1}^{nb(s_2)}$ and $j = \min\{i : B_i^1 \neq B_i^2\}$. Hence, there are two cases:

Case 1: $B_j^1 \cap B_j^2 \neq \emptyset$. Then, from (2) of Theorem 21, B_j^1 and B_j^2 are positive strongly connected components of (G, lab_{s_1}). On the other hand, because the decomposition of (G, lab_{s_1}) in positive strongly connected components is unique, then $B_j^1 = B_j^2$, which is a contradiction.

Case 2: $B_j^1 \cap B_j^2 = \emptyset$. Then, $\exists p \in B_j^1$, $\exists q \in B_j^2$, $p \neq q$: $s_1(p) < s_1(q) \wedge s_2(q) < s_2(p)$. Since G has a tournament of n vertices, then $(p, q) \in A(G) \vee (q, p) \in A(G)$, i.e., $lab_{s_1}(p, q) \neq lab_{s_2}(p, q) \vee lab_{s_1}(q, p) \neq lab_{s_2}(q, p)$, which contradicts that $(G, lab_{s_1}) = (G, lab_{s_2})$.

Therefore, $(G, lab_{s_1}) \neq (G, lab_{s_2})$. $\qquad\qquad\square$

Theorem 23 *Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph. Then, $|U(G)| = |S_n^*|$.*

PROOF. Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph.

Evidently, due to Lemma 22, we have that $|U(G)| \geq |S_n^*|$.

On the other hand, $\forall lab \in U(G)$, $\exists s \in S_n$: $(G, lab) = (G, lab_s)$. Hence, $\exists s' \in [s]_G$ with the maximum number of blocks. Thus, $|U(G)| \leq |S_n^*|$. $\qquad\square$

Observe from Theorem 21 that:

$$S_n^* = \{(B_{\pi(i)})_{i=1}^m : \ P = \{B_i\}_{i=1}^m \in \mathbb{P}_n^* \ \wedge \ \pi \text{ is a permutation over } [\![1, m]\!]\},$$

where $\mathbb{P}_n^*$ is the set of partitions $P = \{B_i\}_{i=1}^m$ of $V(G)$ such that $\forall i \in [\![1, m]\!]$, $G[B_i]$ is a strongly connected subdigraph of G. Thus, we obtain straightforward the following Corollary:

Corollary 24 *Let G be a digraph with $|V(G)| = n$ containing a tournament of n vertices as a subdigraph. Then:*

$$|U(G)| = \sum_{P \in \mathbb{P}_n^*} card(P)!$$

Note that if G is an acyclic tournament of n vertices, the only strongly connected subdigraphs are the trivial ones, i.e., $\mathbb{P}_n^* = \{\{\{1\}, ..., \{n\}\}\}$ and there-

fore,

$$|U(G)| = \sum_{P \in \mathbb{P}_n^*} card(P)! = card(\{\{1\}, ..., \{n\}\})! = n!,$$

giving the result (b) of Theorem 19.

On the other hand, if G is a complete digraph with $|V(G)| = n$, then every subset of vertices defines a strongly connected subdigraph of G. In this way, $\mathbb{P}_n^*$ is the set of all the possible partitions over $V(G)$ and therefore, $|U(G)| = T_n = \sum_{k=0}^{n-1} \binom{n}{k} T_k$, with $T_0 \equiv 1$, since it was obtained in (Demongeot et al., 2008; Aracena et al., 2011). Moreover, if G_1 and G_2 are two orientations of a complete undirected graph satisfying the conditions of Theorem 6, then a strongly connected subdigraph of G_1 is also a strongly connected subdigraph of G_2. Therefore, Corollary 24 implies that $|U(G_1)| \leq |U(G_2)|$, as established in Theorem 6.

5   Acknowledgments

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