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Abstract

The notion of reversibility has been intensively studied in the field of cellular automata (CA), for several reasons. However, a related notion found in physical theories has been so far neglected, not only in CA, but generally in discrete dynamical systems. This is the notion of time-symmetry, which refers to the inability of distinguishing between backward and forward time directions. Here we formalize it in the context of CA, and study some of its basic properties. We also show how some well-known CA fit into the class of time-symmetric CA, and provide a number of results on the relation between this and other classes of CA. The existence of an intrinsically universal time-symmetric CA within the class of reversible CA is proved. Finally, we show the undecidability of time-symmetry for CA of dimension 2 or higher, even within the class of reversible CA. The case of dimension 1 is one of several open questions discussed in the conclusions.

Keywords: Cellular Automata, Time-symmetry, Reversibility, Universality, Decidability.

1. Introduction

An important property that may be present or not in physical or abstract dynamical systems is reversibility; consequently, it has also been an active topic of research in the context of cellular automata [1]. At least two particular reasons for this interest are often mentioned: on one hand, if CA are seen as

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models for massive distributed computation, then Landauer's principle suggests that we should focus on reversible cases. On the other hand, reversibility is often observed in real systems; it is therefore desirable in models of them [2]. Furthermore, a number of interesting results (like the dimension-sensitive difficulty of deciding reversibility [3]) have kept reversible CA in sight over the years.

However, there is one aspect of reversibility, as seen in real systems, which has been mostly neglected when considering cellular automata (in fact, for discrete dynamics in general): the dynamical laws governing physical reality seem to be not only reversible, but *time-symmetric*. For Newtonian mechanics, relativity or quantum mechanics, we can go back in time by applying the same dynamics, provided that we change the sense of time's arrow through a specific transformation of phase-space. In the simplest example, Newtonian mechanics, the transformation leaves masses and positions unchanged, but reverses the sign of momenta.

In the most general sense, we say that a dynamical system (X, T) is time-symmetric if there exists a reversible $R : X \to X$ such that $R \circ T \circ R^{-1} = T^{-1}$ [4](notice that this applies to systems with discrete or continuous time). However, time-symmetries observed in physical systems follow usually a more restricted definition, in which $R^{-1} = R$, and therefore R is an *involution* on X. This is a natural restriction, which follows whenever there is no way to distinguish where the arrow of time is heading. Apparent irreversibility (Loschmidt's paradox) comes only from macroscopic (*i.e.*, coarse-grained) differences in entropy.

Here we study time-symmetric cellular automata, defined as those CA F for which there exists an involution H (which is a CA itself) such that

$$F^{-1} \;=\; H \circ F \circ H \tag{1}$$

Requiring H to be a CA is somewhat arbitrary, since for other systems the time-reversing transformation is not necessarily of the same nature as the dynamics (in fact, the physical theories discussed above are continuous in time). The reason for this restriction is that we expect reversibility (including the particular case of time-symmetry) to be a *local* property. The case when H is *not* a CA may be an interesting direction for future studies.

1.1. Elementary definitions

A cellular automaton (CA) is a function $F : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$, for some finite set of states S and some dimension d, which is defined through a *local function* $f : S^N \to S$ for some finite *neighborhood* $N \subseteq \mathbb{Z}^d$, so that $F(x)_i = f(x_{i+N}) \forall i \in \mathbb{Z}^d$; the function F is then referred to as the *global function*. A particular kind of CA are the *shifts*, for which $|N| = 1$ and f is the identity; when $d = 1$ the term "shift" refers by default to $\sigma(x)_i = x_{i+1}$. If we put the product topology on $\mathbb{Z}^d$, then CA can alternatively be defined as those continuous functions which commute with the shifts. A *subshift* is a closed set $A \subseteq S^{\mathbb{Z}^d}$ which is stable under the shifts.

For $d = 1$ we may assume a neighborhood of the form $\{-r, \dots, r\}$; r is then called the *radius* of the CA. Radius 0 CA are called *autarkic*. Common neighborhood choices for $d = 2$ are Moore's neighborhood $\{-1, 0, 1\}^2$ and von Neumann's neighborhood $\{(-1, 0), (0, 1), (1, 0), (0, -1), (0, 0)\}$.

We will denote with w^∞ the bi-infinite repetition of a finite word $w \in S^*$; semi-infinite repetitions will be denoted with ${}^\omega w$ or w^ω, according to their direction.

Definition 1. *Given two CA F and G on $S_F^{\mathbb{Z}^d}$ and $S_G^{\mathbb{Z}^d}$, respectively, we will distinguish three notions of conjugation.*

- *F and G are* conjugated *if there exists a continuous bijection $\phi : S_G^{\mathbb{Z}^d} \to S_F^{\mathbb{Z}^d}$ such that $F \circ \phi = \phi \circ G$.*
- *If in addition ϕ commutes with a power of the shifts, i. e., if there exist n, m such that $\phi \circ \sigma^n = \sigma^m \circ \phi$, then we say that F and G are* block conjugated.
- *If F and G are block conjugated with $n = m = 1$, we say that they are* CA conjugated.

A technical note: CA are usually defined over a full shift $S^{\mathbb{Z}}$, but they can also be studied over (stable) subshifts. We remark that in this case, for time-symmetry to apply, the subshift must be stable for both F and H. This may cause some problems, since subsystems of a time-symmetric CA cannot be assumed to be time-symmetric too, even if they are stable for F.

2. Some motivating examples

Not only our models of physical reality turn out to exhibit time-symmetry; it is also found in some well known reversible discrete dynamical systems. We show in this section how it applies to a couple of 2D systems, Margolus' billiard and Langton's ant. As a technical note, notice that in both cases the system is not originally described as a cellular automaton in the strict sense; therefore, we describe for each of them a CA that contains them as particular case for a subshift of valid configurations. This should -in principle- be followed by an extension of the rule to the full shift, and such that the system remains time-symmetric; however, doing that is not really required, since we want to show the time-symmetry of the original system; we hence restrict ourselves to the valid subshift.

Margolus' billiard ball CA. A well known example of time-symmetric CA is the Billiard ball model of Margolus [5]. It is not a proper CA, but rather a so-called *partitioned CA*, where the space $\mathbb{Z}^2$ is partitioned in 2×2 blocks of cells in two different ways (see Figure 1(a)). A transformation is applied to each block of each partition alternately. It is easy to see that such an automaton is reversible if and only if its local transformation is one-to-one. The rule used by Margolus is shown in Figure 1(b). It tries to emulate balls that move in straight lines, colliding elastically with each other or with static obstacles. The importance of this model comes from its Turing-universality [5].

We can express Margolus' system in terms of a CA with alphabet $\{white, black\} \times \{\nearrow, \searrow, \swarrow, \nwarrow\}$ and Moore neighborhood. Here the first layer (the white/black component) represents the states of the original Margolus model, and the other represents the current partition, along with the relative position of the cell within its current block. This layer must be initialized in an appropriate way in order to work correctly (see Figure 1)(c).

Notice that reversing the arrows makes the partition flip to the alternative one. At each time step, each cell computes its next white/black state by applying Margolus' rule to the quadrant indicated by its arrow, and then reverses its arrow. Each of this actions -the first on the first layer, the second on the second- is an involution. Furthermore, if at one time step we omit any of them, further iterations will make the automaton evolve back in time.

Langton's ant. Langton's ant was introduced in [6] together with several models emulating different life properties. It was also defined in physics as a model for particles presenting self correlated trajectories [7]. The model can be seen as a Turing machine working on a 2-dimensional tape. Its internal state is an arrow that represents its last movement direction. At each step, the ant turns to the left or to the right depending on the cell color (*white* or *black*), it flips this color and moves one cell forward (see Figure 2(a)). Besides being Turing-universal [8], its celebrity is due mostly to its particular behavior over finite initial configurations. Simulations show that it always falls eventually into a repetitive movement -of period 104- that makes it propagate unboundedly (see Figure 2(b)); this assertion has not been proved, and appears to be very difficult despite the simplicity of the transition rule.

Langton's ant can also be described in terms of a CA with Moore neighborhood and state set $\{head, tail, empty\} \times \{white, black\}$. We represent the arrow through two adjacent cells, one in state *head* and the other in state *tail*. The cell in state *tail* always becomes *empty*, while the cell in state *head* always becomes *tail* and flip its color. Cells adjacent to a *head* can decide to become *head* themselves by looking at the tail position and the color of the head cell. The system simulates Langton's ant only if it starts with only one ant.

Here again, we can define the involution consisting in exchanging *tails* and *heads*. This immediately makes the ant come back to the cell it just had left, which it finds in the color opposite to the one it had found before, causing the ant to turn in the opposite direction, which in turn makes it again go to a previously visited cell, and so on: the ant will forever retrace (and undo) its past trajectory.

3. Preliminary remarks

We begin with some basic observations.

Proposition 1. *Let F be a CA. Then the following are equivalent:*

(i) F *is time-symmetric.*

(ii) *There exists an involution* H *such that* $(F \circ H)$ *is an involution.*

(iii) F *is the composition of two involutions.*

Proof.
$1 \implies 2:$ Let F and H be the CA satisfying (1). Then

$$(F \circ H)^2 \;=\; F \circ H \circ F \circ H \;=\; F \circ F^{-1} \;=\; id$$

$2 \implies 3:$ Take H from *(ii)* and let $G = F \circ H$ which is an involution. We have

$$F \;=\; F \circ id \;=\; F \circ (H \circ H) \;=\; (F \circ H) \circ H \;=\; G \circ H$$

$3 \implies 1:$ Let G and H be involutions such that $F = G \circ H$. Then

$$F^{-1} \;=\; (G \circ H)^{-1} \;=\; H^{-1} \circ G^{-1} \;=\; H \circ G \;=\; H \circ G \circ H \circ H \;=\; H \circ F \circ H$$

□

Remarks 1. *The following additional facts are noteworthy:*

1. *If* F *is time-symmetric, then so is its inverse* F^{-1}*. Moreover, if* $F = G \circ H$ *is a decomposition into involutions, then* $F^{-1} = H \circ G$ *is a decomposition for the inverse. If* H *was the involution verifying (1), then* G *plays that role for* F^{-1}*.*

2. *If* H *and* G *are involutions and* $F = H \circ G$*, then* F *is an involution if and only if* H *and* G *commute.*

3. *For any* $i \in \mathbb{Z}$*,* F^i *is also time-symmetric with the same* H*, and* $H = F^i \circ H \circ F^i$*.*

4. *The identity is a (trivial) involution; from there and the third condition we have that any involution is trivially time-symmetric.*

5. *Not every reversible CA is time-symmetric. For example,* σ *(the shift): if for some* H*,* $(\sigma \circ H) \circ (\sigma \circ H) = id$*, since any CA commutes with the shift, we would have* $\sigma^2 = id$*, which is a contradiction.*

The following diagram commutes:

$$\begin{array}{ccc} X & \overset{H}{\longleftrightarrow} & X \\ F\big\downarrow\big\uparrow F^{-1} & & F^{-1}\big\downarrow\big\uparrow F \\ X & \overset{H}{\longleftrightarrow} & X \end{array}$$

Moreover, if we use $F = G \circ H$ to decompose the dynamics into the alternate applications of the involutions, so that successive configurations are computed as $c'_t = H(c_t)$, $c_{t+1} = G(c'_t)$, we get a dynamics c_0, c'_0, c_1, c'_1, ..., where both F and F^{-1} are being iterated: $c_{t+1} = F(c_t)$ and $c'_{t+1} = F^{-1}(c'_t)$. This curious situation is represented in Figure 3.

It is important to notice here the preservation of time-symmetry under CA conjugacy. For weaker notions of conjugacy time-symmetry is probably not preserved.

Proposition 2. *If F is CA conjugated to T and T is time-symmetric, then F is also time-symmetric.*

Proof. From time-symmetry, there is an involution H such that $T^{-1} = H \circ T \circ H$. From conjugacy, there is a bijective, continuous, shift-commuting ϕ such that $T = \phi \circ F \circ \phi^{-1}$. Then we have (removing the composition symbol, for clarity) that

$$F^{-1} \;=\; \phi^{-1} T^{-1} \phi \;=\; \phi^{-1} H T H \phi \;=\; \phi^{-1} H \phi F \phi^{-1} H \phi \;=\; GFG$$

and $G = \phi^{-1} H \phi$ is clearly an involution, making F time-symmetric. □

4. Universality

The examples given in Section 2 correspond to Turing-complete systems. The following results show that, indeed, the whole range of reversible dynamical behaviors can be observed in time-symmetric CA.

Theorem 1. *Let F be a reversible CA. Then there exists a CA $\tilde{F}$ which is time-symmetric and simulates F in real time.*

Proof. Let f be the local rule of F and denote with f^{-1} the local rule of its inverse F^{-1}; let $N \subset \mathbb{Z}^d$ be large enough to encompass both the neighborhoods of f and f^{-1}; finally, let S be the set of states. We define the CA $\tilde{F}$ with neighborhood N and states S^2, through the local rule

$$\tilde{F}\left((x,y)\right)_i \;=\; \left(f\left(x_{i+N}\right), f^{-1}\left(y_{i+N}\right)\right)$$

where we abuse notation by denoting configurations in $(S^2)^{\mathbb{Z}^d}$ as pairs $(x,y) \in (S^{\mathbb{Z}^d})^2$. $\tilde{F}$ simulates F in real time: to project the space-time diagram of $\tilde{F}$ into that of F, we just discard the second component of the ordered pairs. By discarding the first component instead, we note that $\tilde{F}$ simulates F^{-1} as well.

Let H be the autarkic involution given by the local rule $h(x,y) = (y,x)$. Then we have

$$\tilde{F} \circ H(x,y) \;=\; \tilde{F}(y,x) \;=\; (F(y), F^{-1}(x))$$

and

$$(\tilde{F} \circ H)^2(x,y) \;=\; \tilde{F} \circ H(F(y), F^{-1}(x)) \;=\; \tilde{F}(F^{-1}(x), F(y)) \;=\; (x,y)$$

which is condition *(ii)* in Proposition 1; thus, $\tilde{F}$ is time-symmetric. □

Cellular automata are said to be intrinsically universal if they are able to simulate any other CA. The details vary according to the accepted notion of *simulation*, from which there is a variety. Delorme *et al* [9] have recently reviewed and completed the study of three of these, *surjective, injective* and *mixed* simulation, and shown that for every pair of CA F and G, the product $F \times G$ simulates both F and G in all three senses.

Corollary 1. *There exist time-symmetric CA which are intrinsically universal within the class of reversible CA.*

Proof. This follows from the previous results and comment, and from the existence of reversible intrinsically universal CA (see for example [10]). □

Notice that reversible CA cannot simulate arbitrary CA: intrinsic universality is therefore limited to the reversible class, and time-symmetric CA are as general as reversible CA can get. Turing-universality is not limited by reversibility (information can be "swept away" to preserve it and maintain reversibility) and hence is implied by reversible intrinsic universality.

5. Relations with other CA classes

In Section 3 we gave the shift as an example of a reversible CA which is not time-symmetric. The proof was straightforward; in the general case, however, proving non-time-symmetry seems (so far) to be quite difficult. One possible route is to consider the group $(Aut(A), \circ)$ of reversible CA over the state set A under the composition $\circ$, and define some group homomorphism $\varphi : (Aut(A), \circ) \longrightarrow (G, \cdot)$ into a group $(G, \cdot)$ all of whose elements $g \neq 1_G$ have infinite order. Then it is clear that all periodic CA are in the kernel of φ, and therefore all time-symmetric CA are in the kernel as well.

One such homomorphism $\varphi_A : (Aut(A), \circ) \longrightarrow (\mathbb{Q}, \cdot)$ into the multiplicative group of rational numbers was defined for one dimensional CA in [11], for any state set A. We will not reproduce here the construction, but just remark that it has the properties that $\varphi_A(\sigma) = |A|$ and $\varphi_{A\times B}(F \times G) = \varphi_A(F)\varphi_B(G)$ for any reversible CA F and G over the state sets A and B, respectively.

The kernel of φ_A contains all time-symmetric CA, but also some CA that are not time-symmetric. We see an example in Proposition 2 where we show that there are periodic CA that are not time-symmetric. Another, simpler example is the product of three shifts $F = \sigma^{-2} \times \sigma \times \sigma$. Its state set is $A = \{0, 1\}^3$. This CA is in the kernel of φ_A because it is the composition of three componentwise shifts with φ_A values equal to 2^{-2}, 2 and 2 respectively. But it is not time-symmetric: Let us suppose that an involution H exists such that $F^{-1} = H \circ F \circ H$. Let H be defined by a radius-r local rule. Then, for every $i \in \mathbb{N}$, the composition $H \circ F^i \circ H$ can be defined using a neighborhood whose maximum element is $r + i + r$. But for $i > 2r$ this contradicts the facts that $H \circ F^i \circ H = F^{-i}$ and that the neighborhood of F^{-i} necessarily contains element $2i$.

This example, incidentally, shows that time-symmetric CA are *not closed under composition*: $\sigma^{-2} \times \sigma \times \sigma = (\sigma^{-1} \times \mathrm{id} \times \sigma) \circ (\sigma^{-1} \times \sigma \times \mathrm{id})$, and these are time-symmetric by using the involution that swaps the shifting layers.

Proposition 3. *Every reversible autarkic CA is time-symmetric.*

Proof. Let f be the local rule of a reversible autarkic CA, and let S be its set of states. Suppose first that $f : S \to S$ is a cyclic permutation and, without loss of generality, that $S = \{0, .., n-1\}$ and $f(i) = i+1 \mod n$. Let us remark first that any function of the form $h_a(i) = a - i \mod n$ is an involution. Now if $g_a = h_a \circ f$, then $g_a(i) = h_a(i+1) = a - i - 1 = h_{a-1}(i)$, it is also an involution. Therefore, by Proposition 1, f is time-symmetric and any of the h_a works.

If f decomposes into more than one cycle, we define h and g as before over each of them, obtaining again a decomposition into involutions. □

Since reversible autarkic CA (which are necessarily periodic) are time-symmetric, a natural question is whether every periodic CA is time-symmetric. Something similar happens: every periodic CA is conjugated to a reversible autarkic CA *over a subshift*. To see this, let F be a p-periodic CA with states S and define $\varphi : S^{\mathbb{Z}} \to (S^p)^{\mathbb{Z}}$ as $\varphi(x)_i = (x_i, F(x)_i, .., F^{p-1}(x)_i)$. This φ is continuous and injective, and the induced CA F' in $X \subseteq (S^p)^{\mathbb{Z}}$ is autarkic and period p; its local rule is $f'(a_0, a_1, .., a_{p-1}) = (a_1, a_2, .., a_{p-1}, a_0)$. However, this subshift is in general not stable for the involution constructed in the proof, and time-symmetry cannot be concluded. And it could not, as the following proposition attests.

Theorem 2. *There exists a one-dimensional periodic CA F such that F and F^{-1} are not CA conjugated. In particular, F is not time-symmetric.*

Proof. CA F has the state set $(A \cup \overline{A}) \times B$, where $A = \{0, 1.., 6\}$, $\overline{A} = \{\overline{0}, \overline{1}, .., \overline{6}\}$ and $B = \{odd, even\}$. We think of it as having two layers, the first with states in $A \cup \overline{A}$ and the second in B. The action of F on the first layer ignores the second and is autarkic: it just rotates the states cyclically, $a \mapsto a+1$ and $\overline{a} \mapsto \overline{a+1}$, for all $a \in A$. Additions and subtractions here and in the rest of the proof are modulo 7.

Given a configuration, we say that a cell i is *inactive* if the first layer states at cells i and $i+1$ are in one of the following combinations, for some $a \in \{0, 1, ..., 6\}$:

$$a, a \quad \text{or} \quad \overline{a}, \overline{a} \quad \text{or} \quad a, \overline{a} \quad \text{or} \quad a, \overline{a+1} \quad \text{or} \quad a, \overline{a+3}.$$

Otherwise a cell is called active. A whole configuration is called inactive if all the cells are inactive. Notice that activity of cells (and hence, of configurations) is preserved by F.

The rule for the second layer is the following: the second layer state is not changed at inactive cells, but it alternates ($odd \leftrightarrow even$) at active ones. Inactive configurations have therefore period 7, while configurations that contain an active cell do not have period 7 but they have period 14.

Consider the following configurations, for any $p \in \{odd, even\}$ and $a \in A$:

$$x(a, p) = \binom{{}^{\omega}a.a^{\omega}}{{}^{\omega}p.p^{\omega}} \quad \text{and} \quad y(a, p) = \binom{{}^{\omega}\overline{a}.\overline{a}^{\omega}}{{}^{\omega}p.p^{\omega}}.$$

Notice that $x(a,p)$ and $y(a,p)$ are uniform and inactive, and that they are the only uniform and inactive configurations.

Suppose that F is conjugated to F^{-1}, and let H be a reversible CA that carries the conjugacy, that is, $F^{-1} = H \circ F \circ H^{-1}$. Because H must preserve periods of configurations, it must map inactive configurations into inactive ones. Since H also preserved uniformity, we see that H act as a permutation among configurations of type $x(a,p)$ and $y(a,p)$.

Claim 1. $H(x(a,p)) \neq y(b,q)$, for every a, b, p and q.
Proof. Suppose $H(x(a,p)) = y(b,q)$. By the pigeon hole principle, there are also a', p', b' and q' such that $H(y(a',p')) = x(b',q')$.

From $H = F^i \circ H \circ F^i$ we obtain, with $i = a - a'$, that

$$\begin{aligned} H(x(a',p)) &= F^{a-a'}(H(F^{a-a'}(x(a',p)))) \\ &= F^{a-a'}(H(x(a,p))) \\ &= F^{a-a'}(y(b,q)) \\ &= y(b+a-a',q). \end{aligned}$$

Let us now consider the configuration

$$g = \begin{pmatrix} {}^{\omega}a'.\overline{a'}^{\omega} \\ {}^{\omega}p.p'^{\omega} \end{pmatrix}.$$

It is inactive and therefore $H(g)$ should be inactive, too; however, $H(g)$ is left-asymptotic with $y(b+a-a',q)$ and right-asymptotic with $x(b',q')$. This contradicts its inactivity, because no inactive configuration can have elements of $\overline{A}$ to the left of A on its first layer.

Claim 2. There exist constants $a_x \in A$ and $p_x \in \{odd, even\}$ such that for all $a \in A$ it holds $H(x(a, odd)) = x(a_x - a, p_x)$. Analogously, there exist constants a_y and p_y such that for all $a \in A$ we have $H(y(a, odd)) = y(a_y - a, p_y)$.
Proof. From Claim 1 we know that $H(x(0, odd)) = x(a_x, p_x)$ for some $a_x \in A$ and p_x. Applying $H = F^i \circ H \circ F^i$ with $i = -a$ we obtain that

$$\begin{aligned} H(x(a, odd)) &= F^{0-a}(H(F^{0-a}(x(a, odd)))) \\ &= F^{0-a}(H(x(0, odd))) \\ &= F^{0-a}(x(a_x, p_x)) \\ &= x(a_x - a, p_x), \end{aligned}$$

for all $a \in A$. Analogously for $y(a, odd)$.

Now we are ready to finish the proof of the proposition. Denote $a = 0$ and $p = odd$, and consider the following three inactive configurations

$$z_0 = \begin{pmatrix} {}^{\omega}a.\overline{a}^{\omega} \\ {}^{\omega}p.p^{\omega} \end{pmatrix}, \qquad z_1 = \begin{pmatrix} {}^{\omega}a.\overline{a+1}^{\omega} \\ {}^{\omega}p.p^{\omega} \end{pmatrix} \quad \text{and} \quad z_3 = \begin{pmatrix} {}^{\omega}a.\overline{a+3}^{\omega} \\ {}^{\omega}p.p^{\omega} \end{pmatrix}.$$

Consider first the configuration z_0. It is left asymptotic with $x(0, odd)$ and right asymptotic with $y(0, odd)$. According to Claim 2, $H(z_0)$ is left-asymptotic with

$x(a_x, p_x)$ and right-asymptotic with $y(a_y, p_y)$, and since it should be inactive, either $a_y = a_x$ or $a_y = a_x + 1$ or $a_y = a_x + 3$. Therefore, $a_y - a_x \in \{0, 1, 3\}$. The analogous reasoning using z_1 shows that, because $H(z_1)$ is left-asymptotic with $x(a_x, p_x)$ and right-asymptotic with $y(a_y - 1, p_y)$, we must have $(a_y - 1) - a_x \in \{0, 1, 3\}$, so that $a_y - a_x \in \{1, 2, 4\}$. Finally, using configuration z_3 we see that $a_y - a_x \in \{3, 4, 6\}$. Since no value of $a_y - a_x$ satisfies the three cases, H cannot exist. □

With some more technical work this last proof can be extended to the case where H is a block conjugation. For the weakest notion of conjugacy, however, the result is no longer true: F is conjugated to F^{-1} through $H(r, s)_i = (r_i - 2r_0, s_i)$, for every $r \in (A \cup \overline{A})^{\mathbb{Z}}$ and $s \in B^{\mathbb{Z}}$. Here the operation $r_i - 2r_0$ acts only on the "numerical" aspect of r_i, preserving the "overline" (or its absence) in each cell.

6. Decidability

Reversibility of CA is decidable in dimension one [12] but undecidable in dimension two or higher [3, 13]. This last result is obtained by reducing the tiling problem [14] to the problem of non-reversibility. In what follows we show how the construction in [13] can be easily adapted to the undecidability of time-symmetry in dimension 2; a further (and less trivial) variation on the idea will show that this extends even to the case when the CA is known to be reversible.

6.1. Tiling definitions and general constructions for the proofs

Definition 2. *A* tile set $\mathcal{T} = (T; N; R)$ *consists of a finite set T whose elements are the* tiles*, a* neighborhood $N \subset \mathbb{Z}^2$*, and a local* matching rule $R \subset T^N$ *which gives a relation specifying which tilings are considered valid.* Tilings *are configurations $x \in T^{\mathbb{Z}^2}$. A tiling is* valid at cell $(i, j) \in \mathbb{Z}^2$ *if and only if $x_{(i,j)+N} \in R$, that is, the neighborhood of (i, j) contains a matching combination of tiles. A tiling x is called* valid *if it is valid at all positions in $\mathbb{Z}^2$, and we say that the tile set $\mathcal{T}$ then* admits *tiling x.*

It is undecidable whether a given set of tiles can tile the plane [14]. It is important to remark that, by compacity, if a set of tiles cannot tile the plane, then neither can it tile arbitrarily large squares, that is, there is a constant M such that no square larger that M can be tiled.

The notion of paths on a tiling has been very useful in proving undecidability of several CA properties.

Definition 3. *A* tile set with paths *is a tuple $\mathcal{D} = (T; N; R; S; P)$ where $(T; N; R)$ is a tile set, and $S, P : T \to N$ are functions defining for each tile a follower and a predecessor within the neigborhood, and every pattern x in R satisfies $P(b) = -S(a)$ where $a = x(0)$ and $b = x(S(a))$.*

A given tiling x defines the follower relation *in $\mathbb{Z}^2$ by saying that a cell n is the follower of m if $n = S(x(m)) + m$; the predecessor relation is defined* *analogously. In a valid tiling, the predecessor relation is the inverse of the follower relation.*

We talk about *paths* in a tiling x when referring to sequences $p_0p_1p_2...$ of cells $p_i \in \mathbb{Z}^2$ such that p_{i+1} is the follower of p_i, for all i. Paths are used to embed a one dimensional CA rule into a tiling, but this cannot be related with tilability without the *plane filling property*.

Definition 4. *A tile set with paths is said to have the* plane filling property *if it satisfies the following two conditions:*

- *There exists $x \in T^{\mathbb{Z}^2}$ and a one-way infinite path $p_0p_1p_2...$ such that the tiling x is valid at p_i for all $i \in \mathbb{N}$.*
- *For all such x and $p_0p_1p_2...$, there are arbitrarily large squares of cells such that all cells of the squares are on the path.*

A set of tiles T_0 that has the plane filling property is constructed in [13]. This tile set forces some additional properties on the space-filling path, which will be used below:

- Every cell on an infinite path belongs to squares of size 2^n which the path completely covers in contiguous steps; this happens for arbitrarily large n.
- These squares can be described recursively and within them the path follows the Hilbert curve. (More details follow in the proof of Lemma 1.)
- An infinite path is infinite in both directions.

For the proofs that follow, a state from some 1D CA F is appended to each tile in T_0, obtaining in this way a new set of tiles, say A_0. To reduce tilability to non-reversibility the idea is to define, for a given set of tiles T, a 2D CA F_T with states $T \times A_0$ whose rule consists of iterating the one dimensional CA rule F on the paths running through valid cells. Let us call a cell *active* if the tilings by T and A_0 are valid at the cell. In F_T, all active cells apply the local rule of F along the path. Cells that see a tiling error in the T- or A_0-tiling are *inactive* and they do not change their state.

6.2. Undecidability of time-symmetry in 2D

Theorem 3. *The following two classes of 2D CA are recursively inseparable:*

- *periodic and time-symmetric, and*
- *non-reversible*

Proof. Here we will consider F as the addition modulo 2 with 1D neighborhood $\{0,1\}$, as in [13]. It is easy to see that $F(1^\infty) = F(0^\infty)$. If a valid tiling in T exists, there will be two configurations in $T \times A_0$ with an infinite valid path, one with 1 in each tile and the other with 0. They both lead to the same image, proving the non-reversibility of F_T. This proves that tilability by T implies non-reversibility of F_T.

Now let us see what happens when F_T works on a finite path of active cells that ends in an inactive cell. On such paths, the rule evolves as F would over a finite segment with fixed boundary states. But the dynamics of F in such a case is reversible, even periodic. If T cannot tile the plane, then the size of tilable squares is bounded, and no infinite valid path can appear. In fact, there is a global constant bound on the lengths of valid paths that depends only on T (if arbitrarily long paths could appear, by compacity infinite paths would also be possible).

We see that if T does not admit a tiling of the plane then the F_T is reversible, and even periodic. Moreover, the dynamics within paths are independent, so they can be considered as independent finite and periodic subsystems, and using the technique in the proof of Proposition 3, an involution can be defined that makes the CA time-symmetric within each path. This involution is local because each cell can find the limits of its path within a fixed neighborhood. Thus the CA is time-symmetric and periodic. □

6.3. Undecidability of time-symmetry in reversible 2D CA

Theorem 4. *The following two classes of CA are recursively inseparable:*

- *periodic and time-symmetric, and*
- *reversible but non-time-symmetric*

Proof. We consider now an adaptation of the 1D CA rule $F = \sigma^{-2} \times \sigma \times \sigma$ shown in the previous section to be non-time-symmetric. The behavior is the same as then, except at cells at the boundary of the active path, where a different rule is used that makes the information "bounce back": depending on the direction of the path interruption, the bits will be copied from the fast layer into the slow ones, or vice versa (see Figure 4).

Formally, the rule is defined as follows. First, each cell determines whether it is active by verifying the correctness of both tilings. The validity is tested inside a radius-2 neighborhood around the cell, so that for all active cells at least two consecutive followers and successors are uniquely defined. If a cell is inactive, it does not change its state. Otherwise, for $i \in \{-2, -1, 0, 1\}$, let $(a_i, b_i, c_i) \in \{0, 1\}^3$ be the CA F states in the two predecessor tiles, in the tile itself and in its follower, respectively; in addition, let boolean value v_i be true if the corresponding cell is active, and false otherwise. Then the new state in the cell is

$$(a', b', c') = \begin{cases} (a_{-2}, b_1, c_1) & \text{if } v_{-2} \wedge v_{-1} \wedge v_1; \\ (a_{-2}, a_{-1}, a_0) & \text{if } v_{-2} \wedge v_{-1} \wedge \neg v_1; \\ (b_{-1}, b_1, c_1) & \text{if } \neg v_{-2} \wedge v_{-1} \wedge v_1; \\ (c_0, b_1, c_1) & \text{if } \neg v_{-1} \wedge v_1; \\ (b_{-1}, a_{-1}, a_0) & \text{if } \neg v_{-2} \wedge v_{-1} \wedge \neg v_1; \\ (c_0, b_0, a_0) & \text{if } \neg v_{-1} \wedge \neg v_1 \end{cases}$$

Notice that on a finite active path segment, bordered by inactive cells, the bits cycle within the segment. Thus, if T cannot tile the plane, the involution that symmetrically swaps the bits within each such finite path makes the 2D CA time-symmetric. As in the previous theorem, it is important to notice that these paths have bounded length, so that a T-fixed bounded neighborhood completely encloses the path.

On the other hand, if T can tile the plane, we will prove that F_T cannot be time-symmetric. By contradiction, let us suppose that F_T is time-symmetric, and let H be its corresponding involution. Let x_1 be a valid tiling of T, x_2 a valid tiling of T_0, and x_3 equal to $(0, 0, 0)$ on every cell. Now let x^p be equal to $x = (x_1, x_2, x_3)$ except at position p where its numeric component is $(0, 1, 0)$, and let $(p_t)_{t\in\mathbb{Z}}$ be the infinite path of x_2. We have that $F_T^t(x^{p_0}) = x^{p_{-t}}$, which combined with time-symmetry (which states $H \circ F_T^t = F_T^{-t} \circ H$) we get

$$H(x^{p_{-t}}) = F_T^{-t}(H(x^{p_0})) \tag{2}$$

Since the difference between x and x^p is only at position p, the difference between $H(x)$ and $H(x^p)$ is only within a neighborhood of radius, say r, of p. Thus, from equation (2), $H(x^{p_0})$ has a valid path that passes at distance r from every cell in $\{p_t\}_{t\in\mathbb{Z}}$. Such a path needs to be infinite, and therefore, it must be a Hilbert plane filling path, thus $H(x^{p_0})$ is active on all of its cells. Notice also that the locality of differences at time t implies that the differences between $H(x)$ and $H(x^{p_0})$ around p_0 are removed by the iteration of F_T; therefore, they must be located in the third component.

We may then choose q_0 as a cell where configurations $H(x)$ and $H(x^{p_0})$ differ. $H(x)$ and $H(x^{p_0})$ have both a plane filling path, which thus can be written as $(q_t)_{t\in\mathbb{Z}}$. The difference at q_0 cannot be simultaneously on the fast and on the slow layers of the numerical component: F_T would then shift the differences in opposite directions along the path, making the local difference noticed above impossible. Hence, we have only two cases:

Case 1. The difference between $H(x^{p_0})(q_0)$ and $H(x)(q_0)$ lies on the first layer of the numeric component (the fast layer).

In this case, equation (2) tell us that $d(q_{2t}, p_t) \leq r$ for every $t \in \mathbb{Z}$. From the plane filling property, for every n a square of side 2^n can be found around p_0 that corresponds exactly to consecutive tiles p_{-j} to p_l along the path. Thus cells $(q_t)_{t=-2j}^{2l}$ must be contained in a square of side $n + 2r$, which is impossible if we pick n large enough: if $(n + 2r)^2 < 2n^2$, we get a contradiction with the injectivity of the paths.

Case 2. The difference between $H(x^{p_0})(q_0)$ and $H(x)(q_0)$ lies on the last two layers of the numeric component (the slow layers).

In this case, equation (2) tell us that $d(q_{-t}, p_t) \leq r$ for every $t \in \mathbb{Z}$. Here the particular properties of T_0 become crucial: it has directed paths, and they fill squares according to one of four patterns, none of which is the reverse of another. Thus, the path $(q_{-t})_{t\in\mathbb{Z}}$ cannot draw a Hilbert curve like $(q_t)_{t\in\mathbb{Z}}$ and $(p_t)_{t\in\mathbb{Z}}$, but rather, it must follow the reverse patterns, which for large enough squares turn out to be incompatible with following the forward patterns of $(p_t)_{t\in\mathbb{Z}}$ within a distance r. This is proved in the next lemma, and completes this proof. □

Lemma 1. *Let $(p_t)_{t\in\mathbb{Z}}$ and $(q_t)_{t\in\mathbb{Z}}$ be two Hilbert curves built with tile set T_0. Then $\forall r > 0$, $\exists m : d(p_m, q_{-m}) > r$.*

Proof. By contradiction. Hilbert curves satisfy that in every infinite path, for every natural n, every cell is contained in a square of side 2^n which the path completely covers in contiguous steps. These squares can be described recursively, and are traversed in one of four possible ways. Using the first step of recursion we can describe the four cases as a sequence of sixteen smaller squares, visited according to their indices as shown in Figure 5.

Let us choose n such that $2^n > 8r$ and let us consider a square of side 2^n, say A, filled by $(p_t)_{t=1}^{2m}$, for $m = 2^{2n-1}$. Without loss of generality, we will assume that A is in the leftmost case of Figure 5; p_1 and p_{2m} correspond to the grey and black cells. Notice that p_m is exactly the last cell of square 8 before entering square 9. Consider q_{-m}, which lies at distance at most r from p_m. It must belong to some square B of side 2^n in the (q_t) path; since $2m$ cells are needed to fill B, either q_{-1} or q_{-2m} must be in B too. We will suppose that q_{-1} is in B (the other case is analogous).

A first observation is that squares A and B must overlap horizontally up to a margin of size r on each side. This follows from the fact that $(q_{-t})_{t=1}^{m}$ shadows (from a distance bounded by r) the path $(p_t)_{t=1}^{m}$, which spans A horizontally.

The path $(q_{-t})_{t=1}^{m}$ contains $m = 2^{2n-1}$ cells, and hence must cover at least one complete quarter of B. Let C be this square of side 2^{n-1}, and let q_{-l} be its first cell. Since $-m \leq -l$ and $-l + 2^{2n-2} - 1 \leq -1$, p_l must be in one of the squares 8, 7, 6 or 5. The different cases are illustrated in Figure 6.

- If p_l is in square 8 (Figure 6, left): For C to shadow (from a distance r) the squares 5, 6 and 7, the only corner of C which could correspond to q_{-l} would be the upper right one. However, none of the ways to visit a square starts there.

- If p_l is in square 5 (Figure 6, center): Similar to the previous case; here the visit to C would have to start at the bottom left corner, which is impossible.

- If p_l is in square 7 or 6 (Figure 6, right): The horizontal overlap of A and B (up to a margin r), along with the fact that C is either on the left or the right of B, makes it impossible for C to shadow either the leftmost cells of square 4, or the rightmost cells of square 5. □

7. A note on involutions

Most of the easy examples of involutive CA that one can imagine are of radius 0, or nearly so; like any periodic CA, information cannot travel far, and a suspicion arises that perhaps all involutions are, in a sense, radius 0 CA. The following example goes against this intuition.

Example. A CA involution H which is not block conjugated to any radius 0 CA can be defined for the alphabet $A = \{0, 1, 2\}$ as follows:

$$h(ab) = \begin{cases} a & \text{if } b \neq 0 \vee a = 0 \\ 2 & \text{if } b = 0 \vee a = 1 \\ 1 & \text{if } b = 0 \vee a = 2 \end{cases}$$

Proof. Let $\phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ be a continuous, bijective function such that $\sigma^n \circ \phi = \phi \circ \sigma^m$ for some n and m. Clearly, ϕ^{-1} commutes with the same shifts and is continuous too.

Mirroring Hedlund's argument we can see that ϕ is determined by a local function: continuity implies the existence of r such that the image in cells $[0, n[$ depends only on cells $[-r, r[$ –furthermore, we can assume that r is a multiple of m. We have then $\phi(x)_{[0,n[} = f(x_{-r}, .., x_r)$ for some f, which along with commutativity gives, for any k, $\phi(x)_{[kn,(k+1)n[} = f(x_{km-r}, .., x_{km+r})$.

Suppose now that ϕ is a conjugacy between H and a radius 0 CA Π: $\phi \circ H = \Pi \circ \phi$. Let $\pi : B \to B$ be the local function of Π, which must be an involutive permutation of B.

Notice that the homogeneous configurations ${}^{\omega}a.a^{\omega}$ are fixed points of H for all $a \in A$. Thus u_a defined through $u_a = f(a^{2r+1}) \in B^n$ verify ${}^{\omega}u_a.u_a^{\omega} = \phi({}^{\omega}a.a^{\omega})$, for each a. Since they are fixed points for Π, each character in them is a fixed point for π.

It follows that $y = {}^{\omega}u_1.u_0^{\omega}$ must be a fixed point for Π and hence $x = \phi^{-1}(y)$ must be one for H. However, x is left-asymptotic with 1 and right-asymptotic with 0, and must therefore contain the $a0$ combination, with $a \neq 0$, which gives it period 2 under H. □

8. Conclusions

As shown by the examples in Section 2, time-symmetric CA are actually quite familiar to CA researchers, and have appeared in different contexts. Some cases are very explicit, like the automata constructed in the proof of undecidability of periodicity [15], which actually include an "arrow of time" toggle. Moreover, there are ways of constructing CA rules that make the construction of time-symmetric CA straightforward. For instance, Margolus' billiard is an example of a block automata, *i.e.*, a system which is a composition of two functions applied to independent blocks of the configuration. By incorporating the current function and block to be applied into the configuration, a block automaton can always be expressed as a CA. Defining an involution that toggles the current block is a good idea to prove time-symmetry, but it only works if both block functions are also involutions. What must be stressed is that this is only a *sufficient* condition; the system may be time-symmetric by means of an entirely different involution.

Likewise, partitioned CA (in the sense of Morita [16]) can easily give birth to time-symmetric CA. In that case, cells are partitioned into sub-cells, one for each neighbors; iteration proceeds by the alternation between an exchange step, where cells exchange the contents of the sub-cells associated to each other, and a step which applies a block transformation on the cell. This scheme was succesfully used to construct reversible CA (all we need is a reversible block transformation), and can produce time-symmetric CA as well if the block transformation is chosen as an involution: the exchange step already is one. Again, what we want to stress is that this is a sufficient condition: we could have a partitioned CA which is time-symmetric while having a non-involutive block transformation, if the decomposition happens to be another one.

There are several interesting questions that should probably be addressed next, and have appeared along this text. Two of them are:

- Is there a constructive characterization of CA involutions that can make their enumeration practical? Right now the only way we have to find the involutions is to test all CA exhaustively; some trivial necessary conditions can be used to reduce the search, but they are not enough to make it efficient.

- Is time-symmetry a decidable property in 1D? Since the definition calls for the existence of an involution that verifies a condition, a computable bound on the necessary neighborhood for the involution would be enough to ensure decidability.

The answers to these questions may be related: a better understanding of the structure of involutions may be useful for bounding the required neighborhood and thus deciding time-symmetry.

Another natural couple of questions arises when considering the result of composing, not just 2, but any number of involutions. Since time-symmetric CAs were shown not to be closed under composition, we know that the composition of involutions can go beyond them. On the other hand, the result will always be within the kernel of Kari's φ_A homomorphism (see Section 5). Is every CA in that kernel a composition of involutions? If not, then: is perhaps every periodic CA a composition of involutions?

A different line to explore was suggested in personal communication by G. Theyssier, and is related to the "Open Problem 1" formulated in [9]. There he and his coauthors considered several notions of CA simulation, each one of them defining a partial order and equivalence classes among CA. They show that two equivalent reversible CA will have equivalent inverse CA; the classes of equivalence of the direct and inverse CA can, in principle, be different. The question they ask is: *What reversible CA are equivalent to their inverses?*. The class of time-symmetric CA may be a good starting point towards answering this question, even if the existence of an involutive CA that alternates between the space-time diagrams of F and F^{-1} does not imply (at least not directly) the equivalence of F and F^{-1} in the sense of [9], where only autarkic transformations between CA are considered. On the other hand, periodic CA are all in the same equivalence class for [9], while we have proved that some of them are not even CA conjugated with their inverse. Delorme *et al* remark that so far no example of non-equivalence between a CA and its inverse has been found. Generally speaking, the relation between time-symmetry and other notions of equivalence between a CA and its inverse is unknown territory.

A further direction for future work may be the study of time-symmetry in other discrete dynamical systems. In each case, an important issue is to precise what kind of involution is to be applied. Generally speaking, what we need is an involutive and hopefully local transformation of the system's configuration. That transformation may not be, in general, an object of the same kind as the dynamics itself: that was the case for CA because of the special nature of CA, which transform the whole configuration in discrete time too, and will be the case for automata networks in general. In other cases, like for instance Turing machines, it is not only difficult (the composition of two Turing machines moves the head two steps, and is no longer a Turing machine unless we extend the definitions) but also not expected; rather, for Turing machines, the involution would likely be a transformation on the tape (a CA involution?) along with a change in the current state of the machine. Finally, the locality of the time-reversing involution is not completely granted either: even in CA, it would be interesting to see what happens if that requirement is removed.

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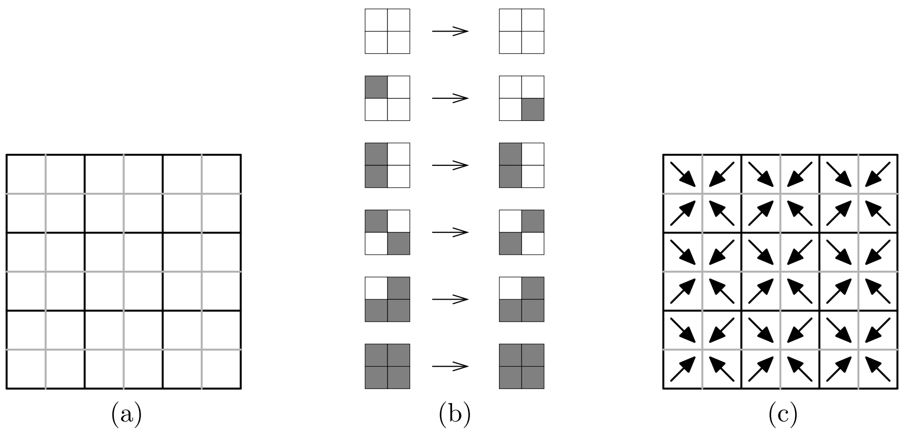

Figure 1: (a) The two partitions of the Margolus model are shown, one with solid lines and the other with dashed lines. (b) The Billiard Ball Model is defined through a permutation over 2×2 blocks of cells. (c) The current partition is obtained by grouping the four cells that point to the same point; reversing the arrows gives the alternate partition.

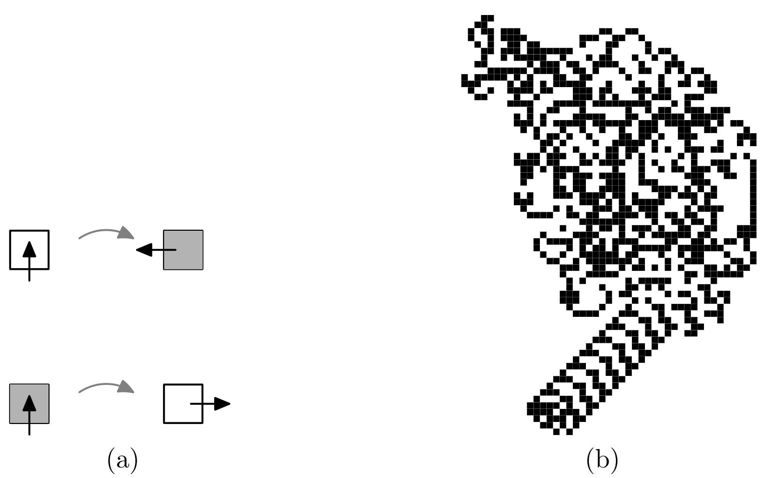

Figure 2: (a) Langton's ant rule. (b) Space configuration at iteration 10,837 after starting with every cell in *white* color.

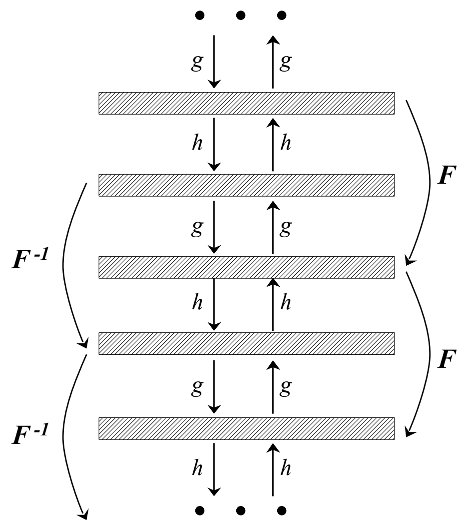

Figure 3: The decomposition of time-symmetric CA into alternating involutions creates a situation where both F and its inverse can be read from the space-time diagram as time moves forward (or backward).

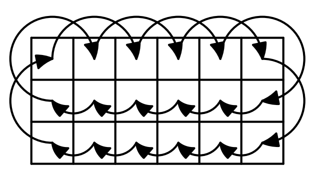

Figure 4: The adapted rule on a doubly bounded path of length 6.

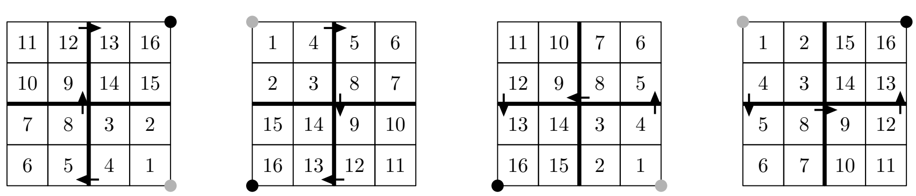

Figure 5: The path must start at the grey corner, visit the small squares in the indicated order by passing from one to the next though their corners as indicated by the arrows, to finally exit through the black corner.

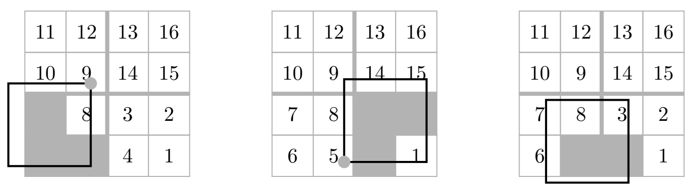

Figure 6: The grey squares are completely covered by path $(p_t)_{t=l}^{l-2^{2n-2}+1}$. Square C is marked with thick black lines. Three cases appear depending on the location of p_l, none of which admits a valid C within a r neighborhood of the grey area.

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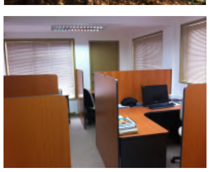

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