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The Asymptotic behaviour of the linear transmission problem in viscoelasticity

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Abstract

Here we consider the transmission problem with localized Kelvin Voigt's viscoelastic damping. Our main result is to show that the corresponding semigroup $e^{\mathcal{A}t}$ is not exponentially stable, but the solution decays polynomially to zero as $1/(1+t)^2$, when the initial data is taken over the Domain $\mathcal{D}(\mathcal{A})$. Moreover we prove that this rate of decay is optimal. Finally using a second order scheme that ensures the decay of energy (Newmark-β method), we give some numerical examples which demonstrate this polynomial asymptotic behavior.

Keywords: Kelvin Voigt; Viscoelastic damping; Transmission problem; Semigroup; Polynomial decay; Newmark-β method.

1 Introduction

Localized frictional damping was studied for several authors in one and several space dimension, as can be seem in [24, 26, 27, 28, 29, 30] to quote but a few. The main result of the above articles is that localized frictional damping produce exponential decay in time of the solution. The more general result occurs in one dimensional space where the solution always decays exponentially to zero for any localized frictional damping effective over any open subset of the domain. This result is no longer valid for materials configurated over bounded domains $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ where the possition of the frictional effect is important. See for example [31], where necessary and sufficient conditions are given to get stabilization of the wave equation with localized frictional damping. That is to say,

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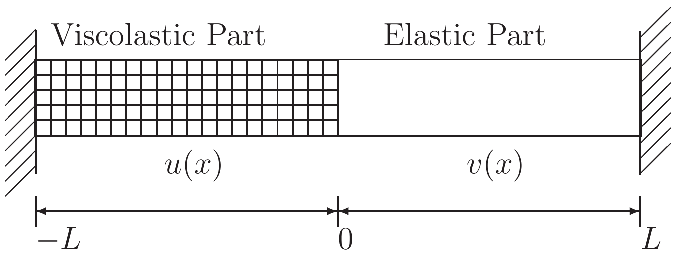
to get the exponential stability the damping mechanism must be effective in a sufficient large neighborhood of the boundary, see also [28].

On the other hand, it is very well know that the viscoelastic Kelvin Voigt's damping when effective in the whole domain is stronger than the frictional damping. This damping mechanism not only produce exponential stability but also turns the corresponding semigroup into an analityc semigroup, which in particular implies that the system is exponentially stable among other important properties, see Zheng-Liu's book [15]. But contradictorily when localized the Kelvin Voigt's damping is weaker than the frictional damping, in the sense that the corresponding semigroup is not exponentially stable as proved in [13].

In this paper we will consider the transmission problem of the wave equation with localized viscoelasticity of Kelvin Voigt type configured as in the following picture.



Here we consider the system

$$\rho_1 u_{tt} - \kappa_1 u_{xx} - \kappa_2 u_{xxt} = 0 \quad \text{in} \quad]-L, 0[\times]0, \infty[\tag{1.1}$$

$$\rho_2 v_{tt} - \kappa_3 v_{xx} = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.2}$$

where the functions $u = u(x, t)$ and $v = v(x, t)$ represents the fraction field of a constituent. κ_1, κ_2 and κ_3 are positive constants. ρ_1, ρ_2 are the mass density functions. The boundary conditions

$$\begin{aligned} &u(-L, t) = 0, \quad v(L, t) = 0, \qquad t \geq 0 \\ &u(0, t) = v(0, t), \qquad \kappa_1 u_x(0, t) + \kappa_2 u_{xt}(0, t) = \kappa_3 v_x(0, t) \end{aligned} \tag{1.3}$$

and initial data

$$\begin{aligned} &u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad]-L, 0[\\ &v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x) \quad \text{in} \quad]0, L[\end{aligned} \tag{1.4}$$

Denoting by $\mathcal{E}$ the energy

$$\mathcal{E}(t) = \frac{1}{2}\left[\rho_1 \int_{-L}^{0} u_t^2\,dx + \rho_2 \int_{0}^{L} v_t^2\,dx + \kappa_1 \int_{-L}^{0} u_x^2\,dx + \kappa_3 \int_{0}^{L} v_x^2\,dx\right].$$

It is not difficult to see that

$$\frac{d}{dt}\mathcal{E}(t) = -\,\kappa_2 \int_{-L}^{0} u_{xt}^2\,dx \tag{1.5}$$

Our main result is that the solution decays exponentially to zero as t^{-2}. Moreover we will prove that this rate of decay is optimal when we take initial data over the domain of the infinitesimal operator $\mathcal{A}$ associated to the semigroup $S_{\mathcal{A}}(t)$ that defines the solution of the transmission problem. That is we will prove that there exists a positive constant c_k such that

$$\|S(t)U_0\| \le \frac{c_k}{t^{2k}}\|U_0\|_{D(\mathcal{A}^k)}, \quad \forall k \in \mathbb{N}$$

The remain part of this article is defined as follows. In the next section 2 we show the existence result in the framework of semigroup. In section 3 we show the polynomial decays as well as the optimality. Finally in section 4, using a second order scheme that ensures the decay of energy (Newmark-β method), we give some numerical examples which demonstrate this polynomial asymptotic behavior.

2 The semigroup setting

In this section, we use the semigroup approach to show the well-posedness of system (1.1)-(1.3). Let us denote by

$$\mathbb{H}^m = H^m(-L,\,0) \times H^m(0,\,L), \qquad \mathbb{L}^2 = L^2(-L,\,0) \times L^2(0,\,L).$$

$$\mathbb{H}^1_L = \left\{(u,v) \in \mathbb{H}^1;\;\; u(-L) = v(L) = 0, \quad u(0) = v(0)\right\}$$

Under the above conditions we have that the phase space is given by

$$\mathcal{H} = \mathbb{H}^1_L \times \mathbb{L}^2$$

Note that this space equipped with the inner product

$$\begin{aligned}\langle (u_1,\,v_1,\,\eta_1,\,\mu_1),\,(u_2,\,v_2,\,\eta_2,\,\mu_2)\rangle_{\mathcal{H}} \;&=\; \kappa_1 \int_{-L}^{0} u_{1x}\,\overline{u}_{2x}\,dx + \kappa_3 \int_{0}^{L} v_{1x}\,\overline{v}_{2x}\,dx \\ &\quad + \rho_1 \int_{-L}^{0} \eta_1\,\overline{\eta}_2\,dx + \rho_2 \int_{0}^{L} \mu_1\,\overline{\mu}_2\,dx\end{aligned}$$

Is a Hilbert space. We also consider the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ \mu \end{pmatrix} = \begin{pmatrix} \eta \\ \mu \\ \frac{1}{\rho_1}(\kappa_1 u_{xx} + \kappa_2 \eta_{xx}) \\ \frac{\kappa_3}{\rho_2} v_{xx} \end{pmatrix},$$

whose domain $\mathcal{D}(\mathcal{A})$ is given by

$$\mathcal{D}(\mathcal{A}) = \big\{U \in \mathcal{H};\ (\eta,\, \mu) \in \mathbb{H}_L\ \ (\kappa_1 u + \kappa_2 \eta, v) \in \mathbb{H}^2, \quad \kappa_1 u_x(0) + \kappa_2 \eta_x(0) = \kappa_3 v_x(0)\big\}$$

where $U = (u,\, v,\, \eta,\, \mu)^T$. Taking $u_t = \eta$ and $v_t = \mu$, (1.1)-(1.2) can be reduced to the following abstract initial value problem for a first-order evolution equation

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \qquad U(0) = U_0, \qquad \forall\, t > 0$$

with $U(t) = (u,\, v,\, u_t,\, v_t)^T$ and $U_0 = (u_0,\, v_0,\, u_1,\, v_1)^T$. Next, we show that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions over $\mathcal{H}$.

Proposition 2.1 *The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and $0 \in \varrho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Then our conclusion will follow using the well known Lumer-Phillips theorem (see [19]). We observe that if $U = (u,\, v,\, \eta,\, \mu) \in \mathcal{D}(\mathcal{A})$ then

$$\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = \kappa_1 \int_{-L}^{0} \eta_x\, \overline{u}_x\, dx + \kappa_3 \int_0^L \mu_x\, \overline{v}_x\, dx + \int_{-L}^{0} (\kappa_1\, u + \kappa_2\, \eta)_{xx}\, \overline{\eta}\, dx + \kappa_3 \int_0^L v_{xx}\, \overline{\mu}\, dx.$$

Integrating by parts, using (1.3), and performing straightforward calculations we obtain

$$Re\, \langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = -\,\kappa_2 \int_{-L}^{0} |\eta_x|^2\, dx. \tag{2.1}$$

Hence $\mathcal{A}$ is a dissipative operator. To show that $0 \in \varrho(\mathcal{A})$ let us take $F = (f,\, g,\, p,\, q) \in \mathcal{H}$. We will show that there exists a unique $U = (u,\, v,\, \eta,\, \mu)$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$, that is,

$$\eta = f \quad \text{in} \quad H^1(-L,\, 0) \tag{2.2}$$

$$\mu = g \quad \text{in} \quad H^1(0,\, L) \tag{2.3}$$

$$\kappa_1\, u_{xx} + \kappa_2\, \eta_{xx} = \rho_1\, p \quad \text{in} \quad L^2(-L,\, 0) \tag{2.4}$$

$$\kappa_3\, v_{xx} = \rho_2\, q \quad \text{in} \quad L^2(0,\, L). \tag{2.5}$$

Replacing (2.2) in (2.4) we have

$$\kappa_1\, u_{xx} = \kappa_2\, f_{xx} + \rho_1\, p \in H^{-1}(-L,\, 0). \tag{2.6}$$

It is not difficult to see that the trasmission problem given by (2.5)-(2.6) is well posed Therefore, we conclude that $0 \in \varrho(\mathcal{A})$.

Theorem 2.2 *For any $U_0 \in \mathcal{H}$ there exists a unique solution $U(t) = (u,\, v,\, u_t,\, v_t)$ of (1.1)-(1.4) satisfying*

$$(u, v, u_t, v_t) \in C([0,\, \infty[:\, \mathbb{H}^1_L \times \mathbb{L}^2)$$

If $U_0 \in \mathcal{D}(\mathcal{A})$, then

$$(u, v, u_t, v_t) \in C^1([0,\, \infty[:\, \mathbb{H}^1_L \times \mathbb{L}^2) \cap C([0,\, \infty[:\, D(\mathcal{A}))$$

3 Polynomial decay and optimality

In this section we will show the polynomial decay of the solutions. To do so we will use the characterization due to A. Borichev and Y. Tomilov [2].

Theorem 3.1 *Let $\mathcal{S}(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\,\mathbb{R} \subset \varrho(\mathcal{A})$. Then*

$$\frac{1}{|\lambda|^\alpha}\, \|(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall\, \lambda \in \mathbb{R} \quad \Longleftrightarrow \quad \|\mathcal{S}(t)\mathcal{A}^{-1}\|_{\mathcal{D}(\mathcal{A})} \leq \frac{C}{t^{1/\alpha}}.$$

In fact, given $\lambda \in \mathbb{R}$ and $F = (f,\, g,\, p,\, q) \in \mathcal{H}$, there exist $U = (u,\, v,\, \eta,\, \mu) \in \mathcal{D}(\mathcal{A})$, such that $i\,\lambda\, U - \mathcal{A}U = F$, that is,

$$\begin{aligned} i\,\lambda\, u - \eta \;&=\; f \quad \text{in} \quad H^1(-L,\, 0) && (3.1)\\ i\,\lambda\, v - \mu \;&=\; g \quad \text{in} \quad H^1(0,\, L) && (3.2)\\ i\,\lambda\, \eta - \kappa_1\, u_{xx} - \kappa_2\, \eta_{xx} \;&=\; \rho_1\, p \quad \text{in} \quad L^2(-L,\, 0) && (3.3)\\ i\,\lambda\, \mu - \kappa_3\, v_{xx} \;&=\; \rho_2\, q \quad \text{in} \quad L^2(0,\, L). && (3.4) \end{aligned}$$

From (2.1), note that

$$Re\,\langle (i\,\lambda\, I - \mathcal{A})U,\, U\rangle_{\mathcal{H}} = Re\,\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = \kappa_2 \int_{-L}^0 |\eta_x|^2\, dx = Re\,\langle F,\, U\rangle_{\mathcal{H}}\,.$$

Thus

$$\kappa_2 \int_{-L}^{0} |\eta_x|^2\, dx \leq \|U\|_{\mathcal{H}}\, \|F\|_{\mathcal{H}}. \tag{3.5}$$

From (3.1) and (3.5) we obtain

$$|\lambda|^2 \int_{-L}^{0} |u_x|^2\, dx \leq C\, \|U\|_{\mathcal{H}}\, \|F\|_{\mathcal{H}} + C\, \|F\|_{\mathcal{H}}^2. \tag{3.6}$$

Theorem 3.2 *Under the above notations we have that the semigroup associated to the transmission problem, decays polynomially as*

$$\|e^{\mathcal{A}\,t}\, U_0\|_{\mathcal{H}} \leq \frac{C_k}{t^{2k}}\, \|U_0\|_{\mathcal{D}(\mathcal{A}^k)}.$$

Proof. From (3.3) we have

$$|\lambda|\, \|\eta\|_{-1} \leq C\, \|u_x\| + C\, \|\eta_x\| + C\, \|F\|_{\mathcal{H}} \leq C\, \|U\|_{\mathcal{H}}^{1/2}\, \|F\|_{\mathcal{H}}^{1/2} + C\, \|F\|_{\mathcal{H}}.$$

Using interpolation and inequality (3.5) we get

$$\begin{aligned} \|\eta\|_{L^2(-L,0)}^2 &\leq C\, \|\eta\|_{-1}\, \|\eta\|_1 \leq \frac{C}{|\lambda|} \left[\|U\|_{\mathcal{H}}^{1/2}\, \|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}} \right] \|\eta\|_1 \\ &\leq \frac{C}{|\lambda|} \left[\|U\|_{\mathcal{H}}\, \|F\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\, \|F\|_{\mathcal{H}}^{3/2} \right]. \end{aligned} \tag{3.7}$$

Multiplying equation (3.3) by $(x+L)\, (\overline{\kappa_1\, u_x + \kappa_2\, \eta_x})$ and taking real part we have

$$\begin{aligned} & Re\, i\, \lambda \int_{-L}^{0} \eta\, (x+L)\, (\overline{\kappa_1\, u_x + \kappa_2\, \eta_x})\, dx - \frac{1}{2} \int_{-L}^{0} (x+L)\, \frac{d}{dx}\, |\kappa_1\, u_x + \kappa_2\, \eta_x|^2\, dx \\ = \;& \rho_1\, Re \int_{-L}^{0} p\, (x+L)\, (\overline{\kappa_1\, u_x + \kappa_2\, \eta_x}). \end{aligned}$$

Using (3.1), note that

$$\kappa_1\, Re\, i\, \lambda \int_{-L}^{0} \eta\, (x+L)\, \overline{u}_x\, dx = -\frac{L}{2}\, \kappa_1\, |\eta(0)|^2 + \frac{1}{2}\, \kappa_1 \int_{-L}^{0} |\eta|^2\, dx - \kappa_1 \int_{-L}^{0} (x+L)\, \eta\, \overline{f}\, dx.$$

We denote the functional

$$I_u = \frac{1}{2} \left[\kappa_1\, |\eta(0)|^2 + |\kappa_1\, u_x(0) + \kappa_2\, \eta_x(0)|^2 \right].$$

From where it follows that

$$\begin{aligned} I_u &= \kappa_2\, Re\, i\, \lambda \int_{-L}^{0} (x+L)\, \eta\, \overline{\eta}_x\, dx + \frac{1}{2}\, \kappa_1 \int_{-L}^{0} |\eta|^2\, dx + \frac{1}{2} \int_{-L}^{0} |\kappa_1\, u_x + \kappa_2\, \eta_x|^2\, dx \\ &\quad - \rho_1\, Re \int_{-L}^{0} p\, (x+L)\, (\overline{\kappa_1\, u_x + \kappa_2\, \eta_x})\, dx - \kappa_1 \int_{-L}^{0} (x+L)\, \eta\, \overline{f}\, dx \\ &\leq C \int_{-L}^{0} \left(|\lambda|\, |\eta_x|\, |\eta| + \eta_x^2 + u_x^2 \right) dx + C\, \|U\|_{\mathcal{H}}\, \|F\|_{\mathcal{H}} \\ &\leq C\, |\lambda|^{1/2} \int_{-L}^{0} |\eta_x| \left(|\lambda|^{1/2}\, |\eta| \right) dx + C\, \|U\|_{\mathcal{H}}\, \|F\|_{\mathcal{H}}. \end{aligned}$$

Using (3.7) we get

$$I_u \le C\,|\lambda|^{1/2}\left(\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right), \tag{3.8}$$

for λ large enough. On the other hand, multiplying equation (3.4) by $(x-L)\,\overline{v}_x$ we get

$$i\,\lambda\,\rho_2\int_0^L \mu\,(x-L)\,\overline{v}_x\,dx - \kappa_3\int_0^L v_{xx}\,(x-L)\,\overline{v}_x\,dx = \rho_2\int_0^L (x-L)\,q\,\overline{v}_x\,dx.$$

Taking the real part and using (3.2) we obtain

$$\begin{aligned}\frac{1}{2}\rho_2\int_0^L\left(|\mu|^2+\frac{\kappa_3}{\rho_2}|v_x|^2\right)dx \;&=\; \frac{1}{2}\rho_2\,L\left(|\mu(0)|^2+\frac{\kappa_3}{\rho_2}|v_x(0)|^2\right)+\rho_2\,Re\int_0^L(x-L)\,q\,\overline{v}_x\,dx\\ &\quad+\rho_2\,Re\int_0^L(x-L)\,\mu\,\overline{g}_x\,dx\end{aligned}$$

Using (1.3), and performing straightforward estimates follows that

$$\begin{aligned}\frac{1}{2}\int_0^L\left(\rho_2\,|\mu|^2+\kappa_3\,|v_x|^2\right)dx \;&\le\; \frac{1}{2}\rho_2\,L\left(|\mu(0)|^2+\frac{\kappa_3}{\rho_2}|v_x(0)|^2\right)+C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\\ &\le\; C\left[\eta(0)|^2+|\kappa_1\,u_x(0)+\kappa_2\,\eta_x(0)|^2\right]+C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}.\end{aligned}$$

Using inequality (3.6) we get

$$\int_0^L\left(|\mu|^2+|v_x|^2\right)dx \;\le\; C\,|\lambda|^{1/2}\left(\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right).$$

From (3.5)-(3.8) we conclude that

$$\|U\|_{\mathcal{H}}^2 \;\le\; C\,|\lambda|^{1/2}\left(\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right).$$

Thus

$$\|U\|_{\mathcal{H}} \;\le\; C\,|\lambda|^{1/2}\,\|F\|_{\mathcal{H}}$$

for λ large enough. The theorem follows.

Finally we prove the optimality result obtained in section 3.

Theorem 3.3 *The rate of decay obtained in Theorem 3.2 is optimal over the domain of* $\mathcal{D}(\mathcal{A})$.

Proof. Given $\lambda \in \mathbb{R}$ and $F = (0,\, 0,\, 0,\, q) \in \mathcal{H}$, there exist $U = (u,\, v,\, \eta,\, \mu) \in \mathcal{D}(\mathcal{A})$ such that, $(i\,\lambda\, I - \mathcal{A})U = F$, that is,

$$
\begin{aligned}
i\,\lambda\, u - \eta &= 0 \quad \text{in} \quad]-l,0[&& (3.9)\\
i\,\rho_1\,\lambda\,\eta - \kappa_1\, u_{xx} - \kappa_2\,\eta_{xx} &= 0 \quad \text{in} \quad]-l,0[&& (3.10)\\
i\,\lambda\, v - \mu &= 0 \quad \text{in} \quad]0,l[&& (3.11)\\
i\,\rho_2\,\lambda\,\mu - \kappa_3\, v_{xx} &= q \quad \text{in} \quad]0,l[. && (3.12)
\end{aligned}
$$

Replacing (3.9) in (3.10) we have

$$u_{xx} + \alpha^2\, u = 0, \quad u(-l) = 0. \tag{3.13}$$

where

$$\alpha^2 = \frac{\rho_1\,\lambda^2}{\kappa_1 + i\,\kappa_2\,\lambda}.$$

It is easy to see that

$$u(x) = \frac{u(0)}{\sinh(i\alpha(l))}\sinh(i\alpha(x+l))$$

Note that

$$\alpha^2 = \frac{\rho_1\,\lambda^2}{\sqrt{\kappa_1^2 + \kappa_2^2\,\lambda^2}}\left(\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2\,\lambda^2}} - i\frac{\kappa_2\lambda}{\sqrt{\kappa_1^2 + \kappa_2^2\,\lambda^2}}\right) := \frac{\rho_1\,\lambda^2}{\sqrt{\kappa_1^2 + \kappa_2^2\,\lambda^2}}\left(\cos\theta + i\sin\theta\right).$$

Therefore

$$\sin\theta \to -1, \quad \cos\theta \to 0$$

So we have that

$$\alpha = \frac{\rho_1^{1/2}\,\lambda}{\sqrt[4]{\kappa_1^2 + \kappa_2^2\,\lambda^2}}e^{i\theta/2} \quad \text{with} \quad e^{i\theta/2} \to \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

as $\lambda \to \infty$. Similarly we have

$$v_{xx} + \beta^2\, v = q, \quad v(l) = 0. \tag{3.14}$$

$$\beta^2 = \frac{\rho_2}{\kappa_3}\lambda^2, \quad \lambda \in \mathbb{R}$$

From where we have that

$$v(x) = u(0)\frac{\sin\beta(l-x)}{\sin\beta l} - \frac{\sin\beta(l-x)}{\beta\sin\beta l}\int_0^l q(s)\sin\beta(l-s)\; ds + \frac{1}{\beta}\int_0^x q(s)\sin\beta(x-s)\; ds$$

Using the trasnmission conditions

$$\kappa_1\, u_x(0) + \kappa_2\, \eta_x(0) = \kappa_3\, v_x(0)$$

From where it follows that

$$(\kappa_1 + i\kappa_2\lambda)u_x(0) = \kappa_3\, v_x(0)$$

from where it follows that

$$\frac{\alpha(\kappa_1 + i\kappa_2\lambda)u(0)}{\sinh(i\alpha(l))}\cosh(i\alpha l) = \beta u(0)\frac{\cos\beta l}{\sin\beta l} - \frac{\cos\beta l}{\sin\beta l}\int_0^l q(s)\sin\beta(l-s)\, ds$$

$$\alpha u(0)(\kappa_1 + i\kappa_2\lambda)\coth(i\alpha l) = \beta u(0)\frac{\cos\beta l}{\sin\beta l} - \frac{\cos\beta l}{\sin\beta l}\int_0^l q(s)\sin\beta(l-s)\, ds$$

$$\alpha u(0)(\kappa_1 + i\kappa_2\lambda)\coth(i\alpha l)\sin\beta l = \beta u(0)\cos\beta l - \cos\beta l\int_0^l q(s)\sin\beta(l-s)\, ds$$

$$u(0)[\alpha(\kappa_1 + i\kappa_2\lambda)\coth(i\alpha l)\sin\beta l - \beta\cos\beta l] = -\cos\beta l\int_0^l q(s)\sin\beta(l-s)\, ds$$

$$u(0) = -\frac{\cos\beta l}{\alpha(\kappa_1 + i\kappa_2\lambda)\coth(i\alpha l)\sin\beta l - \beta\cos\beta l}\int_0^l q(s)\sin\beta(l-s)\, ds$$

Let us take

$$\beta l = 2n\pi + \frac{1}{\sqrt{n}}$$

So we have that

$$\beta \approx \frac{2}{l\pi}n, \quad \sin\beta l \approx \frac{1}{\sqrt{n}}, \quad \alpha\sin\beta l \approx c_0, \quad \coth(i\alpha l) \approx 1$$

as $n \to \infty$ and $0 \neq c_0 \in \mathbb{C}$. This implies that

$$\frac{\cos\beta l}{\alpha(\kappa_1 + i\kappa_2\lambda)\coth(i\alpha l)\sin\beta l - \beta\cos\beta l} \approx \frac{c_1}{\lambda}$$

For $0 \neq c_1 \in \mathbb{C}$. Note that the expression

$$\beta v(x) = \beta u(0)\frac{\sin\beta(l-x)}{\sin\beta l} - \frac{\sin\beta x}{\sin\beta l}\int_0^l q(s)\sin\beta(l-s)\, ds + \int_0^x q(s)\sin\beta(x-s)\, ds$$

can be written as

$$\beta v(x) = \left(c_2 \frac{\sin\beta(l-x)}{\sin\beta l} - \frac{\sin\beta x}{\sin\beta l} \right) \int_0^l q(s)\sin\beta(l-s)\,ds + \int_0^x q(s)\sin\beta(x-s)\,ds$$

$$\beta v(x) = \frac{c_2 \sin\beta(l-x) - \sin\beta x}{\sin\beta l} \int_0^l q(s)\sin\beta(l-s)\,ds + \underbrace{\int_0^x q(s)\sin\beta(x-s)\,ds}_{:=Q(x)}$$

$$\beta v(x) = \left[c_2\cos\beta x - (c_2\cos\beta l + 1)\frac{\sin\beta x}{\sin\beta l} \right] Q(l) + Q(x)$$

Taking $q(s) = \sin\beta s$ and squaring and integrating we have

$$\begin{aligned} Q(x) &= \int_0^x \sin\beta s \sin\beta x \cos\beta s - \sin^2\beta(s)\cos\beta x\,ds \\ &= \sin\beta x \int_0^x \sin\beta s\cos\beta s\,ds - \cos\beta x \int_0^x \sin^2\beta s\,ds \\ &= -\frac{\sin^3\beta x}{2\beta l} - \cos\beta x \int_0^x \sin^2\beta x\,ds \\ &= -\frac{\sin^3\beta x}{2\beta l} - \frac{x\cos\beta x}{2} + \frac{\cos\beta x\sin(2\beta x)}{2\beta} \end{aligned} \tag{3.15}$$

Therefore

$$Q(l) = -\frac{\pi}{n^{5/2}} - \frac{l\cos\beta}{2} + \frac{\cos\beta l}{n^{3/2}} \approx -\frac{l}{2}$$

Note that

$$\int_0^l |Q(s)|^2\,ds \geq \int_0^l \frac{x^2\cos^2\beta x}{8}\,dx - \frac{c}{\beta^2} \geq \frac{l^3}{48} - \frac{c}{|\beta|} \tag{3.16}$$

Finally,

$$\begin{aligned} &\int_0^l \left| c_2\cos\beta x - (c_2\cos\beta l + 1)\frac{\sin\beta x}{\sin\beta l} \right|^2 ds \\ &\geq \frac{|c_2\cos\beta l + 1|}{2\sin^2\beta l} \int_0^l \sin^2\beta x\,dx - c_0 \\ &\approx c_1 n - c_0 \end{aligned} \tag{3.17}$$

Inserting inequalities (3.16) and (3.17) into (3.15) we get that tehre exists a positive constant C such that

$$\int_0^l |\beta v(x)|^2\,dx \geq -C + Cn$$

for n large, that is

$$\frac{1}{n}\int_0^l |\beta v(x)|^2\,dx \geq C_0 \tag{3.18}$$

In particular we have that

$$\|U\|^2 \geq \int_0^l |\beta v(x)|^2$$

If the rate of decay can be improved then we have that $\frac{1}{n^{1-\epsilon}}\|U\|^2$ must be bounded. Using the two above inequalities we get

$$\frac{1}{n^{1-\epsilon}}\|U\|^2 \geq \int_0^l |\beta v(x)|^2 \geq C_0 n^{\epsilon}$$

Which is contradictory to our assumption. From where our conclusion follows

4 Numerical approximations

In this section we show the polynomial decay numerically proved in the previous sections. It is important to note that any numerical approximation is a finite-dimensional simplification of the original problem. Thus, any numerical method used, decay exponentially for large enough times, and this because of its restrictive nature of the finite dimensional space approach. In this regard, we have a robust numerical method of high order which in turn ensures a natural way (without additional artificial viscosity for example) the decay of energy with the same terms prescribed in identity (1.5).

4.1 Linear equation of Motion

First, we approximate the displacement vector $[u, v]^\top$ in space using a conservative finite difference method (called also Finite Volume Method [4]). For $J \in \mathbb{N}$ and $\delta x = L/J$, we define $x_{j+\frac{1}{2}}$, with $j = -J, \ldots, J$, as a uniform discretization of the interval $(-L, L)$. Then, we define $x_j = \frac{x_{j+\frac{1}{2}}+x_{j-\frac{1}{2}}}{2}$, the points of approximation $[u, v]^\top$, for $j = -J-1, \ldots, J$, obtaining a vector $[\mathbf{u}_\delta(t), \mathbf{v}_\delta(t)]^\top$ approximation of $[u, v]^\top$ in $\mathbb{R}^J \times \mathbb{R}^J$. Additionally, let us define $[\eta_\delta(t), \mu_\delta(t)]^\top$ the approximation of the velocity $[\eta, \mu]^\top$, where $\eta_\delta(t) = \dot{\mathbf{u}}_\delta(t)$ and $\mu_\delta(t) = \dot{\mathbf{v}}_\delta(t)$. Integrating (1.1) and (1.2) in each interval $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ (called control volumes in the context of finite volumes method), using the boundary and transmission condition, we easily obtain the linear equation of motion

$$\mathbf{M}\begin{bmatrix} \dot{\eta}_h \\ \dot{\mu}_h \end{bmatrix} + \frac{1}{\delta x^2}\mathbf{C}\begin{bmatrix} \eta_h \\ \mu_h \end{bmatrix} + \frac{1}{\delta x^2}\mathbf{K}\begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \end{bmatrix} \quad = \quad \mathbf{0}, \tag{4.1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system in $\mathcal{M}_{2J}(\mathbb{R})$. Using standard finite differences to approximate the flows on the edges of control volumes, and since the discretization is uniform, we get easily that $\mathbf{M} = \begin{pmatrix} \rho_1 & & & & & \\ & \ddots & & & & \\ & & \rho_1 & & & \\ & & & \rho_2 & & \\ & & & & \ddots & \\ & & & & & \rho_2 \end{pmatrix}$,

$$\mathbf{K} = \begin{pmatrix} 2\kappa_1 & -\kappa_1 & & & & & \\ & \ddots & \ddots & & & & \\ & -\kappa_1 & \kappa_1+\kappa_3 & -\kappa_3 & & & \\ & & -\kappa_3 & 2\kappa_3 & -\kappa_3 & & \\ & & & \ddots & \ddots & & \\ & & & & & & 2\kappa_3 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 2\kappa_2 & -\kappa_2 & & & \\ & \ddots & & & \\ & -\kappa_2 & \kappa_2 & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

On the other hand, and in the context now of finite differences, we know that to be a centered scheme is of second order in space.

4.2 Time discretization

Regarding now to the time discretization, it is desirable that the algorithm has at least second-order accuracy too, and because the spatial discretization used in structural dynamics often leads to inclusion of high-frequency modes in the model, it is also desirable to have unconditional stability. The method consists of updating the displacement, velocity and acceleration vectors at current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$, a small time interval δt later. The Newmark algorithm [18] is based on a set of two relations expressing the forward displacement $[\mathbf{u}_\delta^{n+1}, \mathbf{v}_\delta^{n+1}]^\top$ and velocity $[\eta_\delta^{n+1}, \mu_\delta^{n+1}]^\top$ in terms of their current values and the forward and current values of the acceleration,

$$\eta_\delta^{n+1} \quad = \quad \eta_\delta^n + (1-\gamma)\delta t\, \dot{\eta}_\delta^n + \gamma \delta t\, \dot{\eta}_\delta^{n+1} \tag{4.2}$$

$$\mathbf{u}_\delta^{n+1} \quad = \quad \mathbf{u}_\delta^n + \left(\frac{1}{2} - \beta\right) \delta t^2\, \dot{\eta}_\delta^n + \beta \delta t^2\, \dot{\eta}_\delta^{n+1} \tag{4.3}$$

$$\mu_\delta^{n+1} \quad = \quad \mu_\delta^n + (1-\gamma)\delta t\, \dot{\mu}_\delta^n + \gamma \delta t\, \dot{\mu}_\delta^{n+1} \tag{4.4}$$

$$\mathbf{v}_\delta^{n+1} \quad = \quad \mathbf{v}_\delta^n + \left(\frac{1}{2} - \beta\right) \delta t^2\, \dot{\mu}_\delta^n + \beta \delta t^2\, \dot{\mu}_\delta^{n+1}, \tag{4.5}$$

where β and γ are parameters of the methods that will be fixed later. Replacing (4.2)-(4.5) in the equation of motion (4.1), we obtain

$$\begin{aligned}\left(\mathbf{M}+\gamma\delta t\,\mathbf{C}+\beta\delta t^2\mathbf{K}\right)\left[\begin{array}{c}\dot\eta_\delta^{n+1}\\ \dot\mu_\delta^{n+1}\end{array}\right] &= -\mathbf{C}\left(\left[\begin{array}{c}\eta_\delta^{n}\\ \mu_\delta^{n}\end{array}\right]+(1-\gamma)\delta t\left[\begin{array}{c}\dot\eta_\delta^{n}\\ \dot\mu_\delta^{n}\end{array}\right]\right)\\ &\quad -\mathbf{K}\left(\left[\begin{array}{c}\mathbf{u}_\delta^{n}\\ \mathbf{v}_\delta^{n}\end{array}\right]+\delta t\left[\begin{array}{c}\eta_\delta^{n}\\ \mu_\delta^{n}\end{array}\right]+\left(\frac{1}{2}-\beta\right)\delta t^2\left[\begin{array}{c}\dot\eta_\delta^{n}\\ \dot\mu_\delta^{n}\end{array}\right]\right). \end{aligned} \tag{4.6}$$

The acceleration $[\dot\eta_\delta^{n+1},\dot\mu_\delta^{n+1}]^\top$ is found from (4.6), and the velocity $[\eta_\delta^{n+1},\mu_\delta^{n+1}]^\top$ follow from (4.2) and (4.4), respectively, and finally displacement $[\mathbf{u}_\delta^{n+1},\mathbf{v}_\delta^{n+1}]^\top$ follow from (4.3) and (4.5), respectively by simple vector operations.

4.3 Energy balance of the Newmark algorithm

We define the discrete energy as

$$\mathcal{E}_\delta^n \;:=\; \frac{1}{2}\left[\eta_\delta^\top,\mu_\delta^\top\right]\mathbf{M}\left[\begin{array}{c}\eta_\delta\\ \mu_\delta\end{array}\right]+\frac{1}{2}\left[\mathbf{u}_\delta^\top,\mathbf{v}_\delta^\top\right]\mathbf{K}\left[\begin{array}{c}\mathbf{u}_\delta\\ \mathbf{v}_\delta\end{array}\right]$$

which is an approximation of that defined in (1.5) for the continuous case. The increment of this energy can be expressed in terms of mean values and increments of the displacement and velocity by the following identity:

$$\begin{aligned}\mathcal{E}_\delta^{n+1}-\mathcal{E}_\delta^{n} &= \left[\frac{1}{2}\left[\eta_\delta^\top,\mu_\delta^\top\right]\mathbf{M}\left[\begin{array}{c}\eta_\delta\\ \mu_\delta\end{array}\right]+\frac{1}{2}\left[\mathbf{u}_\delta^\top,\mathbf{v}_\delta^\top\right]\mathbf{K}\left[\begin{array}{c}\mathbf{u}_\delta\\ \mathbf{v}_\delta\end{array}\right]\right]_n^{n+1}\\ &= \left[\begin{array}{c}\eta_\delta^{n+\frac{1}{2}}\\ \mu_\delta^{n+\frac{1}{2}}\end{array}\right]^\top\mathbf{M}\left[\begin{array}{c}\Delta\eta_\delta\\ \Delta\mu_\delta\end{array}\right]+\left[\begin{array}{c}\mathbf{u}_\delta^{n+\frac{1}{2}}\\ \mathbf{v}_\delta^{n+\frac{1}{2}}\end{array}\right]^\top\mathbf{K}\left[\begin{array}{c}\Delta\mathbf{u}_\delta\\ \Delta\mathbf{v}_\delta\end{array}\right]\end{aligned}$$

where $\mathbf{u}^{n+\frac{1}{2}}=\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}$ and $\Delta\mathbf{u}=\mathbf{u}^{n+1}-\mathbf{u}^n$. Now, in order to derive the required energy estimates, we rely on calculations and notations similar to S. Krenk [9] to finally obtain

$$\begin{aligned}&\left[\frac{1}{2}\left[\begin{array}{c}\eta_h\\ \mu_h\end{array}\right]^\top\mathbf{M}_*\left[\begin{array}{c}\eta_h\\ \mu_h\end{array}\right]+\frac{1}{2}\left[\begin{array}{c}\mathbf{u}_h\\ \mathbf{v}_h\end{array}\right]^\top\mathbf{K}\left[\begin{array}{c}\mathbf{u}_h\\ \mathbf{v}_h\end{array}\right]+\left(\beta-\frac{1}{2}\gamma\right)\frac{\delta t^2}{2}\left[\begin{array}{c}\dot\eta_h\\ \dot\mu_h\end{array}\right]^\top\mathbf{M}_*\left[\begin{array}{c}\dot\eta_h\\ \dot\mu_h\end{array}\right]\right]_n^{n+1}\\ &= \left(\gamma-\frac{1}{2}\right)\left\{\left[\begin{array}{c}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\end{array}\right]^\top\mathbf{K}\left[\begin{array}{c}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\end{array}\right]+\left(\beta-\frac{1}{2}\gamma\right)\delta t^2\left[\begin{array}{c}\Delta\dot\eta_h\\ \Delta\dot\mu_h\end{array}\right]^\top\mathbf{M}_*\left[\begin{array}{c}\Delta\dot\eta_h\\ \Delta\dot\mu_h\end{array}\right]\right\}\\ &\quad -\frac{1}{2}\delta t\left\{\delta t^{-2}\left[\begin{array}{c}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\end{array}\right]^\top\mathbf{C}\left[\begin{array}{c}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\end{array}\right]+\left[\begin{array}{c}\eta_h^{n+\frac{1}{2}}\\ \mu_h^{n+\frac{1}{2}}\end{array}\right]^\top\mathbf{C}\left[\begin{array}{c}\eta_h^{n+\frac{1}{2}}\\ \mu_h^{n+\frac{1}{2}}\end{array}\right]\right\}\\ &\quad +\frac{1}{2}\left(\beta-\frac{1}{2}\gamma\right)^2\delta t^3\left[\begin{array}{c}\Delta\dot\eta_h\\ \Delta\dot\mu_h\end{array}\right]^\top\mathbf{C}\left[\begin{array}{c}\Delta\dot\eta_h\\ \Delta\dot\mu_h\end{array}\right]\end{aligned}$$

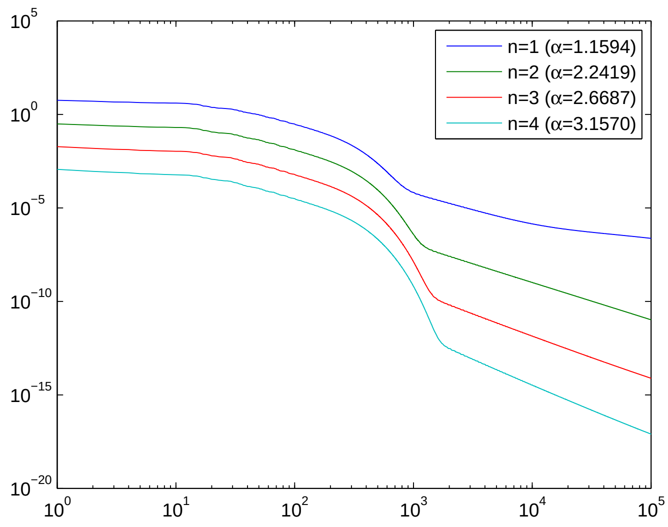


Figure 1: Energy decay for initial conditions with different smoothness. For a graph in log-log scale, there is a decay in order $t^{-\alpha}$.

where $\mathbf{M}_* = \mathbf{M} + \left(\gamma - \frac{1}{2}\right)\delta t\mathbf{C}$. Then, we choose $\gamma = \frac{1}{2}$ and $\beta = \frac{\gamma}{2}$, reducing the above expression to

$$\left[\frac{1}{2}\left[\begin{array}{c}\eta_h\\ \mu_h\end{array}\right]^\top \mathbf{M}\left[\begin{array}{c}\eta_h\\ \mu_h\end{array}\right] + \frac{1}{2}\left[\begin{array}{c}\mathbf{u}_h\\ \mathbf{v}_h\end{array}\right]^\top \mathbf{K}\left[\begin{array}{c}\mathbf{u}_h\\ \mathbf{v}_h\end{array}\right]\right]_n^{n+1}$$
$$= -\frac{1}{2}\delta t\left\{\frac{\Delta\mathbf{u}_h^\top}{\delta t}\widetilde{\mathbf{C}}\frac{\Delta\mathbf{u}_h}{\delta t} + \eta_h^{n+\frac{1}{2},\top}\widetilde{\mathbf{C}}\eta_h^{n+\frac{1}{2}}\right\} \leqslant 0 \qquad (4.7)$$

where $\widetilde{\mathbf{C}} \in \mathcal{M}_J(\mathbb{R})$ represents de reduced matrix of $\mathbf{C} \in \mathcal{M}_{2J}(\mathbb{R})$ does not take into account the null rows and null columns, that is $\widetilde{\mathbf{C}} = \begin{pmatrix} 2\kappa_2 & -\kappa_2 & & \\ -\kappa_2 & 2\kappa_2 & -\kappa_2 & \\ & & \ddots & \\ & & -\kappa_2 & \kappa_2 \end{pmatrix}$. We remark that (4.7) corresponds to the discrete version of (2.4), but more than that, the term on the right is precisely the expected amount, corresponding to a discretization of the right side of (2.3). With this, we expect the rate of decay of energy in the discrete case is an accurate reflection of what happens in the continuous case.

4.4 Numerical examples

Now we present two examples to illustrate graphically the polynomial energy decay.

4.4.1 Example 1. Initial conditions with different smoothness

Let us suppose here that $L = 2$ and $T = 100000$. We will study the asymptotic behavior for a family of initial conditions of the form

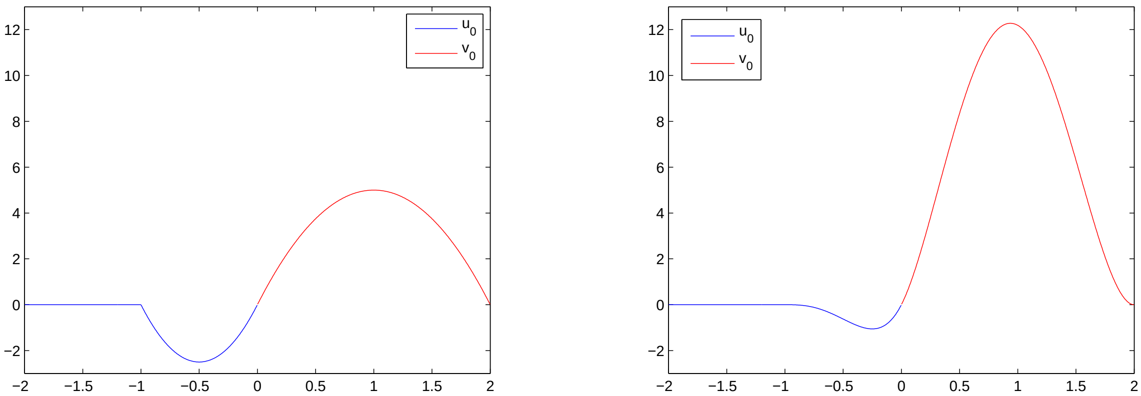
$$u_0 = \begin{cases} 10(x+1)^n x^n & \text{if } x \in (-1,0) \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

at rest, that is $v_0 = \eta_0 = \mu_0 = 0$. We suppose additionally that $\kappa_1 = \kappa_3 = 10$, $\kappa_2 = 1$. Finally, the discretization is given by $J = 10000$ and $N = 100000$, that is $\delta x = L/J = 2.10^{-4}$ and $\delta t = T/N = 1$. Figure 1 shows the asymptotic behavior of the energy plotted in log-log scale, so that all behavior expressed graphically displayed polynomial with straight lines. In this case we see that such polynomial behavior is reinforced from time $t = 1000$, which actually represents 99% of total time (100000 [sec]).

By simple linear regression (least squares) rates obtained numerically for each case. For every exponent n of the initial condition (4.8), we have a rate α different (which would normally be optimal 2 according to the theory). First we see in Figure 1 that the rate α increases as the initial condition becomes more and more regular (n grows), which is expected. But on the other hand, we observe in the same Figure 1 that for $n = 1$, we have a value of α less than 2 ($\alpha = 1.1594$). This does not contradict the theory, and the background is not so surprising, if we observe that the case $n = 1$ corresponds to an initial condition which is definitely not in $D(A)$, hypotheses need to have the optimal polynomial decay. This put into evidence that the hypothesis of belonging of the initial condition in the domain $D(A)$ is relevant is clearer in the second example.

4.4.2 Example 2. When initial condition $U_0 \notin \mathcal{D}(\mathcal{A})$ and when $U_0 \in \mathcal{D}(\mathcal{A})$

Here, we take again $L =$ and $T = 100000$. In this second case, we see the importance that the initial condition is in $D(A)$ to obtain expected polynomial decay of rate t^{-2}. We look at two initial conditions that are on the brink of this situation, which they are described in Figure 2. In the picture on the left it is described the case when $U_0 \in \mathcal{D}(\mathcal{A})$ is not verified. Indeed, at $x = -1$, the function u_0 is continuous but not of class $\mathcal{C}^1$, and hence, it is not in $H^2(-L, 0)$. Regarding the transmission term, the initial condition is of class $\mathcal{C}^1$, but not of class $\mathcal{C}^2$ at $x = 0$, considering v_0, as an extension of u_0 at this point. On the other hand, in the picture on the right, the solution is entirely in $D(A)$, but without



$$u_0 = \begin{cases} 10(x+1)x & \text{if } x \in (-1,0) \\ 0 & \text{otherwise} \end{cases}$$
$$v_0 = \begin{cases} 5(2-x)x & \text{if } x \in (0,2) \\ 0 & \text{otherwise} \end{cases}$$
$$\eta_0 = \mu_0 = 0$$

$$u_0 = \begin{cases} 10(x+1)^3 x & \text{if } x \in (-1,0) \\ 0 & \text{otherwise} \end{cases}$$
$$v_0 = \begin{cases} \frac{5}{16}(2-x)^2 x(-x^2+32x+8) & \text{if } x \in (0,2) \\ 0 & \text{otherwise} \end{cases}$$
$$\eta_0 = \mu_0 = 0$$

Figure 2: Comparison between two initial conditions. Picture on the left: $U_0 \notin \mathcal{D}(\mathcal{A})$; picture on the right: $U_0 \in \mathcal{D}(\mathcal{A})$.

additional regularity beyond that. In fact, the function u_0 is of class $\mathcal{C}^1$ on $x = -1$ but no more than that. However, the transmission term is regular enough.

The result is not expected. Under the same conditions of discretization of the previous example, we see that there is an effective polynomial decay, with rate $t^{-\alpha}$ and $\alpha = 2.022$ when the initial condition satisfies the hypothesis (see green curve in Figure 3). On the other hand, when the initial condition does not meet the hypothesis of belonging to $D(A)$, the rate is polynomial with $\alpha = 0.6611$, obtaining this value by least squares, and even being questioned according to the graph if it is actually polynomial (blue curve , in Figure 3).

The fact that the discretization is a finite dimensional problem, unlike the original semigroup, which in theory makes an exponential decay should be observed for all cases for large enough times. Aware of this finite-dimensional restriction, our examples were made for times large but reasonable in the framework of what we wanted to highlight.

Finally in Figure 4, we see the evolution of displacement behavior of (u, v) and velocities (η, μ) when the decay energy is polynomial, with rate t^{-2}. It is observed that the critical point of decay (in which the left side, although very fast decay, affects the right side that does not decay as quickly) is the transmission term, which is consistent with the

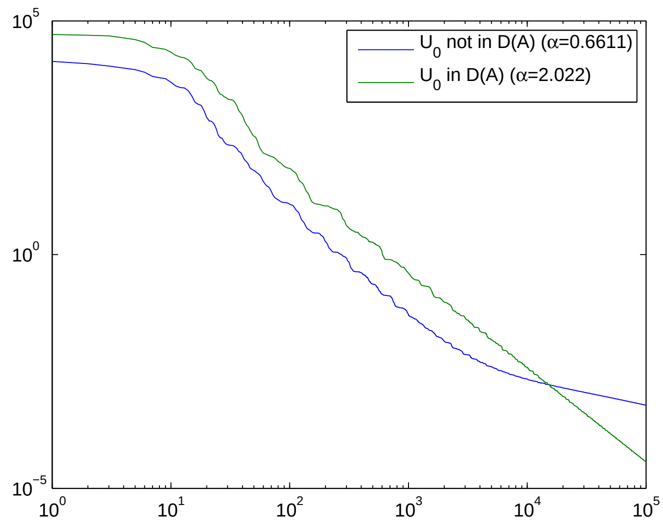
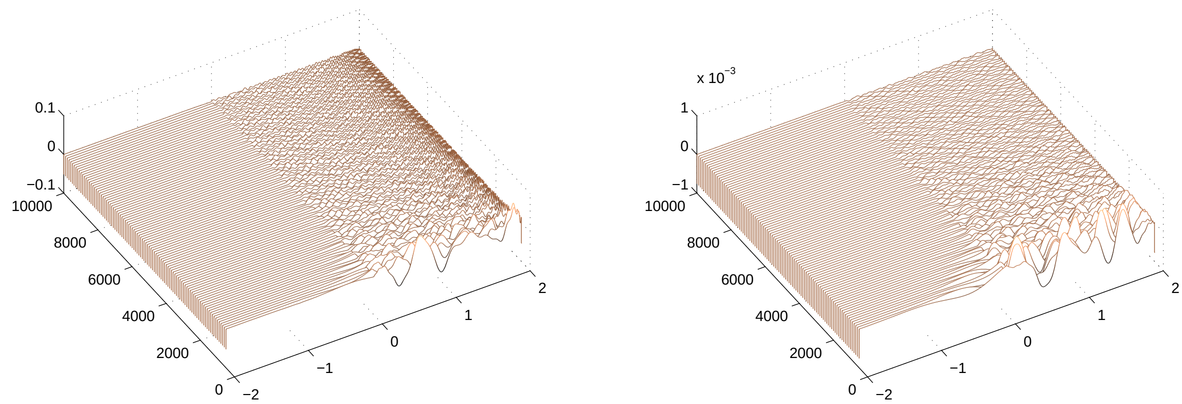


Figure 3: Energy with polinomial decaying $\leqslant C\,(1/t)^2$ when $U_0 \in \mathcal{D}(\mathcal{A})$.

estimate (3.3) which is key to the proof of Theorem 3.2 of our Section 3.



(a) Displacements (u, v)

(b) Velocities (η, μ)

Figure 4: Solution behaviour in space and time. Case of energy with polinomial decaying $\leqslant C\,(1/t)^2$ when $U_0 \in \mathcal{D}(\mathcal{A})$

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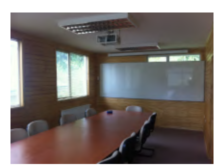


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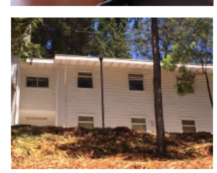
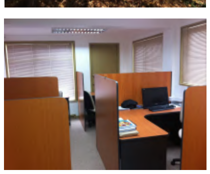

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