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STABLE NUMERICAL METHODS FOR TWO CLASSES OF SDEs WITH MULTIPLICATIVE NOISE: BILINEAR AND SCALAR

H. A. MARDONES *† AND C. M. MORA *‡

Abstract. We develop new numerical schemes for bilinear systems of stochastic differential equations (SDEs). To this end, we present a new methodology for solving stiff multidimensional SDEs with multiplicative noise, as well as, we introduce a new stable numerical method for non-linear scalar SDEs. The rate of weak convergence of the new schemes is linear. Moreover, they preserve the possible exponential stability of the unknown solutions for any step-size. Four numerical experiments illustrate the good performance of the proposed algorithms.

Key words. stochastic differential equations, stable numerical schemes, weak errors, rate of convergence, Lyapunov exponents, bilinear SDEs, locally Lipschitz SDEs

AMS subject classifications. 60H35, 65C30, 60H10, 65C20.

1. Introduction.

This paper introduces a new approach to solve stiff stochastic differential equations (SDEs) with multiplicative noise, namely, SDEs of the form

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \sum_{k=1}^{m} \int_0^t \sigma^k(X_s)\, dW_s^k \tag{1.1}$$

whose numerical solutions by the Euler-Maruyama scheme exhibit incorrect behaviors. Here, $W^1, \ldots, W^m$ are independent real valued Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\right)$, X_t is an adapted $\mathbb{R}^d$-valued stochastic process, $b, \sigma^k : \mathbb{R}^d \to \mathbb{R}^d$ have continuous first-order partial derivatives, and $X_0 \in L^2(\Omega, \mathbb{P})$. For simplicity, we only consider autonomous SDEs. Our final goal is to have a set of schemes that preserve dynamical properties of (1.1), like asymptotic stability, for any step size Δ of the time discretization.

Section 2 is devoted to scalar SDEs. More precisely, we develop a new scheme for (1.1) with $d = 1$. In case $b(0) = 0$ and $\sigma^k(0) = 0$, the new numerical method preserves, for any step size $\Delta > 0$, both the sign of X_0 and the possible exponential stability of X_t, that is, the property $\limsup_{t\to\infty} \frac{1}{t} \ln \|X_t\| \leq -\lambda < 0$. This paves way for the main objective of this article: to provide stable schemes for computing the mean value of $f(X_t)$, where $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and, by abuse of notation,

$$X_t = X_0 + \int_0^t B X_s \; ds + \sum_{k=1}^{m} \int_0^t \sigma^k X_s \; dW_s^k, \tag{1.2}$$

with $B, \sigma^1, \ldots, \sigma^m$ real matrices of dimension $d \times d$. In this research direction, Subsection 3.1 presents a general technique for constructing stable methods for (1.1) with $d > 1$. Then, in Subsections 1.2 and 3.2 we use the new ideas to derive promising weak schemes for (1.2). The bilinear SDE (1.2) arises, for example, from the spatial discretization of some stochastic partial differential equations (see, e.g., [1, 12]), and describes important dynamical features of non-linear SDEs via the linearization

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around their equilibrium points (see, e.g., [6, 34]). Furthermore, the new numerical methods for (1.2) guide us in the application of our general methodology to systems of non-linear SDEs.

1.1. Previous works. In many cases, the semi-implicit and explicit Euler methods preserve dynamical properties of the underlying SDEs provided that the step size of the discretization is small enough. Recall that (see, e.g., [17, 19])

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|X_t\| \leq -\lambda \qquad a.s. \tag{1.3}$$

whenever

$$\|b(x)\| \leq K_n \|x\| \qquad \forall \|x\| \leq n \text{ and } \forall n \in \mathbb{N},$$

$$\left\|\sigma^k(x)\right\| \leq K\|x\| \qquad \forall x \in \mathbb{R}^d, \text{ and} \tag{1.4}$$

$$-\lambda := \sup_{x\in\mathbb{R}^d, x\neq 0} \left(\frac{\langle x, b(x)\rangle + \frac{1}{2}\sum_{k=1}^m \left\|\sigma^k(x)\right\|^2}{\|x\|^2} - \frac{\sum_{k=1}^m \langle x, \sigma^k(x)\rangle^2}{\|x\|^4} \right) < 0. \tag{1.5}$$

Here and subsequently, K stands for a generic positive constant and $K_n > 0$. If Conditions (1.4) and (1.5) hold, together with $b(0) = 0$, then Higham, Mao and Yuan [17] proved that the Euler-Maruyama scheme

$$E_{n+1} = E_n + b(E_n)\Delta + \sum_{k=1}^m \sigma^k(E_n)\left(W^k_{(n+1)\Delta} - W^k_{n\Delta}\right)$$

is almost sure exponentially stable for sufficiently small step sizes $\Delta > 0$ in case

$$\|b(x)\| \leq K\|x\| \qquad \forall x \in \mathbb{R}^d.$$

At the cost of solving systems of algebraic equations, the backward Euler method

$$\bar{E}_{n+1} = \bar{E}_n + b\left(\bar{E}_{n+1}\right)\Delta + \sum_{k=1}^m \sigma^k\left(\bar{E}_n\right)\left(W^k_{(n+1)\Delta} - W^k_{n\Delta}\right) \tag{1.6}$$

improves the numerical stability of E_n (see, e.g., [16, 17, 35]). For instance, Higham, Mao and Yuan [17] established that there exists $\Delta_0 > 0$ such that for any $\Delta \in]0, \Delta_0]$, $\bar{E}_n$ is almost sure exponentially stable provided that (1.4) is valid, $b(0) = 0$, and

$$\sup_{x\neq y} \frac{\langle x-y, b(x) - b(x)\rangle}{\|x-y\|^2} + \sup_{x\neq 0}\left(\frac{\sum_{k=1}^m \left\|\sigma^k(x)\right\|^2}{2\|x\|^2} - \frac{\sum_{k=1}^m \langle x, \sigma^k(x)\rangle^2}{\|x\|^4}\right) < 0. \tag{1.7}$$

The Euler schemes E_n and $\bar{E}_n$ exhibit a poor numerical performance in situations where, for example, some partial derivatives of the diffusion coefficients σ^k are not small. A simple model problem for such SDEs is

$$dX_t = \lambda X_t dW_t, \tag{1.8}$$

with $\lambda > 0$ (see, e.g., [22, 23]); the trajectories of E_n and $\bar{E}_n$ applied to (1.8) blow up unless Δ is very small. Using (1.8) as a motivational problem, Milstein, Platen and Schurz [22] introduced the general formulation of the balanced implicit methods, a class of fully implicit schemes for (1.1) whose implementation depends of the choice of certain weights (see, e.g., [2, 15, 22]). To the best of our knowledge, the reported balanced schemes have good asymptotic stability properties and low speed of weak convergence.

Other implicit integrators for (1.1), together with their predictor-corrector versions, arise from the Itô-Taylor expansions of X_t (see, e.g., [13, 23, 27, 28, 30, 36]). In particular, Kloeden and Platen [18] proposed the following class of weak implicit schemes:

$$\begin{aligned}\tilde{E}_{n+1} = \tilde{E}_n &+ \left(\alpha\, c\left(\tilde{E}_{n+1}\right) + (1-\alpha)\, c\left(\tilde{E}_n\right)\right)\Delta \\ &+ \sum_{k=1}^{m}\left(\eta\sigma^k\left(\tilde{E}_{n+1}\right) + (1-\eta)\,\sigma^k\left(\tilde{E}_n\right)\right)\sqrt{\Delta}\xi_n^k,\end{aligned} \tag{1.9}$$

where $\alpha, \eta \in [0,1]$, $\{\xi_n^k : k = 1, \dots, m \text{ and } n \geq 0\}$ is a collection of independent random variables taking values ± 1 with probability $1/2$, and

$$c(x) = b(x) - \eta \sum_{k=1}^{m}\sum_{j=1}^{d} \sigma^{j,k}(x)\, \frac{\partial \sigma^k}{\partial x^j}(x).$$

We have $\tilde{E}_n \approx X_{n\Delta}$. Applying (1.9) with $\alpha = \eta = 1$ to (1.8) yields a fully implicit method which is almost sure asymptotically stable, but converges to 0 as $n \to \infty$ too much faster than X_t. Taking $\alpha = \eta = 1/2$ gives a trapezoidal scheme with good asymptotically stability properties, nevertheless it fails to preserve the sign of X_0 in the numerical solution of (1.8).

Finally, numerical methods adapted to specific types of SDEs with multiplicative noise have been developed, for instance, in [3, 9, 23, 24, 25]. The numerical integration of mean-square stable SDEs has been treated, for example, in [1, 10, 11, 15, 31]. To the best of our knowledge, this paper is the first work to present numerical methods for SDEs that preserve, for any step size $\Delta > 0$, the asymptotic stability of the solutions of relevant classes of SDEs (see, e.g., [16, 17]).

1.2. New numerical method for bilinear SDEs. It follows from

$$X_t = \frac{X_t}{\|X_t\|}\|X_t\|$$

that we can divide the computation of the solution of (1.2) into:

(i) The approximation of $\hat{X}_t := X_t / \|X_t\|$ by an adapted stochastic process taking values in the unit sphere.

(ii) The numerical simulation of $\|X_t\|$ by means of a scheme that preserves the dynamical properties of $\|X_t\|$.

Since (1.2) is bilinear, using Itô's formula yields

$$\hat{X}_t = \hat{X}_0 + \int_0^t B\left(\hat{X}_s\right)\hat{X}_s ds + \sum_{k=1}^{m}\int_0^t \left(\sigma^k - \left\langle \hat{X}_s, \sigma^k \hat{X}_s\right\rangle\right)\hat{X}_s dW_s^k \tag{1.10}$$

(see Subsection 3.1 for details), where, by abuse of notation,

$$B\left(\hat{X}_s\right) = B - \left\langle \hat{X}_s, B\hat{X}_s\right\rangle + \sum_{k=1}^{m}\left(\frac{3}{2}\left\langle \hat{X}_s, \sigma^k\hat{X}_s\right\rangle^2 - \left\langle \hat{X}_s, \sigma^k \hat{X}_s\right\rangle\sigma^k - \frac{1}{2}\left\|\sigma^k\hat{X}_s\right\|^2\right).$$

Although (1.10) is a locally Lipschitz SDE, its solution has norm 1 for all $t \geq 0$, and hence we can improve the performance of the numerical schemes applied to (1.10) by projecting on the unit sphere at each discretization step. This projection procedure has been used with success in the numerical solution of the non-linear Schödinger equations (see, e.g., [24, 26]).

Moreover, we utilize Itô's formula to obtain

$$\|X_t\| = \|X_0\| + \int_0^t \left(\frac{\langle X_s, BX_s\rangle + \frac{1}{2}\sum_{k=1}^m \left\|\sigma^k X_s\right\|^2}{\|X_s\|} - \frac{1}{2}\sum_{k=1}^m \frac{\langle X_s, \sigma^k X_s\rangle^2}{\|X_s\|^3} \right) ds$$
$$+ \sum_{k=1}^m \int_0^t \frac{\langle X_s, \sigma^k X_s\rangle}{\|X_s\|} dW_s^k \tag{1.11}$$

(see Subsection 3.1 for details). In Section 2 we develop a stable scheme for scalar stiff SDEs that shows very good performance in numerical experiments. A close look at this numerical method leads us to divide and multiply each integrand of (1.11) by $\|X_s\|$, and so (1.11) becomes

$$\|X_t\| = \|X_0\| + \int_0^t \left(\langle \hat{X}_s, B\hat{X}_s\rangle + \frac{1}{2}\sum_{k=1}^m \left(\left\|\sigma^k \hat{X}_s\right\|^2 - \langle \hat{X}_s, \sigma^k \hat{X}_s\rangle^2 \right) \right) \|X_s\| \, ds$$
$$+ \sum_{k=1}^m \int_0^t \langle \hat{X}_s, \sigma^k \hat{X}_s\rangle \|X_s\| \, dW_s^k. \tag{1.12}$$

We propose to approximate numerically X_t by solving the system formed by (1.10) and (1.12). Next, we present a simple way to do this. For simplicity, from now on we consider the equidistant time discretization $T_n = n\Delta$, where $\Delta > 0$ and $n = 0, 1, \ldots$ Suppose that $\bar{X}_n \approx \hat{X}_{T_n}$. Applying the weak Euler approximation to (1.10) we get $\hat{X}_{T_{n+1}} \approx \bar{Z}_{n+1}$ with

$$\bar{Z}_{n+1} = \bar{X}_n + B\left(\bar{X}_n\right)\bar{X}_n\Delta + \sum_{k=1}^m \left(\sigma^k - \left\langle \bar{X}_n, \sigma^k \bar{X}_n\right\rangle\right)\bar{X}_n\sqrt{\Delta}\,\xi_n^k$$

where $\xi_0^1, \xi_0^2, \ldots, \xi_0^m, \xi_1^1, \ldots$ are independent random variables taking values ± 1 with probability 1/2. Projecting $\bar{Z}_{n+1}$ on the unit sphere gives

$$\hat{X}_{T_{n+1}} \approx \bar{X}_{n+1} := \bar{Z}_{n+1} / \left\|\bar{Z}_{n+1}\right\|. \tag{1.13}$$

From (1.12) we deduce that $\|X_t\|$, with $t \in [T_n, T_{n+1}]$, is well approximated by the solution of the linear scalar SDE

$$\eta_t = \bar{\eta}_n + \int_{T_n}^t \left(\langle \bar{X}_n, B\bar{X}_n\rangle + \frac{1}{2}\sum_{k=1}^m \left(\left\|\sigma^k \bar{X}_n\right\|^2 - \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle^2 \right) \right) \eta_s ds \tag{1.14}$$
$$+ \sum_{k=1}^m \int_{T_n}^t \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle \eta_s dW_s^k,$$

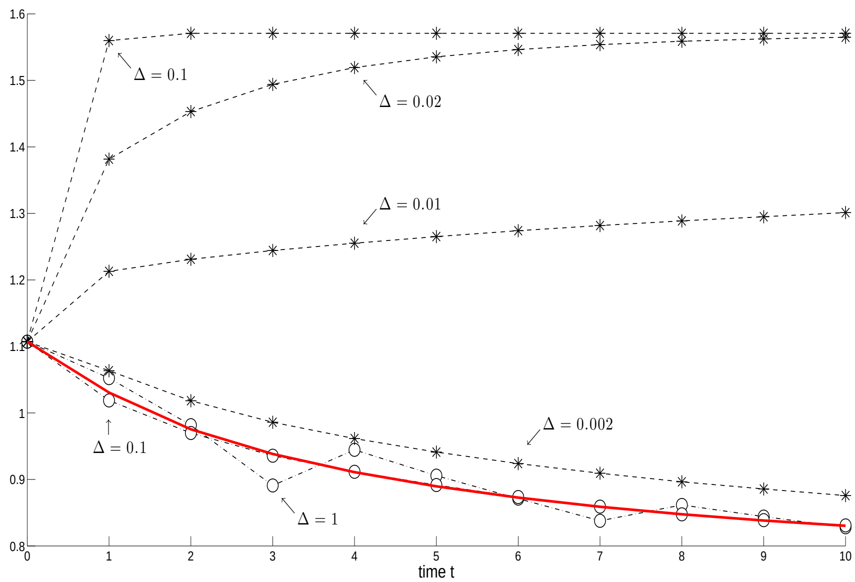


FIG. 1. *Computation of* $\mathbb{E}\arctan\left(1+\left(X_t^1\right)^2\right)$, *where* $t \in [0,10]$ *and* X_t *solves* (1.16) *with* $b=-2$, $\sigma=4$ *and* $\epsilon=4$. *The true values are plotted with a solid line. The circles (resp. stars) represent the approximations of* $\mathbb{E}\arctan\left(1+\left(X_t^1\right)^2\right)$, *with* $t=0,1,\ldots,10$, *given by* $\bar{\eta}_n\bar{X}_n$ *(resp. the backward Euler method* $\bar{E}_n$*).*

where $\bar{\eta}_n \approx \|X_{T_n}\|$. Replacing $W^k_{T_{n+1}} - W^k_{T_n}$ by $\sqrt{\Delta}\xi^k_n$ in the explicit solution of (1.14) we get

$$\bar{\eta}_{n+1} = \bar{\eta}_n \exp\left(\left(\langle \bar{X}_n, B\bar{X}_n\rangle + \frac{1}{2}\sum_{k=1}^m \left\|\sigma^k \bar{X}_n\right\|^2 - \sum_{k=1}^m \langle \bar{X}_n, \sigma^k\bar{X}_n\rangle^2\right)\Delta + \sum_{k=1}^m \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle \sqrt{\Delta}\xi^k_n\right), \quad (1.15)$$

and so $\bar{\eta}_{n+1} \approx \left\|X_{T_{n+1}}\right\|$. Iterating (1.13) and (1.15) gives the recursive scheme $\bar{\eta}_n\bar{X}_n$.

In Section 3, we prove that $\bar{\eta}_n\bar{X}_n$ approximates X_{T_n} with weak rate of convergence equal to 1, and we establish that for any $\Delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n\Delta} \ln\left\|\bar{\eta}_n\bar{X}_n\right\| \leq -\lambda \qquad \mathbb{P}-a.s.$$

whenever (1.2) satisfies the condition (1.5). Moreover, Section 3 develops versions of $\bar{\eta}_n\bar{X}_n$ adapted to systems where B is very ill-conditioned.

We will illustrate the numerical behavior of $\bar{\eta}_n\bar{X}_n$ by means of the test equation

$$X_t = X_0 + \int_0^t \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} X_s ds + \int_0^t \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} X_s dW^1_s + \int_0^t \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X_s dW^2_s, \quad (1.16)$$

which has been proposed in [8, 11]. We take $X_0 = (1,2)^\top$, $b=-2$, $\sigma=4$ and $\epsilon=4$, and so $\lambda=-2$. Hence we are dealing with an exponentially stable SDE. To avoid variance problems, we calcule $\mathbb{E}\arctan\left(1+\left(X_t^1\right)^2\right)$, whose reference values (solid lines) have been obtained by sampling 10^8 times the explicit solution of (1.16). Here $X_t = \left(X_t^1, X_t^2\right)^\top \in \mathbb{R}^2$. Figure 1 compares the computation of $\mathbb{E}\arctan\left(1+\left(X_t^1\right)^2\right)$ by using $\bar{\eta}_n \bar{X}_n$ (represented by circles) with that produced by the backward Euler method $\bar{E}_n$ defined by (1.6) with $W^k_{(n+1)\Delta} - W^k_{n\Delta}$ replaced by $\sqrt{\Delta}\xi_n^k$ (represented by stars); all the sample sizes are equal to 10^6. Figure 1 shows the very good qualitative behavior of $\bar{\eta}_n \bar{X}_n$ in the numerical solution of (1.16). We can observe that the first coordinate of $\bar{\eta}_n \bar{X}_n$ decays to 0 with the same speed that the true solution, even for large step-sizes. In contrast, the trajectories of $\bar{E}_n$ blows up when $\Delta = 0.1$ and $\Delta = 0.02$. Moreover, $\bar{\eta}_n \bar{X}_n$ achieves an excellent accuracy in cases $\Delta = 1$ and $\Delta = 0.1$.

2. Stable schemes for scalar SDEs. In this section, we restrict our attention to stiff scalar SDEs, that is, we focus on

$$X_t = X_0 + \int_0^t b\left(X_s\right) ds + \sum_{k=1}^m \int_0^t \sigma^k\left(X_s\right) dW_s^k,$$

where $b, \sigma^k : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions.

2.1. Derivation of the numerical method. We begin by assuming $b(0) = \sigma^1(0) = \cdots = \sigma^m(0) = 0$. Then

$$X_t = X_0 + \int_0^t \frac{b\left(X_s\right)}{X_s} X_s ds + \sum_{k=1}^m \int_0^t \frac{\sigma^k\left(X_s\right)}{X_s} X_s dW_s^k,$$

where by abuse of notation, we write $b(0)/0$ and $\sigma^k(0)/0$ instead of the derivatives at 0 of b and σ^k respectively.

Suppose that $\bar{X}_n$ is an $\mathfrak{F}_{T_n}$-measurable random variable such that $\bar{X}_n \approx X_{T_n}$, here and subsequently, $T_j = j\Delta$ for all $j = 0, 1, \ldots$ Then, for all $t \in [T_n, T_{n+1}]$,

$$X_t \approx \bar{X}_n + \int_{T_n}^t \frac{b\left(X_s\right)}{X_s} X_s ds + \sum_{k=1}^m \int_{T_n}^t \frac{\sigma^k\left(X_s\right)}{X_s} X_s dW_s^k. \tag{2.1}$$

Since $x \mapsto b(x)/x$ and $x \mapsto \sigma^k(x)/x$ are continuous functions, (2.1) leads to

$$X_t \approx \bar{X}_n + \int_{T_n}^t \frac{b\left(\bar{X}_n\right)}{\bar{X}_n} X_s ds + \sum_{k=1}^m \int_{T_n}^t \frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n} X_s dW_s^k.$$

Hence X_t is approximated by the solution of the linear scalar SDE

$$Y_t = \bar{X}_n + \int_{T_n}^t \frac{b\left(\bar{X}_n\right)}{\bar{X}_n} Y_s ds + \sum_{k=1}^m \int_{T_n}^t \frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n} Y_s dW_s^k, \tag{2.2}$$

and so $X_{T_{n+1}} \approx Y_{T_{n+1}}$. Solving explicitly (2.2) gives

$$X_{T_{n+1}} \approx \bar{X}_n \exp\left(\left(\frac{b\left(\bar{X}_n\right)}{\bar{X}_n} - \frac{1}{2}\sum_{k=1}^m \left(\frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n}\right)^2\right)\Delta + \sum_{k=1}^m \frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n}\left(W^k_{T_{n+1}} - W^k_{T_n}\right)\right). \tag{2.3}$$

As we are interested in weak approximations of X_t, we replace $W^k_{T_{n+1}} - W^k_{T_n}$ in (2.3) by $\sqrt{\Delta}\hat{W}^k_n$, where $\hat{W}^1_0, \hat{W}^2_0, \ldots, \hat{W}^m_0, \hat{W}^1_1, \ldots$ are independent identically distributed (i.i.d.) random variables with symmetric law and variance 1. This gives the numerical scheme

$$\bar{X}_{n+1} = \bar{X}_n \exp\left(\left(\frac{b\left(\bar{X}_n\right)}{\bar{X}_n} - \frac{1}{2}\sum_{k=1}^{m}\left(\frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n}\right)^2\right)\Delta + \sum_{k=1}^{m}\frac{\sigma^k\left(\bar{X}_n\right)}{\bar{X}_n}\sqrt{\Delta}\hat{W}^k_n\right). \tag{2.4}$$

From (2.4) it follows that $\bar{X}_n$ preserves the sign of the initial data. Furthermore, we next establish that $\bar{X}_n$ is almost sure exponential stable for any $\Delta > 0$ provided that $\sigma^1, \ldots, \sigma^m$ are globally Lipschitz and

$$-\lambda := \sup_{x\in\mathbb{R}, x\neq 0}\left(b\left(x\right)/x - \sum_{k=1}^{m}\left(\sigma^k\left(x\right)/x\right)^2/2\right) < 0.$$

Hence $\bar{X}_n$ preserves two important dynamical properties of the solution of (1.1).

Theorem 2.1. *Consider $\left(\bar{X}_n\right)_{n\geq 0}$ given by the recursive formula (2.4). Suppose that (1.4) and (1.5) hold, together with $b\left(0\right) = 0$ and $\mathbb{E}\left(\bar{X}_0\right)^2 < \infty$. Then*

$$\limsup_{n\to\infty}\frac{1}{n\Delta}\ln\left|\bar{X}_n\right| \leq -\lambda \qquad \mathbb{P} - a.s. \tag{2.5}$$

Proof. Deferred to Section 2.3.1. ∎

The high performance achieved by $\bar{X}_n$ in our numerical experiments, together with its good theoretical properties, motivates us to adapt $\bar{X}_n$ to the framework where $b\left(0\right), \sigma^1\left(0\right), \ldots, \sigma^m\left(0\right)$ are not necessarily equal to 0. To this end, we rewrite (1.1) in the form

$$\begin{aligned} X_t = X_0 &+ \int_0^t\left(\frac{b\left(X_s\right) - b\left(0\right)}{X_s}X_s + b\left(0\right)\right)ds \\ &+ \sum_{k=1}^{m}\int_0^t\left(\frac{\sigma^k\left(X_s\right) - \sigma^k\left(0\right)}{X_s}X_s + \sigma^k\left(0\right)\right)dW^k_s, \end{aligned}$$

and so for all $t \in [T_n, T_{n+1}]$,

$$X_t \approx \bar{X}_n + \int_{T_n}^t\left(\mu\left(\bar{X}_n\right)X_s + b\left(0\right)\right)ds + \sum_{k=1}^{m}\int_{T_n}^t\left(\lambda^k\left(\bar{X}_n\right)X_s + \sigma^k\left(0\right)\right)dW^k_s,$$

where $\bar{X}_n$ is an $\mathfrak{F}_{T_n}$-measurable random variable approximating X_{T_n},

$$\mu\left(x\right) = \begin{cases} \frac{b(x)-b(0)}{x}, & \text{if } x\neq 0 \\ b'\left(0\right), & \text{if } x = 0 \end{cases} \quad \text{and } \lambda^k\left(x\right) = \begin{cases} \left(\sigma^k\left(x\right) - \sigma^k\left(0\right)\right)/x, & \text{if } x\neq 0 \\ \left(\sigma^k\right)'\left(0\right), & \text{if } x = 0 \end{cases}.$$

This leads to approximate X_t by the solution of

$$Y_t = \bar{X}_n + \int_{T_n}^t\left(\mu\left(\bar{X}_n\right)Y_s + b\left(0\right)\right)ds + \sum_{k=1}^{m}\int_{T_n}^t\left(\lambda^k\left(\bar{X}_n\right)Y_s + \sigma^k\left(0\right)\right)dW^k_s. \tag{2.6}$$

The explicit solution of (2.6) is

$$Y_t = \Phi_t \left(\bar{X}_n + \left(b\left(0\right) - \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right) \sigma^k \left(0\right) \right) \int_{T_n}^{t} \Phi_s^{-1} ds + \sum_{k=1}^{m} \sigma^k \left(0\right) \int_{T_n}^{t} \Phi_s^{-1} dW_s^k \right),$$

where

$$\Phi_t = \exp \left(\left(\mu\left(\bar{X}_n\right) - \frac{1}{2} \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right)^2 \right) \left(t - T_n\right) + \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right) \left(W_t^k - W_{T_n}^k\right) \right).$$

Since $\Phi_s^{-1} \approx \Phi_{T_n}^{-1}$ for all $s \in [T_n, T_{n+1}]$, $X_{T_{n+1}}$ is approximated by

$$\Phi_{T_{n+1}} \left(\bar{X}_n + \left(b\left(0\right) - \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right) \sigma^k \left(0\right) \right) \Delta + \sum_{k=1}^{m} \sigma^k \left(0\right) \left(W_{T_{n+1}}^k - W_{T_n}^k \right) \right).$$

This yields the weak scheme

$$\bar{X}_{n+1} = \bar{\Phi}_{n+1} \left(\bar{X}_n + \left(b\left(0\right) - \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right) \sigma^k \left(0\right) \right) \Delta + \sum_{k=1}^{m} \sigma^k \left(0\right) \sqrt{\Delta} \hat{W}_n^k \right), \quad (2.7)$$

where $\hat{W}_0^1, \hat{W}_0^2, \ldots, \hat{W}_0^m, \hat{W}_1^1, \ldots$ are i.i.d. random variables with symmetric law and variance 1, and

$$\bar{\Phi}_{n+1} = \exp \left(\left(\mu\left(\bar{X}_n\right) - \frac{1}{2} \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right)^2 \right) \Delta + \sum_{k=1}^{m} \lambda^k \left(\bar{X}_n\right) \sqrt{\Delta} \hat{W}_n^k \right).$$

The following theorem establishes that $\bar{X}$, given by (2.7), converges weakly to X with order $O\left(\Delta\right)$ under a basic set of assumptions. In case $b\left(0\right) = \sigma^1\left(0\right) = \ldots = \sigma^m\left(0\right) = 0$, (2.7) becomes (2.4), and so the rate of weak convergence of $\bar{X}$, defined by (2.4), is equal to 1.

Notation 2.1. *We will use the same symbol $K\left(\cdot\right)$ (resp. K) for different positive increasing functions (resp. positive real numbers) having the common property to be independent of Δ. Similarly, q denotes generic constants greater than or equal to 2. We write $\mathcal{C}_p^\ell\left(\mathbb{R}^d, \mathbb{R}\right)$ for the set of all ℓ-times continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that f and all its partial derivatives of orders $1, 2, \ldots, \ell$ have at most polynomial growth.*

Theorem 2.2. *Let $b, \sigma^1, \ldots, \sigma^m$ be Lipschitz continuous functions belonging to $C_p^4(\mathbb{R}, \mathbb{R})$ such that $\left|b\left(x\right)\right| + \left|\sigma^1\left(x\right)\right| + \cdots + \left|\sigma^m\left(x\right)\right| \leq K\left(1 + \left|x\right|\right)$ for all $x \in \mathbb{R}$. Fix $T > 0$ and $f \in C_p^4(\mathbb{R}, \mathbb{R})$. Consider the scheme $\bar{X}$ described by (2.7) with $\Delta = T/N$, where $N \in \mathbb{N}$. Assume that $\mathbb{E} \exp\left(r \hat{W}_n^k\right) < \infty$ for all $r > 0$, X_0 has finite moments of any order, and that for every $g \in C_p^4(\mathbb{R}, \mathbb{R})$,*

$$\left|\mathbb{E}g(\bar{X}_0) - \mathbb{E}g(X_0)\right| \leq K\left(1 + \mathbb{E}\left|X_0\right|^q\right) T/N \qquad \forall N \in \mathbb{N}.$$

Then for all $N \in \mathbb{N}$,

$$\left|\mathbb{E}f\left(X_T\right) - \mathbb{E}f\left(\bar{X}_N\right)\right| \leq K\left(T\right)\left(1 + \mathbb{E}\left|X_0\right|^q\right) T/N. \quad (2.8)$$

Proof. Deferred to Section 2.3.2. □

REMARK 2.1. *If $\hat{W}_n^k$ are standard Normal random variables, then*

$$\mathbb{E}\exp\left(r\hat{W}_n^k\right) = \exp\left(r^2/2\right).$$

In case $\hat{W}_n^k$ are bounded random variables we also have $\mathbb{E}\exp\left(r\hat{W}_n^k\right) < \infty$.

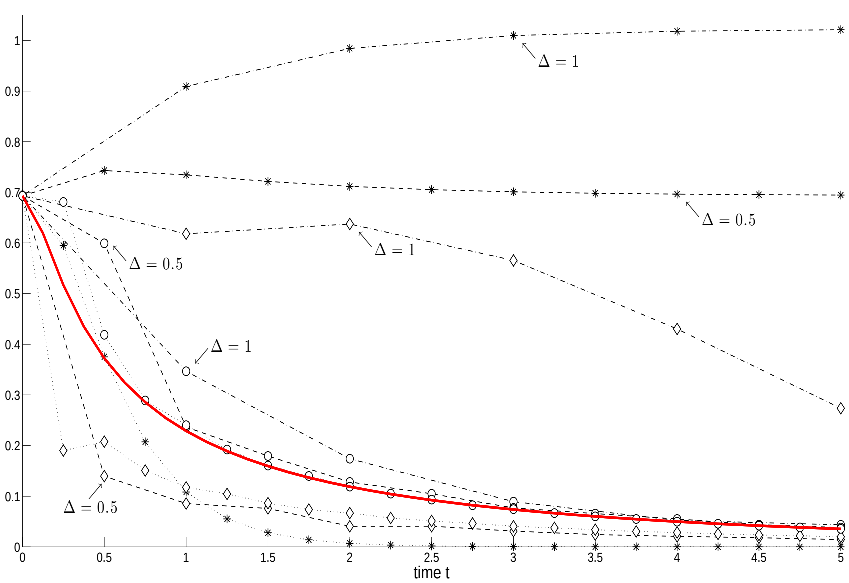
2.2. Numerical experiment. In this subsection, we illustrate the behavior of $\bar{X}_n$ by means of the locally Lipschitz SDE

$$X_t = X_0 + \int_0^t \left(aX_s - b\left(X_s\right)^3\right) ds + \int_0^t \sigma X_s dW_s^1, \tag{2.9}$$

where b, σ are real positive numbers, $a \in \mathbb{R}$, and $X_0 = 1$. This scalar cubic SDE is known as the stochastic Ginzburg-Landau equation, and constitutes a classical test equation in the theory of stochastic bifurcation (see, e.g., [4, 7]). Let $\xi_0^1, \xi_0^2, \ldots, \xi_0^m$, $\xi_1^1, \ldots$ be independent random variables taking the values ± 1 each with probability $1/2$. Then, we will solve numerically (2.9) using three schemes: $\bar{X}_n$ defined by (2.4) with $\hat{W}_n^k$ replaced by ξ_n^k, the backward Euler method $\bar{E}_n$ given by (1.6) with $\sqrt{\Delta}\xi_n^k$ in place of $W_{(n+1)\Delta}^k - W_{n\Delta}^k$, and

$$\begin{cases} \tilde{Z}_{n+1/2}^s = \tilde{Z}_n^s \exp\left((a-\sigma^2/2)\Delta + \sigma\sqrt{\Delta}\xi_n^1\right) \\ \tilde{Z}_{n+1}^s = \tilde{Z}_{n+1/2}^s \left(1 - b\Delta\left(\tilde{Z}_{n+1/2}^s\right)^2/2\right) / \left(1 + b\Delta\left(\tilde{Z}_{n+1/2}^s\right)^2/2\right) \end{cases}.$$

FIG. 2. *Computation of $\mathbb{E}\ln\left(1+(X_t)^2\right)$, where $t \in [0,5]$ and X_t satisfies (2.9) with $a = b = 1$ and $\sigma = 2$. The "true" values are plotted with a solid line. The circles, stars and diamonds stand for the schemes $\bar{X}_n$, $\bar{E}_n$ and $\tilde{Z}^s$ respectively. The step sizes 1, 0.5 and 0.25 are represented by dashdot, dashed and dotted lines respectively.*



It is worth pointing out that $\bar{E}_n$ entails the solution of a nonlinear equation at each step, and that $\tilde{Z}_n^s$ is a weak version of the splitting-step algorithm for (2.9) introduced by Subsection 4.2 of [25].

Figures 2, 3 and Table 2.1 show features of the computation of $\mathbb{E}\ln\left(1+(X_t)^2\right)$ obtained from the sample means of 10^8 observations of $\bar{X}_n$, $\bar{E}_n$ and $\tilde{Z}_n^s$. The solid line identifies the "true" values gotten by sampling 10^8 times $\bar{E}_n$ with $\Delta = 2^{-11} \approx 0,000488$. The lengths of all the 99% confidence intervals are at least of order 10^{-3}, they have been estimated following [18].

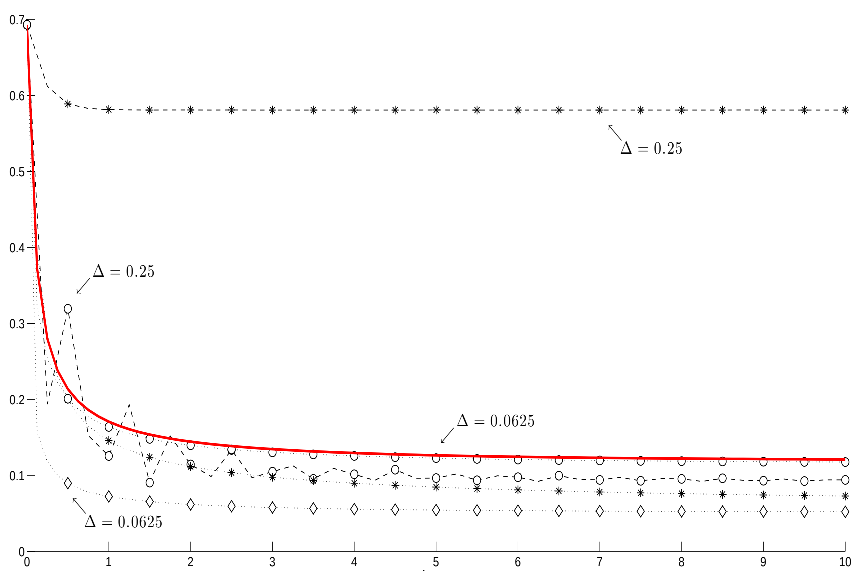
First, we take $a = b = 1$ and $\sigma = 2$, which is the motivating example of [17]. Since $b(x)/x - \left(\sigma^1(x)/x\right)^2/2 \leq -1$,

$$\limsup_{t\to\infty} \frac{1}{t}\ln|X_t| \leq -1 \qquad a.s.$$

From [17] we have that the Euler-Maruyama scheme applied to (2.9) blows up, with positive probability, at a geometric rate. In numerical experiments, we saw that $\bar{E}_n$ and $\tilde{Z}_n^s$ fail to preserve the sign of X_0 for $\Delta = 1$ and $\Delta = 0.5$. To this end, we computed $10\mathbb{E}\left(\pi/2 - \arctan\left(10^3 X_t + 10^2\right)\right)$.

Figure 2 presents the numerical solution of (2.9), with $a = b = 1$ and $\sigma = 2$, using the step sizes $\Delta = 0.25, 0.5, 1$. Figure 2 suggests us that $\bar{X}_n$ replicate very well the long time behavior of X_t, even for $\Delta = 1$. Moreover, we can see that the accuracy of $\bar{X}_n$ is very good, even for large step sizes; $\bar{X}_n$ achieves significantly lower errors than $\tilde{Z}^s$, which is a method adapted to the characteristics of (2.9).

Fig. 3. *Computation of* $\mathbb{E}\ln\left(1+(X_t)^2\right)$*, where* $t \in [0,10]$ *and* X_t *solves* (2.9) *with* $a = 6$, $b = 9$ *and* $\sigma = 3$. *The "true" values are plotted with a solid line. The circles, stars and diamonds represent the schemes* $\bar{X}_n$, $\bar{E}_n$ *and* $\tilde{Z}^s$ *respectively. The step sizes* 0.25 *and* 0.0625 *are denoted by dashed and dotted lines respectively.*



Δ	$\left\lvert \mathbb{E}f\left(X_T\right) - \mathbb{E}f\left(\bar{E}_{T/\Delta}\right)\right\rvert$	$\left\lvert \mathbb{E}f\left(X_T\right) - \mathbb{E}f\left(\tilde{Z}^s_{T/\Delta}\right)\right\rvert$	$\left\lvert \mathbb{E}f\left(X_T\right) - \mathbb{E}f\left(\bar{X}_{T/\Delta}\right)\right\rvert$
1	0.51106	29.0077	0.074617
1/2	0.51504	26.1167	0.05854
1/4	0.45996	21.5922	0.026918
1/8	0.0089311	4.1708	0.0078809
2^{-4}	0.048116	0.068864	0.003364
2^{-5}	0.015164	0.041187	0.0029212
2^{-6}	0.0056366	0.022123	0.0017523
2^{-7}	0.0023573	0.011367	0.00095535

Table 2.1

Estimation of errors involved in the computation of $\mathbb{E}f\left(X_T\right)$ for $T = 10$ and $f\left(x\right) = \ln\left(1 + x^2\right)$. Here, X_t verifies (2.9) with $a = 6$, $b = 9$, $\sigma = 3$ and $X_0 = 1$.

Second, we choose $a = 6$, $b = 9$ and $\sigma = 3$. Then

$$-\lambda = \sup_{x\in\mathbb{R},x\neq 0} \left(a - bx^2 - \sigma^2/2\right) > 0,$$

and so Condition (1.5) does not hold. In this case, (2.9) has three invariant forward Markov measures (see, e.g., [4]). Calculating $10\mathbb{E}\left(\pi/2 - \arctan\left(10^3 X_t + 10^2\right)\right)$ we observe that $\bar{E}_n$ (resp. $\tilde{Z}^s_n$) can take negative values when $\Delta \geq 1/8$ (resp. $\Delta \geq 1/16$).

Figure 3 displays the numerical approximation of $\mathbb{E}\ln\left(1 + \left(X_t\right)^2\right)$ by means of $\bar{X}_n$ and $\bar{E}_n$ with step sizes $\Delta = 0.25$ and $\Delta = 0.0625$, as well as by using $\tilde{Z}^s_n$ with $\Delta = 0.0625$. Moreover, Table 2.1 shows errors made in the weak numerical integration of (2.9), where the reference value of $\mathbb{E}\ln\left(1 + \left(X_{10}\right)^2\right)$ was obtained by sampling 10^8 times $\bar{E}_n$ with $\Delta = 2^{-11}$. In this test problem, the new scheme $\bar{X}_n$ again provides very good approximations of $\mathbb{E}f\left(X_t\right)$, even for large values of Δ.

2.3. Proofs.

2.3.1. Proof of Theorem 2.1. From (2.4) it follows that

$$\ln\left\lvert\bar{X}_{n+1}\right\rvert = \ln\left\lvert\bar{X}_0\right\rvert + \sum_{j=0}^{n}\left(\frac{b\left(\bar{X}_j\right)}{\bar{X}_j} - \frac{1}{2}\sum_{k=1}^{m}\left(\frac{\sigma^k\left(\bar{X}_j\right)}{\bar{X}_j}\right)^2\right)\Delta + S_n,$$

with $S_n = \sum_{j=0}^{n}\sum_{k=1}^{m}\sigma^k\left(\bar{X}_j\right)/\bar{X}_j\ \sqrt{\Delta}\hat{W}^k_j$. By (1.5), we thus get

$$\frac{1}{n+1}\ln\left\lvert\bar{X}_{n+1}\right\rvert \leq \frac{1}{n+1}\ln\left\lvert\bar{X}_0\right\rvert - \lambda\Delta + \frac{1}{n+1}S_n. \tag{2.10}$$

Using (1.4) gives $\mathbb{E}\left(\sum_{k=1}^{m}\sigma^k\left(\bar{X}_j\right)/\bar{X}_j\ \sqrt{\Delta}\hat{W}^k_j\right)^2 \leq K\Delta$, and so

$$\sum_{j=0}^{\infty}\frac{1}{\left(j+1\right)^2}\mathbb{E}\left(\sum_{k=1}^{m}\frac{\sigma^k\left(\bar{X}_j\right)}{\bar{X}_j}\sqrt{\Delta}\hat{W}^k_j\right)^2 < \infty.$$

Since

$$\mathbb{E}\left(\sum_{k=1}^{m}\frac{\sigma^k\left(\bar{X}_j\right)}{\bar{X}_j}\sqrt{\Delta}\hat{W}^k_j \diagup \sigma\left(\bar{X}_0, \hat{W}^1_0, \ldots, \hat{W}^m_0, \ldots, \hat{W}^1_{j-1}, \ldots, \hat{W}^m_{j-1}\right)\right) = 0,$$

applying a generalized law of large numbers we deduce that $S_n/(n+1) \to 0$ a.s. (see, e.g., p. 243 of [14]). Then, letting $n \to \infty$ in (2.10) we obtain (2.5). □

2.3.2. Proof of Theorem 2.2. To shorten notation, for any $n \geq 0$ we set

$$f_n := \left(\mu\left(\bar{X}_n\right) - \frac{1}{2}\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)^2\right)\Delta + \sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\sqrt{\Delta}\hat{W}_n^k$$

and $g_n := \left(b(0) - \sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\sigma^k(0)\right)\Delta + \sum_{k=1}^m \sigma^k(0)\sqrt{\Delta}\hat{W}_n^k$. Then

$$\bar{X}_{n+1} = \exp(f_n)\left(\bar{X}_n + g_n\right) = \bar{X}_n + \left(e^{f_n} - 1 - f_n\right)\bar{X}_n + \left(e^{f_n} - 1\right)g_n + f_n\bar{X}_n + g_n,$$

and hence

$$\bar{X}_{n+1} = \bar{X}_0 + \sum_{k=0}^{n}\left(e^{f_k} - 1 - f_k\right)\bar{X}_k + \sum_{k=0}^{n}\left(e^{f_k} - 1\right)g_k + \sum_{k=0}^{n}\left(f_k\bar{X}_k + g_k\right).$$

Let $q \geq 2$. By

$$\left|\exp(x) - \sum_{j=0}^{k-1} x^j/(j!)\right| \leq |x|^k \exp(|x|),$$

using Hölder's inequality yields

$$\begin{aligned}
\left|\bar{X}_{n+1}\right|^q \leq{}& K\left|\bar{X}_0\right|^q + K(n+1)^{q-1}\left(\sum_{k=0}^{n}|f_k|^{2q}e^{q|f_k|}\left|\bar{X}_k\right|^q + \sum_{k=0}^{n}|f_k|^q|g_k|^q e^{q|f_k|}\right)\\
&+K(n+1)^{q-1}\sum_{k=0}^{n}\Delta^q\left|\mu\left(\bar{X}_k\right) - \frac{1}{2}\sum_{j=1}^{m}\lambda^j\left(\bar{X}_k\right)^2\right|^q\left|\bar{X}_k\right|^q\\
&+K(n+1)^{q-1}\sum_{k=0}^{n}\Delta^q\left|b(0) - \sum_{j=1}^{m}\lambda^j\left(\bar{X}_k\right)\sigma^j(0)\right|^q\\
&+K\Delta^{q/2}\sum_{j=1}^{m}\left|\sum_{k=0}^{n}\left(\lambda^j\left(\bar{X}_k\right)\bar{X}_k + \sigma^j(0)\right)\hat{W}_k^j\right|^q.
\end{aligned} \tag{2.11}$$

For any $t > 0$,

$$\mathbb{E}\exp\left(t\left|\hat{W}_k^j\right|\right) \leq \mathbb{E}\exp\left(t\hat{W}_k^j\right) + \mathbb{E}\exp\left(-t\hat{W}_k^j\right) < \infty, \tag{2.12}$$

and so for all $\ell \in \mathbb{N}$,

$$\mathbb{E}\left(\left|\hat{W}_k^j\right|^\ell\right) < \ell!\,\mathbb{E}\exp\left(\left|\hat{W}_k^j\right|\right) < \infty. \tag{2.13}$$

Since μ and λ^k are bounded functions, we use (2.12), (2.13) and the Burkholder-Davis-Gundy inequality to obtain from (2.11) that

$$\mathbb{E}\left|\bar{X}_{n+1}\right|^q \leq K\mathbb{E}\left|\bar{X}_0\right|^q + K(T) + K(T)\,\Delta\sum_{k=0}^{n}\mathbb{E}\left|\bar{X}_k\right|^q,$$

with $n = 0, \ldots, N-1$. Applying a discrete Gronwall lemma (see, e.g., [4]) we deduce that for all $n = 0, \ldots, N$,

$$\mathbb{E}\left|\bar{X}_n\right|^q \leq K(T)\left(1+\mathbb{E}\left|\bar{X}_0\right|^q\right). \tag{2.14}$$

Consider again $q \geq 2$. Using

$$\left|\bar{X}_{n+1}-\bar{X}_n\right| \leq \left|e^{f_n}-1\right|\left|\bar{X}_n\right| + e^{f_n}\left|g_n\right| \leq \left|f_n\right|\exp\left(\left|f_n\right|\right)\left|\bar{X}_n\right| + \exp\left(f_n\right)\left|g_n\right|,$$

together with (2.12) and (2.13), we get

$$\mathbb{E}\left(\left|\bar{X}_{n+1}-\bar{X}_n\right|^q \diagup \mathfrak{F}_{T_n}\right) \leq K(T)\,\Delta^{q/2}\left(1+\left|\bar{X}_n\right|^q\right). \tag{2.15}$$

Here, we assume without loss of generality that $\hat{W}_n^1, \ldots \hat{W}_n^m$ are $\mathfrak{F}_{T_{n+1}}$-measurable and independent of $\mathfrak{F}_{T_n}$.

Since $\left|\bar{X}_{n+1} - \left(1+f_n+f_n^2/2+f_n^3/6\right)\left(\bar{X}_n+g_n\right)\right| \leq \left|f_n\right|^4\exp\left(\left|f_n\right|\right)\left|\bar{X}_n+g_n\right|$,

$$\begin{aligned}
\bar{X}_{n+1} = {} & \bar{X}_n + \sqrt{\Delta}\sum_{k=1}^m \left(\lambda^k\left(\bar{X}_n\right)\bar{X}_n + \sigma^k(0)\right)\hat{W}_n^k \\
& +\Delta\bar{X}_n\left(\mu\left(\bar{X}_n\right) - \frac{1}{2}\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)^2 + \frac{1}{2}\left(\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^2\right) \\
& +\Delta\left(b(0) - \sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\sigma^k(0) + \sum_{j,k=1}^m \sigma^j(0)\,\lambda^k\left(\bar{X}_n\right)\hat{W}_n^j\hat{W}_n^k\right) \\
& +\Delta^{3/2}\bar{X}_n\left(\mu\left(\bar{X}_n\right) - \frac{1}{2}\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)^2\right)\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\hat{W}_n^k \\
& +\Delta^{3/2}\bar{X}_n\left(\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^3/6 \\
& +\Delta^{3/2}\left(\mu\left(\bar{X}_n\right) - \frac{1}{2}\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)^2 + \frac{1}{2}\left(\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^2\right)\sum_{k=1}^m \sigma^k(0)\,\hat{W}_n^k \\
& +\Delta^{3/2}\left(b(0) - \sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\sigma^k(0)\right)\sum_{k=1}^m \lambda^k\left(\bar{X}_n\right)\hat{W}_n^k + \mathcal{R}_n\left(\Delta,\bar{X}_n\right),
\end{aligned}$$

where $\left|\mathcal{R}_n\left(\Delta,\bar{X}_n\right)\right| \leq \left|f_n\right|^4\exp\left(\left|f_n\right|\right)\left|\bar{X}_n+g_n\right| + K(T)\,\Delta^2\left(1+\left|\bar{X}_n\right|\right)$. This gives

$$\begin{aligned}
& \left|\mathbb{E}\left(\left(\bar{X}_{n+1}-\bar{X}_n\right)^\ell - \left(b\left(\bar{X}_n\right)\Delta + \sum_{k=1}^m \sigma^k\left(\bar{X}_n\right)\left(W_{\Delta(n+1)}^k - W_{\Delta n}^k\right)\right)^\ell \diagup \mathfrak{F}_{T_n}\right)\right| \\
& \leq K(T)\,\Delta^2\left(1+\left|\bar{X}_n\right|^q\right)
\end{aligned} \tag{2.16}$$

provided that $\ell = 1, 2, 3$.

From (2.14), (2.15) and (2.16) we obtain (2.8). To this end, we can apply the classical methodology introduced by Milstein [20] and Talay [32, 33], or we can directly use Theorem 9.1 of [21] (see also Theorem 14.5.2 of [18]). □

3. Stable schemes for systems of SDEs.

3.1. General methodology. Since the coefficients of (1.1) are locally Lipschitz functions, (1.1) has a unique continuous strong solution up to an explosion time (see, e.g., [29]), which we assume to be $+\infty$ a.s. Let $b(0) = \sigma^1(0) = \cdots = \sigma^m(0) = 0$. Then, without loss of generality we can suppose $X_0 \neq 0$ a.s. Hence, almost surely, $X_t \neq 0$ for all $t > 0$.

As we pointed out in Subsection 1.2, we divide the numerical approximation of X_t into the computations of $\|X_t\|$ and $\hat{X}_t := X_t / \|X_t\|$. Since, almost surely, X_t will never reach the origin,

$$\tau_j := \inf\{t > 0 : \|X_t\| < 1/j\} \longrightarrow_{j\to\infty} \infty. \tag{3.1}$$

Applying Itô's formula to $\sqrt{\left\|X_{t\wedge\tau_j}\right\|^2}$ we obtain

$$\begin{aligned}\left\|X_{t\wedge\tau_j}\right\| &= \|X_0\| + \sum_{k=1}^m \int_0^{t\wedge\tau_j} \frac{\langle X_s, \sigma^k(X_s)\rangle}{\|X_s\|} dW_s^k \\ &+ \int_0^{t\wedge\tau_j} \left(\frac{\langle X_s, b(X_s)\rangle + \frac{1}{2}\sum_{k=1}^m \left\|\sigma^k(X_s)\right\|^2}{\|X_s\|} - \frac{1}{2}\sum_{k=1}^m \frac{\langle X_s, \sigma^k(X_s)\rangle^2}{\|X_s\|^3} \right) ds,\end{aligned}$$

and so taking limit as $j \to \infty$ gives

$$\begin{aligned}\|X_t\| &= \|X_0\| + \sum_{k=1}^m \int_0^t \frac{\langle X_s, \sigma^k(X_s)\rangle}{\|X_s\|} dW_s^k \\ &+ \int_0^t \left(\frac{\langle X_s, b(X_s)\rangle + \frac{1}{2}\sum_{k=1}^m \left\|\sigma^k(X_s)\right\|^2}{\|X_s\|} - \frac{1}{2}\sum_{k=1}^m \frac{\langle X_s, \sigma^k(X_s)\rangle^2}{\|X_s\|^3} \right) ds.\end{aligned} \tag{3.2}$$

Moreover, using Itô's formula, together with τ_j, we obtain after a long calculation that

$$\begin{aligned}\frac{X_t}{\|X_t\|} &= \frac{X_0}{\|X_0\|} + \int_0^t \left(\frac{b(X_s)}{\|X_s\|} - \left\langle \frac{X_s}{\|X_s\|}, \frac{b(X_s)}{\|X_s\|} \right\rangle \frac{X_s}{\|X_s\|} \right) ds \\ &+ \frac{1}{2}\sum_{k=1}^m \int_0^t \left(3\left\langle \frac{X_s}{\|X_s\|}, \frac{\sigma^k(X_s)}{\|X_s\|} \right\rangle^2 - \left\langle \frac{\sigma^k(X_s)}{\|X_s\|}, \frac{\sigma^k(X_s)}{\|X_s\|} \right\rangle \right) \frac{X_s}{\|X_s\|} ds \\ &- \sum_{k=1}^m \int_0^t \left\langle \frac{X_s}{\|X_s\|}, \frac{\sigma^k(X_s)}{\|X_s\|} \right\rangle \frac{\sigma^k(X_s)}{\|X_s\|} ds \\ &+ \sum_{k=1}^m \int_0^t \left(\frac{\sigma^k(X_s)}{\|X_s\|} - \left\langle \frac{X_s}{\|X_s\|}, \frac{\sigma^k(X_s)}{\|X_s\|} \right\rangle \frac{X_s}{\|X_s\|} \right) dW_s^k.\end{aligned} \tag{3.3}$$

There are at least two general ways of computing $\hat{X}_{T_{n+1}} := X_{T_{n+1}} / \left\|X_{T_{n+1}}\right\|$. For example, we can approximate $X_{T_{n+1}}$ by $\tilde{Z}_{n+1}$, the numerical solution at time T_{n+1} of

$$Z_t = \bar{\eta}_n \bar{X}_n + \int_{T_n}^t b(Z_s)\, ds + \sum_{k=1}^m \int_{T_n}^t \sigma^k(Z_s)\, dW_s^k,$$

where $\bar{\eta}_n$ and $\bar{X}_n$ are $\mathfrak{F}_{T_n}$-measurable random variables such that $\bar{\eta}_n \approx \|X_{T_n}\|$ and $\bar{X}_n \approx \hat{X}_{T_n} := X_{T_n} / \|X_{T_n}\|$; as in Sections 1 and 2, for simplicity, we take $T_n = n\Delta$ for all $n = 0, 1, \ldots$ Then, taking care of round-off errors we set $\bar{X}_{n+1} := \tilde{Z}_{n+1} / \left\| \tilde{Z}_{n+1} \right\|$, and, in consequence, $\bar{X}_{n+1} \approx \hat{X}_{T_{n+1}}$. Alternatively, we can solve numerically (3.3), which is the method used in Subsections 1.2 and 3.2. In the non-linear case, we can, for instance, approximate the right hand side of (3.3) in each time interval $[T_n, T_{n+1}]$.

Now, we provide a manner of handling $\|X_t\|$. The good performance of the numerical method developed in Section 2 motivates us to rewrite (3.2) in the form

$$\begin{aligned} \|X_t\| = \|X_0\| &+ \sum_{k=1}^{m} \int_0^t \frac{\langle X_s, \sigma^k(X_s) \rangle}{\|X_s\|^2} \|X_s\| \, dW_s^k \\ &+ \int_0^t \left(\frac{\langle X_s, b(X_s) \rangle + \frac{1}{2} \sum_{k=1}^m \left\| \sigma^k(X_s) \right\|^2}{\|X_s\|^2} - \frac{1}{2} \sum_{k=1}^m \frac{\langle X_s, \sigma^k(X_s) \rangle^2}{\|X_s\|^4} \right) \|X_s\| \, ds, \end{aligned} \tag{3.4}$$

and, further, to approximate $\|X_t\|$ on the time interval $[T_n, T_{n+1}]$ by the solution of the scalar SDE

$$\begin{aligned} \eta_t = \bar{\eta}_n &+ \sum_{k=1}^{m} \int_{T_n}^t \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle}{\left\| \tilde{X}_n \right\|^2} \eta_s \, dW_s^k \\ &+ \int_{T_n}^t \left(\frac{\langle \tilde{X}_n, b\left(\tilde{X}_n\right) \rangle + \frac{1}{2} \sum_{k=1}^m \left\| \sigma^k\left(\tilde{X}_n\right) \right\|^2}{\left\| \tilde{X}_n \right\|^2} - \frac{1}{2} \sum_{k=1}^m \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle^2}{\left\| \tilde{X}_n \right\|^4} \right) \eta_s \, ds, \end{aligned} \tag{3.5}$$

where $\bar{\eta}_n \approx \|X_{T_n}\|$ and $\tilde{X}_n$ is an $\mathfrak{F}_{T_n}$-measurable random variable approximating X_s on $[T_n, T_{n+1}]$. Using the explicit solution of (3.5) we find

$$\begin{aligned} \left\| X_{T_{n+1}} \right\| \approx \bar{\eta}_n \exp\Bigg(&\left(\frac{\langle \tilde{X}_n, b\left(\tilde{X}_n\right) \rangle + \frac{1}{2} \sum_{k=1}^m \left\| \sigma^k\left(\tilde{X}_n\right) \right\|^2}{\left\| \tilde{X}_n \right\|^2} - \sum_{k=1}^m \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle^2}{\left\| \tilde{X}_n \right\|^4} \right) \Delta \\ &+ \sum_{k=1}^m \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle}{\left\| \tilde{X}_n \right\|^2} \left(W_{T_{n+1}}^k - W_{T_n}^k \right) \Bigg). \end{aligned}$$

We substitute $W_{T_{n+1}}^k - W_{T_n}^k$ by $\sqrt{\Delta} \hat{W}_n^k$, the random variables $\hat{W}_0^1, \ldots, \hat{W}_0^m, \hat{W}_1^1, \ldots$ being i.i.d. with symmetric law and variance 1. This leads to $\left\| X_{T_{n+1}} \right\| \approx \bar{\eta}_{n+1}$, where $\bar{\eta}_{n+1}$ is given by the recursive formula

$$\begin{aligned} \bar{\eta}_{n+1} = \bar{\eta}_n \exp\Bigg(&\left(\frac{\langle \tilde{X}_n, b\left(\tilde{X}_n\right) \rangle + \frac{1}{2} \sum_{k=1}^m \left\| \sigma^k\left(\tilde{X}_n\right) \right\|^2}{\left\| \tilde{X}_n \right\|^2} - \sum_{k=1}^m \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle^2}{\left\| \tilde{X}_n \right\|^4} \right) \Delta \\ &+ \sum_{k=1}^m \frac{\langle \tilde{X}_n, \sigma^k\left(\tilde{X}_n\right) \rangle}{\left\| \tilde{X}_n \right\|^2} \sqrt{\Delta} \hat{W}_n^k \Bigg). \end{aligned} \tag{3.6}$$

From (3.6) we have that $\bar{\eta}_n > 0$ for all $n \in \mathbb{N}$, in case $\bar{\eta}_0 > 0$. Moreover, if (1.4) and (1.5) hold, then we next assert that $\bar{\eta}_n \bar{X}_n$ is almost sure exponential stable for all $\Delta > 0$ provided that $\bar{X}_n$ is a $\mathfrak{F}_{T_n}$-measurable random variable of norm 1.

THEOREM 3.1. *Let $\bar{\eta}_n$ be as in (3.6). Suppose that $b(0)=0$, $\bar{\eta}_0>0$, and that $\mathbb{E}(\bar{\eta}_0)^2<\infty$. Assume that the inequalities (1.4) and (1.5) hold. Then*

$$\limsup_{n\to\infty}\frac{1}{n\Delta}\ln(\bar{\eta}_n)\leq-\lambda \qquad \mathbb{P}-a.s. \tag{3.7}$$

Proof. Deferred to Section 3.4.1. □

3.2. Bilinear SDEs. We will construct numerical schemes for bilinear SDEs based on (3.2) and (3.3). This fleshes out the new technique.

In this subsection, we assume that X_t satisfies (1.2). Then, (3.3) becomes (1.10), and so $\hat{X}_t$ is approximated in $[T_n,T_{n+1}]$ by $Z_t/\|Z_t\|$, where

$$Z_t=\bar{X}_n+\int_{T_n}^t B\left(\hat{Y}_n\right)Z_s ds+\sum_{k=1}^m\int_{T_n}^t\left(\sigma^k-\left\langle\hat{Y}_n,\sigma^k\hat{Y}_n\right\rangle\right)Z_s dW_s^k, \tag{3.8}$$

with $\bar{X}_n\approx\hat{X}_{T_n}$ and

$$B\left(\hat{Y}_n\right)=B-\left\langle\hat{Y}_n,B\hat{Y}_n\right\rangle+\sum_{k=1}^m\left(\frac{3}{2}\left\langle\hat{Y}_n,\sigma^k\hat{Y}_n\right\rangle^2-\left\langle\hat{Y}_n,\sigma^k\hat{Y}_n\right\rangle\sigma^k-\frac{1}{2}\left\|\sigma^k\hat{Y}_n\right\|^2\right).$$

Moreover, (3.6) takes the form

$$\bar{\eta}_{n+1}=\bar{\eta}_n\exp\left(\left(\langle\hat{Y}_n,B\hat{Y}_n\rangle+\frac{1}{2}\sum_{k=1}^m\left\|\sigma^k\hat{Y}_n\right\|^2-\sum_{k=1}^m\langle\hat{Y}_n,\sigma^k\hat{Y}_n\rangle^2\right)\Delta+\sum_{k=1}^m\langle\hat{Y}_n,\sigma^k\hat{Y}_n\rangle\sqrt{\Delta}\hat{W}_n^k\right), \tag{3.9}$$

where $\hat{Y}_n$ is an $\mathfrak{F}_{T_n}$-measurable random variable of norm 1 that approximates $\hat{X}_s$ on $[T_n,T_{n+1}]$. Thus, we have to specify both $\hat{Y}_n$ and a way of computing $Z_{T_{n+1}}$. The simplest selection is $\hat{Y}_n=\bar{X}_n$, together with the use of the Euler-Maruyama method to integrate numerically (3.8). This leads to

SCHEME 3.1. *Define recursively $\bar{X}_{n+1}=\bar{Z}_{n+1}/\left\|\bar{Z}_{n+1}\right\|$, where*

$$\bar{Z}_{n+1}=\bar{X}_n+B\left(\bar{X}_n\right)\bar{X}_n\Delta+\sum_{k=1}^m\left(\sigma^k-\left\langle\bar{X}_n,\sigma^k\bar{X}_n\right\rangle\right)\bar{X}_n\sqrt{\Delta}\hat{W}_n^k. \tag{3.10}$$

Here $\hat{W}_0^1,\hat{W}_0^2,\ldots,\hat{W}_0^m,\hat{W}_1^1,\ldots$ are i.i.d. symmetric random variables having variance 1. Let $\bar{\eta}_{n+1}$ be given by (3.9) with $\hat{Y}_n=\bar{X}_n$.

Scheme 3.1 reduces to the method defined by (1.13) and (1.15) when $\hat{W}_n^k=\xi_n^k$. Applying Theorem 3.1 we obtain that Scheme 3.1 is almost sure exponential stable for all $\Delta>0$ under condition (1.5). We next establish that it converges weakly with order 1.

THEOREM 3.2. *Consider $T>0$ and $f\in C_p^4(\mathbb{R}^d,\mathbb{R})$. Let $\bar{\eta}_n\bar{X}_n$ be described by Scheme 3.1 with $\Delta=T/N$, where $N\in\mathbb{N}$. Assume that $\hat{W}_n^k$ are bounded random variables, X_0 have finite moments of any order, and that for every $g\in C_p^4(\mathbb{R}^d,\mathbb{R})$,*

$$\left|\mathbb{E}g(\bar{\eta}_0\bar{X}_0)-\mathbb{E}g(X_0)\right|\leq K\left(1+\mathbb{E}\left|X_0\right|^q\right)T/N \qquad \forall N\in\mathbb{N}.$$

Then for all $N \in \mathbb{N}$,

$$\left|\mathbb{E}f\left(X_T\right)-\mathbb{E}f\left(\bar{\eta}_N\bar{X}_N\right)\right| \leq K\left(T\right)\left(1+\mathbb{E}\left\|X_0\right\|^q\right)T/N. \tag{3.11}$$

Proof. Deferred to Section 3.4.2. ∎

We now focus on (1.2) with B ill-conditioned. If the matrix B have very different eigenvalues, then we should carefully approximate the term $\langle \hat{X}_s, B\hat{X}_s\rangle$ in (1.12). This leads us to the problem of finding good candidates for the random variable $\hat{Y}_n$ presented in (3.8) and (3.9) (see Remark 3.1).

We now develop an alternative strategy to use (3.8) and (3.9), which has yielded promising results in our numerical experiments. Suppose that $\bar{X}_n$ and $\bar{\rho}_n$ are $\mathfrak{F}_{T_n}$-measurable random variables such that $\bar{X}_n \approx \hat{X}_{T_n}$, $\left\|\bar{X}_n\right\| = 1$, and $\bar{\rho}_n \approx \left\|X_{T_n}\right\|$. Return to (1.10) and (1.12), and consider the ordinary differential equation

$$Y_n\left(t\right)=\bar{\rho}_n\bar{X}_n+\int_{T_n}^{t}BY_n\left(s\right)ds \qquad \forall t\in\left[T_n,T_{n+1}\right]. \tag{3.12}$$

Since $\hat{Y}_n\left(t\right) := Y_n\left(t\right)/\left\|Y_n\left(t\right)\right\|$ satisfies

$$\hat{Y}_n\left(t\right)=\bar{X}_n+\int_{T_n}^{t}\left(B-\left\langle\hat{Y}_n\left(s\right),B\hat{Y}_n\left(s\right)\right\rangle\right)\hat{Y}_n\left(s\right)ds,$$

from (1.10) we obtain that for any $t \in [T_n, T_{n+1}]$,

$$\hat{X}_t \approx \hat{Y}_n\left(t\right)+\int_{T_n}^{t}\Psi\left(\hat{X}_s\right)\hat{X}_s ds+\sum_{k=1}^{m}\int_{T_n}^{t}\left(\sigma^k-\left\langle\hat{X}_s,\sigma^k\hat{X}_s\right\rangle\right)\hat{X}_s dW_s^k,$$

where $\Psi\left(\hat{X}_s\right)=\sum_{k=1}^m\left(3\left\langle\hat{X}_s,\sigma^k\hat{X}_s\right\rangle^2/2-\left\langle\hat{X}_s,\sigma^k\hat{X}_s\right\rangle\sigma^k-\left\|\sigma^k\hat{X}_s\right\|^2/2\right)$. Hence

$$\hat{X}_t \approx \hat{Y}_n\left(t\right)+\int_{T_n}^{t}\Psi\left(\bar{X}_n\right)\hat{X}_s ds+\sum_{k=1}^{m}\int_{T_n}^{t}\left(\sigma^k-\left\langle\bar{X}_n,\sigma^k\bar{X}_n\right\rangle\right)\hat{X}_s dW_s^k.$$

In the spirit of the Euler-exponential schemes given by [24] we consider

$$\tilde{X}_t = \hat{Y}_n\left(t\right)+\int_{T_n}^{t}\Psi\left(\bar{X}_n\right)\tilde{X}_s ds+\sum_{k=1}^{m}\int_{T_n}^{t}\left(\sigma^k-\left\langle\bar{X}_n,\sigma^k\bar{X}_n\right\rangle\right)\bar{X}_n dW_s^k, \tag{3.13}$$

and so $\tilde{X}_t \approx \hat{X}_t$. Approximating the explicit solution of (3.13) we deduce that $\tilde{X}_{T_{n+1}} \approx U_{n+1}$, where

$$U_{n+1}=\exp\left(\Psi\left(\bar{X}_n\right)\Delta\right)\left(\tilde{Y}_n/\left\|\tilde{Y}_n\right\|+\sum_{k=1}^{m}\left(\sigma^k-\langle\bar{X}_n,\sigma^k\bar{X}_n\rangle\right)\bar{X}_n\sqrt{\Delta}\hat{W}_n^k\right). \tag{3.14}$$

Here $\hat{W}_0^1, \hat{W}_0^2, \ldots, \hat{W}_0^m, \hat{W}_1^1, \ldots$ are i.i.d. symmetric random variables having variance 1, and $\tilde{Y}_n = \exp\left(B\Delta\right)\bar{X}_n$. This implies $\tilde{Y}_n/\left\|\tilde{Y}_n\right\| = \hat{Y}_n\left(T_{n+1}\right)$.

Similarly, using (1.12) and $\left\|\hat{Y}_n(t)\right\| = \left\|\bar{X}_n\right\| + \int_{T_n}^{t} \langle \hat{Y}_n(s), B\hat{Y}_n(s)\rangle \left\|\hat{Y}_n(s)\right\| ds$ we can assert that $\|X_t\| \approx \rho_t$ for all $t \in [T_n, T_{n+1}]$, where

$$\rho_t = \left\|\hat{Y}_n(t)\right\| + \int_{T_n}^{t} \left(\sum_{k=1}^{m} \left(\left\|\sigma^k \bar{X}_n\right\|^2 / 2 - \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle^2 / 2 \right) \right) \rho_s ds \qquad (3.15)$$
$$+ \sum_{k=1}^{m} \int_{T_n}^{t} \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle \rho_s dW_s^k.$$

Approximating the explicit solution of (3.15) we get $\rho_{T_{n+1}} \approx \bar{\rho}_{n+1}$, with

$$\bar{\rho}_{n+1} = \bar{\rho}_n \left\|\tilde{Y}_n\right\| \exp\left(\sum_{k=1}^{m} \left(\left\|\sigma^k \bar{X}_n\right\|^2 / 2 - \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle^2 \right) \Delta + \sum_{k=1}^{m} \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle \sqrt{\Delta} \hat{W}_n^k \right). \qquad (3.16)$$

We have derived the following numerical method.

Scheme 3.2. *Let $\left(\bar{X}_n\right)_{n\geq 0}$ be given by the recursive formula*

$$\bar{X}_{n+1} = U_{n+1} / \left\|U_{n+1}\right\|,$$

where U_{n+1} is as in (3.14). *Moreover, $\left(\bar{\rho}_n\right)_{n\geq 0}$ is defined recursively by* (3.16).

On the other hand, applying the Euler approximation to (3.13) we obtain that $\tilde{X}_{T_{n+1}} \approx V_{n+1}$, where

$$V_{n+1} = \tilde{Y}_n / \left\|\tilde{Y}_n\right\| + \Psi\left(\bar{X}_n\right) \bar{X}_n \Delta + \sum_{k=1}^{m} \left(\sigma^k - \langle \bar{X}_n, \sigma^k \bar{X}_n\rangle\right) \bar{X}_n \sqrt{\Delta} \hat{W}_n^k. \qquad (3.17)$$

Then, we can replace U_{n+1} by V_{n+1} in Scheme 3.2. This yields

Scheme 3.3. *Define recursively $\bar{X}_{n+1} = V_{n+1} / \|V_{n+1}\|$, where V_{n+1} is given by* (3.17). *Let $\left(\bar{\rho}_n\right)_{n\geq 0}$ be described by* (3.16).

We next establish the exponential stability of Schemes 3.2 and 3.3 for any step-size Δ, under Condition (1.7) applied to (1.2). Here, for brevity reasons, we do not address the rate of convergence of Schemes 3.2 and 3.3.

Theorem 3.3. *Consider Scheme 3.2, or Scheme 3.3, with $\bar{\rho}_0 > 0$ and $\mathbb{E}\left(\bar{\rho}_0\right)^2 < \infty$. Suppose that*

$$-\tilde{\lambda} := \sup_{\|x\|=1} \langle x, Bx\rangle + \sup_{\|x\|=1} \sum_{k=1}^{m} \left(\left\|\sigma^k x\right\|^2 / 2 - \langle x, \sigma^k(x)\rangle^2 \right) < 0. \qquad (3.18)$$

Then

$$\limsup_{n\to\infty} \frac{1}{n\Delta} \ln\left(\bar{\rho}_n\right) \leq -\tilde{\lambda} \qquad \mathbb{P} - a.s.$$

Proof. Deferred to Section 3.4.3. □

Remark 3.1. *Applying the midpoint rule to the solution of* (3.12) *we obtain*

$$X_s \approx \bar{Y}_n := \frac{\exp\left(B\Delta/2\right) \bar{X}_n}{\left\|\exp\left(B\Delta/2\right) \bar{X}_n\right\|}, \qquad (3.19)$$

and so we can approximate $\langle \hat{X}_s, B\hat{X}_s\rangle$ by $\langle \bar{Y}_n, B\bar{Y}_n\rangle$ whenever $s \in [T_n, T_{n+1}]$. Moreover, $\langle \hat{X}_s, \sigma^k \hat{X}_s\rangle \approx \langle \bar{Y}_n, \sigma^k \bar{Y}_n\rangle$ and $\left\|\sigma^k \hat{X}_s\right\|^2 \approx \left\|\sigma^k \bar{Y}_n\right\|^2$ for all $s \in [T_n, T_{n+1}]$.

Therefore, the choice $\hat{Y}_n = \bar{Y}_n$ *in* (3.8) *and* (3.9) *yields* $\hat{X}_{T_{n+1}} \approx Z_{T_{n+1}} / \left\| Z_{T_{n+1}} \right\|$ *and* $\left\| X_{T_{n+1}} \right\| \approx \bar{\eta}_{n+1}$. *We can solve* (3.8) *by using the Euler-exponential method introduced in [24]. This gives the recursive scheme* $\bar{X}_{n+1} := \bar{V}_{n+1} / \left\| \bar{V}_{n+1} \right\|$, *where*

$$\bar{V}_{n+1} = \exp\left(B\left(\bar{Y}_n\right)\Delta\right)\left(\bar{X}_n + \sum_{k=1}^{m}\left(\sigma^k - \left\langle \bar{Y}_n, \sigma^k \bar{Y}_n \right\rangle\right)\bar{X}_n \sqrt{\Delta}\hat{W}_n^k\right),$$

with $\hat{W}_n^k$ *as in Scheme 3.1. Finally, we take* $\bar{\eta}_{n+1}$ *defined by* (3.9) *with* $\hat{Y}_n = \bar{Y}_n$. *A good alternative to* $\bar{V}_{n+1}$ *is to apply the backward Euler method to* (3.8).

3.3. Numerical experiments.

3.3.1. Lyapunov exponents. We will illustrate the potential of Scheme 3.1 to compute the Lyapunov exponents of (1.2) (see [4, 34] for classical theoretical and numerical references). To this end, we calcule $\ell\left(X_0\right) := \lim_{t\to\infty} \frac{1}{t} \ln \left\| X_t \right\|$, where

$$X_t = X_0 + \int_0^t \begin{pmatrix} a - \frac{\sigma^2}{2} & 0 \\ 0 & b - \frac{\sigma^2}{2} \end{pmatrix} X_s ds + \int_0^t \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X_s dW_s^1, \tag{3.20}$$

with $a, b \in \mathbb{R}$, $\sigma > 0$ and $X_0 \neq 0$. In this well-known test problem [5, 18, 34], $\ell\left(X_0\right)$ does not depend on the initial condition X_0, and further

$$\ell = \frac{a+b}{2} + \frac{a-b}{2} \frac{\int_0^{2\pi} \cos\left(2\theta\right) \exp\left(\frac{a-b}{2\sigma^2}\cos\left(2\theta\right)\right) d\theta}{\int_0^{2\pi} \exp\left(\frac{a-b}{2\sigma^2}\cos\left(2\theta\right)\right) d\theta}. \tag{3.21}$$

Following [34] we take $a = 1$ and $b = -2$. We choose $X_0 = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^\top$. In the context of (1.2), there is no loss of generality in assuming $\left\| X_0 \right\| = 1$, because $\ell\left(X_0\right) = \ell\left(X_0 / \left\| X_0 \right\|\right)$ for bilinear SDEs.

In case B is well-conditioned, we can approximate the Lyapunov exponent ℓ by

$$\tilde{\ell}_N\left(\Delta\right) := \frac{1}{N\Delta} \ln \left\| \bar{\eta}_N \bar{X}_N \right\| = \frac{1}{N\Delta} \sum_{n=0}^{N-1} \ln\left(\frac{\left\| \bar{\eta}_{n+1} \bar{X}_{n+1} \right\|}{\left\| \bar{\eta}_n \bar{X}_n \right\|}\right),$$

where $\bar{\eta}_n \bar{X}_n$ is given by Scheme 3.1 and N is sufficiently large. From (3.9) we have $\tilde{\ell}_N\left(\Delta\right) = \frac{1}{N\Delta} \sum_{n=0}^{N-1} \mathcal{L}_n\left(\bar{X}_n\right)$, with

$$\mathcal{L}_n\left(x\right) = \left(\langle x, Bx \rangle + \frac{1}{2}\sum_{k=1}^{m} \left\| \sigma^k x \right\|^2 - \sum_{k=1}^{m} \langle x, \sigma^k x \rangle^2\right)\Delta + \sum_{k=1}^{m} \langle x, \sigma^k x \rangle \sqrt{\Delta} \hat{W}_n^k.$$

For concreteness, we consider $\hat{W}_n^k$ uniformly distributed on $\left[-\sqrt{3}, \sqrt{3}\right]$. Then

$$\tilde{\ell}_{n+1}\left(\Delta\right) = \tilde{\ell}_n\left(\Delta\right)\left(1 - \frac{1}{n+1}\right) + \frac{\mathcal{L}_n\left(\bar{X}_n\right)}{\left(n+1\right)\Delta}.$$

In order to evaluate the performance of $\tilde{\ell}_n$, we also compute ℓ by means of the algorithm $\hat{\ell}_n$ introduced in pages 1155 and 1156 of [34].

Table 3.1 compares the average of 20 realizations of both $\tilde{\ell}_N\left(\Delta\right)$ and $\hat{\ell}_N\left(\Delta\right)$ applied to (3.20) with $\sigma = 10$ and $\sigma = 20$. We have actually computed $\frac{1}{500} \ln\left(\left\| X_{500} \right\|\right)$ since we choose $N = 500/\Delta$. Integrating numerically (3.21) we obtain the reference

Δ	0.1	0.05	0.01	0.002	0.001	0.0001
$\sigma = 10,\ \ell = -0.48875$						
$\hat{\ell}_N(\Delta)$	30.09701	26.77603	-3.71372	-0.57889	-0.48503	-0.48811
$\tilde{\ell}_N(\Delta)$	-0.52693	-0.50851	-0.49928	-0.49132	-0.48930	-0.48937
$\sigma = 20,\ \ell = -0.49719$						
$\hat{\ell}_N(\Delta)$	86.56572	114.26780	93.37706	-9.63927	-2.96739	-0.48334
$\tilde{\ell}_N(\Delta)$	-0.55164	-0.51945	-0.50120	-0.49883	-0.49798	-0.49703

Table 3.1

Computed values for a final integration time $T = N\Delta = 500$ of the Lyapunov exponent ℓ for (3.20) with $a = 1$, $b = -2$, $X_0 = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^\top$ and different parameters σ.

value $\ell = -0.48875$ for $\sigma = 10$, as well as that $\ell = -0.49719$ whenever $\sigma = 20$. Table 3.1 shows the very good accuracy of $\tilde{\ell}_N$. In case $\sigma = 10$, the relative error $\left|\tilde{\ell}_N(0.1) - \ell\right| / |\ell|$ is equal to 0.0781, and so $\tilde{\ell}_N(\Delta)$ provides a good approximation of ℓ even if $\Delta = 0.1$. If we increase the noise to $\sigma = 20$, then $\left|\tilde{\ell}_N(0.05) - \ell\right| / |\ell| = 0.0448$. In contrast, $\hat{\ell}_N(0.001)$ produces the value -2.96739.

3.3.2. Ill-conditioned matrix B. This subsection addresses the test problem (1.16), but with more general drift coefficient B. Indeed, we deal with the numerical solution of the SDE

$$X_t = X_0 + \int_0^t \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} X_s ds + \int_0^t \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} X_s dW_s^1 + \int_0^t \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X_s dW_s^2. \quad (3.22)$$

In order to study cases where B is ill-conditioned, we take $b_1 = -100$ and $b_2 = 2$. We also choose $\sigma = 4$, $\epsilon = 1$ and $X_0 = (1,2)^\top$. We have that X_t converges exponentially fast to 0 since

$$\begin{aligned} -\tilde{\lambda} &= \sup_{\|x\|=1} \langle x, Bx\rangle + \sup_{\|x\|=1} \left(\frac{1}{2} \sum_{k=1}^m \left\|\sigma^k x\right\|^2 - \sum_{k=1}^m \langle x, \sigma^k x\rangle^2 \right) \\ &= \max\{b_1, b_2\} + \left(\epsilon^2 - \sigma^2\right)/2 = -11/2. \end{aligned}$$

Similar to Subsection 1.2, we compute the mean value of the bounded random variable $\arctan\left(1 + \left(X_t^2\right)^2\right)$, where $X_t = \left(X_t^1, X_t^2\right)$. In Figure 4, the solid line provides the reference values for $\mathbb{E}\arctan\left(1 + \left(X_t^2\right)^2\right)$, which have been obtained by sampling 10^8 times the Euler-Maruyama scheme E_n applied to (3.22) with step-size $\Delta = 2^{-14} \approx 0.0000610$ and $W_{(n+1)\Delta}^k - W_{n\Delta}^k$ replaced by $\sqrt{\Delta}\xi_n^k$. Here, $\xi_0^1, \xi_0^2, \ldots, \xi_0^m, \xi_1^1, \ldots$ are independent random variables taking values ± 1 with probability 1/2. Moreover, the circles (resp. stars) represent the estimated values of $\mathbb{E}\arctan\left(1 + \left(X_t^2\right)^2\right)$ produced by averaging over 10^6 random realizations of Scheme 3.2 with $\hat{W}_n^k = \xi_n^k$ (resp. the backward Euler method $\bar{E}_n$ given by (1.6) with $W_{(n+1)\Delta}^k - W_{n\Delta}^k$ replaced by $\sqrt{\Delta}\xi_n^k$). Figure 4 shows that the second coordinate of $\bar{E}_n$ does not converge to 0 when $\Delta = 0.25, 0.125$, and it goes too fast to 0 for $\Delta = 0.0625$. On the contrary, Scheme 3.2 is very accurate, even with $\Delta = 0.25$.

Table 3.2 presents the errors produced at time $T = 0.5$ by the new numerical methods. We assign the weak error $\epsilon(Y, \Delta) := \left|\mathbb{E}f(X_T) - \mathbb{E}f\left(Y_{T/\Delta}\right)\right|$ to every

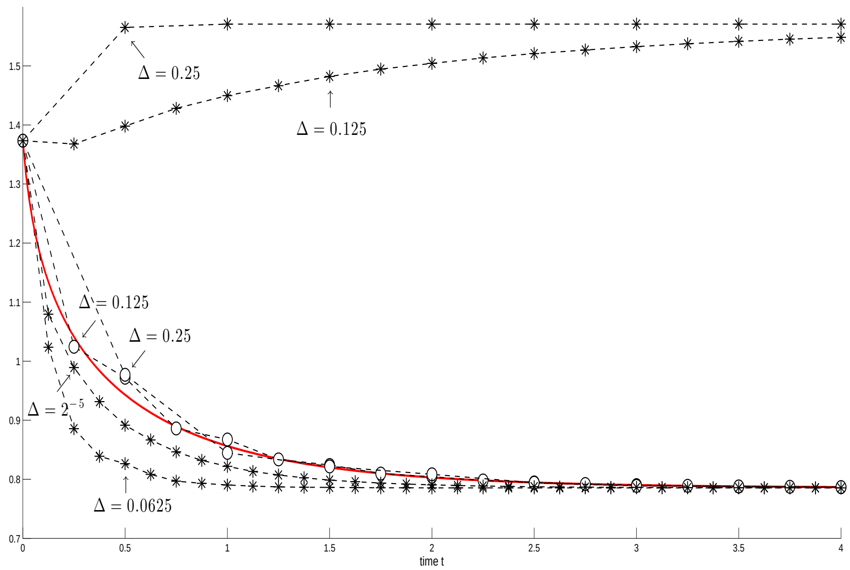


FIG. 4. *Computation of* $\mathbb{E}\arctan\left(1+\left(X_t^2\right)^2\right)$*, where* $t \in [0,4]$ *and* X_t *solves* (3.22) *with* $b_1=-10$*,* $b_2=2$*,* $\sigma=4$*,* $\epsilon=1$ *and* $X_0=(1,2)^\top$*. The true values are plotted with a solid line. The circles and stars represent the approximations obtained by Scheme 3.2 and* $\bar{E}_n$ *respectively.*

scheme Y with step-size Δ, where $f(x_1,x_2)=\arctan\left(1+(x_2)^2\right)$. Table 3.2 compares estimates of $\epsilon(Y,\Delta)$ obtained by sampling 10^6 times the backward Euler method $\bar{E}_n$,

TABLE 3.2

Absolute errors $\epsilon(Y,\Delta) = \left|\mathbb{E}f(X_T)-\mathbb{E}f\left(Y_{T/\Delta}\right)\right|$ *involved in the computation of* $\mathbb{E}f(X_T)$*, where* X_t *verifies (3.22) with* $b_1=-100$*,* $b_2=2$*,* $\sigma=4$*,* $\epsilon=1$ *and* $X_0=(1,2)^\top$*. Here* $T=0.5$ *and* $f(x_1,x_2)=\arctan\left(1+(x_2)^2\right)$.

Δ	$\epsilon(Y,\Delta)$				
	$\bar{E}$	Scheme 3.2	Remark 3.1	Scheme 3.1	Scheme 3.3
1/2	–	0.11463	0.16262	0.15767	0.097492
1/4	0.62209	0.033856	0.039728	0.15767	0.033187
1/8	0.45494	0.028848	0.055127	0.15767	0.027073
1/16	0.1167	0.00092922	0.031429	0.15767	0.00001519
1/32	0.051448	0.0034819	0.02738	0.15765	0.0030137
1/64	0.022656	0.0013437	0.022279	0.057997	0.001135
2^{-7}	0.010789	0.0007307	0.016977	0.015613	0.00065061
2^{-8}	0.0052678	0.00037132	0.011165	0.0061579	0.00034013
2^{-9}	0.0031195	0.00034469	0.0059087	0.0032785	0.0003577
2^{-10}	0.001551	0.00017328	0.0032079	0.001571	0.00017914
2^{-11}	0.00045271	0.00023427	0.001994	0.00044898	0.0002315
2^{-12}	0.00031327	0.000029582	0.00092728	0.00030817	0.000028237

Schemes 3.1-3.3 and the method introduced in Remark 3.1. We take $\hat{W}_n^k = \xi_n^k$, and the length of all the 99%-confidence intervals are at least of order 10^{-3} (see, e.g., [18]). Table 3.2 shows that the values of $\epsilon\left(Scheme\ 3.2, \Delta\right)$ and $\epsilon\left(Scheme\ 3.3, \Delta\right)$ are quite similar. We can also see that Schemes 3.2 and 3.3 are very accurate. Moreover, the second coordinate given by Scheme 3.1 decays too fast to 0 when $\Delta \geq 1/32$. Finally, the numerical method given by Remark 3.1 has exponentially stable trajectories, but tends to 0 slightly more slowly than the 'true' solution.

3.4. Proofs.

3.4.1. Proof of Theorem 3.1. Using (3.6) yields

$$\begin{aligned}\ln\left(\bar{\eta}_{n+1}\right) = & \ln\left(\bar{\eta}_0\right) + S_n \\ & + \sum_{j=0}^{n} \left(\frac{\langle \tilde{X}_j, b\left(\tilde{X}_j\right)\rangle + \frac{1}{2}\sum_{k=1}^{m} \left\| \sigma^k\left(\tilde{X}_j\right)\right\|^2}{\left\|\tilde{X}_j\right\|^2} - \sum_{k=1}^{m} \frac{\langle \tilde{X}_j, \sigma^k\left(\tilde{X}_j\right)\rangle^2}{\left\|\tilde{X}_j\right\|^4} \right)\Delta,\end{aligned}$$

where $S_n = \sum_{j=0}^{n}\sum_{k=1}^{m} \frac{\langle \tilde{X}_j, \sigma^k\left(\tilde{X}_j\right)\rangle}{\left\|\tilde{X}_j\right\|^2}\sqrt{\Delta}\hat{W}_n^k$. Then, (1.5) leads to

$$\frac{1}{n+1}\ln\left(\bar{\eta}_{n+1}\right) \leq \frac{1}{n+1}\ln\left(\bar{\eta}_0\right) + \frac{1}{n+1}S_n - \lambda\Delta. \tag{3.23}$$

From (1.4) it follows that

$$\mathbb{E}\left(\sum_{k=1}^{m} \frac{\langle \tilde{X}_j, \sigma^k\left(\tilde{X}_j\right)\rangle}{\left\|\tilde{X}_j\right\|^2}\sqrt{\Delta}\hat{W}_n^k\right)^2 \leq \mathbb{E}\left(\sum_{k=1}^{m} \frac{\left\|\sigma^k\left(\tilde{X}_j\right)\right\|}{\left\|\tilde{X}_j\right\|}\sqrt{\Delta}\left|\hat{W}_n^k\right|\right)^2 \leq K\Delta,$$

and so applying a generalized law of large numbers, as in the proof of Theorem 2.1, we obtain that $S_n/\left(n+1\right) \to 0$ a.s. By (3.23),

$$\limsup_{n\to\infty} \frac{1}{n+1}\ln\left(\bar{\eta}_{n+1}\right) \leq -\lambda\Delta \qquad a.s. \qquad \square$$

3.4.2. Proof of Theorem 3.2. We first establish that for an arbitrary $q \geq 2$,

$$\mathbb{E}\left\|\bar{\eta}_n\bar{X}_n\right\|^q \leq K\left(T\right)\mathbb{E}\left|\bar{\eta}_0\right|^q \qquad \forall n = 0,\ldots,N. \tag{3.24}$$

To shorten notation, we set $\lambda^k\left(\bar{X}_n\right) = \left\langle \bar{X}_n, \sigma^k\bar{X}_n\right\rangle$ and

$$\mu\left(\bar{X}_n\right) = \left\langle \bar{X}_n, B\bar{X}_n\right\rangle + \sum_{k=1}^{m}\left(\left\|\sigma^k\bar{X}_n\right\|^2 - \left\langle \bar{X}_n, \sigma^k\bar{X}_n\right\rangle^2\right)/2.$$

Similar to the proof of (2.14), we rewrite $\bar{\eta}_{n+1}$ as $\exp\left(h_n\right)\bar{\eta}_n$, and so

$$\bar{\eta}_{n+1} = \bar{\eta}_n + \left(\exp\left(h_n\right) - 1 - h_n\right)\bar{\eta}_n + h_n\bar{\eta}_n,$$

with $h_n := \left(\mu\left(\bar{X}_n\right) - \frac{1}{2}\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)^2\right)\Delta + \sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\sqrt{\Delta}\hat{W}_n^k$. Thus

$$\bar{\eta}_{n+1} = \bar{\eta}_0 + \sum_{k=0}^{n}\left(\exp\left(h_k\right) - 1 - h_k\right)\bar{\eta}_k + \sum_{k=0}^{n} h_k\bar{\eta}_k.$$

Using $\left|\exp\left(h_k\right)-1-h_k\right|\le\left|h_k\right|^2\exp\left(\left|h_k\right|\right)$ we obtain

$$\begin{aligned}\left|\bar{\eta}_{n+1}\right|^q \le & K\left|\bar{\eta}_0\right|^q+K\left(n+1\right)^{q-1}\sum_{k=0}^{n}\left|h_k\right|^{2q}e^{q\left|h_k\right|}\left|\bar{\eta}_k\right|^q+K\Delta^{q/2}\sum_{j=1}^{m}\left|\sum_{k=0}^{n}\lambda^j\left(\bar{X}_k\right)\bar{\eta}_k\hat{W}_k^j\right|^q \\ & +K\left(n+1\right)^{q-1}\sum_{k=0}^{n}\Delta^q\left|\mu\left(\bar{X}_k\right)-\frac{1}{2}\sum_{j=1}^{m}\lambda^j\left(\bar{X}_k\right)^2\right|^q\left|\bar{\eta}_k\right|^q. \end{aligned} \tag{3.25}$$

For any $k\in\mathbb{Z}_+$, $\left\|\mu\left(\bar{X}_k\right)\right\|\le K$ and $\left\|\lambda^j\left(\bar{X}_k\right)\right\|\le K$, because $\left\|\bar{X}_k\right\|=1$. We also have that $\hat{W}_0^1$ is a bounded random variable. Then, applying the Burkholder-Davis-Gundy inequality we deduce from (3.25) that

$$\mathbb{E}\left|\bar{\eta}_{n+1}\right|^q\le K\mathbb{E}\left|\bar{\eta}_0\right|^q+K\left(T\right)\Delta\sum_{k=0}^{n}\mathbb{E}\left|\bar{\eta}_k\right|^q$$

for all $n=0,\dots,N-1$. A discrete Gronwall lemma (see, e.g., [4]) now leads to (3.24).

We proceed to find a truncated asymptotic expansion of $\bar{\eta}_{n+1}\bar{X}_{n+1}$ as Δ goes to 0. In what follows, we use the same symbol $\mathcal{O}\left(\cdot\right)$ for different random functions from $[0,T]$ to $\mathbb{R}$ or $\mathbb{R}^{d\times d}$ such that $\left\|\mathcal{O}\left(x\right)\right\|\le K\left(T\right)x$. Since

$$\left|\exp\left(x\right)-1+x+x^2/2+x^3/6\right|\le x^4\exp\left(\left|x\right|\right),$$

$$\begin{aligned}\bar{\eta}_{n+1}=&\bar{\eta}_n\left(1+\left(\mu\left(\bar{X}_n\right)-\frac{1}{2}\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)^2\right)\Delta+\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\sqrt{\Delta}\right) \\ &+\bar{\eta}_n\left(\mu\left(\bar{X}_n\right)-\frac{1}{2}\sum_{j=1}^{m}\lambda^j\left(\bar{X}_n\right)^2\right)\left(\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)\Delta^{3/2} \\ &+\frac{\bar{\eta}_n}{2}\left(\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^2\Delta+\frac{\bar{\eta}_n}{6}\left(\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^3\Delta^{3/2}+\bar{\eta}_n\mathcal{O}\left(\Delta^2\right).\end{aligned} \tag{3.26}$$

Multiplying the right hand sides of (3.10) and (3.26) yields

$$\begin{aligned}\bar{\eta}_{n+1}\bar{Z}_{n+1}=&\left(1+B\Delta+\sum_{k=1}^{m}\sigma^k\hat{W}_n^k\sqrt{\Delta}\right)\bar{\eta}_n\bar{X}_n+\Gamma_n\Delta^{3/2}\bar{\eta}_n\bar{X}_n \\ &+\Delta\sum_{k=1}^{m}\left(\sigma^k-\lambda^k\left(\bar{X}_n\right)/2\right)\lambda^k\left(\bar{X}_n\right)\left(\left(\hat{W}_n^k\right)^2-1\right)\bar{\eta}_n\bar{X}_n \\ &+\Delta\sum_{k\neq j}\left(\sigma^j-\lambda^j\left(\bar{X}_n\right)/2\right)\lambda^k\left(\bar{X}_n\right)\hat{W}_n^j\hat{W}_n^k\ \bar{\eta}_n\bar{X}_n+\mathcal{O}\left(\Delta^2\right)\bar{\eta}_n\bar{X}_n,\end{aligned} \tag{3.27}$$

where Γ_n is a random matrix such that $\left\|\Gamma_n\right\|\le K$ and $\mathbb{E}\left(\Gamma_n\diagup\mathfrak{F}_{T_n}\right)=0$; throughout the proof, we assume without loss of generality that $\hat{W}_n^1,\dots\hat{W}_n^m$ are $\mathfrak{F}_{T_{n+1}}$-measurable and independent of $\mathfrak{F}_{T_n}$. Indeed,

$$\begin{aligned}\Gamma_n=&\left(B+\frac{1}{2}\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)^2-\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\sigma^k\right)\sum_{j=1}^{m}\lambda^j\left(\bar{X}_n\right)\hat{W}_n^j \\ &+\frac{1}{6}\left(\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)\hat{W}_n^k\right)^3+\left(\mu\left(\bar{X}_n\right)-\frac{1}{2}\sum_{k=1}^{m}\lambda^k\left(\bar{X}_n\right)^2\right)\sum_{k=1}^{m}\left(\sigma^k-\lambda^k\left(\bar{X}_n\right)\right)\hat{W}_n^k.\end{aligned}$$

From (3.10) we have

$$\left\|\bar{Z}_{n+1}\right\|^{2}=1+\Delta\sum_{k=1}^{m}\left(\lambda^{k}\left(\bar{X}_{n}\right)^{2}-\left\|\sigma^{k}\bar{X}_{n}\right\|^{2}\right)\left(1-\left(\hat{W}_{n}^{k}\right)^{2}\right) \tag{3.28}$$
$$+\Delta\sum_{k\neq j}\left(\left\langle\sigma^{k}\bar{X}_{n},\sigma^{j}\bar{X}_{n}\right\rangle-\lambda^{j}\left(\bar{X}_{n}\right)\lambda^{k}\left(\bar{X}_{n}\right)\right)\hat{W}_{n}^{k}\hat{W}_{n}^{j}$$
$$+2\Delta^{3/2}\sum_{k=1}^{m}\left\langle B\left(\bar{X}_{n}\right)\bar{X}_{n},\sigma^{k}\bar{X}_{n}-\lambda^{k}\left(\bar{X}_{n}\right)\bar{X}_{n}\right\rangle\hat{W}_{n}^{k}+\mathcal{O}\left(\Delta^{2}\right).$$

Hence, there exists $\Delta_0 > 0$ such that $\left|1-\left\|\bar{Z}_{n+1}\right\|^{2}\right| \leq 1/2$ for all $\Delta \leq \Delta_0$, because $\left|\hat{W}_{n}^{k}\right| \leq K$. Using the power series expansion of $x \mapsto (1+x)^{-1/2}$ we get

$$\frac{1}{\|z\|}=1+\frac{1}{2}\left(1-\|z\|^{2}\right)+\left(1-\|z\|^{2}\right)^{2}\sum_{k=2}^{\infty}\frac{(2k)!}{(k!)^{2}4^{k}}\left(1-\|z\|^{2}\right)^{k-2}$$

whenever $\left|1-\|z\|^{2}\right|<1$. This, together with (3.28), implies that for all $\Delta \leq \Delta_0$,

$$1/\left\|\bar{Z}_{n+1}\right\|=1+\frac{\Delta}{2}\sum_{k=1}^{m}\left(\lambda^{k}\left(\bar{X}_{n}\right)^{2}-\left\|\sigma^{k}\bar{X}_{n}\right\|^{2}\right)\left(\left(\hat{W}_{n}^{k}\right)^{2}-1\right) \tag{3.29}$$
$$-\frac{\Delta}{2}\sum_{k\neq j}\left(\left\langle\sigma^{k}\bar{X}_{n},\sigma^{j}\bar{X}_{n}\right\rangle-\lambda^{j}\left(\bar{X}_{n}\right)\lambda^{k}\left(\bar{X}_{n}\right)\right)\hat{W}_{n}^{k}\hat{W}_{n}^{j}$$
$$-\Delta^{3/2}\sum_{k=1}^{m}\left\langle B\left(\bar{X}_{n}\right)\bar{X}_{n},\sigma^{k}\bar{X}_{n}-\lambda^{k}\left(\bar{X}_{n}\right)\bar{X}_{n}\right\rangle\hat{W}_{n}^{k}+\mathcal{O}\left(\Delta^{2}\right).$$

Combining (3.27) with (3.29) gives

$$\bar{\eta}_{n+1}\bar{X}_{n+1}=\left(1+B\Delta+\sum_{k=1}^{m}\sigma^{k}\hat{W}_{n}^{k}\sqrt{\Delta}\right)\bar{\eta}_{n}\bar{X}_{n}+\tilde{\Gamma}_{n}\Delta^{3/2}\bar{\eta}_{n}\bar{X}_{n} \tag{3.30}$$
$$+\Delta\sum_{k=1}^{m}\left(\lambda^{k}\left(\bar{X}_{n}\right)\sigma^{k}-\left\|\sigma^{k}\bar{X}_{n}\right\|^{2}/2\right)\left(\left(\hat{W}_{n}^{k}\right)^{2}-1\right)\bar{\eta}_{n}\bar{X}_{n}$$
$$+\Delta\sum_{k\neq j}\left(\lambda^{k}\left(\bar{X}_{n}\right)\sigma^{j}-\left\langle\sigma^{k}\bar{X}_{n},\sigma^{j}\bar{X}_{n}\right\rangle/2\right)\hat{W}_{n}^{j}\hat{W}_{n}^{k}\ \bar{\eta}_{n}\bar{X}_{n}+\mathcal{O}\left(\Delta^{2}\right)\bar{\eta}_{n}\bar{X}_{n},$$

where $\Delta \leq \Delta_0$ and $\tilde{\Gamma}_n$ is a random matrix satisfying $\left\|\tilde{\Gamma}_{n}\right\| \leq K$ and $\mathbb{E}\left(\tilde{\Gamma}_{n}\diagup\mathfrak{F}_{T_{n}}\right)=0$.

By (3.24), it is sufficient to prove that (3.11) holds for all $N \geq T/\Delta_0$. Then, from now we suppose $\Delta \leq \Delta_0$. Looking at (3.30) we easily see that $\left\|\bar{\eta}_{n+1}\bar{X}_{n+1}-\bar{\eta}_{n}\bar{X}_{n}\right\| \leq K(T)\Delta^{1/2}\left\|\bar{\eta}_{n}\bar{X}_{n}\right\|$, and so

$$\mathbb{E}\left(\left\|\bar{\eta}_{n+1}\bar{X}_{n+1}-\bar{\eta}_{n}\bar{X}_{n}\right\|^{q}\diagup\mathfrak{F}_{T_{n}}\right)\leq K(T)\Delta^{q/2}\left(1+\left\|\bar{\eta}_{n}\bar{X}_{n}\right\|^{q}\right). \tag{3.31}$$

Moreover, (3.30) leads to

$$\left\|\mathbb{E}\left(\bar{\eta}_{n+1}\bar{X}_{n+1}-\bar{\eta}_{n}\bar{X}_{n}-\left(B\Delta+\sum_{k=1}^{m}\sigma^{k}\left(W_{T_{n+1}}^{k}-W_{T_{n}}^{k}\right)\right)\bar{\eta}_{n}\bar{X}_{n}\diagup\mathfrak{F}_{T_{n}}\right)\right\|$$
$$\leq K(T)\Delta^{2}\left(1+\left\|\bar{\eta}_{n}\bar{X}_{n}\right\|\right).$$

Using again (3.30) we deduce that, up to terms of order $\mathcal{O}\left(\Delta^2\right)\left\|\bar{\eta}_n\bar{X}_n\right\|^q$, the second and third moments of $\bar{\eta}_{n+1}\bar{X}_{n+1}-\bar{\eta}_n\bar{X}_n$ coincide with that of

$$B\bar{\eta}_n\bar{X}_n\Delta+\sum_{k=1}^{m}\sigma^k\bar{\eta}_n\bar{X}_n\hat{W}_n^k\sqrt{\Delta},$$

and then with that of $\left(B\Delta+\sum_{k=1}^{m}\sigma^k\left(W^k_{T_{n+1}}-W^k_{T_n}\right)\right)\bar{\eta}_n\bar{X}_n$. Therefore, combining classical arguments [20, 32, 33] with (3.24) and (3.31) we can assert that (3.11) holds for all $T/N\leq\Delta_0$ (see also Theorem 14.5.2 of [18]). □

3.4.3. Proof of Theorem 3.3. From (3.16) we have

$$\ln\left(\bar{\rho}_{n+1}\right)=\ln\left(\bar{\rho}_0\right)+\sum_{j=0}^{n}\left(\ln\left(\left\|e^{B\Delta}\bar{X}_j\right\|\right)+\sum_{k=1}^{m}\left(\frac{1}{2}\left\|\sigma^k\bar{X}_j\right\|^2-\langle\bar{X}_j,\sigma^k\bar{X}_j\rangle^2\right)\Delta\right)+S_n,$$

where $S_n=\sum_{j=0}^{n}\sum_{k=1}^{m}\langle\bar{X}_j,\sigma^k\bar{X}_j\rangle\sqrt{\Delta}\hat{W}_n^k$. Since $Y_n\left(t\right):=\exp\left(B\left(t-T_n\right)\right)\bar{X}_n$ satisfies (3.12), using $\left\|\bar{X}_j\right\|=1$ we get

$$\ln\left(\left\|Y_n\left(t\right)\right\|\right)=\int_{T_n}^{t}\langle\frac{Y_s}{\left\|Y_s\right\|},B\frac{Y_s}{\left\|Y_s\right\|}\rangle ds\leq\left(t-T_n\right)\sup_{\|x\|=1}\langle x,Bx\rangle,$$

and so $\ln\left(\left\|\exp\left(B\Delta\right)\bar{X}_j\right\|\right)\leq\Delta\sup_{\|x\|=1}\langle x,Bx\rangle$. We now apply (3.18) to obtain

$$\ln\left(\bar{\rho}_{n+1}\right)\leq\ln\left(\bar{\rho}_0\right)-\left(n+1\right)\Delta\tilde{\lambda}+S_n,$$

As in the proof of Theorems 2.1 and 3.1, a generalized law of large numbers gives $S_n/\left(n+1\right)\rightarrow 0$ *a.s.*, and the theorem follows. □

4. Conclusion. We introduce a new idea to solve numerically multidimensional SDEs with multiplicative noise, whose dynamical properties can be meaningfully influenced by the noise terms. Applying the new methodology we obtain stable numerical methods for bilinear systems of SDEs, which have good numerical performances. Bilinear SDEs appears in applications such as dynamical studies of the motion of helicopter rotor blades in a turbulent wind. We also develop a promising stable numerical scheme for scalar SDEs with multiplicative noise.

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