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Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

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Alfredo Bermúdez, Dolores Gómez,
Rodolfo Rodríguez, Pablo Venegas

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Mathematical and numerical analysis of a transient non-linear axisymmetric eddy current model

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Abstract This paper deals with the analysis and computation of transient electromagnetic fields in conductive non-linear magnetic media. We analyze a weak formulation of the resulting problem in the axisymmetric case, with the source term given by means of a non-homogeneous Dirichlet boundary condition. Existence and uniqueness of solution is proved for this problem under fairly general assumptions on the non-linear constitutive relation between the magnetic field H and the magnetic induction B. The technique we use is based on implicit time discretization, a priori estimates and passage to the limit by compactness. For the numerical approximation, we propose a backward Euler time-discretization and prove its well-posedness and error estimates. Subsequently, we combine it with a finite element method for space discretization. Stability and error estimates are also obtained for this full-discretization. Finally, some numerical results, which confirm the theoretically predicted behavior of the method, are reported.

Keywords transient eddy current · axisymmetric problem · non-linear partial differential equations · non-homogeneous Dirichlet boundary condition · finite elements.

Mathematics Subject Classification (2000) 65N30 (78A55 78M10)

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Alfredo Bermúdez · M. Dolores Gómez
Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, 15706, Santiago de Compostela, Spain
E-mail: alfredo.bermudez@usc.es, mdolores.gomez@usc.es

Rodolfo Rodríguez · Pablo Venegas
CI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail: rodolfo@ing-mat.udec.cl, pvenegas@ing-mat.udec.cl

1 Introduction

An important challenge in the analysis and design of electrical machines is the accurate computation of power losses in the ferromagnetic components of the core due to hysteresis and eddy-current effects. These losses determine the efficiency of the device and have a significant influence on its operating cost. There are numerous publications devoted to obtain analytical simplified expressions to approximate their different components, which are only valid under certain assumptions that do not hold in many practical situations. Numerical modeling is an interesting alternative to overcome these limitations (see [11, 16, 21]).

We focus on the eddy current losses. Their numerical computation requires solving the Maxwell quasi-static partial differential equations, which is the aim of this paper. For linear magnetic materials this is a well established subject, even for three-dimensional (3D) models where edge finite elements are very useful (see, for instance, [1, 7, 13]). The non-linear case was studied in [24, 26] in terms of the magnetic field, where a transient eddy current model is considered on a 3D bounded conducting domain under homogeneous Dirichlet boundary conditions. A time semi-discretization scheme to approximate this problem is proposed and analyzed in these references; however, current sources are not taken into account, the only non vanishing data being the initial condition.

More recently, a non-linear axisymmetric transient eddy current model was analyzed in [6] under rather general assumptions on the non-linear constitutive relation between the magnetic field H and the magnetic induction B (i.e., the so called H-B curve). In this case, the source term enters in the model by setting the magnetic flux across a meridian section of the device. Existence of solution was obtained by applying an abstract result which needs, in particular, the strong monotonicity of the H-B curve. A full discretization to approximate this problem was also proposed in this reference and error estimates were obtained.

In the present paper we also focus on the axisymmetric eddy current model defined in a non-linear magnetic device. However, we consider the case in which the source term enters in the model by setting the magnetic field on the boundary, which results in a non-homogeneous Dirichlet boundary condition. This is shown to be a mathematically suitable condition for the problem to be well posed and, at the same time, it is physically realistic in the sense that there are industrial applications where it can be readily obtained from easily measurable quantities. This is the case, for instance, of the numerical simulation of eddy currents in metallurgical electrodes [4, 5, 14], induction heating systems [8] or current losses in a toroidal laminated core [18, 20]. In all these applications the Dirichlet boundary data for the magnetic field can be obtained from the current intensity. We notice that the non-homogeneous character of the boundary condition brings some technical complications in both the mathematical and the numerical analysis with respect to previous works on the subject [6, 24, 26].

In our case, the behavior of the material is defined by means of a general continuous and monotone non-linear H-B curve which (unlike in references [24] and [26]) may also depend on the position, what allows us to deal with heterogeneous media. Moreover, we also consider a time and space dependent electrical conductivity. This is important in practical applications, because this quantity is typically a function of temperature, which in its turn is a time dependent field.

By using classical weighted two-dimensional Sobolev spaces for axisymmetric problems, we prove the existence of a solution to a weak formulation in terms of the magnetic field. The technique used for this purpose (commonly known as the Rothe's method, see [22]) consists of introducing an implicit time discretization, obtaining a priori estimates and then passing to the limit as the time-step goes to zero. Let us remark that, to the best of the authors' knowledge, this problem does not fit in other existing results because of the time dependence of the coefficients as well as the non-homogeneous character of the boundary condition.

Under further assumptions, we also prove the uniqueness of solution and perform the numerical analysis of the problem. For the numerical solution, first the problem is discretized in time with a backward Euler scheme, which is proved to be well posed. Then, a full discrete approximation is introduced by using continuous piecewise linear functions on triangular meshes. Under appropriate assumptions, we analyze both, the semi- and the fully discrete schemes. For the former our analysis is based on [26]. For the latter we adapt the classical theory of linear parabolic equations (see, for instance, [28]), whereas to deal with the non-homogeneous Dirichlet boundary condition we resort to some arguments from [2]. Therefore, for the fully discrete problem, we obtain an L^2-like estimate without assuming any additional regularity of the solution. Moreover, under appropriate smoothness assumptions, we also obtain an optimal-order error estimate.

The paper is organized as follows. First, in Section 2, we describe the transient axisymmetric eddy current model and introduce the non-linear parabolic partial differential equation to be solved. In Section 3, we recall some functional spaces, establish a weak formulation of the problem and study its well posedness. Section 4 is devoted to the numerical analysis of the semi-discrete problem arising from a backward Euler time-discretization. In Section 5, we combine it with a finite element method for space discretization and prove stability and error estimates of the resulting full discretization. Finally, in Section 6, we report a numerical test which confirms the theoretical results.

2 The transient eddy current model

Eddy currents are usually modeled by the so called low-frequency Maxwell's equations (see, for instance, [1]):

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J}, \\ \frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0}, \\ \operatorname{div} \boldsymbol{B} &= 0. \end{aligned}$$

We have used above standard notations in electromagnetism: $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ the magnetic induction, $\boldsymbol{H}$ the magnetic field and $\boldsymbol{J}$ the current density. To obtain a closed system we need to add constitutive laws. On one hand, assuming that the materials are electrically linear, Ohm's law in conductors reads

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

where σ is the electrical conductivity, which is supposed to be bounded above and below away from zero. On the other hand, assuming that the magnetic materials are soft and hysteresis effects can be neglected, we may consider that $\boldsymbol{B}$ and $\boldsymbol{H}$ are related as follows:

$$\boldsymbol{B} = \boldsymbol{\mathcal{B}}(\boldsymbol{H}), \tag{2.1}$$

where $\boldsymbol{\mathcal{B}}$ is a non-linear mapping.

The above equations lead to the partial differential equation in conductors

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\left(\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\right) = \mathbf{0}, \tag{2.2}$$

which has to be solved together with the non-linear equation (2.1) and appropriate boundary and initial conditions.

2.1 Axisymmetric case

We restrict our attention to the case where a 3D conducting domain $\widetilde{\Omega}$ has cylindrical symmetry and all fields are independent of the angular variable θ. Then, in order to reduce the dimension and thereby the computational effort, it is convenient to consider a cylindrical coordinate system (r, θ, z). Let us denote by $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ the corresponding unit vectors of the local orthonormal basis.

We assume that the magnetic field is of the form

$$\boldsymbol{H}(r, z, t) = H(r, z, t)\boldsymbol{e}_\theta.$$

Then, assuming an isotropic behavior of the material, the magnetic induction $\boldsymbol{B}$ will be of the same form,

$$\boldsymbol{B}(r, z, t)) = B(r, z, t)\boldsymbol{e}_\theta,$$

and hence automatically divergence-free. Therefore, a scalar non-linear model

$$B(r, z, t) = \mathcal{B}(r, z, H(r, z, t)),$$

with $\mathcal{B}(r, z, \cdot)$ a non-linear mapping in $\mathbb{R}$, may be used to describe the H-B-relation.

Taking into account that

$$\mathbf{curl}\,\boldsymbol{H}(r, z, t) = -\frac{\partial H}{\partial z}(r, z, t)\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH)(r, z, t)\boldsymbol{e}_z, \tag{2.3}$$

it is straightforward to check that (2.2) is equivalent to the scalar partial differential equation

$$\frac{\partial B}{\partial t} - \frac{\partial}{\partial r}\left(\frac{1}{\sigma r}\frac{\partial (rH)}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H}{\partial z}\right) = 0,$$

which holds in any meridian section Ω of $\widetilde{\Omega}$ and for all time $t \in [0, T]$. In order to have a well-posed problem, we add an initial condition

$$B(r, z, 0) = B_0(r, z) \qquad \text{in } \Omega,$$

and suitable boundary condition on the boundary $\Gamma := \partial\Omega$. In view of applications, we consider a non-homogeneous Dirichlet boundary condition

$$H(r, z, t) = g(r, z, t) \qquad \text{on } \Gamma \times [0, T],$$

where g is a given function. For applications of this model, we refer for instance to [4, 8, 14], where this kind of problem arises in the simulation of metallurgical heating processes. We also refer to [18, 20], where it is shown how g can be obtained from the current intensity along the coil of a toroidal solenoid.

Altogether, the resulting axisymmetric problem consists of finding scalar fields $H(r, z, t)$ and $B(r, z, t)$ such that,

$$\frac{\partial B}{\partial t} - \frac{\partial}{\partial r}\left(\frac{1}{\sigma r}\frac{\partial (rH)}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H}{\partial z}\right) = f \qquad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$B = \mathcal{B}(H) \qquad \text{in } \Omega \times (0, T), \tag{2.5}$$

$$H = g \qquad \text{in } \Gamma \times (0, T), \tag{2.6}$$

$$B|_{t=0} = B_0 \qquad \text{in } \Omega, \tag{2.7}$$

where $\sigma(r, z, t)$, $f(r, z, t)$, $g(r, z, t)$ and $B_0(r, z)$ are given data. Notice that although most of the variables and coefficients are function of the space variables (r, z) and the time t, when there is no possibility of confusion we will not write explicitly this dependence, as in the equations above.

Remark 2.1 We have allowed a general right-hand side f in (2.4) in order to consider a more general parabolic problem, although in the eddy current model f is null.

3 Mathematical analysis

In this section, we make a precise statement of the problem to be solved by means of a weak formulation suitable for its mathematical analysis. Then, we prove the existence and, under additional assumptions, the uniqueness of a solution. First, we introduce some preliminary results which will be used along the paper.

3.1 Functional spaces and preliminary results

We define weighted Sobolev spaces appropriate for the mathematical analysis of the problem and recall some of their properties. For compactness of notation, from now on, the partial derivatives will be denoted by ∂_r, ∂_z and ∂_t.

Let $\Omega \subset \{(r, z) \in \mathbb{R}^2 : r > 0\}$ be a bounded domain with a Lipschitz boundary Γ. We denote by $\boldsymbol{n} = n_r \boldsymbol{e}_r + n_z \boldsymbol{e}_z$ and $\boldsymbol{t} = t_r \boldsymbol{e}_r + t_z \boldsymbol{e}_z$ (with $t_r := -n_z$ and $t_z := n_r$) the outer normal and tangent vectors to Ω. Let $\mathrm{L}^2_r(\Omega)$ denote the weighted Lebesgue space of all measurable functions u defined in Ω for which

$$\|u\|^2_{\mathrm{L}^2_r(\Omega)} := \int_\Omega |u|^2 \, r \, dr \, dz < \infty.$$

Given $k \in \mathbb{N}$, the weighted Sobolev space $\mathrm{H}^k_r(\Omega)$ consists of all functions in $\mathrm{L}^2_r(\Omega)$ whose derivatives up to order k are also in $\mathrm{L}^2_r(\Omega)$. We define the norms and semi-norms of these spaces in the standard way; for instance,

$$|u|^2_{\mathrm{H}^1_r(\Omega)} := \int_\Omega \left(|\partial_r u|^2 + |\partial_z u|^2\right) r \, dr \, dz.$$

Let

$$\widetilde{\mathrm{H}}^1_r(\Omega) := \mathrm{H}^1_r(\Omega) \cap \mathrm{L}^2_{1/r}(\Omega),$$

where $\mathrm{L}^2_{1/r}(\Omega)$ denotes the set of all measurable functions u defined in Ω for which

$$\|u\|^2_{\mathrm{L}^2_{1/r}(\Omega)} := \int_\Omega \frac{|u|^2}{r}\, dr\, dz < \infty.$$

$\widetilde{\mathrm{H}}^1_r(\Omega)$ is a Hilbert space with the norm

$$\|u\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} := \|u\|^2_{\mathrm{H}^1_r(\Omega)} + \|u\|^2_{\mathrm{L}^2_{1/r}(\Omega)}.$$

Let $\widetilde{\mathrm{H}}^2_r(\Omega) := \left\{ u \in \widetilde{\mathrm{H}}^1_r(\Omega) :\ \|u\|_{\widetilde{\mathrm{H}}^2_r(\Omega)} < \infty \right\}$, where

$$\|u\|^2_{\widetilde{\mathrm{H}}^2_r(\Omega)} := \|u\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + \left| \frac{1}{r}\partial_r(ru) \right|^2_{\mathrm{H}^1_r(\Omega)} + \|\partial_z u\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}.$$

Finally we recall from [12, Section 3] that functions in $\widetilde{\mathrm{H}}^1_r(\Omega)$ have traces on Γ. We denote

$$\widetilde{\mathrm{H}}^{1/2}_r(\Gamma) := \left\{ v|_\Gamma :\ v \in \widetilde{\mathrm{H}}^1_r(\Omega) \right\},$$

endowed with the norm

$$\|g\|_{\widetilde{\mathrm{H}}^{1/2}_r(\Gamma)} := \inf \left\{ \|v\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} :\ v \in \widetilde{\mathrm{H}}^1_r(\Omega) \text{ with } v|_\Gamma = g \right\},$$

which makes the trace operator $v \to v|_\Gamma$ continuous from $\widetilde{\mathrm{H}}^1_r(\Omega)$ onto $\widetilde{\mathrm{H}}^{1/2}_r(\Gamma)$.

3.2 Weak formulation

In order to establish a weak formulation of the above problem, we consider the following subspace of $\widetilde{\mathrm{H}}^1_r(\Omega)$:

$$\mathcal{U} := \left\{ G \in \widetilde{\mathrm{H}}^1_r(\Omega) :\ G|_\Gamma = 0 \right\}.$$

We multiply equation (2.4) by rG, with G being a test function in $\mathcal{U}$, integrate in Ω and use a Green's formula, to obtain the following weak formulation of (2.4)–(2.7):

Problem 3.1 Given $g \in \mathrm{L}^\infty(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))$, $f \in \mathrm{L}^\infty(0,T;\mathcal{U}')$ and $B_0 \in \mathrm{L}^2_r(\Omega)$, find $H \in \mathrm{L}^\infty(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$ and $B \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_r(\Omega))$ with $\partial_t B \in \mathrm{L}^\infty(0,T;\mathcal{U}')$, such that

$$\begin{aligned} \langle \partial_t B, G\rangle + a_t(H,G) &= \langle f, G\rangle && \forall G \in \mathcal{U}, \quad \text{a.e. in } [0,T],\\ B &= \mathcal{B}(H) && \text{in } \Omega \times (0,T),\\ H &= g && \text{in } \Gamma \times (0,T),\\ B|_{t=0} &= B_0 && \text{in } \Omega. \end{aligned}$$

In the first equation above, $a_t :\ \widetilde{\mathrm{H}}^1_r(\Omega) \times \widetilde{\mathrm{H}}^1_r(\Omega) \to \mathbb{R}$ denotes the bilinear form defined by

$$a_t(G_1,G_2) := \int_\Omega \frac{1}{\sigma(\cdot,t)r} \Big(\partial_r(rG_1)\,\partial_r(rG_2) + \partial_z(rG_1)\,\partial_z(rG_2) \Big)\, dr\, dz$$

and $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathcal{U}$ and its dual space $\mathcal{U}'$.

3.3 Existence and uniqueness

We introduce the following hypotheses that will be used to prove the existence of a solution to the above problem:

H.1: The mapping $\mathcal{B}: \ \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, namely,
- $\mathcal{B}(\cdot, u): \ \Omega \to \mathbb{R}$ is measurable for each $u \in \mathbb{R}$,
- $\mathcal{B}(r, z, \cdot): \ \mathbb{R} \to \mathbb{R}$ is continuous for each $(r, z) \in \Omega$.

H.2: $\mathcal{B}(r, z, u)$ is monotone with respect to u, namely,

$$(\mathcal{B}(r,z,u) - \mathcal{B}(r,z,v))\,(u-v) \geq 0 \qquad \forall u, v \in \mathbb{R}, \quad \text{a.e. } (r,z) \in \Omega.$$

H.3: There exist $a_0 \in \mathrm{L}^2_r(\Omega)$ and $b_0 \geq 0$ such that

$$|\mathcal{B}(\cdot, v)| \leq a_0(\cdot) + b_0\,|v| \qquad \forall v \in \mathbb{R}.$$

H.4: The electrical conductivity $\sigma: \ \Omega \times (0,T) \to \mathbb{R}$ belongs to $\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^\infty(\Omega))$ and there exist strictly positive constants σ_* and σ^* such that

$$\sigma_* \leq \sigma(r,z,t) \leq \sigma^* \qquad \text{a.e. } (r,z,t) \in \Omega \times (0,T).$$

H.5: There exists $H_0 \in \widetilde{\mathrm{H}}^1_r(\Omega)$ such that

$$B_0(r,z) = \mathcal{B}(r,z,H_0(r,z)) \qquad \text{a.e. } (r,z) \in \Omega.$$

H.6: There holds $g \in \mathrm{H}^2(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))$ and $f \in \mathrm{H}^1(0,T;\mathcal{U}')$.

From the boundedness assumption on σ, we derive the following result.

Lemma 3.1 *The bilinear forms a_t are continuous uniformly in $t \in [0,T]$. Moreover, they are elliptic also uniformly in $t \in [0,T]$; namely,*

$$a_t(G,G) \geq \gamma\,\|G\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \qquad \forall G \in \mathcal{U},$$

where γ is a positive constant depending only on σ^* and Ω.

Proof For the continuity, it is immediate to check that, for all $G_1, G_2 \in \widetilde{\mathrm{H}}^1_r(\Omega)$, $a_t(G_1,G_2) \leq \frac{2}{\sigma_*}\,\|G_1\|_{\widetilde{\mathrm{H}}^1_r(\Omega)}\,\|G_2\|_{\widetilde{\mathrm{H}}^1_r(\Omega)}$. The ellipticity follows from the fact that

$$a_t(G,G) \geq \frac{1}{\sigma^*}\left(|G|^2_{\mathrm{H}^1_r(\Omega)} + \|G\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + 2\int_\Omega G\,(\partial_r G)\;dr\,dz\right)$$

and

$$2\int_\Omega G\,(\partial_r G)\;dr\,dz = \int_\Omega \partial_r(G^2)\,dr\,dz = \int_\Gamma G^2 n_r\,dS = 0 \qquad \forall G \in \mathcal{U}.$$

□

Now, for each $t \in [0,T]$, let $H_g(t) \in \widetilde{\mathrm{H}}^1_r(\Omega)$ be the unique solution of the Dirichlet problem

$$\begin{aligned} (H_g(t), w)_{\widetilde{\mathrm{H}}^1_r(\Omega)} &= 0 \qquad && \forall w \in \mathcal{U}, \\ H_g(t) &= g(t) && \text{on } \Gamma, \end{aligned}$$

where $(\cdot,\cdot)_{\widetilde{\mathrm{H}}^1_r(\Omega)}$ denotes the Hilbert product in $\widetilde{\mathrm{H}}^1_r(\Omega)$. It is easy to check that $\|H_g(t)\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} = \|g(t)\|_{\widetilde{\mathrm{H}}^{1/2}_r(\Gamma)}$ for all $t \in [0,T]$ and, by virtue of H.6, $H_g \in \mathrm{H}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$ with

$$\|H_g\|_{\mathrm{H}^k(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))} = \|g\|_{\mathrm{H}^k(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))}, \qquad k = 0,1,2. \tag{3.1}$$

In order to prove that Problem 3.1 has a solution, we write $H = H_u + H_g$, with H_g as defined above. Clearly $H_u \in \mathcal{U}$ for all $t \in [0,T]$. Then, Problem 3.1 is equivalent to finding $H_u \in \mathrm{L}^\infty(0,T;\mathcal{U})$ and $B \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_r(\Omega))$ with $\partial_t B \in \mathrm{L}^\infty(0,T;\mathcal{U}')$ such that

$$\langle \partial_t B, G\rangle + a_t(H_u, G) = \langle F(t), G\rangle \qquad \forall G \in \mathcal{U}, \quad \text{a.e. in } [0,T], \tag{3.2}$$

$$B = \widetilde{\mathcal{B}}(H_u, t) \qquad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$H_u|_{t=0} = H_0 - H_g(0) \qquad \text{in } \Omega, \tag{3.4}$$

where $\widetilde{\mathcal{B}} :\ \Omega \times \mathbb{R} \times [0,T] \to \mathbb{R}$ is defined by

$$\widetilde{\mathcal{B}}(r,z,v,t) := \mathcal{B}(r,z,v + H_g(r,z,t)), \qquad (r,z) \in \Omega, \quad t \in [0,T], \quad v \in \mathbb{R},$$

and $F :\ [0,T] \to \mathcal{U}'$ by

$$\langle F(t), G\rangle := \langle f(t), G\rangle - a_t(H_g(t), G) \qquad \forall G \in \mathcal{U}, \quad t \in [0,T].$$

It easy to check that $\widetilde{\mathcal{B}}$ satisfies a monotonicity property similar to H.2, namely,

$$(\widetilde{\mathcal{B}}(r,z,v,t) - \widetilde{\mathcal{B}}(r,z,w,t))(v-w) \geq 0 \quad \forall v,w \in \mathbb{R} \quad \forall (r,z,t) \in \Omega \times [0,T]. \tag{3.5}$$

Moreover, from the regularity of H_g, we have that $\widetilde{\mathcal{B}}(\cdot,t)$ is a Carathéodory function for all $t \in [0,T]$ (cf. Hypothesis H.1) and that there exist $b_0 \geq 0$ (the same as in H.3) and

$$\widetilde{a}_0(\cdot,t) := a_0(\cdot) + b_0\,|H_g(\cdot,t)| \in \mathrm{L}^2_r(\Omega), \quad t \in [0,T],$$

such that

$$\left|\widetilde{\mathcal{B}}(\cdot,v,t)\right| \leq \widetilde{a}_0(\cdot,t) + b_0\,|v| \qquad \forall (v,t) \in \mathbb{R} \times [0,T]. \tag{3.6}$$

To prove the existence of solution, we proceed by classical arguments of time discretization, a priori estimates and passing to the limit. First, we introduce the linear operators $A(t) :\ \widetilde{\mathrm{H}}^1_r(\Omega) \to \widetilde{\mathrm{H}}^1_r(\Omega)'$ induced by $a_t(\cdot,\cdot)$ (i.e., $A(t)G := a_t(G,\cdot)$, $G \in \widetilde{\mathrm{H}}^1_r(\Omega)$), which by virtue of Lemma 3.1 are bounded uniformly in $t \in [0,T]$.

3.3.1 Time discretization

Let us fix $m \in \mathbb{N}$ and set $\Delta t := T/m$. For $i = 0, \ldots, m$, we define $t^i := i\Delta t$, $H_g^i(r,z) := H_g(r,z,t^i)$, $f^i(r,z) := f(r,z,t^i)$, $\sigma^i(r,z) := \sigma(r,z,t^i)$, $A^i := A(t^i)$ and $F^i := F(t^i)$. Notice that all these terms are well defined because σ, f and g are continuous in time, as a consequence of H.4 and H.6. Moreover, there holds

$$\langle F^i, G\rangle = \langle f^i, G\rangle - \langle A^i H_g^i, G\rangle \qquad \forall G \in \mathcal{U}, \qquad i = 0, \ldots, m. \tag{3.7}$$

A time discretization of (3.2)–(3.4) based on a backward Euler scheme reads as follows: find $H_u^i \in \mathcal{U}$ and $B^i \in \mathcal{U}'$, $i = 0, \ldots, m$, satisfying

$$\bar{\partial} B^{i+1} + A^{i+1} H_u^{i+1} = F^{i+1} \qquad \text{in } \mathcal{U}', \quad i = 0, \ldots, m-1, \tag{3.8}$$

$$B^i = \widetilde{\mathcal{B}}(H_u^i, t^i), \qquad i = 0, \ldots, m, \tag{3.9}$$

$$H_u^0 = H_0 - H_g^0 \qquad \text{in } \Omega, \tag{3.10}$$

where $\bar{\partial} B^{i+1}$ denotes the difference quotient $\bar{\partial} B^{i+1} := \left(B^{i+1} - B^i\right)/\Delta t$.

The existence of a weak solution to the problem above at each time step is guaranteed by the following lemma.

Lemma 3.2 *There exists a unique solution of* (3.8)–(3.10).

Proof First, for each $j = 0, \ldots, m$, let us define $\widetilde{\mathcal{B}}^j : \mathrm{L}_r^2(\Omega) \to \mathrm{L}_r^2(\Omega)$ as follows: given $G \in \mathrm{L}_r^2(\Omega)$, $\widetilde{\mathcal{B}}^j(G)(r,z) := \widetilde{\mathcal{B}}(r,z,G(r,z),t^j)$, $(r,z) \in \Omega$. From (3.6) and the fact that $\widetilde{\mathcal{B}}(\cdot,t)$ is a Carathéodory function for all $t \in [0,T]$, we have that $\widetilde{\mathcal{B}}^j$ is continuous (see, for instance, [15, Lemma 16.1]).

Next, we notice that H_u^{i+1} is a solution of (3.8)–(3.9) if and only if it is a solution of the following non-linear problem:

$$Z(H_u^{i+1}) := \frac{\widetilde{\mathcal{B}}^{i+1}(H_u^{i+1})}{\Delta t} + A^{i+1} H_u^{i+1} = F^{i+1} + \frac{\widetilde{\mathcal{B}}^{i}(H_u^{i})}{\Delta t} \qquad \text{in } \mathcal{U}',$$

Since $\widetilde{\mathcal{B}}^{i+1}$ is monotone (cf. (3.5)), continuous and $A^{i+1} : \mathcal{U} \to \mathcal{U}'$ is linear, bounded and elliptic, it is easy to check that $Z : \mathcal{U} \to \mathcal{U}'$ is strongly monotone, coercive and continuous. Thus, from the theory of monotone operators, it follows that the equation above has a unique solution (see, for instance, [23, Theorem 2.18]). ☐

3.3.2 A priori estimates

The next goal is to prove an a priori estimate for the solution of (3.8)–(3.10). Notice that if $\mathcal{B}$ were strongly monotone and Lipschitz continuous, then the results from [27, Lemma 3.1] could be applied with this purpose. Since this is not our case, the proof will follow an alternative path.

Here and thereafter C with or without subscripts will be used for positive constants not necessarily the same at each occurrence, but always independent of the time-step Δt and, in the following section, of the mesh-size h, too.

Lemma 3.3 *There exists $C > 0$ such that, for all $l = 0, \ldots, m-1$,*

$$\left\|\bar{\partial} B^{l+1}\right\|_{\mathcal{U}'}^2 + \left\|H_u^{l+1}\right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2 \leq C.$$

Proof We apply (3.8) to $\left(H_u^{i+1} - H_u^i\right) \in \mathcal{U}$. From the monotonicity property H.2, it is straightforward to obtain for $l = 0, \ldots, m-1$,

$$\sum_{i=0}^{l} \langle A^{i+1} H_u^{i+1}, H_u^{i+1} - H_u^i \rangle \leq \sum_{i=0}^{l} \langle F^{i+1}, H_u^{i+1} - H_u^i \rangle - \sum_{i=0}^{l} \int_\Omega \bar{\partial} B^{i+1} \left(H_g^{i+1} - H_g^i\right) r\, dr\, dz. \tag{3.11}$$

To bound the term on the left-hand side of the above equation, first we use the classical identity $2(p-q)p = p^2 + (p-q)^2 - q^2$ to write

$$\begin{aligned} 2\langle A^{i+1} H_u^{i+1}, H_u^{i+1} - H_u^i \rangle &\geq \langle A^{i+1} H_u^{i+1}, H_u^{i+1} \rangle - \langle A^{i+1} H_u^i, H_u^i \rangle \\ &= \langle A^{i+1} H_u^{i+1}, H_u^{i+1} \rangle - \langle A^i H_u^i, H_u^i \rangle + \langle (A^i - A^{i+1}) H_u^i, H_u^i \rangle. \end{aligned} \tag{3.12}$$

Now, for the last term on the right-hand side we have that

$$\begin{aligned} \left| \langle (A^i - A^{i+1}) H_u^i, H_u^i \rangle \right| &= \left| \int_\Omega \frac{\sigma^{i+1} - \sigma^i}{\sigma^{i+1}\sigma^i} \frac{1}{r} \left(\left|\partial_r (r H_u^i)\right|^2 + \left|\partial_z (r H_u^i)\right|^2 \right) dr\, dz \right| \\ &\leq \frac{1}{\sigma_*^2} \left\| \partial_t \sigma \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \Delta t \left\| H_u^i \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}, \end{aligned} \tag{3.13}$$

where we have used that $\sigma^{i+1} - \sigma^i = \int_{t^i}^{t^{i+1}} \partial_t \sigma(s)\, ds$ and assumption H.4. Therefore, summing up (3.12) and using (3.13) and Lemma 3.1, it follows that

$$\begin{aligned} \sum_{i=0}^{l} \langle A^{i+1} H_u^{i+1}, H_u^{i+1} - H_u^i \rangle \geq \frac{\gamma}{2} \left\| H_u^{l+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} - \frac{1}{2\sigma_*} \left\| H_u^0 \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \\ - \frac{\left\| \partial_t \sigma \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}}{2\sigma_*^2} \Delta t \sum_{i=0}^{l} \left\| H_u^i \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}. \end{aligned} \tag{3.14}$$

On the other hand, for the first term on the right-hand side of (3.11), by summation by parts and using Young's inequality, we obtain for all $\eta > 0$

$$\begin{aligned} &\left| \sum_{i=0}^{l} \langle F^{i+1}, H_u^{i+1} - H_u^i \rangle \right| \\ &= \left| \langle F^{l+1}, H_u^{l+1} \rangle - \langle F^1, H_u^0 \rangle - \sum_{i=0}^{l-1} \langle F^{i+2} - F^{i+1}, H_u^{i+1} \rangle \right| \\ &\leq \frac{1}{2\eta} \left\| F^{l+1} \right\|^2_{\mathcal{U}'} + \frac{\eta}{2} \left\| H_u^{l+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + \left\| F^1 \right\|_{\mathcal{U}'} \left\| H_u^0 \right\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \\ &\quad + \Delta t \sum_{i=0}^{l-1} \left\| H_u^{i+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + C \left\| \partial_t \sigma \right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \Delta t \sum_{i=0}^{l-1} \left\| H_g^{i+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \\ &\quad + C \Delta t \sum_{i=0}^{l-1} \left\| \bar{\partial} H_g^{i+2} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + C \Delta t \sum_{i=0}^{l-1} \left\| \bar{\partial} f^{i+2} \right\|^2_{\mathcal{U}'}, \end{aligned} \tag{3.15}$$

where the last three terms are derived by proceeding as in (3.13) from the following inequality (cf. (3.7)):

$$\left\|\bar{\partial}F^{i+2}\right\|_{\mathcal{U}'} \leq \left\|\bar{\partial}f^{i+2}\right\|_{\mathcal{U}'} + \left\|A^{i+2}\bar{\partial}H_g^{i+2}\right\|_{\mathcal{U}'} + \left\|\bar{\partial}A^{i+2}H_g^{i+1}\right\|_{\mathcal{U}'}.$$

Similarly, for the last term of (3.11), by summation by parts and using Young's inequality and (3.6), it follows that for all $\eta > 0$

$$\begin{aligned}
&\left|\sum_{i=0}^{l}\int_{\Omega}\bar{\partial}B^{i+1}\left(H_g^{i+1}-H_g^{i}\right)r\,dr\,dz\right| \\
&\quad = \left|\sum_{i=0}^{l}\int_{\Omega}\left(B^{i+1}-B^{i}\right)\bar{\partial}H_g^{i+1}r\,dr\,dz\right| \\
&\quad \leq \int_{\Omega}\left|B^{l+1}\bar{\partial}H_g^{l+1}\right|r\,dr\,dz + \int_{\Omega}\left|B^{0}\bar{\partial}H_g^{1}\right|r\,dr\,dz \\
&\qquad + \sum_{i=0}^{l-1}\int_{\Omega}\left|B^{i+1}\left(\bar{\partial}H_g^{i+2}-\bar{\partial}H_g^{i+1}\right)\right|r\,dr\,dz \\
&\quad \leq \frac{\eta}{2}\left\|H_u^{l+1}\right\|^2_{\mathrm{L}^2_r(\Omega)} + C\eta\left\|H_g^{l+1}\right\|^2_{\mathrm{L}^2_r(\Omega)} + \frac{C}{\eta}\left\|\bar{\partial}H_g^{l+1}\right\|^2_{\mathrm{L}^2_r(\Omega)} \\
&\qquad + \left\|B^{0}\right\|_{\mathrm{L}^2_r(\Omega)}\left\|\bar{\partial}H_g^{1}\right\|_{\mathrm{L}^2_r(\Omega)} + \Delta t\sum_{i=0}^{l-1}\left\|H_u^{i+1}\right\|^2_{\mathrm{L}^2_r(\Omega)} + C\Delta t\sum_{i=0}^{l-1}\left\|H_g^{i+1}\right\|^2_{\mathrm{L}^2_r(\Omega)} \\
&\qquad + C + C\Delta t\sum_{i=0}^{l-1}\left\|\frac{\bar{\partial}H_g^{i+2}-\bar{\partial}H_g^{i+1}}{\Delta t}\right\|^2_{\mathrm{L}^2_r(\Omega)}.
\end{aligned} \tag{3.16}$$

Whence, by replacing (3.14)–(3.16) into (3.11), choosing $\eta := \gamma/4$ and using that the last terms in (3.15) and (3.16) are respectively bounded by $\|f\|^2_{\mathrm{H}^1(0,T;\mathcal{U}')}$ and $\|g\|^2_{\mathrm{H}^2(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))}$ (cf. (3.1)), we obtain

$$\frac{\gamma}{4}\left\|H_u^{l+1}\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C + \Delta t\sum_{i=0}^{l-1}\left\|H_u^{i+1}\right\|^2_{\mathrm{L}^2_r(\Omega)}.$$

Therefore, using a discrete Gronwall's lemma we arrive at

$$\left\|H_u^{l+1}\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C,$$

with a constant C depending on $\|H_0\|_{\widetilde{\mathrm{H}}^1_r(\Omega)}$, $\|f\|_{\mathrm{H}^1(0,T;\mathcal{U}')}$, $\|g\|_{\mathrm{H}^2(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))}$ and $\|\sigma\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^\infty(\Omega))}$.

Finally, to end the theorem, we bound $\left\|\bar{\partial}B^{l+1}\right\|^2_{\mathcal{U}'}$ by using (3.8) and the above inequality. □

3.3.3 Convergence

The next step is to define approximate solutions to (3.2)–(3.4) and prove its weak convergence to an actual solution of this problem. With this aim, we introduce some notation. Let $B_{\Delta t}: [0,T] \to \mathcal{U}'$ be the piecewise linear continuous in time function given by

$$B_{\Delta t}(t^0) := \widetilde{\mathcal{B}}(H_u^0, t^0);$$
$$B_{\Delta t}(t) := \widetilde{\mathcal{B}}(H_u^{i-1}, t^{i-1}) + (t - t^{i-1})\,\bar{\partial}\widetilde{\mathcal{B}}(H_u^i, t^i), \quad t \in (t^{i-1}, t^i], \quad i = 1, \dots, m.$$

Notice that, by virtue of (3.6), $B_{\Delta t}$ actually takes values in $\mathrm{L}^2_r(\Omega)$. We also consider the step function $\overline{H}_{u\Delta t}: [0,T] \to \mathcal{U}$ defined as follows:

$$\overline{H}_{u\Delta t}(t^0) := H_u^0; \qquad \overline{H}_{u\Delta t}(t) := H_u^i, \quad t \in (t^{i-1}, t^i], \quad i = 1, \dots, m.$$

Step functions $\overline{B}_{\Delta t}$, $\overline{H}_{g\Delta t}$, $\overline{A}_{\Delta t}$, $\overline{f}_{\Delta t}$ and $\overline{\sigma}_{\Delta t}$ are defined in a similar way.

Using the above notation and (3.7), we rewrite equation (3.8) as follows:

$$\partial_t B_{\Delta t} + \overline{A}_{\Delta t}\overline{H}_{u\Delta t} = \overline{f}_{\Delta t} - \overline{A}_{\Delta t}\overline{H}_{g\Delta t} \qquad \text{in } \mathcal{U}', \qquad \text{a.e. in } (0,T). \tag{3.17}$$

From Lemma 3.3, (3.6) and (3.1), we deduce that there exists $C > 0$ such that

$$\left\|\overline{B}_{\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2_r(\Omega))} + \left\|\partial_t B_{\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\mathcal{U}')} + \left\|\overline{A}_{\Delta t}\overline{H}_{u\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\widetilde{\mathrm{H}}^1_r(\Omega)')} + \left\|\overline{H}_{u\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))} \le C. \tag{3.18}$$

This allows us to conclude that there exists H_u, B and X such that

$$\overline{H}_{u\Delta t} \to H_u \qquad \text{in } \mathrm{L}^\infty(0,T;\mathcal{U}) \text{ weakly star,} \tag{3.19}$$
$$B_{\Delta t} \to B \qquad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2_r(\Omega)) \text{ weakly star,} \tag{3.20}$$
$$\partial_t B_{\Delta t} \to \partial_t B \qquad \text{in } \mathrm{L}^\infty(0,T;\mathcal{U}') \text{ weakly star,} \tag{3.21}$$
$$\overline{A}_{\Delta t}(\overline{H}_{u\Delta t} + \overline{H}_{g\Delta t}) \to X \qquad \text{in } \mathrm{L}^\infty(0,T;\mathcal{U}') \text{ weakly star.} \tag{3.22}$$

Hence, taking limit in (3.17), it follows that

$$\partial_t B + X = f \qquad \text{in } \mathcal{U}', \quad \text{a.e. in } (0,T), \tag{3.23}$$

because $\overline{f}_{\Delta t} \to f$ in $\mathrm{L}^2(0,T;\mathcal{U}')$, for $f \in \mathrm{H}^1(0,T;\mathcal{U}')$. Next step is to derive that $B = \mathcal{B}(H_u + H_g)$ and $X = A(H_u + H_g)$. With this end, first we prove the following.

Lemma 3.4 *$B_{\Delta t} \to B$ strongly in $\mathcal{C}([0,T];\mathcal{U}')$.*

Proof As a consequence of Lemma 3.3, it is easy to check that the family of functions $\left\{B_{\Delta t}: [0,T] \to \mathcal{U}'\right\}_{\Delta t}$ is equicontinuous. Moreover, $\{B_{\Delta t}(t)\}_{\Delta t}$ is relatively compact in $\mathcal{U}'$ for each $t \in [0,T]$. In fact, because of (3.18), $\{B_{\Delta t}(t)\}_{\Delta t}$ is a bounded set in $\mathrm{L}^2_r(\Omega)$, which is compactly included in $\mathcal{U}'$ (the latter because the inclusion $\widetilde{\mathrm{H}}^1_r(\Omega) \subset \mathrm{L}^2_r(\Omega)$ is compact; see, for instance, [19]). Therefore, by applying the Ascoli's theorem (see, for instance, [17]), we obtain that $\left\{B_{\Delta t}: [0,T] \to \mathcal{U}'\right\}_{\Delta t}$ is relatively compact in $\mathcal{C}([0,T];\mathcal{U}')$. This together with (3.20) allow us to conclude that the convergence $B_{\Delta t} \to B$ is strong in $\mathcal{C}([0,T];\mathcal{U}')$. □

Now we are in a position to prove the following two lemmas.

Lemma 3.5 *Let H_u and B be the weak star limits defined in* (3.19) *and* (3.20), *respectively. Then,*

$$B = \mathcal{B}(H_u + H_g) \qquad a.e.\ in\ \Omega \times [0,T].$$

Proof From Lemmas 3.3 and 3.4, we have that

$$\left\|\overline{B}_{\Delta t} - B\right\|_{\mathrm{L}^\infty(0,T;\mathcal{U}')} \leq \left\|\overline{B}_{\Delta t} - B_{\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\mathcal{U}')} + \left\|B_{\Delta t} - B\right\|_{\mathrm{L}^\infty(0,T;\mathcal{U}')} \to 0.$$

From the latter and the weak star convergence of $\overline{H}_{u\Delta t}$, it follows that

$$\int_0^T \langle \overline{B}_{\Delta t}, \overline{H}_{u\Delta t}\rangle \, dt \to \int_0^T \langle B, H_u\rangle \, dt.$$

On the other hand, from the monotonicity of $\mathcal{B}$ and the fact that $\overline{B}_{\Delta t} = \mathcal{B}(\overline{H}_{u\Delta t} + \overline{H}_{g\Delta t})$, we have that

$$\int_0^T \langle \overline{B}_{\Delta t} - \mathcal{B}(G + \overline{H}_{g\Delta t}), \overline{H}_{u\Delta t} - G\rangle \, dt \geq 0 \qquad \forall G \in \mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega)).$$

Since $\overline{H}_{g\Delta t}$ converges to H_g in $\mathrm{L}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$ and also a.e. in $\Omega \times [0,T]$, because of hypothesis H.1 we have that $\mathcal{B}(G + \overline{H}_{g\Delta t})$ converges to $\mathcal{B}(G + H_g)$ a.e. in $\Omega \times [0,T]$. Hence, this convergence also holds strongly in $\mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega))$ because of hypothesis H.3 and the Lebesgue dominated convergence theorem. Thus, we obtain

$$\int_0^T \int_\Omega (B - \mathcal{B}(G + H_g))(H_u - G)\, r\, dr\, dz\, dt \geq 0 \qquad \forall G \in \mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega)).$$

Now, by taking $G := H_u + \epsilon U$, for any $U \in \mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega))$ and $\epsilon > 0$, we arrive at

$$\int_0^T \int_\Omega (B - \mathcal{B}(H_u + H_g - \epsilon U))U\, r\, dr\, dz\, dt \leq 0.$$

By taking $\epsilon \to 0$ and choosing $U := B - \mathcal{B}(H_u + H_g)$, it follows that $B = \mathcal{B}(H_u + H_g)$ a.e. in $\Omega \times [0,T]$ and we obtain the result. □

Lemma 3.6 *Let H_u and X be the weak star limits defined in* (3.19) *and* (3.22), *respectively. Then,*

$$X = A(H_u + H_g) \qquad a.e.\ in\ [0,T].$$

Proof First notice that for all $G \in \mathrm{L}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$ and all $U \in \mathcal{U}$

$$\begin{aligned}\langle \overline{A}_{\Delta t} G - AG, U\rangle &= \int_\Omega \frac{\sigma - \overline{\sigma}_{\Delta t}}{\sigma \overline{\sigma}_{\Delta t}} \Big(\partial_r(rG)\,\partial_r(rU) + \partial_z(rG)\,\partial_r(rU)\Big)\frac{1}{r}\, dr\, dz \\ &\leq C_\sigma \left\|\sigma - \overline{\sigma}_{\Delta t}\right\|_{\mathrm{L}^\infty(\Omega)} \left\|G\right\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \left\|U\right\|_{\widetilde{\mathrm{H}}^1_r(\Omega)}.\end{aligned}$$

Moreover, since $\sigma \in \mathrm{W}^{1,\infty}(0,T;\mathrm{L}^\infty(\Omega))$, it follows that

$$\left\|\sigma - \overline{\sigma}_{\Delta t}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \leq \Delta t \left\|\partial_t \sigma\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}$$

and, therefore,

$$\left\|\overline{A}_{\Delta t}G - AG\right\|^2_{\mathrm{L}^2(0,T;\mathcal{U}')} \leq C_\sigma \Delta t^2 \left\|\partial_t\sigma\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \left\|G\right\|^2_{\mathrm{L}^2(0,T,\widetilde{\mathrm{H}}^1_r(\Omega))}.$$

Finally, from the above result, Lemma 3.3, (3.19) and the fact that $\overline{H}_{g\Delta t} \to H_g$ in $\mathrm{L}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$, we obtain for all $V \in \mathrm{L}^2(0,T;\mathcal{U})$

$$\begin{aligned}
&\left|\int_0^T \langle \overline{A}_{\Delta t}(\overline{H}_{u\Delta t} + \overline{H}_{g\Delta t}) - A(H_u + H_g), V\rangle \, dt\right| \\
&\qquad \leq \left\|(\overline{A}_{\Delta t} - A)(\overline{H}_{u\Delta t} + \overline{H}_{g\Delta t})\right\|_{\mathrm{L}^2(0,T;\mathcal{U}')} \left\|V\right\|_{\mathrm{L}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))} \\
&\qquad\quad + \left|\int_0^T \langle A(\overline{H}_{u\Delta t} - H_u), V\rangle \, dt\right| + \left|\int_0^T \langle A(\overline{H}_{g\Delta t} - H_g), V\rangle \, dt\right| \longrightarrow 0.
\end{aligned}$$

Whence, from (3.22), $X = A(H_u + H_g)$ a.e. in $[0,T]$ and we end the proof. □

Now we are in a position to conclude that Problem 3.1 has a solution.

Theorem 3.1 *Under assumptions H.1–H.6, Problem 3.1 has a solution.*

Proof Let $H := H_u + H_g$. It follows from (3.23) and Lemma 3.6 that

$$\langle \partial_t B, G\rangle + a_t(H,G) = \langle f, G\rangle \qquad \forall G \in \mathcal{U}, \quad \text{a.e. in } (0,T).$$

and, from Lemma 3.5, that $B = \mathcal{B}(H)$. On the other hand, since $H_u \in \mathcal{U}$, we have that $H|_\Gamma = g$. Finally, as a consequence of Lemma 3.4 we have that $B_{\Delta t}(0) \to B(0)$ in $\mathcal{U}'$. Hence, since $B_{\Delta t}(0) = \widetilde{\mathcal{B}}(H_u^0, t^0) = \mathcal{B}(H_u^0 + H_g(0)) = \mathcal{B}(H_0) = B_0$, we conclude that $B(0) = B_0$. Therefore (H,B) is a solution to Problem 3.1. □

In order to prove that Problem 3.1 has a unique solution, we will assume from now on the following strengthened forms of hypotheses H.2 and H.4:

H.2*: $\mathcal{B}(r,z,u)$ is strongly monotone with respect to u uniformly in Ω; namely, there exists $\beta > 0$ such that

$$\left(\mathcal{B}(r,z,v) - \mathcal{B}(r,z,w)\right)(v-w) \geq \beta \left|v-w\right|^2 \qquad \forall v,w \in \mathbb{R}, \quad \text{a.e. } (r,z) \in \Omega.$$

H.4*: σ does not depend on time and there exist strictly positive constants σ_* and σ^* such that $\sigma_* \leq \sigma(r,z) \leq \sigma^*$ a.e. in Ω.

Hypothesis H.2* is a recurrent assumption in electromagnetism which covers a large number of models of physical interest (see [6, 24–26]). On the other hand, notice that from H.4* and the definition of $a_t(\cdot,\cdot)$, it follows that this bilinear form is also time independent. Thus, from now on, we will denote it $a(\cdot,\cdot)$.

As a first consequence of these hypotheses we can prove further regularity of the solution to Problem 3.1 and its uniqueness.

Theorem 3.2 *Under assumptions H.1, H.2*, H.3, H.4*, H.5 and H.6, Problem 3.1 has a unique solution* (H,B) *and there holds* $H \in \mathrm{H}^1(0,T;\mathrm{L}^2_r(\Omega))$.

Proof The existence of solution follows from Theorem 3.1. The uniqueness is a consequence of [10, Theorem 4]. For the additional regularity, we notice that being $\mathcal{B}$ strongly monotone (H.2*), by applying (3.8) to $(H_u^{i+1} - H_u^i)$ it is straightforward to prove that

$$\Delta t \sum_{i=0}^{l} \left\| \bar{\partial} H_u^{i+1} \right\|_{\mathrm{L}_r^2(\Omega)}^2 \leq C, \qquad l = 0, \ldots, m-1. \tag{3.24}$$

The rest of the proof consists of adapting the previous arguments and using the a priori estimate above. Let us remark that this additional regularity result actually does not need of σ being time independent. □

4 Numerical analysis. Time semi-discrete problem

The aim of this section is to derive error estimates for the semi-discrete in time scheme introduced in Section 3.3.1 to approximate Problem 3.1. With this end, we will use the following norm:

$$\left(\int_0^T \|G\|_{\mathrm{L}_r^2(\Omega)}^2 \, dt + \left\| \int_0^T G \, dt \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2 \right)^{1/2} \qquad G \in \mathrm{L}^2(0,T; \widetilde{\mathrm{H}}_r^1(\Omega)). \tag{4.1}$$

Let us remark that a similar norm appears in the analysis of other nonlinear problems in electromagnetism (see, for instance, [25]).

To obtain the estimates we will follow the techniques introduced in [26]. However, our approach is slightly different, mainly because of the presence of the non-homogeneous Dirichlet boundary condition. With this aim, we will further assume that the dependence of B on H is Lipschitz continuous. More precisely, from now on, we assume the following strengthened form of hypothesis H.1:

H.1*: The mapping $\mathcal{B}: \ \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, uniformly Lipschitz continuous with respect to the third variable; namely:
- ◦ $\mathcal{B}(\cdot, u): \ \Omega \to \mathbb{R}$ is measurable for each $u \in \mathbb{R}$;
- ◦ $\exists L > 0: \quad |\mathcal{B}(r,z,u) - \mathcal{B}(r,z,v)| \leq L\,|u - v| \quad \forall u, v \in \mathbb{R}, \quad \forall (r,z) \in \Omega.$

Remark 4.1 For $\mathcal{B}$ satisfying hypotheses H.1* and H.2*, the a priori estimate (3.24) as well as that from Lemma 3.3 hold true even for $g \in \mathrm{H}^1(0,T; \widetilde{\mathrm{H}}_r^{1/2}(\Gamma))$. Therefore, under assumptions H.1*, H.2*, H.3, H.4*, H.5 and a weaker form of H.6 with $g \in \mathrm{H}^1(0,T; \widetilde{\mathrm{H}}_r^{1/2}(\Gamma))$ (instead of g in $\mathrm{H}^2(0,T; \widetilde{\mathrm{H}}_r^{1/2}(\Gamma))$), Problem 3.1 also has a unique solution (H, B) and there holds $H \in \mathrm{H}^1(0,T; \mathrm{L}_r^2(\Omega))$. Indeed, all the forthcoming results remain valid for $g \in \mathrm{H}^1(0,T; \widetilde{\mathrm{H}}_r^{1/2}(\Gamma))$.

We consider the backward Euler time discretization of Problem 3.1 that we have introduced in Section 3.3.1. We keep the notation defined therein. The resulting discrete problem written now in terms of the main variable H^{i+1}, reads as follows:

Problem 4.1 For $i = 0, \ldots, m-1$, find $H^{i+1} \in \widetilde{\mathrm{H}}_r^1(\Omega)$ satisfying

$$\begin{aligned} \int_\Omega \bar{\partial}\mathcal{B}(H^{i+1})\, G\, r\, dr\, dz + a(H^{i+1}, G) &= \langle f^{i+1}, G \rangle \qquad && \forall G \in \mathcal{U}, \\ H^{i+1}|_\Gamma &= g^{i+1} && \text{on } \Gamma, \\ H^0 &= H_0 && \text{in } \Omega. \end{aligned}$$

The existence and the uniqueness of a weak solution at each time step follow from Lemma 3.2 by writing $H^i := H_u^i + H_g^i$, $i = 0, \dots, m$ (with H_g^i as defined in Section 3.3). The following result yields an a priori estimate for the solution of the above problem.

Lemma 4.1 *There exists $C > 0$ such that, for all $l = 0, \dots, m-1$,*

$$\left\| \bar{\partial} \mathcal{B}(H^{l+1}) \right\|_{\mathcal{U}'}^2 + \Delta t \sum_{i=0}^{l} \left\| \bar{\partial} H^{i+1} \right\|_{\mathrm{L}_r^2(\Omega)}^2 + \left\| H^{l+1} \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2 \leq C.$$

Proof Since $H^i := H_u^i + H_g^i$, $i = 0, \dots, m$, the proof follows from Lemma 3.3, the a priori estimate (3.24) (established in the proof of Theorem 3.2) and the regularity of H_g (cf. (3.1)). □

4.1 Error estimates for the time discretization

To derive an error estimate for the solution to Problem 4.1, first we notice that the piecewise linear function $B_{\Delta t}$ written in terms of H^i reads as follows:

$$B_{\Delta t}(t^0) = \mathcal{B}(H_0));$$
$$B_{\Delta t}(t) = \mathcal{B}(H^{i-1}) + (t - t^{i-1}) \bar{\partial} \mathcal{B}(H^i), \quad t \in (t^{i-1}, t^i], \quad i = 1, \dots, m.$$

Also we define the step function $\overline{H}_{\Delta t} : [0, T] \to \widetilde{\mathrm{H}}_r^1(\Omega)$ as in Section 3.3.3:

$$\overline{H}_{\Delta t}(t^0) := H_0; \qquad \overline{H}_{\Delta t}(t) := H^i, \quad t \in (t^{i-1}, t^i], \quad i = 1, \dots, m, \tag{4.2}$$

so that $\overline{H}_{\Delta t} = \overline{H}_{u \Delta t} + \overline{H}_{g \Delta t}$. Using this notation we rewrite the first equation from Problem 4.1 as follows:

$$\int_{\Omega} \partial_t B_{\Delta t} G \, r \, dr \, dz + a(\overline{H}_{\Delta t}, G) = \langle \overline{f}_{\Delta t}, G \rangle \qquad \forall G \in \mathcal{U}, \qquad \text{a.e. in } [0, T], \tag{4.3}$$

and we identify the solution of Problem 4.1 with its piecewise constant interpolant $\overline{H}_{\Delta t}$. Now, we are in a position to prove the following error estimate:

Theorem 4.1 *Let H and $\overline{H}_{\Delta t}$ be the solutions to Problems 3.1 and 4.1, respectively. Under assumptions H.1*, H.2*, H.3, H.4*, H.5 and H.6, there holds*

$$\int_0^T \left\| H - \overline{H}_{\Delta t} \right\|_{\mathrm{L}_r^2(\Omega)}^2 dt + \left\| \int_0^T (H - \overline{H}_{\Delta t}) \, dt \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2 \leq C \Delta t^2 \left\{ 1 + \|g\|_{\mathrm{H}^1(0,T;\widetilde{\mathrm{H}}_r^{1/2}(\Gamma))}^2 + \|f\|_{\mathrm{H}^1(0,T;\mathcal{U}')}^2 \right\}.$$

Proof First, we subtract (4.3) from the first equation of Problem 3.1 and integrate with respect to time. Thus, we obtain for all $G \in \mathcal{U}$

$$\int_{\Omega} (B - B_{\Delta t})(t) \, G \, r \, dr \, dz + a\left(\int_0^t (H - \overline{H}_{\Delta t})(s) \, ds, G \right) = \left\langle \int_0^t (f - \overline{f}_{\Delta t})(s) \, ds, G \right\rangle.$$

Next, we take $G = (e - e_g)(t)$ in the above equation, with $e := H - \overline{H}_{\Delta t}$ and $e_g := H_g - \overline{H}_{g\Delta t}$, and integrate in time. Thus, we arrive at

$$\int_0^T \int_\Omega (B - B_{\Delta t})(t)\,(e - e_g)(t)\, r\, dr\, dz\, dt + \int_0^T a\left(\int_0^t e(s)\, ds, (e - e_g)(t)\right) dt = \int_0^T \left\langle \int_0^t \left(f - \overline{f}_{\Delta t}\right)(s)\, ds, (e - e_g)(t) \right\rangle dt$$

or, equivalently,

$$\begin{aligned}\int_0^T \int_\Omega \left(\mathcal{B}(H) - \mathcal{B}(\overline{H}_{\Delta t})\right) e\, r\, dr\, dz\, dt + \int_0^T a\left(\int_0^t (e - e_g)(s)\, ds, e - e_g\right) dt \\ = \int_0^T \left\langle \int_0^t (f - \overline{f}_{\Delta t})(s)\, ds, e - e_g \right\rangle dt - \int_0^T a\left(\int_0^t e_g(s)\, ds, e - e_g\right) dt \\ + \int_0^T \int_\Omega \left(B_{\Delta t} - \mathcal{B}(\overline{H}_{\Delta t})\right)(e - e_g)\; r\, dr\, dz\, dt \\ + \int_0^T \int_\Omega \left(\mathcal{B}(H) - \mathcal{B}(\overline{H}_{\Delta t})\right) e_g\, r\, dr\, dz\, dt.\end{aligned}$$

We rewrite the second term on the left-hand side above as follows

$$\int_0^T a\left(\int_0^t (e - e_g)(s)\, ds, e - e_g\right) dt = \frac{1}{2} a\left(\int_0^T (e - e_g)\; dt, \int_0^T (e - e_g)\; dt\right).$$

Then, from the ellipticity of $a(\cdot, \cdot)$ (cf. Lemma 3.1) and the strong monotonicity of $\mathcal{B}$ (cf. H.2*), we have that

$$\begin{aligned}\beta \int_0^T \|e\|^2_{\mathrm{L}^2_r(\Omega)}\; dt + \frac{\gamma}{2} \left\| \int_0^T (e - e_g)\; dt \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \\ \leq \left| \int_0^T \left\langle \int_0^t (f - \overline{f}_{\Delta t})(s)\, ds, e - e_g \right\rangle dt \right| + \left| \int_0^T a\left(\int_0^t e_g(s)\, ds, e - e_g\right) dt \right| \\ + \left| \int_0^T \int_\Omega \left(B_{\Delta t} - \mathcal{B}(\overline{H}_{\Delta t})\right)(e - e_g)\; r\, dr\, dz\, dt \right| \\ + \left| \int_0^T \int_\Omega \left(\mathcal{B}(H) - \mathcal{B}(\overline{H}_{\Delta t})\right) e_g\, r\, dr\, dz\, dt \right|.\end{aligned} \tag{4.4}$$

The next step is to bound each term on the right-hand side of the above equation. First, from the Lipschitz continuity of $\mathcal{B}$ (cf. H.1*) and Young's inequality, the last term is easily bounded as follows,

$$\left| \int_0^T \int_\Omega \left(\mathcal{B}(H) - \mathcal{B}(\overline{H}_{\Delta t})\right) e_g\, r\, dr\, dz\, dt \right| \leq \eta \int_0^T \|e\|^2_{\mathrm{L}^2_r(\Omega)}\; dt + \frac{C}{\eta} \int_0^T \|e_g\|^2_{\mathrm{L}^2_r(\Omega)}\; dt \tag{4.5}$$

for all $\eta > 0$. On the other hand, by using again the Lipschitz continuity of $\mathcal{B}$, we have that $\left|B_{\Delta t}(t) - \mathcal{B}(\overline{H}_{\Delta t}(t))\right| \leq \Delta t \left|\bar{\partial}\mathcal{B}(H^i)\right| \leq L\Delta t \left|\bar{\partial} H^i\right|$ for all $t \in (t^{i-1}, t^i]$.

Then, from this and Lemma 4.1, it follows that also for all $\eta > 0$

$$\left|\int_0^T \int_\Omega \left(B_{\Delta t} - \mathcal{B}(\overline{H}_{\Delta t})\right)(e - e_g)\, r\, dr\, dz\, dt\right| \\ \leq L\Delta t \sum_{i=1}^m \int_{t^{i-1}}^{t^i} \left\|\bar{\partial} H^i\right\|_{\mathrm{L}^2_r(\Omega)} \|e - e_g\|_{\mathrm{L}^2_r(\Omega)}\, dt \\ \leq \frac{C}{\eta}\Delta t^2 + \eta \int_0^T \|e\|^2_{\mathrm{L}^2_r(\Omega)}\, dt + \eta \int_0^T \|e_g\|^2_{\mathrm{L}^2_r(\Omega)}\, dt. \tag{4.6}$$

Finally, for the two remaining terms on the right-hand side of (4.4), from integration by parts and Young's inequality we arrive at

$$\left|\int_0^T \left\langle \int_0^t (f - \overline{f}_{\Delta t})(s)\, ds, e - e_g \right\rangle dt\right| \\ = \left|\left\langle \int_0^T (f - \overline{f}_{\Delta t})\, dt, \int_0^T (e - e_g)\, dt\right\rangle - \int_0^T \left\langle f - \overline{f}_{\Delta t}, \int_0^t (e - e_g)(s)\, ds\right\rangle dt\right| \\ \leq \alpha \left\|\int_0^T (e - e_g)\, dt\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + \int_0^T \left\|\int_0^t (e - e_g)(s)\, ds\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} dt \\ + C_\alpha \int_0^T \left\|f - \overline{f}_{\Delta t}\right\|^2_{\mathcal{U}'}\, dt \tag{4.7}$$

for all $\alpha > 0$ and, similarly,

$$\int_0^T a\left(\int_0^t e_g(s)\, ds, e - e_g\right) dt \\ \leq \alpha \left\|\int_0^T (e - e_g)\, dt\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + \int_0^T \left\|\int_0^t (e - e_g)(s)\, ds\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} dt \\ + C_\alpha \int_0^T \|e_g\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}\, dt, \tag{4.8}$$

for all $\alpha > 0$, too. Then, by replacing (4.5)–(4.8) into (4.4) with $\eta = \beta/4$ and $\alpha = \gamma/8$, we obtain that there exists $C > 0$ such that

$$\frac{\beta}{2} \int_0^T \|e\|^2_{\mathrm{L}^2_r(\Omega)}\, dt + \frac{\gamma}{4} \left\|\int_0^T (e - e_g)\, dt\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \\ \leq C\left\{\Delta t^2 + \|e_g\|^2_{\mathrm{L}^2(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))} + \left\|f - \overline{f}_{\Delta t}\right\|^2_{\mathrm{L}^2(0,T;\mathcal{U}')}\right\} \\ + 2\int_0^T \left\|\int_0^t (e - e_g)(s)\, ds\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} dt.$$

Since this inequality actually holds with T substituted by τ for any $\tau \in (0, T]$, the result follows from Gronwall's lemma, classical interpolation results and (3.1). □

Remark 4.2 Under the same assumptions as above, but with $\mathcal{B}$ satisfying hypothesis H.2 instead of H.2* (namely, continuous instead of Lipschitz continuous), we have the following error estimate:

$$\int_0^T \left\|H - \overline{H}_{\Delta t}\right\|^2_{\mathrm{L}^2_r(\Omega)} dt + \left\|\int_0^T (H - \overline{H}_{\Delta t})\, dt\right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C\Delta t.$$

(See Theorem 3.2 from [26] for a similar result in a 3D problem.) In fact, the Lipschitz continuity of $\mathcal{B}$ was only used to prove (4.5) and (4.6). Then, it is enough to bound the corresponding left-hand sides without using the Lipschitz continuity. For that in (4.5), we notice that

$$\left|\int_0^T \int_\Omega \left(B_{\Delta t} - \mathcal{B}(\overline{H}_{\Delta t})\right)(e - e_g)\; r\, dr\, dz\, dt\right| \\ \leq \Delta t \left(\sum_{i=1}^{m-1} \Delta t \left\|\bar{\partial}\mathcal{B}(H^{i+1})\right\|^2_{\mathcal{U}'}\right) \left(\int_0^T \|e - e_g\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} dt\right)^{1/2}.$$

Hence, from Lemma 4.1, the fact that $H \in \mathrm{L}^\infty(0,T;\widetilde{\mathrm{H}}^1_r(\Omega))$ and the regularity of H_g (cf. (3.1)), it follows that

$$\left|\int_0^T \int_\Omega \left(B_{\Delta t} - \mathcal{B}(\overline{H}_{\Delta t})\right)(e - e_g)\; r\, dr\, dz\, dt\right| \leq C\Delta t.$$

On the other hand, for the left-hand side in (4.6), we obtain from hypothesis H.3, Lemma 4.1, a classical interpolation result and (3.1)

$$\left|\int_0^T \int_\Omega \left(\mathcal{B}(H) - \mathcal{B}(\overline{H}_{\Delta t})\right) e_g\, r\, dr\, dz\, dt\right| \leq C\Delta t\, \|g\|_{\mathrm{H}^1(0,T;\widetilde{\mathrm{H}}^{1/2}_r(\Gamma))},$$

which allows us to conclude the remark.

5 Numerical analysis. Fully discrete problem

In this section, we will introduce a space discretization of Problem 4.1 and obtain error estimates for the fully discrete approximation. First, we will estimate the error in the $\mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega))$-norm without assuming any additional regularity of the solution. With this aim, we will derive an estimate for the difference between the fully and the semi-discrete problems and will use the results of the previous section (Lemma 4.1 and Theorem 4.1). Subsequently, by assuming further regularity of the solution H, we will also derive error estimates in a discrete version of the norm (4.1).

From now on, we assume that Ω is a polygonal domain. Let Γ_0 be the intersection between Γ and the symmetry axis ($r = 0$) and $\Gamma_1 := \Gamma \setminus \Gamma_0$. We consider a family of regular, quasi-uniform partitions $\{\mathcal{T}_h\}_{h>0}$ of Ω into triangles, where h denotes the mesh-size (i.e., the maximal length of the sides of the triangulation). Let $\mathcal{L}_h$ be the space of piecewise linear continuous finite elements,

$$\mathcal{L}_h := \{G_h \in \mathcal{C}(\Omega) :\; G_h|_T \in \mathcal{P}_1\ \forall T \in \mathcal{T}_h\},$$

and $\mathcal{V}_h$ the subspace of functions vanishing on Γ_0:

$$\mathcal{V}_h := \{G_h \in \mathcal{L}_h : \ G_h|_{\Gamma_0} = 0\}.$$

Notice that $\mathcal{V}_h \subset \widetilde{\mathrm{H}}^1_r(\Omega)$. We also consider the finite-dimensional subspace

$$\mathcal{U}_h := \mathcal{V}_h \cap \mathcal{U} = \{G_h \in \mathcal{L}_h : \ G_h|_{\Gamma} = 0\}.$$

Finally, we denote by $\mathcal{V}_h(\Gamma)$ the space of traces on Γ of functions in $\mathcal{V}_h$:

$$\mathcal{V}_h(\Gamma) := \{G_h|_{\Gamma} : G_h \in \mathcal{V}_h\}.$$

Notice too that for all $G_h \in \mathcal{V}_h(\Gamma)$, $G_h|_{\Gamma_0} = 0$.

In order to define a discrete approximation on Γ for the Dirichlet boundary data, we introduce the Sobolev space

$$\mathrm{L}^2_r(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} : \ \int_{\Gamma} v^2 \, r \, dS < \infty \right\}$$

and the orthogonal projector $\Pi^h_\Gamma : \ \mathrm{L}^2_r(\Gamma) \to \mathcal{V}_h(\Gamma)$ defined for all $v \in \mathrm{L}^2_r(\Gamma)$ by

$$\Pi^h_\Gamma v \in \mathcal{V}_h(\Gamma) : \qquad \int_{\Gamma_1} \frac{1}{\sigma} \left(\Pi^h_\Gamma v - v \right) v_h \, r \, dS = 0 \qquad \forall v_h \in \mathcal{V}_h(\Gamma).$$

We propose the following Galerkin discretization of Problem 4.1 as the fully discrete approximation of Problem 3.1:

Problem 5.1 For $i = 0, \dots, m-1$, find $H_h^{i+1} \in \mathcal{V}_h$ satisfying

$$\int_\Omega \bar{\partial} \mathcal{B}(H_h^{i+1}) \, G_h \, r \, dr \, dz + a(H_h^{i+1}, G_h) = \left\langle f^{i+1}, G_h \right\rangle \qquad \forall G_h \in \mathcal{U}_h,$$
$$H_h^{i+1}|_\Gamma = \Pi^h_\Gamma g^{i+1},$$
$$H_h^0 = H_{0h}.$$

In principle $H_{0h} \in \mathcal{V}_h$ is any arbitrary approximation of H_0; see Remarks 5.2 and 5.3 below for a discussion about a convenient choice. The existence and the uniqueness of solution follow by applying similar techniques as those in the proof of Lemma 3.2. The following lemma yields an a priori estimate for the solution of Problem 5.1.

Lemma 5.1 *There exists $C > 0$ such that, for all $l = 0, \dots, m-1$,*

$$\left\| \bar{\partial} \mathcal{B}(H_h^{l+1}) \right\|^2_{\mathcal{U}'} + \Delta t \sum_{i=0}^{l} \left\| \bar{\partial} H_h^{i+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \left\| H_h^{l+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C.$$

Proof It follows by applying the same techniques as in the proof of Lemma 4.1. □

5.1 Finite element approximation properties

In order to derive error estimates for the proposed numerical scheme, first we will establish several approximation properties of the finite element spaces.

We consider the Clément-type operator $I_h : \widetilde{\mathrm{H}}^1_r(\Omega) \to \mathcal{V}_h$ defined in [3, Eq. (36)]. In Theorem 2 from this reference it is proved that, for all $u \in \widetilde{\mathrm{H}}^1_r(\Omega)$,

$$\|u - I_h u\|_{\mathrm{L}^2_r(\Omega)} + h\,\|u - I_h u\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq Ch\,\|u\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \tag{5.1}$$

and, for all $u \in \mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^1_r(\Omega)$,

$$\|u - I_h u\|_{\mathrm{L}^2_r(\Omega)} + h\,\|u - I_h u\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq Ch^2\,\|u\|_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)}\,. \tag{5.2}$$

Let $\mathcal{N}$ be the set of all vertices of $\mathcal{T}_h$. For any $P \in \mathcal{N}$, ω_P denotes the union of all elements sharing P and $h_P := \sup_{T\subset\omega_P} h_T$, with h_T being the diameter of T. Let $\{\varphi_P :\ P \in \mathcal{N}\}$ be the standard nodal basis of $\mathcal{L}_h$.

Next, we establish a discrete lifting result that will be used in the sequel.

Lemma 5.2 *For all $u \in \widetilde{\mathrm{H}}^1_r(\Omega)$, there exists $v_h \in \mathcal{V}_h$ which satisfies*

$$v_h = \Pi^h_\Gamma u - I_h u \qquad \text{on } \Gamma$$

and

$$\|v_h\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C\,\|u\|_{\widetilde{\mathrm{H}}^1_r(\Omega)}\,.$$

Moreover, if $u \in \mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^1_r(\Omega)$, then

$$\|v_h\|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq Ch\,\|u\|_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)}\,.$$

Proof We define $v_h := \sum_{P\in\mathcal{N}\cap\Gamma_1} (\Pi^h_\Gamma u - I_h u)(P)\,\varphi_P$. Notice that $\operatorname{supp} v_h \subset \bigcup\{T \in \mathcal{T}_h :\ T \cap \Gamma_1 \neq \emptyset\}$. A straightforward computation allows us to show that $\|v_h\|^2_{\mathrm{L}^2_r(\omega_P)} \leq Ch_P \left\|\Pi^h_\Gamma u - I_h u\right\|^2_{\mathrm{L}^2_r(\partial\omega_P\cap\Gamma_1)}$ for all $P \in \mathcal{N} \cap \Gamma_1$. Hence, using weighted inverse inequalities (see [3, Lemmas 3 & 4]), we obtain

$$\|v_h\|^2_{\widetilde{\mathrm{H}}^1_r(\omega_P)} \leq Ch_P^{-2}\,\|v_h\|^2_{\mathrm{L}^2_r(\omega_P)} \leq Ch_P^{-1}\left\|\Pi^h_\Gamma u - I_h u\right\|^2_{\mathrm{L}^2_r(\partial\omega_P\cap\Gamma_1)}\,.$$

Summing for all $P \in \mathcal{N} \cap \Gamma_1$ and using the quasi-uniformity of the meshes lead to

$$\|v_h\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq Ch^{-1}\left\|\Pi^h_\Gamma u - I_h u\right\|^2_{\mathrm{L}^2_r(\Gamma_1)}\,. \tag{5.3}$$

Moreover, since $\left\|\Pi^h_\Gamma u\right\|_{\mathrm{L}^2_r(\Gamma_1)} \leq \frac{\sigma^*}{\sigma_*}\,\|u\|_{\mathrm{L}^2_r(\Gamma_1)}$ and $\Pi^h_\Gamma I_h u = I_h u$ on Γ, we have

$$\left\|\Pi^h_\Gamma u - I_h u\right\|^2_{\mathrm{L}^2_r(\Gamma_1)} \leq \left(\frac{\sigma^*}{\sigma_*}\right)^2 \sum_{\ell\subset\Gamma_1} \|u - I_h u\|^2_{\mathrm{L}^2_r(\ell)}\,. \tag{5.4}$$

Now, from [9, Lemma 4] it follows that

$$\|u - I_h u\|^2_{\mathrm{L}^2_r(\ell)} \leq C\left\{h_T^{-1}\,\|u - I_h u\|^2_{\mathrm{L}^2_r(T)} + h_T\,\|u - I_h u\|^2_{\mathrm{H}^1_r(T)}\right\}, \tag{5.5}$$

where $T \in \mathcal{T}_h$ is such that $\ell \subset \partial T$. If $u \in \widetilde{\mathrm{H}}^1_r(\Omega)$, then, from the latter and (5.1), we obtain

$$\sum_{\ell \subset \Gamma_1} \|u - I_h u\|^2_{\mathrm{L}^2_r(\ell)} \leq Ch \|u\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} . \tag{5.6}$$

Therefore, the first inequality of the lemma follows from (5.3), (5.4) and (5.6). On the other hand, for $u \in \mathrm{H}^2_r(T) \cap \widetilde{\mathrm{H}}^1_r(\Omega)$ we proceed analogously but applying (5.2) instead of (5.1) to bound (5.5). Thus, we conclude the proof. □

Let us introduce the elliptic projector $P_h : \widetilde{\mathrm{H}}^1_r(\Omega) \to \mathcal{V}_h$ defined for all $u \in \widetilde{\mathrm{H}}^1_r(\Omega)$ as follows:

$$P_h u \in \mathcal{V}_h : \qquad a(P_h u, w_h) = a(u, w_h) \qquad \forall w_h \in \mathcal{U}_h, \tag{5.7}$$

$$P_h u = \Pi^h_\Gamma(u|_\Gamma) \qquad \text{on } \Gamma. \tag{5.8}$$

To obtain an error estimate for this projector, first we prove the following lemma.

Lemma 5.3 *Let $\boldsymbol{p} := (p_r, p_z) \in \mathrm{H}^1_r(\Omega)^2$ be such that $p_z \in \mathrm{L}^2_{1/r}(\Omega)$ and $\boldsymbol{p} \cdot \boldsymbol{t} = 0$ on Γ. Then, there exists $\boldsymbol{p}_h \in \mathcal{L}^2_h$ such that $\boldsymbol{p}_h \cdot \boldsymbol{t} = 0$ on Γ, $\boldsymbol{p}_h \cdot \boldsymbol{n}$ is continuous on Γ, and*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{L}^2_r(\Omega)^2} + h\,|\boldsymbol{p} - \boldsymbol{p}_h|_{\mathrm{H}^1_r(\Omega)^2} \leq Ch \left\{ \|\boldsymbol{p}\|_{\mathrm{H}^1_r(\Omega)^2} + \|p_z\|_{\mathrm{L}^2_{1/r}(\Omega)} \right\}. \tag{5.9}$$

Proof We will us a Clément-type interpolant of $\boldsymbol{p}$. We define its values at each node $P \in \mathcal{N}$ differently according to its location:

- If $P \notin \Gamma$, then we set $\boldsymbol{p}_P := |\omega_P|^{-1} \int_{\omega_P} \boldsymbol{p}\, r\, dr\, dz$.
- If $P \in \Gamma$ is not a vertex of the polygon Ω, then the two edges ℓ_1 and ℓ_2 sharing P have the same tangent and normal vectors which we denote $\boldsymbol{t}_P$ and $\boldsymbol{n}_P$, respectively. In this case, we set $\boldsymbol{p}_P := (\widetilde{\boldsymbol{p}}_P \cdot \boldsymbol{n}_P)\, \boldsymbol{n}_P$, where $\widetilde{\boldsymbol{p}}_P := |\omega_P|^{-1} \int_{\omega_P} \boldsymbol{p}\, r\, dr\, dz$.
- If P is a vertex of Ω, then we set $\boldsymbol{p}_P := \boldsymbol{0}$.

Finally, we define $\boldsymbol{p}_h := \sum_{P \in \mathcal{N}} \boldsymbol{p}_P \varphi_P$.

By construction $\boldsymbol{p}_h \in \mathcal{L}^2_h$ and $\boldsymbol{p}_h \cdot \boldsymbol{t} = 0$ on Γ. To prove (5.9), first we notice that, since $\sum_{P \in \mathcal{N}} \varphi_P = 1$, we have

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|^2_{\mathrm{L}^2_r(\Omega)^2} = \int_\Omega (\boldsymbol{p} - \boldsymbol{p}_h) \sum_{P \in \mathcal{N}} \varphi_P (\boldsymbol{p} - \boldsymbol{p}_P)\, r\, dr\, dz.$$

Hence, by using Cauchy-Schwartz inequality, it is easy to check that

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{L}^2_r(\Omega)^2} \leq C \left(\sum_{P \in \mathcal{N}} \|\boldsymbol{p} - \boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} \right)^{1/2} . \tag{5.10}$$

Similar arguments allow us to write

$$\begin{aligned} |\boldsymbol{p} - \boldsymbol{p}_h|_{\mathrm{H}^1_r(\Omega)^2} &\leq C \left\{ \left\| \sum_{P \in \mathcal{N}} \nabla \varphi_P\, (\boldsymbol{p} - \boldsymbol{p}_P)^{\mathrm{t}} \right\|_{\mathrm{L}^2_r(\Omega)^{2\times 2}} + \left\| \sum_{P \in \mathcal{N}} \varphi_P \nabla \boldsymbol{p} \right\|_{\mathrm{L}^2_r(\Omega)^{2\times 2}} \right\} \\ &\leq C \left\{ |\boldsymbol{p}|^2_{\mathrm{H}^1_r(\Omega)^2} + \sum_{P \in \mathcal{N}} h_P^{-2} \|\boldsymbol{p} - \boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} \right\}^{1/2} , \end{aligned} \tag{5.11}$$

where we have also used that, for regular meshes, $\|\nabla\varphi_P\|_{\mathrm{L}^2_r(\Omega)^2} \leq Ch_P^{-1}$.

Thus, there only remains to estimate $\|\boldsymbol{p}-\boldsymbol{p}_P\|_{\mathrm{L}^2_r(\omega_P)^2}$ for all $P\in\mathcal{N}$. To do this, we distinguish again the same three cases as above:

- If $P\notin\Gamma$, since $\boldsymbol{p}_P$ is the mean value of $\boldsymbol{p}$ in $\mathrm{L}^2_r(\omega_P)^2$, then, from [3, Lemma 6],

$$\|\boldsymbol{p}-\boldsymbol{p}_P\|_{\mathrm{L}^2_r(\omega_P)^2} = \inf_{\boldsymbol{q}\in\mathbb{P}_0(\omega_P)^2} \|\boldsymbol{p}-\boldsymbol{q}\|_{\mathrm{L}^2_r(\omega_P)^2} \leq Ch_P \|\boldsymbol{p}\|_{\mathrm{H}^1_r(\omega_P)^2}. \tag{5.12}$$

- If $P\in\Gamma$ is not a vertex of Ω, since $\boldsymbol{p}_P\cdot\boldsymbol{t}_P=0$, it follows that

$$\|\boldsymbol{p}-\boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} = \|\boldsymbol{p}\cdot\boldsymbol{t}_P\|^2_{\mathrm{L}^2_r(\omega_P)} + \|(\boldsymbol{p}-\boldsymbol{p}_P)\cdot\boldsymbol{n}_P\|^2_{\mathrm{L}^2_r(\omega_P)}. \tag{5.13}$$

Now, since $\|(\boldsymbol{p}-\boldsymbol{p}_P)\cdot\boldsymbol{n}_P\|_{\mathrm{L}^2_r(\omega_P)} = \|(\boldsymbol{p}-\widetilde{\boldsymbol{p}}_P)\cdot\boldsymbol{n}_P\|_{\mathrm{L}^2_r(\omega_P)}$, with $\widetilde{\boldsymbol{p}}_P$ being the mean value of $\boldsymbol{p}$ in $\mathrm{L}^2_r(\omega_P)^2$, by proceeding as in (5.12) we obtain

$$\|(\boldsymbol{p}-\boldsymbol{p}_P)\cdot\boldsymbol{n}_P\|_{\mathrm{L}^2_r(\omega_P)} \leq Ch_P \|\boldsymbol{p}\|_{\mathrm{H}^1_r(\omega_P)^2}. \tag{5.14}$$

To bound the other term on the right-hand side of (5.13), we use that $\boldsymbol{p}\cdot\boldsymbol{t}_P$ vanishes on $\ell_1\cup\ell_2\subset\partial\omega_P$ and consider also three cases:

– If $P\in\Gamma_1$ and $\omega_P\cap\Gamma_0=\emptyset$, then $\max_{\omega_P} r/\min_{\omega_P} r\leq C$, with C being a constant which only depends on the regularity of the mesh. In such a case, from the classical Poincaré inequality and a scaling argument we have

$$\begin{aligned}\|\boldsymbol{p}\cdot\boldsymbol{t}_P\|^2_{\mathrm{L}^2_r(\omega_P)} &\leq \max_{\omega_P} r \int_{\omega_P} |\boldsymbol{p}\cdot\boldsymbol{t}_P|^2\, dr\, dz \\ &\leq Ch_P^2 \max_{\omega_P} r \int_{\omega_P} |\nabla(\boldsymbol{p}\cdot\boldsymbol{t}_P)|^2\, dr\, dz \\ &\leq Ch_P^2 \frac{\max_{\omega_P} r}{\min_{\omega_P} r} \int_{\omega_P} |D\boldsymbol{p}|^2\, r\, dr\, dz \leq Ch_P^2 |\boldsymbol{p}|^2_{\mathrm{H}^1_r(\omega_P)^2}. \end{aligned} \tag{5.15}$$

– If $P\in\Gamma_1$ and $\omega_P\cap\Gamma_0\neq\emptyset$, then let K_P be the smallest closed parallelogram such that $\omega_P\subset K_P\subset\overline{\Omega}$, with one edge on Γ_0 and other one on Γ_1, as shown in Fig. 1 (for the existence of such K_P, we may need to assume that the mesh is sufficiently fine).

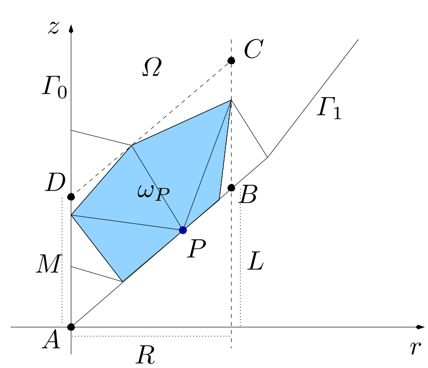

Fig. 1 Parallelogram K_P of vertices A, B, C, D satisfying $\omega_P\subset K_P\subset\overline{\Omega}$.

We use the notation from Fig. 1. In particular, the slope of the edge $\overline{AB}$ is $m := L/R$ and the length of the edge $\overline{AD}$ is M. Notice that $M \leq Ch_P$, with C a constant which only depends on the regularity of the mesh. For simplicity, we consider a coordinate system (r, z) centered at the vertex A. Given $\boldsymbol{p} \in \mathcal{C}^\infty(K_P)^2$, let $v := \boldsymbol{p} \cdot \boldsymbol{t}_P$. Then,

$$v(r,z) = \int_{mr}^{z} \partial_z v(r,s)\, ds, \qquad mr \leq z \leq mr + M, \qquad 0 \leq r \leq R.$$

Hence,

$$\begin{aligned}\int_{\omega_P} |v(r,z)|^2\, r\, dr\, dz &\leq \int_{K_P} |v(r,z)|^2\, r\, dr\, dz \\ &= \int_0^R \left[\int_{mr}^{mr+M} \left| \int_{mr}^{z} \partial_z v(r,s)\, ds \right|^2 dz \right] r\, dr \\ &\leq \int_0^R \left\{ \int_{mr}^{mr+M} M \left[\int_{mr}^{mr+M} |\partial_z v(r,s)|^2\, ds \right] dz \right\} r\, dr \\ &\leq M^2 \int_{K_P} |\nabla v(r,z)|^2\, r\, dr\, dz.\end{aligned}$$

Therefore,

$$\|\boldsymbol{p} \cdot \boldsymbol{t}_P\|^2_{\mathrm{L}^2_r(\omega_P)} \leq Ch_P^2\, |\boldsymbol{p} \cdot \boldsymbol{t}_P|^2_{\mathrm{H}^1_r(K_P)} \leq Ch_P^2\, |\boldsymbol{p}|^2_{\mathrm{H}^1_r(K_P)^2} \tag{5.16}$$

for all $\boldsymbol{p} \in \mathcal{C}^\infty(K_P)^2$. Since this space is dense in $\mathrm{H}^1_r(K_P)^2$ (cf. [19, Theorem 4.3(ii)]), the inequality above holds for all $\boldsymbol{p} \in \mathrm{H}^1_r(K_P)^2$, too.

- Finally, if $P \in \Gamma_0$ (and is not a vertex of Ω), then $\ell_1, \ell_2 \subset \Gamma_0$ and $\boldsymbol{p} \cdot \boldsymbol{t}_P = p_z$. Since $p_z \in \mathrm{L}^2_{1/r}(\Omega)$ and hence $p_z \in \widetilde{\mathrm{H}}^1_r(\Omega)$, it is easy to check that $r^{1/2} p_z \in \mathrm{H}^1(\omega_P)$. Now, this last term vanishes on $\Gamma_0 \supset \ell_1 \cup \ell_2$, so that we can apply a scaling argument and the classical Poincaré inequality to write

$$\begin{aligned}\|\boldsymbol{p} \cdot \boldsymbol{t}_P\|^2_{\mathrm{L}^2_r(\omega_P)} &= \int_{\omega_P} (r^{1/2} p_z)^2\, dr\, dz \leq Ch_P^2 \int_{\omega_P} \left| \nabla (r^{1/2} p_z) \right|^2 dr\, dz \\ &\leq Ch_P^2 \left\{ |p_z|^2_{\mathrm{H}^1_r(\omega_P)} + \|p_z\|^2_{\mathrm{L}^2_{1/r}(\omega_P)} \right\}.\end{aligned} \tag{5.17}$$

Therefore, by replacing (5.14) and (5.15), (5.16) or (5.17), as corresponds, into (5.13), we have that

$$\|\boldsymbol{p} - \boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} \leq Ch_P^2 \left\{ |\boldsymbol{p}|^2_{\mathrm{H}^1_r(\widetilde{\omega}_P)^2} + \|p_z\|^2_{\mathrm{L}^2_{1/r}(\omega_P)} \right\}, \tag{5.18}$$

where $\widetilde{\omega}_P := K_P$, if $P \in \Gamma_1$ and $\omega_P \cap \Gamma_0 \neq \emptyset$, and $\widetilde{\omega}_P := \omega_P$, otherwise.

∘ If P is a vertex of Ω, then $\boldsymbol{p}_P = \boldsymbol{0}$ and the unit vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, tangent to the respective edges ℓ_1 and ℓ_2 on Γ sharing P, form a basis of $\mathbb{R}^2$. Therefore,

$$\|\boldsymbol{p} - \boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} = \|\boldsymbol{p}\|^2_{\mathrm{L}^2_r(\omega_P)^2} \leq C \left\{ \|\boldsymbol{p} \cdot \boldsymbol{t}_1\|^2_{\mathrm{L}^2_r(\omega_P)} + \|\boldsymbol{p} \cdot \boldsymbol{t}_2\|^2_{\mathrm{L}^2_r(\omega_P)} \right\}.$$

Since $\boldsymbol{p} \cdot \boldsymbol{t}_1|_{\ell_1} = 0$ and $\boldsymbol{p} \cdot \boldsymbol{t}_2|_{\ell_2} = 0$, a similar analysis to that leading to (5.16) and (5.17), yields

$$\|\boldsymbol{p} - \boldsymbol{p}_P\|^2_{\mathrm{L}^2_r(\omega_P)^2} \leq Ch_P \left\{ |\boldsymbol{p}|^2_{\mathrm{H}^1_r(\widetilde{\omega}_P)^2} + \|p_z\|^2_{\mathrm{L}^2_{1/r}(\omega_P)} \right\}. \tag{5.19}$$

Whence, by replacing (5.12), (5.18) and (5.19) into (5.10) and (5.11), we obtain (5.9). On the other hand, we notice that by construction $\boldsymbol{p}_h$ vanishes at the vertices of Ω, so that $\boldsymbol{p}_h \cdot \boldsymbol{n}$ is continuous along the boundary Γ and we end the proof. $\square$

Now we are in a position to prove an error estimate for the projector P_h. This proof relies on a duality argument for which we will need additional regularity of the solution of the corresponding adjoint problem. This is the reason why, from now on, we also make the following assumption:

H.7: Given $w \in \mathrm{L}^2_r(\Omega)$, the unique solution $\varphi \in \mathcal{U}$ of the elliptic problem

$$a(v, \varphi) = \int_\Omega v w \, r \, dr \, dz \qquad \forall v \in \mathcal{U} \tag{5.20}$$

satisfies $\varphi \in \mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^2_r(\Omega)$ and

$$\|\varphi\|_{\mathrm{H}^2_r(\Omega)} + \|\varphi\|_{\widetilde{\mathrm{H}}^2_r(\Omega)} \leq C \, \|w\|_{\mathrm{L}^2_r(\Omega)} \, .$$

This assumption is fulfilled, for instance, when σ is constant and Ω is a rectangle (cf. [12, Theorem 4.1]).

Before proving an error estimate for the projector P_h, we establish the following auxiliary result which follows easily from assumption H.7.

Lemma 5.4 *Given $w \in \mathrm{L}^2_r(\Omega)$, let $\varphi \in \mathcal{U}$ be the solution to* (5.20). *Then,*

$$a(v, \varphi) = -\int_\Omega \operatorname{div}\left(\frac{1}{\sigma r} \nabla (r\varphi)\right) v \, r \, dr \, dz + \int_{\Gamma_1} \frac{1}{\sigma r} \nabla (r\varphi) \cdot \boldsymbol{n} \, v \, r \, dS \qquad \forall v \in \mathrm{H}^1_r(\Omega). \tag{5.21}$$

Proof First notice that both integrals on the right-hand side above are well defined. In fact, on one hand, by testing (5.20) with $v \in \mathcal{D}(\Omega)$ it follows that $-\operatorname{div}(1/(\sigma r) \nabla (r\varphi)) = w \in \mathrm{L}^2_r(\Omega)$. On the other hand, for the last integral we use that for $\varphi \in \widetilde{\mathrm{H}}^2_r(\Omega)$, there holds $(1/r)\nabla(r\varphi) = ((1/r)\, \partial_r (r\varphi), \partial_z \varphi) \in \mathrm{H}^1_r(\Omega)^2$ and, hence, $(1/(\sigma r))\, \nabla (r\varphi) \cdot \boldsymbol{n} \in \mathrm{L}^2_r(\Gamma_1)$, because of a trace result (see, for instance, [9, Lemma 4]) and the fact that σ is bounded below away from zero.

Therefore, to prove (5.21), it is enough to check it with $v \in \mathcal{C}^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ_0, since the set of such functions is dense in $\mathrm{H}^1_r(\Omega)$ (see [19, Theorem 4.3(ii)]). For such a function v, let $\varepsilon > 0$ be such that $\operatorname{supp} v \subset \Omega_\varepsilon := \left\{(r, z) \in \overline{\Omega} : \ r > \varepsilon\right\}$. Then,

$$\begin{aligned} a(v, \varphi) &= \int_{\Omega_\varepsilon} \frac{1}{\sigma r} \nabla (r\varphi) \cdot \nabla (rv) \, dr \, dz \\ &= -\int_{\Omega_\varepsilon} \operatorname{div}\left(\frac{1}{\sigma r} \nabla (r\varphi)\right) v \, r \, dr \, dz + \int_{\partial \Omega_\varepsilon} \frac{1}{\sigma r} \nabla (r\varphi) \cdot \boldsymbol{n} \, v \, r \, dS \\ &= -\int_{\Omega} \operatorname{div}\left(\frac{1}{\sigma r} \nabla (r\varphi)\right) v \, r \, dr \, dz + \int_{\Gamma_1} \frac{1}{\sigma r} \nabla (r\varphi) \cdot \boldsymbol{n} \, v \, r \, dS. \end{aligned}$$

Thus, we conclude the proof. $\square$

The following lemma provides an optimal-order error estimate for $(u - P_h u)$.

Lemma 5.5 *For all* $u \in \mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^1_r(\Omega)$

$$\|u - P_h u\|_{\mathrm{L}^2_r(\Omega)} \leq Ch^2 \|u\|_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)} .$$

Proof First, we prove an estimate in the norm induced by $a(\cdot,\cdot)$. From the definition of P_h we have that

$$a(u - P_h u, u - P_h u) \leq C \|u - P_h u - y_h\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \qquad \forall y_h \in \mathcal{U}_h.$$

Then, taking $y_h := I_h u - P_h u + v_h$, with $v_h \in \mathcal{V}_h$ as in Lemma 5.2, it follows that

$$a(u - P_h u, u - P_h u) \leq C \left\{ \|u - I_h u\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} + \|v_h\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \right\}.$$

Hence, from (5.2) and Lemma 5.2 we obtain

$$a(u - P_h u, u - P_h u) \leq Ch^2 \|u\|^2_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)} . \tag{5.22}$$

Next, we resort to a duality argument. Let $\varphi \in \mathcal{U}$ be the solution of

$$a(v, \varphi) = \int_\Omega v\,(u - P_h u)\, r\, dr\, dz \qquad \forall v \in \mathcal{U}.$$

Hence, according to hypothesis H.7, $\varphi \in \mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^2_r(\Omega)$ and

$$\|\varphi\|_{\mathrm{H}^2_r(\Omega)} + \|\varphi\|_{\widetilde{\mathrm{H}}^2_r(\Omega)} \leq C \|u - P_h u\|_{\mathrm{L}^2_r(\Omega)} . \tag{5.23}$$

Moreover, by taking $v \in \mathcal{D}(\Omega)$ in the equation above, we have that

$$-\operatorname{div}\left(\frac{1}{\sigma r}\nabla(r\varphi)\right) = u - P_h u \qquad \text{in } \Omega.$$

By multiplying this equation by $(u - P_h u)$ and using Lemma 5.4 and the definition of P_h (cf. (5.7)–(5.8)), we obtain for all $\varphi_h \in \mathcal{U}_h$,

$$\|u - P_h u\|^2_{\mathrm{L}^2_r(\Omega)} = a(u - P_h u, \varphi - \varphi_h) - \int_{\Gamma_1} \frac{1}{\sigma r}\nabla(r\varphi)\cdot\boldsymbol{n}\left(u - \Pi^h_\Gamma u\right) r\, dS. \tag{5.24}$$

Next, we estimate the two terms on the right-hand side above. For the first one, we choose $\varphi_h = I^0_h \varphi$, where $I^0_h : \mathcal{U} \to \mathcal{U}_h$ is another Clément-type interpolant operator defined in [3, Eq. (37)]. Then, (5.22) and Theorem 2 from [3] lead to

$$a(u - P_h u, \varphi - \varphi_h) \leq Ch^2 \|u\|_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)} \|\varphi\|_{\mathrm{H}^2_r(\Omega)\cap\widetilde{\mathrm{H}}^1_r(\Omega)} . \tag{5.25}$$

To estimate the other term, we define $\boldsymbol{p} = (p_r, p_z) := (1/r)\nabla(r\varphi)$. For $\varphi \in \widetilde{\mathrm{H}}^2_r(\Omega)$, $\boldsymbol{p} \in \mathrm{H}^1_r(\Omega)^2$ and $p_z \in \mathrm{L}^2_{1/r}(\Omega)$. Moreover, $\boldsymbol{p}\cdot\boldsymbol{t} = 0$ on Γ. In fact, since $\varphi \in \mathcal{U}$, we have $\boldsymbol{p}\cdot\boldsymbol{t}|_{\Gamma_0} = p_z|_{\Gamma_0} = (\partial_z\varphi)|_{\Gamma_0} = 0$, and $\boldsymbol{p}\cdot\boldsymbol{t}|_{\Gamma_1} = ((1/r)\varphi t_r)|_{\Gamma_1} + (\nabla\varphi\cdot\boldsymbol{t})|_{\Gamma_1} = 0$, too. Thus, $\boldsymbol{p}$ satisfies the hypothesis of Lemma 5.3. Hence, let $\boldsymbol{p}_h \in \mathcal{L}^2_h$ be as in that lemma. Let w_h be defined on Γ by $w_h|_{\Gamma_1} := \boldsymbol{p}_h\cdot\boldsymbol{n}$ and $w_h|_{\Gamma_0} := 0$. Since $\boldsymbol{p}_h\cdot\boldsymbol{n}$ vanishes at the vertices of Ω (because $\boldsymbol{p}_h\cdot\boldsymbol{t} = 0$ on Γ and $\boldsymbol{p}_h\cdot\boldsymbol{n}$ is continuous on Γ), we have that $w_h \in \mathcal{V}_h(\Gamma)$. Whence, from the definition of Π^h_Γ we have that

$$\left|\int_{\Gamma_1} \frac{1}{\sigma r}\nabla(r\varphi)\cdot\boldsymbol{n}\left(u - \Pi^h_\Gamma u\right) r\, dS\right| = \left|\int_{\Gamma_1} \frac{1}{\sigma}\left(\boldsymbol{p}\cdot\boldsymbol{n} - w_h\right)\left(u - \Pi^h_\Gamma u\right) r\, dS\right| \\ \leq \frac{1}{\sigma_*}\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{L}^2_r(\Gamma_1)^2}\left\|u - \Pi^h_\Gamma u\right\|_{\mathrm{L}^2_r(\Gamma_1)}. \tag{5.26}$$

Now, by proceeding as in Lemma 5.2 (cf. (5.4), (5.5)) and using (5.2), we obtain

$$\left\| u - \Pi_\Gamma^h u \right\|_{\mathrm{L}_r^2(\Gamma_1)} \leq C \left\| u - I_h u \right\|_{\mathrm{L}_r^2(\Gamma_1)} \leq C h^{3/2} \left\| u \right\|_{\mathrm{H}_r^2(\Omega) \cap \widetilde{\mathrm{H}}_r^1(\Omega)} . \tag{5.27}$$

On the other hand, using again [9, Lemma 4], we write for all edges $\ell \subset \Gamma_1$

$$\left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|_{\mathrm{L}_r^2(\ell)^2}^2 \leq C \left\{ h_T^{-1} \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|_{\mathrm{L}_r^2(T)^2}^2 + h_T \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|_{\mathrm{H}_r^1(T)^2}^2 \right\},$$

with $T \in \mathcal{T}_h$ such that $\ell \subset \partial T$. Therefore, from Lemma 5.3, we obtain

$$\left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|_{\mathrm{L}_r^2(\Gamma_1)^2}^2 \leq C h \left\{ \left\| \boldsymbol{p} \right\|_{\mathrm{H}_r^1(\Omega)^2}^2 + \left\| p_z \right\|_{\mathrm{L}_{1/r}^2(\Omega)}^2 \right\} \leq C h \left\| \varphi \right\|_{\widetilde{\mathrm{H}}_r^2(\Omega)}^2 . \tag{5.28}$$

Then, the result follows from (5.24)–(5.28) and (5.23). □

Remark 5.1 Using similar arguments, it is straightforward to prove that

$$\left\| u - P_h u \right\|_{\mathrm{L}_r^2(\Omega)} \leq C h \left\| u \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)} \qquad \forall u \in \widetilde{\mathrm{H}}_r^1(\Omega).$$

In fact, the only differences are that we use $a(u - P_h u, u - P_h u) \leq C \left\| u \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2$ instead of (5.22) and, instead of (5.27), we use $\left\| u - \Pi_\Gamma^h u \right\|_{\mathrm{L}_r^2(\Gamma_1)} \leq C h^{1/2} \left\| u \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}$ (which follows by the same arguments that (5.27), but using (5.1) instead of (5.2)).

5.2 Error estimates for the full discretization

The following auxiliary result yields an estimate for the difference between the fully and the semi-discrete problems.

Lemma 5.6 *Let H^{i+1} and H_h^{i+1}, $i = 0, \ldots, m$, be the solutions to Problems 4.1 and 5.1, respectively. Then,*

$$\Delta t \sum_{i=1}^{m} \left\| H^{i+1} - H_h^{i+1} \right\|_{\mathrm{L}_r^2(\Omega)}^2 \leq C \left(h^2 + \left\| H_0 - H_{0h} \right\|_{\mathrm{L}_r^2(\Omega)}^2 \right).$$

Proof We split the quantities to estimate into two terms:

$$H^{i+1} - H_h^{i+1} = (H^{i+1} - P_h H^{i+1}) + (P_h H^{i+1} - H_h^{i+1}) \tag{5.29}$$

The first one is a projection error that can be bounded by using the results from the previous section. The second one is a purely discrete term, which we denote

$$\rho_h^{i+1} := P_h H^{i+1} - H_h^{i+1}, \qquad i = 0, \ldots, m-1.$$

Notice that $\rho_h^{i+1} \in \mathcal{U}_h$, because $(P_h H^{i+1})|_\Gamma = \Pi_\Gamma^h (H^{i+1}|_\Gamma) = H_h^{i+1}|_\Gamma$ (cf. (5.8) and the second equation from Problem 5.1).

A calculation from the first equations of Problems 4.1 and 5.1 and (5.7) yields

$$\int_\Omega \left(\bar{\partial} \mathcal{B}(H^{i+1}) - \bar{\partial} \mathcal{B}(H_h^{i+1}) \right) G_h \, r \, dr \, dz + a(\rho_h^{i+1}, G_h) = 0 \qquad \forall G_h \in \mathcal{U}_h.$$

Summing up the above equations, we obtain

$$\int_{\Omega} \Big(\mathcal{B}(H^{l+1}) - \mathcal{B}(H_h^{l+1})\Big)\, G_h \, r \, dr \, dz + \Delta t \, a\left(\sum_{i=0}^{l} \rho_h^{i+1}, G_h\right) = \int_{\Omega} \left(\mathcal{B}(H_0) - \mathcal{B}(H_{0h})\right) G_h \, r \, dr \, dz$$

for $l = 0, \ldots, m-1$, or, equivalently,

$$\int_{\Omega} \Big(\mathcal{B}(P_h H^{l+1}) - \mathcal{B}(H_h^{l+1})\Big)\, G_h \, r \, dr \, dz + \Delta t \, a\left(\sum_{i=0}^{l} \rho_h^{i+1}, G_h\right) = \int_{\Omega} \left(\mathcal{B}(H_0) - \mathcal{B}(H_{0h})\right) G_h \, r \, dr \, dz + \int_{\Omega} \Big(\mathcal{B}(P_h H^{l+1}) - \mathcal{B}(H^{l+1})\Big)\, G_h \, r \, dr \, dz.$$

Hence, choosing $G_h = \rho_h^{l+1}$, using the strong monotonicity and Lipschitz continuity of $\mathcal{B}$ (cf. H.2* and H.1*), Cauchy-Schwartz and Young's inequalities, we obtain

$$\frac{\beta}{2} \left\| \rho_h^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \Delta t \, a\left(\sum_{i=0}^{l} \rho_h^{i+1}, \rho_h^{l+1}\right) \leq \frac{C}{\beta} \left\| H_0 - H_{0h} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \frac{C}{\beta} \left\| P_h H^{l+1} - H^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)}.$$

Now, summing up the above equations multiplied by Δt and using Remark 5.1,

$$\frac{\beta}{2} \sum_{l=0}^{m-1} \Delta t \left\| \rho_h^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \Delta t^2 \sum_{l=0}^{m-1} a\left(\sum_{i=0}^{l} \rho_h^{i+1}, \rho_h^{l+1}\right) \leq \frac{CT}{\beta} \left\| H_0 - H_{0h} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \frac{C\Delta t}{\beta} \sum_{l=0}^{m-1} h^2 \left\| H^{l+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}.$$

On the other hand, writing $\rho_h^{l+1} = \sum_{i=0}^{l} \rho_h^{i+1} - \sum_{i=0}^{l-1} \rho_h^{i+1}$ and using the identity $2(p-q)p = p^2 + (p-q)^2 - q^2$ and the ellipticity of $a(\cdot,\cdot)$ (cf. Lemma 3.1), it is easy to obtain the following inequality:

$$\Delta t^2 \sum_{l=0}^{m-1} a\left(\sum_{i=0}^{l} \rho_h^{i+1}, \rho_h^{l+1}\right) \geq \frac{\gamma}{2} \left\| \Delta t \sum_{i=0}^{m-1} \rho_h^{i+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)}. \tag{5.30}$$

Hence, substituting this inequality into the previous one, we have that

$$\sum_{l=0}^{m-1} \Delta t \left\| \rho_h^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \left\| \Delta t \sum_{i=0}^{m-1} \rho_h^{i+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \leq C \left\{ \left\| H_0 - H_{0h} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \Delta t \sum_{l=0}^{m-1} h^2 \left\| H^{l+1} \right\|^2_{\widetilde{\mathrm{H}}^1_r(\Omega)} \right\}.$$

Whence, from Lemma 4.1 we obtain

$$\sum_{l=0}^{m-1} \Delta t \left\| \rho_h^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} \leq C \left\{ \| H_0 - H_{0h} \|^2_{\mathrm{L}^2_r(\Omega)} + h^2 \right\}.$$

Thus, the result follows from the decomposition (5.29), the above inequality, Remark 5.1 and Lemma 4.1 again. □

Remark 5.2 If the initial data is taken as $H_{0h} := I_h H_0$, with I_h being the Clément-type interpolant operator used in the previous section, then, because of (5.1),

$$\left(\Delta t \sum_{i=1}^{m} \left\| H^{i+1} - H_h^{i+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} \right)^{1/2} \leq Ch \left\{ 1 + \| H_0 \|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \right\}.$$

The following result, whose proof follows immediately from Lemma 5.6 and Theorem 4.1, yields an error estimate for the fully discrete problem.

Theorem 5.1 *Let H and H_h^{i+1}, $i = 0, \ldots, m$, be the solutions to Problems 3.1 and 5.1, respectively. Let $\overline{H}^h_{\Delta t}$ be the step function defined by*

$$\overline{H}^h_{\Delta t}(t^0) := H_h^0; \qquad \overline{H}^h_{\Delta t}(t) := H_h^i, \quad t \in (t^{i-1}, t^i], \quad i = 1, \ldots, m.$$

Then, under hypotheses H.1, H.2*, H.3, H.4*, H.5, H.6 and H.7,*

$$\left\| H - \overline{H}^h_{\Delta t} \right\|_{L^2(0,T;\mathrm{L}^2_r(\Omega))} \leq C \left\{ h + \Delta t + \| H_0 - H_{0h} \|_{\widetilde{\mathrm{H}}^1_r(\Omega)} \right\}.$$

Notice that the above result does not require any additional regularity assumption on the solution of the continuous problem H. However, the order $\mathcal{O}(h)$ in the error estimate is not necessarily optimal for regular solutions. Our next goal is to show that this order can be improved when the solution to Problem 3.1 is assumed to be more regular.

Theorem 5.2 *Let H and H_h^{i+1}, $i = 0, \ldots, m$, be the solutions to Problems 3.1 and 5.1, respectively. Under hypotheses H.1*, H.2*, H.3, H.4*, H.5, H.6 and H.7, if $H \in \mathrm{H}^1(0,T;\mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^1_r(\Omega))$, then*

$$\left(\sum_{i=0}^{m-1} \Delta t \left\| H(t^{i+1}) - H_h^{i+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} \right)^{1/2}$$
$$\leq C \left\{ (\Delta t + h^2) \| H \|_{\mathrm{H}^1(0,T;\mathrm{H}^2_r(\Omega) \cap \widetilde{\mathrm{H}}^1_r(\Omega))} + \| H_0 - H_{0h} \|_{\mathrm{L}^2_r(\Omega)} + \Delta t \| f \|_{\mathrm{H}^1(0,T;\mathcal{U}')} \right\}.$$

Proof Once more, we split the error into two terms,

$$H(t^{i+1}) - H_h^{i+1} = (H(t^{i+1}) - P_h H(t^{i+1})) + (P_h H(t^{i+1}) - H_h^{i+1}), \tag{5.31}$$

where the first one is a projection error that can be bounded by using Lemma 5.5 and the second one is a purely discrete term that we denote

$$\widehat{\rho}_h^{i+1} := P_h H(t^{i+1}) - H_h^{i+1}, \qquad i = 0, \ldots, m-1.$$

Notice that $\widehat{\rho}_h^{i+1} \in \mathcal{U}_h$, because $(P_h H(t^{i+1}))|_\Gamma = \Pi_\Gamma^h(g(t^{i+1})) = H_h^{i+1}|_\Gamma$ (cf. (5.8) and the second equations from Problems 3.1 and 5.1).

To estimate this term, we integrate from 0 to t^{l+1} the first equation of Problem 3.1 and use (5.7) to obtain for all $G_h \in \mathcal{U}_h$

$$\int_\Omega \mathcal{B}(H(t^{l+1}))\, G_h\, r\, dr\, dz + \Delta t\, a\left(\sum_{i=0}^{l} P_h H(t^{i+1}), G_h\right) = a\left(\int_0^{t^{l+1}} (\widehat{H}_{\Delta t} - H)\, dt, G_h\right) + \left\langle \int_0^{t^{l+1}} f\, dt, G_h \right\rangle + \int_\Omega \mathcal{B}(H_0)\, G_h\, r\, dr\, dz,$$

where $\widehat{H}_{\Delta t}$ denotes the step function defined by

$$\widehat{H}_{\Delta t}(t^0) := H(t^0), \qquad \widehat{H}_{\Delta t}(t) := H(t^i), \quad t \in (t^{i-1}, t^i]. \quad i = 1, \ldots, m.$$

(Notice that we have used a different notation this time, since $\overline{H}_{\Delta t}$ was already used in (4.2) for another step function.)

On the other hand, by summing up the first equation of Problem 5.1 for $i = 0, \ldots, l$, it follows that for all $G_h \in \mathcal{U}_h$

$$\int_\Omega \mathcal{B}(H_h^{l+1})\, G_h\, r\, dr\, dz + \Delta t\, a\left(\sum_{i=0}^{l} H_h^{i+1}, G_h\right) = \left\langle \Delta t \sum_{i=0}^{l} f^{i+1}, G_h \right\rangle + \int_\Omega \mathcal{B}(H_{0h})\, G_h\, r\, dr\, dz.$$

Subtracting this equation from the previous one, we obtain for all $G_h \in \mathcal{U}_h$

$$\begin{aligned} &\int_\Omega \left(\mathcal{B}(P_h H(t^{l+1})) - \mathcal{B}(H_h^{l+1})\right) G_h\, r\, dr\, dz + \Delta t\, a\left(\sum_{i=0}^{l} \widehat{\rho}_h^{i+1}, G_h\right) \\ &\quad = \int_\Omega (\mathcal{B}(H_0) - \mathcal{B}(H_{0h}))\, G_h\, r\, dr\, dz \\ &\qquad + \int_\Omega \left(\mathcal{B}(P_h H(t^{l+1})) - \mathcal{B}(H(t^{l+1}))\right) G_h\, r\, dr\, dz \\ &\qquad + a\left(\int_0^{t^{l+1}} (\widehat{H}_{\Delta t} - H)\, dt, G_h\right) + \left\langle \int_0^{t^{l+1}} \left(f - \overline{f}_{\Delta t}\right)\, dt, G_h \right\rangle. \end{aligned}$$

At this point we proceed as in the proof of Lemma 5.6. We choose $G_h = \widehat{\rho}_h^{l+1}$ and use the strong monotonicity and Lipschitz continuity of $\mathcal{B}$ (cf. H.2* and H.1*), Cauchy-Schwartz and Young's inequalities, to write

$$\begin{aligned} &\frac{\beta}{2} \left\| \widehat{\rho}_h^{l+1} \right\|^2_{\mathrm{L}^2_r(\Omega)} + \Delta t\, a\left(\sum_{i=0}^{l} \widehat{\rho}_h^{i+1}, \widehat{\rho}_h^{l+1}\right) \\ &\quad \le \frac{C}{\beta} \|H_0 - H_{0h}\|^2_{\mathrm{L}^2_r(\Omega)} + \frac{C}{\beta} \left\| P_h H(t^{l+1}) - H(t^{l+1}) \right\|^2_{\mathrm{L}^2_r(\Omega)} \\ &\qquad + a\left(\int_0^{t^{l+1}} (\widehat{H}_{\Delta t} - H)\, dt, \widehat{\rho}_h^{l+1}\right) + \left\langle \int_0^{t^{l+1}} \left(f - \overline{f}_{\Delta t}\right)\, dt, \widehat{\rho}_h^{l+1} \right\rangle. \end{aligned}$$

Then, we sum up the above equations multiplied by Δt and obtain

$$\frac{\beta}{2}\sum_{l=0}^{m-1}\Delta t\left\|\widehat{\rho}_h^{l+1}\right\|_{\mathrm{L}_r^2(\Omega)}^2+\Delta t^2\sum_{l=0}^{m-1}a\left(\sum_{i=0}^{l}\widehat{\rho}_h^{i+1},\widehat{\rho}_h^{l+1}\right)$$
$$\leq\frac{CT}{\beta}\left\|H_0-H_{0h}\right\|_{\mathrm{L}_r^2(\Omega)}^2+\frac{C\Delta t}{\beta}\sum_{l=0}^{m-1}\left\|P_hH(t^{l+1})-H(t^{l+1})\right\|_{\mathrm{L}_r^2(\Omega)}^2$$
$$+\Delta t\sum_{l=0}^{m-1}a\left(\int_0^{t^{l+1}}(\widehat{H}_{\Delta t}-H)\,dt,\widehat{\rho}_h^{l+1}\right)+\Delta t\sum_{l=0}^{m-1}\left\langle\int_0^{t^{l+1}}\left(f-\overline{f}_{\Delta t}\right)\,dt,\widehat{\rho}_h^{l+1}\right\rangle. \tag{5.32}$$

We estimate the second term on the left-hand side above also as we did in the proof of Lemma 5.6 (cf. (5.30)):

$$\Delta t^2\sum_{l=0}^{m-1}a\left(\sum_{i=0}^{l}\widehat{\rho}_h^{i+1},\widehat{\rho}_h^{l+1}\right)\geq\frac{\gamma}{2}\left\|\Delta t\sum_{i=0}^{m-1}\widehat{\rho}_h^{i+1}\right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2. \tag{5.33}$$

On the other hand, it is easy to prove by summation by parts that

$$\Delta t\sum_{l=0}^{m-1}a\left(\int_0^{t^{l+1}}(\widehat{H}_{\Delta t}-H)\,dt,\widehat{\rho}_h^{l+1}\right)$$
$$=a\left(\int_0^{T}(\widehat{H}_{\Delta t}-H)\,dt,\Delta t\sum_{l=0}^{m-1}\widehat{\rho}_h^{l+1}\right)-\sum_{l=0}^{m-2}a\left(\int_{t^{l+1}}^{t^{l+2}}(\widehat{H}_{\Delta t}-H)\,dt,\Delta t\sum_{i=0}^{l}\widehat{\rho}_h^{i+1}\right)$$

and

$$\Delta t\sum_{l=0}^{m-1}\left\langle\int_0^{t^{l+1}}\left(f-\overline{f}_{\Delta t}\right)\,dt,\widehat{\rho}_h^{l+1}\right\rangle$$
$$=\left\langle\int_0^{T}\left(f-\overline{f}_{\Delta t}\right)\,dt,\Delta t\sum_{l=0}^{l}\widehat{\rho}_h^{l+1}\right\rangle-\sum_{l=0}^{m-2}\left\langle\int_{t^{l+1}}^{t^{l+2}}\left(f-\overline{f}_{\Delta t}\right)\,dt,\Delta t\sum_{i=0}^{l}\widehat{\rho}_h^{i+1}\right\rangle.$$

Now, by replacing these two equations and (5.33) into (5.32) and using the continuity of $a(\cdot,\cdot)$ and Young's inequality, it follows that

$$\sum_{l=0}^{m-1}\Delta t\left\|\widehat{\rho}_h^{l+1}\right\|_{\mathrm{L}_r^2(\Omega)}^2+\left\|\Delta t\sum_{i=0}^{m-1}\widehat{\rho}_h^{i+1}\right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2$$
$$\leq C\left\{\left\|H_0-H_{0h}\right\|_{\mathrm{L}_r^2(\Omega)}^2+\Delta t\sum_{l=0}^{m-1}\left\|P_hH(t^{l+1})-H(t^{l+1})\right\|_{\mathrm{L}_r^2(\Omega)}^2\right.$$
$$\left.+\left\|f-\overline{f}_{\Delta t}\right\|_{\mathrm{L}^2(0,T;\mathcal{U}')}^2+\left\|\widehat{H}_{\Delta t}-H\right\|_{\mathrm{L}^2(0,T;\widetilde{\mathrm{H}}_r^1(\Omega))}^2+\sum_{l=0}^{m-2}\left\|\Delta t\sum_{i=0}^{l}\widehat{\rho}_h^{i+1}\right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2\right\}.$$

Hence, by using a discrete Gronwall's lemma, classical interpolation results and Lemma 5.5, we obtain

$$\sum_{l=0}^{m-1} \Delta t \left\| \widehat{\rho}_h^{l+1} \right\|_{\mathrm{L}_r^2(\Omega)}^2 + \left\| \Delta t \sum_{i=0}^{m-1} \widehat{\rho}_h^{i+1} \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}^2$$
$$\leq C \Big\{ \|H_0 - H_{0h}\|_{\mathrm{L}_r^2(\Omega)}^2 + \Delta t^2 \|f\|_{\mathrm{H}^1(0,T;\mathrm{L}_r^2(\Omega))}^2$$
$$+ (\Delta t^2 + h^4) \|H\|_{\mathrm{H}^1(0,T;\mathrm{H}_r^2(\Omega)\cap\widetilde{\mathrm{H}}_r^1(\Omega))}^2 \Big\}. \tag{5.34}$$

Therefore, the result follows from (5.31), this estimate and Lemma 5.5. □

Remark 5.3 When $H_0 \in \mathrm{H}_r^2(\Omega) \cap \widetilde{\mathrm{H}}_r^1(\Omega)$, we can use, for instance, $H_{0h} := I_h H_0$, with I_h being again the Clément-type interpolant operator used in the previous section. In such a case, from (5.2) we have

$$\left(\sum_{i=0}^{m-1} \Delta t \left\| H(t^{i+1}) - H_h^{i+1} \right\|_{\mathrm{L}_r^2(\Omega)}^2 \right)^{1/2}$$
$$\leq C \Big\{ (\Delta t + h^2) \|H\|_{\mathrm{H}^1(0,T;\mathrm{H}_r^2(\Omega)\cap\widetilde{\mathrm{H}}_r^1(\Omega))}$$
$$+ h^2 \|H_0\|_{\mathrm{H}_r^2(\Omega)\cap\widetilde{\mathrm{H}}_r^1(\Omega)} + \Delta t \|f\|_{\mathrm{H}^1(0,T;\mathrm{L}_r^2(\Omega))} \Big\}.$$

Remark 5.4 Let us further assume that Ω is a rectangle. In such a case, the following error estimate holds:

$$\max_{1\leq l\leq m} \left\| \sum_{i=0}^{l-1} \Delta t (H(t^{i+1}) - H_h^{i+1}) \right\|_{\widetilde{\mathrm{H}}_r^1(\Omega)}$$
$$\leq C \Big\{ (\Delta t + h) \|H\|_{\mathrm{H}^1(0,T;\mathrm{H}_r^2(\Omega)\cap\widetilde{\mathrm{H}}_r^1(\Omega))}$$
$$+ \|H_0 - H_{0h}\|_{\mathrm{L}_r^2(\Omega)} + \Delta t \|f\|_{\mathrm{H}^1(0,T;\mathrm{L}_r^2(\Omega))} \Big\}.$$

In fact, a similar error estimate, but in the norm induced by $a(\cdot,\cdot)$ holds for any convex domain as a consequence of (5.22), (5.31) and (5.34). Hence, the estimate above follows from the equivalence between both norms in rectangles proved in [12, Proposition 3.1].

6 Numerical experiments

We have developed a FORTRAN code which implements the fully discrete numerical scheme analyzed in the previous section. To solve the non-linear systems we have used Newton's method.

In order to test the error estimate proved for the numerical scheme (cf. Theorem 5.2), we have used a problem with a known analytical solution. Let $\Omega := (0,1) \times (-1,1)$, $T = 1$ and the electrical conductivity $\sigma = 1$. We have considered a non-linear H-B curve given by

$$B(H) = H + \arctan(H).$$

Finally, we have chosen the right-hand side f, the boundary condition g and the initial data B_0 so that the solution is

$$H(r,z,t) = e^t \sin(\pi r/2)\sin(\pi z/2).$$

The method has been used on several successively refined meshes and time-steps, both chosen in a convenient way to analyze the convergence with respect to these discretization parameters. The numerical approximations have been compared with the analytical solution by computing the percentage error for H in a discrete $\mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega))$-norm as follows:

$$E_h^{\Delta t}(H) := 100 \frac{\left(\sum_{i=0}^{m-1} \Delta t \left\| H(t^{i+1}) - H_h^{i+1} \right\|^2_{\mathrm{L}^2_r(\Omega)}\right)^{1/2}}{\left(\sum_{i=0}^{m-1} \Delta t \left\| H(t^{i+1}) \right\|^2_{\mathrm{L}^2_r(\Omega)}\right)^{1/2}}.$$

We have also computed the percentage error for the eddy current $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$ (cf. (2.3)) in the analogous discrete $\mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega)^2)$-norm:

$$E_h^{\Delta t}(\boldsymbol{J}) := 100 \frac{\left(\sum_{i=0}^{m-1} \Delta t \left\| \mathbf{curl}\,\boldsymbol{H}(t^{i+1}) - \mathbf{curl}\,\boldsymbol{H}_h^{i+1} \right\|^2_{\mathrm{L}^2_r(\Omega)^2}\right)^{1/2}}{\left(\sum_{i=0}^{m-1} \Delta t \left\| \mathbf{curl}\,\boldsymbol{H}(t^{i+1}) \right\|^2_{\mathrm{L}^2_r(\Omega)^2}\right)^{1/2}},$$

where $\boldsymbol{H}_h^{i+1} := H_h^{i+1}\boldsymbol{e}_\theta$.

Table 1 shows the percentage errors $E_h^{\Delta t}(H)$ for the magnetic field at different levels of discretization. Taking a small enough time-step Δt, one can observe the behavior of the error with respect to the space discretization (see, for instance, the last row of the table). On the other hand, by considering a small enough mesh-size h, one can inspect the order of convergence with respect to Δt (see, for instance, the last column). In this example, we observe an order of convergence $\mathcal{O}(h^2 + \Delta t)$, which coincides with that predicted by the theoretical analysis (cf. Remark 5.3).

Table 1 Percentage errors of the computed magnetic field: $E_h^{\Delta t}(H)$; $\Delta t_0 = 0.2$, $h_0 = \sqrt{2}/2$.

Δt	h_0	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$	$h_0/32$
Δt_0	11.303186	2.776175	0.649365	0.244530	0.244530	0.257269
$\Delta t_0/2$	11.319166	2.813540	0.672449	0.169753	0.122235	0.131386
$\Delta t_0/4$	11.327962	2.834780	0.692873	0.161602	0.063345	0.065098
$\Delta t_0/8$	11.332618	2.846108	0.705389	0.167654	0.042866	0.031640
$\Delta t_0/16$	11.335037	2.852116	0.712470	0.173253	0.040402	0.016118
$\Delta t_0/32$	11.336268	2.855402	0.716551	0.177379	0.042634	0.010994
$\Delta t_0/64$	11.336871	2.858055	0.720272	0.181395	0.046589	0.013961

In Table 2 we report the percentage errors $E_h^{\Delta t}(\boldsymbol{J})$ for the current density. As in the previous table, one can observe the behavior of the error with respect to space and time discretization by taking small enough time-step Δt and mesh-size h, respectively. In this case we observe an order of convergence $\mathcal{O}(h + \Delta t)$. Although such behavior has not been proved, the reported numerical results agree with what can be expected from Remark 5.4.

Table 2 Percentage errors of the computed current density: $E_h^{\Delta t}(\boldsymbol{J})$; $\Delta t_0 = 0.5$, $h_0 = \sqrt{2}/16$.

Δt	h_0	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$	$h_0/32$
Δt_0	2.295415	1.460797	1.164167	1.077472	1.054695	1.048924
$\Delta t_0/2$	2.115802	1.165048	0.763272	0.623870	0.583859	0.573421
$\Delta t_0/4$	2.055560	1.057179	0.588016	0.391297	0.323992	0.304855
$\Delta t_0/8$	2.037246	1.024542	0.528749	0.295576	0.198324	0.165304
$\Delta t_0/16$	2.031716	1.015408	0.511773	0.264467	0.148224	0.099893
$\Delta t_0/32$	2.029951	1.012872	0.507225	0.255804	0.132270	0.074239

Finally, we report simultaneous dependence on h and Δt for the errors in both quantities, the magnetic field and the current density: $E_h^{\Delta t}(H)$ and $E_h^{\Delta t}(\boldsymbol{J})$, respectively. With this aim, we proceed in the following way: first, in each case, we choose initial values of h and Δt so that the time and the space discretization errors are both of approximately the same size; then, for each of the successively refined meshes, we take values of Δt proportional to h^2 in the first case and to h in the second one (see the values within boxes in Tables 1 and 2, respectively).

Fig. 2 shows log-log plots of the corresponding percentage errors. The slopes of the curves show clear orders of convergence $\mathcal{O}(h^2) = \mathcal{O}(h^2 + \Delta t)$ for $E_h^{\Delta t}(H)$ and $\mathcal{O}(h) = \mathcal{O}(h + \Delta t)$ for $E_h^{\Delta t}(\boldsymbol{J})$.

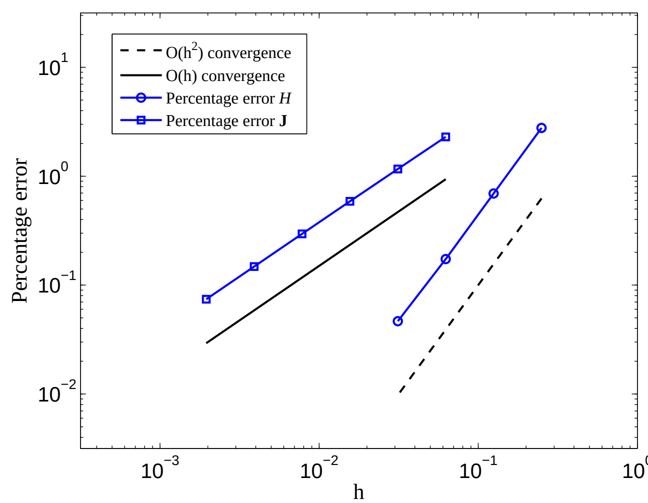

Fig. 2 Percentage errors $E_h^{\Delta t}(H)$ and $E_h^{\Delta t}(\boldsymbol{J})$ versus the mesh-size h (log-log scale), with Δt proportional to h^2 for the former and to h for the latter.

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