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Fabián Flores-Bazán

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Abstract

An alternative-type version of the Fritz John optimality conditions is established, which covers situations where no result appearing elsewhere is applicable. As a by product, a versatile formulation of this necessary Fritz John optimality conditions along with a simple proof is provided. This encompasses several versions appearing in the literature.

Key words. Nonlinear Programming, Fritz John conditions.

1 Introduction

The Fritz John necessary optimality conditions along with the Karush-Kuhn-Tucker have proved to be one of the fundamental pieces in the development of nonlinear optimization. There is not a standard way in the presentation of these results in most Textbooks devoted to students coming from mathematics departament and even from engineering, see Bazaraa et al. (2006); Bector et al. (2005). The main goal of this note is to provide an alternative-type version of the Fritz John optimality condition, whose proof uses simple separation theorems on convex sets and properties of polar cones. Our formulation is versatile and requires the notion of contingent cone. Its versatility is shown by recovering various versions appearing in the common literature, and it is suitable for expository purposes.

*Departamento de Ingeniería Matemática and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile (fflores@ing-mat.udec.cl). This research was supported in part by CONICYT-Chile through FONDECYT 112-0980, BASAL Projects, CMM, Universidad de Chile.

Section 2 starts with basic definitions of contingent cone, polar cone, and a new characterization of a disjunction in term of pointedness. In addition, a Gordan-type alternative theorem suitable for our purpose is established. The alternative-type version of the Fritz John optimality conditions is established in Section 3 (see Theorem 3.1). Particular situations appearing in the literature are dicussed in Section 4.

2 Basic definitions and preliminary results

In what follows given a set $A \subseteq \mathbb{R}^n$, its closure is denoted by $\overline{A}$; its convex hull by $\mathrm{co}(A)$ which is the smallest convex set containing A; its topological interior by $\mathrm{int}\ A$. We set $\mathrm{cone}(A) \doteq \bigcup_{t\geq 0} tA$, $\mathrm{cone}_+(A) \doteq \bigcup_{t>0} tA$ and $\overline{\mathrm{cone}}(A) \doteq \overline{\bigcup_{t\geq 0} tA}$.

Definition 2.1. *Let $\emptyset \neq K \subseteq \mathbb{R}^n$ and $\bar{x} \in \overline{K}$, the Contingent cone of K at $\bar{x}$, denoted by $T(K;\bar{x})$, is the set*

$$T(K;\bar{x}) \doteq \Big\{ v \in \mathbb{R}^n :\ \exists\, t_k > 0, \exists\, x_k \in K,\ x_k \to \bar{x},\ t_k(x_k - \bar{x}) \to v \Big\}.$$

When K is convex and $\bar{x} \in K$, then it is not hard to prove that

$$T(K;\bar{x}) = \overline{\bigcup_{t\geq 0} t(K-\bar{x})}.$$

By $\langle \cdot,\cdot \rangle$ we stand for the inner or scalar product in $\mathbb{R}^n$, whose elements in $\mathbb{R}^n$ are considered column vectors. Thus, $\langle a,b \rangle = a^\top b$ for all $a, b \in \mathbb{R}^n$.
Given a nonempty set $P \subseteq \mathbb{R}^n$, its polar cone, P^*, is defined as

$$P^* = \{\xi \in \mathbb{R}^n :\ \langle \xi, p \rangle \geq 0 \ \ \forall\, p \in P\}.$$

It well known that, whenever P is a closed convex cone, we have (the bipolar theorem) $P = P^{**} \doteq (P^*)^*$, and in general we have $P^{**} = \overline{\mathrm{co}}(\mathrm{cone}\ P)$.
We say that a (not necessarily convex) cone, P, is pointed if $\mathrm{co}\ P \cap (-\mathrm{co}\ P) = \{0\}$. Notice that

$$\mathrm{cone}(A) \text{ is pointed } \iff \mathrm{cone}(\mathrm{co}\ A) \text{ is pointed.} \tag{1}$$

Given a convex set $A \subseteq \mathbb{R}^n$, the (outward) normal cone (in the sense of convex analysis) of A at $\bar{x} \in A$, is the set

$$N(A;\bar{x}) \doteq \{\xi \in \mathbb{R}^n :\ \langle \xi, x - \bar{x} \rangle \leq 0, \ \ \forall\, x \in A\}.$$

Part of the next theorem appears, in a more general framework, in Theorem 3.2 of [8].

Theorem 2.2. *Let $P \subseteq \mathbb{R}^m$ be a convex closed cone such that* $\text{int } P \neq \emptyset$*, and $A \subseteq \mathbb{R}^m$ be any nonempty set. Then the following assertions are equivalent:*

(a) $\exists\ \lambda^* \in P^* \setminus \{0\}$, $\langle \lambda^*, a\rangle \geq 0, \quad \forall\ a \in A$;

(b) $\text{cone}(A + \text{int } P)$ *is pointed;*

(c) $\text{co}(A) \cap (-\text{int } P) = \emptyset$.

Proof. $(a) \Longrightarrow (b)$: Suppose that $0 = \sum_{i=1}^{l} x_i$ with $x_i \in \text{cone}(A + \text{int } P)$, we shall prove that $x_i = 0$ for all i. By choice, $x_i = t_i(a_i + p_i)$ with $t_i \geq 0$, $a_i \in A$, $p_i \in \text{int } P$ for $i = 1, \ldots, l$. This implies that $\sum_{i=1}^{l} t_i a_i \in -\text{int } P$. This yields a contradiction if $\sum_{i=1}^{l} t_i > 0$ under (a), since the inequality in (a) also holds for all $a \in \text{co}(A)$, and

$$\text{int } P = \{p \in P :\ \langle q, p\rangle > 0\ \ \forall\ q \in P^*,\ q \neq 0\}.$$

$(b) \Longrightarrow (c)$: By (1), $\text{cone}(\text{co } A + \text{int } P)$ is pointed. Assume on the contrary that $\text{co}(A) \cap (-\text{int } P) \neq \emptyset$. Then, $0 \in \text{co } A + \text{int } P$. This implies that $\text{cone}(\text{co } A + \text{int } P) = \mathbb{R}^m$, contradicting (b).
$(c) \Longrightarrow (a)$: By applying a standard theorem on separation of convex sets, we get the existence of $p \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle p, z\rangle \geq \alpha,\quad \forall\ z \in \text{co } A,\ \text{ and } \langle p, w\rangle \leq \alpha,\quad \forall\ w \in -\overline{\text{int } P} = -P. \tag{2}$$

From the first inequality of (2) it follows that $\langle p, a\rangle \geq \alpha$ for all $a \in A$, and from the second inequality we get $\alpha \geq 0$. Hence $p \in P^*$, proving the desired result. $\square$

Remark 2.3. *It is not difficult to check that for any set $A \subseteq \mathbb{R}^m$,*

$$A \cap (-\text{int } P) = \emptyset \Longleftrightarrow \overline{A} \cap (-\text{int } P) = \emptyset \Longleftrightarrow (A + P) \cap (-\text{int } P) = \emptyset$$

$$\Longleftrightarrow \overline{\text{cone}}(A + P) \cap (-\text{int } P) = \emptyset.$$

We can go further when A is the image of a subset $C \subseteq \mathbb{R}^n$ through a linear transformation $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$. The next proposition can be considered as a Gordan-type alternative theorem, and it has its origin in Proposition 2.7 of [7].

Proposition 2.4. *Let $\mathcal{F}$ be a real matrix of order $m \times n$, and write $\mathcal{F}^\top = (\mathcal{F}_1^\top \ \cdots \ \mathcal{F}_m^\top)$, where $\mathcal{F}_i$ is the $i-$th row of $\mathcal{F}$. Let $C \subseteq \mathbb{R}^n$ be any nonempty set. Then*

$$\mathcal{F}(C) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset \Longleftrightarrow \max_{1 \le i \le m} \langle \mathcal{F}_i^\top, v \rangle \ge 0 \quad \forall \ v \in \overline{C},$$

and the following statements are equivalent:

(*a*) $\text{cone}(\mathcal{F}(C) + \text{int } \mathbb{R}^m_+)$ *is pointed;*

(*b*) $\mathcal{F}(\overline{\text{co}}(C)) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset$;

(*c*) $\mathcal{F}(\text{co}(C)) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset$;

(*d*) $\max_{1 \le i \le m} \langle \mathcal{F}_i^\top, v \rangle \ge 0 \quad \forall \ v \in \overline{\text{co}}(C)$;

(*e*) $\text{co}(\{\mathcal{F}_i^\top : \ i = 1, \dots, m\}) \cap C^* \neq \emptyset$.

Proof. The first part is straightforward. By the previous theorem

$$\text{cone}(\mathcal{F}(C) + \text{int } \mathbb{R}^m_+) \text{ is pointed } \Longleftrightarrow \text{co}(\mathcal{F}(C)) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset.$$

It is easy to check that $\text{co}(\mathcal{F}(C)) = \mathcal{F}(\text{co}(C))$ and

$$\begin{aligned} \mathcal{F}(\text{co}(C)) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset &\Longleftrightarrow \mathcal{F}(\overline{\text{co}}(C)) \cap (-\text{int } \mathbb{R}^m_+) = \emptyset \\ &\Longleftrightarrow \overline{\mathcal{F}(\text{co}(C))} \cap (-\text{int } \mathbb{R}^m_+) = \emptyset. \end{aligned}$$

Both relations along with the fact that (a) of Theorem 2.2 amounts to writing

$$\text{co}(\{\mathcal{F}_i^\top : \ i = 1, \dots, m\}) \cap (C)^* \neq \emptyset,$$

allow us to conclude with all the remaining equivalences. □

3 The Fritz-John optimality condition of the alternative-type

Let us consider the minimization problem with inequality constraints:

$$\begin{cases} \min f(x) \\ g_i(x) \le 0, \quad i = 1, \dots, m \\ x \in X, \end{cases} \tag{3}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are given functions, and $X \subseteq \mathbb{R}^n$ is any nonempty set. Define the feasible set to (3) as

$$K \doteq \{x \in X : \ g_i(x) \leq 0, \quad i = 1, \ldots, m\}.$$

For fixed $\bar{x} \in K$, we associate its active index set,

$$I = I(\bar{x}) \doteq \{i : \quad g_i(\bar{x}) = 0\}. \tag{4}$$

Based on Proposition 2.4 we establish a new version, as an alternative-type result, of the Fritz John optimality conditions, which is new in the literature.

Theorem 3.1. (*Fritz John necessary optimality conditions of alternative-type*) *Let us consider problem (3) and $\bar{x} \in K$, with $X \subseteq \mathbb{R}^n$. Let f, g_i, $i \in I$, be differentiable at $\bar{x}$. Then, exactly one of the following two assertions hold:*

(*a*) *there exists $v \in \mathbb{R}^n$ such that*

$$\begin{aligned} \langle \nabla f(\bar{x}), v \rangle < 0, \qquad & v \in \overline{\text{co}}[T(X; \bar{x})]; \\ \langle \nabla g_i(\bar{x}), v \rangle < 0, \qquad & i \in I, \end{aligned} \tag{5}$$

(*b*) *there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, satisfying*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in [T(X; \bar{x})]^*, \tag{6}$$

or equivalently, $\max_{i \in I} \{\langle \nabla f(\bar{x}), v \rangle, \langle \nabla g_i(\bar{x}), v \rangle\} \geq 0, \quad \forall \ v \in T(X; \bar{x}).$

Furthermore, if each g_i is differentiable at $\bar{x}$, condition (6) can be written as

$$\begin{cases} \lambda_0 \nabla f(\bar{x}) + \displaystyle\sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) \in [T(X; \bar{x})]^*; \\ \lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m. \end{cases} \tag{7}$$

Proof. This is a direct application of Proposition 2.4 to

$$\mathcal{F} \doteq \begin{pmatrix} \nabla f(\bar{x})^\top \\ \nabla g_I(\bar{x})^\top \end{pmatrix},$$

where $\nabla g_I(\bar{x})^\top$ is the matrix having as rows $\nabla g_i(\bar{x})^\top$ for $i \in I$. □

We now compare with a result, which in our opinion, is the more general one concerning the validity of the Fritz John optimality conditions. It appears in the book by Giorgi, Guerraggio and Therfelder [11, Theorem 3.6.5], but its origin goes back to [1]. Some recent remarks on Fritz John optimality conditions in the same direction as in [11] were presented in [9, Theorem 13].

Theorem 3.2. ([*11, Theorem* 3.6.5], [*9, Theorem* 13]) *Let $\bar{x} \in K$ be a local solution to problem (3) with $X \subseteq \mathbb{R}^n$. Let f, g_i, $i \in I$, be differentiable at $\bar{x}$. Then, for every convex subcone T_1 of $T(X;\bar{x})$ there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, satisfying*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i\in I} \lambda_i \nabla g_i(\bar{x}) \in [T_1]^*. \tag{8}$$

Next example shows an instance where our previous theorem is applicable providing a sharper result, whereas Theorem 3.2 yields less information.

Example 3.3. *Let us take $f(x_1,x_2) = x_1$, $g(x_1,x_2) = x_2$, and*

$$X = \{(x_1,x_2) : \ x_1x_2 = 0, \ x_1 \geq 0, \ x_2 \geq 0\}, \quad \bar{x} = (0,0).$$

Then, $T(X;\bar{x}) = X$, which is nonconvex. It follows that $[T(X;\bar{x})]^* = \mathbb{R}^2_+ = \overline{\text{co}}[T(X;\bar{x})]$, and $\bar{x} = (0,0)$ is a minimum of f on $\{(x_1,x_2) \in X : \ g(x_1,x_2) \leq 0\}$. Easy computations show that the corresponding system (5) has no solution, and therefore there exist λ_0, $\lambda_1 \geq 0$, not both zero, such that

$$\lambda_0 \nabla f(\bar{x}) + \lambda_1 \nabla g(\bar{x}) \in [T(X;\bar{x})]^*. \tag{9}$$

In this case, any $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ satisfies such conditions.
Being $T(X;\bar{x})$ nonconvex, the only non-trivial convex subcones are

$$T_1 = \{(x_1,0) : \ x_1 \geq 0\}, \ T_2 = \{(0,x_2) : \ x_2 \geq 0\}.$$

Any of these cones provide, via (8), less information than (9). Other candidates for T_1 are: the Clarke tangent cone of X at $\bar{x}$, $T_C(X;\bar{x})$, which is always convex, is $\{(0,0)\}$ in our case; the open cone of interior directions to X at $\bar{x}$, $I(X;\bar{x})$ ([1, Theorem 3.1]), which in our example is empty; the open cone of quasi-interior directions to X at $\bar{x}$, $Q(X;\bar{x})$ ([10]), which is also empty here.

It is well-known that besides differentiability of g_i at $\bar{x}$, for $i \in I$, continuity of g_i at $\bar{x}$ for $i \notin I$, local minimality of $\bar{x}$ imply that the system

$$\begin{aligned} &\langle \nabla f(\bar{x}), v \rangle < 0, \qquad v \in T(X; \bar{x}); \\ &\langle \nabla g_i(\bar{x}), v \rangle < 0, \qquad i \in I, \end{aligned} \tag{10}$$

has no solution (this will be proved in the next corollary by completeness), that is, (a) of Theorem 3.1 does not hold and therefore (b) is true, provided $T(X; \bar{x})$ is convex. This is expressed in the following result, whose formulation is very versatile as we shall show in Section 4, and it encompasses many recent results appearing in the literature.

Corollary 3.4. (*Fritz John necessary optimality conditions*) *Let us consider problem (3) and $\bar{x} \in K$. Let $X \subseteq \mathbb{R}^n$ such that $T(X; \bar{x})$ is convex. Let f, g_i, $i \in I$, be differentiable at $\bar{x}$; g_i, $i \notin I$, be continuous at $\bar{x}$. If $\bar{x}$ is a local solution to (3), then there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, satisfying*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in [T(X; \bar{x})]^*. \tag{11}$$

Proof. We claim that (a) of Theorem 3.1 does not hold, proving that the system (10) has no solution. The proof even if standard, we shall provide it just for convenience of the reader. Suppose, on the contrary, that v is a solution to (10) there exist sequences $\lambda_k > 0$, $x_k \in X$, $x_k \to \bar{x}$, satisfying $\lambda_k(x_k - \bar{x}) \to v$. By differentiability at $\bar{x}$

$$f(x_k) - f(\bar{x}) = \langle \nabla f(\bar{x}), x_k - \bar{x} \rangle + \|x_k - \bar{x}\| o(\|x_k - \bar{x}\|)$$

with $o(t) \to 0$ as $t \to 0$. On multiplying this equality by λ_k, letting $k \to +\infty$ and using the first inequality of (10), we get the existence of k_1 such that

$$f(x_k) < f(\bar{x}), \qquad \forall\ k \geq k_1. \tag{12}$$

It only remains to check that x_k is feasible for all k sufficiently large to reach a contradiction.

Let $i \in I$. We get similarly as for f

$$g_i(x_k) - g_i(\bar{x}) = \langle \nabla g_i(\bar{x}), x_k - \bar{x} \rangle + \|x_k - \bar{x}\| o(\|x_k - \bar{x}\|)$$

On multiplying by λ_k this equality and letting $k \to +\infty$, we obtain, for some k_2,

$$g_i(x_k) < 0, \qquad \forall\ i \in I, \ \ \forall\ k \geq k_2. \tag{13}$$

Since g_i is continuous for $i \notin I$, there exists k_3 such that

$$g_i(x_k) < 0, \quad \forall\, i \notin I, \ \ \forall\, k \geq k_3. \tag{14}$$

On combining (13) and (14), we conclude that x_k is feasible for all k sufficiently large. Hence $\bar{x}$ cannot be a local solution to (3), showing that (a) of the previous theorem is not valid, and therefore (b) holds, and the corollary follows. □

4 Some particular situations

Let us show some interesting specializations appearing in the literature where $T(X;\bar{x})$ is convex.

4.1 The set X is not convex with $T(X;\bar{x})$ convex

Let us consider the problem with an additional quadratic equality constraint:

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0, \quad i = 1, \ldots, m, \\ q(x) = 0, \\ x \in \mathbb{R}^n, \end{cases} \tag{15}$$

where q is any quadratic function of the form

$$q(x) \doteq \frac{1}{2} x^\top A x + a^\top x + \alpha,$$

with A being a (real) symmetric matrix, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Clearly the

$$X \doteq \{x \in \mathbb{R}^n : \ q(x) = 0 \ \}$$

is not necessarily convex even if q is convex. Let $\bar{x}$ feasible for Problem (15). It is not difficult to find that (see for instance [6, Theorem 2.1])

$$T(X;\bar{x}) = \Big\{ v \in \mathbb{R}^n : \ (A\bar{x} + a)^\top v = 0 \ \Big\} \ \text{ if } \ A\bar{x} + a \neq 0;$$

whereas

$$T(X;\bar{x}) = \Big\{ v \in \mathbb{R}^n : \ v^\top A v = 0 \Big\} \ \text{ if } \ A\bar{x} + a = 0.$$

This set, en general, is nonconvex. If additionally, q is convex, that is, A is positive semidefinite, a more precise formulation may be obtained since $v^\top Av \Longleftrightarrow Av = 0$:

$$T(X;\bar{x}) = \begin{cases} (A\bar{x}+a)^\perp & \text{if } A\bar{x}+a \neq 0; \\ \ker A & \text{if } A\bar{x}+a = 0. \end{cases} \tag{16}$$

Thus,

$$[T(X;\bar{x})]^* = \begin{cases} \mathbb{R}(A\bar{x}+a) & \text{if } A\bar{x}+a \neq 0; \\ (\ker A)^\perp = A(\mathbb{R}^n) & \text{if } A\bar{x}+a = 0. \end{cases}$$

Hence, the Fritz-John conditions (11) reduces to the existence of λ_0, λ_i, $i \in I$, not all zero, $\lambda \in \mathbb{R}$, $y \in \mathbb{R}^n$, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i\in I} \lambda_i \nabla g_i(\bar{x}) = \begin{cases} \lambda(A\bar{x}+a) & \text{if } A\bar{x}+a \neq 0; \\ Ay & \text{if } A\bar{x}+a = 0. \end{cases}$$

4.2 The set X is convex

In this case, $T(X;\bar{x})$ is convex, and since $T(X;\bar{x}) = \overline{\bigcup_{t\geq 0} t(X-\bar{x})}$, we get $[T(X;\bar{x})]^* = -N(X;\bar{x})$, and so (11) can be written as

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i\in I} \lambda_i \nabla g_i(\bar{x}) \in -N(X;\bar{x}).$$

Thus, Theorem 3.2.2 in Bector et al. (2005) is obtained.

4.3 The set X is open or $\bar{x} \in \text{int } X$

In this situation, $T(X;\bar{x}) = \mathbb{R}^n$, and therefore condition (11) reduces to

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i\in I} \lambda_i \nabla g_i(\bar{x}) = 0.$$

This is nothing else than Theorem 4.2.8 in Bazaraa et al. (2006), and Theorem 3.2.1 in Bector et al. (2005) when $X = \mathbb{R}^n$.

4.4 The set X is an affine subspace

This case deals with $X = \{x \in \mathbb{R}^n : \ Hx = d\} = \bar{x} + \ker H$, with H being a real $p \times n$ matrix and $\bar{x} \in X$. Thus, we obtain $T(X;\bar{x}) = \ker H$ and $[T(X;\bar{x})]^* = (\ker H)^* = (\ker H)^\perp$. Hence (11) is expressed as

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i\in I} \lambda_i \nabla g_i(\bar{x}) \in (\ker H)^\perp = H^\top(\mathbb{R}^p),$$

that is, there exists $y_i \in \mathbb{R}$, $i = 1, \dots, p$, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + H^\top y = 0.$$

4.5 The set X is polyhedral

Let us consider (see for instance Birbil et al., 2007)

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0, \quad i = 1, \dots, m, \\ h_j^\top x \leq d_j, \quad j = 1, \dots, p, \\ x \in \mathbb{R}^n. \end{cases} \tag{17}$$

In this case, we can take $X = \{x \in \mathbb{R}^n : \ Hx \leq d\}$ with H being a $p \times n$ matrix, and refine (11). More precisely, by denoting $h_j^\top$ to be the rows of H and setting $J \doteq \{j : \ h_j^\top \bar{x} = d\}$, the conclusion of Theorem 3.4 reduces to the existence of $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, and $u_j \geq 0$, $j \in J$, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} u_j h_j = 0. \tag{18}$$

Indeed, by setting $C \doteq \{v \in \mathbb{R}^n : \ h_j^\top v \leq 0, \ j \in J\}$, which is a closed convex cone, one can easily check that $T(X; \bar{x}) = C$ (see Lemma 5.1.4 in Bazaraa et al., 2006) By applying Proposition 2.4, we get the existence of $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) \in C^*.$$

The conclusion follows once we notice that $C^* = \{H_J u : \ u \leq 0\}$, with H_J is the matrix with columns h_j for $j \in J$. Thus we recovered the Fritz-John conditions as appears in Birbil et al. (2007).

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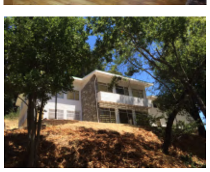

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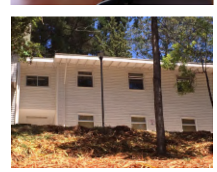
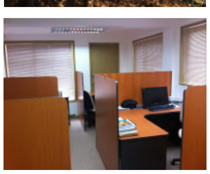

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