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## Centro de Investigación en Ingeniería Matemática (CI$^2$MA)

**Analysis of a mixed-FEM for the pseudostress-velocity formulation of the Stokes problem with varying density**

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# ANALYSIS OF A MIXED-FEM FOR THE PSEUDOSTRESS-VELOCITY FORMULATION OF THE STOKES PROBLEM WITH VARYING DENSITY

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**Abstract.** We propose and analyse a mixed finite element method for the nonstandard pseudostress-velocity formulation of the Stokes problem with varying density $\rho$ in $\mathbb{R}^d$, $d \in \{2,3\}$. Since the resulting variational formulation does not have the standard dual-mixed structure, we reformulate the continuous problem as an equivalent fixed-point problem. Then, we apply the classical Babuška-Brezzi theory to prove that the associated mapping $\mathbb{T}$ is well defined, and assuming that $\|\frac{\nabla\rho}{\rho}\|_{\mathbf{L}^\infty(\Omega)}$ is sufficiently small, we show that $\mathbb{T}$ is a contraction mapping, which implies that the variational formulation is well-posed. Under the same hypothesis on $\rho$ we prove stability of the continuous problem. Next, adapting to the discrete case the arguments of the continuous analysis, we are able to establish suitable hypotheses on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. A feasible choice of subspaces is given by Raviart-Thomas elements of order $k \geq 0$ for the pseudostress and polynomials of degree $k$ for the velocity. Finally, several numerical results illustrating the good performance of the method with these discrete spaces, and confirming the theoretical rate of convergence, are provided.

**Key words.** Stokes, varying density, pseudostress-velocity formulation, mixed finite elements, a priori error analysis.

**AMS subject classifications.** 65N30, 65N50, 74B05, 65N15.

## 1. Introduction.

The numerical simulation of incompressible fluid flow problems, modelled by the Stokes equations, has been widely studied during the last decades. Different formulations (velocity-pressure, vorticity-velocity-pressure and pseudostress-velocity, among others) and different numerical methods (conforming and nonconforming methods) have been introduced and analyzed, all of them with different advantages and disadvantages. In particular, the study of numerical methods for the stress- and pseudostress-based fomulations for the Stokes problem has become a very active research area during the last decade (see e.g. [5], [6], [7], [9], [12],[13], [15]), motivated by the fact that they provide a direct approximation of the stress or pseudostress tensor (besides the approximation of the velocity and/or pressure). This kind of formulations have been also extended to the case of quasi-Newtonian flows and multiphysics problems such as the Stokes-Darcy coupled problem (see e.g.[11], [14] and [16]).

Now, concerning the fluid flow problem studied in this paper, the first work in studying conforming finite element methods for the Stokes problem with varying density is [3], where the authors propose and analyse two variational formulations to solve the fluid flow problem. The first one is a velocity-pressure formulation which yields a nonsymetric saddle point formulation, whereas the second one is a momentum-pressure formulation which yields a standard saddle-point formulation. Well-posedness of the velocity-pressure formulation is analyzed by using a generalization of the Babuška-Brezzi theory introduced in [18] (see also [2]) whereas the classical Babuška-Brezzi theory is applied to prove well-posedness of the momentum-pressure formulation. It is important to notice that, in both cases, existence and uniqueness of solution of the continuous and discrete problems are

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attained by assuming that $\frac{\nabla\rho}{\rho}$ is not too large. Additionally, well-posedness of the discrete problem is fulfilled by assuming that the discretization parameter $h$ is sufficiently small.

In this paper, we introduce and analyze a pseudostress-velocity formulation for the Stokes problem with varying density analyzed in [3]. Since the resulting variational formulation does not have the standard dual-mixed structure, we reformulate the continuous problem as an equivalent fixed-point problem. Then, we apply the classical Babuška-Brezzi theory to prove that the associated mapping $\mathbb{T}$ is well defined, and assuming that $\|\frac{\nabla\rho}{\rho}\|_{\mathbf{L}^\infty(\Omega)}$ is sufficiently small, we show that $\mathbb{T}$ is a contraction mapping, which implies that the variational formulation is well-posed. We observe that this assumption is consistent with the approach in [3]. Next, we adapt the theory developed for the continuous problem to the discrete case, and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. We mention here that, unlike the approach in [3], we prove that our Galerkin scheme becomes well-posed without requiring any assumption on the discretization parameter $h$.

The rest of this work is organized as follows. In Section 2 we introduce the model problem and derive the mixed variational formulation. In Section 3 we analyse the well-posedness of the continuous problem. For the existence and uniqueness of solution we introduce an equivalent fixed-point problem and we prove that it has a unique solution, assuming that $\|\frac{\nabla\rho}{\rho}\|_{\mathbf{L}^\infty(\Omega)}$ is sufficiently small. Under a similar assumption we prove that the solution is stable. Next, in Section 4 we define the Galerkin scheme and derive general hypotheses on the finite element subspaces ensuring that, on the one hand, the discrete scheme becomes well-posed, and on the other hand, it satisfies a Cea estimate. Specific choices of finite element subspaces satisfying these assumptions are introduced in Section 5. Finally, numerical results illustrating the performance of the method are reported in Section 6.

## 2. Continuos problem.

In this section we introduce and analyze a weak dual-mixed formulation for the Stokes problem with varying density analyzed in [3]. In particular, we discuss existence, uniqueness and stability of solution. We start by introducing some definitions and fixing some notations.

### 2.1. Preliminaries.

Given a vector field $\mathbf{v} := (v_1, \ldots, v_d)$ and a tensor field $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,\ldots,d}$, with $d = 2, 3$, we define the operators:

$$\nabla\mathbf{v} = \left(\frac{\partial v_i}{\partial x_j}\right), \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \big(\operatorname{div}(\tau_{i1}, \ldots, \tau_{id})\big),$$

where div is the usual divergence operator acting on vector fields.

Now, let $\mathcal{O}$ be a domain in $\mathbf{R}^d$, with Lipschitz boundary $\Gamma$. For $r \geq 0$ and $p \in [1, \infty]$, we denote by $L^p(\mathcal{O})$ and $W^{r,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{W^{r,p}(\mathcal{O})}$, respectively.

Note that $W^{0,p}(\mathcal{O}) = L^p(\mathcal{O})$. If $p = 2$, we write $H^r(\mathcal{O})$ in place of $W^{r,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{r,\mathcal{O}}$, respectively. We define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^d, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{d\times d}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^d.$$

Also, we shall make use of the Hilbert space

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \operatorname{div}\,\mathbf{w} \in L^2(\mathcal{O})\right\},$$

which is standard in the realm of mixed problems (see [4] or [17] for instance). This spaces is endowed with the norm

$$\|\mathbf{w}\|^2_{\operatorname{div},\mathcal{O}} = \|\mathbf{w}\|^2_{0,\mathcal{O}} + \|\operatorname{div}\,\mathbf{w}\|^2_{0,\mathcal{O}}.$$

The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$ endowed with the norm $\|\cdot\|_{\mathbf{div},\mathcal{O}}$, which can be characterized as

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \left\{\boldsymbol{\tau}\in\mathbb{L}^2(\mathcal{O}) : \mathbf{c}^t\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}\,;\mathcal{O}), \quad \forall\,\mathbf{c}\in\mathbf{R}^d\right\}.$$

Note that if $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau}\in\mathbf{L}^2(\mathcal{O})$.

Next, for the sake of simplicity, we will also use the notations:

$$(u,v)_\Omega := \int_\Omega u\,v, \qquad (\mathbf{u},\mathbf{v})_\Omega := \int_\Omega \mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_\Omega := \int_\Omega \boldsymbol{\sigma}:\boldsymbol{\tau}\,,$$

where $\boldsymbol{\sigma}:\boldsymbol{\tau} = \mathrm{tr}\,(\boldsymbol{\sigma}^t\boldsymbol{\tau}) = \sum_{i,j=1}^{2}\sigma_{ij}\tau_{ij}$, with $\boldsymbol{\tau}^t = (\tau_{ji})$ and $\mathrm{tr}\,\boldsymbol{\tau} = \sum_{i=1}^{d}\tau_{ii}$, for any tensor $\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij})$. In addition, we denote by

$$\boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{d}\mathrm{tr}\,(\boldsymbol{\tau})\mathrm{I}$$

the deviatoric part of tensor $\boldsymbol{\tau}$, where I is the identity matrix in $\mathbf{R}^{d\times d}$. It is not difficult to see that there hold

$$\|\boldsymbol{\tau}^d\|_{0,\Omega}^2 = \|\boldsymbol{\tau}\|_{0,\Omega}^2 - \frac{1}{d}\|\mathrm{tr}\,\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \text{and} \quad \|\mathrm{tr}\,\boldsymbol{\tau}\|_{0,\Omega} \le \sqrt{d}\|\boldsymbol{\tau}\|_{0,\Omega}. \tag{2.1}$$

Furthermore, given a non-negative integer $k$, we denote by $\mathbb{P}_k(\mathcal{O})$ the space of polynomials defined on $\mathcal{O}$ of degree $\le k$

In addition, it is easy to see that there holds:

$$\mathbb{H}(\mathbf{div};\mathcal{O}) = \mathbb{H}_0(\mathbf{div};\mathcal{O}) \oplus \mathbb{P}_0(\mathcal{O})\,\mathrm{I}\,, \tag{2.2}$$

where

$$\mathbb{H}_0(\mathbf{div};\mathcal{O}) := \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O}) : \quad \int_{\mathcal{O}}\mathrm{tr}\,\boldsymbol{\tau} = 0\right\}. \tag{2.3}$$

More precisely, each $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O})$ can be decomposed uniquely as:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathrm{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\mathcal{O}) \quad \text{and} \quad c := \frac{1}{d\,|\mathcal{O}|}\int_{\mathcal{O}}\mathrm{tr}\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{2.4}$$

This decomposition will be utilized below to analyze the weak formulation of our problem.

We end this section by mentioning that, throughout the rest of the paper, we shall frequently use the notation $C$ and $c$, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters.

**2.2. Model Problem.** In this paper we shall consider a viscous fluid occupying a bounded polygon or polyhedra domain $\Omega$ in $\mathbf{R}^d$, $d=2,3$, with boundary $\Gamma=\partial\Omega$, governed by the Stokes equations with varying density:

$$\begin{aligned} \boldsymbol{\sigma} &= \nu(\rho\nabla\mathbf{u}) - p\mathrm{I} \quad \text{in} \quad \Omega, \qquad -\mathbf{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega \\ \mathrm{div}\,(\rho\mathbf{u}) &= 0 \quad \text{in} \quad \Omega, \qquad \mathbf{u} = 0 \quad \text{on} \quad \Gamma, \qquad (p,1)_\Omega = 0. \end{aligned} \tag{2.5}$$

Here, the unknowns are the pseudostress tensor $\boldsymbol{\sigma}$, the fluid velocity $\mathbf{u}$ and the pressure $p$. The given data are the external force per unit mass $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the viscosity $\nu > 0$, which is assumed to be constant, and the density function $\rho \in H^1(\Omega) \cap W^{1,\infty}(\Omega)$, satisfying:

$$\frac{\nabla \rho}{\rho} \in \mathbf{L}^\infty(\Omega) \quad \text{and} \quad 0 < \rho_0 < \rho(x) < \rho_1, \text{ a.e. in } \Omega, \tag{2.6}$$

where $\rho_1$ and $\rho_2$ are positive constants.

The model in (2.5), which is derived from the full steady Navier-Stokes equations for viscous fluids, is well-justified if we assume the following assumptions (justifications on the model can be found in [8]):

i) Only the laminar case is considered and the second-order diffusion term in the viscous stress tensor is neglected.
ii) The Mach number is small enough, which implies that the coupling between the pressure and the temperature can be neglected.

In particular, ii) implies that the state law can be chosen as a simple equation linking the density and the temperature, in which the temperature is approximated by a reference one.

Now, in order to rewrite equations (2.5) as a pseudostress-velocity formulation, we first observe that identity $\operatorname{div}(\rho\mathbf{u}) = 0$ in $\Omega$ implies

$$\rho \operatorname{div} \mathbf{u} = -\mathbf{u} \cdot \nabla \rho \quad \text{in} \quad \Omega. \tag{2.7}$$

Then, observing that $\operatorname{tr} \boldsymbol{\sigma} = \nu \rho \operatorname{div} \mathbf{u} - d\, p$, (2.7) implies that the pressure can be written in terms of the pseudostress and the velocity as follows:

$$p = -\frac{1}{d}\left(\nu \mathbf{u} \cdot \nabla \rho + \operatorname{tr} \boldsymbol{\sigma}\right) \quad \text{in} \quad \Omega. \tag{2.8}$$

In this way, we eliminate the pressure from (2.5) and obtain the equivalent system of equations:

$$\begin{gathered} \frac{\nu^{-1}}{\rho}\boldsymbol{\sigma}^d \;=\; \nabla \mathbf{u} + \frac{1}{d}\left(\mathbf{u} \cdot \frac{\nabla \rho}{\rho}\right)\mathrm{I} \quad \text{in} \quad \Omega, \qquad -\mathbf{div}\boldsymbol{\sigma} \;=\; \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} \;=\; 0 \quad \text{on} \quad \Gamma, \qquad (\operatorname{tr} \boldsymbol{\sigma}, 1)_\Omega \;=\; -\nu(\mathbf{u} \cdot \nabla \rho, 1)_\Omega. \end{gathered} \tag{2.9}$$

**2.3. Dual-mixed variational formulation.** Now, we introduce the variational formulation of the model problem (2.9). To do that, we test equations (2.9) by suitable test functions, integrate by parts and use the homogeneous boundary condition to obtain the variational problem: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{L}^2(\Omega)$ such that $(\operatorname{tr} \boldsymbol{\sigma} \,+\, \nu \mathbf{u} \cdot \nabla \rho, 1)_\Omega = 0$ and

$$\begin{aligned} \nu^{-1}(\frac{1}{\rho}\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\boldsymbol{\tau}, \mathbf{u})_\Omega - \frac{1}{d}(\mathbf{u} \cdot \frac{\nabla \rho}{\rho}, \operatorname{tr} \boldsymbol{\tau})_\Omega &= \quad 0, \\ (\mathbf{div}\boldsymbol{\sigma}, \mathbf{v})_\Omega &= \quad -(\mathbf{f}, \mathbf{v})_\Omega. \end{aligned} \tag{2.10}$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{L}^2(\Omega)$

Let us now define the tensor

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \frac{\nu}{d|\Omega|}(\mathbf{u} \cdot \nabla \rho, 1)_\Omega \mathrm{I}. \tag{2.11}$$

It is clear that

$$\boldsymbol{\sigma}_0 \;\in\; \mathbb{H}_0(\mathbf{div}; \Omega) \;\text{ if and only if }\; (\operatorname{tr} \boldsymbol{\sigma} \,+\, \nu \mathbf{u} \cdot \nabla \rho, 1)_\Omega = 0. \tag{2.12}$$

In this way, thanks to (2.11) and (2.4), problem (2.10) can be reformulated equivalently as: Find $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \nu^{-1}(\frac{1}{\rho}\boldsymbol{\sigma}_0^d, \boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\boldsymbol{\tau}, \mathbf{u})_\Omega - \frac{1}{d}(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}, \mathrm{tr}\,\boldsymbol{\tau})_\Omega = &\quad 0, \\ (\mathbf{div}\boldsymbol{\sigma}_0, \mathbf{v})_\Omega = &\quad -(\mathbf{f}, \mathbf{v})_\Omega, \end{aligned} \tag{2.13}$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega)$.

The following lemma establishes that problems (2.10) and (2.13) are in fact equivalent.

Lemma 2.1. *If* $(\boldsymbol{\sigma}, \mathbf{u})$ *is a solution of* (2.10)*, then* $(\boldsymbol{\sigma}_0, \mathbf{u}) := (\boldsymbol{\sigma} + \frac{\nu}{d|\Omega|}(\mathbf{u}\cdot\nabla\rho, 1)_\Omega \mathrm{I}, \mathbf{u})$ *is a solution of* (2.13)*. Conversely, if* $(\boldsymbol{\sigma}_0, \mathbf{u})$ *is a solution of* (2.13)*, then* $(\boldsymbol{\sigma}, \mathbf{u}) := (\boldsymbol{\sigma}_0 - \frac{\nu}{d|\Omega|}(\mathbf{u}\cdot\nabla\rho, 1)_\Omega \mathrm{I}, \mathbf{u})$ *is a solution of* (2.10).

*Proof.* The first assertion is evident. On the other hand, by testing the first equation of (2.13) by $\tau := (\rho - \frac{(\rho,1)_\Omega}{|\Omega|})\mathrm{I} \in \mathbb{H}_0(\mathbf{div};\Omega)$, it follows that $(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}, 1)_\Omega = 0$, which implies the second assertion. ∎

As a consequence of the above, in what follows we focus on analysing problem (2.13).

## 3. Analysis of the continuous problem.

In this section we analyse the well-posedness of problem (2.13), that is, we establish stability, existence and uniqueness of solution. In order to do that, we start by writing our problem in the classical variational setting and state the main properties of the bilinear forms involved.

In the following, when no confusion arises, for the sake of brevity we omit the subscript 0 on $\boldsymbol{\sigma}_0$.

### 3.1. Variational formulation.

First, let us define the spaces $\mathbb{H} := \mathbb{H}(\mathbf{div};\Omega)$, $\mathbb{H}_0 := \mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{Q} := \mathbf{L}^2(\Omega)$ and the product norm

$$\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbb{H}\times\mathbf{Q}} := (\|\boldsymbol{\tau}\|^2_{\mathbf{div},\Omega} + \|\mathbf{v}\|^2_{0,\Omega})^{1/2}.$$

Then, defining the bilinear forms $a(\cdot,\cdot) : \mathbb{H}\times\mathbb{H} \to \mathbf{R}$, $b(\cdot,\cdot) : \mathbb{H}\times\mathbf{Q} \to \mathbf{R}$ and $c(\cdot,\cdot) : \mathbb{H}\times\mathbf{Q} \to \mathbf{R}$, as

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \nu^{-1}(\frac{1}{\rho}\boldsymbol{\sigma}^d, \boldsymbol{\tau}^d)_\Omega, \quad b(\boldsymbol{\tau}, \mathbf{v}) := (\mathbf{div}\boldsymbol{\tau}, \mathbf{v})_\Omega, \quad c(\boldsymbol{\tau}, \mathbf{v}) := \frac{1}{d}(\mathbf{v}\cdot\frac{\nabla\rho}{\rho}, \mathrm{tr}\,\boldsymbol{\tau})_\Omega, \tag{3.1}$$

the variational formulation (2.13) reads: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) - c(\boldsymbol{\tau}, \mathbf{u}) = &\quad 0, \\ b(\boldsymbol{\sigma}, \mathbf{v}) = &\quad -(\mathbf{f}, \mathbf{v})_\Omega, \end{aligned} \tag{3.2}$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0 \times \mathbf{Q}$.

It is clear that assumption (2.6), Hölder's inequality and (2.1) imply the continuity of these bilinear forms:

$$\begin{aligned} |a(\boldsymbol{\sigma}, \boldsymbol{\tau})| &\leq \frac{1}{\nu\rho_0}\|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega}\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}, && \boldsymbol{\sigma},\ \boldsymbol{\tau} \in \mathbb{H}, \\ |b(\boldsymbol{\tau}, \mathbf{v})| &\leq \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\|\mathbf{v}\|_{0,\Omega}, && \boldsymbol{\tau} \in \mathbb{H},\ \mathbf{v} \in \mathbf{Q}, \\ |c(\boldsymbol{\tau}, \mathbf{v})| &\leq \frac{1}{\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\|\mathbf{v}\|_{0,\Omega}, && \boldsymbol{\tau} \in \mathbb{H},\ \mathbf{v} \in \mathbf{Q} \end{aligned} \tag{3.3}$$

Furthermore, it is well known that the bilinear form $b$ satisfies the inf-sup condition (see, for instance [4]):

$$\sup_{\boldsymbol{\tau}\in\mathbb{H}_0\setminus\mathbf{0}} \frac{b(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}} \geq \beta\|\mathbf{v}\|_{0,\Omega} \qquad \forall\, \mathbf{v}\in\mathbf{Q}. \tag{3.4}$$

Finally, the following inequality holds (see for instance, Lemma 3.1 in [1] or Chapter IV in [4]):

$$C_a\|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^d\|_{0,\Omega}^2 + \|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0, \tag{3.5}$$

with $C_a$ only depending on $\Omega$. This inequality, and assumption (2.6) imply the ellipticity of $a$ on the subspace

$$\mathbb{K}_0 := \{\boldsymbol{\tau}\in\mathbb{H}_0 \;:\; \mathbf{div}\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega\},$$

that is

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \frac{C_a}{\nu\rho_1}\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}^2, \quad \forall\,\boldsymbol{\tau}\in\mathbb{K}_0. \tag{3.6}$$

**3.2. Stability.** Now, we establish the stability of (3.2).

LEMMA 3.1. *Let* $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0\times\mathbf{Q}$ *be a solution to* (3.2). *Assume that*

$$C_{dep}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{1}{2}, \tag{3.7}$$

*with*

$$C_{dep} := \frac{1}{\beta\sqrt{d}}\left(1 + 2\frac{\rho_1}{C_a\rho_0}\right).$$

*Then, there exist constants* $C_{\boldsymbol{\sigma}}$ *and* $C_{\mathbf{u}}$*, only depending on the stability constants in* (3.3), (3.4), (3.5)*, such that*

$$\|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} \leq C_{\boldsymbol{\sigma}}\,\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \|\mathbf{u}\|_{0,\Omega} \leq C_{\mathbf{u}}\,\|\mathbf{f}\|_{0,\Omega}. \tag{3.8}$$

*(Explicit expressions for* $C_{\boldsymbol{\sigma}}$ *and* $C_{\mathbf{u}}$ *can be found in* (3.13) *and* (3.14)*).*

*Proof.* Let $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0\times\mathbf{Q}$ be a solution to (3.2). First, we observe that from the second equation of (3.2), it is easy to conclude that $\mathbf{div}\boldsymbol{\sigma} = -\mathbf{f}$, which implies

$$\|\mathbf{div}\boldsymbol{\sigma}\|_{0,\Omega} = \|\mathbf{f}\|_{0,\Omega}. \tag{3.9}$$

Now, from the inf-sup condition in (3.4), the first equation of (3.2), Hölder's inequality, the inequality in (2.1), and the continuity of the bilinear forms $a$ and $c$ in (3.3), we observe that

$$\begin{aligned}\|\mathbf{u}\|_{0,\Omega} \quad &\leq \frac{1}{\beta}\sup_{\boldsymbol{\tau}\in\mathbb{H}_0\setminus\mathbf{0}}\frac{b(\boldsymbol{\tau},\mathbf{u})}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}} = \frac{1}{\beta}\sup_{\boldsymbol{\tau}\in\mathbb{H}_0\setminus\mathbf{0}}\frac{|-a(\boldsymbol{\sigma},\boldsymbol{\tau}) + c(\boldsymbol{\tau},\mathbf{u})|}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}},\\ &\leq \frac{1}{\nu\rho_0\beta}\|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} + \frac{1}{\beta\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}\|_{0,\Omega}.\end{aligned} \tag{3.10}$$

Then, thanks to assumption (3.7), we obtain

$$\|\mathbf{u}\|_{0,\Omega} \leq \frac{2}{\nu\rho_0\beta}\|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega}. \tag{3.11}$$

On the other hand, from the first equation of (3.2) with $\boldsymbol{\tau} = \boldsymbol{\sigma}$, there holds

$$a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = -b(\boldsymbol{\sigma}, \mathbf{u}) + c(\boldsymbol{\sigma}, \mathbf{u}) = (\mathbf{f}, \mathbf{u})_\Omega + c(\boldsymbol{\sigma}, \mathbf{u}),$$

which together to assumption (2.6), the continuity of $c$ in (3.3), and Hölder's inequality, implies

$$\|\boldsymbol{\sigma}^d\|_{0,\Omega}^2 \leq \nu\rho_1 \|\mathbf{u}\|_{0,\Omega} \|\mathbf{f}\|_{0,\Omega} + \frac{\nu\rho_1}{\sqrt{d}} \|\mathbf{u}\|_{0,\Omega} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega}. \tag{3.12}$$

Hence, adding $(1 + C_a)\|\mathbf{div}\boldsymbol{\sigma}\|_{0,\Omega}^2$ on both sides of (3.12), and using (3.5), (3.9) and (3.11), we get

$$\begin{aligned} \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega}^2 &\leq \frac{\nu\rho_1}{C_a} \|\mathbf{u}\|_{0,\Omega} \left( \|\mathbf{f}\|_{0,\Omega} + \frac{1}{\sqrt{d}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} \right) + \frac{(1 + C_a)}{C_a} \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} \|\mathbf{f}\|_{0,\Omega}, \\ &\leq \left( \frac{2\rho_1}{C_a\rho_0\beta} + \frac{1 + C_a}{C_a} \right) \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} \|\mathbf{f}\|_{0,\Omega} + \frac{2\rho_1}{C_a\rho_0\beta\sqrt{d}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega}^2. \end{aligned}$$

In this way, from assumption (3.7) it follows that

$$\|\boldsymbol{\sigma}\|_{\mathbf{div},\Omega} \leq 2 \left( \frac{2\rho_1}{C_a\rho_0\beta} + \frac{1 + C_a}{C_a} \right) \|\mathbf{f}\|_{0,\Omega}, \tag{3.13}$$

which together to (3.11) implies

$$\|\mathbf{u}\|_{0,\Omega} \leq \frac{4}{\nu\rho_0\beta} \left( \frac{2\rho_1}{C_a\rho_0\beta} + \frac{1 + C_a}{C_a} \right) \|\mathbf{f}\|_{0,\Omega}, \tag{3.14}$$

which completes the proof. ◻

**3.3. Existence and uniqueness of solution.** As mentioned before, in order to prove existence and uniqueness of solution, we now introduce the linear mapping:

$$\mathbb{T} : (\boldsymbol{\xi}, \mathbf{z}) \in \mathbb{H}_0 \times \mathbf{Q} \to (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q},$$

as the solution to the following variation of problem (3.2): Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) &= c(\boldsymbol{\tau}, \mathbf{z}), \\ b(\boldsymbol{\sigma}, \mathbf{v}) &= -(\mathbf{f}, \mathbf{v})_\Omega, \end{aligned} \tag{3.15}$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0 \times \mathbf{Q}$. With the stability properties in Section 3.1, it is not difficult to see that problem (3.15) is uniquely solvable, and hence the operator $\mathbb{T}$ is well defined (see Theorem 2.1 in [15]).

The following lemma establishes that $\mathbb{T}$ is a contraction mapping and hence, according to the Banach fixed point theorem, it has a unique fixed point in $\mathbb{H}_0 \times \mathbf{Q}$.

Lemma 3.2. *Assume that*

$$C_{\mathbb{T}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} < 1, \tag{3.16}$$

*with*

$$C_{\mathbb{T}} := \frac{1}{\beta\sqrt{d}} \left( 1 + \frac{\rho_1}{C_a\rho_0} \right) + \frac{\rho_1\nu}{C_a\sqrt{d}}. \tag{3.17}$$

*Then,* $\mathbb{T}$ *is a contraction mapping in* $\mathbb{H}_0 \times \mathbf{Q}$*.*

*Proof.* Let $(\boldsymbol{\sigma}_1, \mathbf{u}_1)$, $(\boldsymbol{\sigma}_2, \mathbf{u}_2)$, $(\boldsymbol{\xi}_1, \mathbf{z}_1)$, $(\boldsymbol{\xi}_2, \mathbf{z}_2)$ in $\mathbb{H}_0 \times \mathbf{Q}$, such that

$$\mathbb{T}(\boldsymbol{\xi}_1, \mathbf{z}_1) = (\boldsymbol{\sigma}_1, \mathbf{u}_1) \quad \text{and} \quad \mathbb{T}(\boldsymbol{\xi}_2, \mathbf{z}_2) = (\boldsymbol{\sigma}_2, \mathbf{u}_2).$$

From the definition of $\mathbb{T}$ in (3.15), it follows that

$$\begin{aligned} a(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}_1 - \mathbf{u}_2) = & \quad c(\boldsymbol{\tau}, \mathbf{z}_1 - \mathbf{z}_2), \\ b(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{v}) = & \quad 0, \end{aligned} \tag{3.18}$$

for all $(\boldsymbol{\tau}, \mathbf{v})$ in $\mathbb{H}_0 \times \mathbf{Q}$, which implies

$$\mathbf{div}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = 0, \tag{3.19}$$

and

$$a(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = c(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{z}_1 - \mathbf{z}_2). \tag{3.20}$$

Then, from (3.19), (3.20), the ellipticity of $a$ on $\mathbb{K}_0$ in (3.6), and the continuity of $c$ in (3.3), there holds

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbf{div},\Omega} \leq \frac{\rho_1 \nu}{C_a \sqrt{d}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\Omega}. \tag{3.21}$$

Now, from (3.3), (3.4), and the first equation of (3.18), we obtain

$$\begin{aligned} \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega} \quad & \leq \frac{1}{\beta} \sup_{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) \setminus \mathbf{0}} \frac{|b(\boldsymbol{\tau}, \mathbf{u}_1 - \mathbf{u}_2)|}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}}, \\ & = \frac{1}{\beta} \sup_{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) \setminus \mathbf{0}} \frac{|c(\boldsymbol{\tau}, \mathbf{z}_1 - \mathbf{z}_2) - a(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1, \boldsymbol{\tau})|}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}}, \\ & \leq \frac{1}{\beta \sqrt{d}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\Omega} + \frac{1}{\nu \rho_0 \beta} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbf{div},\Omega}, \end{aligned}$$

which together to (3.21) implies

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega} \leq \frac{1}{\beta \sqrt{d}} \left(1 + \frac{\rho_1}{C_a \rho_0}\right) \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\Omega}. \tag{3.22}$$

In this way, from (3.21) and (3.22), there holds

$$\begin{aligned} \|\mathbb{T}(\boldsymbol{\xi}_1, \mathbf{z}_1) - \mathbb{T}(\boldsymbol{\xi}_2, \mathbf{z}_2)\|_{\mathbb{H} \times \mathbf{Q}} \leq & \quad \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbf{div},\Omega} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega}, \\ \leq & \quad C_{\mathbb{T}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\Omega}, \\ \leq & \quad C_{\mathbb{T}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2, \mathbf{z}_1 - \mathbf{z}_2)\|_{\mathbb{H} \times \mathbf{Q}}. \end{aligned}$$

Therefore, according to assumption (3.16), we obtain that $\mathbb{T}$ is a contraction mapping, which concludes the proof. □

Now we establishes the main result of this section.

Theorem 3.3. *Assume that*

$$C_{WP}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{1}{2}, \tag{3.23}$$

*with*

$$C_{WP} := \frac{1}{\beta\sqrt{d}}\left(1+2\frac{\rho_1}{C_a\rho_0}\right) + \frac{\rho_1\nu}{C_a\sqrt{d}}. \tag{3.24}$$

*Then, there exist a unique* $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ *satisfying* (3.2)*. Moreover, the solution is stable in the sense that, it satisfies inequalities* (3.8)*.*

*Proof.* It is clear that $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ is the unique solution of problem (3.2) if and only if it is the unique fixed point of the mapping $\mathbb{T}$. Then, noting that $C_{\mathbb{T}} \leq C_{WP}$, from Lemma 3.2 and the classical Banach fixed point theorem, it follows that $\mathbb{T}$ has a unique fixed point in $\mathbb{H}_0 \times \mathbf{Q}$, which implies the first assertion.

In turn, since $C_{dep} \leq C_{WP}$, the stability of $(\boldsymbol{\sigma},\mathbf{u})$ follows from Lemma 3.1. □

**4. The mixed finite element scheme.** In this section we introduce the Galerkin scheme of problem (3.2) and analyze its well-posedness by establishing suitable assumptions on the finite element subspaces involved. Then, we provide specific examples for these subspaces, satisfying the required hypotheses.

**4.1. Preliminaries.** We start by selecting the following arbitrary discrete spaces:

$$\mathbf{H}_h \subseteq \mathbf{H}(\mathbf{div};\Omega), \qquad Q_h \subseteq L^2(\Omega). \tag{4.1}$$

Then we define the subspaces

$$\begin{aligned} \mathbb{H}_h &:= \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\ :\ \mathbf{c}^t\boldsymbol{\tau} \in \mathbf{H}_h \quad \forall\, \mathbf{c} \in \mathbf{R}^d\right\}, \\ \mathbb{H}_{h,0} &:= \mathbb{H}_h \cap \mathbb{H}_0(\mathbf{div};\Omega), \\ \mathbf{Q}_h &:= Q_h^d. \end{aligned} \tag{4.2}$$

In this way, the Galerkin scheme for (3.2) reduces to: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h,\mathbf{u}_h) - c(\boldsymbol{\tau}_h,\mathbf{u}_h) &= 0, \\ b(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= -(\mathbf{f},\mathbf{v}_h)_\Omega, \end{aligned} \tag{4.3}$$

for all $(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$.

Now, we establish general hypotheses on the finite element subspaces (4.2), ensuring later on the well-posedness of (4.3). We start by observing that in order to have meaningful space $\mathbb{H}_{h,0}$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h$. This request is certainly satisfied if we assume that:

**(H.0)** $[\mathbb{P}_0(\Omega)]^{d\times d} \subseteq \mathbb{H}_h$.

Then, it follows that $\mathrm{I} \in \mathbb{H}_h$, for all $h$, and hence there holds the decomposition:

$$\mathbb{H}_h = \mathbb{H}_{h,0} \oplus \mathbb{P}_0(\Omega)\mathrm{I}.$$

Now, we look at the discrete kernel on $b$, which is defined by:

$$\mathbb{K}_{h,0} := \{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} \ : \ b(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v} \in \mathbf{Q}_h\}.$$

In order to have a more explicit definition of $\mathbb{K}_{h,0}$ we introduce the following assumption:

$(\mathbf{H.1})$ $\operatorname{div} \mathbf{H}_h \subseteq Q_h$.

Then, it follows from the definition of $b$ that

$$\mathbb{K}_{h,0} := \{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} \ : \ \mathbf{div}\boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega\} \subseteq \mathbb{K}_0.$$

Next, we assume that the discrete version of (3.4) holds, that is:

$(\mathbf{H.2})$ There exists $\hat{\beta} > 0$, independent of $h$, such that

$$\sup_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} \setminus \mathbf{0}} \frac{b(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}} \geq \hat{\beta} \|\mathbf{v}_h\|_{0,\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h. \tag{4.4}$$

**4.2. Well-posedness of the discrete problem.** In this section, we adapt the analysis from Section 3 to the discrete case to prove the well-posedness of (4.3). First we observe that, since we are considering conforming finite element subspaces, the continuity of the bilinear forms $a$, $b$, and $c$ (cf. (3.3)) are inherited from the continuous case, with the exact same constants. Moreover, since $\mathbb{K}_{h,0} \subseteq \mathbb{K}_0$, we deduce that the ellipticity of $a$ on $\mathbb{K}_{h,0}$ holds:

$$a(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \frac{C_a}{\nu \rho_1} \|\boldsymbol{\tau}_h\|^2_{\mathbf{div},\Omega} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{K}_{h,0}. \tag{4.5}$$

In this way, according to (3.3), (4.4), (4.5), and the classical Babuška-Brezzi theory, and similarly to the analysis of the continuous problem, we are able to introduce the well-defined linear mapping

$$\hat{\mathbb{T}} : (\boldsymbol{\xi}_h, \mathbf{z}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h \to (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$$

as the solution to the problem: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \ c(\boldsymbol{\tau}_h, \mathbf{z}_h), \\ b(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= \ -(\mathbf{f}, \mathbf{v}_h)_\Omega, \end{aligned} \tag{4.6}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$.

Remark 4.1. *It is easy to see that $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is the solution of (4.3), if and only if, $\hat{\mathbb{T}}(\boldsymbol{\sigma}_h, \mathbf{u}_h) = (\boldsymbol{\sigma}_h, \mathbf{u}_h)$. In this way, in order to prove that (4.3) is well-posed, we proceed analogously to Section 3.3, and prove that $\hat{\mathbb{T}}$ has a unique fixed-point in $\mathbb{H}_{h,0} \times \mathbf{Q}_h$.*

Theorem 4.2. *Assume that hypotheses $(\mathbf{H.0})$, $(\mathbf{H.1})$ and $(\mathbf{H.2})$ hold. In addition, assume that*

$$\hat{C}_{WP} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{1}{2}, \tag{4.7}$$

*with*

$$\hat{C}_{WP} := \frac{1}{\hat{\beta}\sqrt{d}} \left(1 + 2\frac{\rho_1}{C_a \rho_0}\right) + \frac{\rho_1 \nu}{C_a \sqrt{d}}. \tag{4.8}$$

*Then there exists a unique* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ *satisfying* (4.3). *Moreover, there exist positive constants* $\hat{C}_{\boldsymbol{\sigma}}$ *and* $\hat{C}_{\mathbf{u}}$, *only depending on the stability constants in* (3.3), (4.4) *and* (4.5), *such that*

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} \leq \hat{C}_{\boldsymbol{\sigma}} \|\mathbf{f}\|_{0,\Omega} \quad \text{and} \quad \|\mathbf{u}_h\|_{0,\Omega} \leq \hat{C}_{\mathbf{u}} \|\mathbf{f}\|_{0,\Omega}. \tag{4.9}$$

*(Explicit expressions for* $\hat{C}_{\boldsymbol{\sigma}}$ *and* $\hat{C}_{\mathbf{u}}$ *can be found in* (4.10) *and* (4.11)*).*

*Proof.* Let

$$\hat{C}_{\hat{\mathbb{T}}} := \frac{1}{\hat{\beta}\sqrt{d}}\left(1 + \frac{\rho_1}{C_a \rho_0}\right) + \frac{\rho_1 \nu}{C_a \sqrt{d}}.$$

It is clear that $\hat{C}_{\hat{\mathbb{T}}} \leq \hat{C}_{WP}$. Then, we proceed analogously to Lemma 3.2, to prove that the mapping $\hat{\mathbb{T}}$ has a unique fixed point $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ which, according to Remark 4.1, is also the unique solution of (4.3).

Next, we let

$$\hat{C}_{dep} := \frac{1}{\hat{\beta}\sqrt{d}}\left(1 + 2\frac{\rho_1}{C_a \rho_0}\right),$$

and observe that $\hat{C}_{dep} \leq \hat{C}_{WP}$. Then, noting that from the second equation of (4.3) the holds

$$\|\mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega} \leq \|\mathbf{f}\|_{0,\Omega},$$

we proceed as in the proof of Lemma 3.1 to obtain that

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} \leq 2\left(\frac{2\rho_1}{C_a \rho_0 \hat{\beta}} + \frac{1 + C_a}{C_a}\right)\|\mathbf{f}\|_{0,\Omega}, \tag{4.10}$$

and

$$\|\mathbf{u}_h\|_{0,\Omega} \leq \frac{4}{\nu \rho_0 \hat{\beta}}\left(\frac{2\rho_1}{C_a \rho_0 \hat{\beta}} + \frac{1 + C_a}{C_a}\right)\|\mathbf{f}\|_{0,\Omega}. \tag{4.11}$$

which concludes the proof. □

**4.3. A priori error estimate.** Now, we establish the corresponding Cea a priori error estimate. To that end, we first introduce some notations and state some previous results. We begin by defining the set:

$$\mathbb{H}_h^{\mathbf{f}} := \{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} : b(\boldsymbol{\tau}_h, \mathbf{v}_h) = -(\mathbf{f}, \mathbf{v}_h)_\Omega, \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\},$$

which is clearly non-empty, since (4.4) hold. Also, it is not difficult to see that, due to the inf-sup condition (4.4), the following inequality holds (see for instance [4]):

$$\inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\mathbf{f}}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega} \leq \left(1 + \frac{1}{\hat{\beta}}\right) \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}. \tag{4.12}$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h$ and $\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$. As usual, we shall then decompose these errors into

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}} = (\boldsymbol{\sigma} - \hat{\boldsymbol{\tau}}_h) + (\hat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h), \tag{4.13}$$

$$\mathbf{e}_{\mathbf{u}} = \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}} = (\mathbf{u} - \hat{\mathbf{v}}_h) + (\hat{\mathbf{v}}_h - \mathbf{u}_h), \tag{4.14}$$

for a given $(\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h) \in \mathbb{H}_h^{\mathbf{f}} \times \mathbf{Q}_h$.

Finally, we observe that the Galerkin orthogonality holds:

$$\begin{aligned} a(\mathbf{e}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{e}_{\mathbf{u}}) - c(\boldsymbol{\tau}_h, \mathbf{e}_{\mathbf{u}}) = &\quad 0, \\ b(\mathbf{e}_{\boldsymbol{\sigma}}, \mathbf{v}_h) = &\quad 0, \end{aligned} \tag{4.15}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$.

We now establish the main result of this section.

THEOREM 4.3. *Assume that hypotheses* (**H.0**)*,* (**H.1**) *and* (**H.2**) *hold. In addition, assume that*

$$\max\{C_{WP}, \hat{C}_{WP}\} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{1}{2}, \tag{4.16}$$

*with $C_{WP}$ and $\hat{C}_{WP}$ defined in* (3.24) *and* (4.8)*, respectively. Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ be the unique solutions of the continuous and discrete problems* (3.2) *and* (4.3)*, respectively. Then there exists $C_{cea} > 0$, independent of $h$, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq C_{cea} \left\{ \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega} + \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega} \right\}. \tag{4.17}$$

*Proof.* Let $(\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h) \in \mathbb{H}_h^{\mathbf{f}} \times \mathbf{Q}_h$, and define $\boldsymbol{\xi}_{\boldsymbol{\sigma}}$, $\boldsymbol{\xi}_{\mathbf{u}}$, $\boldsymbol{\chi}_{\boldsymbol{\sigma}}$ and $\boldsymbol{\chi}_{\mathbf{u}}$, as in (4.13) and (4.14). It is easy to see that the first equation of (4.15) can be rewritten as

$$a(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\chi}_{\mathbf{u}}) - c(\boldsymbol{\tau}_h, \boldsymbol{\chi}_{\mathbf{u}}) = -a(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) - b(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}) + c(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}) \quad \forall \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}. \tag{4.18}$$

Then, from the inf-sup condition (4.4), the continuity of $a$, $b$ and $c$ in (3.3) and (4.18), it follows that

$$\begin{aligned} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{0,\Omega} \leq &\quad \frac{1}{\hat{\beta}} \sup_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \frac{b(\boldsymbol{\tau}_h, \boldsymbol{\chi}_{\mathbf{u}})}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}}, \\ = &\quad \frac{1}{\hat{\beta}} \sup_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \frac{-a(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) - a(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) - b(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}) + c(\boldsymbol{\tau}_h, \boldsymbol{\chi}_{\mathbf{u}}) + c(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}})}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}}, \\ \leq &\quad \frac{1}{\nu \rho_0 \hat{\beta}} (\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega}) + \frac{1}{\hat{\beta}} \left( 1 + \frac{1}{\sqrt{d}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \right) \|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega}, \\ &\quad + \frac{1}{\hat{\beta}\sqrt{d}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{0,\Omega}, \end{aligned}$$

which, together to assumption (4.16) implies

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{0,\Omega} \leq \frac{2}{\nu \rho_0 \hat{\beta}} (\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega}) + \frac{2}{\hat{\beta}} \left( 1 + \frac{1}{\sqrt{d}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \right) \|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega}. \tag{4.19}$$

In turn, since $\hat{\boldsymbol{\tau}}_h \in \mathbb{H}_h^{\mathbf{f}}$, we observe that $\boldsymbol{\chi}_{\boldsymbol{\sigma}} \in \mathbb{K}_{h,0}$, and then, from (4.18) with $\boldsymbol{\tau}_h = \boldsymbol{\chi}_{\boldsymbol{\sigma}}$, we obtain

$$a(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\chi}_{\boldsymbol{\sigma}}) = -a(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\chi}_{\boldsymbol{\sigma}}) + c(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\xi}_{\mathbf{u}}) + c(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\chi}_{\mathbf{u}}),$$

and using the continuity of $a$ and $c$ in (3.3), and the ellipticity of $a$ in (4.5), we get

$$\begin{aligned}\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} \leq & \frac{\rho_1}{C_a\rho_0}\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \frac{\nu\rho_1}{C_a\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega}, \\ & + \frac{\nu\rho_1}{C_a\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{0,\Omega}.\end{aligned} \tag{4.20}$$

In this way, combining (4.19) and (4.20) it follows that

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} \leq \frac{k_1}{2}\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \frac{k_2}{2}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega} + \frac{2\rho_1}{\rho_0 C_a\hat{\beta}\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{0,\Omega},$$

which together to assumption (4.16), yields

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} \leq k_1\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + k_2\|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega}, \tag{4.21}$$

with

$$\begin{aligned} k_1 :=\ & \frac{\rho_1}{C_a\rho_0}\left(1 + \frac{2}{\hat{\beta}\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\right), \\ k_2 :=\ & \frac{\nu\rho_1}{C_a\sqrt{d}}\left(1 + \frac{2}{\hat{\beta}} + \frac{2}{\hat{\beta}\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\right)\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}. \end{aligned}$$

As a consequence, combining (4.19) and (4.21), we get

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{0,\Omega} \leq k_3\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + k_4\|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega}, \tag{4.22}$$

with

$$\begin{aligned} k_3 :=\ & \frac{2}{\nu\rho_0\hat{\beta}}(1 + k_1), \\ k_4 :=\ & \frac{2}{\hat{\beta}}\left(1 + \frac{1}{\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)} + \frac{k_2}{\nu\rho_0}\right). \end{aligned}$$

Therefore, according to the triangle inequality, from (4.21) and (4.22), we obtain

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{0,\Omega} \leq (1 + k_1 + k_3)\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega}\ +\ (1 + k_2 + k_4)\|\boldsymbol{\xi}_{\mathbf{u}}\|_{0,\Omega},$$

and since $(\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h) \in \mathbb{H}_h^{\mathbf{f}} \times \mathbf{Q}_h$ is arbitrary, we get

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div},\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{0,\Omega} \leq (1 + k_1 + k_3)\inf_{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\mathbf{f}}}\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}\ +\ (1 + k_2 + k_4)\inf_{\mathbf{v}_h\in\mathbf{Q}_h}\|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega},$$

which together to (4.12), concludes the proof. □

**4.4. Approximating the pressure and the original pseudostress.** First, we propose a post-processing procedure to approximate the pressure. To do that, we observe that if $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ is the unique solution of (3.2), then, according to (2.8) and (2.11), it is possible to recover the pressure $p \in L_0^2(\Omega)$ from the identity

$$p = -\frac{\nu}{d}\left(\mathbf{u}\cdot\nabla\rho\ -\ \frac{1}{|\Omega|}(\mathbf{u}, \nabla\rho)_\Omega\right) - \frac{1}{d}\mathrm{tr}\,\boldsymbol{\sigma}. \tag{4.23}$$

In this way, if $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ is the unique solution of (4.3), it is reasonable to think that the function

$$p_h = -\frac{\nu}{d}\left(\mathbf{u}_h \cdot \nabla\rho \,-\, \frac{1}{|\Omega|}(\mathbf{u}_h, \nabla\rho)_\Omega\right) - \frac{1}{d}\mathrm{tr}\,\boldsymbol{\sigma}_h, \tag{4.24}$$

is a good approximation of the pressure. This result is established next.

Corollary 4.4. *Assume that hypotheses of Theorem 4.3 hold. Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete problems* (3.2) *and* (4.3)*, respectively. Then, there exists* $C > 0$*, independent of* $h$*, such that*

$$\|p - p_h\|_{0,\Omega} \leq C\left\{\inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega} + \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega}\right\}.$$

*Proof.* From (4.23) and (4.24), Hölder and the triangle inequalities, it follows that

$$\begin{aligned}\|p - p_h\|_{0,\Omega} \leq\;& \frac{\nu}{d}\|(\mathbf{u} - \mathbf{u}_h)\cdot\nabla\rho\|_{0,\Omega} \,+\, \frac{\nu}{d|\Omega|^{1/2}}\,|(\mathbf{u} - \mathbf{u}_h, \nabla\rho)_\Omega| + \frac{1}{d}\|\mathrm{tr}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \\ \leq\;& \frac{\nu}{d}\|\rho\|_{W^{1,\infty}(\Omega)}\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,+\, \frac{\nu}{d|\Omega|^{1/2}}\|\rho\|_{1,\Omega}\,\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \frac{1}{d}\|\mathrm{tr}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}.\end{aligned}$$

Then, the result follows from Theorems 5.1 and 5.3. □

In what follows, for the sake of clarity we again make the difference between $\boldsymbol{\sigma}_0 \in \mathbb{H}_0$ and $\boldsymbol{\sigma} \in \mathbb{H}$, as in Section 2.

In order to approximate the original pseudostress in (2.11), let us remind that in Section 2, Lemma 2.1, we have proved that formulations (2.10) and (2.13) are equivalent. That is, we have proved that $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H} \times \mathbf{Q}$ is the unique solution of (2.10) if and only if $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ is the unique solution of (2.13), where $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}$ are related by

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}_0 - \frac{\nu}{d|\Omega|}(\mathbf{u}, \nabla\rho)_\Omega \mathrm{I}. \tag{4.25}$$

In turn, in this section we have propose a mixed finite element method to approximate the solution of (2.13) (or equivalently (3.2)).

As a result, if $(\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ is the unique solution of (4.3), it is easy to see that the tensor

$$\boldsymbol{\sigma}_h := \boldsymbol{\sigma}_{h,0} - \frac{\nu}{d|\Omega|}(\mathbf{u}_h, \nabla\rho)_\Omega \mathrm{I} \tag{4.26}$$

approximates $\boldsymbol{\sigma} \in \mathbb{H}$ defined by (4.25). This result is established in the following Corollary.

Corollary 4.5. *Assume that hypotheses of Theorem 4.3 hold. Let* $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ *and* $(\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete problems* (3.2) *and* (4.3)*, respectively. In addition, let* $\boldsymbol{\sigma} \in \mathbb{H}$ *and* $\boldsymbol{\sigma}_h \in \mathbb{H}_h$ *be the tensors defined by* (4.25) *and* (4.26)*, respectively. Then, there exists* $C > 0$*, independent of* $h$*, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} \leq C\left\{\inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma}_0 - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega} + \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega}\right\}. \tag{4.27}$$

*Proof.* First, from (4.25), (4.26) and the triangle inequality, it is easy to see that

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} = &\quad \|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{h,0}-\frac{\nu}{d|\Omega|}(\mathbf{u}-\mathbf{u}_h,\nabla\rho)_\Omega \mathrm{I}\|_{\mathbf{div},\Omega}\\ \leq &\quad \|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{h,0}\|_{\mathbf{div},\Omega}+\frac{\nu}{d^{1/2}|\Omega|^{1/2}}|(\mathbf{u}-\mathbf{u}_h,\nabla\rho)_\Omega|\\ \leq &\quad \|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{h,0}\|_{\mathbf{div},\Omega}+\frac{\nu\|\nabla\rho\|_{0,\Omega}}{d^{1/2}|\Omega|^{1/2}}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}.\end{aligned}$$

Then, the result is a direct application of Theorem 4.3. □

**5. Particular choices of discrete spaces.** We now specify examples of finite elements subspaces satisfying the hypotheses (**H.0**), (**H.1**) and (**H.2**). To this end we let $\mathcal{T}_h$ be a regular family of triangulations of the polygonal region $\overline{\Omega}$ by triangles $T$ of diameter $h_T$ such that $\overline{\Omega}=\cup\{T: T\in\mathcal{T}_h\}$ and define $h:=\max\{h_T: T\in\mathcal{T}_h\}$. Now, given an integer $l\geq 0$ and a subset $S$ of $\mathbf{R}^d$, we denote by $\mathbb{P}_l(S)$ the space of polynomials of total degree at most $l$ defined on $S$.

**5.1. The Raviart-Thomas element.** For each integer $k\geq 0$ and for each $T\in\mathcal{T}_h$, we define the local Raviart-Thomas space of order $k$ (see, for instance [4]):

$$\mathbf{RT}_k(T):=[\mathbb{P}_k(T)]^d\oplus\mathbb{P}_k(T)\mathbf{x},$$

where $\mathbf{x}:=(x_1,\cdots,x_d)^t$ is a generic vector of $\mathbf{R}^d$. Then, we specify the discrete spaces in (4.2) by defining:

$$\begin{aligned}\mathbf{H}_h := &\quad \{\tau\in\mathbf{H}(\mathrm{div}\,;\Omega): \quad \tau|_T\in\mathbf{RT}_k(T), \quad \forall T\in\mathcal{T}_h\},\\ Q_h := &\quad \left\{v\in L^2(\Omega): \quad v|_T\in\mathbb{P}_k(T), \quad \forall T\in\mathcal{T}_h\right\}.\end{aligned}\tag{5.1}$$

It is well known that these subspaces satisfy the following approximation properties (see, e.g. Theorem 3.16 in [19]):

For each $s\in(0,k+1]$ and for each $\tau\in\mathbf{H}^s(\Omega)$, with $\mathrm{div}\,\tau\in H^s(\Omega)$, there exists $\tau_h\in\mathbf{H}_h$, such that

$$\|\tau-\tau_h\|_{\mathrm{div}\,,\Omega}\leq Ch^s\left\{\|\tau\|_{s,\Omega}+\|\mathrm{div}\,\tau\|_{s,\Omega}\right\}.\tag{5.2}$$

For each $s\in[0,k+1]$ and for each $v\in H^s(\Omega)$ there exists $v_h\in Q_h$ such that

$$\|v-v_h\|_{0,\Omega}\leq Ch^s\|v\|_{s,\Omega}.\tag{5.3}$$

Moreover, it is easy to see that the corresponding discrete spaces $\mathbb{H}_h$ and $\mathbf{Q}_h$ satisfy assumptions (**H.0**), (**H.1**) and (**H.2**). In particular, the proof of the inf-sup condition (4.4) can be found in [15, Lemma 2.4].

According to the above, and Theorem 4.3, we are able to establish the convergence of the Galerkin scheme (4.3) for this particular choice of spaces.

THEOREM 5.1. *Assume that*

$$\max\{C_{WP},\hat{C}_{WP}\}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\leq\frac{1}{2},\tag{5.4}$$

*with $C_{WP}$ and $\hat{C}_{WP}$ defined in* (3.24) *and* (4.8), *respectively. In addition, let $\mathbb{H}_{h,0}$ and $\mathbf{Q}_h$ be the finite element subspaces defined by* (4.2) *in terms of the specific discrete spaces given by* (5.1).

*Then, the Galerkin scheme* (4.3) *has a unique solution* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{Q}_h$ *and there exists* $C_1 > 0$, *independent of* $h$, *such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}} \leq C_1 \|\mathbf{f}\|_{0,\Omega}.$$

*Moreover, let* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ *be the unique solution of the continuous problem* (3.2) *and assume that* $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, *and* $\mathbf{u} \in \mathbf{H}^s(\Omega)$, *for some* $s \in (0, k+1]$. *Then there exists* $C_2 > 0$, *independent of* $h$, *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq C_2 h^s \left\{\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega}\right\}.$$

*Proof.* Since the finite element subspaces $\mathbb{H}_{h,0}$ and $\mathbf{Q}_h$ satisfy hypotheses (**H.0**), (**H.1**) and (**H.2**), then the proof is a straightforward application of Theorem 4.2 and properties (5.2) and (5.3). □

Finally, from Corollary 4.4 and Theorem 5.1 we obtain the optimal convergence of the post-processed pressure introduced in (4.24).

COROLLARY 5.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0 \times \mathbf{Q}$ *be the unique solution of the continuous problem* (3.2), *and* $p \in L^2_0(\Omega)$ *given by* (4.23). *In addition let* $p_h$ *be the discrete pressure computed by the post-processing formula* (4.24). *Assume that hypotheses of Theorem 5.1 hold. Then, there exists* $C > 0$, *independent of* $h$, *such that*

$$\|p - p_h\|_{0,\Omega} \leq C h^s \left\{\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega}\right\}.$$

**5.2. The Brezzi-Douglas-Marini element.** Now, for each integer $k \geq 0$, we introduce the following discrete spaces in (4.2):

$$\begin{aligned} \mathbf{H}_h &:= \left\{\tau \in \mathbf{H}(\mathrm{div}\,;\Omega) : \quad \tau|_T \in \mathbb{P}_{k+1}(T), \quad \forall\, T \in \mathcal{T}_h\right\}, \\ Q_h &:= \left\{v \in L^2(\Omega) : \quad v|_T \in \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}_h\right\}. \end{aligned} \tag{5.5}$$

We remark that the product space $\mathbf{H}_h \times Q_h$ constitutes the finite element approximation for mixed problem introduced by Brezzi, Douglas and Marini (BDM) (see, e.g. [4]).

Again, it is well known that these subspaces satisfy the following approximation properties (see, e.g. Theorem 3.16 in [19]):

For each $s \in (0, k+1]$ and for each $\tau \in \mathbf{H}^s(\Omega)$, with $\mathrm{div}\,\tau \in H^s(\Omega)$, there exists $\tau_h \in \mathbf{H}_h$, such that

$$\|\tau - \tau_h\|_{\mathrm{div}\,,\Omega} \leq C h^s \left\{\|\tau\|_{s,\Omega} + \|\mathrm{div}\,\tau\|_{s,\Omega}\right\}. \tag{5.6}$$

For each $s \in [0, k+1]$ and for each $v \in H^s(\Omega)$ there exists $v_h \in Q_h$ such that

$$\|v - v_h\|_{0,\Omega} \leq C h^s \|v\|_{s,\Omega}. \tag{5.7}$$

Moreover, the corresponding discrete spaces $\mathbb{H}_h$ and $\mathbf{Q}_h$ satisfy assumptions (**H.0**), (**H.1**) and (**H.2**). For the proof of the inf-sup condition (4.4) in (**H.2**), we just comment that it follows analogously to the Raviart-Thomas case (see, again [15, Lemma 2.4]), recalling that it is also possible to construct a Fortin operator by using the BDM-projection.

Now, we establish the convergence of the Galerkin scheme (4.3) for this particular choice of spaces.

THEOREM 5.3. *Assume that*

$$\max\{C_{WP},\hat{C}_{WP}\}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\leq\frac{1}{2},\tag{5.8}$$

*with* $C_{WP}$ *and* $\hat{C}_{WP}$ *defined in* (3.24) *and* (4.8), *respectively. In addition, let* $\mathbb{H}_{h,0}$ *and* $\mathbf{Q}_h$ *be the finite element subspaces defined by* (4.2) *in terms of the specific discrete spaces given by* (5.5). *Then, the Galerkin scheme* (4.3) *has a unique solution* $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_{h,0}\times\mathbf{Q}_h$ *and there exists* $C_1>0$, *independent of* $h$, *such that*

$$\|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}}\leq C_1\|\mathbf{f}\|_{0,\Omega}.$$

*Moreover, let* $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0\times\mathbf{Q}$ *be the unique solution of the continuous problem* (3.2) *and assume that* $\boldsymbol{\sigma}\in\mathbb{H}^s(\Omega)$, $\mathbf{div}\boldsymbol{\sigma}\in\mathbf{H}^s(\Omega)$, *and* $\mathbf{u}\in\mathbf{H}^s(\Omega)$, *for some* $s\in(0,k+1]$. *Then there exists* $C_2>0$, *independent of* $h$, *such that*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\leq C_2h^s\left\{\|\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{u}\|_{s,\Omega}\right\}.$$

*Proof.* Since the finite element subspaces $\mathbb{H}_{h,0}$ and $\mathbf{Q}_h$ satisfy hypotheses (**H.0**), (**H.1**) and (**H.2**), then the proof is a straightforward application of Theorem 4.2 and properties (5.6) and (5.7). ∎

We end this section by establishing the rate of convergence of the post-processed pressure computed by formula (4.24). Its proof follows from Corollary 4.4 and Theorem 5.3.

COROLLARY 5.4. *Let* $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0\times\mathbf{Q}$ *be the unique solution of the continuous problem* (3.2), *and* $p\in L_0^2(\Omega)$ *given by* (4.23). *In addition let* $p_h$ *be the discrete pressure computed by the post-processing formula* (4.24). *Assume that hypotheses of Theorem 5.3 hold. Then, there exists* $C>0$, *independent of* $h$, *such that*

$$\|p-p_h\|_{0,\Omega}\leq Ch^s\left\{\|\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{u}\|_{s,\Omega}\right\}.$$

## 6. Numerical results.

In this section we present a numerical example in $\mathbf{R}^2$, illustrating the performance of the mixed finite element scheme (4.3). Here we consider the specific finite element subspaces $\mathbb{H}_{0,h}$ and $\mathbf{Q}_h$ defined in terms of the specific discrete spaces given by (5.1) with $k=0$. In addition, the zero integral mean condition for tensors in the space $\mathbb{H}_{0,h}$ is imposed via a real Lagrange multiplier. In what follows, $N$ stands for the total number of degrees of freedom defining $\mathbb{H}_{0,h}\times\mathbf{Q}_h$. Denoting by $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0\times\mathbf{Q}$ and $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_{h,0}\times\mathbf{Q}_h$, the solutions of (3.2) and (4.3), respectively, the individual errors are defined by

$$e(\boldsymbol{\sigma}):=\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega},\quad e(\mathbf{u}):=\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\quad e(p):=\|p-p_h\|_{0,\Omega},$$

where the approximate pressure $p_h$ is computed by the post-processing formula (4.24). Furthermore, we define the experimental rates of convergence

$$r(\boldsymbol{\sigma}):=\frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')},\quad r(\mathbf{u}):=\frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')},\quad r(p):=\frac{\log(e(p)/e'(p))}{\log(h/h')},$$

where $h$ and $h'$ are two consecutive meshsizes with errors $e$ and $e'$.

In what follows, we consider a region $\Omega := (-1,1)\times(-1,1)$ and define the density function

$$\rho(x_1,x_2) := \exp(\mu(x_1+x_2)) \qquad \forall\,(x_1,x_2)\in\Omega,$$

where $\mu$ is a parameter in $\mathbf{R}$. We notice that

$$\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)} = |\mu|, \tag{6.1}$$

and then, as we shall see in Figure 6.4, and as predicted in (4.16), the good performance of our method depends strongly on the choice of $\mu$.

In turn, we choose the datum $\mathbf{f}$ so that the exact solution is given by the smooth functions

$$\begin{aligned} \mathbf{u}(x_1,x_2) = & \quad \frac{\operatorname{curl}(\sin^2(\pi x_1)\sin^2(\pi x_1))}{\rho(x_1,x_2)} && \forall(x_1,x_2)\in\Omega,\\ p(x_1,x_2) = & \quad x_1\sin(x_2) && \forall(x_1,x_2)\in\Omega, \end{aligned}$$

where $\operatorname{curl}\varphi := \left(\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right)$, for any sufficiently smooth function $\varphi$.

The numerical results shown below were obtained in a *Pentium Xeon computer with dual processor*, using a MATLAB code. In Table 6.1 we summarize the convergence history of our mixed finite element scheme (4.3), with $\mu = 2$ and for a set of shape-regular triangulations of the computational domain $\Omega$. We observe there that, looking at the experimental rates of convergence, the $O(h)$ predicted by Theorem 5.1, with $s = 1$ is attained in all the unknowns. In order to emphasize the good performance of our scheme, in Figures 6.1 and 6.2 we display some components of the approximate (left) and exact (right) solutions of our example for $N = 82177$. We also display in Figure 6.3 the approximate (left) and exact (right) pressures. It is clear from these Figures that the finite element subspaces employed provide very accurate approximations to the unknowns. In addition, we observe that the discrete pressure $p_h$, computed via the post-processing formula (4.24), presents some oscillations, perhaps because $\nabla\rho$ is not a polynomial function on each element. Despite this minor issue, Table 6.1 confirms the fact that $p_h$ converges to $p$ with optimal rate of convergence as predicted.

Finally, having in mind assumption (4.16), in Figure 6.4 we display the relation between $\mu$ (cf. (6.1)) and the condition number of the global matrix given by the left hand side of (4.3) computed with the command *condest* in MATLAB, considering a fixed mesh of size $h = 1/4$. We observe here that the condition number is stable for $|\mu|\leq 6$ and blows up for $|\mu| > 6$. This phenomenon shows that assumption (4.16), beyond of being just a theoretical hypothesis, in practise, it ensures the good performance of the numerical method for small values of $\|\frac{\nabla\rho}{\rho}\|_{\mathbf{L}^\infty(\Omega)}$.

Table 6.1
*degrees of freedom, meshsizes, errors, and rates of convergence.*

| $N$ | $h$ | $e(\boldsymbol{\sigma})$ | $r(\boldsymbol{\sigma})$ | $e(\mathbf{u})$ | $r(\mathbf{u})$ | $e(p)$ | $r(p)$ |
|---|---|---|---|---|---|---|---|
| 89 | 1 | 4.530E+01 | – | 9.770E+00 | – | 3.795E+00 | – |
| 337 | 1/2 | 1.232E+01 | 1.956 | 5.657E+00 | 0.821 | 3.353E+00 | 0.186 |
| 1313 | 1/4 | 4.301E+00 | 1.547 | 3.367E+00 | 0.763 | 1.798E+00 | 0.916 |
| 5185 | 1/8 | 1.873E+00 | 1.211 | 1.743E+00 | 0.959 | 9.026E-01 | 1.004 |
| 20609 | 1/16 | 8.954E-01 | 1.070 | 8.775E-01 | 0.995 | 4.493E-01 | 1.011 |
| 82177 | 1/32 | 4.423E-01 | 1.020 | 4.394E-01 | 1.000 | 2.242E-01 | 1.005 |
| 328193 | 1/64 | 2.204E-01 | 1.006 | 2.198E-01 | 1.001 | 1.121E-01 | 1.002 |

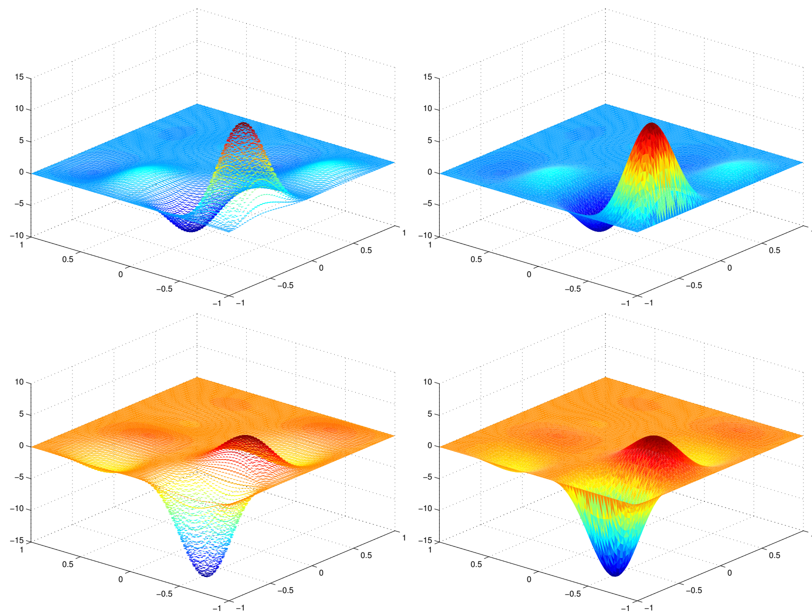

FIG. 6.1. *first (top) and second (bottom) components of* $\mathbf{u}_h$ *and* $\mathbf{u}$ *for* $N = 82177$

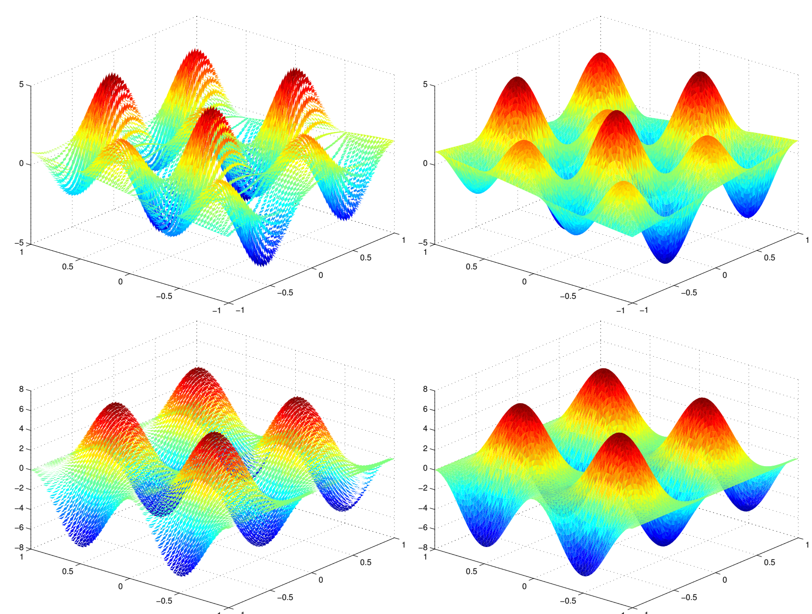

FIG. 6.2. $\sigma_{11}$, $\sigma_{11,h}$ *(top) and* $\sigma_{21}$, $\sigma_{21,h}$ *(bottom) for* $N = 82177$

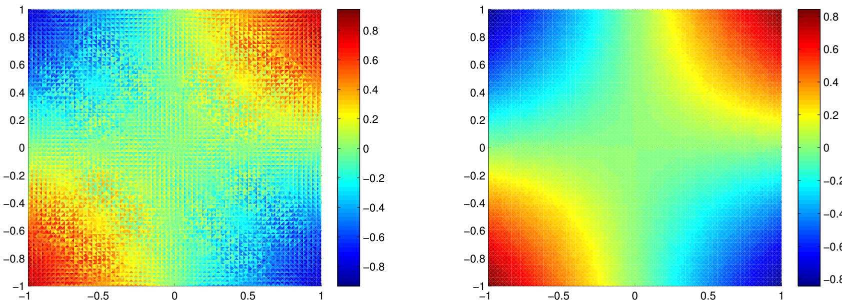

FIG. 6.3. *$p$ and $p_h$ for N=82177*

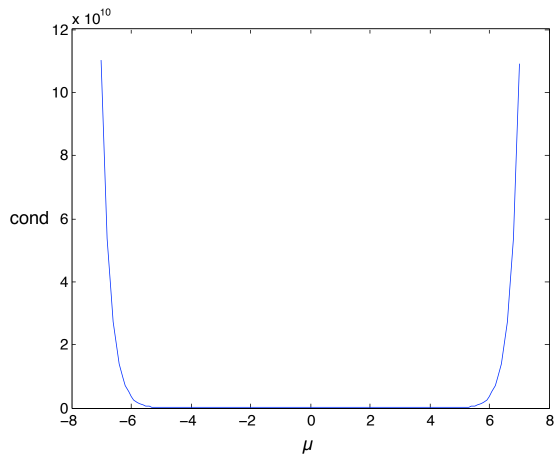


FIG. 6.4. *$\mu$ v/s condition number*

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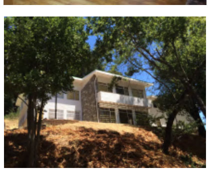

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