UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

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Aníbal Coronel, Patricio Cumsille, Mauricio Sepúlveda

PREPRINT 2013-24

SERIE DE PRE-PUBLICACIONES

CONVERGENCE OF A LEVEL-SET ALGORITHM IN SCALAR CONSERVATION LAWS

ANÍBAL CORONEL†, PATRICIO CUMSILLE†, AND MAURICIO SEPÚLVEDA‡

Abstract. This paper is concerned with the convergence of the level-set algorithm introduced by Aslam (2001, *J. Comput. Phy.* 167, 413-438) for tracking the discontinuities in scalar conservation laws in the case of linear or strictly convex flux function. The numerical method is deduced by an appropriate level-set representation of the entropy solution: the zero of a level-set function is used as an indicator of the discontinuity curves and two auxiliary states, which are assumed continuous through the discontinuities, are introduced. We rewrite the numerical level-set algorithm as a procedure consisting of three big steps: initialization, evolution and reconstruction. In the initialization step we choose an entropy admissible level-set representation of the initial condition. In the evolution step we solve at each iteration step an uncoupled system of three equations (two conservation laws for the auxiliary states and the level-set equation for the approximation of the level set function) and select the entropy admissible level-set representation of the solution profile at the end of the time iteration, which is used as the initial condition by the next iteration. The reconstruction is naturally given by the recuperation of the entropy solution by using the level-set representation with auxiliary states and the level-set function determined at the evolution step. We prove the convergence of the numerical solution to the entropy solution in L^p_{loc} for every $p \geq 1$, using L^∞-weak BV estimates and a cell entropy inequality. In addition, some numerical examples focused on the elementary wave interaction are presented.

1. Introduction

In this paper we study the convergence of a level-set approximation for tracking the admissible discontinuities of the following Cauchy problem:

$$u_t + (f(u))_x = 0, \quad (x,t) \in Q := \mathbb{R} \times \mathbb{R}_0^+, \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

under the following data assumptions

$$u_0 \in L^\infty(\mathbb{R}), \text{ i.e. } u_0(x) \in [u_m, u_M] = [\operatorname{ess\,inf}_{\mathbb{R}}(u_0), \operatorname{ess\,sup}_{\mathbb{R}}(u_0)] \text{ for a.e. } x \in \mathbb{R}; \text{ and} \tag{1.2}$$

$$f(u) = au \ (a \text{ constant}) \text{ or } f \in C^2(\mathbb{R},\mathbb{R}) \text{ and } f''(u) \geq \alpha > 0, \forall u \in \mathbb{R} \text{ and some } \alpha > 0, \tag{1.3}$$

based on the method of lines [2] and a level set technique [3, 4, 5, 6]. T. Aslam [1] has numerically studied the algorithm and also make an extension to the system of gas dynamics [7]. However, he lefts out the theoretical convergence analysis and in our knowledge that kind of results are still missing.

We recall that under (1.3), in the case of strictly convexity of the flux, the admissible discontinuities or shock waves are characterized by the Rankine-Hugoniot jump condition

$$s(t) = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{1.4}$$

and the Lax entropy inequality

$$f'(u_L) > s(t) > f'(u_R), \tag{1.5}$$

Date: October 28, 2013.

† GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile, E-mail: acoronel@ubiobio.cl, pcumsill@gmail.com.

‡ CI^{2}MA and DIM, Universidad de Concepción, Chile, E-mail: mauricio@ing-mat.udec.cl,

where s is the speed of the shock curve, u_L and u_R are the values of u just on the left and right of the shock at time t, respectively, see for instance [8, 9]. We note that, the hypothesis (1.3) in the linear case, implies that the discontinuities in the weak solutions satisfies only (1.4) with $s(t) = a$ and the condition (1.5) is not required.

Now, we briefly discuss, at continuous level, the key ideas motivating the level-set algorithm. Let us consider that u admits the representation $u = \mathscr{H}(p, v, w)$, where the change of variable $\mathscr{H}$ and the auxiliary states p, v and w will be specified later. Assuming that the entropy solution profile $u(\cdot, t_1)$ is given, then the construction of the entropy admissible profile $u(\cdot, t_2)$, for $t_2 > t_1$, via the original level-set algorithm can be interpreted at continuous level as a procedure consisting of three stages, as described below:

(i) *Selection of initial states at $t = t_1$.* We select three functions $p_1, v_1, w_1 : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}
&p_1(c) = 0 \text{ on each } c \text{ where } u(\cdot, t_1) \text{ is discontinuous;}\\
&p_1, v_1 \text{ and } w_1 \text{ are continuous through the zeros of } p_1\text{; and}\\
&u(\cdot, t_1) = \mathscr{H}\Big(p_1(\cdot), v_1(\cdot), w_1(\cdot)\Big) \text{ satisfies (1.4)-(1.5).}
\end{aligned} \tag{1.6}$$

(ii) *Calculus of the states at $t = t_2$.* By the relation $u = \mathscr{H}(p, v, w)$, we can perform the following formal calculus

$$\begin{aligned}
u_t + f(u)_x \;=\;& \partial_1\mathscr{H}(p, v, w)\Big[p_t + F_p(p)_x\Big] + \partial_1\mathscr{H}(p, v, w)p_x\Big[f'(u) - F_p'(p)\Big]\\
&+\partial_2\mathscr{H}(p, v, w)\Big[v_t + F_v(v)_x\Big] + \partial_2\mathscr{H}(p, v, w)v_x\Big[f'(u) - F_v'(v)\Big]\\
&+\partial_3\mathscr{H}(p, v, w)\Big[w_t + F_w(w)_x\Big] + \partial_3\mathscr{H}(p, v, w)w_x\Big[f'(u) - F_w'(w)\Big].
\end{aligned} \tag{1.7}$$

If we select, F_p, F_v, F_w and $\mathscr{H}$, such that

$$\begin{aligned}
&F_p(p(x, t)) = s(t_1) \text{ for each } x = c \text{ zero of } p_1, \quad F_v = F_w = f,\\
&\partial_1\mathscr{H}(p, v, w) = 0 \text{ for } p \neq 0,\\
&\partial_2\mathscr{H}(p, v, w) = 0 \text{ for } p > 0 \text{ and } F_v' = f' \text{ for } p \leq 0,\\
&\partial_3\mathscr{H}(p, v, w) = 0 \text{ for } p \leq 0 \text{ and } F_w' = f' \text{ for } p > 0,
\end{aligned} \tag{1.8}$$

it is enough to demand that $p(\cdot, t_2), v(\cdot, t_2)$ and $w(\cdot, t_2)$ must be solutions of

$$\left\{\begin{aligned} z_t + F_z(z)_x &= 0, \quad (x, t) \in \mathbb{R}\times]t_1, t_2[,\\ z(x, t_1) &= z_1(x) \quad x \in \mathbb{R}, \end{aligned}\right. \qquad \text{with } z_1 \in \{p_1, v_1, w_1\}, \tag{1.9}$$

to deduce that $u(\cdot, t_2)$ is a weak solution of (1.1).

(iii) *Definition of an entropy admissible states at $t = t_2$.* The information about the discontinuities of $u(\cdot, t_2)$ are given by the zeros of $p(\cdot, t_2)$, then we can select $p_2 = p(\cdot, t_2)$. The construction of v_2 and w_2 are done in terms of $p_2, v(\cdot, t_2)$ and $w(\cdot, t_2)$ and in order to have that $u(\cdot, t_2) = \mathscr{H}\Big(p_2(\cdot), v_2(\cdot), w_2(\cdot)\Big)$ satisfies (1.4)-(1.5).

We note that, when the flux function is linear the relation (1.7) is satisfied with $F_p = F_v = F_w = a$ and the step (iii) is dropped.

The numerical level-set algorithm can be stated as an iterative process where each iteration is the discrete analogue of the steps (i)-(iii). A systematical analysis of this process implies that we can define the algorithm in three main steps: initialization, evolution and reconstruction. Indeed, let $\mathscr{H}$ the natural level representation of u with the level set function p, see (3.1). First, we remark that, the step (i) is applied only for the initial condition, since in the next iterations the condition (1.6) is naturally satisfied by the construction of the step (iii) at the previous iteration. Then, in the numerical level-set algorithm, we define the first step called initialization to select the entropy admissible initial states p_0, v_0 and w_0 representing to u_0. Second, we solve numerically the equations (1.9) at each level of time and define the entropy admissible auxiliary states v and w, which are used as initial conditions by the next time level. Thus, we introduce the second step of the numerical level-set algorithm called evolution. Finally, we remark that the numerical entropy solution of (1.1) can be reconstructed in a natural way by the function $\mathscr{H}$, this third step of the level-set algorithm is called the reconstruction step.

The aim of this paper is the development of a convergence theory for the level-set numerical algorithm, considering that the equations (1.9) (in the evolution step) are solved by monotone finite volume methods. In this sense, we recall that the first convergence theory of monotone finite volume methods for scalar multidimensional conservation laws on cartesian triangulations was introduced in a seminal paper by Crandall and Majda [10], where the three central key issues are: (a) The regularity assumptions: $f \in \mathrm{Lip}(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$; (b) The compactness proof of the numerical scheme is derived from uniform estimate of the bounded variation norm of the numerical solution and Helly's theorem; and (c) The convergence proof of the numerical solution towards the unique entropy solution is deduced from a single cell entropy inequality. In general, the main drawback, for application of this theory, for instance in the case of general triangulations, is the uniform control of the bounded variation norm. In particular, we note that, a straightforward application of Crandall-Majda theory to study the convergence of the level-set algorithm is not possible, since the required bounded variation estimates are difficult to get even in the case of linear flux function with upwind approximation of the auxiliary states and the level set function. Afterwards, in order to overcome this difficulty, two theories have been introduced. First, maybe the second big approach to prove convergence of finite volume methods without using the bounded variation estimate, was given by Coquel and Lefloch [11], where the proof is developed using the notion of measured valued solutions which was defined by Szepessy [12] inspired in the work of Diperna [12], for further details and the application of this theory to monotone finite volume methods we refer to the work of Cockburn, Coquel and Lefloch [13]. Second and more recently, Eymard, Gallouët and Herbin [8] gave a new methodology to prove convergence of finite volume methods using a weak bounded variation estimate and a new notion of generalized solutions to conservation laws called entropy process solutions. In this paper, we make an application (adaptation and improvements) of the ideas given by Eymard, Gallouët and Herbin.

The outline of this paper is as follows. In Section 2, we recall some concepts and results used through of the paper. In Section 3, we describe at continuous level the fundamental ideas of the numerical level set method. Based on the described ideas on Section 3 we introduce in detail the discrete level set scheme on Section 4. In section Section 5, we show that the level set scheme used for approximation of the linear equation converge to the weak solution. In Section 6, assuming strictly convexity flux function, we prove that the numerical scheme solution converges towards the entropy solution. In Section 7, we give some numerical examples focused on the elementary wave interaction. Finally, in Section 8, we make the conclusion of the paper.

2. Preliminaries

In this section, we recall the notions of entropy process solutions and nonlinear weak-$\star$ convergence, the notation and concepts concerning to monotone finite volume methods, the general ideas of level set methods and some technical results used in the proof of the convergence.

2.1. Entropy process solutions and nonlinear weak-$\star$ convergence.

The entropy process solutions can be defined for any convex entropy-entropy flux pairs, see [8]. However, it is well known that it can be characterized in terms of Kružkov's entropies [8, 14]. Then, in this paper we consider that characterization as definition of entropy process solutions.

Definition 2.1. *A function $\mu \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+ \times]0,1[)$ is called an entropy process solution of the cauchy problem* (1.1) *if the following inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_0^+} \int_0^1 \Big\{ |\mu(x,t,\alpha) - k| \varphi_t(x,t) + \Big(f(\mu(x,t,\alpha) \top k) - f(\mu(x,t,\alpha) \perp k) \Big) \varphi_x(x,t) \Big\} dx dt d\alpha$$
$$+ \int_{\mathbb{R}} |u_0(x) - k| \varphi_t(x,0) dx \geq 0, \qquad \forall k \in \mathbb{R}, \quad \forall \varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+), \tag{2.1}$$

is satisfied. Here $a \top b = \max\{a,b\}$ and $a \perp b = \min\{a,b\}$ for all $a,b \in \mathbb{R}$.

The convergence towards the entropy process solution is related with the following notion of convergence and compactness result in $L^\infty(\Omega)$, see [8].

Definition 2.2. *Let Ω be an open subset of $\mathbb{R}^d$, $\{u_n\} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega\times]0,1[)$. The sequence $\{u_n\}$ converge towards u in the nonlinear weak-$\star$ sense if*

$$\int_\Omega \theta(u_n(x))\varphi(x)dx \to \int_0^1 \int_\Omega \theta(u(x,\alpha))\varphi(x)dxd\alpha, \quad \text{as } n \to \infty,$$

for all $\varphi \in L^1(\Omega)$ and $\theta \in C(\mathbb{R},\mathbb{R})$.

Proposition 2.1. *Let Ω be an open subset of $\mathbb{R}^d$ and $\{u_n\}$ a bounded sequence of $L^\infty(\Omega)$. Then there exists a subsequence of $\{u_n\}$, which will still be denoted by $\{u_n\}$, and a function $u \in L^\infty(\Omega\times]0,1[)$ such that the subsequence $\{u_n\}$ converges towards u in the the nonlinear weak-$\star$ sense.*

We comment that the function θ in the definition 2.2 in the convergence analysis is taking as the entropy or entropy fluxes, i.e. $\theta = |\cdot - k|$ or $\theta = f(\cdot \top k) - f(\cdot \bot k)$.

2.2. **Finite volume method and Monotone flux schemes.** Let us consider the standard notation of an homogeneous discretization of $\mathbb{R} \times \mathbb{R}_+$, i. e.

$$\mathbb{R} \times \mathbb{R}_+ = \bigcup_{j\in\mathbb{Z}} \bigcup_{n\in\mathbb{N}} [x_{j-1/2}, x_{j+1/2}[\times[t_n, t_{n+1}[, \quad x_{j+1/2} = (j+1/2)\Delta x, \quad t_n = n\Delta t, \tag{2.2}$$

where Δx and Δt are the given spatial and temporal step sizes, respectively. We recall that, see [8, 9], the basis of finite volume schemes for (1.1a) is given by its integral form:

$$\int_{k_1}^{k_2} u(x,\tau_2)dx - \int_{k_1}^{k_2} u(x,\tau_1)dx + \int_{\tau_1}^{\tau_2} f(u(k_2,t))dt - \int_{\tau_1}^{\tau_2} f(u(k_1,t))dt = 0, \tag{2.3}$$

where $[k_1, k_2[\subset \mathbb{R}$ is a generic computational cell (control volume) and $[\tau_1, \tau_2[\subset \mathbb{R}_0^+$ is the evolution interval. The first two terms in (2.3) can be rewritten using the average of $u(\cdot, \tau_i)$ on the control volume $[k_1, k_2]$ and the last two terms can be approximated by choosing a suitable function called the numerical flux function. To be precise, if we select $k_1 = x_{j-1/2}, k_2 = x_{j+1/2}, \tau_1 = t_n$ and $\tau_2 = t_{n+1}$, the relation (2.3) implies that the finite volume discretization of (1.1) is given by

$$u_j^{n+1} - u_j^n + \lambda(f_{j+1/2}^n - f_{j-1/2}^n) = 0, \qquad (j,n) \in \mathbb{Z}\times\mathbb{N}, \tag{2.4a}$$

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, \quad j \in \mathbb{Z}, \tag{2.4b}$$

where

$$\lambda = \frac{\Delta t}{\Delta x}, \quad u_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)dx \quad \text{and} \quad f_{j+1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt.$$

In the case of $2p+1$ points $f_{j+1/2}^n$ may be written as follows

$$f_{j+1/2}^n = g(u_{j-p+1}^n, \dots, u_{j+p}^n) \quad \text{where} \quad g \in \mathrm{Lip}(\mathbb{R}^{2p},\mathbb{R}) \quad \text{and} \quad g(u,\dots,u) = f(u). \tag{2.5}$$

The function g is called the numerical flux function and the last property in (2.5) is known as the consistence for the finite volume scheme (2.4). A particular and interesting case are the well-known monotone flux schemes, where $g : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumptions [8]:

- (G1) *Lip-regularity.* g is locally lipschitz with respect to each of its variables on $[u_m, u_M]^2$,
- (G2) *Consistence.* $g(u,u) = f(u)$, for all $u \in [u_m, u_M]$,
- (G3) *Monotonicity.* g is non-decreasing with respect to its first variable and non-increasing with respect to its second variable on $[u_m, u_M]^2$.

The explicit monotone flux schemes have played a very important role in the development of numerical analysis for conservation laws due to its good properties: consistence in the finite volume sense, L^∞-stability, BV-stability and convergence of the numerical solution to the entropy solution under a CFL condition, see [8, 9] for further details.

2.3. **Level set method.** The level set method was introduced by S. Osher and J. A. Sethian in [3] to give an answer to the following situation: given an interface Γ in $\mathbb{R}^d$ of co-dimension one, bounding an open region Ω, analyse and compute its subsequent motion under a velocity field $\mathbf{v}$ depending of the geometry and physics. The main idea of the level set methodology is to capture the propagating interface as the zero level set of a higher dimensional smooth (at least Lipschitz continuous) function $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ and the bounded region Ω is such that $\varphi > 0$. The function φ is called the level set function and has the following properties:

$$\begin{aligned} \varphi(x,t) &> 0, && \text{for } x \in \Omega(t), \\ \varphi(x,t) &< 0, && \text{for } x \notin \overline{\Omega}(t), \\ \varphi(x,t) &= 0, && \text{for } x \in \partial\Omega = \Gamma(t). \end{aligned}$$

Thus, the interface $\Gamma(t)$ is captured for all later time by solving the level set equation

$$\frac{\partial \varphi}{\partial t} + v_N |\nabla \varphi| = 0, \quad \text{where} \quad v_N = \mathbf{v} \cdot \frac{\nabla \varphi}{|\nabla \varphi|}, \tag{2.6}$$

and locating the set where φ vanishes, for a detailed discussion see [4, 5, 6]. Here $|\nabla\varphi|$ is the euclidian norm of φ. We remark that, the numerical approximation of a level set equation (2.6) should be done following the ideas of [5], where the finite volume method for general Hamilton-Jacobi equations is developed.

2.4. **Some technical results.** In this subsection we consider two results which are needed in the proof of the convergence.

Lemma 2.1. *Consider $g : \mathbb{R} \to \mathbb{R}$ a monotonic lipschitz continuous function with Lipschitz constant $G > 0$. Then the following inequality holds*

$$\left| \int_c^d \Big(g(x) - g(c) \Big) dx \right| \geq \frac{1}{2G} (g(d) - g(c))^2 \tag{2.7}$$

for all $c, d \in \mathbb{R}$.

We refer to [8] for the proof of Lemma 2.1.

Proposition 2.2. *Consider $\Phi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Phi(s) = \int_A^s \tau \frac{d}{d\tau} F(\tau,\tau) d\tau,$$

for some given $A \in \mathbb{R}$ and $F \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then

$$\Phi(b) - \Phi(a) = b\Big[F(b,b) - F(a,b)\Big] - a\Big[F(a,a) - F(a,b)\Big] - \int_a^b \Big[F(\tau,\tau) - F(a,b)\Big] d\tau,$$

for all $a, b \in [A, \infty[$.

Proof. If we make an integration by parts, we see that Φ can be written as

$$\Phi(s) = sF(s,s) - A\,F(A,A) - \int_A^s F(\tau,\tau) d\tau.$$

Now, for $a, b \in [A, \infty[$, we have that

$$\begin{aligned} \Phi(b) - \Phi(a) &= \left\{ bF(b,b) - A\,F(A,A) - \int_A^b F(\tau,\tau)d\tau \right\} - \left\{ aF(a,a) - A\,F(A,A) - \int_A^a F(\tau,\tau)d\tau \right\} \\ &= bF(b,b) - aF(a,a) - \int_a^b F(\tau,\tau)d\tau, \end{aligned}$$

which implies the desired identity. $\square$

3. Motivation of the level set numerical approximation of (1.1)

In this section we consider a contextualized interpretation of the steps (i)-(iii) given at the introduction. Asumming that the flux satisfying (1.3), we consider first the case of strictly convex flux function case and the case of linear flux function.

3.1. **Strictly convex flux function.** Let us consider that u (solution of (1.1)) is represented, using three unknown functions p, v and $w : \mathbb{R}\times\mathbb{R}_0^+ \to \mathbb{R}$, as follows

$$u(x,t) \quad = \quad \begin{cases} w(x,t), & p(x,t) > 0,\\ v(x,t), & p(x,t) \leq 0,\end{cases} \tag{3.1}$$

where $p(.,t) \in C^0(\mathbb{R})$ and denotes a level set function such that $p = 0$ on the discontinuity curves of u, and w and v are artificial states such that $v(.,t)$ and $w(.,t)$ are continuous through the zeros of p. Furthermore, in the representation (3.1), we require that u must be the entropy admissible solution. Then the selected (or calculated) p, v and w are not at all arbitrary and must be done such that u satisfies the entropy condition (1.5) extended to everywhere of Q as follows:

$$f'\big(u_L(x,t)\big) > f'\big(u_R(x,t)\big), \quad (x,t) \in Q, \tag{3.2}$$

where

$$u_L(x,t) = \begin{cases} w(x,t), & p_x(x,t) > 0,\\ v(x,t), & p_x(x,t) \leq 0,\end{cases} \qquad \text{and} \qquad u_R(x,t) = \begin{cases} v(x,t), & p_x(x,t) \leq 0,\\ w(x,t), & p_x(x,t) > 0.\end{cases} \tag{3.3}$$

Note that the definition of u_L and u_R is quite natural. Indeed, let c such that $p(c,t) = 0$, then, if the profile $p(\cdot,t)$ is decreasing in a neighbourhood of c, by (3.1), we have that $u_L(x,t) = w(x,t)$ and $u_R(x,t) = v(x,t)$, whereas that, if $p(\cdot,t)$ is increasing in a neighbourhood of c we deduce that $u_L(x,t) = v(x,t)$ and $u_R(x,t) = w(x,t)$.

In the rest of the section, we consider that the entropy profile solution $u(\cdot,\tau)$ is given and we introduce systematically the formal calculus of the entropy profile solution $u(\cdot,\tau+\Delta t)$, via the level-set technique:

($\mathscr{C}_1$) Let us consider that the functions p_τ, v_τ and $w_\tau : \mathbb{R}\to\mathbb{R}$ are known and such that: $p_\tau = 0$ on the discontinuities of $u(\cdot,\tau)$; v_τ and w_τ are continuous through the points where $p_\tau = 0$; and that $u(\cdot,\tau)$ defined via (3.1) satisfies (3.2).

($\mathscr{C}_2$) For the evolution, we start by noticing that (3.1) formally implies the following equivalence

$$u_t + (f(u))_x = 0 \quad \Leftrightarrow \quad \begin{cases} w_t + (f(w))_x = 0, & p(x,t) > 0,\\ v_t + (f(v))_x = 0, & p(x,t) \leq 0.\end{cases} \tag{3.4}$$

Then, in a natural way, the artificial states $w(\cdot,\tau+\Delta t)$ and $v(\cdot,\tau+\Delta t)$ are calculated by solving everywhere $x \in \mathbb{R}$ the equations defined locally in (3.4) $\big(p_\tau(x) > 0$ or $p_\tau(x) \leq 0\big)$. In other words and more precisely $w(\cdot,\tau+\Delta t)$ and $v(\cdot,\tau+\Delta t)$ are determined by solving

$$\begin{cases} w_t + (f(w))_x &= 0, \quad \text{for } (x,t) \in \mathbb{R}\times[\tau,\tau+\Delta t],\\ w(x,\tau) &= w_\tau(x), \quad \text{for } x \in \mathbb{R},\end{cases} \tag{3.5}$$

and

$$\begin{cases} v_t + (f(v))_x &= 0, \quad \text{for } (x,t) \in \mathbb{R}\times[\tau,\tau+\Delta t]\\ v(x,\tau) &= v_\tau(x), \quad \text{for } x \in \mathbb{R}.\end{cases} \tag{3.6}$$

Now, the determination of the level set function profile $p(\cdot,\tau+\Delta t)$ is done by solving the level set equation:

$$\begin{cases} p_t + s(\cdot,\tau)p_x &= 0, \quad \text{for } (x,t) \in \mathbb{R}\times[\tau,\tau+\Delta t]\\ p(x,\tau) &= p_\tau(x), \quad \text{for } x \in \mathbb{R},\end{cases} \tag{3.7}$$

where $s(\cdot,\tau)$ is calculated by extendeding everywhere in Q the shock speed (1.4), recasted as follows:

$$s(x,t) = \frac{f\big(u_L(x,t)\big) - f\big((u_R(x,t)\big)}{u_L(x,t) - u_R(x,t)}, \text{ with } u_L \text{ and } u_R \text{ given by (3.3).} \tag{3.8}$$

($\mathscr{C}_3$) In order to obtain an entropy admissible solution profile $u(\cdot,\tau+\Delta t)$, the new profiles $v(\cdot,\tau+\Delta t)$ and $w(\cdot,\tau+\Delta t)$ are corrected by selecting the entropy admissible discontinuities using the information given by the zero level of $p(\cdot,\tau+\Delta t)$. In this way, the new level set function profile is given by $p_{\tau+\Delta t}(x)=p(x,\tau+\Delta t)$ and the new auxiliary profiles v and w at $t=\tau+\Delta t$ are defined by

$$w_{\tau+\Delta t}(x)=\begin{cases} w(x,\tau+\Delta t), & p_{\tau+\Delta t}(x)>0 \text{ or } (u_L,u_R) \text{ does not satisfies (3.2)},\\ v(x,\tau+\Delta t), & p_{\tau+\Delta t}(x)\leq 0\ (u_L,u_R) \text{ satisfies (3.2)},\end{cases} \tag{3.9}$$

and

$$v_{\tau+\Delta t}(x)=\begin{cases} v(x,\tau+\Delta t), & p_{\tau+\Delta t}(x)<0 \text{ or } (u_L,u_R) \text{ does not satisfies (3.2)},\\ w(x,\tau+\Delta t), & p_{\tau+\Delta t}(x)\geq 0 \text{ and } (u_L,u_R) \text{ satisfies (3.2)},\end{cases} \tag{3.10}$$

where $u_L=u_L(x,\tau+\Delta t)$ and $u_R=u_R(x,\tau+\Delta t)$ are calculated by (3.3).

We remark that some calculus and assumptions in ($\mathscr{C}_1$)-($\mathscr{C}_3$) are only formal, in the sense that we cannot rigourosly prove these steps. Maybe p can be discontinuos in space since the velocity s defined by (3.8) is not always a continuous function. However, some properties can be rigourosly stated, for instance the fact that the real state solution w for $p>0$ never reaches the solution of the ghost state w for $p\leq 0$, we refer to [1] for further details.

3.2. **Linear flux function.** In the case of the linear flux function, we assume that the weak solution u is represented by (3.1) for some functions p,v and w. Noticing that the entropy condition it is not required, the steps ($\mathscr{C}_1$)-($\mathscr{C}_3$) are replaced by:

($\mathscr{L}_1$) Let us consider that the functions p_τ, v_τ and $w_\tau:\mathbb{R}\to\mathbb{R}$ are known and such that: $p_\tau=0$ on the discontinuities of $u(\cdot,\tau)$; and v_τ and w_τ are continuous through the points where $p_\tau=0$.

($\mathscr{L}_2$) For the evolution, we note that (3.5), (3.6) and (3.7) are replaced by

$$\begin{cases} w_t+(aw)_x &= 0, \quad \text{for } (x,t)\in\mathbb{R}\times[\tau,\tau+\Delta t],\\ w(x,\tau) &= w_\tau(x), \quad \text{for } x\in\mathbb{R},\end{cases} \tag{3.11}$$

$$\begin{cases} v_t+(av)_x &= 0, \quad \text{for } (x,t)\in\mathbb{R}\times[\tau,\tau+\Delta t]\\ v(x,\tau) &= v_\tau(x), \quad \text{for } x\in\mathbb{R},\end{cases} \tag{3.12}$$

and

$$\begin{cases} p_t+ap_x &= 0, \quad \text{for } (x,t)\in\mathbb{R}\times[\tau,\tau+\Delta t]\\ p(x,\tau) &= p_\tau(x), \quad \text{for } x\in\mathbb{R}\end{cases} \tag{3.13}$$

respectively. Then $p_{\tau+\Delta t}(x)=p\big(x-(\tau+\Delta t)a\big)$, $v_{\tau+\Delta t}(x)=v\big(x-(\tau+\Delta t)a\big)$ and $w_{\tau+\Delta t}(x)=w\big(x-(\tau+\Delta t)a\big)$.

We note that the equations (3.12)-(3.13) are uncoupled, which implies that these equations can be separated solved.

4. The numerical method

The hybrid scheme introduced by Aslam [1], based on level set and finite volume methods, can be described as a numerical algorithm consisting of three big steps, called initialization, evolution and reconstruction steps, which will be fully detailed on subsections 4.2, 4.3 and 4.4, respectively.

4.1. **Notation.** Let us first introduce some notation. We recall the notation $\mathrm{sgn}^{\pm}$ for the applications from $\mathbb{R}$ to $\{0,1\}$ defined by $\mathrm{sgn}^+(x)=\mathbb{1}_{\mathbb{R}^+}(x)$ and $\mathrm{sgn}^-(x)=-\mathrm{sgn}^+(-x)$, where $\mathbb{1}_A:X\to\{0,1\}$ is the well known indicator function defined by $\mathbb{1}_A(x)=1$ for $x\in A$ and $\mathbb{1}_A(x)=0$ for $x\in X-A$. As usual we set the notation $a^+=\max\{a,0\}$ and $a^-=\min\{a,0\}$ and note that $a^+=a\mathrm{sgn}^+(a)$ and $a^-=(-a)^+$. Additionally, we define the indicators of the monotonic behaviour of the level set function p denoted by $\mathscr{P}_j^n$ and the entropy verification denoted by $\mathscr{E}_j^n$, as follows:

$$\mathscr{P}_j^n=\mathrm{sgn}^+\left(\frac{p_{j+1}^n-p_{j-1}^n}{2\Delta x}\right) \quad\text{and}\quad \mathscr{E}_j^n=1-\mathrm{sgn}^-\Big(f'(u_{L,j}^n)-f'(u_{R,j}^n)\Big), \quad (j,n)\in\mathbb{Z}\times\mathbb{N}. \tag{4.1}$$

Note that $\mathscr{E}_j^n = 1$ when the Lax entropy condition is satisfied and 0 otherwise.

4.2. **Initialization step: Calculus of $\mathbf{u}^0, \mathbf{p}^0, \mathbf{w}^0$ and $\mathbf{v}^0$.** In this step we introduce the discretization corresponding to the initial condition, the initial level set function and the initial admissible states for the level set representation of the initial condition. In addition, we calculate the extended initial speed of propagation for discontinuities. More precisely, we proceed as follows. First, we discretize the initial condition u_0 by applying (2.4b). Then, we consider a continuous function $p_0 : \mathbb{R} \to \mathbb{R}$ such that it vanishes over the control volumes where the function u_0 is discontinuous, and in the natural way of the finite volume methodology, we introduce the following initial level set function discretization

$$p_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} p_0(x)dx, \quad j \in \mathbb{Z}. \tag{4.2}$$

Later, we can make the calculus of entropy admissible states $\mathbf{w}^0$ and $\mathbf{v}^0$ and the initial speed $\mathbf{s}^0$ as follows

$$w_j^0 = w_j^{-1/2} + \Big(v_j^{-1/2} - w_j^{-1/2}\Big)\Big(1 - \mathrm{sgn}^+(p_j^0)\Big)\mathscr{E}_j^0, \tag{4.3}$$

$$v_j^0 = v_j^{-1/2} + \Big(w_j^{-1/2} - v_j^{-1/2}\Big)\mathrm{sgn}^+(p_j^0)\mathscr{E}_j^0 \quad \text{and} \tag{4.4}$$

$$s_j^0 = \frac{f(u_{L,j}^0) - f(u_{R,j}^0)}{u_{L,j}^0 - u_{R,j}^0}, \tag{4.5}$$

where

$$w_j^{-1/2} = u_j^0 \mathrm{sgn}^+(p_j^0) + \tilde{u}_j\Big(1 - \mathrm{sgn}^+(p_j^0)\Big), \quad v_j^{-1/2} = u_j^0\Big(1 - \mathrm{sgn}^+(p_j^0)\Big) + \tilde{u}_j \mathrm{sgn}^+(p_j^0),$$
$$u_{L,j}^0 = \mathscr{P}_j^0\, w_j^{-1/2} + (1 - \mathscr{P}_j^0)\, v_j^{-1/2} \quad \text{and} \quad u_{R,j}^0 = (1 - \mathscr{P}_j^0)\, w_j^{-1/2} + \mathscr{P}_j^0\, v_j^{-1/2}.$$

Here $\tilde{u}_j$ denotes an arbitrary discretization of a local continuous extension of u_0, in order to have that w and v are continuous across of discontinuities of u_0.

For the convergence analysis we assume the following hypothesis

$$\begin{aligned} &p_0 \in L^\infty(\mathbb{R}) \text{ with } p_0(x) \in [p_m, p_M] = [\mathrm{ess\,inf}_{\mathbb{R}}(p_0), \mathrm{ess\,sup}_{\mathbb{R}}(p_0)] \text{ for a.e. } x \in \mathbb{R}; \\ &v_0 \in L^\infty(\mathbb{R}) \text{ with } v_0(x) \in [u_m, u_M] \text{ for a.e. } x \in \mathbb{R}; \text{ and} \\ &w_0 \in L^\infty(\mathbb{R}) \text{ and } w_0(x) \in [u_m, u_M] \text{ for a.e. } x \in \mathbb{R}, \end{aligned} \tag{4.6}$$

where u_m and u_M are defined in (1.2). We note that, the hypothesis given on (4.6) are the natural regularity required by the methodology of entropy process solutions [8].

4.3. **Evolution step: Calculus of $\mathbf{p}^n, \mathbf{w}^n$ and $\mathbf{v}^n$ for $n = 1, \ldots, N$.** Let us consider that for some arbitrary $n \in \{0, \ldots, N-1\}$, the states $\mathbf{u}^n, \mathbf{p}^n, \mathbf{w}^n, \mathbf{v}^n$ and the speed $\mathbf{s}^n$, are given. We calculate the new states $\mathbf{u}^{n+1}, \mathbf{p}^{n+1}, \mathbf{w}^{n+1}, \mathbf{v}^{n+1}$ and the new speed $\mathbf{v}^{n+1}$ via four sequential iterative steps. First, we calculate the intermediate states $\mathbf{w}^{n+1/2}$ and $\mathbf{v}^{n+1/2}$ by applying a finite volume scheme for conservation laws. Subsequently, by a finite volume scheme for Hamilton-Jacobi equations we obtain $\mathbf{p}^{n+1}$. Afterwards, we update the speed of discontinuities propagation using a discrete version of Rankine-Hugoniot jump condition. Then, we characterize the entropy satisfying discontinuities in order to define $\mathbf{w}^{n+1}$ and $\mathbf{v}^{n+1}$ and update n before to apply again the sequence of the four evolution stages. More precisely, in this paper, we apply iteratively on $n \in \{0, \ldots, N-1\}$ the following calculations:

(E.i) *Intermediate states.* The intermediate states $\mathbf{w}^{n+1/2}$ and $\mathbf{v}^{n+1/2}$ are calculated by applying a monotone scheme with numerical flux g (see section 2.2):

$$w_j^{n+1/2} = w_j^n - \lambda\Big(g(w_j^n, w_{j+1}^n) - g(w_{j-1}^n, w_j^n)\Big) \quad \text{and} \tag{4.7}$$

$$v_j^{n+1/2} = v_j^n - \lambda\Big(g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n)\Big). \tag{4.8}$$

(E.ii) *Level set evolution.* The level set equation state $\mathbf{p}^{n+1}$ is calculated by applying a generalized upwind scheme:

$$p_j^{n+1} = p_j^n - \lambda(s_j^n)^+(p_j^n - p_{j-1}^n) - \lambda(s_j^n)^-(p_{j+1}^n - p_j^n). \tag{4.9}$$

(E.iii) *Extended shock speed.* Using the notation $\mathscr{P}_j^n$ defined on (4.1), we introduce the discrete left-right states and the extended discrete shock speed as follows:

$$u_{L,j}^{n+1} = \mathscr{P}_j^{n+1}\, w_j^{n+1/2} + (1 - \mathscr{P}_j^{n+1})\, v_j^{n+1/2}, \tag{4.10}$$

$$u_{R,j}^{n+1} = (1 - \mathscr{P}_j^{n+1})\, w_j^{n+1/2} + \mathscr{P}_j^{n+1}\, v_j^{n+1/2} \quad \text{and} \tag{4.11}$$

$$s_j^{n+1} = \frac{f(u_{L,j}^{n+1}) - f(u_{R,j}^{n+1})}{u_{L,j}^{n+1} - u_{R,j}^{n+1}}. \tag{4.12}$$

(E.iv) *New artificial entropy satisfying states.* Using the indicator $\mathscr{E}_j^n$ defined on (4.1), we introduce the states $\mathbf{w}^{n+1}$ and $\mathbf{v}^{n+1}$ such that $\mathbf{u}^{n+1}$ is consistent with the entropy condition:

$$w_j^{n+1} = w_j^{n+1/2} + \Big(v_j^{n+1/2} - w_j^{n+1/2}\Big)\Big(1 - \mathrm{sgn}^+(p_j^n)\Big)\mathscr{E}_j^n \quad \text{and} \tag{4.13}$$

$$v_j^{n+1} = v_j^{n+1/2} + \Big(w_j^{n+1/2} - v_j^{n+1/2}\Big)\mathrm{sgn}^+(p_j^n)\mathscr{E}_j^n. \tag{4.14}$$

4.4. **Reconstruction step: Calculus of $\mathbf{u}^{n+1}$.** In this we apply the definition of level set representation to reconstruct $\mathbf{u}^{n+1}$ from $\mathbf{p}^{n+1}, \mathbf{w}^{n+1}$ and $\mathbf{v}^{n+1}$, i.e.

$$u_j^{n+1} = \mathrm{sgn}^+(p_j^{n+1})\, w_j^{n+1} + \Big(1 - \mathrm{sgn}^+(p_j^{n+1})\Big)\, v_j^{n+1}. \tag{4.15}$$

5. Convergence analysis for the linear flux function

In this section, we study the convergence analysis of the numerical scheme described on section 4, in the case of a linear flux function. Furthermore, without loss of generality, we assume that the velocity is positive. To be precise we assume that

$$f(u) = au, \qquad a > 0. \tag{5.1}$$

The assumption (5.1) implies that the evolution and reconstruction steps can be rewrited in the following short form:

$$w_j^{n+1} = w_j^n - \lambda a(w_j^n - w_{j-1}^n), \tag{5.2a}$$

$$v_j^{n+1} = v_j^n - \lambda a(v_j^n - v_{j-1}^n), \tag{5.2b}$$

$$p_j^{n+1} = p_j^n - \lambda a(p_j^n - p_{j-1}^n), \tag{5.2c}$$

$$u_j^{n+1} = \mathrm{sgn}^+(p_j^{n+1})\, w_j^{n+1} + \Big(1 - \mathrm{sgn}^+(p_j^{n+1})\Big)\, v_j^{n+1}. \tag{5.2d}$$

Here, we note that the schemes are uncoupled and can be separated solved.

Lemma 5.1. *Consider f given by* (5.1)*, u_0 with the regularity required by the hypothesis* (1.2)*; p_0, v_0 and w_0 satisfying* (4.6) *and the requirements specified by the initialization step (see equations* (4.2)-(4.5)*); and λ satisfying the following CFL condition*

$$\lambda a < 1 - \xi, \qquad \text{with} \quad \xi \in]0, 1[. \tag{5.3}$$

Then w_j^n, v_j^n, p_j^n and u_j^n obtained by the scheme (5.2) *satisfies the following properties:*

(a) *(L^∞-estimates) The functions $w_\Delta, v_\Delta, p_\Delta$ and u_Δ defined from $\mathbb{R} \times \mathbb{R}_0^+$ to $\mathbb{R}$ by*

$$w_\Delta(x,t) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} w_j^n \mathbb{1}_{Q_j^n}(x,t), \qquad v_\Delta(x,t) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} v_j^n \mathbb{1}_{Q_j^n}(x,t),$$

$$p_\Delta(x,t) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} p_j^n \mathbb{1}_{Q_j^n}(x,t) \quad \text{and} \quad u_\Delta(x,t) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} u_j^n \mathbb{1}_{Q_j^n}(x,t), \tag{5.4}$$

where $Q_j^n := [x_{j-1/2}, x_{j+1/2}[\times[t_n, t_{n+1}[$, *belong to* $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$. *Furthermore,* $w_\Delta, v_\Delta, u_\Delta \in [u_m, u_M]$ *and* $p_\Delta \in [p_m, p_M]$ *in the almost everywhere sense.*

(b) *(Weak BV-estimates) Let* $i_0, i_1 \in \mathbb{Z}$ *and* $R \in \mathbb{R}^+$ *such that* $-R \in]x_{i_0-1/2}, x_{i_0+1/2}[$ *and* $R \in]x_{i_1-1/2}, x_{i_1+1/2}[$. *Then, there exists* $C \in \mathbb{R}^+$ *only depending on* R, T, u_0 *and* ξ *such that*

$$\sqrt{\Delta x}\sum_{j=i_0}^{i_1}\sum_{n=0}^{N} a\Delta t|w_j^n - w_{j-1}^n| \le C. \tag{5.5}$$

A similar estimate is valid for v_j^n *and* p_j^n.

Proof. To prove the lemma we apply the standard arguments detailed in [8]. □

Theorem 5.1. *Consider the hypothesis of Lemma 5.1 and* u_j^n *for* $(n,j)\in\mathbb{N}\times\mathbb{Z}$ *obtained by the scheme* (5.2). *Then, the approximate solution* u_Δ *defined by* (5.4) *converges to the weak solution* u *of* (1.1) *in* $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ *for the weak-*$\star$ *topology when* $\Delta x\to 0$.

Proof. By Lemma 5.1-(a) and Proposition 2.1, we deduce that $w_\Delta \to w, v_\Delta \to v, p_\Delta \to p$ and $u_\Delta \to u$ in $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ for the weak-$\star$ topology, when $\Delta x \to 0$.

Let us consider $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+,\mathbb{R})$ and we select the indexes $J_{\min}, J_{\max}, N_{\max}\in\mathbb{N}$ such that $\text{supp}(\varphi)\subset[x_{J_{\min}}, x_{J_{\max}}]\times[0, t_{N_{\max}}[$. We multiply (5.2a) by $\text{sgn}^+(p_j^{n+1})\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx$ and summing over $j\in\mathbb{Z}$ and $n\in\mathbb{N}$ we get

$$A + B = 0, \tag{5.6}$$

where

$$\begin{aligned} A &= \sum_{j=J_{\min}}^{J_{\max}}\sum_{n=0}^{N_{\max}-1}\Big(w_j^{n+1}-w_j^n\Big)\text{sgn}^+(p_j^{n+1})\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx \quad \text{and}\\ B &= \sum_{j=J_{\min}}^{J_{\max}}\sum_{n=0}^{N_{\max}-1}\lambda a\Big(w_j^n - w_{j-1}^n\Big)\text{sgn}^+(p_j^{n+1})\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx. \end{aligned}$$

We analyse separately the convergence of A and B. In the case of A, by summing by parts we deduce that A can be rewrite as follows

$$\begin{aligned} A = &\sum_{j=J_{\min}}^{J_{\max}}\sum_{n=0}^{N_{\max}-1} w_j^n\left\{\text{sgn}^+(p_j^n)\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_n)dx - \text{sgn}^+(p_j^{n+1})\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx\right\}\\ &+\sum_{j=J_{\min}}^{J_{\max}}\left\{w_j^{N_{\max}}\text{sgn}^+(p_j^{N_{\max}})\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{N_{\max}})dx - w_j^0\text{sgn}^+(p_j^0)\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,0)dx\right\}\\ = &-\sum_{j=J_{\min}}^{J_{\max}}\sum_{n=0}^{N_{\max}-1} w_j^n\text{sgn}^+(p_j^{n+1})\int_{x_{j-1/2}}^{x_{j+1/2}}\Big\{\varphi(x,t_{n+1})-\varphi(x,t_n)\Big\}dx\\ &+\sum_{j=J_{\min}}^{J_{\max}}\sum_{n=0}^{N_{\max}-1} w_j^n\Big(\text{sgn}^+(p_j^{n+1})-\text{sgn}^+(p_j^n)\Big)\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_n)dx\\ &-\sum_{j=i_0}^{i_1} w_j^0\text{sgn}^+(p_j^0)\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,0)dx\\ := &\, A^1 + A^2 + A^3, \end{aligned}$$

which implies that

$$A \to -\int_{\mathbb{R}}\int_{\mathbb{R}^+} w(x,t)\text{sgn}^+(p(x,t))\varphi_t(x,t)dxdt - \int_{\mathbb{R}} w_0(x)\text{sgn}^+(p_0(x))\varphi(x,0)dxdt, \tag{5.7}$$

when $\Delta x \to 0$, since

$$\begin{aligned}
&A^1 \to -\int_{\mathbb{R}}\int_{\mathbb{R}^+} w(x,t)\mathrm{sgn}^+(p(x,t))\varphi_t(x,t)dxdt,\\
&A^2 \to 0 \quad \text{and}\\
&A^3 \to -\int_{\mathbb{R}} w(x,0)\mathrm{sgn}^+(p(x,0))\varphi(x,0)dxdt.
\end{aligned}$$

For the term B, by applying the rectangle rule and summation by parts, we can make the following calculus

$$\begin{aligned}
&B + \int_{\mathbb{R}}\int_{\mathbb{R}^+} aw(x,t)\mathrm{sgn}^+(p(x,t))\varphi_x(x,t)dxdt\\
&= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} \lambda a\Big(w_j^n - w_{j-1}^n\Big)\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x,t_{n+1})dx\\
&\quad + \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} aw_j^n \mathrm{sgn}^+(p_j^n)\Big(\varphi_{j+1/2}^n - \varphi_{j-1/2}^n\Big)\Delta t\\
&= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} \lambda a\Big(w_j^n - w_{j-1}^n\Big)\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x,t_{n+1})dx\\
&\quad + \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} a\Delta t\Big(w_{j-1}^n - w_j^n\Big)\mathrm{sgn}^+(p_j^n)\varphi_{j-1/2}^n\\
&\quad + \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} a\Delta t w_{j-1}^n\Big(\mathrm{sgn}^+(p_{j-1}^n) - \mathrm{sgn}^+(p_j^n)\Big)\varphi_{j-1/2}^n\\
&\quad + \sum_{n=0}^{N_{\max}-1} \Big\{ w_{J_{\max}}^n \mathrm{sgn}^+(p_{J_{\max}}^n)\varphi_{J_{\max}+1/2}^n - w_{J_{\min}-1}^n \mathrm{sgn}^+(p_{J_{\min}-1}^n)\varphi_{J_{\min}-1/2}^n \Big\}\\
&= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} \lambda a\Big(w_j^n - w_{j-1}^n\Big)\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \Big(\varphi(x,t_{n+1}) - \varphi_{j-1/2}^n\Big)dx\\
&\quad + \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}-1} a\Delta t w_{j-1}^n\Big(\mathrm{sgn}^+(p_{j-1}^n) - \mathrm{sgn}^+(p_j^n)\Big)\varphi_{j-1/2}^n\\
&:= B_1 + B_2.
\end{aligned}$$

Now, since $B_1 \to 0$ when $\Delta x \to 0$ by weak BV-estimate for w_Δ and $B_2 \to 0$ when $\Delta x \to 0$ by the L^∞-estimate for p_Δ (see Lemma 5.1), we deduce that

$$B \to -\int_{\mathbb{R}}\int_{\mathbb{R}^+} aw(x,t)\mathrm{sgn}^+(p(x,t))\varphi_x(x,t)dxdt, \qquad \text{when } \Delta x \to 0. \tag{5.8}$$

Replacing (5.7)-(5.8) in (5.6) we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} w\ \mathrm{sgn}^+(p)\Big(\varphi_t + a\varphi_x\Big)dxdt + \int_{\mathbb{R}} w_0\ \mathrm{sgn}^+(p_0)\varphi(\cdot,0)dx = 0. \tag{5.9}$$

On the other hand, proceeding in a similar way with (5.2b) we obtain that the limit v of v_Δ satisfies

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} v\Big(1-\mathrm{sgn}^+(p)\Big)\Big(\varphi_t + a\varphi_x\Big)dxdt + \int_{\mathbb{R}} v_0\Big(1-\mathrm{sgn}^+(p_0)\Big)\varphi(\cdot,0)dx = 0. \tag{5.10}$$

Summing (5.9) and (5.10) we deduce that u is a weak solution of (1.1) with f defined by (5.1), which concludes the proof of the theorem. □

6. Convergence analysis for the non-linear convex flux function

6.1. L^∞ estimate.

Lemma 6.1. *Consider the assumptions*

- $(\mathscr{A}_1)$ *f satisfies the hypothesis of strictly convexity given on* (1.3)*;*
- $(\mathscr{A}_2)$ *u_0 satisfies the hypothesis* (1.2)*;*
- $(\mathscr{A}_3)$ *p_0, v_0 and w_0 satisfies* (4.6) *and the requirements specified by the initialization step (see equations* (4.2)-(4.5)*);*
- $(\mathscr{A}_4)$ *g is a monotone flux;*
- $(\mathscr{A}_5)$ *The functions $w_\Delta, v_\Delta, p_\Delta$ and u_Δ defined from $\mathbb{R}\times\mathbb{R}_0^+$ are determined by the correspondence rules given at* (5.4) *with u_j^n, p_j^n, v_j^n and w_j^n obtained by* (4.15), (4.9), (4.14) *and* (4.13)*, respectively; and*
- $(\mathscr{A}_6)$ *λ satisfies the following CFL condition*

$$\lambda\|f'\|_{L^\infty([u_m,u_M])} < 1-\xi, \qquad \text{with } \xi \in]0,1[. \tag{6.1}$$

Then $u_\Delta, p_\Delta, v_\Delta$, and w_Δ belong to $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$. Furthermore, $w_\Delta, v_\Delta, u_\Delta \in [u_m, u_M]$ and $p_\Delta \in [p_m, p_M]$ in the almost everywhere sense.

Proof. The proof of the required properties for p_Δ, v_Δ and w_Δ is a straightforward application of the standard arguments, see [8]. Then, we omit the details and only prove the result for u_Δ.

We proceed by induction on n. For $n=0$ the assertion is valid by definition of the discretization. Before giving the proof for $n\geq 1$, we comment two points. Firstly, we notice that the identity

$$\begin{aligned}
u_j^{n+1} &= r_{w,j}^n\, w_j^{n+1/2} + r_{v,j}^n\, v_j^{n+1/2}, \quad \text{where}\\
r_{w,j}^n &= \mathrm{sgn}^+(p_j^{n+1})\Big[1-\Big(1-\mathrm{sgn}^+(p_j^n)\Big)\mathscr{E}_j^n\Big] + \Big(1-\mathrm{sgn}^+(p_j^{n+1})\Big)\mathrm{sgn}^+(p_j^n)\mathscr{E}_j^n\\
r_{v,j}^n &= \mathrm{sgn}^+(p_j^{n+1})\Big(1-\mathrm{sgn}^+(p_j^n)\Big)\mathscr{E}_j^n + \Big(1-\mathrm{sgn}^+(p_j^{n+1})\Big)\Big(1-\mathrm{sgn}^+(p_j^n)\mathscr{E}_j^n\Big)
\end{aligned} \tag{6.2}$$

is satisfied. In the second place, we have that

$$\begin{aligned}
w_j^{n+1/2} &= \lambda\mathbb{A}(w_{j-1}^n, w_j^n)w_{j-1}^n\\
&\quad +(1-\lambda\mathbb{A}(w_{j-1}^n, w_j^n)+\lambda\mathbb{B}(w_j^n, w_{j+1}^n))w_{j-1}^n - \lambda\mathbb{B}(w_j^n, w_{j+1}^n)w_j^n \quad \text{and}\\
v_j^{n+1/2} &= \lambda\mathbb{A}(v_{j-1}^n, v_j^n)v_{j-1}^n\\
&\quad +\Big[1-\lambda\mathbb{A}(v_{j-1}^n, v_j^n)+\lambda\mathbb{B}(v_j^n, v_{j+1}^n)\Big]v_{j-1}^n + \lambda\mathbb{B}(v_j^n, v_{j+1}^n)v_j^n,
\end{aligned} \tag{6.3}$$

where $\mathbb{A}$ and $\mathbb{B}$ denote the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined as follows

$$\begin{aligned}
\mathbb{A}(x_1,x_2) &= \begin{cases} \dfrac{g(x_1,x_2)-f(x_2)}{x_1-x_2}, & x_1\neq x_2,\\ 0, & \text{otherwise,}\end{cases} \qquad \text{and}\\
\mathbb{B}(x_1,x_2) &= \begin{cases} \dfrac{g(x_1,x_2)-f(x_1)}{x_2-x_1}, & x_1\neq x_2,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Thus, for $n\geq 1$ we assume that $w_j^n, v_j^n \in [u_m,u_M]$ and by the monotonicity of g, the CFL condition (6.1) and (6.3), we have that $w_j^{n+1/2}, v_j^{n+1/2} \in [u_m,u_M]$. Then by (6.2), noticing that $r_{w,j}^n, r_{v,j}^n \in \{0,1\}$ and $r_{w,j}^n + r_{v,j}^n = 1$ (see Appendix A) we can deduce the following estimates

$$u_m = (r_{w,j}^n + r_{v,j}^n)u_m \leq (r_{w,j}^n w_j^{n+1/2} + r_{v,j}^n v_j^{n+1/2}) \leq (r_{w,j}^n + r_{v,j}^n)u_M = u_M,$$

which concludes the proof of the lemma. □

6.2. **Weak BV estimate.**

Lemma 6.2. *Consider the hypothesis of lemma 6.1. Then for $i_0, i_1 \in \mathbb{Z}$ and $N_1 \in \mathbb{N}$ there exists $C > 0$, independent of Δx and Δt, such that w_Δ satisfies the inequality*

$$\sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Delta t \Big[\max_{(a,b)\in \mathscr{W}_j^n} \Big(g(a,b) - f(a) \Big) + \max_{(a,b)\in \mathscr{W}_j^n} \Big(g(a,b) - f(b) \Big) \Big] \leq \frac{C}{\Delta x}, \tag{6.4}$$

where $\mathscr{W}_j^n = [w_{j-1}^n \bot w_j^n, w_{j-1}^n \top w_j^n]$. Furthermore, the inequality (6.4) is valid for v_Δ.

Proof. Multiplying (4.7) by $\Delta x w_j^n$ and summing over $j \in \{i_0, \ldots, i_1\}$ and $n \in \{0, \ldots, N_1\}$, we deduce that

$$\tilde{A} + \tilde{B} = 0, \tag{6.5}$$

where

$$\tilde{A} = \Delta x \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big(w_j^{n+1/2} - w_j^n \Big) w_j^n \quad \text{and} \quad \tilde{B} = \Delta t \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big(g(w_j^n, w_{j+1}^n) - g(w_{j-1}^n, w_j^n) \Big) w_j^n.$$

Now, the proof is reduced to get a lower bounds for $\tilde{A}$ and $\tilde{B}$, since it implies that exists $C > 0$, independent of Δx and Δt, satisfying

$$\mathbb{L} := \Delta t \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big[\Big(g(w_{j-1}^n, w_j^n) - f(w_{j-1}^n) \Big)^2 + \Big(g(w_{j-1}^n, w_j^n - f(w_j^n) \Big)^2 \Big] \leq C. \tag{6.6}$$

Then, by the Cauchy-Schwarz inequality applied to the left side of (6.4) and using (6.6), we get the desired estimate (6.4).

Bound for $\tilde{A}$. We consider the algebraic identity: $2(x-y)y = -(x-y)^2 + x^2 - y^2$ for all $x, y \in \mathbb{R}$, to rewrite $\tilde{A}$ and from the finite volume scheme (4.7) together with the CFL condition (6.1), we see that

$$\begin{aligned}
\tilde{A} &= -\frac{\Delta x}{2} \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big(w_j^{n+1/2} - w_j^n \Big)^2 + \frac{\Delta x}{2} \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big[\Big(w_j^{n+1/2} \Big)^2 - \Big(w_j^n \Big)^2 \Big] \\
&\geq -\frac{1-\xi}{4M} \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Delta t \Big(g(w_j^n, w_{j+1}^n) - g(w_{j-1}^n, w_j^n) \Big)^2 + \frac{\Delta x}{2} \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big[\Big(w_j^{n+1/2} \Big)^2 - \Big(w_j^n \Big)^2 \Big] \\
&:= -\frac{1-\xi}{4M} \; \tilde{A}_2 + \; \tilde{A}_1.
\end{aligned} \tag{6.7}$$

Additionally, the inequality $(x+y)^2 \leq 2(x^2 + y^2)$ leads to

$$\begin{aligned}
\tilde{A}_2 &\leq 2\Delta t \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big[\Big(g(w_j^n, w_{j+1}^n) - f(w_j^n) \Big)^2 + \Big(g(w_{j-1}^n, w_j^n) - f(w_j^n) \Big)^2 \Big] \\
&= 2\Delta t \sum_{j=i_0}^{i_1} \sum_{n=0}^{N_1} \Big[\Big(g(w_{j-1}^n, w_j^n) - f(w_{j-1}^n) \Big)^2 + \Big(g(w_{j-1}^n, w_j^n) - f(w_j^n) \Big)^2 \Big] \\
&\quad + 2\Delta t \sum_{n=0}^{N_1} \Big[\Big(g(w_{i_1}^n, w_{i_1+1}^n) - f(w_{i_1+1}^n) \Big)^2 + \Big(g(w_{i_0-1}^n, w_{i_0}^n) - f(w_{i_0-1}^n) \Big)^2 \Big] \\
&:= 2\mathbb{L} + \tilde{A}_3.
\end{aligned} \tag{6.8}$$

Then, from (6.7) and (6.8), we get the following bound for $\tilde{A}$

$$\tilde{A} \geq -\frac{1-\xi}{2M} \mathbb{L} - \frac{1-\xi}{2M} \tilde{A}_3 + \tilde{A}_1. \tag{6.9}$$

Bound for $\tilde{B}$. Summing by parts and considering the identity of Proposition 2.2 with g instead of F, we deduce that

$$\begin{aligned}
\tilde{B} &= \Delta t \sum_{j=i_0}^{i_1}\sum_{n=0}^{N_1}\Big(\Big[g(w_{j-1}^n,w_j^n)-g(w_{j-1}^n,w_{j-1}^n)\Big]w_{j-1}^n-\Big[g(w_{j-1}^n,w_j^n)-g(w_j^n,w_j^n)\Big]w_j^n\Big)\\
&\quad+\Delta t \sum_{j=i_0}^{i_1}\sum_{n=0}^{N_1}\Big(\Big[g(w_{i_1}^n,w_{i_1+1}^n)-g(w_{i_1}^n,w_{i_1}^n)\Big]w_{i_1}^n-\Big[g(w_{i_0-1}^n,w_{i_0}^n)-g(w_{i_0}^n,w_{i_0}^n)\Big]w_{i_0}^n\Big)\\
&= \Delta t \sum_{j=i_0}^{i_1}\sum_{n=0}^{N_1}\Big(\Phi(w_j^n)-\Phi(w_{j-1}^n)+\int_{w_{j-1}^n}^{w_j^n}\Big[g(\tau,\tau)-g(w_{j-1}^n,w_j^n)\Big]d\tau\Big)+\tilde{B}_1\\
&= \Delta t \sum_{j=i_0}^{i_1}\sum_{n=0}^{N_1}\int_{w_{j-1}^n}^{w_j^n}\Big[g(\tau,\tau)-g(w_{j-1}^n,w_j^n)\Big]d\tau+\Delta t\sum_{n=0}^{N_1}\Big[\Phi(w_{i_1}^n)-\Phi(w_{i_0-1}^n)\Big]+\tilde{B}_1\\
&:= \tilde{B}_3+\tilde{B}_2+\tilde{B}_1.
\end{aligned}$$

Here we note that, the application of Lemma 2.1 and the CFL condition (6.1) implies the inequality $\tilde{B}_3 \geq \mathbb{L}/2M$. Then $\tilde{B}$ satisfies

$$\tilde{B} \geq \frac{1}{2M}\mathbb{L}+\tilde{B}_2+\tilde{B}_1. \tag{6.10}$$

Proof of (6.6)*.* From (6.5), (6.9) and (6.10), and the bounds

$$\begin{aligned}
&-\tilde{A}_1 \leq \frac{N}{2}\max\{(w_m)^2,(w_M)^2\}:=C_1, \quad \tilde{A}_3\leq 2Mt_{N_1}\max\{|w_{i_0-1}^n|,|w_{i_0}^n|,|w_{i_1}^n|,|w_{i_1+1}^n|\}:=C_2\\
&-\tilde{B}_2 \leq := C_3, \quad -\tilde{B}_3 \leq := C_4,
\end{aligned}$$

we have that

$$\mathbb{L}\leq \frac{2M}{\xi}\left(\frac{1-\xi}{2M}\tilde{A}_3-\tilde{A}_1-\tilde{B}_1-\tilde{B}_2\right)\leq \frac{2M}{\xi}\left(\frac{1-\xi}{2M}C_1+C_2+C_3+C_4\right):=C,$$

which proves (6.6). The proof of the Lemma is completed. □

6.3. **Cell entropy inequality.**

Lemma 6.3. *Consider the hypothesis of Lemma 6.1 and the notation* $a\top b:=\max\{a,b\}$ *and* $a\bot b:=\min\{a,b\}$ *for* $a,b\in\mathbb{R}$*. Then,* w_Δ *satisfies the following local o cell entropy inequality*

$$\eta_k(w_j^{n+1/2})-\eta_k(w_j^n)+\lambda\Big(G_k(w_j^n,w_{j+1}^n)-G_k(w_{j-1}^n,w_j^n)\Big)\leq 0,\quad \forall (j,n)\in\mathbb{Z}\times\mathbb{N},\ \forall k\in\mathbb{R}, \tag{6.11}$$

where $\eta_k:\mathbb{R}\to\mathbb{R}$ *and* $G_k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are defined by*

$$\eta_k(x)=|x-k| \text{ and } G_k(x,y)=g(x\top k,y\top k)-g(x\bot k,y\bot k).$$

Furthermore, a similar inequality to (6.11) *is valid for* v_Δ*.*

Proof. From (4.7) and the monotonic behaviour of g, we deduce that

$$\begin{aligned}
&w_j^n\top k-w_j^n\top k+\lambda\Big(g(w_j^n\top k,w_{j+1}^n\top k)-g(w_{j-1}^n\top k,w_j^n\top k)\Big)\geq w_j^{n+1/2}\top k \quad\text{and}\\
&w_j^n\bot k-w_j^n\bot k+\lambda\Big(g(w_j^n\bot k,w_{j+1}^n\bot k)-g(w_{j-1}^n\bot k,w_j^n\bot k)\Big)\leq w_j^{n+1/2}\bot k.
\end{aligned}$$

Then, using the algebraic identity: $a\top b-a\bot b:=|a-b|$ for all $a,b\in\mathbb{R}$, we deduce that (6.11) is satisfied. □

6.4. **Convergence towards the entropy process solution.**

Lemma 6.4. *Consider the hypothesis of lemma 6.1. Then,*

(a) *There exists $p \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+)$ such that p_Δ converges towards p in $L^\infty(\mathbb{R} \times \mathbb{R}_0^+)$ for the weak-$\star$ topology when $\Delta x \to 0$ and p is the weak solution of the level-set equation.*

(b) *There exists $\mu_w \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+ \times]0,1[)$ such that w_Δ converges towards μ_w in the nonlinear weak-$\star$ sense when $\Delta x \to 0$ and μ_w satisfies the following inequality*

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}_0^+}\int_0^1 \Big\{&\eta_k\Big(\mu_w(x,t,\alpha)\Big)\mathrm{sgn}^+\Big(p(x,t)\Big)\varphi_t(x,t) + \\ &\Big(f(\mu_w(x,t,\alpha)\top k) - f(\mu_w(x,t,\alpha)\bot k)\Big)\mathrm{sgn}^+\Big(p(x,t)\Big)\varphi_x(x,t)\Big\}dxdt \\ &+ \int_{\mathbb{R}} |w_0(x) - k|\mathrm{sgn}^+\Big(p_0(x)\Big)\varphi_t(x,0)dx \geq 0, \\ &\forall k \in [u_m, u_M], \quad \forall \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+).\end{aligned} \tag{6.12}$$

(c) *There exists $\mu_v \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+ \times]0,1[)$ such that v_Δ converges towards μ_v in the nonlinear weak-$\star$ sense when $\Delta x \to 0$ and μ_v satisfies the following inequality*

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}_0^+}\int_0^1 \Big\{&\eta_k\Big(\mu_v(x,t,\alpha)\Big)\Big(1-\mathrm{sgn}^+\Big(p(x,t)\Big)\Big)\varphi_t(x,t) + \\ &\Big(f(\mu_v(x,t,\alpha)\top k) - f(\mu_v(x,t,\alpha)\bot k)\Big)\Big(1-\mathrm{sgn}^+\Big(p(x,t)\Big)\Big)\varphi_x(x,t)\Big\}dxdt \\ &+ \int_{\mathbb{R}} |v_0(x) - k|\Big(1-\mathrm{sgn}^+ p_0(x)\Big)\varphi_t(x,0)dx \geq 0, \\ &\forall k \in [u_m, u_M], \quad \forall \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+).\end{aligned} \tag{6.13}$$

(d) *There exists $\mu \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+ \times]0,1[)$ such that u_Δ converges towards μ in the nonlinear weak-$\star$ sense when $\Delta x \to 0$ and μ is the entropy process solution of* (1.1).

Proof. For (a), by Lemma 6.1 and Proposition 2.1, we note that p_Δ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$, then there exists $p \in L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ such that p_Δ converges towards p in $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ for the weak-$\star$ topology when $\Delta x \to 0$. Now, we prove that p is a weak solution of the level-set equation by standard arguments as in [8].

For (b), by Lemma 6.1 and Proposition 2.1, we note that w_Δ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$. This fact implies the existence of $\mu_w \in L^\infty(\mathbb{R} \times \mathbb{R}_0^+ \times]0,1[)$ such that $w_\Delta \to \mu_w$ in the nonlinear weak-$\star$ sense when $\Delta x \to 0$. Now, we prove that μ_w satisfies (6.12). First we rewrite (6.11) as follows

$$\eta_k(w_j^{n+1}) - \eta_k(w_j^n) + \lambda\Big(G_k(w_j^n, w_{j+1}^n) - G_k(w_{j-1}^n, w_j^n)\Big) \leq \eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2}). \tag{6.14}$$

Then, we consider $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$, such that $\mathrm{supp}(\varphi) \subset [x_{J_{\min}}, x_{J_{\max}}] \times [0, t_{N_{\max}}[$ for some $J_{\min}, J_{\max} \in \mathbb{Z}$ and $N_{\max} \in \mathbb{N}$. Now, if we multiply (6.14) by $\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1})dx$ and summing over $(j,n) \in \mathbb{Z}\times\mathbb{N}$, we get

$$A + B \leq C, \tag{6.15}$$

where

$$\begin{aligned} A &= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[\eta_k(w_j^{n+1}) - \eta_k(w_j^n)\Big]\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1})dx, \\ B &= \lambda \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[G_k(w_j^n, w_{j+1}^n) - G_k(w_{j-1}^n, w_j^n)\Big]\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1})dx \quad \text{and} \end{aligned}$$

$$C = \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[\eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2})\Big] \mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx.$$

Next, we analyse the limit of each term in (6.15) and obtain the desired inequality.

Analysis of A. By, a summation by parts, we have that A can be rewrited equivalently as follows

$$\begin{aligned}
A &= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[\eta_k(w_j^{n+1}) - \eta_k(w_j^{n})\Big] \mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx \\
&= -\sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \eta_k(w_j^{n}) \mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \Big[\varphi(x, t_{n+1}) - \varphi(x, t_n)\Big] dx \\
&\quad - \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \eta_k(w_j^{n}) \Big[\mathrm{sgn}^+(p_j^{n+1}) - \mathrm{sgn}^+(p_j^{n})\Big] \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_n) dx \\
&\quad + \sum_{j=J_{\min}}^{J_{\max}} \Big[\eta_k(w_j^{N_{\max}-1}) \mathrm{sgn}^+(p_j^{N_{\max}}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{N_{\max}}) dx \\
&\qquad\qquad - \eta_k(w_j^{0}) \mathrm{sgn}^+(p_j^{0}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, 0) dx\Big] \\
&= -\int_{\mathbb{R}} \int_{\mathbb{R}_0^+} \eta_k(w_\Delta(x,t)) \mathrm{sgn}^+(p(x,t)) \varphi_t(x,t) dx dt + A_\Delta^{Ext} \\
&\quad - \int_{\mathbb{R}} \eta_k(w_\Delta(x,0)) \mathrm{sgn}^+(p(x,0)) \varphi_t(x,0) dx.
\end{aligned}$$

Letting $\Delta x \to 0$, we obtain that

$$\begin{aligned}
A \to &-\int_{\mathbb{R}} \int_{\mathbb{R}_0^+} \int_0^1 \eta_k(\mu_w(x,t,\alpha)) \mathrm{sgn}^+(p(x,t)) \varphi_t(x,t) dx dt \\
&- \int_{\mathbb{R}} \eta_k(w(x,0)) \mathrm{sgn}^+(p(x,0)) \varphi_t(x,0) dx. \qquad (6.16)
\end{aligned}$$

since A_Δ^{Ext} converges to 0.

Analysis of B. The consistence property of the numerical flux and a rearrangement of B leads to

$$\begin{aligned}
B &= \lambda \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[f(w_j^n \top k) - f(w_j^n \bot k)\Big] \\
&\qquad \Big[\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx - \mathrm{sgn}^+(p_{j+1}^{n+1}) \int_{x_{j+1/2}}^{x_{j+3/2}} \varphi(x, t_{n+1}) dx\Big] \\
&\quad + \lambda \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[\Big\{g(w_j^n \top k, w_j^n \top k) - f(w_j^n \bot k)\Big\} + \Big\{f(w_j^n \bot k) - g(w_j^n \bot k, w_j^n \bot k)\Big\}\Big] \\
&\qquad \Big[\mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx - \mathrm{sgn}^+(p_{j+1}^{n+1}) \int_{x_{j+1/2}}^{x_{j+3/2}} \varphi(x, t_{n+1}) dx\Big] \\
&\quad + \lambda \sum_{n=0}^{N_{\max}} \Big\{G_k(w_{J_{\max}}^n, w_{J_{\max}+1}^n) \mathrm{sgn}^+(p_{J_{\max}}^{n+1}) \int_{x_{J_{\max}-1/2}}^{x_{J_{\max}+1/2}} \varphi(x, t_{n+1}) dx \\
&\qquad\qquad - G_k(w_{J_{\min}}^n, w_{J_{\min}+1}^n) \mathrm{sgn}^+(p_{J_{\min}}^{n+1}) \int_{x_{J_{\min}-1/2}}^{x_{J_{\min}+1/2}} \varphi(x, t_{n+1}) dx\Big\} \\
&:= B_1 + B_2 + B_3. \qquad (6.17)
\end{aligned}$$

Now, we get three bounds related to B_1, B_2 and B_3. First, for B_1, we note that

$$\begin{aligned}
&\left| B_1 + \int_{\mathbb{R}} \int_{\mathbb{R}_0^+} \Big[f(w_\Delta(x,t) \top k) - f(w_\Delta(x,t) \perp k) \Big] \mathrm{sgn}^+(p_\Delta(x,t)) \varphi_x(x,t) dt dx \right| \\
&= \left| B_1 + \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[f(w_j^n \top k) - f(w_j^n \perp k) \Big] \mathrm{sgn}^+(p_j^n) \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi_x(x,t) dt dx \right| \\
&= \Bigg| \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[f(w_j^n \top k) - f(w_j^n \perp k) \Big] \\
&\qquad \Big[\lambda \Big\{ \mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx - \mathrm{sgn}^+(p_{j+1}^{n+1}) \int_{x_{j+1/2}}^{x_{j+3/2}} \varphi(x, t_{n+1}) dx \Big\} \\
&\qquad\qquad + \mathrm{sgn}^+(p_j^n) \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi_x(x,t) dt dx \Big] \Bigg| \\
&= \Bigg| \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[f(w_j^n \top k) - f(w_j^n \perp k) \Big] \Big[\mathrm{sgn}^+(p_j^{n+1}) - \mathrm{sgn}^+(p_{j+1}^{n+1}) + \mathrm{sgn}^+(p_j^n) \Big] \\
&\qquad \Big[\int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \Big\{ - \frac{\varphi(x+\Delta x, t_{n+1}) - \varphi(x, t_{n+1})}{\Delta x} + \varphi_x(x,t) \Big\} dt dx \Big] \Bigg| \\
&\leq 2 \Delta x \, t_{N_{\max}} \left(x_{J_{\max}} - x_{J_{\min}} \right) \|f\|_{\mathrm{Lip}(\mathbb{R})} \, \|\varphi_{xx}\|_{L^\infty(\mathbb{R} \times \mathbb{R}_0^+)}. \qquad (6.18)
\end{aligned}$$

Later, for B_2, we apply the weak-BV estimate (Lemma 6.2), and obtain the following bound

$$|B_2| \leq C \|\varphi_x(\cdot, t)\|_{L^\infty(\mathbb{R})} \sqrt{\Delta x}. \qquad (6.19)$$

Meanwhile, for B_3, by the selection of $J_{\min}$ and $J_{\max}$, we clearly have that

$$|B_3| \leq t_{N_{\max}} \|g\|_{L^\infty([u_m, u_M]^2)} \Big\{ |\varphi(x_{J_{\max}}, t)| + |\varphi(x_{J_{\min}}, t)| \Big\} = 0. \qquad (6.20)$$

Then, from (6.18), (6.19) and (6.20), when $\Delta x \to 0$, we obtain that

$$B \to - \int_{\mathbb{R}} \int_{\mathbb{R}_0^+} \int_0^1 \Big[f(\mu_w(x,t,\alpha) \top k) - f(\mu_w(x,t,\alpha) \perp k) \Big] \mathrm{sgn}^+(p_\Delta(x,t)) \varphi_x(x,t) dt dx d\alpha. \quad (6.21)$$

Analysis of C. We remark that (see Appendix B)

$$\begin{aligned}
&\Big[\eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2}) \Big] \mathrm{sgn}^+(p_j^{n+1}) \\
&= \begin{cases} \eta_k(v_j^{n+1/2}) - \eta_k(w_j^{n+1/2}), & \mathrm{sgn}^+(p_j^{n+1}) = 1, \ \mathrm{sgn}^+(p_j^n) = 0 \text{ and } \mathscr{E}_j^n = 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

This means that the term C is simplified to the discontinuity curves (since $\mathrm{sgn}^+(p_j^n) = 0$ and $\mathrm{sgn}^+(p_j^{n+1}) = 1$) satisfying the entropy condition (since $\mathscr{E}_j^n = 1$). Then,

$$\begin{aligned}
C &= \sum_{j=J_{\min}}^{J_{\max}} \sum_{n=0}^{N_{\max}} \Big[\eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2}) \Big] \mathrm{sgn}^+(p_j^{n+1}) \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx. \\
&= \sum_{(j,n) \in \mathbb{E}} \Big[\eta_k(v_j^{n+1/2}) - \eta_k(w_j^{n+1/2}) \Big] \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(x, t_{n+1}) dx. \qquad (6.22)
\end{aligned}$$

where

$$\mathbb{E} = \Big\{ (j,n) \in \{J_{\min}, \ldots, J_{\max}\} \times \{0, \ldots, N_{\max}\} \quad : \quad \mathrm{sgn}^+(p_j^{n+1}) = \mathscr{E}_j^n = 1 \text{ and } \mathrm{sgn}^+(p_j^n) = 0 \Big\}.$$

Now, by the strictly convexity of the flux function and the definition of $u_{L,j}^{n+1}$ and $u_{R,j}^{n+1}$ given by (4.10) and (4.11), respectively, for each $(j,n+1)\in\mathbb{E}$ we have that

$$\begin{aligned}
\mathscr{E}_j^{n+1}=1 \quad &\Rightarrow \quad f'\Big(u_{L,j}^{n+1}\Big) > f'\Big(u_{R,j}^{n+1}\Big)\\
&\Rightarrow \quad u_{L,j}^{n+1} > u_{R,j}^{n+1}\\
&\Rightarrow \quad (1-2\mathscr{P}_j^{n+1})(v_j^{n+1/2}-w_j^{n+1/2})>0\\
&\Rightarrow \quad \Big(v_j^{n+1/2}>w_j^{n+1/2}\ \wedge\ \mathscr{P}_j^{n+1}=0\Big)\vee\Big(v_j^{n+1/2}<w_j^{n+1/2}\ \wedge\ \mathscr{P}_j^{n+1}=1\Big)
\end{aligned}$$

Then, we can introduce the partition $\{\mathbb{E}_1,\mathbb{E}_2\}$ of $\mathbb{E}$, where

$$\begin{aligned}
\mathbb{E}_1 &= \Big\{(j,n+1)\in\mathbb{E} \quad : \quad v_j^{n+1/2}>w_j^{n+1/2}\ \wedge\ \mathscr{P}_j^{n+1}=0\Big\} \quad \text{and}\\
\mathbb{E}_2 &= \Big\{(j,n+1)\in\mathbb{E} \quad : \quad v_j^{n+1/2}<w_j^{n+1/2}\ \wedge\ \mathscr{P}_j^{n+1}=1\Big\}.
\end{aligned}$$

Here, for $k\in[u_m,u_M]$, we note that

$$\begin{aligned}
(j,n+1)\in\mathbb{E}_1 \quad &\Rightarrow \quad 0\le |u_m-k|<|w_j^{n+1/2}-k|<|v_j^{n+1/2}-k|<|u_M-k|\\
&\Rightarrow \quad 0<\eta_k(v_j^{n+1/2})-\eta_k(w_j^{n+1/2})<\eta_k(u_M)-\eta_k(u_m)
\end{aligned}\tag{6.23}$$

and

$$\begin{aligned}
(j,n+1)\in\mathbb{E}_2 \quad &\Rightarrow \quad 0\le |u_m-k|<|v_j^{n+1/2}-k|<|w_j^{n+1/2}-k|<|u_M-k|\\
&\Rightarrow \quad \eta_k(v_j^{n+1/2})-\eta_k(w_j^{n+1/2})<0.
\end{aligned}\tag{6.24}$$

From (6.22)-(6.24) and since $\varphi\ge 0$, we deduce that

$$\begin{aligned}
C &= \sum_{(j,n)\in\mathbb{E}_1}\Big[\eta_k(v_j^{n+1/2})-\eta_k(w_j^{n+1/2})\Big]\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx\\
&\quad+\sum_{(j,n)\in\mathbb{E}_2}\Big[\eta_k(v_j^{n+1/2})-\eta_k(w_j^{n+1/2})\Big]\int_{x_{j-1/2}}^{x_{j+1/2}}\varphi(x,t_{n+1})dx\\
&\le \Big[\eta_k(u_M)-\eta_k(u_m)\Big]\|\varphi\|_{L^\infty}\,T_{max}\,N_s\,\Delta x,
\end{aligned}$$

where N_s is the number of shock curves of u, which implies that

$$C\le 0, \quad \text{when} \quad \Delta x\to 0. \tag{6.25}$$

We conclude the proof of this item by noticing that the equations (6.15), (6.16), (6.21) and (6.25) imply (6.12).

For (c), to prove this item we proceed in a similar way as for the proof of (b).

For (d), by Lemma 6.1 and Proposition 2.1, we note that u_Δ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$. This fact implies the existence of $\mu\in L^\infty(\mathbb{R}\times\mathbb{R}_0^+\times]0,1[)$ such that $u_\Delta\to\mu$ in the nonlinear weak-$\star$ sense when $\Delta x\to 0$. Furthermore, from the recently proved items (a), (b), (c) and by (4.15) we deduce that

$$\mu(x,t,\alpha)=\mathrm{sgn}^+(p(x,t))\mu_w(x,t,\alpha)+\Big(1-\mathrm{sgn}^+(p(x,t))\Big)\mu_v(x,t,\alpha),$$

which implies

$$\begin{aligned}
&\eta_k(\mu(x,t,\alpha))=\eta_k(\mu_w(x,t,\alpha))\mathrm{sgn}^+(p(x,t))+\eta_k(\mu_v(x,t,\alpha))\Big(1-\mathrm{sgn}^+(p(x,t))\Big) \quad \text{and}\\
&f(\mu(x,t,\alpha)\top k)-f(\mu(x,t,\alpha)\bot k)\\
&\qquad=\Big(f(\mu_w(x,t,\alpha)\top k)-f(\mu_w(x,t,\alpha)\bot k)\Big)\mathrm{sgn}^+(p(x,t))
\end{aligned}$$

$$+\Big(f(\mu_v(x,t,\alpha)\top k)-f(\mu_v(x,t,\alpha)\bot k)\Big)\Big(1-\mathrm{sgn}^+(p(x,t))\Big).$$

Then, using this identities and adding the inequalities (6.12) and (6.13) we conclude that μ is a entropy process solution of (1.1). □

6.5. **Convergence towards the Kružkov's entropy solution.** The prove the convergence of u_Δ to the entropy solution we first deduce the link between the notions of entropy process solutions and Kružkov entropy solutions for (1.1) and then prove the convergence. More precisely we have the following two results.

Theorem 6.1. *Under the assumptions* (1.2)-(1.3), *the entropy process solution of the problem* (1.1) *is unique. Moreover there exists a function* $u\in L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ *such that* $u(x,t)=\mu(x,t,\alpha)$ *for almost every* $(x,t,\alpha)\in\mathbb{R}\times\mathbb{R}_0^+\times]0,1[$.

Theorem 6.2. *Consider the hypothesis of lemma 6.1. Then* u_Δ *converge to* u*, the entropy solution of* (1.1)*, in the strong topology of* $L^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_0^+)$ *for all* $p>1$ *when* $\Delta x\to 0$.

To prove Theorems 6.1 and 6.2 we can follow line by line the proofs of the results given in [8] for a scalar multidimensional conservation law. The proof of theorem 6.1 is based on the doubling of variables technique introduced by Kružkov in [14]. Meanwhile, the proof of theorem 6.1 is given by a contradiction argument.

7. Numerical examples for Burgers equation

In this section, we present three numerical examples with the 1D Burgers flux function $f(u)=u^2/2$ and 2D Burgers flux functions $f(u)=g(u)=u^2/2$. The initial condition in the first two examples are focused on the 1D elementary wave interaction. We consider the third example for a 2D extension of the level set method over cartesian grids.

In all numerical examples the function g on the implementation of (4.7) and (4.8) is the Godunov numerical flux function, see [9].

7.1. **Example 1: Rarefaction-Schock interaction.** We consider the initial condition

$$u_0(x)=\begin{cases}1, & x\in[-1,1],\\ 0, & \text{otherwise.}\end{cases}\tag{7.1}$$

The initial increasing discontinuity at $x=-1$ generates a rarefaction wave, and the initial decreasing discontinuity at $x=1$ becomes an entropic shock wave. Both waves interact at $(x,t)=(3,4)$ forming a shock wave shaped curved line. More, precisely the analytical entropy solution is given by

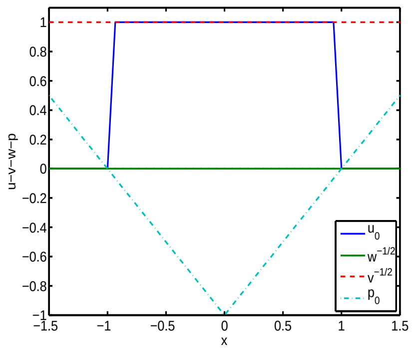
$$u(x,t)=\left\{\begin{array}{lll}0, & x\in]-\infty,-1]\cup](t+2)/2,\infty[, & \\ (x+1)/t, & x\in]-1,t-1], & t\le 4\\ 1, & x\in[t-1,(t+2)/2], & \\ 0, & x\in]-\infty,-1]\cup]3\sqrt{t}/2,\infty[, & t>4\\ (x+1)/t, & x\in[-1,3\sqrt{t}/2], & \end{array}\right.$$

According to the criterion of the first stage of the algorithm (1.6), a natural selection for the initial level set would be (see Figure 1(a))

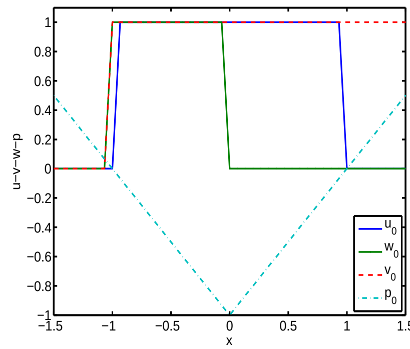
$$p_0(x)=\begin{cases}-x-1, & x\in]-\infty,0],\\ x-1, & x\in]0,\infty[,\end{cases}\qquad v^{-1/2}(x)=1,\quad\text{and}\quad w^{-1/2}(x)=0.\tag{7.2}$$

However, this selection of initial level set does not satisfies the entropy condition required in (3.2). That is why we need to rectify this initial level set by redefining $v^{-1/2}$ and $w^{-1/2}$, as in Figure 1(b).

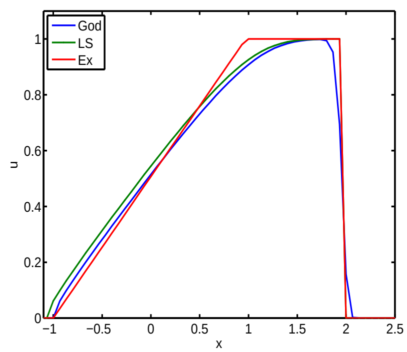
For a discretization of $\Delta t=0.0266$, $\Delta x=0.0667$, we perform simulations with our level-set method to the time $T=2$ (see Figure 1(c) and Figure 1(d)) and time $T=5$ (see Figure 1(e) and Figure 1(f)). We note that $T=2$ and $T=5$ are times before and after of the interaction, respectively. We compare our results with the exact solution and the numerical solution using the Godunov scheme. We consider a discretization so not too thin to observe graphically the differences between the two methods of order 1, and compare (on a finer discretization is not



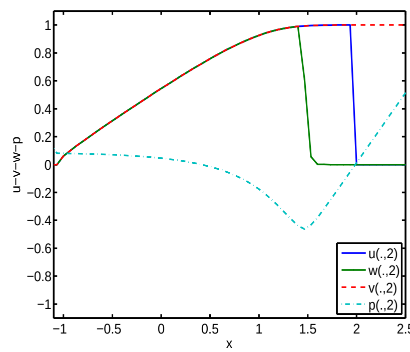
(a) p_0, $w^{-1/2}$ and $v^{-1/2}$



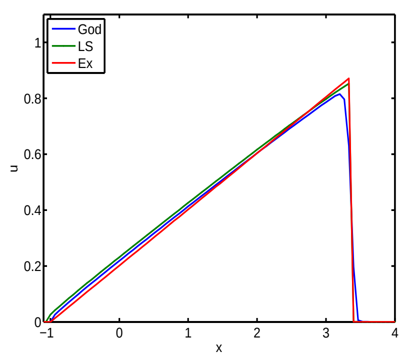
(b) p, w, v and u at $T = 0$



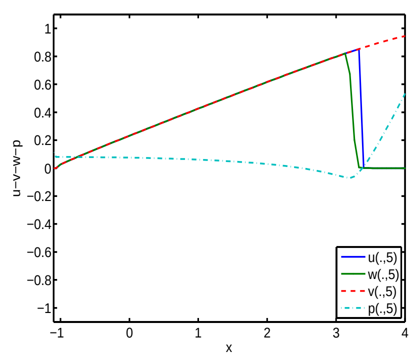
(c) u at $T = 2$



(d) p, w, v and u at $T = 2$



(e) u at $T = 5$



(f) p, w, v and u at $T = 5$

FIGURE 1. Example 1. (a) The selected initial level set function p_0 and the auxiliar states $w^{-1/2}$ and $v^{-1/2}$, see (7.2). (b) The inital level set profile $p_\Delta(\cdot, 0)$, the initial conditión $u_\Delta(\cdot, 0)$, and the entropy satisfying auxiliar states profiles $w_\Delta(\cdot, 0)$ and $v_\Delta(\cdot, 0)$. (c)-(e) Godunov (God), Level set (LS) approximate solutions and Exact (Ex) solution at $T = 2$ and $T = 5$, respectively. (d)-(f) The level set profile $p_\Delta(\cdot, T)$, the level set aproximate profile $u_\Delta(\cdot, T)$, and the entropy satisfying auxiliar states profiles $w_\Delta(\cdot, T)$ and $v_\Delta(\cdot, T)$, at $T = 2$ and $T = 5$, respectively.

visually appreciate the differences). Our level set method, is based and is characterized by better capturing shocks. In this regard, we note a better accuracy of the method level set at time $T = 2$

(see Figure 1(e)). In addition, in the right graphics Figure 1(b), Figure 1(d) and Figure 1(f), we observe the evolution of each of the entropy admissible variables used in the level-set scheme for $T \in \{0, 2, 5\}$.

7.2. **Example 2: Schock-Schock interaction.** We consider the initial condition

$$u_0(x) = \begin{cases} 2, & x \in]-\infty, -1], \\ 1, & x \in]-1, 1], \\ 0, & x \in]1, \infty[. \end{cases} \tag{7.3}$$

In this case a discontinuity is initially located at $x = -1$ which propagates as an entropic shock with velocity $\sigma = 3/2$, and a discontinuity initially located at the point $x = 1$ which propagates with velocity $\sigma = 1/2$. Both shock lines intersect and interact at $(x, t) = (2, 2)$, generating a new shock with velocity $\sigma = 1$. For the initial condition (7.3) the analytical entropy solution of the burger equation is defined as follows

$$u(x,t) = \begin{cases} \left.\begin{array}{ll} 2, & x \in]-\infty, (3/2)t - 1], \\ 1, & x \in](3/2)t - 1, (1/2)t - 1], \\ 0, & x \in [(1/2)t - 1, \infty[, \end{array}\right\} & t \leq 2, \\ \left.\begin{array}{ll} 2, & x \in]-\infty, t], \\ 1, & x \in [t, \infty[, \end{array}\right\} & t > 2. \end{cases}$$

Now, applying the first stage of the algorithm (1.6), we chose the following initial level set representation of u_0

$$p_0(x) = \begin{cases} x+1, & x \in]-\infty, 0], \\ x-1, & x \in]0, \infty[, \end{cases} \quad w^{-1/2}(x) = \begin{cases} 2, & x \in]-\infty, 0], \\ 0, & x \in]0, \infty[, \end{cases} \quad \text{and } v^{-1/2}(x) = 1, \tag{7.4}$$

which results verify the desired entropy inequality (3.2). Thus ensure an approximation consistent with the entropy solution, see Figure 2(a) and Figure 2(b).

For a discretization of $\Delta t = 0.0133$, $\Delta x = 0.0667$, we perform simulations to the time $T = 1$ (see Figure 2(c) and Figure 2(d)) and time $T = 3$ (see Figure 2(e) and Figure 2(f)). We note that the interaction occurs at $t = 2$, then $T = 1$ and $T = 3$ are before and after of the interaction, respectively. We compare level-set results with the exact solution and the Godunov numerical solution. In this example, Godunov method also captures well the shocks, and both methods of order 1, give very similar results. In addition, in the right graphics Figure 2(b), Figure 2(d) and Figure 2(f), we observe the evolution of each of the variables used in the level-set scheme, consistent with the approximate solution at $T \in \{0, 1, 3\}$.

7.3. **Example 3: 2D Burgers equation.** We consider the initial condition

$$u_0(x, y) = \begin{cases} 1, & |x| + |y| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

At the boundary $x+y = 1$ for $x, y \in [0, 1]$ generates an entropic shock and the boundary $x+y = -1$ for $x, y \in [-1, 0]$ becomes a rarefaction wave.

The main modifications of the 1D algorithm described on section 4 are the definition of 2D coherent definition of 1D entropy condition given on (3.2). Here, in this simulation, we consider that 3.2 is reemplaced by the following condition

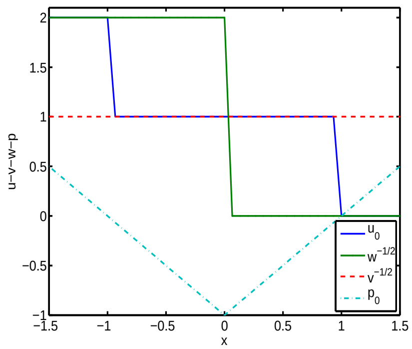
$$\Big(f'(u_L) - f'(u_R)\Big)\partial_x p + \Big(g'(u_L) - g'(u_R)\Big)\partial_y p \geq 0, \quad \text{in } Q, \tag{7.5}$$

where

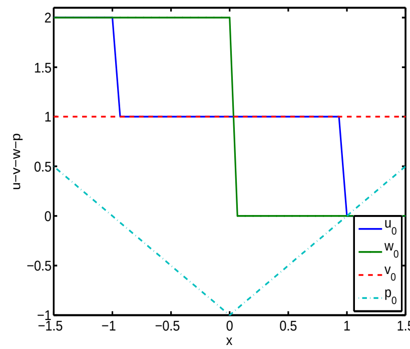
$$\begin{aligned} u_L(x,y,t) &= \begin{cases} w(x,y,t), & \big(\partial_x p + \partial_y p\big)(x,y,t) \geq 0, \\ v(x,y,t), & \big(\partial_x p + \partial_y p\big)(x,y,t) < 0, \end{cases} \quad \text{and} \\ u_R(x,y,t) &= \begin{cases} v(x,y,t), & \big(\partial_x p + \partial_y p\big)(x,y,t) < 0, \\ w(x,y,t), & \big(\partial_x p + \partial_y p\big)(x,y,t) \geq 0. \end{cases} \end{aligned}$$

Now, according to the criterion of the first stage of the algorithm (1.6), we consider the initial level set function and auxiliar states defined as follows (see Figure 3)

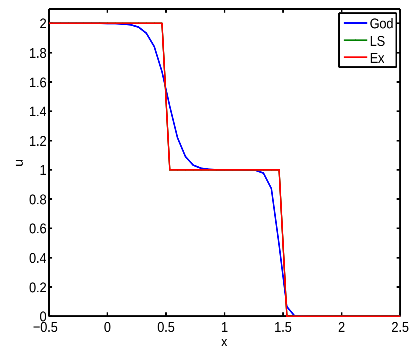
$$p_0(x, y) = |x| + |y| - 1, \qquad v^{-1/2}(x, y) = 1, \quad \text{and} \quad w^{-1/2}(x, y) = 0. \tag{7.6}$$



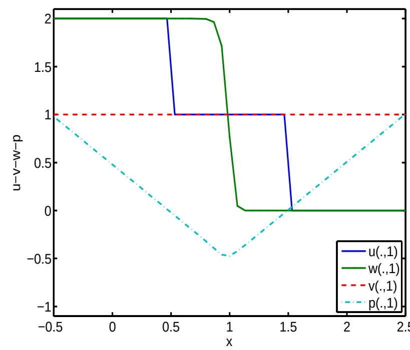
(a) p_0, $w^{-1/2}$ and $v^{-1/2}$



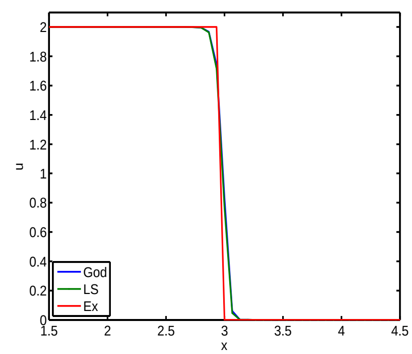
(b) p, w, v and u at $T = 0$



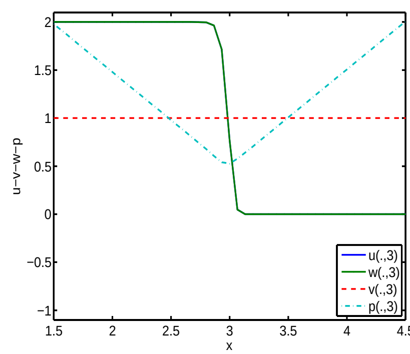
(c) u at $T = 2$



(d) p, w, v and u at $T = 2$



(e) u at $T = 3$

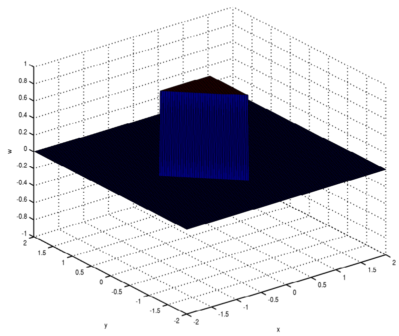


(f) p, w, v and u at $T = 3$

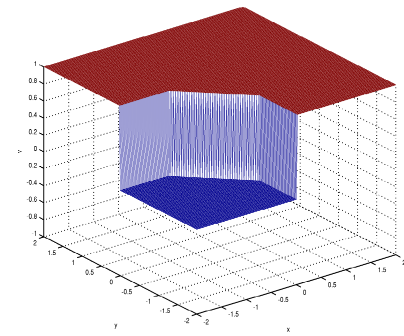
FIGURE 2. Example 2. (a) The selected initial level set function p_0 and the auxiliar states $w^{-1/2}$ and $v^{-1/2}$, see (7.4). (b) The inital level set profile $p_\Delta(\cdot, 0)$, the initial conditión $u_\Delta(\cdot, 0)$, and the entropy satisfying auxiliar states profiles $w_\Delta(\cdot, 0)$ and $v_\Delta(\cdot, 0)$. (c)-(e) Godunov (God), Level set (LS) approximate solutions and Exact (Ex) solution at $T = 2$ and $T = 5$, respectively. (d)-(f) The level set profile $p_\Delta(\cdot, T)$, the level set aproximate profile $u_\Delta(\cdot, T)$, and the entropy satisfying auxiliar states profiles $w_\Delta(\cdot, T)$ and $v_\Delta(\cdot, T)$, at $T = 2$ and $T = 3$, respectively.

However, the selection (7.6) of initial auxiliar states does not satisfies the entropy condition required in (7.5). Then, before start the evolution of the algorithm we redefine $v^{-1/2}$ and $w^{-1/2}$, as given in Figure 3(a) and Figure 3(b).

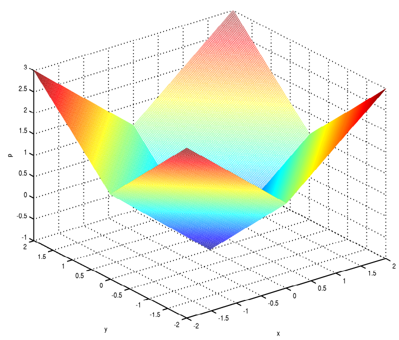
For a discretization of $\Delta t = 0.008$, $\Delta x = \Delta y = 0.02$, we perform a simulation with our level-set method to the time $T = 1$ (see Figure 4). We compare our results with the numerical solution using the Godunov scheme see Figure 5. In addition, in Figure 6 we show the profile $u_\Delta(x, x, 1)$ of the numerical solutions obtained by Godunov and level set methods with $\Delta t = 0.0312$ and $\Delta x = \Delta y = 0.08$.



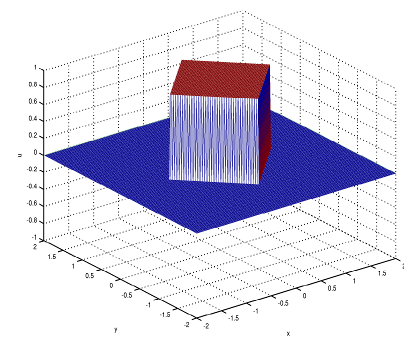
(a) w at $T = 0$



(b) v at $T = 0$



(c) p at $T = 0$

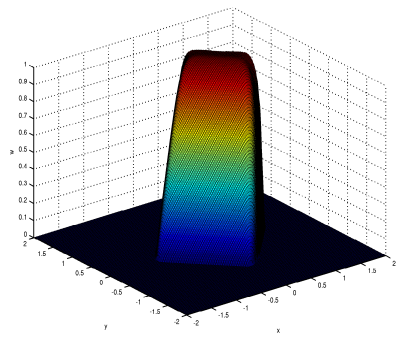


(d) u at $T = 0$

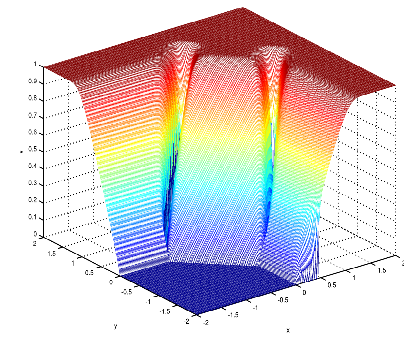
FIGURE 3. Example 3. Initialization step of the level set algorithm. Left: (a) The auxiliar state $w^{-1/2}$, (b) The auxiliar state $v^{-1/2}$ and (e) The selected initial level set function p_0, see (7.6). Right: (b)-(d) The inital entropy satisfying auxiliar states profiles $w_\Delta(\cdot, 0)$ and $v_\Delta(\cdot, 0)$; and (f) The initial condition $u_\Delta(\cdot, 0)$.

8. Conclusions

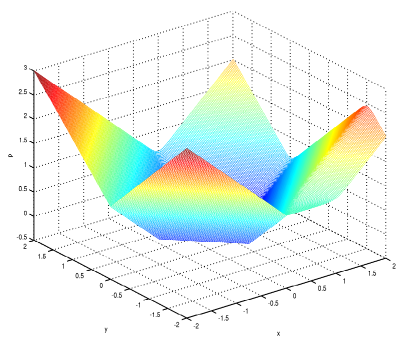
In this paper we gave a convergence proof of the level-set algorithm introduced by Aslam [1] for tracking discontinuities in scalar conservation laws with linear or strictly convex flux function. To do so, we adapt and improve the ideas introduced by Eymard et al. [8], who have developed a new approach to show convergence of finite volume methods by using a weak bounded variation estimate together with a new notion of generalized solutions to conservation laws called entropy process solutions. Roughly speaking, this method consists in showing that the numerical solution satisfies a weak bounded variation estimate, then showing that this implies that the numerical solution converges to a entropy process solution to the conservation law, and finally that this solution is unique and that the numerical solution strongly converges in suitable spaces. Additionally, we have fully described the overall numerical method, and some numerical examples dealing with the elementary wave interaction were given. The numerical examples show that the level-set algorithm is competitive with respect to the well-known WENO5 method, since in presence of discontinuities



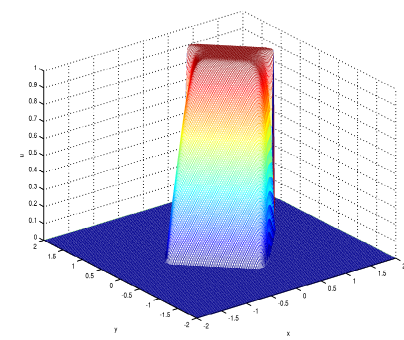
(a) $w_\Delta(x,y,1)$



(b) $v_\Delta(x,y,1)$



(c) $p_\Delta(x,y,1)$



(d) $u_\Delta(x,y,1)$

Figure 4. Example 3. End profiles obtained of the level set algorithm. The auxiliar states $w_\Delta(\cdot,\cdot,1)$ and $v_\Delta(\cdot,\cdot,1)$, the level set function $p_\Delta(\cdot,\cdot,1)$, and aproximate solution $u_\Delta(\cdot,\cdot,1)$ obtained by the level set method with $\Delta x = \Delta y = 0.02$ and $\Delta t = 0.008$. See Figure 5(b) for the level curves of $u_\Delta(\cdot,\cdot,1)$ plotted on (d).

it is well known that this last one is not of order 5 of accuracy, while the first one suitably tracks the discontinuities of the solution without requiring a high order formal accuracy in smooth regions, and at the same time that it retains stable, non-oscillatory and sharp discontinuity transitions. In addition, the level-set algorithm has the advantage of being easy to implement and computationally effective, because it requires relatively few grid points in order to that the numerical solution be satisfactory.

Acknowledgement

The first author thanks for the support of the research projects 124109 3/R, 104709 01 F/E and 121909 GI/C at Universidad del Bío-Bío, Chile. The third author thanks for the support of Fondecyt project 1110540, CONICYT project Anillo ACT1118 (ANANUM), Red Doctoral REDOC.CTA, project UCO1202 at Universidad de Concepción, and Basal, CMM, Universidad de Chile.

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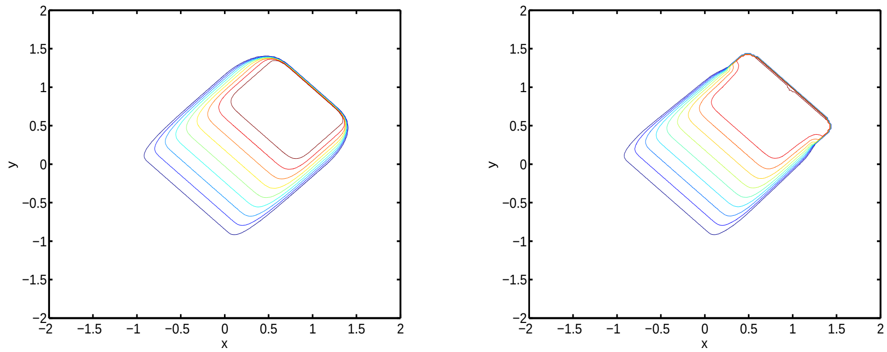

(a) u_Δ at $T = 1$ (b) u_Δ at $T = 1$

FIGURE 5. Example 3. (a) Level curves of the aproximate solution $u_\Delta(\cdot,\cdot,1)$ obtained by Godunov method. (b) Level curves of the aproximate solution $u_\Delta(\cdot,\cdot,1)$ obtained by Level set method. In both cases $\Delta x = \Delta y = 0.02$ and $\Delta t = 0.008$.

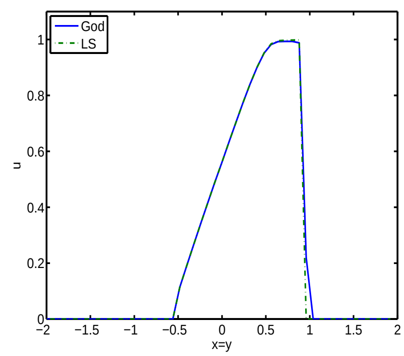

FIGURE 6. Example 3. Plot of the profile $u_\Delta(x, x, 1)$ for Godunov (God) and Level set (LS) approximate solutions with $\Delta x = \Delta y = 0.08$ and $\Delta t = 0.312$.

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Appendix A. Coefficients of (6.2)

By definition of sgn^+ we have that $\mathrm{sgn}^+(p_j^{n+1}), \mathrm{sgn}^+(p_j^{n+1}), \mathscr{E}_j^n \in \{0,1\}$. This fact implies that the coefficients $r_{w,j}^n$ and $r_{v,j}^n$ in (6.2), are such that $r_{v,j}^n, r_{v,j}^n \in \{0,1\}$ and $r_{v,j}^n + r_{v,j}^n = 1$, see Table 1.

$\mathrm{sgn}^+(p_j^{n+1})$	$\mathrm{sgn}^+(p_j^{n+1})$	$\mathscr{E}_j^n$	$r_{v,j}^n$	$r_{w,j}^n$	$r_{v,j}^n + r_{w,j}^n$
1	1	1	1	0	1
1	1	0	1	0	1
1	0	1	0	1	1
1	0	0	1	0	1
0	1	1	1	0	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

Table 1. Posibles coefficients $r_{v,j}^n$ and $r_{w,j}^n$ of (6.2) and its sum

Appendix B. Analysis of $\tilde{C}$ in the proof of Lemma (6.4)

We note that

$$\begin{aligned}
I_j^n &:= \Big\{\eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2})\Big\}\mathrm{sgn}^+(p_j^{n+1}) \\
&= \Big\{\eta_k(w_j^{n+1}) - \eta_k(w_j^{n+1/2})\Big\}\mathrm{sgn}^+(p_j^{n+1}) \\
&= \Big\{\Big|w_j^{n+1} - k\Big| - \Big|w_j^{n+1/2} - k\Big|\Big\}\mathrm{sgn}^+(p_j^{n+1}) \\
&= \Big\{\Big|w_j^{n+1/2} + (v_j^{n+1/2} - w_j^{n+1/2})(1 - \mathrm{sgn}^+(p_j^n))\mathscr{E}_j^n - k\Big| - \Big|w_j^{n+1/2} - k\Big|\Big\}\mathrm{sgn}^+(p_j^{n+1}).
\end{aligned}$$

Then, using that $\mathrm{sgn}^+(p_j^{n+1}), \mathrm{sgn}^+(p_j^n), \mathscr{E}_j^n \in \{0,1\}$ we deduce that (see Table 2):

$$I_j^n = = \begin{cases} |v_j^{n+1/2} - k| - |w_j^{n+1/2} - k|, & \mathrm{sgn}^+(p_j^{n+1}) = 1,\ \mathrm{sgn}^+(p_j^n) = 0 \text{ and } \mathscr{E}_j^n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

.

$\mathrm{sgn}^+(p_j^{n+1})$	$\mathrm{sgn}^+(p_j^n)$	$\mathscr{E}_j^n$	I_j^n
1	1	1	0
1	1	0	0
1	0	1	$\lvert v_j^{n+1/2} - k\rvert - \lvert w_j^{n+1/2} - k\rvert$
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Table 2. Posibles combinations of $\mathrm{sgn}^+(p_j^{n+1}), \mathrm{sgn}^+(p_j^n), \mathscr{E}_j^n \in \{0,1\}$ and the simplication of I_j^n

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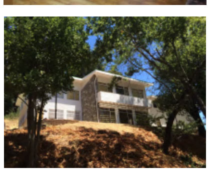
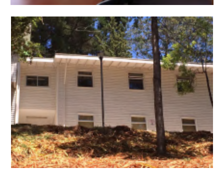

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