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FINITE ELEMENT ANALYSIS FOR A PRESSURE-STRESS FORMULATION OF A FLUID-STRUCTURE INTERACTION SPECTRAL PROBLEM

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ABSTRACT. The aim of this paper is to analyze an elastoacoustic vibration problem employing a dual-mixed formulation in the solid domain. The Cauchy stress tensor and the rotation are the primary variables in the elastic structure while the standard pressure formulation is considered in the acoustic fluid. The resulting mixed eigenvalue problem is approximated by a conforming Galerkin scheme based on the lowest order Lagrange and Arnold-Falk-Winther finite element subspaces in the fluid and solid domains respectively. We show that the scheme provides a correct approximation of the spectrum and prove quasi-optimal error estimates. Finally, we report some numerical experiments.

1. INTRODUCTION

The dynamic interaction between a fluid and a structure plays an important role in many engineering fields (cf. [24]). In this paper, we are concerned with an elastoacoustic problem. We aim to compute the free vibration modes of an elastic structure in contact with a compressible fluid, which may have a free surface subject to gravity oscillations (sloshing). Under the assumption of small displacements in the solid, the problem reduces to the coupling of a scalar-valued equation describing the propagation of a pressure wave and a vector-valued equation modeling the propagation of waves in an elastic medium. The focus of this study is to determine the free vibration modes of the overall coupled system.

The choice of the main variables in each media gives rise to different variational formulations for the elastoacoustic vibration problem. Traditionally, a primal formulation is used in the solid, i.e., the displacement is used in the elastic structure. In early works [25], the acoustic wave equation is written in terms of the pressure, which leads to non-symmetric eigenvalue problems (see also [6]). Alternatively, the fluid can be modeled by using the displacement [22, 4, 8, 3], a displacement potential on its own [7, 24] or combined with the pressure [10]. We refer to [7] for a comparison of these different formulations.

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More recently, dual-mixed formulations have been considered in the solid for the elastoacoustic source problem (see, e.g., [18] and [19]). In such an approach, the unknowns in the solid domain are the Cauchy stress tensor and the rotation while the pressure is maintained as the only variable in the fluid. The resulting formulation is symmetric, it delivers direct finite element approximations of the stresses and avoids the locking phenomenon that arises in the nearly incompressible case. The aim of this paper is to analyze the elastoacoustic eigenvalue problem corresponding to this formulation.

We define a Galerkin scheme by approximating the unknowns of the fluid and the solid by the lowest-order Lagrange and Arnold-Falk-Winther [2] finite elements, respectively. The latter consist of piecewise linear approximations for the stress and piecewise constant functions for the rotation (as well as for the displacement, which will not appear as an unknown in the present problem). The symmetry of the stress tensor is imposed weakly by means of a suitable Lagrange multiplier (the rotation). Therefore, the spectral problem we have to deal with has a saddle point structure.

When we undertook the analysis of this formulation, we realized that it does not fit in any of the existing theories for mixed eigenvalue problems (see Part 3 of [11] and the references therein). Actually, we had to pave the way for the present study by first considering in [23] the dual-mixed eigenvalue formulation of the standalone elasticity problem with reduced symmetry. In such a case, we proved that the mixed Galerkin approximation is spectrally correct and provided asymptotic error estimates for the eigenvalues and eigenfunctions by adapting results from [15, 16]. The analysis given here is a generalization of the results obtained in [23] to the elastoacoustic eigenvalue problem.

The paper is organized as follows. In Section 2 we introduce a mixed formulation with reduced symmetry of the eigenvalue elastoacoustic problem and define the corresponding solution operator. Sections 3 and 4 are devoted to the characterization of the spectrum of the solution operator. In Section 5 we introduce the technical finite element results that are used in Section 6 to describe the spectrum of the discrete solution operator. In Section 7 we show that the numerical scheme provides a correct spectral approximation and establish asymptotic error estimates for the eigenvalues and eigenfunctions. Finally, we present in Section 8 a numerical test which confirms that the method is not polluted with spurious modes and show that the experimental rates of convergence are in accordance with the theoretical ones.

We end this section with some notation which will be used below. Given any Hilbert space $\mathcal{V}$, let $\mathcal{V}^n$ and $\mathcal{V}^{n\times n}$ denote, respectively, the space of vectors and tensors of order n ($n = 2$ or 3) with entries in $\mathcal{V}$. In particular, $\boldsymbol{I}$ is the identity matrix of $\mathbb{R}^{n\times n}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^n \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^n \tau_{ij}\sigma_{ij}$.

Let Ω be a Lipschitz bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. For $s \geq 0$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ or $\mathrm{H}^s(\Omega)^{n\times n}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define the Hilbert space $\mathrm{H}(\mathbf{div};\Omega) := \left\{\boldsymbol{\tau} \in \mathrm{L}^2(\Omega)^{n\times n} : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{L}^2(\Omega)^n\right\}$, whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$.

Given two Hilbert spaces $\mathbb{S}$ and $\mathbb{T}$ and a bounded bilinear form $c : \mathbb{S} \times \mathbb{T} \to \mathbb{R}$, we will denote

$$\ker(c) := \{s \in \mathbb{S} : \ c(s,t) = 0 \quad \forall t \in \mathbb{T}\}.$$

We will also say that c satisfies the inf-sup condition for the pair $\{\mathbb{S}, \mathbb{T}\}$, whenever there exists $\beta > 0$ such that

$$\sup_{0 \neq s \in \mathbb{S}} \frac{c(s,t)}{\|s\|_{\mathbb{S}}} \geq \beta \|t\|_{\mathbb{T}} \qquad \forall t \in \mathbb{T}.$$

Let $\{\mathbb{S}_h\}_{h>0}$ and $\{\mathbb{T}_h\}_{h>0}$ be families of finite dimensional subspaces of the Hilbert spaces $\mathbb{S}$ and $\mathbb{T}$, respectively. The discrete kernel of c is the set

$$\ker_h(c) := \{s_h \in \mathbb{S}_h : \ c(s_h, t_h) = 0 \quad \forall t_h \in \mathbb{T}_h\}.$$

We say that c satisfies a uniform (discrete) inf-sup condition for the pair $\{\mathbb{S}_h, \mathbb{T}_h\}$ when there exists $\beta^* > 0$, independent of h, such that

$$\sup_{0 \neq s_h \in \mathbb{S}_h} \frac{c(s_h, t_h)}{\|s_h\|_{\mathbb{S}}} \geq \beta^* \|t_h\|_{\mathbb{T}} \qquad \forall t_h \in \mathbb{T}_h.$$

Finally, we employ $\mathbf{0}$ to denote a generic null vector or tensor and C to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The spectral problem

Our aim is to compute the free vibration modes of an elastic structure in contact with a compressible, inviscid, and homogeneous fluid, with mass density ρ_{F} and acoustic speed c. The solid is supposed to be isotropic and linearly elastic with a constant mass density ρ_{S} and Lamé coefficients λ_{S} and μ_{S}. The gravity acceleration is denoted by g.

Let Ω_{F} and Ω_{S} be polyhedral Lipschitz bounded domains occupied by the fluid and the solid, respectively, as shown in Figure 1. The boundary $\partial\Omega_{\mathrm{F}}$ of the fluid domain is split into two parts: the interface Σ between the fluid and the solid and the open boundary of the fluid Γ_0 (the case $\Gamma_0 = \emptyset$ is not excluded). The boundary $\partial\Omega_{\mathrm{S}}$ of the solid domain is the union of Σ, $\Gamma_{\mathrm{D}} \neq \emptyset$, and Γ_{N}, the structure being fixed on Γ_{D} and free of stress on Γ_{N}. We assume that Σ is oriented by the unit

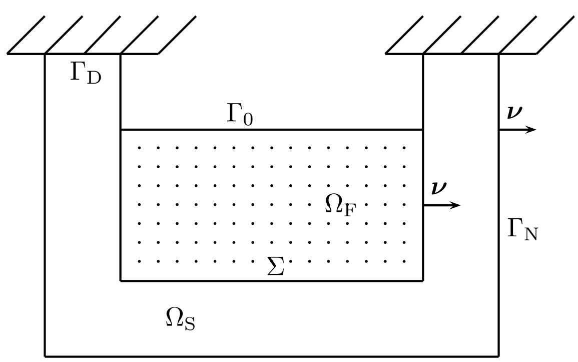

Figure 1. Fluid and solid domains

normal vector $\boldsymbol{\nu}$ outward to the boundary of Ω_{F}. The outward unit normal vector to $\partial\Omega_{\mathrm{S}} \setminus \Sigma$ is also denoted by $\boldsymbol{\nu}$.

We introduce the elasticity operator $\mathcal{C}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathcal{C}\boldsymbol{\tau} := \lambda_{\mathrm{S}}\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu_{\mathrm{S}}\boldsymbol{\tau}.$$

The constitutive equation relating the solid displacement field $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega_{\mathrm{S}},$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ is the linearized strain tensor.

Under the hypothesis of small oscillations, the classical approximation yields the following eigenvalue problem for the free vibration modes of the coupled system and the corresponding natural frequencies $\omega > 0$ (see, for instance, [7, 24]):

$$\boldsymbol{\sigma} - \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{0} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{2.1}$$

$$\mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_{\mathrm{S}}\boldsymbol{u} = \boldsymbol{0} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{2.2}$$

$$\Delta p + \frac{\omega^2}{c^2}p = 0 \qquad \text{in } \Omega_{\mathrm{F}}, \tag{2.3}$$

$$\boldsymbol{\sigma\nu} + p\boldsymbol{\nu} = \boldsymbol{0} \qquad \text{on } \Sigma, \tag{2.4}$$

$$\frac{\partial p}{\partial\boldsymbol{\nu}} - \omega^2\rho_{\mathrm{F}}\boldsymbol{u}\cdot\boldsymbol{\nu} = 0 \qquad \text{on } \Sigma, \tag{2.5}$$

$$\frac{\partial p}{\partial\boldsymbol{\nu}} - \frac{\omega^2}{g}p = 0 \qquad \text{on } \Gamma_0, \tag{2.6}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2.7}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{0} \qquad \text{on } \Gamma_{\mathrm{N}}. \tag{2.8}$$

We follow [19] and employ primal and dual-mixed approaches in the fluid Ω_{F} and the solid Ω_{S}, respectively, to derive a full continuous variational formulation of the problem. This procedure is dual to the approach adopted in [4, 8]. In particular, the transmission condition (2.5) that is essential in [4, 8] becomes natural in our formulation, while (2.4) changes to an essential condition that will be incorporated directly into the definition of the space to which the unknowns $\boldsymbol{\sigma}$ and p will belong.

Therefore, the main unknown $\boldsymbol{\sigma}$ in the solid should belong to the space

$$\boldsymbol{\mathcal{W}} := \{\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}}) :\ \boldsymbol{\tau\nu} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{N}}\},$$

whose subspace

$$\boldsymbol{\mathcal{W}}_{\Sigma} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} :\ \boldsymbol{\tau\nu} = \boldsymbol{0} \text{ on } \Sigma\}$$

will also be useful in the following. The rotation $\boldsymbol{r} := \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ will intervene in our variational formulation as a Lagrange multiplier. It will be sought in the space

$$\boldsymbol{\mathcal{Q}} := \left\{\boldsymbol{s} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^{n\times n} :\ \boldsymbol{s}^{\mathtt{t}} = -\boldsymbol{s}\right\}$$

of skew-symmetric tensors. Using this new variable $\boldsymbol{r}$, the constitutive equation becomes

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla\boldsymbol{u} - \boldsymbol{r}.$$

Testing this equation with $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$ and integrating by parts yield

$$\int_{\Omega_{\mathrm{S}}} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} = -\int_{\Omega_{\mathrm{S}}} \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} - \langle\boldsymbol{\tau\nu},\boldsymbol{u}\rangle_{\Sigma} - \int_{\Omega_{\mathrm{S}}} \boldsymbol{\tau} : \boldsymbol{r},$$

where $\langle\cdot,\cdot\rangle_\Sigma$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Sigma)^n := \left[\mathrm{H}^{1/2}(\Sigma)^n\right]'$ and $\mathrm{H}^{1/2}(\Sigma)^n$ with respect to the $\mathrm{L}^2(\Sigma)^n$-inner product. Let us remark that since $\boldsymbol{\tau\nu} = \mathbf{0}$ on Γ_N, it is easy to check that $\boldsymbol{\tau\nu}$ is well defined in $\mathrm{H}^{-1/2}(\Sigma)^n$. We now eliminate the displacement field by substituting back $\boldsymbol{u} = -\frac{1}{\omega^2\rho_\mathrm{S}}\,\mathbf{div}\,\boldsymbol{\sigma}$ (cf. (2.2)) into the last equation:

$$\int_{\Omega_\mathrm{S}} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_{\Omega_\mathrm{S}} \frac{1}{\omega^2\rho_\mathrm{S}}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \langle\boldsymbol{\tau\nu},\boldsymbol{u}\rangle_\Sigma + \int_{\Omega_\mathrm{S}} \boldsymbol{\tau} : \boldsymbol{r} = 0. \tag{2.9}$$

In its turn, the variational formulation in Ω_F is given by

$$\int_{\Omega_\mathrm{F}} \nabla p\cdot\nabla q - \int_{\Omega_\mathrm{F}} \frac{\omega^2}{c^2} pq - \left\langle \frac{\partial p}{\partial \boldsymbol{\nu}}, q\right\rangle_{\partial\Omega_\mathrm{F}} = 0,$$

which is obtained by multiplying equation (2.3) by $q \in \mathrm{H}^1(\Omega_\mathrm{F})$ and integrating by parts. Moreover, using (2.5) and (2.6), we have that

$$\left\langle \frac{\partial p}{\partial \boldsymbol{\nu}}, q\right\rangle_{\partial\Omega_\mathrm{F}} = \int_\Sigma \omega^2\rho_\mathrm{F}\boldsymbol{u}\cdot\boldsymbol{\nu} q + \int_{\Gamma_0} \frac{\omega^2}{g} pq.$$

Hence, assuming that the test functions $\boldsymbol{\tau}$ and q satisfy $\boldsymbol{\tau\nu} = -q\boldsymbol{\nu}$ on Σ, we end up with

$$\int_{\Omega_\mathrm{F}} \frac{1}{\omega^2\rho_\mathrm{F}}\nabla p\cdot\nabla q - \int_{\Omega_\mathrm{F}} \frac{1}{\rho_\mathrm{F} c^2} pq + \langle\boldsymbol{\tau\nu},\boldsymbol{u}\rangle_\Sigma - \int_{\Gamma_0} \frac{1}{\rho_\mathrm{F} g} pq = 0. \tag{2.10}$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is imposed weakly through the following equation:

$$\int_{\Omega_\mathrm{S}} \boldsymbol{\sigma} : \boldsymbol{s} = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{2.11}$$

We introduce the closed subspace of $\boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_\mathrm{F})$

$$\boldsymbol{\mathcal{Y}} := \left\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_\mathrm{F}) :\ \boldsymbol{\tau\nu} + q\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Sigma\right\},$$

endowed with the norm

$$\|(\boldsymbol{\tau}, q)\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega_\mathrm{S})} + \|q\|^2_{1,\Omega_\mathrm{F}}.$$

For commodity we will also denote the Hilbertian product norm in $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ by

$$|||((\boldsymbol{\tau}, q), \boldsymbol{s})|||^2 := \|(\boldsymbol{\tau}, q)\|^2 + \|\boldsymbol{s}\|^2_{0,\Omega_\mathrm{S}}.$$

For $(\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$, $\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}$, and $\boldsymbol{v} \in \mathrm{L}^2(\Omega_\mathrm{S})^n$, we introduce the bounded bilinear forms

$$\begin{aligned}
a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= \int_{\Omega_\mathrm{S}} \frac{1}{\rho_\mathrm{S}}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega_\mathrm{F}} \frac{1}{\rho_\mathrm{F}}\nabla p\cdot\nabla q,\\
d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= \int_{\Omega_\mathrm{S}} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_{\Omega_\mathrm{F}} \frac{1}{\rho_\mathrm{F} c^2} pq + \int_{\Gamma_0} \frac{1}{\rho_\mathrm{F} g} pq,\\
b((\boldsymbol{\tau}, q), \boldsymbol{s}) &:= \int_{\Omega_\mathrm{S}} \boldsymbol{\tau} : \boldsymbol{s},\\
A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)),\\
B((\boldsymbol{\tau}, q), (\boldsymbol{s}, \boldsymbol{v})) &:= b((\boldsymbol{\tau}, q), \boldsymbol{s}) + \int_{\Omega_\mathrm{S}} \mathbf{div}\,\boldsymbol{\tau}\cdot\boldsymbol{v}.
\end{aligned}$$

Then, subtracting (2.9) from (2.10) and imposing (2.11), we arrive at the following variational eigenvalue problem in which $\lambda := \omega^2$.

Problem 1. *Find* $\lambda \in \mathbb{R}$, $(\boldsymbol{\sigma}, p) \in \boldsymbol{\mathcal{Y}}$, *and* $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$ *such that* $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \neq \mathbf{0}$ *and*

$$\begin{aligned} a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &= \lambda \left[d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) \right] \qquad && \forall (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}, \\ \lambda b((\boldsymbol{\sigma}, p), \boldsymbol{s}) &= 0 && \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \end{aligned}$$

Notice that the symmetry constraint (2.11) has been multiplied by the eigenvalue λ to obtain a symmetric variational eigenvalue problem. Therefore, the symmetry of the stress $\boldsymbol{\sigma}$ is lost for $\lambda = 0$, which is an eigenvalue of Problem 1. However, this is not relevant in practice, because $\lambda = 0$ would be present as an spurious eigenvalue even though the last equation were not multiplied by λ.

By means of a shift argument, this eigenvalue problem can be rewritten as follows.

Problem 2. *Find* $\lambda \in \mathbb{R}$, $(\boldsymbol{\sigma}, p) \in \boldsymbol{\mathcal{Y}}$, *and* $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$ *such that* $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \neq \mathbf{0}$ *and*

$$\begin{aligned} A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) &= (\lambda + 1) \left[d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) \right] \\ b((\boldsymbol{\sigma}, p), \boldsymbol{s}) &= (\lambda + 1)\, b((\boldsymbol{\sigma}, p), \boldsymbol{s}) \end{aligned}$$

for all $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ *and* $\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}$.

The solution operator corresponding to this eigenvalue problem is

$$\begin{aligned} \boldsymbol{T} :\ & \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}, \\ & ((\boldsymbol{F}, f), \boldsymbol{g}) \longmapsto \boldsymbol{T}((\boldsymbol{F}, f), \boldsymbol{g}) := ((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*), \end{aligned}$$

where $((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*)$ is the solution of the following source problem:

$$A((\boldsymbol{\sigma}^*, p^*), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}^*) = d((\boldsymbol{F}, f), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{g}) \quad \forall (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}, \tag{2.12}$$

$$b((\boldsymbol{\sigma}^*, p^*), \boldsymbol{s}) = b((\boldsymbol{F}, f), \boldsymbol{s}) \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{2.13}$$

In order to show that this problem is well posed, we begin by noticing that the identity

$$\mathcal{C}^{-1} \boldsymbol{\tau} : \boldsymbol{\tau} = \frac{1}{n\lambda_{\mathrm{S}} + 2\mu_{\mathrm{S}}} (\operatorname{tr} \boldsymbol{\tau})^2 + \frac{1}{2\mu_{\mathrm{S}}} \boldsymbol{\tau}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} \tag{2.14}$$

yields

$$\int_{\Omega_{\mathrm{S}}} \mathcal{C}^{-1} \boldsymbol{\tau} : \boldsymbol{\tau} + \int_{\Omega_{\mathrm{S}}} \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\tau} \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} \geq \frac{1}{2\mu_S} \left\| \boldsymbol{\tau}^{\mathtt{D}} \right\|_{0,\Omega_{\mathrm{S}}}^2 + \frac{1}{\rho_S} \left\| \operatorname{\mathbf{div}} \boldsymbol{\tau} \right\|_{0,\Omega_{\mathrm{S}}}^2 \tag{2.15}$$

for all $\boldsymbol{\tau} \in \mathrm{H}(\operatorname{\mathbf{div}}; \Omega_{\mathrm{S}})$. This inequality is the starting point in the proof of the ellipticity of $A(\cdot, \cdot)$ on $\boldsymbol{\mathcal{Y}}$.

Lemma 2.1. *There exists a constant* $\alpha > 0$ *independent of* λ_{S} *such that*

$$A((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)) \geq \alpha \left\| (\boldsymbol{\tau}, q) \right\|^2 \qquad \forall (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}.$$

Proof. The result follows easily from (2.15) and [23, Lemma 2.1] (see also [17, Lemma 2.2]). □

The following inf-sup condition will be repeatedly used in the forthcoming analysis.

Lemma 2.2. *There exists a constant* $\beta > 0$ *such that*

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}} \frac{B((\boldsymbol{\tau}, q), (\boldsymbol{s}, \boldsymbol{v}))}{\left\| (\boldsymbol{\tau}, q) \right\|} \geq \beta \left(\left\| \boldsymbol{s} \right\|_{0,\Omega_{\mathrm{S}}} + \left\| \boldsymbol{v} \right\|_{0,\Omega_{\mathrm{S}}} \right) \qquad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n.$$

Proof. The inclusion $\mathcal{W}_\Sigma \times \{0\} \subset \mathcal{Y}$ yields

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},q)\in\mathcal{Y}} \frac{B((\boldsymbol{\tau},q),(\boldsymbol{s},\boldsymbol{v}))}{\|(\boldsymbol{\tau},q)\|} \geq \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{W}_\Sigma} \frac{B((\boldsymbol{\tau},0),(\boldsymbol{s},\boldsymbol{v}))}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}}.$$

The result is then a direct consequence of the inf-sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{W}_\Sigma} \frac{\displaystyle\int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\tau}\cdot\boldsymbol{v} + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\tau} : \boldsymbol{s}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}} \geq \beta\left(\|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{v}\|_{0,\Omega_{\mathrm{S}}}\right) \qquad \forall(\boldsymbol{s},\boldsymbol{v}) \in \mathcal{Q}\times \mathrm{L}^2(\Omega_{\mathrm{S}})^n, \tag{2.16}$$

which can be found, for instance, in [12]. □

In particular, we deduce that b satisfies the inf-sup condition for the pair $\{\mathcal{Y},\mathcal{Q}\}$ with the same constant β appearing in (2.16). Moreover, Lemma 2.1 shows that $A(\cdot,\cdot)$ is elliptic in particular on $\ker(b)$. Hence, the Babuška-Brezzi theory implies that the linear operator $\boldsymbol{T}$ is well defined and bounded. Moreover, the norm of this operator remains bounded in the nearly incompressible case (i.e., when $\lambda_{\mathrm{S}} \to \infty$). Notice that $(\lambda,(\boldsymbol{\sigma},p),\boldsymbol{r}) \in \mathbb{R}\times\mathcal{Y}\times\mathcal{Q}$ solves Problem 2 if and only if $(1/(1+\lambda),((\boldsymbol{\sigma},p),\boldsymbol{r}))$, is an eigenpair of $\boldsymbol{T}$, i.e., if and only if $((\boldsymbol{\sigma},p),\boldsymbol{r}) \neq \mathbf{0}$ and

$$\boldsymbol{T}((\boldsymbol{\sigma},p),\boldsymbol{r}) = \frac{1}{1+\lambda}\left((\boldsymbol{\sigma},p),\boldsymbol{r}\right).$$

Our description of the spectrum of the solution operator begins with the identification of the kernel of $\boldsymbol{I}-\boldsymbol{T}$, where $\boldsymbol{I}$ is the identity in $\mathcal{Y}\times\mathcal{Q}$. Let $\mathcal{Y}_{\mathbb{R}}$ be the closed subspace of $\mathcal{Y}$ given by

$$\mathcal{Y}_{\mathbb{R}} := \{(\boldsymbol{\tau},\xi) \in \mathcal{W}\times\mathbb{R} :\ \boldsymbol{\tau\nu} + \xi\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\}.$$

Then, it is straightforward to check that

$$\ker(a) = \{(\boldsymbol{\tau},\xi) \in \mathcal{Y}_{\mathbb{R}} :\ \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega_{\mathrm{S}}\}.$$

By virtue of the definition of $\boldsymbol{T}$, we have that $\boldsymbol{T}|_{\ker(a)\times\mathcal{Q}} :\ [\ker(a)\times\mathcal{Q}] \to [\ker(a)\times\mathcal{Q}]$ reduces to the identity. Thus, $\mu = 1$ is an eigenvalue of $\boldsymbol{T}$. Moreover, if $((\boldsymbol{\sigma},p),\boldsymbol{r})$ is an associated eigenfunction, then

$$a((\boldsymbol{\sigma},p),(\boldsymbol{\tau},q)) = 0 \qquad \forall(\boldsymbol{\tau},q)\in\mathcal{Y},$$

which shows that $((\boldsymbol{\sigma},p),\boldsymbol{r}) \in \ker(a)\times\mathcal{Q}$. Therefore, we have proved the following result.

Lemma 2.3. *$\mu = 1$ is an eigenvalue of $\boldsymbol{T}$ with associated eigenspace* $\ker(a)\times\mathcal{Q}$.

3. The auxiliary operator $\boldsymbol{P}$

Given $q \in \mathrm{H}^1(\Omega_{\mathrm{F}})$, let $\widehat{\boldsymbol{u}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^n$ and $\widehat{\boldsymbol{\sigma}} \in \mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})$ be the solution of the following linear elasticity problem:

$$\begin{aligned}
-\mathbf{div}\,\widehat{\boldsymbol{\sigma}} &= \mathbf{0} && \text{in } \Omega_{\mathrm{S}},\\
\widehat{\boldsymbol{\sigma}} &= \mathcal{C}\boldsymbol{\varepsilon}(\widehat{\boldsymbol{u}}) && \text{in } \Omega_{\mathrm{S}},\\
\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu} &= q\boldsymbol{\nu} && \text{on } \Sigma,\\
\widehat{\boldsymbol{u}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{D}},\\
\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{N}}.
\end{aligned}$$

Let us define the bounded linear operator

$$\boldsymbol{E}:\ \mathrm{H}^1(\Omega_{\mathrm{F}}) \longrightarrow \boldsymbol{\mathcal{W}},\qquad q \longmapsto \boldsymbol{E}q := -\widehat{\boldsymbol{\sigma}},$$

which provides a symmetric divergence-free extension of a given pressure field q to the solid domain. Classical regularity results for the elasticity equations in polyhedral (polygonal) domains (cf. [14, 20]) ensure the existence of $t_{\mathrm{S}} \in (0,1]$, which depends on the geometry of Ω_{S} and the Lamé coefficients, such that $\boldsymbol{E}q \in \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$ and

$$\|\boldsymbol{E}q\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} \le C\,\|q\|_{1,\Omega_{\mathrm{F}}} \qquad \forall q \in \mathrm{H}^1(\Omega_{\mathrm{F}}). \tag{3.1}$$

We also define

$$\widehat{\boldsymbol{E}}q := (\boldsymbol{E}q, q)\,,$$

which clearly belongs to $\boldsymbol{\mathcal{Y}}$.

In what follows, $\bar{q} := q - \frac{1}{|\Omega_{\mathrm{F}}|}\int_{\Omega_{\mathrm{F}}} q$ stands for the zero mean value component of functions q from $\mathrm{H}^1(\Omega_{\mathrm{F}})$. We introduce the auxiliary operator

$$\boldsymbol{P}:\ \boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}},\qquad ((\boldsymbol{\sigma},p),\boldsymbol{r}) \longmapsto \boldsymbol{P}((\boldsymbol{\sigma},p),\boldsymbol{r}) := ((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}}),$$

where $(\widetilde{\boldsymbol{\sigma}},\widetilde{p}) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}} + \widehat{\boldsymbol{E}}\bar{p}$ and $(\widetilde{\boldsymbol{r}},\widetilde{\boldsymbol{u}}) \in \boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega_{\mathrm{S}})^n$ solve the problem

$$d((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),(\boldsymbol{\tau},\xi)) + B((\boldsymbol{\tau},\xi),(\widetilde{\boldsymbol{r}},\widetilde{\boldsymbol{u}})) = 0 \qquad \forall(\boldsymbol{\tau},\xi)\in\boldsymbol{\mathcal{Y}}_{\mathbb{R}}, \tag{3.2}$$

$$B((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),(\boldsymbol{s},\boldsymbol{v})) = \int_{\Omega_{\mathrm{S}}} \operatorname{\mathbf{div}}\boldsymbol{\sigma}\cdot\boldsymbol{v} \quad \forall(\boldsymbol{s},\boldsymbol{v})\in\boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega_{\mathrm{S}})^n. \tag{3.3}$$

We notice that, by definition, $\boldsymbol{P}((\boldsymbol{\sigma},p),\boldsymbol{r})$ is independent of the last variable $\boldsymbol{r}$.

Taking into account that $\boldsymbol{E}\bar{p}$ is symmetric and divergence-free in Ω_{S}, we have that (3.2)–(3.3) hold true if and only if $(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c}) := (\widetilde{\boldsymbol{\sigma}},\widetilde{p}) - \widehat{\boldsymbol{E}}\bar{p} \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ and $(\widetilde{\boldsymbol{r}},\widetilde{\boldsymbol{u}}) \in \boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega_{\mathrm{S}})^n$ satisfy

(3.4)

$$d((\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c}),(\boldsymbol{\tau},\xi)) + B((\boldsymbol{\tau},\xi),(\widetilde{\boldsymbol{r}},\widetilde{\boldsymbol{u}})) = -d(\widehat{\boldsymbol{E}}\bar{p},(\boldsymbol{\tau},\xi)) \quad \forall(\boldsymbol{\tau},\xi)\in\boldsymbol{\mathcal{Y}}_{\mathbb{R}},$$

$$B((\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c}),(\boldsymbol{s},\boldsymbol{v})) = \int_{\Omega_{\mathrm{S}}} \operatorname{\mathbf{div}}\boldsymbol{\sigma}\cdot\boldsymbol{v} \qquad \forall(\boldsymbol{s},\boldsymbol{v})\in\boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega_{\mathrm{S}})^n. \tag{3.5}$$

Since $\boldsymbol{\mathcal{W}}_\Sigma\times\{0\}\subset\boldsymbol{\mathcal{Y}}_{\mathbb{R}}$, we deduce from (2.16) that the bilinear form B satisfies the inf-sup condition for the pair $\left\{\boldsymbol{\mathcal{Y}}_{\mathbb{R}},\boldsymbol{\mathcal{Q}}\times\mathrm{L}^2(\Omega_{\mathrm{S}})^n\right\}$. Moreover, the ellipticity in the kernel of d is an immediate consequence of Lemma 2.1 and the fact that $\ker(B)\cap\boldsymbol{\mathcal{Y}}_{\mathbb{R}}\subset\ker(a)$. Therefore, the Babuška-Brezzi theory implies that $\boldsymbol{P}$ is a well posed bounded linear operator.

Moreover, it is straightforward to check that the solution $((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}}))$ to (3.2)–(3.3) satisfies

$$-\operatorname{\mathbf{div}}\widetilde{\boldsymbol{\sigma}} = -\operatorname{\mathbf{div}}\boldsymbol{\sigma} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{3.6}$$

$$\widetilde{\boldsymbol{\sigma}} = \mathcal{C}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) \qquad \text{in } \Omega_{\mathrm{S}}, \tag{3.7}$$

$$\widetilde{\boldsymbol{\sigma}}\boldsymbol{\nu} = -\widetilde{p}\boldsymbol{\nu} \qquad \text{on } \Sigma, \tag{3.8}$$

$$\widetilde{\boldsymbol{u}} = \boldsymbol{0} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{3.9}$$

$$\widetilde{\boldsymbol{\sigma}}\boldsymbol{\nu} = \boldsymbol{0} \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{3.10}$$

and

$$\widetilde{\boldsymbol{r}} = \frac{1}{2}\left[\nabla\widetilde{\boldsymbol{u}} - (\nabla\widetilde{\boldsymbol{u}})^{\mathtt{t}}\right] \qquad \text{in } \Omega_{\mathrm{S}}.$$

Owing to the regularity result for the classical elasticity problem (see again [14, 20]), we know that $\widetilde{\boldsymbol{u}} \in \mathrm{H}^{1+t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^n$ for the same $t_{\mathrm{S}} \in (0,1]$ as in (3.1) and there exists $C > 0$, independent of $\boldsymbol{\sigma}$ and p, such that

$$\|\widetilde{\boldsymbol{u}}\|_{1+t_{\mathrm{S}},\Omega_{\mathrm{S}}} \leq C\left(\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}} + \|\widetilde{p}\|_{1,\Omega_{\mathrm{F}}}\right) \leq C\left(\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}} + \|p\|_{1,\Omega_{\mathrm{F}}}\right).$$

The following lemma summarizes these additional regularity results.

Lemma 3.1. *There exists $C > 0$ such that, for all $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$,*

$$\|\widetilde{\boldsymbol{\sigma}}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{u}}\|_{1+t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{r}}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\widetilde{p}\|_{1,\Omega_{\mathrm{F}}} \leq C\left(\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}} + \|p\|_{1,\Omega_{\mathrm{F}}}\right),$$

where $((\widetilde{\boldsymbol{\sigma}}, \widetilde{p}), (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \left[\boldsymbol{\mathcal{Y}}_{\mathbb{R}} + \widehat{\boldsymbol{E}}\bar{p}\right] \times \left[\mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \boldsymbol{\mathcal{Q}}\right]$ is the solution to (3.2)–(3.3)*. Consequently, $\boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \subset \left[\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n} \times \mathrm{H}^1(\Omega_{\mathrm{F}})\right] \times \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$.*

We recall that, by construction, p and $\widetilde{p}$ have the same zero mean value component, i.e., $\widetilde{p} - \frac{1}{|\Omega_{\mathrm{F}}|}\int_{\Omega_{\mathrm{F}}} \widetilde{p} = \bar{p}$. Moreover, according to (3.6), $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}$ in Ω_{S}. It follows that the operator $\boldsymbol{P}$ is idempotent and that its kernel is given by $\ker(\boldsymbol{P}) = \ker(a) \times \boldsymbol{\mathcal{Q}}$. Therefore, being $\boldsymbol{P}$ a projector, we have that $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} = [\ker(a) \times \boldsymbol{\mathcal{Q}}] \oplus \boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}})$. Our aim now is to show that $\boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}})$ is an invariant subspace of $\boldsymbol{T}$. To this end, let us us rewrite the equations of Problem 2 as follows:

$$\mathbb{A}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) = (\lambda + 1)\,\mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) \qquad \forall((\boldsymbol{\tau}, q), \boldsymbol{s}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}},$$

where $\mathbb{A}$ and $\mathbb{B}$ are the bounded bilinear forms in $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ defined by

$$\mathbb{A}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) := A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) + b((\boldsymbol{\sigma}, p), \boldsymbol{s}),$$

$$\mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) := d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) + b((\boldsymbol{\sigma}, p), \boldsymbol{s}).$$

We introduce the orthogonal complement to $\ker(a) \times \boldsymbol{\mathcal{Q}}$ in $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ with respect to the bilinear form $\mathbb{B}$ by

$$[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} := \{((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} :\ \mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) = 0 \quad \forall((\boldsymbol{\tau}, q), \boldsymbol{s}) \in \ker(a) \times \boldsymbol{\mathcal{Q}}\}.$$

Our next goal is to prove that $[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} = \boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}})$. The first step is the following result.

Lemma 3.2. $[\ker(a) \times \boldsymbol{\mathcal{Q}}] \cap [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} = \{\mathbf{0}\}$.

Proof. Let $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in [\ker(a) \times \boldsymbol{\mathcal{Q}}] \cap [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$. We have that $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \ker(a) \times \boldsymbol{\mathcal{Q}}$ solves

$$\begin{aligned} d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) &= 0 \qquad \forall(\boldsymbol{\tau}, q) \in \ker(a), \\ b((\boldsymbol{\sigma}, p), \boldsymbol{s}) &= 0 \qquad \forall\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \end{aligned}$$

As a consequence of Lemmas 2.1 and 2.2, d is elliptic on $\ker(a)$ and b satisfies the inf-sup condition for the pair $\{\ker(a), \boldsymbol{\mathcal{Q}}\}$. Therefore, $((\boldsymbol{\sigma}, p), \boldsymbol{r}) = \mathbf{0}$ is the unique solution of this problem, which yields the result. □

Lemma 3.3. $\boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) = [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$.

Proof. Let us first show that $\boldsymbol{P}(\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}})\subset[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$. Given $((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}})\in\boldsymbol{P}(\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}})$, by virtue of (3.2)–(3.3),

$$\mathbb{B}(((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}}),((\boldsymbol{\tau},q),\boldsymbol{s}))=0\qquad\forall((\boldsymbol{\tau},q),\boldsymbol{s})\in\ker(a)\times\boldsymbol{\mathcal{Q}},$$

which means that $((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$.

Conversely, let $((\boldsymbol{\sigma},p),\boldsymbol{r})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ and let $((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}})=\boldsymbol{P}((\boldsymbol{\sigma},p),\boldsymbol{r})$. We have just proved that $((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$, so that $((\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{p}-p),\widetilde{\boldsymbol{r}}-\boldsymbol{r})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$, too. Moreover, from the definition of $\boldsymbol{P}$, we have that $\mathbf{div}(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})=\mathbf{0}$ in Ω_{S} and $\widetilde{p}-p\in\mathbb{R}$; in other words, $((\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{p}-p),\widetilde{\boldsymbol{r}}-\boldsymbol{r})\in\ker(a)\times\boldsymbol{\mathcal{Q}}$. Hence, according to Lemma 3.2, $((\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma},\widetilde{p}-p),\widetilde{\boldsymbol{r}}-\boldsymbol{r})=\mathbf{0}$, so that $((\boldsymbol{\sigma},p),\boldsymbol{r})=((\widetilde{\boldsymbol{\sigma}},\widetilde{p}),\widetilde{\boldsymbol{r}})=\boldsymbol{P}((\boldsymbol{\sigma},p),\boldsymbol{r})\in\boldsymbol{P}(\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}})$ and we conclude the proof. □

4. The spectral characterization of $\boldsymbol{T}$

The first result of this section concerns a regularity result for $\boldsymbol{T}|_{[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}}$.

Proposition 4.1. *The subspace $[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ is invariant for $\boldsymbol{T}$. Moreover, there exist $t_{\mathrm{S}},t_{\mathrm{F}}\in(0,1]$ such that, for all $((\boldsymbol{F},f),\boldsymbol{g})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$, if $((\boldsymbol{\sigma}^*,p^*),\boldsymbol{r}^*)=\boldsymbol{T}((\boldsymbol{F},f),\boldsymbol{g})$, then $\boldsymbol{\sigma}^*,\boldsymbol{r}^*\in\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$, $\mathbf{div}\,\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega_{\mathrm{S}})^n$, $p^*\in\mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})$, and there exists $C>0$ such that*

$$\|\boldsymbol{\sigma}^*\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}}+\|\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega_{\mathrm{S}}}+\|\boldsymbol{r}^*\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}}+\|p^*\|_{1+t_{\mathrm{F}},\Omega_{\mathrm{F}}}\le C\,\|((\boldsymbol{F},f),\boldsymbol{g})\|.$$

Consequently, the operator $\boldsymbol{T}|_{[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}}:\ [\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}\to[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ is compact.

Proof. It is straightforward to check that $\boldsymbol{T}([\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}})\subset[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ (see [23, Proposition A.1]), which guarantees that $\boldsymbol{T}|_{[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}}:\ [\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}\to[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ is correctly defined.

Let $((\boldsymbol{F},f),\boldsymbol{g})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ and $((\boldsymbol{\sigma}^*,p^*),\boldsymbol{r}^*)=\boldsymbol{T}((\boldsymbol{F},f),\boldsymbol{g})$. We already know from Lemmas 3.3 and 3.1 that $((\boldsymbol{\sigma}^*,p^*),\boldsymbol{r}^*)\in\boldsymbol{T}([\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}})\subset[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}=\boldsymbol{P}(\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}})\subset\left[\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}\times\mathrm{H}^1(\Omega_{\mathrm{F}})\right]\times\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$. On the other hand, testing (2.12) with $(\boldsymbol{\tau},0),(\mathbf{0},q)\in\mathcal{D}(\Omega_{\mathrm{S}})^{n\times n}\times\mathcal{D}(\Omega_{\mathrm{F}})\subset\boldsymbol{\mathcal{Y}}$ yields

$$\begin{aligned}\mathcal{C}^{-1}\boldsymbol{\sigma}^*-\nabla\left(\frac{1}{\rho_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}^*\right)+\boldsymbol{r}^*=\mathcal{C}^{-1}\boldsymbol{F}+\boldsymbol{g}\qquad&\text{in }\Omega_{\mathrm{S}},\\ -c^2\Delta p^*+p^*=f\qquad&\text{in }\Omega_{\mathrm{F}}.\end{aligned}$$

Then, since ρ_{S} is constant, we have from the first equation that $\mathbf{div}\,\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega_{\mathrm{S}})^n$. Hence, the equation posed in Ω_{F} and the boundary condition

$$\frac{\partial p^*}{\partial\boldsymbol{\nu}}=\begin{cases}\dfrac{1}{g}(f-p)&\text{on }\Gamma_0,\\[2mm]\dfrac{\rho_{\mathrm{F}}}{\rho_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}^*\cdot\boldsymbol{\nu}&\text{on }\Sigma,\end{cases}$$

(obtained by testing this time (2.12) with appropriate $(\boldsymbol{\tau},q)\in\boldsymbol{\mathcal{Y}}$ and integrating by parts) allow us to deduce from classical regularity results for the Poisson problem in polyhedral (polygonal) domains (see again [14, 20]) that there exists $t_{\mathrm{F}}\in(0,1]$ such that $p^*\in\mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})$. Moreover, it is easy to check by using again Lemma 3.1 that the estimate of the proposition holds true.

Finally, the compactness of $\boldsymbol{T}|_{[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}}$ follows from the fact that the space

$$\left\{(\boldsymbol{\sigma}^*,p^*),\boldsymbol{r}^*)\in\left[\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}\times\mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})\right]\times\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}:\ \mathbf{div}\,\boldsymbol{\sigma}^*\in\mathrm{H}^1(\Omega_{\mathrm{S}})^n\right\}$$

is compactly included in $\big[\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})\times\mathrm{H}^1(\Omega_{\mathrm{F}})\big]\times\mathrm{L}^2(\Omega_{\mathrm{S}})^{n\times n}$. □

As shown in [23, Proposition A.2], the following result ensures that the eigenvalues of $\boldsymbol{T}$ are non-defective. Another immediate consequence of this result is that $\mu=0$ is not an eigenvalue of $\boldsymbol{T}$.

Lemma 4.2. *For all non-vanishing* $((\boldsymbol{\sigma},p),\boldsymbol{r})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$

$$\mathbb{A}(((\boldsymbol{\sigma},p),\boldsymbol{r}),((\boldsymbol{\sigma},p),\boldsymbol{r}))\geq\mathbb{B}(((\boldsymbol{\sigma},p),\boldsymbol{r}),((\boldsymbol{\sigma},p),\boldsymbol{r}))>0.$$

Proof. The first inequality follows from the definition of the bilinear forms $\mathbb{A}$ and $\mathbb{B}$ and the fact that a is a positive semi-definite bilinear form. To prove the second, we have that $\boldsymbol{\sigma}$ is symmetric for all $((\boldsymbol{\sigma},p),\boldsymbol{r})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$, because of the definition of this space. Hence, by virtue of (2.14), we have that

$$\begin{aligned}\mathbb{B}(((\boldsymbol{\sigma},p),\boldsymbol{r}),((\boldsymbol{\sigma},p),\boldsymbol{r}))&=\int_{\Omega_{\mathrm{S}}}\mathcal{C}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\int_{\Omega_{\mathrm{F}}}\frac{1}{\rho_{\mathrm{F}}c^2}p^2+\int_{\Gamma_0}\frac{1}{\rho_{\mathrm{F}}g}p^2\\&\geq\min\left\{\frac{n}{n\lambda_{\mathrm{S}}+2\mu_{\mathrm{S}}},\frac{1}{2\mu_{\mathrm{S}}}\right\}\|\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}}^2+\frac{1}{\rho_{\mathrm{F}}c^2}\|p\|_{0,\Omega_{\mathrm{F}}}^2\geq0.\end{aligned}$$

Moreover, the expression above cannot vanish; otherwise $(\boldsymbol{\sigma},p)=\mathbf{0}$ and, hence, $((\boldsymbol{\sigma},p),\boldsymbol{r})\in[\ker(a)\times\boldsymbol{\mathcal{Q}}]\cap[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}=\{\mathbf{0}\}$ (cf. Lemma 3.2). Thus, we conclude the proof. □

We end this section with the spectral characterization of $\boldsymbol{T}$.

Theorem 4.3. *The spectrum of* $\boldsymbol{T}$ *decomposes as follows:* $\mathrm{sp}(\boldsymbol{T})=\{0,1\}\cup\{\mu_k\}_{k\in\mathbb{N}}$, *where:*

i) $\mu=1$ *is an infinite-multiplicity eigenvalue of* $\boldsymbol{T}$ *and its associated eigenspace is* $\ker(a)\times\boldsymbol{\mathcal{Q}}$*;*
ii) $\{\mu_k\}_{k\in\mathbb{N}}\subset(0,1)$ *is a sequence of finite-multiplicity eigenvalues of* $\boldsymbol{T}$ *which converge to* 0 *and the corresponding eigenspaces lie on* $[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$*; moreover, the ascent of each of these eigenvalues is* 1*;*
iii) $\mu=0$ *is not an eigenvalue of* $\boldsymbol{T}$.

Proof. Since $\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}}=[\ker(a)\times\boldsymbol{\mathcal{Q}}]\oplus[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ (cf. Lemmas 3.2 and 3.3), $\boldsymbol{T}|_{\ker(a)\times\boldsymbol{\mathcal{Q}}}:[\ker(a)\times\boldsymbol{\mathcal{Q}}]\to[\ker(a)\times\boldsymbol{\mathcal{Q}}]$ is the identity and $\boldsymbol{T}|_{[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}}:[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}\to[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ is compact (cf. Proposition 4.1), we have that the decomposition of $\mathrm{sp}(\boldsymbol{T})$ follows from the spectral characterization of compact operators. Property (i) was established in Lemma 2.3. Finally, properties (ii) and (iii) follow from Lemma 4.2 and [23, Propositions A1, A2]. □

As an immediate consequence of Proposition 4.1 we have the following additional regularity result for the eigenfunctions of $\boldsymbol{T}$ lying on $[\ker(a)\times\boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$.

Corollary 4.4. *Let* $((\boldsymbol{\sigma},p),\boldsymbol{r})\in\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}}$ *be an eigenfunction of* $\boldsymbol{T}$ *associated to an eigenvalue* $\mu\in(0,1)$*. Then,* $\boldsymbol{\sigma},\boldsymbol{r}\in\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$*,* $\mathbf{div}\,\boldsymbol{\sigma}\in\mathrm{H}^1(\Omega_{\mathrm{S}})^n$*,* $p\in\mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})$*, with* $t_{\mathrm{S}},t_{\mathrm{F}}\in(0,1]$ *as in Proposition 4.1, and*

$$\|\boldsymbol{\sigma}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}}+\|\mathbf{div}\,\boldsymbol{\sigma}\|_{1,\Omega_{\mathrm{S}}}+\|\boldsymbol{r}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}}+\|p\|_{1+t_{\mathrm{F}},\Omega_{\mathrm{F}}}\leq C\,\|((\boldsymbol{\sigma},p),\boldsymbol{r})\|,$$

with $C>0$ *depending on the eigenvalue.*

5. The discrete spaces

Let $\{\mathcal{T}_h(\Omega_{\rm S})\}_{h>0}$ and $\{\mathcal{T}_h(\Omega_{\rm F})\}_{h>0}$ be shape-regular families of triangulations of the polyhedral (polygonal) regions $\bar{\Omega}_{\rm S}$ and $\bar{\Omega}_{\rm F}$, respectively, by tetrahedrons (triangles) T of diameter h_T, with mesh size $h := \max\{h_T : \quad T \in \mathcal{T}_h(\Omega_{\rm S}) \cup \mathcal{T}_h(\Omega_{\rm F})\}$. We assume that the vertices of $\{\mathcal{T}_h(\Omega_{\rm S})\}_{h>0}$ and $\{\mathcal{T}_h(\Omega_{\rm F})\}_{h>0}$ coincide on Σ. In what follows, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^n$, $\mathcal{P}_k(S)$ denotes the space of polynomial functions defined in S of total degree $\leq k$.

We define

$$\begin{aligned}\boldsymbol{\mathcal{W}}_h &:= \left\{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}} : \ \boldsymbol{\tau}_h|_T \in \mathcal{P}_1(T)^{n\times n} \ \forall T \in \mathcal{T}_h(\Omega_{\rm S})\right\},\\ \mathcal{V}_h &:= \left\{q_h \in \mathrm{H}^1(\Omega_{\rm F}) : \ q_h|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h(\Omega_{\rm F})\right\}\end{aligned}$$

and introduce the finite element subspaces of $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Q}}$ given respectively by

$$\begin{aligned}\boldsymbol{\mathcal{Y}}_h &:= \{(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{W}}_h \times \mathcal{V}_h : \ \boldsymbol{\tau}_h\boldsymbol{\nu} + q_h\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\} = (\boldsymbol{\mathcal{W}}_h \times \mathcal{V}_h) \cap \boldsymbol{\mathcal{Y}},\\ \boldsymbol{\mathcal{Q}}_h &:= \left\{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}} : \ \boldsymbol{s}_h|_T \in \mathcal{P}_0(T)^{n\times n} \ \forall T \in \mathcal{T}_h(\Omega_{\rm S})\right\}.\end{aligned}$$

In addition, for the analysis below we will also use the space

$$\boldsymbol{\mathcal{U}}_h := \left\{\boldsymbol{v}_h \in \mathrm{L}^2(\Omega_{\rm S})^n : \ \boldsymbol{v}_h|_T \in \mathcal{P}_0(T)^n \ \forall T \in \mathcal{T}_h(\Omega_{\rm S})\right\}.$$

Notice that $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$ is the lowest-order mixed finite element of the family introduced for linear elasticity by Arnold, Falk and Winther (see [1, 2]).

Let us now recall some well-known approximation properties of the finite element spaces introduced above. Given $s \in (0,1]$, let $\boldsymbol{\Pi}_h : \mathrm{H}^s(\Omega_{\rm S})^{n\times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$ be the usual lowest-order Brezzi-Douglas-Marini (BDM) interpolation operator (see [13]), which is characterized by the identities

$$\int_F (\boldsymbol{\Pi}_h\boldsymbol{\tau})\,\boldsymbol{\nu}_F \cdot \boldsymbol{\zeta} = \int_F \boldsymbol{\tau}\boldsymbol{\nu}_F \cdot \boldsymbol{\zeta} \qquad \forall \boldsymbol{\zeta} \in \mathcal{P}_1(F)^n$$

for all faces (edges) F of elements $T \in \mathcal{T}_h(\Omega_{\rm S})$, with $\boldsymbol{\nu}_F$ being a unit vector normal to the face (edge) F. It is well known that $\boldsymbol{\Pi}_h$ is a bounded linear operator and that the following commuting diagram property holds true (cf. [13]):

$$\mathbf{div}(\boldsymbol{\Pi}_h\boldsymbol{\tau}) = \boldsymbol{L}_h(\mathbf{div}\,\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\Omega_{\rm S})^{n\times n} \cap \mathrm{H}(\mathbf{div};\Omega_{\rm S}), \tag{5.1}$$

where $\boldsymbol{L}_h : \ \mathrm{L}^2(\Omega_{\rm S})^n \to \boldsymbol{\mathcal{U}}_h$ is the $\mathrm{L}^2(\Omega_{\rm S})^n$-orthogonal projector. In addition, it is well-known that the arguments leading to [21, Theorem 3.16] allow showing that there exists $C > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega_{\rm S}} \leq Ch^s\left(\|\boldsymbol{\tau}\|_{s,\Omega_{\rm S}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega_{\rm S}}\right) \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\Omega_{\rm S})^{n\times n} \cap \mathrm{H}(\mathbf{div};\Omega_{\rm S}). \tag{5.2}$$

Finally, we denote by $\boldsymbol{R}_h : \ \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $\mathrm{L}^2(\Omega_{\rm S})^{n\times n}$-norm and by $\pi_h : \ \mathrm{H}^1(\Omega_{\rm F}) \to \mathcal{V}_h$ the orthogonal projector with respect to the $\mathrm{H}^1(\Omega_{\rm F})$-norm. Then, for any $s \in (0,1]$, we have

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\rm S})} \leq Ch^s\,\|\boldsymbol{\tau}\|_{\mathrm{H}^s(\mathbf{div};\Omega_{\rm S})} \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\mathbf{div};\Omega_{\rm S}) \cap \boldsymbol{\mathcal{W}}, \tag{5.3}$$

$$\|\boldsymbol{r} - \boldsymbol{R}_h\boldsymbol{r}\|_{0,\Omega_{\rm S}} \leq Ch^s\,\|\boldsymbol{r}\|_{s,\Omega_{\rm S}} \qquad \forall \boldsymbol{r} \in \mathrm{H}^s(\Omega_{\rm S})^{n\times n} \cap \boldsymbol{\mathcal{Q}}, \tag{5.4}$$

$$\|\boldsymbol{v} - \boldsymbol{L}_h\boldsymbol{v}\|_{0,\Omega_{\rm S}} \leq Ch^s\,\|\boldsymbol{v}\|_{s,\Omega_{\rm S}} \qquad \forall \boldsymbol{v} \in \mathrm{H}^s(\Omega_{\rm S})^n, \tag{5.5}$$

$$\|q - \pi_h q\|_{1,\Omega_{\rm F}} \leq Ch^s\,\|q\|_{1+s,\Omega_{\rm F}} \qquad \forall q \in \mathrm{H}^{1+s}(\Omega_{\rm F}). \tag{5.6}$$

Notice that (5.3) is actually a straightforward consequence of (5.2), (5.1), and (5.5).

In what follows, we gather some of the technical tools that will be used in the subsequent analysis. Let $\boldsymbol{E}$ be the extension operator defined in Section 3. The following estimate holds true.

Lemma 5.1. *There exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\mathrm{S}}} \leq C \|q\|_{1,\Omega_{\mathrm{F}}} \qquad \forall q \in \mathrm{H}^1(\Omega_{\mathrm{F}}),$$
$$\|\boldsymbol{E} q - \boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\mathrm{S}}} \leq C h^{t_{\mathrm{S}}} \|q\|_{1,\Omega_{\mathrm{F}}} \qquad \forall q \in \mathrm{H}^1(\Omega_{\mathrm{F}}).$$

Proof. First notice that $\boldsymbol{E} q \in \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n} \cap \mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})$ for all $q \in \mathrm{H}^1(\Omega_{\mathrm{F}})$ and (3.1) holds true. Hence, $\boldsymbol{\Pi}_h \boldsymbol{E} q$ is well defined and the first estimate follows from the boundedness of $\boldsymbol{\Pi}_h : \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$. For the second estimate we use (5.2) with $s = t_{\mathrm{S}}$, (3.1) again, and the fact that $\boldsymbol{E} q$ is divergence-free in Ω_{S}. □

Next, we introduce the discrete counterparts of $\boldsymbol{E}$ and $\widehat{\boldsymbol{E}}$, defined for any $q \in \mathrm{H}^1(\Omega_{\mathrm{F}})$ by

$$\boldsymbol{E}_h q := \boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q) \in \boldsymbol{\mathcal{W}}_h \qquad \text{and} \qquad \widehat{\boldsymbol{E}}_h q := (\boldsymbol{E}_h q, \pi_h q)\,. \tag{5.7}$$

It is clear that $\widehat{\boldsymbol{E}}_h q \in \boldsymbol{\mathcal{Y}}_h$ for all $q \in \mathrm{H}^1(\Omega_{\mathrm{F}})$. Moreover, we have the following result.

Lemma 5.2. *There exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{E} q - \boldsymbol{E}_h q\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})} \leq C \left(h^{t_{\mathrm{S}}} \|q\|_{1,\Omega_{\mathrm{F}}} + \|q - \pi_h q\|_{1,\Omega_{\mathrm{F}}} \right) \qquad \forall q \in \mathrm{H}^1(\Omega_{\mathrm{F}}).$$

Proof. Since $\mathbf{div}\, \boldsymbol{E} q = \mathbf{div}\, \boldsymbol{E}_h q = \mathbf{0}$, we only have to estimate the $\mathrm{L}^2(\Omega_{\mathrm{S}})$-norm. To this end, we add and subtract $\boldsymbol{\Pi}_h \boldsymbol{E} q$ and use the triangle inequality to obtain

$$\|\boldsymbol{E} q - \boldsymbol{E}_h q\|_{0,\Omega_{\mathrm{S}}} \leq \|\boldsymbol{E} q - \boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\Pi}_h \boldsymbol{E}(q - \pi_h q)\|_{0,\Omega_{\mathrm{S}}}\,.$$

Hence, the proof follows from the two estimates in Lemma 5.1. □

Our aim now is to show that any $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ sufficiently smooth can be well approximated from $\boldsymbol{\mathcal{Y}}_h$. We define the approximation separately on each subdomain. In Ω_{F} we just take $q_h := \pi_h q$, whereas in Ω_{S} we correct the BDM interpolant $\boldsymbol{\Pi}_h \boldsymbol{\tau}$ in order to obtain a tensor $\boldsymbol{\tau}_h$ satisfying the constraint $\boldsymbol{\tau}_h \boldsymbol{\nu} + q_h \boldsymbol{\nu} = \mathbf{0}$ on Σ from the definition of $\boldsymbol{\mathcal{Y}}_h$. We do this as is shown in the following lemma.

Lemma 5.3. *Let $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\tau} \in \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$ and let*

$$(\boldsymbol{\tau}_h, q_h) := (\boldsymbol{\Pi}_h \boldsymbol{\tau} + (\boldsymbol{E}_h q - \boldsymbol{\Pi}_h \boldsymbol{E} q)\,, \pi_h q)\,.$$

Then, $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ and

$$\|(\boldsymbol{\tau}, q) - (\boldsymbol{\tau}_h, q_h)\| \leq C \left[\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})} + \|q - \pi_h q\|_{1,\Omega_{\mathrm{F}}} \right].$$

Proof. First notice that

$$\boldsymbol{\tau}_h \boldsymbol{\nu} + q_h \boldsymbol{\nu} = \boldsymbol{\Pi}_h (\boldsymbol{\tau} - \boldsymbol{E} q)\, \boldsymbol{\nu} + (\boldsymbol{E}_h q + \pi_h q)\, \boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Sigma.$$

In fact, from the definition of $\boldsymbol{E}_h$ (cf. (5.7)), it is clear that $(\boldsymbol{E}_h q + \pi_h q)\, \boldsymbol{\nu}$ vanishes on Σ and so does $\boldsymbol{\Pi}_h (\boldsymbol{\tau} - \boldsymbol{E} q)\, \boldsymbol{\nu}$, because $(\boldsymbol{\tau} - \boldsymbol{E} q)\, \boldsymbol{\nu} = \boldsymbol{\tau}\boldsymbol{\nu} + q\boldsymbol{\nu} = \mathbf{0}$ on Σ for $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$.

To prove the estimate, we use again the definition of $\boldsymbol{E}_h$ to write

$$\begin{aligned}\|(\boldsymbol{\tau}, q) - (\boldsymbol{\tau}_h, q_h)\| &\\ \leq \|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})} &+ \|\boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q - q)\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})} + \|q - \pi_h q\|_{1,\Omega_{\mathrm{F}}}\,.\end{aligned}$$

Then, the result follows from the first inequality in Lemma 5.1 and the fact that $\boldsymbol{E}(\pi_h q - q)$ is divergence-free and, hence, so is $\boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q - q)$. $\square$

6. The discrete problem

The discrete counterpart of Problem 2 reads as follows.

Problem 3. *Find* $\lambda_h \in \mathbb{R}$, $(\boldsymbol{\sigma}_h, p_h) \in \boldsymbol{\mathcal{Y}}_h$, *and* $\boldsymbol{r}_h \in \boldsymbol{\mathcal{Q}}_h$ *such that* $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \neq \mathbf{0}$ *and*

$$\begin{aligned} &A((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h) \\ &\qquad\qquad = (\lambda_h + 1)\left[d((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h)\right], \\ &b((\boldsymbol{\sigma}_h, p_h), \boldsymbol{s}_h) = (\lambda_h + 1)\, b((\boldsymbol{\sigma}_h, p_h), \boldsymbol{s}_h) \end{aligned}$$

for all $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ *and* $\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$.

The discrete version of the operator $\boldsymbol{T}$ is then given by

$$\begin{aligned} \widetilde{\boldsymbol{T}}_h :\ &\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}, \\ &((\boldsymbol{F}, f), \boldsymbol{g}) \longmapsto \widetilde{\boldsymbol{T}}_h((\boldsymbol{F}, f), \boldsymbol{g}) := ((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*), \end{aligned}$$

where $((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ is the solution of the following discrete source problem:

$$\begin{aligned} A((\boldsymbol{\sigma}_h^*, p_h^*), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h^*) &= d((\boldsymbol{F}, f), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{g}), \\ b((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{s}_h) &= b((\boldsymbol{F}, f), \boldsymbol{s}_h) \end{aligned}$$

for all $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ and $\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$. We can use the classical Babuška-Brezzi theory to prove that $\widetilde{\boldsymbol{T}}_h$ is well defined and bounded uniformly with respect to h. In fact, we already know from Lemma 2.1 that A is elliptic on the whole $\boldsymbol{\mathcal{Y}}$, whereas the discrete inf-sup condition

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h} \frac{b((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)}{\|(\boldsymbol{\tau}_h, q_h)\|} \geq \beta \|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$$

follows immediately (as shown in Lemma 2.2) from the following one provided by [1, Theorem 11.9]: There exists $\beta^* > 0$, independent of h, such that

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h \cap \boldsymbol{\mathcal{W}}_\Sigma} \frac{B((\boldsymbol{\tau}_h, q_h), (\boldsymbol{s}_h, \boldsymbol{v}_h))}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}} \geq \beta^* \left(\|\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}\right) \quad \forall (\boldsymbol{s}_h, \boldsymbol{v}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h. \tag{6.1}$$

Moreover, the following Cea-like estimate holds true: There exists $C > 0$, independent of h, such that for all $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$,

$$\begin{aligned} &\|\boldsymbol{T}((\boldsymbol{\sigma}, p), \boldsymbol{r}) - \widetilde{\boldsymbol{T}}_h((\boldsymbol{\sigma}, p), \boldsymbol{r})\| \\ &\qquad \leq C \inf_{((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h} |||\boldsymbol{T}((\boldsymbol{\sigma}, p), \boldsymbol{r}) - ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)|||\,. \end{aligned} \tag{6.2}$$

The reason why we have called this operator $\widetilde{\boldsymbol{T}}_h$ instead of just $\boldsymbol{T}_h$, is that we preserve this notation for its restriction onto the finite element space. In fact, since $\widetilde{\boldsymbol{T}}_h(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \subset \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, we are allowed to define

$$\boldsymbol{T}_h := \widetilde{\boldsymbol{T}}_h|_{\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h} :\ \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h \longrightarrow \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$$

and it is well-known that $\mathrm{sp}(\widetilde{\boldsymbol{T}}_h) = \mathrm{sp}(\boldsymbol{T}_h) \cup \{0\}$ (see, for instance, [4, Lemma 4.1]).

Once more, as in the continuous case, we have that $(\lambda_h, (\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ solves Problem 3 if and only if $(1/(1+\lambda_h), ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h))$ is an eigenpair of $\boldsymbol{T}_h$, i.e., if and only if $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \neq \boldsymbol{0}$ and

$$\boldsymbol{T}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) = \frac{1}{1+\lambda_h}\,((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)\,.$$

To describe the spectrum of this operator, we will proceed as in the continuous case and decompose $\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ in a convenient direct sum. To this end, let

$$\begin{aligned} \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}} &:= \{(\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{W}}_h \times \mathbb{R} :\ \boldsymbol{\tau}_h \boldsymbol{\nu} + \xi \boldsymbol{\nu} = \boldsymbol{0} \text{ on } \Sigma\}\,, \\ \ker_h(a) &:= \{(\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}} :\ \mathbf{div}\,\boldsymbol{\tau}_h = \boldsymbol{0} \text{ in } \Omega_{\mathrm{S}}\}\,. \end{aligned}$$

Clearly $\boldsymbol{T}_h|_{\ker_h(a)\times\boldsymbol{\mathcal{Q}}_h} :\ [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \to [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]$ reduces to the identity. Thus, $\mu_h = 1$ is an eigenvalue of $\boldsymbol{T}_h$ and, from the definition of $\widetilde{\boldsymbol{T}}_h$, $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)$ is an associated eigenfunction if and only if $(\boldsymbol{\sigma}_h, p_h) \in \ker_h(a)$. Therefore, we have the following discrete analogue to Lemma 2.3.

Lemma 6.1. $\mu_h = 1$ *is an eigenvalue of* $\boldsymbol{T}_h$ *with associated eigenspace* $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$.

Let $\widehat{\boldsymbol{E}}_h$ be the operator defined in (5.7) and

$$\begin{aligned} \boldsymbol{P}_h :\ \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} &\longrightarrow \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h, \\ ((\boldsymbol{\sigma}, p), \boldsymbol{r}) &\longmapsto \boldsymbol{P}_h((\boldsymbol{\sigma}, p), \boldsymbol{r}) := ((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), \widetilde{\boldsymbol{r}}_h), \end{aligned}$$

where $(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h) \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}} + \widehat{\boldsymbol{E}}_h \bar{p}$ and $(\widetilde{\boldsymbol{r}}_h, \widetilde{\boldsymbol{u}}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$ solve the following problem:

(6.3)

$$d((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), (\boldsymbol{\tau}_h, \xi)) + B((\boldsymbol{\tau}_h, \xi), (\widetilde{\boldsymbol{r}}_h, \widetilde{\boldsymbol{u}}_h)) = 0 \qquad \forall (\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}},$$

$$B((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), (\boldsymbol{s}_h, \boldsymbol{v}_h)) = \int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\sigma} \cdot \boldsymbol{v}_h \qquad \forall (\boldsymbol{s}_h, \boldsymbol{v}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h. \tag{6.4}$$

Here again, since $\mathbf{div}\,\boldsymbol{E}_h \bar{p} = \boldsymbol{0}$, (6.3)–(6.4) hold true if and only if $(\widetilde{\boldsymbol{\sigma}}_{h,0}, \widetilde{c}_h) := (\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h) - \widehat{\boldsymbol{E}}_h \bar{p} \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}$ and $(\widetilde{\boldsymbol{r}}_h, \widetilde{\boldsymbol{u}}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$ solve the equations

$$d((\widetilde{\boldsymbol{\sigma}}_{h,0}, \widetilde{c}_h), (\boldsymbol{\tau}_h, \xi)) + B((\boldsymbol{\tau}_h, \xi), (\widetilde{\boldsymbol{r}}_h, \widetilde{\boldsymbol{u}}_h)) = -d(\widehat{\boldsymbol{E}}_h \bar{p}, (\boldsymbol{\tau}_h, \xi)), \tag{6.5}$$

$$B((\widetilde{\boldsymbol{\sigma}}_{h,0}, \widetilde{c}_h), (\boldsymbol{s}_h, \boldsymbol{v}_h)) = \int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\sigma} \cdot \boldsymbol{v}_h - b(\widehat{\boldsymbol{E}}_h \bar{p}, \boldsymbol{s}_h) \tag{6.6}$$

for all $(\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}$ and $(\boldsymbol{s}_h, \boldsymbol{v}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$.

Equations (6.3)–(6.4) are a stable finite element discretization of the mixed problem (3.2)–(3.3) used to define $\boldsymbol{P}$. Indeed, the uniform discrete inf-sup condition of B for the pair $\{\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}, \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h\}$ is an easy consequence of (6.1). Moreover, Lemma 2.1 guarantees the uniform ellipticity of d on $\ker(a) \supset \ker_h(a)$, whereas the fact that $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) \subset \boldsymbol{\mathcal{U}}_h$ implies that $\ker_h(B) \subset \ker_h(a)$. Hence, as a consequence of the Babuška-Brezzi theory, problem (6.5)–(6.6), and a fortiori problem (6.3)–(6.4), are well posed. Furthermore, thanks to the definition of $\widehat{\boldsymbol{E}}_h \bar{p}$, the first estimate from Lemma 5.1, and the fact that $\|\pi_h \bar{p}\|_{1,\Omega_{\mathrm{F}}} \leq \|\bar{p}\|_{1,\Omega_{\mathrm{F}}}$ (since π_h is a projection), we can claim that the operators $\boldsymbol{P}_h$ are bounded uniformly with respect to h and the following Strang-like estimate holds true:

$$
\begin{aligned}
(6.7)\quad & \|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)\| + \|\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_h\|_{0,\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{r}}_h\|_{0,\Omega_{\mathrm{S}}} \\
&\leq C\Bigg[\inf_{(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\boldsymbol{\tau}_h,\xi)\| + \inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{U}}_h}\|\widetilde{\boldsymbol{u}}-\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} + \inf_{\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\|\widetilde{\boldsymbol{r}}-\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}} \\
&\qquad + \sup_{\boldsymbol{0}\neq(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\frac{|d(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},(\boldsymbol{\tau}_h,\xi))|}{\|(\boldsymbol{\tau}_h,\xi)\|} + \sup_{\boldsymbol{0}\neq\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\frac{|b(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},\boldsymbol{s}_h)|}{\|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}}\Bigg],
\end{aligned}
$$

where $((\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c}),(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}}))$ and $((\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h),(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{r}}_h))$ are the solutions to (3.4)–(3.5) and (6.5)–(6.6), respectively.

Lemma 6.2. *There exists $C>0$, independent of h, such that*

$$
\begin{aligned}
|||(\boldsymbol{P}-\boldsymbol{P}_h)((\boldsymbol{\sigma},p),\boldsymbol{r})||| &\leq C\Big[h^{t_{\mathrm{S}}}\Big(\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}}+\|p\|_{1,\Omega_{\mathrm{F}}}\Big) \\
&\qquad + \|\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}-\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}})\|_{0,\Omega_{\mathrm{S}}} + \|p-\pi_h p\|_{1,\Omega_{\mathrm{F}}}\Big].
\end{aligned}
$$

Proof. The triangle inequality yields

$$
\begin{aligned}
\|(\widetilde{\boldsymbol{\sigma}},\widetilde{p})-(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{p}_h)\| &\leq \|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)\| + \Big\|\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p}\Big\| \\
&\leq \|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)\| + \|\boldsymbol{E}\bar{p}-\boldsymbol{E}_h\bar{p}\|_{0,\Omega_{\mathrm{S}}} + \|\bar{p}-\pi_h\bar{p}\|_{1,\Omega_{\mathrm{F}}},
\end{aligned}
$$

which allows us to resort to (6.7). Now, since $\widetilde{\boldsymbol{\sigma}}_0=\widetilde{\boldsymbol{\sigma}}-\boldsymbol{E}\bar{p}\in\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}\cap\boldsymbol{\mathcal{W}}$ (thanks to Lemma 3.1 and (3.1)) and $\pi_h\widetilde{c}=\widetilde{c}$ is also constant, Lemma 5.3 leads to

$$
\inf_{(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\boldsymbol{\tau}_h,\xi)\| \leq C\,\|\widetilde{\boldsymbol{\sigma}}_0-\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}}_0\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}.
$$

Moreover, since $\boldsymbol{E}\bar{p}$ is divergence-free, we have that $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_0=\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}$ and, by virtue of (5.1), $\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}}_0)=\boldsymbol{L}_h(\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_0)=\boldsymbol{L}_h(\mathbf{div}\,\widetilde{\boldsymbol{\sigma}})=\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}})$, so that

$$
\|\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_0-\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}}_0)\|_{0,\Omega_{\mathrm{S}}} = \|\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}-\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}})\|_{0,\Omega_{\mathrm{S}}}.
$$

On the other hand, the Cauchy-Schwarz inequality and the trace inequality in $\mathrm{H}^1(\Omega_{\mathrm{F}})$ yield

$$
\begin{aligned}
&\sup_{\boldsymbol{0}\neq(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\frac{|d(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},(\boldsymbol{\tau}_h,\xi))|}{\|(\boldsymbol{\tau}_h,\xi)\|} + \sup_{\boldsymbol{0}\neq\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\frac{|b(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},\boldsymbol{s}_h)|}{\|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}} \\
&\qquad\qquad \leq C\Big(\|\boldsymbol{E}\bar{p}-\boldsymbol{E}_h\bar{p}\|_{0,\Omega_{\mathrm{S}}}+\|\bar{p}-\pi_h\bar{p}\|_{1,\Omega_{\mathrm{F}}}\Big).
\end{aligned}
$$

Then, from all the above and (6.7), we derive

$$
\begin{aligned}
&\|(\widetilde{\boldsymbol{\sigma}},\widetilde{p})-(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{p}_h)\| + \|\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_h\|_{0,\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{r}}_h\|_{0,\Omega_{\mathrm{S}}} \\
&\quad\leq C\Big[\|\widetilde{\boldsymbol{\sigma}}_0-\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}}_0\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}-\mathbf{div}(\boldsymbol{\Pi}_h\widetilde{\boldsymbol{\sigma}})\|_{0,\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{u}}-\boldsymbol{L}_h\widetilde{\boldsymbol{u}}\|_{0,\Omega_{\mathrm{S}}} \\
&\qquad + \|\widetilde{\boldsymbol{r}}-\boldsymbol{R}_h\widetilde{\boldsymbol{r}}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{E}\bar{p}-\boldsymbol{E}_h\bar{p}\|_{0,\Omega_{\mathrm{S}}} + \|\bar{p}-\pi_h\bar{p}\|_{1,\Omega_{\mathrm{F}}}\Big].
\end{aligned}
$$

Now, we use again that $\widetilde{\boldsymbol{\sigma}}_0\in\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}\cap\boldsymbol{\mathcal{W}}$, Lemma 3.1, and (3.1), to write

$$
\|\widetilde{\boldsymbol{\sigma}}_0\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{u}}\|_{1+t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{r}}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} \leq C\Big(\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega_{\mathrm{S}}}+\|p\|_{1,\Omega_{\mathrm{F}}}\Big).
$$

The result then follows by using the approximation properties (5.2), (5.4), and (5.5), the equality $\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_0=\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}=\mathbf{div}\,\boldsymbol{\sigma}$ (cf. (3.6)), Lemma 5.2, and the facts that $\|\bar{p}\|_{1,\Omega_{\mathrm{F}}}\leq\|p\|_{1,\Omega_{\mathrm{F}}}$ and $\|\bar{p}-\pi_h\bar{p}\|_{1,\Omega_{\mathrm{F}}}=\|p-\pi_h p\|_{1,\Omega_{\mathrm{F}}}$. □

Lemma 6.3. *There exists $C > 0$ independent of h such that:*

i) if $((\boldsymbol{\sigma}, p), \boldsymbol{r})$ is an eigenfunction of $\boldsymbol{T}$ associated to an eigenvalue $\mu \in (0,1)$, then

$$|||(\boldsymbol{P} - \boldsymbol{P}_h)((\boldsymbol{\sigma}, p), \boldsymbol{r})||| \leq Ch^{\min\{t_{\mathrm{S}}, t_{\mathrm{F}}\}} \, |||((\boldsymbol{\sigma}, p), \boldsymbol{r})||| \, ;$$

ii) if $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, then

$$|||(\boldsymbol{P} - \boldsymbol{P}_h)((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)||| \leq Ch^{t_{\mathrm{S}}} \left(\|\mathbf{div}\, \boldsymbol{\sigma}_h\|_{0,\Omega_{\mathrm{S}}} + \|p_h\|_{1,\Omega_{\mathrm{F}}} \right).$$

Proof. Case (i). The estimate follows from Lemma 6.2, (5.3), (5.6), Corollary 4.4, and the fact that $\mathbf{div}\, \widetilde{\boldsymbol{\sigma}} = \mathbf{div}\, \boldsymbol{\sigma}$, because of (3.6).

Case (ii). Let $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, $((\widetilde{\boldsymbol{\sigma}}, \widetilde{p}), \widetilde{\boldsymbol{r}}) = \boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)$ and $((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), \widetilde{\boldsymbol{r}}_h) = \boldsymbol{P}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)$. For $(\boldsymbol{\sigma}_h, p_h) \in \boldsymbol{\mathcal{Y}}_h$ we have that $p_h = \pi_h p_h$. On the other hand, by virtue of (3.6), $\mathbf{div}\, \widetilde{\boldsymbol{\sigma}} = \mathbf{div}\, \boldsymbol{\sigma}_h$. Hence, because of (5.1) and the fact that $\mathbf{div}\, \boldsymbol{\mathcal{W}}_h \subset \boldsymbol{\mathcal{U}}_h$, there also holds

$$\mathbf{div}(\boldsymbol{\Pi}_h \widetilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\, \widetilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\, \boldsymbol{\sigma}_h) = \mathbf{div}\, \boldsymbol{\sigma}_h = \mathbf{div}\, \widetilde{\boldsymbol{\sigma}}.$$

The result follows then directly from Lemma 6.2 and Corollary 4.4. □

For $((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), \widetilde{\boldsymbol{r}}_h) = \boldsymbol{P}_h((\boldsymbol{\sigma}, p), \boldsymbol{r})$, (6.4) implies that $\int_{\Omega_{\mathrm{S}}} \boldsymbol{v}_h \cdot \mathbf{div}\, \widetilde{\boldsymbol{\sigma}}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{v}_h \cdot \mathbf{div}\, \boldsymbol{\sigma}$ for all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h$. Hence, it is easy to check that the operator $\boldsymbol{P}_h$ is idempotent and, then, so is $\boldsymbol{P}_h|_{\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h}$ too, because $\boldsymbol{P}_h(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \subset \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$. Moreover, it is easy to check that $\ker(\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h}) = \ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$. Therefore, being $\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h}$ a projector, we have that $\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h = [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \oplus \boldsymbol{P}_h(\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h)$.

Our next goal is to show that $\boldsymbol{P}_h(\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h) = [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$, where

$$\begin{aligned}
[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}} := \{ &((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h : \\
&\mathbb{B}(((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)) = 0 \ \forall ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \ker_h(a) \times \boldsymbol{\mathcal{Q}}_h \},
\end{aligned}$$

with the bilinear form $\mathbb{B}$ as defined in Section 3. With this end, we repeat the same steps as in that section. In particular, we have the following discrete analogue to Lemma 3.2.

Lemma 6.4. $[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \cap [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}} = \{\mathbf{0}\}$.

Proof. Since the discrete inf-sup condition

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h \cap \boldsymbol{\mathcal{W}}_\Sigma} \frac{\displaystyle\int_{\Omega_{\mathrm{S}}} \boldsymbol{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}} \geq \beta^* \|\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h$$

follows from (6.1), the proof runs almost identically to that of Lemma 3.2. □

Now we are ready to establish the claimed result. We skip its proof since it is almost identical to that of Lemma 3.3.

Lemma 6.5. $\boldsymbol{P}_h(\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h) = [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$.

The proof of the following lemma is almost identical to that of Lemma 4.2.

Lemma 6.6. *For all $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$,*

$$\mathbb{A}(((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h), ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) \geq \mathbb{B}(((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h), ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) > 0.$$

Now, we are in a position to write down a characterization of the spectrum of the operator $\boldsymbol{T}_h$ and, hence, of the solutions to Problem 3.

Theorem 6.7. *The spectrum of* $\boldsymbol{T}_h$ *consists of* $M := \dim(\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h)$ *eigenvalues, repeated accordingly to their respective multiplicities. The spectrum decomposes as follows:* $\operatorname{sp}(\boldsymbol{T}_h) = \{1\} \cup \{\mu_{hk}\}_{k=1}^{K}$. *Moreover,*

i) *the eigenspace associated to* $\mu_h = 1$ *is* $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$*;*
ii) $\mu_{hk} \in (0,1)$, $k = 1, \ldots, K := M - \dim(\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h)$, *are non-defective eigenvalues, repeated accordingly to their respective multiplicities, with associated eigenspaces lying on* $[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$*;*
iii) $\mu_h = 0$ *is not an eigenvalue of* $\boldsymbol{T}_h$.

Proof. Since $\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h = [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \oplus [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$ (cf. Lemmas 6.4 and 6.5), $\boldsymbol{T}_h|_{\ker_h(a)\times\boldsymbol{\mathcal{Q}}_h} : [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \to [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]$ is the identity, and $\boldsymbol{T}_h([\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}) \subset [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$ (cf. [23, Proposition A.1]), we have that the theorem follows from Lemmas 6.1 and 6.6 and [23, Proposition A.2]. □

7. Spectral approximation

To prove that $\boldsymbol{T}_h$ provides a correct spectral approximation of $\boldsymbol{T}$, we will resort to the corresponding theory for non-compact operators from [15]. With this end, for the sake of brevity, we will denote throughout this section $\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{X}}_h := \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$. Moreover, when no confusion can arise, we will use indistinctly $\boldsymbol{x}$, $\boldsymbol{y}$, etc. to denote elements in $\boldsymbol{\mathcal{X}}$ and, analogously, $\boldsymbol{x}_h$, $\boldsymbol{y}_h$, etc. for those in $\boldsymbol{\mathcal{X}}_h$. Moreover, we denote by $|||\cdot|||$ the norm in $\boldsymbol{\mathcal{X}}$ as above, as well as the corresponding induced norm on operators acting from $\boldsymbol{\mathcal{X}}$ into the same space. Finally, we will use $|||\cdot|||_h$ as in [15] to denote the norm of an operator restricted to the discrete subspace $\boldsymbol{\mathcal{X}}_h$; namely, if $\boldsymbol{S} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$, then

$$|||\boldsymbol{S}|||_h := \sup_{\boldsymbol{0} \neq \boldsymbol{x}_h \in \boldsymbol{\mathcal{X}}_h} \frac{|||\boldsymbol{S}\boldsymbol{x}_h|||}{|||\boldsymbol{x}_h|||}.$$

We recall some classical notation for spectral approximation. For $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ closed subspaces of $\boldsymbol{\mathcal{X}}$, we set $\delta(\boldsymbol{x}, \boldsymbol{\mathcal{E}}) := \inf_{\boldsymbol{y} \in \boldsymbol{\mathcal{E}}} |||\boldsymbol{x} - \boldsymbol{y}|||$, $\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}) := \sup_{\boldsymbol{y} \in \boldsymbol{\mathcal{E}}:\, |||\boldsymbol{y}|||=1} \delta(\boldsymbol{y}, \boldsymbol{\mathcal{F}})$, and $\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}) := \max\{\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}), \delta(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{E}})\}$, the latter being the so called *gap* between subspaces $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$.

The first step to adapt the results from [15] to our problem is to establish the following two properties, in which

$$t := \min\{t_{\mathrm{S}}, t_{\mathrm{F}}\}.$$

P1 : There exist strictly positive constants C and h_0 such that, if $h \leq h_0$, then

$$|||\boldsymbol{T} - \boldsymbol{T}_h|||_h \leq C h^t.$$

P2 : For each eigenfunction $\boldsymbol{x}$ of $\boldsymbol{T}$ associated to an eigenvalue $\mu \in (0,1)$, there exist strictly positive constants C and h_0 such that, if $h \leq h_0$, then

$$\delta(\boldsymbol{x}, \boldsymbol{\mathcal{X}}_h) \leq C h^t \, |||\boldsymbol{x}||| \, .$$

The latter (P2) follows immediately from Lemma 5.3, the smoothness of the eigenfunctions established in Corollary 4.4, and the approximation properties of the finite element spaces (5.3), (5.4), and (5.6). The following lemma proves the former (P1).

Lemma 7.1. *There exists* $C > 0$*, independent of* h*, such that*

$$|||\boldsymbol{T} - \boldsymbol{T}_h|||_h \leq C h^t.$$

Proof. For $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{X}}_h = \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, we write

$$\begin{aligned}(\boldsymbol{T} - \boldsymbol{T}_h)((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) &= (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) \\ &\quad + (\boldsymbol{T} - \boldsymbol{T}_h)((\boldsymbol{I} - \boldsymbol{P}_h)((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) \\ &= (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)),\end{aligned}$$

the last equality because $(\boldsymbol{I} - \boldsymbol{P}_h)$ is a projector onto $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$ and $\boldsymbol{T}$ and $\boldsymbol{T}_h$ are both the identity on this subspace. Now,

$$\begin{aligned}&(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) \\ &\qquad = \underbrace{(\boldsymbol{T} - \widetilde{\boldsymbol{T}}_h)((\boldsymbol{P}_h - \boldsymbol{P})((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h))}_{E_1} + \underbrace{(\boldsymbol{T} - \widetilde{\boldsymbol{T}}_h)(\boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h))}_{E_2}.\end{aligned}$$

For the first term we use Lemma 6.3 (ii) to write

$$|||E_1||| \leq \left(\|\boldsymbol{T}\| + \|\widetilde{\boldsymbol{T}}_h\|\right) |||(\boldsymbol{P}_h - \boldsymbol{P})((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)||| \leq Ch^{t_{\mathrm{S}}} \|(\boldsymbol{\sigma}_h, p_h)\|.$$

For the second, by virtue of the Cea-like estimate (6.2), we have that

$$|||E_2||| \leq C \inf_{((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h} |||\boldsymbol{T}(\boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)) - ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)|||.$$

Now, since $\boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ (cf. Lemma 3.3), according to Proposition 4.1, if we denote $((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*) = \boldsymbol{T}(\boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h))$, then we have that $\boldsymbol{\sigma}^*, \boldsymbol{r}^* \in \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$, $\mathbf{div}\, \boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega_{\mathrm{S}})^n$, $p^* \in \mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})$, and

$$\begin{aligned}\|\boldsymbol{\sigma}^*\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\mathbf{div}\, \boldsymbol{\sigma}^*\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{r}^*\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|p^*\|_{1+t_{\mathrm{F}},\Omega_{\mathrm{F}}} &\leq C\, |||\boldsymbol{P}((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)||| \\ &\leq C \|(\boldsymbol{\sigma}_h, p_h)\|.\end{aligned}$$

Then, from the last two inequalities, Lemma 5.3, and the approximation properties (5.3), (5.4), and (5.6), we write

$$|||E_2||| \leq C \inf_{((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h} |||((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*) - ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)||| \leq Ch^t \|(\boldsymbol{\sigma}_h, p_h)\|,$$

which together with the estimate of E_1 and the first two equalities of the proof allow us to conclude the lemma. □

Now we are in a position to apply the spectral approximation theory from [15]. Our first result was proved to follow from property P1 in Theorem 1 from this reference.

Theorem 7.2. *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{sp}(\boldsymbol{T}) = \emptyset$. Then, there exist $h_0 > 0$ and $C > 0$ such that, for all $h < h_0$, $F \cap \mathrm{sp}(\boldsymbol{T}_h) = \emptyset$.*

An immediate consequence of this theorem is that the proposed finite element method does not introduce spurious modes with eigenvalues interspersed among those with a physical meaning (the squares of free vibration frequencies). Let us remark that such a spectral pollution appears in standard finite element discretizations of other formulations of this same problem (see [22, 8]).

The spectral convergence of $\boldsymbol{T}_h$ to $\boldsymbol{T}$ as $h \to 0$ can also be derived by adapting to our problem results from [15, Section 2]. More precisely, by repeating the arguments in the proofs of Theorems 2 and 3 from this reference and using properties P1 and P2, it is easy to prove that for all isolated eigenvalue μ of $\boldsymbol{T}$ with finite multiplicity m (and, hence, $\mu \in (0,1)$), for h small enough, there exist m eigenvalues $\mu_{h,1}, \ldots, \mu_{h,m}$ of $\boldsymbol{T}_h$ (repeated accordingly to their respective multiplicities) which converge to μ

as $h \to 0$. Moreover, if $\boldsymbol{\mathcal{E}}$ is the eigenspace of $\boldsymbol{T}$ corresponding to μ and $\boldsymbol{\mathcal{E}}_h$ is the invariant subspace of $\boldsymbol{T}_h$ spanned by the eigenspaces of $\boldsymbol{T}_h$ corresponding to $\mu_{h,1}, \ldots, \mu_{h,m}$, then $\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leq Ch^t$, for h small enough.

Finally, the arguments from [23, Section 5] can be readily adapted to this coupled fluid-structure eigenvalue problem to prove a double order error estimate. We summarize these results in the following theorem, in which $\lambda := (1/\mu) - 1$ is an eigenvalue of Problem 1 with multiplicity m and $\lambda_{hi} := (1/\mu_{hi}) - 1$, $i = 1, \ldots, m$, are the eigenvalues of Problem 3 (repeated accordingly to their respective multiplicities) converging to λ.

Theorem 7.3. *There exist constants $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leq Ch^t \qquad \text{and} \qquad \max_{1 \leq i \leq m} |\lambda - \lambda_{hi}| \leq Ch^{2t}.$$

8. Numerical results

We report in this section the results of a numerical test carried out with the method proposed in Section 6 which was implemented in a MATLAB code.

We have chosen a two-dimensional test which corresponds to compute the vibration frequencies of an elastic container partially filled with a (compressible) liquid. The geometrical data is shown in Figure 2.

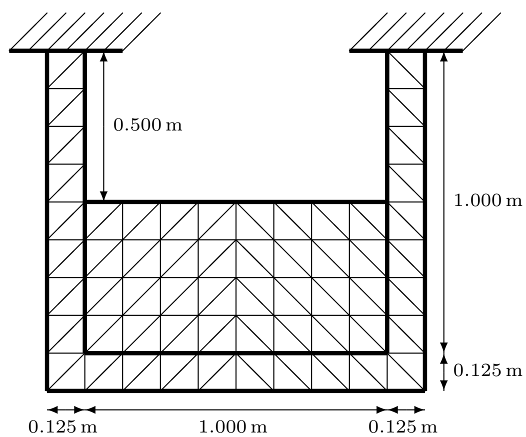

Figure 2. Fluid and solid domains. Coarsest mesh ($N = 1$).

We have used several meshes which are successive uniform refinements of the coarse initial triangulation shown in Figure 2. The refinement parameter N is the number of element layers across the thickness of the solid ($N = 1$ for the mesh in Figure 2).

We have used the following physical parameters, which correspond to steel and water:

- Solid density: $\rho_{\mathrm{S}} = 7700\,\mathrm{kg/m^3}$,
- Young modulus: $E = 1.44 \times 10^{11}$ Pa,
- Poisson ratio: $\nu = 0.35$,
- Fluid density: $\rho_{\mathrm{F}} = 1000\,\mathrm{kg/m^3}$,

- Acoustic speed: $c = 1430\,\mathrm{m/s}$,
- Gravity acceleration: $g = 9.8\,\mathrm{m/s^2}$.

We recall that the Lamé coefficients of a material are defined in terms of the Young modulus E and the Poisson ratio ν as follows: $\lambda_S := E\nu/[(1+\nu)(1-2\nu)]$ and $\mu_S := E/[2(1+\nu)]$.

Let us remark that two completely different type of free vibration modes appear in this kind of problem: the so called *sloshing* and *elastoacoustic* modes. Sloshing modes arise from the gravity oscillations of the liquid free surface which, in absence of resonance, depend very mildly on the physical parameters of the fluid and the structure. Indeed, the size of the lowest sloshing frequencies are typically around $\sqrt{\pi g/\operatorname{length}(\Gamma_0)}$. On the other hand, elastoacoustic modes arise from the natural vibrations of the structural-acoustic coupled system. The lowest elastoacoustic vibration frequencies are in this test around 100 times larger than the lowest sloshing frequencies. We refer to [5, 9] for a more detailed discussion.

Because of this, we report on separate tables the lowest computed sloshing and elastoacoustic vibration frequencies $\omega_h := \sqrt{\lambda_h}$. We report the former ($\omega^S_{h,k}$) in Table 1 and the latter ($\omega^E_{h,k}$) in Table 2. We have used several different meshes with increasing levels of refinement. The table also includes the estimated orders of convergence, as well as more accurate values of the vibration frequencies extrapolated from the computed ones by means of a least-squares fitting. A double order of convergence can be clearly observed in all cases.

We have also solved the same problem with an alternative finite element method for a pure displacement formulation in both media proposed and analyzed in [5, 9]. We report on the last column of both tables the results obtained by extrapolation from the vibration frequencies computed with this method on the same meshes. An excellent agreement can be clearly appreciated.

TABLE 1. Lowest computed sloshing frequencies $\omega^S_{h,k}$ (in rad/s).

Mode	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$	Order	Extrapolated	[5, 9]
$\omega^S_{h,1}$	5.3196	5.3164	5.3153	5.3148	5.3145	2.00	5.3138	5.3138
$\omega^S_{h,2}$	7.8697	7.8490	7.8417	7.8383	7.8365	2.00	7.8324	7.8324
$\omega^S_{h,3}$	9.7135	9.6560	9.6358	9.6264	9.6213	1.99	9.6097	9.6099

TABLE 2. Lowest computed elastoacoustic vibration frequencies $\omega^E_{h,k}$ (in rad/s).

Mode	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$	Order	Extrapolated	[5, 9]
$\omega^E_{h,1}$	446.94	444.76	443.92	443.52	443.29	1.80	442.71	443.01
$\omega^E_{h,2}$	1484.03	1476.47	1473.59	1472.21	1471.46	1.81	1469.45	1468.95
$\omega^E_{h,3}$	2596.19	2586.83	2583.29	2581.61	2580.72	1.84	2578.33	2577.86
$\omega^E_{h,4}$	2790.03	2774.01	2767.89	2764.95	2763.32	1.79	2758.94	2758.63

Finally, Figures 3 and 4 show the deformed structure and the fluid pressure field for the lowest-frequency sloshing and elastoacoustic vibration modes, respectively.

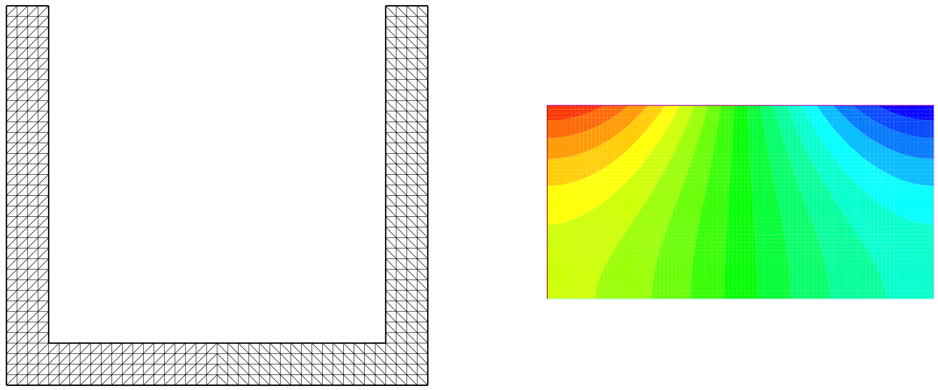

FIGURE 3. Lowest-frequency sloshing mode. Deformed structure (left) and fluid pressure field (right).

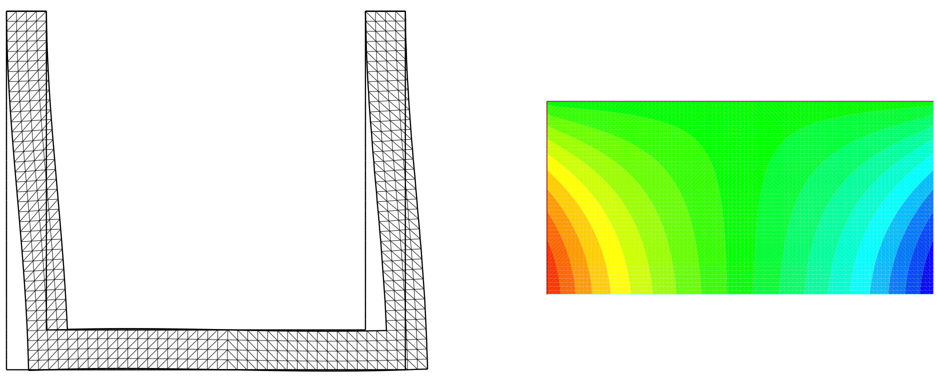

FIGURE 4. Lowest-frequency elastoacoustic vibration mode. Deformed structure (left) and fluid pressure field (right).

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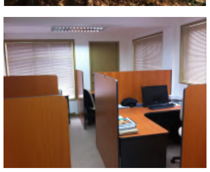

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