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**First-order weak balanced schemes for bilinear stochastic
differential equations**

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First-order weak balanced schemes for bilinear stochastic differential equations *

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Abstract

This paper starts to develop balanced schemes for stochastic differential equations (SDEs) with multiplicative noise based on the addition of stabilizing functions to the drift terms. First, we use the linear scalar SDE as a test problem to show that it is possible to construct efficient almost sure stable first-order weak balanced schemes. Second, we design balanced schemes for bilinear SDEs that achieve the first order of weak convergence, and do not involve the simulation of multiple stochastic integrals. Numerical experiments show a promising performance of the new numerical methods.

1 Introduction

In this paper, we deal with the problem of constructing balanced Euler schemes having the same rate of weak converge as the Euler method. Consider the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t b(X_s) ds + \sum_{k=1}^m \int_0^t \sigma^k(X_s) dW_s^k, \quad (1)$$

where X_t is an adapted $\mathbb{R}^d$ -valued stochastic process, $b, \sigma^k : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are smooth functions and $W^1, \dots, W^m$ are independent standard Wiener processes. For solving (1) in cases the diffusion terms σ^k play an essential role in the dynamics of X_t , Milstein, Platen and Schurz [14] introduced the balanced implicit method

$$\begin{aligned} Z_{n+1} &= Z_n + b(Z_n) \Delta + \sum_{k=1}^m \sigma^k(Z_n) \left(W_{(n+1)\Delta}^k - W_{n\Delta}^k \right) \\ &\quad + \left(c^0(Z_n) \Delta + \sum_{k=1}^m c^k(Z_n) \left| W_{(n+1)\Delta}^k - W_{n\Delta}^k \right| \right) (Z_n - Z_{n+1}), \end{aligned} \quad (2)$$

where $c^0, \dots, c^m$ are weight functions that should be appropriately chosen for each SDE, and $\Delta > 0$. Up to now, the schemes of type (2) use the damping

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functions $c^1, \dots, c^m$ to control the numerical instabilities caused by σ^k (see, e.g., [1, 14, 16, 17, 21]), and hence their rate of weak convergence is equal to $1/2$, which is low. To the best of our knowledge, concrete balanced versions of the Milstein scheme have been developed only in particular cases, like $m = 1$, where the Milstein scheme does not involve multiple stochastic integrals with respect to different Brownian motions [2, 10].

We are interested in the development of efficient 1-order balanced schemes for computing $\mathbb{E}f(X_t)$, with $f : \mathbb{R}^d \rightarrow \mathbb{R}$ smooth. This motivates us to design balanced schemes based only on c_0 . More precisely, we address the general question of whether we can find c^0 such that

$$Z_{n+1} = Z_n + b(Z_n) \Delta + c^0(\Delta, Z_n) (Z_{n+1} - Z_n) \Delta + \sum_{k=1}^m \sigma^k(Z_n) \xi_n^k \quad (3)$$

reproduces the long-time behavior of X_t . Here, $\xi_0^1, \xi_0^2, \dots, \xi_0^m, \xi_1^1, \dots$ are independent random variables satisfying $\mathbf{P}(\xi_n^k = \pm 1) = 1/2$. Section 2 gives a positive answer to this problem when (1) reduces to an almost sure exponentially stable linear scalar SDE. In this test case, we obtain an explicit expression for $c^0(\Delta, \cdot)$ that makes Z_n an almost sure asymptotically stable numerical method for all $\Delta > 0$. Moreover, Section 2 introduces a stabilized trapezoidal scheme. Section 3 focuses on bilinear systems of SDEs. In case $b, \sigma^k : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are linear, we propose an optimization procedure for identifying a suitable weight function c_0 . Section 3 also provides a choice of c_0 based on a heuristic closed formula. Both techniques show good results in our numerical experiments, which encourages further studies of (3). All proofs are deferred to Section 4.

2 The linear scalar SDE

In this section, we assume that X_t satisfies the scalar SDE

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \lambda X_s dW_s^1, \quad (4)$$

where, for simplicity, the real numbers μ, λ satisfy $2\mu - \lambda^2 < 0$. The SDE (4) is a classical test equation for studying the stability properties of the numerical schemes for (1) (see, e.g., [1, 7, 8]).

2.1 Stabilized Euler scheme

Set $T_n = n\Delta$, with $\Delta > 0$ and $n = 0, 1, \dots$. For all $t \in [T_n, T_{n+1}]$ we have

$$X_t = X_{T_n} + \int_{T_n}^t (\mu X_s + a(\Delta) X_s - a(\Delta) X_s) ds + \int_{T_n}^t \lambda X_s dW_s^1,$$

where $a(\Delta)$ is an arbitrary real number. Then

$$X_{T_{n+1}} \approx X_{T_n} + \mu X_{T_n} \Delta + a(\Delta) (X_{T_{n+1}} - X_{T_n}) \Delta + \lambda X_{T_n} (W_{T_{n+1}}^1 - W_{T_n}^1),$$

and so X_t is weakly approximated by the recursive scheme

$$Y_{n+1}^s = Y_n^s + \mu Y_n^s \Delta + a(\Delta) (Y_{n+1}^s - Y_n^s) \Delta + \lambda Y_n^s \xi_n^1, \quad (5)$$

where, from now on, $\xi_0^1, \xi_1^1, \dots$ is a sequence of independent random variables taking values ± 1 with probability $1/2$. In case $a(\Delta) \Delta \neq 1$, we have

$$Y_{n+1}^s = Y_n^s \left(1 + \left(\mu \Delta + \lambda \sqrt{\Delta} \xi_n^1 \right) / (1 - a(\Delta) \Delta) \right).$$

We wish to find a locally bounded function $\Delta \mapsto a(\Delta)$ such that:

P1) Y_n^s preserves a.s. the sign of Y_0^s for all $n \in \mathbb{N}$.

P2) Y_n^s converges almost surely to 0 as $n \rightarrow \infty$ whenever $2\mu - \lambda^2 < 0$.

We check easily that Property P1 holds iff $a(\Delta) \in]-\infty, p_1[\cup]p_2, +\infty[$, with $p_1 := \min \left\{ 1, 1 - |\lambda| \sqrt{\Delta} + \mu\Delta \right\} / \Delta$ and $p_2 := \max \left\{ 1, 1 + |\lambda| \sqrt{\Delta} + \mu\Delta \right\} / \Delta$. A close look at $\mathbb{E} \log \left(1 + \left(\mu\Delta + \lambda\sqrt{\Delta}\xi_n^1 \right) / (1 - a(\Delta)\Delta) \right)$ reveals that:

Lemma 2.1. *Suppose that $a(\Delta)\Delta \neq 1$. Then, a necessary and sufficient condition for Property P1, together with $\lim_{n \rightarrow \infty} Y_n^s = 0$ a.s., is that*

$$\begin{cases} a(\Delta) \in]-\infty, p_1[\cup]p_2, p_3[, & \text{in case } \mu < 0 \\ a(\Delta) \in]-\infty, p_1[\cup]p_2, +\infty[, & \text{in case } \mu = 0 \text{ and } \lambda \neq 0 \\ a(\Delta) \in]p_3, p_1[\cup]p_2, +\infty[, & \text{in case } \mu > 0 \end{cases},$$

where $p_3 := (\mu^2\Delta + 2\mu - \lambda^2) / (2\mu\Delta)$.

Using Lemma 2.1 we deduce that we can choose

$$a(\Delta) = \begin{cases} \mu - \alpha_1(\Delta)\lambda^2, & \text{if } \mu \leq 0 \\ \mu - \alpha_2(\Delta)\lambda^2, & \text{if } \mu > 0 \text{ and } \Delta < 2/\mu \\ \left(1 + |\lambda| \sqrt{\Delta} + \mu\Delta \right) / \Delta + \beta, & \text{if } \mu > 0 \text{ and } \Delta \geq 2/\mu \end{cases}, \quad (6)$$

where $\beta > 0$, $1/4 < \alpha_2(\Delta) \leq 1/4 + (\lambda^2 - 2\mu)(2 - \mu\Delta) / (8\lambda^2)$ and α_1 is a bounded function satisfying $\alpha_1(\Delta) > 1/4$.

Theorem 2.2. *Let $2\mu - \lambda^2 < 0$. Then, Y_n^s with $a(\Delta)$ given by (6) satisfies Properties P1 and P2.*

Notation 2.1. *We denote by $\mathcal{C}_p^\ell(\mathbb{R}^d, \mathbb{R})$ the set of all ℓ -times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$, whose partial derivatives up to order 4 have at most polynomial growth.*

Remark 2.1. *Assume that X_0 have finite moments of any order, together with $2\mu - \lambda^2 < 0$. Suppose that for every $g \in \mathcal{C}_p^4(\mathbb{R}, \mathbb{R})$ there exists $K > 0$ such that $|\mathbb{E}g(X_0) - \mathbb{E}g(Y_0^s)| \leq K(1 + \mathbb{E}|X_0|^q)T/N$ for all $N \in \mathbb{N}$. Let $a(\Delta)$ be given by (6). Since $\Delta \mapsto a(\Delta)$ is a bounded function, using classical arguments (see, e.g., [11, 13, 15, 16, 20]) we can deduce that there exists $N_0 \in \mathbb{N}$ such that for any $f \in \mathcal{C}_p^4(\mathbb{R}, \mathbb{R})$,*

$$|\mathbb{E}f(X_T) - \mathbb{E}f(Y_N^s)| \leq K(T)(1 + \mathbb{E}|X_0|^q)T/N \quad \forall N \geq N_0, \quad (7)$$

where $q \geq 2$ and $T \mapsto K(T)$ is a positive increasing function. Furthermore, from $2\mu - \lambda^2 < 0$ it follows that there exists $\epsilon \in (0, 1)$ such that

$$|1 - a(\Delta)\Delta| > \epsilon \quad \forall \Delta > 0 \quad (8)$$

(see Section 4), and so (7) holds for all $N \in \mathbb{N}$. This is proved by applying, for instance, Theorem 3.2.

Following [14], we now illustrate the behavior of Y_n^s using (4) with $\mu = 0$ and $\lambda = 4$. We take $X_0 = 1$. Since $\mu \leq 0$, we choose $\alpha_1(\Delta) = 1/4 + 1/100$; its convenient to keep the weights as small as possible. Figure 1 displays the computation of $\mathbb{E} \sin(X_t/5)$ obtained from the sample means of $25 \cdot 10^9$ observations of: Y_n^s with $a(\Delta) = -0.26\lambda^2$, the fully implicit method $\tilde{Y}_{n+1} = \tilde{Y}_n / \left(1 + \lambda^2\Delta - \lambda\sqrt{\Delta}\xi_n^1 \right)$ (see p. 497 of [11]), and the balanced scheme

$$\tilde{Z}_{n+1} = \tilde{Z}_n \left(1 + \lambda\sqrt{\Delta}\xi_n^1 + \lambda\sqrt{\Delta} \right) / \left(1 + \lambda\sqrt{\Delta} \right),$$

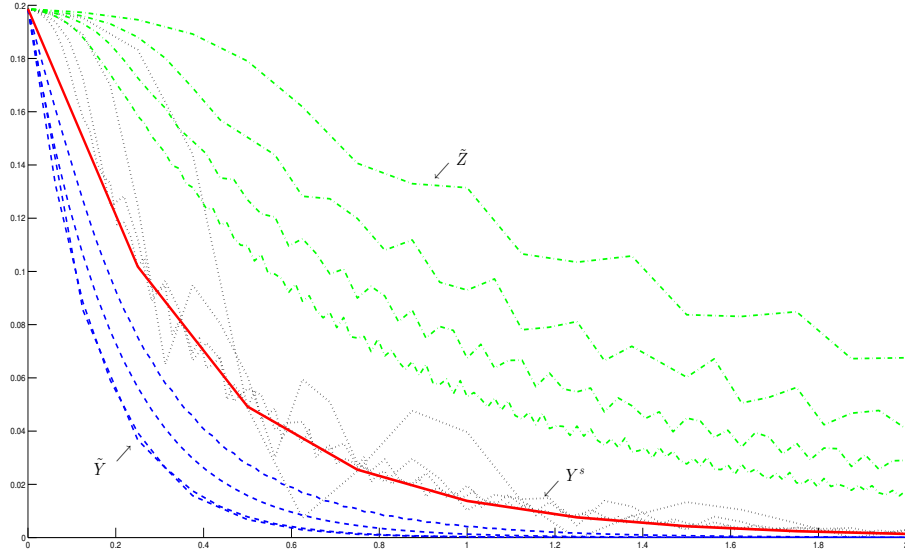


Figure 1: Computation of $\mathbb{E} \sin(X_t/5)$, where $t \in [0, 2]$ and X_t solves (4) with $\mu = 0$, $\lambda = 4$ and $X_0 = 1$. Dashed line: $\tilde{Y}$, dashdot line: $\tilde{Z}$, dotted line: Y^s , and solid line: reference values. Here, Δ takes the values $1/8$, $1/16$, $1/32$ and $1/64$. As we expected, smaller Δ produce better approximations.

which is a weak version of the method developed in Section 2 of [14]. Solid lines identifies the ‘true’ values gotten by sampling $25 \cdot 10^9$ times $\exp(-8t + 4W_t)$. In contrast with the poor performance of the Euler-Maruyama scheme when the step sizes are greater than or equal to $1/16$, Figure 1 suggests us that Y_n^s is an efficient scheme having good qualitative and convergence properties. In this numerical experiment, the accuracy of $\tilde{Z}_n$ is not good, and $\tilde{Y}_n$ decays too fast to 0 as $n \rightarrow \infty$.

2.2 Stabilized trapezoidal method

The trapezoidal scheme (see, e.g., p. 497 of [11])

$$Z_{n+1}^T = Z_n^T + \mu \frac{Z_{n+1}^T + Z_n^T}{2} \Delta - \lambda^2 \frac{Z_{n+1}^T + Z_n^T}{4} \Delta + \lambda \frac{Z_{n+1}^T + Z_n^T}{2} \sqrt{\Delta} \xi_n^1 \quad (9)$$

have a good speed of weak convergence to the solution of (4), but Z_n^T fails to preserve the sign of X_0 . Analysis similar to that in Subsection 2.1 shows the next theorem, which ensures the existence of $a \in \mathbb{R}$ such that

$$\begin{aligned} Y_{n+1}^T &= Y_n^T + \mu (Y_{n+1}^T + Y_n^T) \Delta / 2 - \lambda^2 (Y_{n+1}^T + Y_n^T) \Delta / 4 \\ &\quad + \lambda (Y_{n+1}^T + Y_n^T) \sqrt{\Delta} \xi_n^1 / 2 + (Y_{n+1}^T - Y_n^T) a \Delta \end{aligned} \quad (10)$$

verifies: **(P1’)** Y_n^T has the same sign as Y_0^T a.s. for all $n \in \mathbb{N}$; and **(P2’)** Y_n^T converges a.s. to 0 as $n \rightarrow \infty$ whenever $2\mu - \lambda^2 < 0$. Moreover, as in Remark 2.1 using standard arguments we can prove that Y_n^T have linear rate of weak convergence.

		Δ				
		1/4	1/8	1/16	1/32	1/64
$\epsilon(Y^T)$	$\mu = -1$	0.1510	0.0668	0.0321	0.0172	0.0110
	$\mu = 2$	0.2436	0.1638	0.1074	0.0713	0.0437
$\epsilon(Z^T)$	$\mu = -1$	77.935	9.8110	0.0706	0.0402	0.0218
	$\mu = 2$	78.527	10.047	0.1983	0.1012	0.0514
$\epsilon(Y^s)$	$\mu = -1$	0.3703	0.1250	0.0545	0.0269	0.0133
	$\mu = 2$	0.3633	0.0697	0.0361	0.0178	0.0088
$\epsilon(\tilde{Z})$	$\mu = -1$	0.9259	0.7567	0.5870	0.4353	0.3115
	$\mu = 2$	1.4534	1.3128	1.1220	0.9116	0.7092

Table 1: Estimation of errors involved in the computation of $\mathbb{E}g(X_T)$ and $\mathbb{E}h(X_T)$ for $T = 1$. Here, X_t verifies (4) with $X_0 = 1$ and $\lambda = 4$.

Theorem 2.3. *Let $2\mu - \lambda^2 < 0$. Consider Scheme (10) with $a < \mu/2 - 5\lambda^2/16$. Then Y_n^T satisfies Properties P1' and P2'. Moreover,*

$$Y_{n+1}^T = \frac{4 + (2\mu - \lambda^2 - 4a)\Delta + 2\lambda\sqrt{\Delta}\xi_n^1}{4 - (2\mu - \lambda^2 + 4a)\Delta - 2\lambda\sqrt{\Delta}\xi_n^1} Y_n^T.$$

In order to evaluate the sign-preserving property and the accuracy of Y_n^T , we compute $\mathbb{E}g(X_t)$ and $\mathbb{E}h(X_t)$, where $g(x) = 100(\pi/2 - \arctan(1000x + 100))$ and $h(x) = \log(1 + x^2)$. We choose $X_0 = 1$, $\lambda = 4$, and μ takes the values -1 and 2 . In Table 1 we compare the following schemes: Scheme (10) with $a = \mu/2 - 3\lambda^2/8$, the trapezoidal scheme Z_n^T , the weak balanced scheme

$$\tilde{Z}_{n+1} = \tilde{Z}_n \left(1 + \left(\mu\Delta + \lambda\sqrt{\Delta}\xi_1^n \right) / \left(1 - \mu\Delta/2 + |\lambda|\sqrt{\Delta} \right) \right)$$

and Scheme (5) with $\alpha_1(\Delta) = 0.26$, $\alpha_2(\Delta) = 1/4 + 10^{-4}(\lambda^2 - 2\mu)\mu/(8\lambda^2)$ and $\beta = 0.01$. Indeed, Table 1 presents the errors

$$\epsilon(\hat{Y}) := \left| \mathbb{E}g(X_T) - \mathbb{E}g(\hat{Y}_N) \right| + \left| \mathbb{E}h(X_T) - \mathbb{E}h(\hat{Y}_N) \right|,$$

with $T = 1$ and $N = T/\Delta$. For each numerical method $\hat{Y}$, we estimate $\epsilon(\hat{Y})$ by sampling $25 \cdot 10^9$ times both $\hat{Y}$ and the explicit solution $\exp((\mu - 8)t + 4W_t)$. Table 1 shows that Y_n^T has good qualitative properties, and in addition Y_n^T inherits the good speed of weak convergence of Z_n^T . We can also observe the very good behavior of the stabilized Euler scheme Y^s .

3 A non-commutative system of bilinear SDEs

This section is devoted to the SDE

$$X_t = X_0 + \int_0^t B X_s ds + \sum_{k=1}^m \int_0^t \sigma^k X_s dW_s^k, \quad (11)$$

where $X_t \in \mathbb{R}^d$ and B, σ^k are given real matrices of size $d \times d$. The bilinear SDEs describe dynamical features of non-linear SDEs via the linearization around their equilibrium points. The system of SDEs (11) also appears, for example, in the spatial discretization of stochastic partial differential equations (see, e.g., [6, 9]).

3.1 Heuristic balanced scheme

We now return to (3). Since (11) is bilinear, we restrict c^0 to be linear, and so (3) becomes

$$Z_{n+1} = Z_n + BZ_n\Delta + H(\Delta)(Z_{n+1} - Z_n)\Delta + \sum_{k=1}^m \sigma^k Z_n \sqrt{\Delta} \xi_n^k, \quad (12)$$

with $H :]0, \infty[\rightarrow \mathbb{R}^{d \times d}$ and $\Delta > 0$. The rate of weak convergence of Z_n is equal to 1 provided, for instance, that $H(\Delta)$ and $(I - \Delta H(\Delta))^{-1}$ are bounded on any interval $\Delta \in]0, a]$ (see, e.g., [16]). Generalizing roughly Subsection 2.1 we choose

$$H(\Delta) = B - \sum_{k=1}^m \alpha_k(\Delta) (\sigma^k)^\top \sigma^k,$$

where, for example, $\alpha_k(\Delta) = -0.26$. This gives the recursive scheme

$$\begin{aligned} & \left(I - \Delta B - 0.26 \Delta \sum_{k=1}^m (\sigma^k)^\top \sigma^k \right) Y_{n+1}^s \\ &= Y_n^s - 0.26 \Delta \sum_{k=1}^m (\sigma^k)^\top \sigma^k Y_n^s + \sum_{k=1}^m \sigma^k Y_n^s \sqrt{\Delta} \xi_n^k, \end{aligned} \quad (13)$$

which is a first-order weak balanced version of the semi-implicit Euler method.

3.2 Optimal criterion to select c_0

In case $I - \Delta H(\Delta)$ is invertible, according to (12) we have

$$Z_{n+1} = Z_n + (I - \Delta H(\Delta))^{-1} \left(\Delta B + \sum_{k=1}^m \sqrt{\Delta} \xi_n^k \sigma^k \right) Z_n, \quad (14)$$

where I is the identity matrix. Therefore, a more general formulation of Z_n is given by

$$V_{n+1} = V_n + (I + \Delta M(\Delta)) \left(\Delta B + \sum_{k=1}^m \sqrt{\Delta} \xi_n^k \sigma^k \right) V_n, \quad (15)$$

with $M :]0, \infty[\rightarrow \mathbb{R}^{d \times d}$. In fact, taking $M(\Delta) = ((I - \Delta H(\Delta))^{-1} - I) / \Delta$ we obtain (14) from (15). The following theorem provides a useful estimate of the growth rate of V_n in terms of $\mathbb{E} \log(\|A_0(\Delta, M(\Delta))x\|)$, a quantity that we can compute explicitly in each specific situation.

Theorem 3.1. *Let V_n be defined recursively by (15). Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n\Delta} \log(\|V_n\|) \leq \frac{1}{\Delta} \sup_{x \in \mathbb{R}^d, \|x\|=1} \mathbb{E} \log(\|A_n(\Delta, M(\Delta))x\|), \quad (16)$$

where $A_n(\Delta, M) = I + (I + \Delta M) \left(\Delta B + \sum_{k=1}^m \sqrt{\Delta} \xi_n^k \sigma^k \right)$.

Set $\lambda := \sup_{\|x\|=1} \left(\langle x, Bx \rangle + \frac{1}{2} \sum_{k=1}^m \|\sigma^k x\|^2 - \sum_{k=1}^m \langle x, \sigma^k x \rangle^2 \right)$. Then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|X_t\|) \leq \lambda \quad a.s. \quad (17)$$

Δ	1/2	1/4	1/8	1/16	1/32	1/64
$M_{1,1}(\Delta)$	-1.6099	-5.1036	-4.8804	-7.1499	-1.6758	0.9887
$M_{2,1}(\Delta)$	0.0975	0.2758	0.7667	1.0136	1.1500	0.9918
$M_{1,2}(\Delta)$	-0.0975	-0.2752	-0.8505	-0.1814	-1.0448	-1.9947
$M_{2,2}(\Delta)$	-1.3173	-5.9305	-2.6136	-2.3003	-1.7421	-1.9005
Order	-10	-19	-21	-21	-20	-19

Table 2: Approximate values of the weight matrix $(M_{i,j}(\Delta))_{1 \leq i,j \leq 2}$ for (20) with $\sigma_1 = 7$, $\sigma_2 = 4$ and $\epsilon = 1$, together with the corresponding order of magnitude of the objective function minimum.

(see, e.g., [8]). Fix $\Delta > 0$. We would like that for all $\|x\| = 1$,

$$\frac{1}{\Delta} \mathbb{E} \log (\|A_0(\Delta, M(\Delta))x\|) \approx \langle x, Bx \rangle + \frac{1}{2} \sum_{k=1}^m \|\sigma^k x\|^2 - \sum_{k=1}^m \langle x, \sigma^k x \rangle^2.$$

A simpler problem is to find $M(\Delta)$ for which the upper bounds (16) and (17) are as close as possible, and so we can expect, for instance, that V_n is exponentially stable whenever $\lambda < 0$. Then, we propose to take

$$M(\Delta) \in \arg \min \left\{ \left(\frac{1}{\Delta} \sup_{x \in \mathbb{R}^d, \|x\|=1} \mathbb{E} \log (\|A_0(\Delta, M)x\|) - \lambda \right)^2 : M \in \mathcal{M} \right\}, \quad (18)$$

where $\mathcal{M}$ is a predefined subset $\mathbb{R}^{d \times d}$. Two examples of $\mathcal{M}$ used successfully in our numerical experiments are $\mathbb{R}^{d \times d}$ and

$$\left\{ (M_{i,j})_{1 \leq i,j \leq d} : |M_{i,j}| \leq K \text{ for all } i, j = 1, \dots, d \right\},$$

with K large enough. The next theorem states that V_n converges weakly with order 1.

Theorem 3.2. *Consider $T > 0$ and $f \in C_p^4(\mathbb{R}^d, \mathbb{R})$. Let V_n be given by (15) with $\Delta = T/N$, where $N \in \mathbb{N}$. Assume that X_0 have finite moments of any order, and that for every $g \in C_p^4(\mathbb{R}^d, \mathbb{R})$,*

$$|\mathbb{E}g(X_0) - \mathbb{E}g(V_0)| \leq K(1 + \mathbb{E}\|X_0\|^q)T/N \quad \forall N \in \mathbb{N},$$

with $K > 0$. Let $\Delta \rightarrow M(\Delta)$ be bounded on $[0, T]$. Then

$$|\mathbb{E}f(X_T) - \mathbb{E}f(V_N)| \leq K(T)(1 + \mathbb{E}\|X_0\|^q)T/N \quad \forall N \in \mathbb{N}, \quad (19)$$

where $q \geq 2$ and $K(\cdot)$ is a positive increasing function.

3.3 Numerical experiments

3.3.1 Exponentially stable SDE

We consider the non-commutative test equation

$$dX_t = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} X_t dW_t^1 + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X_t dW_t^2, \quad (20)$$

		Δ					
		1/2	1/4	1/8	1/16	1/32	1/64
$\epsilon(\tilde{Y})$	$T = 1$	6.5497	9.4879	12.733	11.0676	0.15183	0.02365
	$T = 3$	18.814	28.8744	38.9743	34.1327	0.0086188	0.00075718
$\epsilon(\tilde{Z})$	$T = 1$	1.3395	1.1777	0.98272	0.7757	0.58279	0.42137
	$T = 3$	1.0611	0.78255	0.51624	0.30475	0.1643	0.08361
$\epsilon(Y^s)$	$T = 1$	1.1914	0.85936	0.49789	0.15466	0.042484	0.018271
	$T = 3$	0.81853	0.38585	0.10185	0.0096884	0.0013717	0.00055511
$\epsilon(V)$	$T = 1$	1.2544	0.8482	0.36579	0.11998	0.029324	0.0069274
	$T = 3$	0.64867	0.16695	0.035366	0.0065051	0.00068084	0.00031002

Table 3: Estimation of errors involved in the computation of $\mathbb{E} \log \left(1 + \|X_T\|^2 \right)$ for $T = 1$ and $T = 3$. Here, X_t verifies (20) with $\sigma_1 = 7$, $\sigma_2 = 4$, $\epsilon = 1$ and $X_0 = (1, 2)^\top$.

where $\sigma_1 = 7$, $\sigma_2 = 4$, $\epsilon = 1$ and $X_0 = (1, 2)^\top$. Since $0 < \sigma_2 < \sigma_1 < 3\sigma_2$, applying elementary calculus we get $\lambda = (\epsilon^2 - \sigma_2^2)/2 < 0$, and so X_t converges exponentially fast to 0. In order to illustrate the performance of schemes of type (3), we take V_n with $M(\Delta)$ given by (18) and

$$\mathcal{M} = \left\{ (M_{i,j})_{1 \leq i,j \leq 2} : |M_{i,j}| \leq 20 \text{ for all } i, j = 1, 2 \right\}.$$

Table 2 provides four-decimal approximations of the components of $M(\Delta)$, which have been obtained by running (5⁴-times) the MATLAB function `fmincon` for the initial parameters

$$\left\{ (M_{i,j})_{1 \leq i,j \leq 2} : M_{i,j} \in \{-2, -1, 0, 1, 2\} \text{ for all } i, j = 1, 2 \right\}.$$

Figure 2 shows the computation $\mathbb{E} \log \left(1 + \|X_t\|^2 \right)$ by means of V_n (dashed line), Y_n^s (dotted line), and

$$\begin{aligned} \tilde{Z}_{n+1} &= \tilde{Z}_n + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tilde{Z}_n \sqrt{\Delta} \xi_n^1 + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tilde{Z}_n \sqrt{\Delta} \xi_n^2 \\ &\quad + \sqrt{\Delta} \begin{pmatrix} |\sigma_1| + |\epsilon| & 0 \\ 0 & |\sigma_2| + |\epsilon| \end{pmatrix} (\tilde{Z}_n - \tilde{Z}_{n+1}) \end{aligned}$$

(dashdot line). $\tilde{Z}_n$ is a weak version of the balanced scheme proposed in Subsection 5.2 of [16]. The reference values for $\mathbb{E} \log \left(1 + \|X_t\|^2 \right)$ (solid line) have been calculated by using the weak Euler method

$$\tilde{Y}_{n+1} = \tilde{Y}_n + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tilde{Y}_n \sqrt{\Delta} \xi_n^1 + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tilde{Y}_n \sqrt{\Delta} \xi_n^2$$

with step-size $\Delta = 2^{-13} \approx 0.000122$. Indeed, we plot the sample means obtained from 10^8 trajectories of each scheme. Furthermore, Table 3 provides estimates of the errors $\epsilon(\hat{Y}) := \left| \mathbb{E} \log \left(1 + \|X_T\|^2 \right) - \mathbb{E} \log \left(1 + \|\hat{Y}_N\|^2 \right) \right|$, where $T = 1, 3$,

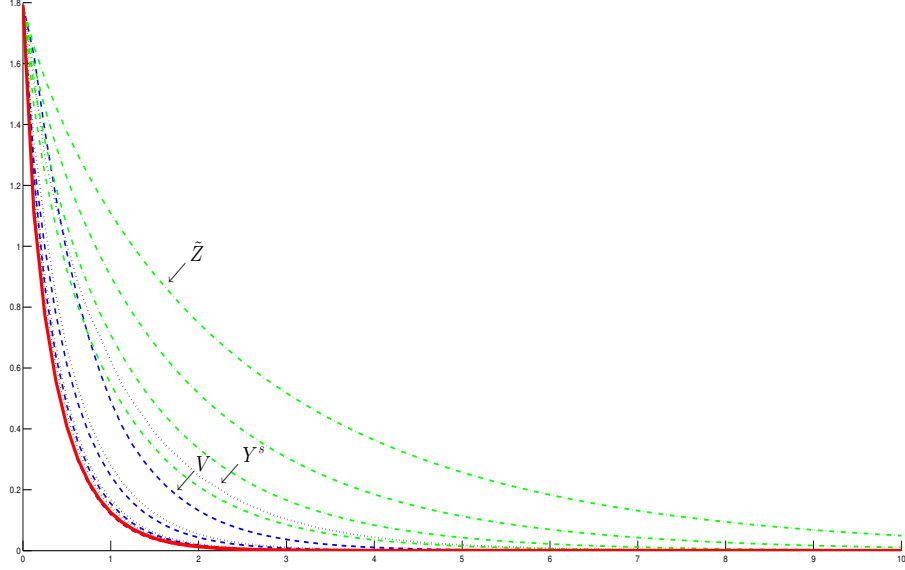


Figure 2: Computation of $\mathbb{E} \log(1 + \|X_t\|^2)$, where $t \in [0, 10]$ and X_t solves (20). Dashed line: V_n , dashdot line: $\tilde{Z}$, dotted line: Y^s , and solid line: reference values. Here, Δ is equal to $1/8$, $1/16$, $1/32$ and $1/64$; smaller deltas produce better approximations.

$N = T/\Delta$, and $\hat{Y}$ represents the numerical methods V_n , Y_n^s , $\tilde{Y}_n$ and $\tilde{Z}_n$. From Table 3 we can see that $\tilde{Y}_n$ blows up for $\Delta \leq 1/16$. Figure 2, together with Table 3, illustrate that $\tilde{Z}_n$ is stable, but presents a slow rate of weak convergence. In contrast, the performance of V_n is very good, V_n mix good stability properties with reliable approximations. The heuristic balanced scheme Y_n^s shows a very good behavior. In fact, the accuracy of Y_n^s is very similar to that of V_n for $\Delta \leq 1/16$, and Y_n^s does not involve any optimization process.

3.3.2 SDE with an unstable equilibrium point

We solve numerically the SDE

$$dX_t = \begin{pmatrix} 0 & b_2 \\ -b_2 & b_1 \end{pmatrix} X_t dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} X_t dW_t^1 + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X_t dW_t^2, \quad (21)$$

with $b_1 = 0.06$, $b_2 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\epsilon = 0.3$ and $X_t = (X_t^1, X_t^2)^\top \in \mathbb{R}^2$. We set $X_0 = (1, -1)^\top / \sqrt{2}$. We have that 0 is unstable equilibrium point of (21). Indeed, $\liminf_{t \rightarrow +\infty} \|X_t\| > 0$, because there exists $\theta \in (0, 1/2)$ such that $\langle x, Bx \rangle + \frac{1}{2} \sum_{k=1}^m \|\sigma^k x\|^2 - (1 + \theta) \sum_{k=1}^m \langle x, \sigma^k x \rangle^2 \geq 0$ for all $\|x\| = 1$ (see, e.g., [3]).

We apply to (21) the scheme V_n , where $M(\Delta)$ is defined by (18) with $\mathcal{M} = \{(M_{i,j})_{1 \leq i,j \leq 2} : |M_{i,j}| \leq 6 \text{ for all } i, j = 1, 2\}$. To this end, we first compute $M(1/128)$ by proceeding as in Subsection 3.3.1 with $M_{i,j} \in \{-1, -0.5, 0, 0.5, 1\}$.

Δ	1/4	1/8	1/16	1/32	1/64	1/128
$M_{1,1}(\Delta)$	-0.6568	-0.6683	-0.6745	-0.6805	-0.6887	-0.6949
$M_{2,1}(\Delta)$	-0.2573	-0.3220	-0.3587	-0.3777	-0.3867	-0.3911
$M_{1,2}(\Delta)$	0.3123	0.3110	0.3112	0.3130	0.3165	0.3193
$M_{2,2}(\Delta)$	-0.6382	-0.6492	-0.6544	-0.6597	-0.6675	-0.6735
Order	-18	-19	-19	-20	-21	-26

Table 4: Approximate weight matrices $(M_{i,j}(\Delta))_{1 \leq i,j \leq 2}$ for (21), together with the corresponding order of magnitude of the objective function minimum.

Then, we solve the optimization problem corresponding to $\Delta = 1/64$ (resp. $\Delta = 1/32, \dots, 1/4$) by running the MATLAB code `fmincon` with initial solution $M(1/128)$ (resp. $M(1/64), \dots, M(1/8)$) (see Table 4).

Figure 3 presents the computation $\mathbb{E} \arctan(1 + (X_t^2)^2)$ estimated by sampling 10^8 trajectories of V_n , the backward Euler scheme

$$\begin{aligned} \tilde{Y}_{n+1} &= \tilde{Y}_n + \begin{pmatrix} 0 & b_2 \\ -b_2 & b_1 \end{pmatrix} \tilde{Y}_{n+1} \Delta + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tilde{Y}_n \sqrt{\Delta} \xi_n^1 \\ &\quad + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tilde{Y}_n \sqrt{\Delta} \xi_n^2 \end{aligned}$$

and the balanced scheme $\tilde{Z}_n$ defined by (2) with $|W_{T_{n+1}}^k - W_{T_n}^k|$ replaced by $\sqrt{\Delta} \xi_n^k$, $c^0 = -\frac{1}{2} \begin{pmatrix} 0 & b_2 \\ -b_2 & b_1 \end{pmatrix}$, $c^1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $c^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ (see [1, 16]). Solid line provides the ‘exact’ values obtained by sampling 10^8 times the weak Euler-Maruyama scheme with step-size $\Delta = 2^{-13} \approx 0.000122$. Moreover, Table 5 provides the errors $\epsilon(\hat{Y}) := |\mathbb{E}f(X_T) - \mathbb{E}f(\hat{Y}_N)|$, where $T = 10, 20$, $N = T/\Delta$, and $\hat{Y}$ stands for the schemes V_n , $\tilde{Y}_n$ and $\tilde{Z}_n$. Figure 3 and Table 5 show the good accuracy of the new scheme V_n . We also see that V_n and $\tilde{Z}_n$ reply the unstable behavior of the exact solution and that $\tilde{Y}_n$ tends to 0 in case $\Delta = 1/4$. Finally, we have checked that the performs of the heuristic scheme Y_n^s is similar to that of the backward Euler scheme $\tilde{Y}_n$.

		Δ					
		1/4	1/8	1/16	1/32	1/64	1/128
$\epsilon(\tilde{Y})$	$T = 10$	0.30305	0.17473	0.089215	0.04452	0.022189	0.011144
	$T = 20$	0.47504	0.29513	0.14533	0.069623	0.033923	0.016778
$\epsilon(\tilde{Z})$	$T = 10$	0.10986	0.12667	0.10127	0.069549	0.044963	0.028632
	$T = 20$	0.056883	0.049722	0.051519	0.036835	0.022017	0.0127
$\epsilon(V)$	$T = 10$	0.084783	0.051834	0.014313	0.0031168	0.00055984	0.00011417
	$T = 20$	0.025462	0.0080347	0.014935	0.011232	0.0062795	0.0032123

Table 5: Estimation of errors involved in the computation of $\mathbb{E} \arctan(1 + (X_T^2)^2)$ for $T = 10$ and $T = 20$, where X_t solves (21).

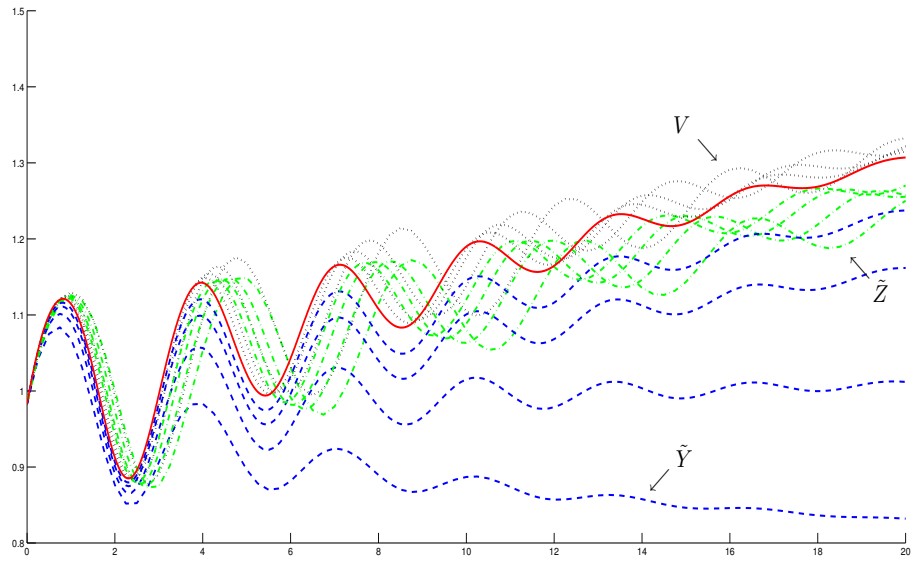


Figure 3: Computation of $\mathbb{E} \arctan(1 + (X_t^2)^2)$, where $t \in [0, 20]$ and X_t solves (21). Dashed line: $\tilde{Y}$, dashdot line: $\tilde{Z}$, dotted line: V , and solid line: reference values. Here, Δ takes the values $1/4$, $1/8$, $1/16$ and $1/32$; smaller values of Δ produce better approximations.

4 Proofs

Proof of Lemma 2.1. We first prove that under Property (P1), $\lim_{n \rightarrow \infty} Y_n^s = 0$ a.s. iff

$$\begin{cases} a(\Delta) < (\mu^2 \Delta + 2\mu - \lambda^2) / (2\mu \Delta), & \text{if } \mu < 0 \\ a(\Delta) \in \mathbb{R}, & \text{if } \mu = 0 \text{ and } \lambda \neq 0 \\ a(\Delta) > (\mu^2 \Delta + 2\mu - \lambda^2) / (2\mu \Delta), & \text{if } \mu > 0 \end{cases} \quad (22)$$

Suppose that Property P1 holds. Applying the strong law of large numbers and the law of iterated logarithm we obtain that $Y_n^s \rightarrow 0$ a.s. as $n \rightarrow \infty$ iff

$$\mathbb{E} \log \left(1 + \left(\mu \Delta + \lambda \sqrt{\Delta} \xi_n^1 \right) / (1 - a(\Delta) \Delta) \right) < 0 \quad (23)$$

(see, e.g., Lemma 5.1 of [7]). Since

$$\mathbb{E} \log \left(1 + \frac{\mu \Delta + \lambda \sqrt{\Delta} \xi_n^1}{1 - a(\Delta) \Delta} \right) = \frac{1}{2} \log \left(\left(1 + \frac{\mu \Delta}{1 - a(\Delta) \Delta} \right)^2 - \frac{\lambda^2 \Delta}{(1 - a(\Delta) \Delta)^2} \right),$$

inequality (23) becomes $2\mu(1 - a(\Delta) \Delta) + \mu^2 \Delta - \lambda^2 < 0$, which is equivalent to (22). This establishes our first claim.

From the assertion of the first paragraph we get that Property P1, together with $\lim_{n \rightarrow \infty} Y_n^s = 0$ a.s., is equivalent to (a) $a(\Delta) \in]-\infty, \min\{p_1, p_3\}[\cup]p_2, p_3[$ for $\mu < 0$; (b) $a(\Delta) \in]-\infty, p_1[\cup]p_2, +\infty[$ for $\mu = 0$ and $\lambda \neq 0$; and $a(\Delta) \in]p_3, p_1[\cup]\max\{p_2, p_3\}, +\infty[$ for $\mu > 0$. This gives the lemma, because $p_1 < p_3$ (resp. $p_2 > p_3$) whenever $\mu < 0$ (resp. $\mu > 0$). $\square$

Proof of Theorem 2.2. In case $\lambda \neq 0$, using differential calculus we obtain that the function $\Delta \mapsto (1 - |\lambda| \sqrt{\Delta} + \mu \Delta) / \Delta$ attains its global minimum at $4/\lambda^2$. Then, for all $\Delta > 0$ and $\lambda \in \mathbb{R}$ we have

$$(1 - |\lambda| \sqrt{\Delta} + \mu \Delta) / \Delta \geq \mu - \lambda^2 / 4. \quad (24)$$

First, we suppose that $\mu \leq 0$ and $\alpha_1(\Delta) > 1/4$. From (24) it follows that $p_1 > \mu - \alpha_1(\Delta) \lambda^2$, which implies $a(\Delta) \in]-\infty, p_1[$. Second, if $\mu > 0$ and $\Delta \geq 2/\mu$, then $a(\Delta) \in]p_2, +\infty[$.

Third, assume that $\mu > 0$ and $\Delta < 2/\mu$. Since $\mu > 0$, for any $\Delta < \lambda^2/\mu^2$ we have $1 - |\lambda| \sqrt{\Delta} + \mu \Delta < 1$. Using $2\mu - \lambda^2 < 0$ we get $\lambda^2/\mu^2 > 2/\mu$, and so $p_1 = (1 - |\lambda| \sqrt{\Delta} + \mu \Delta) / \Delta$ whenever $\Delta < 2/\mu$. Applying (24) gives $p_1 > \mu - \alpha_2(\Delta) \lambda^2$, because $\alpha_2(\Delta) > 1/4$. On the other hand, we have $p_3 < \mu - \alpha_2(\Delta) \lambda^2$ if and only if $2\mu - \lambda^2 < \mu \Delta (2\mu - 4\alpha_2(\Delta) \lambda^2) / 2$, which becomes

$$\frac{2}{\mu} > \Delta \left(1 + (4\alpha_2(\Delta) - 1) \frac{\lambda^2}{\lambda^2 - 2\mu} \right) \quad (25)$$

since $2\mu - \lambda^2 < 0$ and $\mu > 0$. By $2/\mu > \Delta$, (25) holds in case

$$\frac{2}{\mu} \geq \Delta + (4\alpha_2(\Delta) - 1) \frac{\lambda^2}{\lambda^2 - 2\mu} \frac{2}{\mu},$$

which is equivalent to $\alpha_2(\Delta) \leq 1/4 + (\lambda^2 - 2\mu)(2 - \mu \Delta) / (8\lambda^2)$. Then $p_3 < \mu - \alpha_2(\Delta) \lambda^2$, hence $a(\Delta) \in]p_3, p_1[$.

Combining Lemma 2.1 with the above three cases yields Properties P1 and P2. $\square$

Proof of the inequality (8). If $\mu \leq 0$, then $a(\Delta) \leq \mu - \lambda^2/4 \leq 0$, and so $1 - a(\Delta)\Delta \geq 1$. Let $\mu > 0$, together with $\Delta \geq 2/\mu$. Then we have $1 - a(\Delta)\Delta = -|\lambda|\sqrt{\Delta} - \mu\Delta - \beta\Delta \leq -2$.

Finally, suppose that $\mu > 0$ and $\Delta < 2/\mu$. Since $2\mu - \lambda^2 < 0$, there exists $\epsilon \in (0, 1)$ such that $2\mu - \lambda^2 < -2\epsilon\mu$. Hence $\mu - \lambda^2/4 < (1 - \epsilon)\mu/2 < (1 - \epsilon)/\Delta$, which implies

$$\left(\mu - \frac{1 - \epsilon}{\Delta}\right) \frac{1}{\lambda^2} < \frac{1}{4} < \alpha_2(\Delta).$$

We thus get $1 - a(\Delta)\Delta > \epsilon$. $\square$

Proof of Theorem 2.3. We first prove that Properties P1' and P2' hold provided that $2\mu - \lambda^2 < 0$ and

$$a < \min \left\{ 1/\Delta, \mu/2 - \lambda^2/4 + \left(2 - |\lambda|\sqrt{\Delta}\right)/2\Delta \right\}. \quad (26)$$

From (26) we have $4 + (2\mu - \lambda^2 - 4a)\Delta - 2|\lambda|\sqrt{\Delta} > 0$, and so for all $\Delta > 0$,

$$4 + (2\mu - \lambda^2 - 4a)\Delta + 2\lambda\sqrt{\Delta}\xi_n^1 > 0.$$

Since $2\mu - \lambda^2 < 0$, $4 - (2\mu - \lambda^2 + 4a)\Delta - 2|\lambda|\sqrt{\Delta} > 0$. Hence, for any $\Delta > 0$, $4 - (2\mu - \lambda^2 + 4a)\Delta - 2\lambda\sqrt{\Delta}\xi_n^1 > 0$. Therefore Y_n^T satisfies Property P1'. Moreover, as in the proof of Lemma 2.1, using the strong law of large numbers and the law of iterated logarithm we deduce that $Y_n^T \rightarrow 0$ a.s. as $n \rightarrow \infty$ iff

$$\mathbb{E} \log \left(\frac{4 + (2\mu - \lambda^2 - 4a)\Delta + 2\lambda\sqrt{\Delta}\xi_n^1}{4 - (2\mu - \lambda^2 + 4a)\Delta - 2\lambda\sqrt{\Delta}\xi_n^1} \right) < 0. \quad (27)$$

Inequality (27) is equivalent to

$$(4 + (2\mu - \lambda^2 - 4a)\Delta)^2 - 4\lambda^2\Delta < (4 - (2\mu - \lambda^2 + 4a)\Delta)^2 - 4\lambda^2\Delta,$$

which becomes $16(2\mu - \lambda^2)\Delta(1 - a\Delta) < 0$, and so Property P2' holds because $a < 1/\Delta$.

Consider $\lambda = 0$. Then, the claim of the first paragraph guarantees that Properties P1' and P2' holds if $a < \min \{1/\Delta, \mu/2 + 1/\Delta\}$. Since $2\mu - \lambda^2 < 0$ we have $\mu < 0$, and so a sufficient condition for Properties P1' and P2' is $a < \mu/2$.

Finally, suppose that $\lambda \neq 0$ and set $f(\Delta) = (2 - |\lambda|\sqrt{\Delta})/2\Delta$ for all $\Delta > 0$. Then, we get $f'(\Delta) = (|\lambda|/4 - 1/\sqrt{\Delta})/\Delta^{3/2}$. Note that f is increasing or decreasing depending on $\sqrt{\Delta} > 4/|\lambda|$ or $\sqrt{\Delta} < 4/|\lambda|$, respectively. Thus, f attains its global minimum at $\Delta_0 = 16/\lambda^2$. Since $2\mu - \lambda^2 < 0$, we have $\mu/2 - \lambda^2/4 + f(\Delta_0) = \mu/2 - 5\lambda^2/16 < 0$. Then, $\mu/2 - 5\lambda^2/16 \leq \min \{1/\Delta, \mu/2 - \lambda^2/4 + f(\Delta)\}$. Using again the claim of the first paragraph we conclude that Properties P1' and P2' holds under $a < \mu/2 - 5\lambda^2/16$. $\square$

Proof of Theorem 3.1. From (15) it follows that

$$V_n = A_{n-1}(\Delta, M(\Delta)) A_{n-2}(\Delta, M(\Delta)) \cdots A_0(\Delta, M(\Delta)) V_0.$$

Since ξ_n^k are bounded random variables,

$$\sup_{x \in \mathbb{R}^d, \|x\|=1} \mathbb{E} \log_+ (\|A_0(\Delta, M(\Delta))x\|) < \infty,$$

where $\log_+(x)$ stands for the positive part of $\log(x)$. Hence, the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\|V_n\|)$$

exists whenever $V_0 \neq 0$, and only depends on V_0 . Furthermore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\|V_n\|) = \int_{\|x\|=1} \mathbb{E} \log(\|A_0(\Delta, M(\Delta))x\|) \mu(dx),$$

where μ is a probability measure (see, e.g., Theorem 3.1 of [5]). This gives (16). $\square$

Proof of Theorem 3.2. Let $q \geq 2$. Iterating (15) we obtain

$$V_{n+1} = V_0 + (I + \Delta M(\Delta)) \left(\Delta B \sum_{k=0}^n V_k + \sum_{j=1}^m \sum_{k=0}^n \sqrt{\Delta} \xi_k^j \sigma^j V_k \right).$$

Since $\Delta \rightarrow M(\Delta)$ is locally bounded,

$$\|V_{n+1}\|^q \leq K_q(T) \left(\|V_0\|^q + \frac{1}{N} \sum_{k=0}^n \|V_k\|^q + m^{q-1} \sum_{j=1}^m \left\| \sum_{k=0}^n \sqrt{\Delta} \xi_k^j \sigma^j V_k \right\|^q \right),$$

where, from now on, $K_q(\cdot)$ is a generic positive increasing function. By ξ_k^j is bounded, applying the Burkholder-Davis-Gundy inequality yields

$$\mathbb{E} \|V_{n+1}\|^q \leq K_q(T) \left(\|V_0\|^q + \frac{1}{N} \sum_{k=0}^n \|V_k\|^q \right),$$

and so using a discrete Gronwall lemma (see, e.g., [4]) we get

$$\mathbb{E} \|V_n\|^q \leq K_q(T) \mathbb{E} \|V_0\|^q \quad \forall n = 0, \dots, N. \quad (28)$$

According to (15) we have

$$V_{n+1} - V_n = (I + \Delta M(\Delta)) \left(\Delta B + \sum_{k=1}^m \sqrt{\Delta} \xi_n^k \sigma^k \right) V_n. \quad (29)$$

Hence $\|V_{n+1} - V_n\| \leq K(T) \Delta^{1/2} \|V_n\|$, which implies

$$\mathbb{E} (\|V_{n+1} - V_n\|^q \mid \mathfrak{F}_{T_n}) \leq K_q(T) \Delta^{q/2} \|V_n\|^q. \quad (30)$$

From (29) it follows

$$\begin{aligned} & \left\| \mathbb{E} \left(V_{n+1} - V_n - \left(B\Delta + \sum_{k=1}^m \sigma^k \left(W_{(n+1)\Delta}^k - W_{n\Delta}^k \right) \right) V_n \mid \mathfrak{F}_{n\Delta} \right) \right\| \\ & \leq K(T) \Delta^2 (1 + \|V_n\|). \end{aligned}$$

Moreover, using (29) we deduce that the second (resp., third) moments of $V_{n+1} - V_n$ coincide with that of $\left(B\Delta + \sum_{k=1}^m \sigma^k \left(W_{(n+1)\Delta}^k - W_{n\Delta}^k \right) \right) V_n$, except for terms of order $\mathcal{O}(\Delta^2) \|V_n\|^2$ (resp., $\mathcal{O}(\Delta^2) \|V_n\|^3$). Here, $\mathcal{O}(\Delta^2)$ stands for different random functions depending on Δ^2 that are less than $K(T) \Delta^2$. Therefore, combining classical arguments [12, 18, 19] with (28) and (30) we conclude that (19) holds (see also Theorem 14.5.2 of [11]). $\square$

References

- [1] J. Alcock and K. Burrage, *A note on the Balanced method*, BIT **46** (2006), 689–710.
- [2] ———, *Stable strong order 1.0 schemes for solving stochastic ordinary differential equations*, BIT **52** (2012), 539–557.
- [3] J. A. D. Appleby, X. Mao, and A. Rodkina, *Stabilization and destabilization of nonlinear differential equations by noise*, IEEE Trans. Automat. Control **53** (2008), 683–691.
- [4] L. Arnold, *Random dynamical systems*, Springer, Berlin, 1998.
- [5] J. E. Cohen and C. M. Newman, *The stability of large random matrices and their products*, Ann. Probab. **12** (1984), 283–310.
- [6] I. Gyöngy, *Approximations of stochastic partial differential equations*, Stochastic Partial Differential Equations (G. Da Prato and L. Tubaro, eds.), Lecture Notes in Pure and Appl. Math., vol. 227, Dekker, New York, 2002, pp. 287–307.
- [7] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer. Anal. **38** (2000), 753–769.
- [8] D. J. Higham, X. Mao, and C. Yuan, *Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations*, SIAM J. Numer. Anal. **45** (2007), 592–609.
- [9] A. Jentzen and P. Kloeden, *Taylor approximations for stochastic partial differential equations*, SIAM, Philadelphia, 2011.
- [10] C. Kahl and H. Schurz, *Balanced Milstein methods for ordinary SDEs*, Monte Carlo Methods Appl. **12** (2006), 143–170.
- [11] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer-Verlag, Berlin, 1992.
- [12] G. N. Milstein, *Weak approximation of solutions of systems of stochastic differential equations*, Theory Probab. Appl. **30** (1985), 750–766.
- [13] ———, *Numerical integration of stochastic differential equations*, Kluwer Academic Publishers, Dordrecht, 1995.
- [14] G. N. Milstein, E. Platen, and H. Schurz, *Balanced implicit methods for stiff stochastic systems*, SIAM J. Numer. Anal. **35** (1998), 1010–1019.
- [15] C. Mora, *Weak exponential schemes for stochastic differential equations with additive noise*, IMA J. Numer. Anal. **25** (2005), 486–506.
- [16] H. Schurz, *Convergence and stability of balanced implicit methods for systems of SDEs*, Int. J. Numer. Anal. Model. **2** (2005), 197–220.
- [17] ———, *Basic concepts of numerical analysis of stochastic differential equations explained by balanced implicit theta methods*, Stochastic Differential Equations and Processes (Z. Mounir and D. V. Filatova, eds.), Springer, New York, 2012, pp. 1–139.
- [18] D. Talay, *Efficient numerical schemes for the approximation of expectations of functionals of SDE and applications*, Filtering and Control of Random Processes (J. Szpirglas, H. Korezlioglu, and G. Mazziotto, eds.), Lecture Notes in Control and Information Science, vol. 61, Springer, Berlin, 1984, pp. 294–313.
- [19] ———, *Discretisation d’une équation différentielle stochastique et calcul approché d’espérances de fonctionnelles de la solution*, Math. Modelling Numer. Anal. **20** (1986), 141–179.

- [20] ———, *Probabilistic numerical methods for partial differential equations: elements of analysis*, Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995) (Denis Talay and Luciano Tubaro, eds.), Lecture Notes in Mathematics, vol. 1627, Springer, Berlin, 1996, pp. 148–196.
- [21] M. V. Tretyakov and Z. Zhang, *A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications*, SIAM J. Numer. Anal. **51** (2013), 3135–3162.

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