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Analysis of an augmented HDG method for a class of quasi-Newtonian Stokes flows*

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Abstract

In this paper we introduce and analyze a hybridizable discontinuous Galerkin (HDG) method for numerically solving a class of nonlinear Stokes models arising in quasi-Newtonian fluids. Similarly as in previous papers dealing with the application of mixed finite element methods to these nonlinear models, we use the incompressibility condition to eliminate the pressure, and set the velocity gradient as an auxiliary unknown. In addition, we enrich the HDG formulation with two suitable augmented equations, which allows us to apply known results from nonlinear functional analysis, namely a nonlinear version of Babuška-Brezzi theory and the classical Banach fixed-point theorem, to prove that the discrete scheme is well-posed and derive the corresponding a priori error estimates. Then we discuss some general aspects concerning the computational implementation of the method, which show a significant reduction of the size of the linear systems involved in the Newton iterations. Finally, we provide several numerical results illustrating the good performance of the proposed scheme and confirming the optimal order of convergence provided by the HDG approximation.

Key words: nonlinear Stokes model, mixed finite element method, hybridized discontinuous Galerkin method, augmented formulation

1 Introduction

The devising of suitable numerical methods for solving the linear and nonlinear Stokes and related problems has become a very active research area during the last decade. In particular, a mixed finite element method and a suitable augmented version of the latter for a nonlinear Stokes flow problem involving a non-Newtonian fluid, are introduced and analized in [19]. In addition, the velocity-pressure-stress formulation for incompressible flows has gained considerable attention in recent years due to its natural applicability to non-Newtonian flows, where the corresponding constitutive equations are nonlinear. In general, an interesting feature of the mixed methods is given by the fact that, besides the original unknowns, they yield direct approximations of several other quantities of physical interest. For instance, an accurate direct calculation of the stresses is very desirable for flow problems involving interaction with solid structures.

On the other hand, the hybridizable discontinuous Galerkin (HDG) method, introduced in [10] for diffusion problems, is one of the several high-order discretization schemes that benefit from the

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hybridization technique originally applied in [14] to the local discontinuous Galerkin (LDG) method for time dependent convection-diffusion problems. The main advantages of HDG methods include a substantial reduction of the globally coupled degrees of freedom, which was a criticism for the discontinuous Galerkin (DG) methods for elliptic problems during the last decade, and the fact that convergence is obtained even for a polynomial degree $k = 0$. Additionally, the approximate flux converges with order $k+1$ for $k \geq 0$, and an element-by-element computation of a new approximation of the scalar variable is possible, which converges with order $k+2$ for $k \geq 1$ (see e.g. [9, 12, 11]). In the context of the linear Stokes equation, the hybridization for DG methods was initially introduced in [5] and then analyzed in [28, 11]. Lately, an overview of the recent work by Cockburn and co-workers on the devising of hybridizable discontinuous Galerkin (HDG) methods for the Stokes equations of incompressible flow was provided in [13].

Now, the utilization of DG methods to numerically solve nonlinear boundary value problems has been first considered in [3] and [22]. Indeed, the application of the local discontinuous Galerkin (LDG) method to a class of nonlinear diffusion problems was developed in [3], whereas the extension of the interior penalty *hp* DG method to quasilinear elliptic equations was studied in [22]. The results from [3] were generalized in [4], where the a-priori and a-posteriori error analyses of the LDG method as applied to certain type of nonlinear Stokes models (whose kinematic viscosities are nonlinear monotone functions of the gradient of the velocity) were derived. The approach in [4] is based on the introduction of the flux and the tensor gradient of the velocity as further unknowns. A suitable Lagrange multiplier is also needed to ensure that the corresponding discrete variational formulation is well-posed. A two-fold saddle point operator equation is obtained as the resulting LDG mixed formulation, which is then reduced to a dual mixed formulation. A nonlinear version of the well known Babuška-Brezzi theory is applied to prove that the discrete formulation is well-posed and derive the corresponding a priori error analysis. In turn, the analysis from [22] was extended in [15], where the a priori and a posteriori error analysis, with respect to a mesh-dependent energy norm, of a class of interior penalty *hp* DGFEM for the numerical approximation of basically the same quasi-Newtonian fluid flow problems studied in [4], were provided. Furthermore, an HDG approach was employed in [27] for the numerical solution of steady and time-dependent nonlinear convection-diffusion equations. In fact, the approximate scalar variable and corresponding flux are first expressed in [27] in terms of an approximate trace of the scalar variable, and then the jump condition of the numerical fluxes are explicitly enforced across the element boundaries. As a consequence, a global equation system solely in terms of the approximate trace of the scalar variable is obtained at every Newton iteration. At the end, and similarly as in previous papers on HDG, an element-by-element postprocessing scheme is applied to obtain new approximations of the flux and the scalar variable, which converge with order $k+1$ and $k+2$, respectively, in the L^2-norm. Nevertheless, and up to our knowledge, there is still no contribution in the literature concerning HDG for nonlinear Stokes systems.

According to the above discussion, we are interested in this paper in applying the HDG approach to the class of quasi-Newtonian Stokes flows studied in [4, 17, 15] (see also [19, 23]). To this end, we plan to employ the same velocity-pseudostress formulation from [19]. In what follows, given any Hilbert space U, $\mathbf{U} := U^n$ and $\mathbb{U} := U^{n\times n}$ denote, respectively, the space of vector and square matrices of order n, $n \in \{2,3\}$, with entries in U. In order to define the boundary value problem of interest, we now let Ω be a bounded and simply connected polygonal domain in R^n with boundary Γ. As in [19], our goal is to determine the velocity $\mathbf{u}$, the pseudostress tensor $\boldsymbol{\sigma}$, and the pressure p of a steady flow occupying the region Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$, and a scalar field p such that

$$\begin{gathered}\boldsymbol{\sigma} \,=\, \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega, \qquad \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega p \,=\, 0\,,\end{gathered} \tag{1.1}$$

where $\mu : R^+ \to R^+$ is the nonlinear kinematic velocity function of the fluids, $\mathbf{div}$ stands for the usual divergence operator div acting along each row of tensor, $\nabla \mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $R^{n\times n}$, and $\mathbb{I}$ is the identity matrix of $R^{n\times n}$. As required by the incompressibility condition, we assume from now on that the datum $\mathbf{g}$ satisfies the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0$, where $\boldsymbol{\nu}$ stands for the unit outward normal at Γ. The kind of nonlinear Stokes problem given by (1.1) appears in the modeling of a large class of non-Newtonian fluids (see, e.g. [1, 25, 26, 30]). In particular, the Ladyzhenskaya law, is given by $\mu(t) := \mu_0 + \mu_1 t^{\beta-2}\ \forall\, t \in R^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta > 1$, and the Carreau law for viscoplastic flows (see, e.g. [26, 30]) reads $\mu(t) := \mu_0 + \mu_1(1+t^2)^{(\beta-2)/2}\ \forall\, t \in R^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta \geq 1$.

The rest of the work is organized as follows. In Section 2 we introduce the augmented hybridizable discontinuous Galerkin formulation involving the velocity, the pseudostress, the velocity gradient and the trace of the velocity, as unknowns. In Section 3 we show the unique solvability of the augmented HDG scheme by considering an equivalent formulation and then applying a nonlinear version of the Babuška-Brezzi theory and the classical Banach fixed-point Theorem. The corresponding *a priori* error estimates are derived in Section 4. Next, in Section 5 we discuss some general aspects concerning the computational implementation of the HDG method. Finally, several numerical experiments validating the good performance of the method and confirming the rates of convergence derived are reported in Section 6. We end the present section with further notations to be used below. Given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in R^{n\times n}$, we write as usual

$$\mathrm{tr}\,(\boldsymbol{\tau}) \ := \ \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau}^{\mathrm{d}} \ := \ \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} \ := \ \sum_{i,j=1}^n \tau_{ij}\zeta_{ij}.$$

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, null tensor or null operator, and use C, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The augmented HDG method

2.1 The hybridizable discontinuous Galerkin method

We begin by eliminating the pressure. Indeed, we know from [19, Section 2.1] that the pair given by the first and third equations in (1.1) is equivalent to

$$\boldsymbol{\sigma} \ = \ \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - p\mathbb{I} \ \text{ in } \Omega \quad \text{and} \quad p \ = \ -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}) \ \text{ in } \Omega. \tag{2.1}$$

In what follows we let $\psi_{ij} : R^{n\times n} \to R$ be the mapping given by $\psi_{ij}(\mathbf{r}) := \mu(|\mathbf{r}|)r_{ij}$ for all $\mathbf{r} := (r_{ij}) \in R^{n\times n}$, for all $i, j \in \{1, \ldots, n\}$. Then, throughout this paper we assume that μ is of class C^1 and that there exist $\gamma_0, \alpha_0 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in R^{n\times n}$, there holds

$$|\psi_{ij}(\mathbf{r})| \ \leq \ \gamma_0\|\mathbf{r}\|_{R^{n\times n}}, \qquad \left|\frac{\partial}{\partial r_{kl}}\psi_{ij}(\mathbf{r})\right| \ \leq \ \gamma_0, \quad \forall\, i,j,k,l \in \{1,\ldots,n\}, \tag{2.2}$$

and

$$\sum_{i,j,k,l=1}^n \frac{\partial}{\partial r_{kl}}\psi_{ij}(\mathbf{r}) s_{ij} s_{kl} \ \ \geq \ \ \alpha_0\|\mathbf{s}\|^2_{R^{n\times n}}. \tag{2.3}$$

It is easy to check that the Carreau law satisfies (2.2) and (2.3) for all $\mu_0 > 0$, and for all $\beta \in [1, 2]$. In particular, with $\beta = 2$ we recover the usual linear Stokes model.

Observe that we can rewrite (2.1) as

$$\boldsymbol{\sigma} \;=\; \boldsymbol{\psi}(\nabla \mathbf{u}) - p\mathbb{I} \quad \text{in } \Omega \quad \text{and} \quad p \;=\; -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}) \quad \text{in } \Omega,$$

where $\boldsymbol{\psi} : R^{n\times n} \to R^{n\times n}$ is given by $\boldsymbol{\psi}(\mathbf{r}) := (\psi_{ij}(\mathbf{r}))$ for all $\mathbf{r} := (r_{ij}) \in R^{n\times n}$. Hence, replacing p by $-\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma})$ in the first equation of (1.1), and introducing the gradient $\mathbf{t} := \nabla \mathbf{u}$ in Ω as an auxiliary unknown, we arrive at the system

$$\begin{aligned} \boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\sigma}^{\mathrm{d}} \;=\; \mathbf{0} \quad \text{in} \quad \Omega\,, &\qquad \mathbf{t} - \nabla\mathbf{u} \;=\; \mathbf{0} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, &\qquad \operatorname{tr}(\mathbf{t}) \;=\; 0 \quad \text{in} \quad \Omega\,, \\ \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, &\qquad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) \;=\; 0\,. \end{aligned} \tag{2.4}$$

We recall here that a well-posed continuous formulation of (2.4) has been proposed in [19, Section 2], which reads: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X_1 \times M_1 \times \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} \quad &= \quad 0 \quad \forall\, \mathbf{s} \in X_1, \\ -\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}} - \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \quad &= \quad -\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in M_1, \\ -\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \quad &= \quad \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega), \end{aligned} \tag{2.5}$$

where $X_1 := \{\mathbf{s} \in \mathbb{L}^2(\Omega) \;:\; \operatorname{tr}(\mathbf{s}) = 0\}$ and $M_1 = \big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) \;:\; \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\big\}$. The purpose of reminding here (2.5) will become clear in the *a priori* error analysis given below in Section 4.

Next, in order to introduce the HDG method for the system (2.4), we first need some preliminary notations. Let $\mathcal{T}_h$ be a shape-regular triangulation of $\bar{\Omega}$ without the presence of hanging nodes, and let $\mathcal{E}_h$ be the set of faces F of $\mathcal{T}_h$. Then, we set

$$\partial\mathcal{T}_h \;:=\; \cup\{\partial T \;:\; T \in \mathcal{T}_h\}\,,$$

and introduce the inner products:

$$\begin{aligned} (\mathbf{u},\mathbf{v})_{\mathcal{T}_h} \;&:=\; \sum_{T\in\mathcal{T}_h} \int_T \mathbf{u}\cdot\mathbf{v} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\mathcal{T}_h), \\ (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{T}_h} \;&:=\; \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\sigma} : \boldsymbol{\tau} \quad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{T}_h), \\ \langle \mathbf{u},\mathbf{v}\rangle_{\partial\mathcal{T}_h} \;&:=\; \sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathbf{u}\cdot\mathbf{v} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\partial\mathcal{T}_h), \\ \langle \mathbf{u},\mathbf{v}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} \;&:=\; \sum_{T\in\mathcal{T}_h} \sum_{F\in\partial T\setminus\Gamma} \int_F \mathbf{u}\cdot\mathbf{v} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\partial\mathcal{T}_h), \end{aligned}$$

with the induced norm

$$\|\mathbf{v}\|_{\mathcal{T}_h} \;:=\; (\mathbf{v},\mathbf{v})_{\mathcal{T}_h}^{1/2} \quad \forall\, \mathbf{v} \in \mathbf{L}^2(\mathcal{T}_h).$$

In addition, we let $P_k(U)$ be the space of polynomials of total degree at most k defined on the domain U, and denote by $\mathcal{E}_h^i$ and $\mathcal{E}_h^\partial$ the set of interior and boundary faces, respectively, of $\mathcal{E}_h$.

On the other hand, let $\boldsymbol{\nu}^+$ and $\boldsymbol{\nu}^-$ be the outward unit normal vectors on the boundaries of two neighboring elements T^+ and T^-, respectively. We use $(\boldsymbol{\tau}^\pm, \mathbf{v}^\pm)$ to denote the traces of $(\boldsymbol{\tau}, \mathbf{v})$ on

$F := \partial T^+ \cap \partial T^-$ from the interior of $T^{\pm}$, where $\boldsymbol{\tau}$ and $\mathbf{v}$ are second-order tensorial and vectorial functions, respectively. Then, we define the means $\{\!\!\{\cdot\}\!\!\}$ and jumps $[\![\cdot]\!]$ for $F \in \mathcal{E}_h^i$, as follows

$$\begin{aligned} \{\!\!\{\boldsymbol{\tau}\}\!\!\} &:= \frac{1}{2}\left(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-\right), \qquad & \{\!\!\{\mathbf{v}\}\!\!\} &:= \frac{1}{2}\left(\mathbf{v}^+ + \mathbf{v}^-\right), \\ [\![\boldsymbol{\tau}]\!] &:= \boldsymbol{\tau}^+\boldsymbol{\nu}^+ + \boldsymbol{\tau}^-\boldsymbol{\nu}^-, \qquad & [\![\mathbf{v}]\!] &:= \mathbf{v}^+ \otimes \boldsymbol{\nu}^+ + \mathbf{v}^- \otimes \boldsymbol{\nu}^-, \end{aligned}$$

where $\otimes$ denotes the usual dyadic or tensor product. Next, given $k \geq 1$, the finite dimensional discontinuous subspaces are given by

$$\begin{aligned} S_h &:= \left\{\mathbf{s} \in \mathbb{L}^2(\Omega) \;:\; \mathbf{s}|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\right\}, \\ \Sigma_h &:= \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \;:\; \boldsymbol{\tau}|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h, \quad \text{and} \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\right\}, \\ V_h &:= \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) \;:\; \mathbf{v}|_T \in \mathbf{P}_{k-1}(T) \quad \forall\, T \in \mathcal{T}_h\right\}, \\ M_h &:= \left\{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h^i) \;:\; \boldsymbol{\mu}|_F \in \mathbf{P}_k(F) \quad \forall\, F \in \mathcal{E}_h^i\right\}. \end{aligned}$$

In this way, proceeding as in [11], the HDG formulation of (2.4) reduces to: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\lambda}_h) \in S_h \times \Sigma_h \times V_h \times M_h$, such that

$$(\boldsymbol{\psi}(\mathbf{t}_h), \mathbf{s}_h)_{\mathcal{T}_h} - (\mathbf{s}_h, \boldsymbol{\sigma}_h^{\mathrm{d}})_{\mathcal{T}_h} = 0 \quad \forall\, \mathbf{s}_h \in S_h, \tag{2.6a}$$

$$(\mathbf{t}_h, \boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} + (\mathbf{u}_h, \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} - \langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h} = 0 \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \tag{2.6b}$$

$$(\boldsymbol{\sigma}_h, \nabla\mathbf{v}_h)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}_h\boldsymbol{\nu}}, \mathbf{v}_h\rangle_{\partial\mathcal{T}_h} = (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} \quad \forall\, \mathbf{v}_h \in V_h, \tag{2.6c}$$

$$\langle \widehat{\boldsymbol{\sigma}_h\boldsymbol{\nu}}, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0 \quad \forall\, \boldsymbol{\mu}_h \in M_h, \tag{2.6d}$$

where, letting Π_Γ be the $\mathbf{L}^2(\Gamma)$ projection onto the space of piecewise polynomials of degree $\leq k$ on $\mathcal{E}_h^\partial$, we define the numerical fluxes $\widehat{\mathbf{u}}_h$ and $\widehat{\boldsymbol{\sigma}_h\boldsymbol{\nu}}$ as

$$\widehat{\mathbf{u}}_h := \begin{cases} \Pi_\Gamma(\mathbf{g}) & \text{on } \mathcal{E}_h^\partial, \\ \boldsymbol{\lambda}_h & \text{on } \mathcal{E}_h^i, \end{cases} \quad \text{and} \quad \widehat{\boldsymbol{\sigma}_h\boldsymbol{\nu}} := \boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{S}(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \;\text{ on } \partial\mathcal{T}_h, \tag{2.6e}$$

where $\mathbf{S}$ is a stabilization operator to be defined below. Note that the condition $\widehat{\mathbf{u}}_h = \Pi_\Gamma(\mathbf{g})$ on $\mathcal{E}_h^\partial$ is usually imposed in the equivalent way $\langle \widehat{\mathbf{u}}_h, \boldsymbol{\mu}_h\rangle_\Gamma = \langle \mathbf{g}, \boldsymbol{\mu}_h\rangle_\Gamma \;\forall\, \boldsymbol{\mu}_h \in \mathbf{P}(\mathcal{E}_h)$, which is employed to perform the solvability analysis of (2.6). In this sense, note first that problem (2.6) can be reformulated as

$$\begin{aligned} (\boldsymbol{\psi}(\mathbf{t}_h), \mathbf{s}_h)_{\mathcal{T}_h} - (\mathbf{s}_h, \boldsymbol{\sigma}_h^{\mathrm{d}})_{\mathcal{T}_h} &= 0, \\ (\mathbf{t}_h, \boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} + (\mathbf{u}_h, \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} - \langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \boldsymbol{\lambda}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= \langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma, \\ -(\mathbf{v}_h, \mathbf{div}(\boldsymbol{\sigma}_h))_{\mathcal{T}_h} + \langle \mathbf{S}(\mathbf{u}_h - \boldsymbol{\lambda}_h), \mathbf{v}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle \mathbf{S}\mathbf{u}_h, \mathbf{v}_h\rangle_\Gamma &= (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \mathbf{S}\mathbf{g}, \mathbf{v}_h\rangle_\Gamma, \\ \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} - \langle \mathbf{S}(\mathbf{u}_h - \boldsymbol{\lambda}_h), \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0, \end{aligned}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\mu}_h) \in S_h \times \Sigma_h \times V_h \times M_h$, where (2.6c) has been rewritten using that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \nabla\mathbf{v}_h)_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\sigma}_h : \nabla\mathbf{v}_h = \sum_{T\in\mathcal{T}_h} \left\{-\int_T \mathbf{div}(\boldsymbol{\sigma}_h)\cdot\mathbf{v}_h + \int_{\partial T} \boldsymbol{\sigma}_h\boldsymbol{\nu}\cdot\mathbf{v}_h\right\}, \\ &= -(\mathbf{v}_h, \mathbf{div}(\boldsymbol{\sigma}_h))_{\mathcal{T}_h} + \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \mathbf{v}_h\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

We complete the definition of the HDG method by describing the stabilization tensor $\mathbf{S}$. In [11], general conditions for $\mathbf{S}$ were proposed, where in particular $\mathbf{S}^+$ does not necessarily match $\mathbf{S}^-$ for each $F \in \mathcal{E}_h^i$. Here, we consider the special case in which $\mathbf{S}^+ = \mathbf{S}^-$ in each $F \in \mathcal{E}_h^i$, that is, $\mathbf{S}$ has only one value on each $F \in \mathcal{E}_h$. More precisely, given $F \in \mathcal{E}_h$, the tensor $\mathbf{S}$ satisfies the following conditions:

$$\mathbf{S}|_F \text{ is constant, and } \mathbf{S}|_F \text{ is symmetric and positive definite.}$$

Observe that $\mathbf{S}^{-1}$ is well defined and symmetric and positive definite as well on each $F \in \mathcal{E}_h$. In (3.5) below, we select a particular choice for tensor $\mathbf{S}$ in order to establish the well-posedness of (2.9).

2.2 The augmented HDG formulation

In order to establish the unique solvability of the nonlinear problem (2.9), we now enrich the HDG formulation with two augmented equations arising from the constitutive and equilibrium equations, that is

$$\kappa_1(\boldsymbol{\sigma}_h^{\mathrm{d}} - \boldsymbol{\psi}(\mathbf{t}_h), \boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} = 0 \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \tag{2.7}$$

and

$$\kappa_2(\mathbf{div}(\boldsymbol{\sigma}_h), \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} = -\kappa_2(\mathbf{f}, \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \tag{2.8}$$

where $\kappa_1, \kappa_2 > 0$ are parameters to be determined later on. In this way, our problem becomes: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\lambda}_h) \in S_h \times \Sigma_h \times V_h \times M_h$ such that

$$\begin{aligned}
(\boldsymbol{\psi}(\mathbf{t}_h), \mathbf{s}_h)_{\mathcal{T}_h} - (\mathbf{s}_h, \boldsymbol{\sigma}_h^{\mathrm{d}})_{\mathcal{T}_h} &= 0\,, && \text{(2.9a)}\\
(\mathbf{t}_h, \boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} + (\mathbf{u}_h, \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} - \langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \boldsymbol{\lambda}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= \langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma\,, && \text{(2.9b)}\\
-(\mathbf{v}_h, \mathbf{div}(\boldsymbol{\sigma}_h))_{\mathcal{T}_h} + \langle \mathbf{S}(\mathbf{u}_h - \boldsymbol{\lambda}_h), \mathbf{v}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle \mathbf{S}\mathbf{u}_h, \mathbf{v}_h\rangle_\Gamma &= (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \mathbf{S}\mathbf{g}, \mathbf{v}_h\rangle_\Gamma\,, && \text{(2.9c)}\\
\langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} - \langle \mathbf{S}(\mathbf{u}_h - \boldsymbol{\lambda}_h), \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0\,, && \text{(2.9d)}\\
\kappa_1(\boldsymbol{\sigma}_h^{\mathrm{d}} - \boldsymbol{\psi}(\mathbf{t}_h), \boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} &= 0\,, && \text{(2.9e)}\\
\kappa_2(\mathbf{div}(\boldsymbol{\sigma}_h), \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} &= -\kappa_2(\mathbf{f}, \mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h}\,, && \text{(2.9f)}
\end{aligned}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\mu}_h) \in S_h \times \Sigma_h \times V_h \times M_h$. Hence, in what follows we proceed as in [3, 4] and derive an equivalent formulation to (2.9) (see (2.11) below), for which we prove its unique solvability. In addition, the *a priori* error estimates for (2.9) will also be based on the analysis of (2.11). We emphasize, however, that the introduction of this equivalent formulation is just for theoretical purposes and by no means for the explicit computation of the solution of (2.9), which is solved directly as we explain below in Section 5.

First, we consider equation (2.9d) and note that

$$\begin{aligned}
0 &= \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} - \langle \mathbf{S}\mathbf{u}_h, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle \mathbf{S}\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma}\\
&= \sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\setminus\Gamma}\int_F \boldsymbol{\sigma}_h\boldsymbol{\nu}\cdot\boldsymbol{\mu}_h - \sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\setminus\Gamma}\int_F \mathbf{S}\mathbf{u}_h\cdot\boldsymbol{\mu}_h + \sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\setminus\Gamma}\int_F \mathbf{S}\boldsymbol{\lambda}_h\cdot\boldsymbol{\mu}_h\\
&= \sum_{F\in\mathcal{E}_h^i}\int_F [\![\boldsymbol{\sigma}_h]\!]\cdot\boldsymbol{\mu}_h - 2\sum_{F\in\mathcal{E}_h^i}\int_F \Big(\mathbf{S}\{\!\{\mathbf{u}_h\}\!\}\cdot\boldsymbol{\mu}_h - \mathbf{S}\boldsymbol{\lambda}_h\cdot\boldsymbol{\mu}_h\Big)\\
&= \int_{\mathcal{E}_h^i}\Big([\![\boldsymbol{\sigma}_h]\!] - 2\mathbf{S}\{\!\{\mathbf{u}_h\}\!\} + 2\mathbf{S}\boldsymbol{\lambda}_h\Big)\cdot\boldsymbol{\mu}_h \quad \forall\, \boldsymbol{\mu}_h \in M_h.
\end{aligned}$$

Hence, using that $[\![\boldsymbol{\sigma}_h]\!] - 2\mathbf{S}\{\!\{\mathbf{u}_h\}\!\} + 2\mathbf{S}\boldsymbol{\lambda}_h \in M_h$, we find that

$$[\![\boldsymbol{\sigma}_h]\!] - 2\mathbf{S}\{\!\{\mathbf{u}_h\}\!\} + 2\mathbf{S}\boldsymbol{\lambda}_h \;=\; \mathbf{0} \quad \text{on } \mathcal{E}_h^i,$$

which yields

$$\boldsymbol{\lambda}_h \;=\; \{\!\{\mathbf{u}_h\}\!\} - \frac{1}{2}\mathbf{S}^{-1}[\![\boldsymbol{\sigma}_h]\!] \quad \text{on } \mathcal{E}_h^i. \tag{2.10}$$

Observe that (2.10) coincides with the expression for $\widehat{\mathbf{u}}_h$ given in [11]. We now replace $\boldsymbol{\lambda}_h$ from (2.10) in (2.9b) and (2.9c). For this purpose, we first observe that

$$\begin{aligned} -\langle \boldsymbol{\tau}_h\boldsymbol{\nu}, \boldsymbol{\lambda}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= -\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\setminus\Gamma} \boldsymbol{\tau}_h\boldsymbol{\nu}\cdot\boldsymbol{\lambda}_h \;=\; -\int_{\mathcal{E}_h^i}[\![\boldsymbol{\tau}_h]\!]\cdot\boldsymbol{\lambda}_h, \\ &= \frac{1}{2}\int_{\mathcal{E}_h^i}\mathbf{S}^{-1}[\![\boldsymbol{\sigma}_h]\!]\cdot[\![\boldsymbol{\tau}_h]\!] \;-\; \int_{\mathcal{E}_h^i}\{\!\{\mathbf{u}_h\}\!\}\cdot[\![\boldsymbol{\tau}_h]\!], \end{aligned}$$

and

$$\begin{aligned} -\langle \mathbf{S}\boldsymbol{\lambda}_h, \mathbf{v}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= -\langle \mathbf{S}\mathbf{v}_h, \boldsymbol{\lambda}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} \;=\; -\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\setminus\Gamma}\mathbf{S}\mathbf{v}_h\cdot\boldsymbol{\lambda}_h, \\ &= -2\int_{\mathcal{E}_h^i}\mathbf{S}\{\!\{\mathbf{v}_h\}\!\}\cdot\boldsymbol{\lambda}_h \;=\; \int_{\mathcal{E}_h^i}\{\!\{\mathbf{v}_h\}\!\}\cdot[\![\boldsymbol{\sigma}_h]\!] \;-\; 2\int_{\mathcal{E}_h^i}\mathbf{S}\{\!\{\mathbf{u}_h\}\!\}\cdot\{\!\{\mathbf{v}_h\}\!\}. \end{aligned}$$

In this way, the foregoing equations together with (2.9a), (2.9b), (2.9c), (2.9e) and (2.9f) lead to the problem: Find $((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in H_h \times V_h$ such that

$$\begin{aligned} [\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)] \;+\; [\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{u}_h] &= [\mathcal{F}_h,(\mathbf{s}_h,\boldsymbol{\tau}_h)] \quad \forall\,(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h, \\ [\mathcal{B}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{v}_h] \;-\; [\mathcal{S}_h(\mathbf{u}_h),\mathbf{v}_h] &= [\mathcal{G}_h,\mathbf{v}_h] \;+\; [\mathcal{C}_h(\mathbf{u}_h),\mathbf{v}_h] \quad \forall\,\mathbf{v}_h\in V_h, \end{aligned} \tag{2.11}$$

where $H_h := S_h \times \Sigma_h$, and the operators $\mathcal{A}_h : H_h \to H_h'$, $\mathcal{B}_h : H_h \to V_h'$, $\mathcal{S}_h : V_h \to V_h'$ and $\mathcal{C}_h : V_h \to V_h'$, and the functionals $\mathcal{F}_h : H_h \to R$ and $\mathcal{G}_h : V_h \to R$, are defined by

$$\begin{aligned} [\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)] \;:=\;& (\boldsymbol{\psi}(\mathbf{t}_h),\mathbf{s}_h)_{\mathcal{T}_h} - (\mathbf{s}_h,\boldsymbol{\sigma}_h^{\mathrm{d}})_{\mathcal{T}_h} + (\mathbf{t}_h,\boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} + \frac{1}{2}\int_{\mathcal{E}_h^i}\mathbf{S}^{-1}[\![\boldsymbol{\sigma}_h]\!]\cdot[\![\boldsymbol{\tau}_h]\!] \\ &+ \kappa_1(\boldsymbol{\sigma}_h^{\mathrm{d}} - \boldsymbol{\psi}(\mathbf{t}_h),\boldsymbol{\tau}_h^{\mathrm{d}})_{\mathcal{T}_h} + \kappa_2(\mathbf{div}(\boldsymbol{\sigma}_h),\mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h}, \end{aligned} \tag{2.12}$$

$$[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h] \;:=\; (\mathbf{v}_h,\mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} - \int_{\mathcal{E}_h^i}\{\!\{\mathbf{v}_h\}\!\}\cdot[\![\boldsymbol{\tau}_h]\!], \tag{2.13}$$

$$[\mathcal{S}_h(\mathbf{u}_h),\mathbf{v}_h] \;:=\; \langle \mathbf{S}\mathbf{u}_h,\mathbf{v}_h\rangle_{\partial\mathcal{T}_h}, \tag{2.14}$$

$$[\mathcal{C}_h(\mathbf{u}_h),\mathbf{v}_h] \;:=\; -2\int_{\mathcal{E}_h^i}\mathbf{S}\{\!\{\mathbf{u}_h\}\!\}\cdot\{\!\{\mathbf{v}_h\}\!\},$$

$$[\mathcal{F}_h,(\mathbf{s}_h,\boldsymbol{\tau}_h)] \;:=\; \langle\boldsymbol{\tau}_h\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma - \kappa_2(\mathbf{f},\mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h},$$

$$[\mathcal{G}_h,\mathbf{v}_h] \;:=\; -(\mathbf{f},\mathbf{v}_h)_{\mathcal{T}_h} - \langle\mathbf{S}\mathbf{g},\mathbf{v}_h\rangle_\Gamma,$$

where $[\cdot,\cdot]$ stands in each case for the duality pairing induced by the corresponding operators and functionals. Note, for purposes that will become clear below, that one of the unknowns terms, namely $[\mathcal{C}_h(\mathbf{u}_h),\mathbf{v}_h]$, has been placed on the right-hand side of the second equation in (2.11).

3 Solvability analysis

In this section, we establish the unique solvability of the nonlinear problem (2.11). To this end, and following [3, 4], we let $\mathtt{h} \in L^\infty(\mathcal{E}_h)$ be the function related to the local meshsizes, that is

$$\mathtt{h}(x) \;:=\; \begin{cases} \min\{h_{T_1}, h_{T_2}\} & \text{if } x \in \operatorname{int}(\partial T_1 \cap \partial T_2), \\ h_T & \text{if } x \in \operatorname{int}(\partial T \cap \Gamma), \end{cases}$$

and assume that the meshsize is bounded, that is, that there exists a constant $h_0 > 0$ such that

$$h \;:=\; \max_{T\in\mathcal{T}_h}\{h_T\} \;\leq\; h_0. \tag{3.1}$$

The main idea of our analysis consist of redefining (2.11) as a fixed point problem.

3.1 Preliminaries

The analysis below requires the following preliminary results.

Lemma 3.1 (Discrete trace's inequality)**.** *There exists $C_{\mathrm{tr}} > 0$, depending only on the shape regularity of the mesh, such that for each $T \in \mathcal{T}_h$ and $F \in \partial T$ there holds*

$$\|\mathbf{z}\|^2_{0,F} \;\leq\; C_{\mathrm{tr}}\left\{h_T^{-1}\|\mathbf{z}\|^2_{0,T} \;+\; h_T|\mathbf{z}|^2_{1,T}\right\} \quad \forall\, \mathbf{z} \in \mathbf{H}^1(T). \tag{3.2}$$

Proof. The proof uses a trace theorem and a scaling argument (see [8] for details). □

Lemma 3.2. *There exists $c_0 > 0$, independent of h, such that for all $\mathbf{z} \in \mathbf{H}^1(\Omega)$ there holds*

$$\|\mathtt{h}^{1/2}\mathbf{z}\|_{0,\mathcal{E}_h^i} \;\leq\; c_0\|\mathbf{z}\|_{1,\Omega}. \tag{3.3}$$

Proof. Given $\mathbf{z} \in \mathbf{H}^1(\Omega)$, we have

$$\|\mathtt{h}^{1/2}\mathbf{z}\|^2_{0,\mathcal{E}_h^i} \;=\; \int_{\mathcal{E}_h^i}\mathtt{h}|\mathbf{z}|^2 \;=\; \frac{1}{2}\int_{\mathcal{E}_h^i}\mathtt{h}\left(|\mathbf{z}^+|^2 + |\mathbf{z}^-|^2\right) \;\leq\; \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathtt{h}|\mathbf{z}|^2 \;\leq\; C\sum_{T\in\mathcal{T}_h} h_T\|\mathbf{z}\|^2_{0,\partial T},$$

where C depends on the regularity of $\mathcal{T}_h$. Next, using (3.2) and (3.1), we deduce from the foregoing inequalities that

$$\|\mathtt{h}^{1/2}\mathbf{z}\|^2_{0,\mathcal{E}_h^i} \;\leq\; CC_{\mathrm{tr}}\sum_{T\in\mathcal{T}_h} h_T\left\{h_T^{-1}\|\mathbf{z}\|^2_{0,T} + h_T|\mathbf{z}|^2_{1,T}\right\} \;\leq\; CC_{\mathrm{tr}}(1+h^2)\sum_{T\in\mathcal{T}_h}\|\mathbf{z}\|^2_{1,T} \;\leq\; c_0^2\|\mathbf{z}\|^2_{1,\Omega},$$

with $c_0 := (CC_{\mathrm{tr}}(1+h_0))^{1/2}$, which completes the proof. □

Lemma 3.3. *There exists a constant $c_1 > 0$, independent of h, such that*

$$\|\boldsymbol{\tau}_h\|^2_{0,\Omega} \;\leq\; c_1\left\{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|^2_{0,\Omega} \;+\; \|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h} \;+\; \|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i}\right\} \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h.$$

Proof. We follow similarly as in the proof of [2, Proposition 3.1, Chapter IV]. Indeed, given $\boldsymbol{\tau}_h \in \Sigma_h$, we know from [20, Corollary 2.4 in Chapter I] that there is a unique $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $\operatorname{div}(\mathbf{z}) = \operatorname{tr}(\boldsymbol{\tau}_h)$ and

$$\|\mathbf{z}\|_{1,\Omega} \;\leq\; C\|\operatorname{tr}(\boldsymbol{\tau}_h)\|_{0,\Omega}. \tag{3.4}$$

Now, utilizing that $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$, we have that

$$\begin{aligned}
\|\mathrm{tr}\,(\boldsymbol{\tau}_h)\,\|_{0,\Omega}^2 &= \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h)\,\mathrm{div}(\mathbf{z}) \;=\; \int_\Omega \boldsymbol{\tau}_h : \{\mathrm{tr}\,(\nabla\mathbf{z})\,\mathbb{I}\}\,,\\
&= n\int_\Omega \boldsymbol{\tau}_h : (\nabla\mathbf{z} - (\nabla\mathbf{z})^{\mathrm{d}}) \;=\; n\int_\Omega \boldsymbol{\tau}_h : \nabla\mathbf{z} - n\int_\Omega \boldsymbol{\tau}_h^{\mathrm{d}} : \nabla\mathbf{z},\\
&= n\sum_{T\in\mathcal{T}_h}\left\{-\int_T \mathbf{z}\cdot\mathbf{div}(\boldsymbol{\tau}_h) + \int_{\partial T}\boldsymbol{\tau}_h\boldsymbol{\nu}\cdot\mathbf{z}\right\} - n\int_\Omega \boldsymbol{\tau}_h^{\mathrm{d}} : \nabla\mathbf{z},\\
&= -n(\mathbf{z},\mathbf{div}(\boldsymbol{\tau}_h))_{\mathcal{T}_h} + n\int_{\mathcal{E}_h^i}[\![\boldsymbol{\tau}_h]\!]\cdot\mathbf{z} - n\int_\Omega \boldsymbol{\tau}_h^{\mathrm{d}} : \nabla\mathbf{z}.
\end{aligned}$$

Next, applying Cauchy-Schwarz inequality, and then (3.3) and (3.4), we find that

$$\begin{aligned}
\|\mathrm{tr}\,(\boldsymbol{\tau}_h)\,\|_{0,\Omega}^2 &\le n\|\mathbf{z}\|_{0,\Omega}\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h} + n\|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}\|\mathtt{h}^{1/2}\mathbf{z}\|_{0,\mathcal{E}_h^i} + n\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}|\mathbf{z}|_{1,\Omega}\\
&\le n\|\mathbf{z}\|_{0,\Omega}\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h} + nc_0\|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}\|\mathbf{z}\|_{1,\Omega} + n\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}|\mathbf{z}|_{1,\Omega}\\
&\le C\|\mathbf{z}\|_{1,\Omega}\left\{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h}^2 + \|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}^2\right\}^{1/2}\\
&\le C\|\mathrm{tr}\,(\boldsymbol{\tau}_h)\,\|_{0,\Omega}\left\{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h}^2 + \|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}^2\right\}^{1/2},
\end{aligned}$$

which gives

$$\|\mathrm{tr}\,(\boldsymbol{\tau}_h)\,\|_{0,\Omega}^2 \;\le\; C\left\{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h}^2 + \|\mathtt{h}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}^2\right\}.$$

This inequality and the fact that $\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 = \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}^2 + \frac{1}{n}\|\mathrm{tr}\,(\boldsymbol{\tau}_h)\,\|_{0,\Omega}^2$, complete the proof. □

We now realize, thanks to the previous lemma, that for convenience of further analysis, we need to establish a particular choice of the stabilization tensor $\mathbf{S}$. For this purpose, we let $\tau > 0$ be a constant and set the tensor $\mathbf{S}$ as follows

$$\mathbf{S}|_F \;:=\; \tau\,\mathtt{h}\,\mathbb{I} \quad \forall\, F \in \mathcal{E}_h, \tag{3.5}$$

which certainly yields

$$\mathbf{S}^{-1}|_F \;:=\; (\tau\mathtt{h})^{-1}\mathbb{I} \quad \forall\, F \in \mathcal{E}_h. \tag{3.6}$$

In addition, we consider the following definition of a norm onto Σ_h

$$\|\boldsymbol{\tau}_h\|_{\Sigma_h}^2 \;:=\; \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}^2 \;+\; \|\mathbf{div}(\boldsymbol{\tau}_h)\|_{\mathcal{T}_h}^2 \;+\; \|(\tau\mathtt{h})^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{E}_h^i}^2 \quad \forall\,\boldsymbol{\tau}_h \in \Sigma_h$$

which, according to Lemma 3.3, satisfies

$$\|\boldsymbol{\tau}_h\|_{0,\Omega} \;\le\; c_2\|\boldsymbol{\tau}_h\|_{\Sigma_h} \quad \forall\,\boldsymbol{\tau}_h \in \Sigma_h, \tag{3.7}$$

where $c_2 > 0$ depends on c_1 and τ, but is independent of h. Note that the above suggests the following norm on $H_h := S_h \times \Sigma_h$

$$\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h} \;:=\; \left\{\|\mathbf{s}_h\|_{0,\Omega}^2 \;+\; \|\boldsymbol{\tau}_h\|_{\Sigma_h}^2\right\}^{1/2} \quad \forall\,(\mathbf{s}_h,\boldsymbol{\tau}_h) \in H_h.$$

On the other hand, we define the nonlinear operator $\mathbb{A} : S_h \to S_h'$ by

$$[\mathbb{A}(\mathbf{t}_h),\mathbf{s}_h] \;:=\; (\boldsymbol{\psi}(\mathbf{t}_h),\mathbf{s}_h)_{\mathcal{T}_h} \quad \forall\,\mathbf{t}_h,\mathbf{s}_h \in S_h.$$

Then, we have the following result.

Lemma 3.4. *Let γ_0 and α_0 be the constants from* (2.2) *and* (2.3)*, respectively. Then, for all $\mathbf{t}_h, \mathbf{s}_h \in S_h$ there hold*

$$\|\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h)\|_{S_h'} \quad \leq \quad \gamma_0 \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega} \tag{3.8}$$

and

$$[\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h), \mathbf{t}_h - \mathbf{s}_h] \quad \geq \quad \alpha_0 \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}^2. \tag{3.9}$$

Proof. See [19, Lemma 2.1] or [4, Section 3]. $\square$

We are now ready to establish that the nonlinear operator $\mathcal{A}_h$ defining the problem (2.11) is also Lipschitz-continuous and strongly monotone. In particular, the second property will depend on a suitable choice of the parameter κ_1.

Lemma 3.5. *Let $\mathcal{A}_h$ be the nonlinear operator defined by* (2.12)*. Then, there exists a constant $C_{\mathrm{LC}} > 0$, independent of h and τ, such that*

$$\|\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h)\|_{H_h'} \;\leq\; C_{\mathrm{LC}}\, \|(\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|_{H_h} \quad \forall\, (\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h) \in H_h.$$

Proof. Given $(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)$ and $(\mathbf{r}_h, \boldsymbol{\rho}_h) \in H_h$, we obtain from the definition of $\mathbb{A}$ and (3.6) that

$$\begin{aligned} [\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{r}_h, \boldsymbol{\rho}_h)] \;&=\; [\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h), \mathbf{r}_h] - \kappa_1 [\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h), \boldsymbol{\rho}_h^{\mathrm{d}}] \\ &- (\mathbf{r}_h, (\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}})_{\mathcal{T}_h} + (\mathbf{t}_h - \mathbf{s}_h, \boldsymbol{\rho}_h^{\mathrm{d}})_{\mathcal{T}_h} + \frac{1}{2} \sum_{F \in \mathcal{E}_h^i} \int_F (\tau \mathtt{h})^{-1/2} [\![(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)]\!] \cdot (\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\rho}_h]\!] \\ &+ \kappa_1 ((\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}}, \boldsymbol{\rho}_h^{\mathrm{d}})_{\mathcal{T}_h} + \kappa_2 (\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h), \mathbf{div}(\boldsymbol{\rho}_h))_{\mathcal{T}_h}, \end{aligned} \tag{3.10}$$

from which, applying Cauchy-Schwarz inequality and (3.8), it follows that

$$\begin{aligned} [\mathcal{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{r}_h, \boldsymbol{\rho}_h)] \;&\leq\; \|\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h)\|_{S_h'} \|\mathbf{r}_h\|_{0,\Omega} + \kappa_1 \|\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h)\|_{S_h'}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \|\mathbf{r}_h\|_{0,\Omega}\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \frac{1}{2} \|(\tau \mathtt{h})^{-1/2} [\![(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)]\!]\|_{0,\mathcal{E}_h^i}\, \|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\rho}_h]\!]\|_{0,\mathcal{E}_h^i} + \kappa_1 \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}}\|_{0,\Omega}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|_{\mathcal{T}_h}\, \|\mathbf{div}(\boldsymbol{\rho}_h)\|_{\mathcal{T}_h}, \\ \leq\; &\gamma_0 \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}\, \|\mathbf{r}_h\|_{0,\Omega} + \gamma_0 \kappa_1 \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \|\mathbf{r}_h\|_{0,\Omega}\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \frac{1}{2} \|(\tau \mathtt{h})^{-1/2} [\![(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)]\!]\|_{0,\mathcal{E}_h^i}\, \|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\rho}_h]\!]\|_{0,\mathcal{E}_h^i} + \kappa_1 \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\mathrm{d}}\|_{0,\Omega}\, \|\boldsymbol{\rho}_h^{\mathrm{d}}\|_{0,\Omega} \\ &+ \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|_{\mathcal{T}_h}\, \|\mathbf{div}(\boldsymbol{\rho}_h)\|_{\mathcal{T}_h}. \end{aligned}$$

In this way, setting

$$C_{\mathrm{LC}} \;:=\; 3 \max \{1, \gamma_0, \kappa_1, \gamma_0 \kappa_1, \kappa_2\},$$

we conclude that

$$[\mathcal{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{r}_h, \boldsymbol{\rho}_h)] \quad \leq \quad C_{\mathrm{LC}}\, \|(\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|_{H_h}\, \|(\mathbf{r}_h, \boldsymbol{\rho}_h)\|_{H_h},$$

which ends the proof. $\square$

Lemma 3.6. *Let $\mathcal{A}_h$ be the nonlinear operator defined by (2.12), and assume that the parameter κ_1 lies in $\left(0, \frac{2\alpha_0}{\gamma_0^2}\right)$, where α_0 and γ_0 are the positive constants from (2.2) and (2.3). Then, there exists a constant $C_{\rm SM} > 0$, independent of h and τ, such that*

$$[\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)] \;\geq\; C_{\rm SM}\, \|(\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|^2_{H_h},$$

for all $(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h) \in H_h$.

Proof. Given $(\mathbf{t}_h, \boldsymbol{\sigma}_h)$ and $(\mathbf{s}_h, \boldsymbol{\tau}_h) \in H_h$, we take $(\mathbf{r}_h, \boldsymbol{\rho}_h) = (\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)$ in (3.10), to obtain

$$\begin{aligned}
[\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)] \;&=\; [\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h), \mathbf{t}_h - \mathbf{s}_h] \\
&- \kappa_1 [\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\mathbf{s}_h), (\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}] + \frac{1}{2}\|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i} \\
&+ \kappa_1 \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega} + \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h},
\end{aligned}$$

which, according to (3.8) and (3.9), implies that

$$\begin{aligned}
&[\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)] \\
&\geq\; \alpha_0\, \|\mathbf{t}_h - \mathbf{s}_h\|^2_{0,\Omega} - \gamma_0 \kappa_1\, \|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|_{0,\Omega} + \frac{1}{2}\|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i} \\
&\quad +\; \kappa_1\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega} + \kappa_2\, \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h}, \\
&\geq\; \alpha_0\, \|\mathbf{t}_h - \mathbf{s}_h\|^2_{0,\Omega} - \gamma_0 \kappa_1 \left\{ \frac{\|\mathbf{t}_h - \mathbf{s}_h\|^2_{0,\Omega}}{2\delta} \;+\; \frac{\delta\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega}}{2} \right\} \\
&\quad +\; \frac{1}{2}\|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i} \;+\; \kappa_1\, \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega} + \kappa_2\, \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h}, \\
&=\; \left(\alpha_0 - \frac{\gamma_0 \kappa_1}{2\delta}\right) \|\mathbf{t}_h - \mathbf{s}_h\|^2_{0,\Omega} + \kappa_1 \left(1 - \frac{\gamma_0 \delta}{2}\right) \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega} \\
&\quad +\; \kappa_2\, \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h} + \frac{1}{2}\|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i} \quad \forall\ \delta > 0.
\end{aligned}$$

It follows that the constants multiplying the norms above become positive if $\delta \in \left(0, \frac{2}{\gamma_0}\right)$ and $\kappa_1 \in \left(0, \frac{2\alpha_0 \delta}{\gamma_0}\right)$. In particular, for $\delta = \frac{1}{\gamma_0}$ we require $\kappa_1 \in \left(0, \frac{2\alpha_0}{\gamma_0^2}\right)$, whence we find that

$$\begin{aligned}
&[\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h) - \mathcal{A}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)] \\
&\geq\; \left(\alpha_0 - \frac{\gamma_0^2 \kappa_1}{2}\right) \|\mathbf{t}_h - \mathbf{s}_h\|^2_{0,\Omega} + \frac{\kappa_1}{2} \|(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)^{\rm d}\|^2_{0,\Omega} \\
&\quad +\; \kappa_2\, \|\mathbf{div}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h)\|^2_{\mathcal{T}_h} + \frac{1}{2}\|(\tau \mathtt{h})^{-1/2} [\![\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h]\!]\|^2_{0,\mathcal{E}_h^i} \\
&\geq\; C_{\rm SM} \|(\mathbf{t}_h, \boldsymbol{\sigma}_h) - (\mathbf{s}_h, \boldsymbol{\tau}_h)\|^2_{H_h},
\end{aligned}$$

with $C_{\rm SM} := \min\left\{\alpha_0 - \frac{\gamma_0^2 \kappa_1}{2}, \frac{\kappa_1}{2}, \kappa_2, \frac{1}{2}\right\}$, thus completing the proof of the lemma. □

Our next goal is to show the discrete inf-sup condition for the linear operator $\mathcal{B}_h$. More precisely, we have the following result.

Lemma 3.7. *There exists a constant $C_{\rm inf} > 0$, independent of h and τ, such that*

$$\sup_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in H_h \\ (\mathbf{s}_h, \boldsymbol{\tau}_h) \neq \mathbf{0}}} \frac{[\mathcal{B}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), \mathbf{v}_h]}{\|(\mathbf{s}_h, \boldsymbol{\tau}_h)\|_{H_h}} \;\geq\; C_{\rm inf}\, \|\mathbf{v}_h\|_{0,\Omega} \quad \forall\ \mathbf{v}_h \in V_h.$$

Proof. We begin by recalling from (2.13) that $\mathcal{B}_h$ does not depend on $\mathbf{s}_h$, and hence it suffices to show the existence of $C_{\inf} > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \Sigma_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) - \int_{\mathcal{E}_h^i} \{\!\{\mathbf{v}_h\}\!\} \cdot [\![\boldsymbol{\tau}_h]\!]}{\|\boldsymbol{\tau}_h\|_{\Sigma_h}} \;\geq\; C_{\inf}\, \|\mathbf{v}_h\|_{0,\Omega} \quad \forall\, \mathbf{v}_h \in V_h.$$

To this end we let $\mathrm{RT}_{k-1}(\Omega)$ be the global Raviart-Thomas space of degree $k-1$, which is clearly contained in S_h, and note that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \Sigma_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) - \int_{\mathcal{E}_h^i} \{\!\{\mathbf{v}_h\}\!\} \cdot [\![\boldsymbol{\tau}_h]\!]}{\|\boldsymbol{\tau}_h\|_{\Sigma_h}} \;\geq\; \sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{RT}_{k-1}(\Omega)\backslash\{\mathbf{0}\} \\ \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) = 0}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\Sigma_h}}.$$

In this way, and observing that $\|\boldsymbol{\tau}_h\|_{\Sigma_h}$ is equivalent to $\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}$ $\forall\, \boldsymbol{\tau}_h \in \mathrm{RT}_{k-1}(\Omega)$ such that $\int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h) = 0$, with constants independent of h and τ, the rest of the proof follows from classical results from mixed finite element methods (see, e.g. [16, Section 4.2 and Lemma 2.6]). □

The following three lemmas establish the positive semidefiniteness of $\mathcal{S}_h$ and some discrete trace, inverse, and boundedness inequalities to be employed later on.

Lemma 3.8. *The operator $\mathcal{S}_h : V_h \to V_h'$, defined by* (2.14) *is positive semidefinite, that is,*

$$[\mathcal{S}_h(\mathbf{v}_h), \mathbf{v}_h] \;\geq\; 0 \quad \forall\, \mathbf{v}_h \in V_h.$$

Proof. It is clear from (2.14) that

$$[\mathcal{S}_h(\mathbf{v}_h), \mathbf{v}_h] \;=\; \sum_{T \in \mathcal{T}_h} \sum_{F \in \partial T} \int_F \mathbf{S}\mathbf{v}_h \cdot \mathbf{v}_h \quad \forall\, \mathbf{v}_h \in\; V_h,$$

which, thanks to the fact that $\mathbf{S}$ is a positive definite tensor on $\mathcal{E}_h$, completes the proof. □

Lemma 3.9 (Discrete trace's inequality + inverse's inequality)**.** *There exists $C_{\mathrm{inv}} > 0$, depending only on k and the shape regularity of the mesh, such that*

$$\|\mathbf{v}\|^2_{0,\partial T} \;\leq\; C_{\mathrm{inv}}\, h_T^{-1} \|\mathbf{v}\|^2_{0,T} \quad \forall\, \mathbf{v} \in \mathbf{P}_k(T), \quad \forall\, T \in \mathcal{T}_h, \tag{3.11}$$

and

$$\|\boldsymbol{\tau}\|^2_{0,\partial T} \;\leq\; C_{\mathrm{inv}}\, h_T^{-1} \|\boldsymbol{\tau}\|^2_{0,T} \quad \forall\, \boldsymbol{\tau} \in \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}_h. \tag{3.12}$$

Proof. The proof uses the discrete trace inequality from Lemma 3.1 and an inverse inequality. See also [3, Lemma 3.2]. □

Lemma 3.10. *There exist constants $\widehat{C}_1, \widehat{C}_2, \widehat{C}_3 > 0$, independent of h and τ, such that*

i) $\|\mathtt{h}^{1/2}\{\!\{\mathbf{v}_h\}\!\}\|_{0,\mathcal{E}_h^i} \;\leq\; \widehat{C}_1 \|\mathbf{v}_h\|_{0,\Omega} \quad \forall\, \mathbf{v}_h \in V_h.$

ii) $\|\mathtt{h}^{1/2}\mathbf{v}_h\|_{0,\mathcal{E}_h^\partial} \;\leq\; \widehat{C}_2 \|\mathbf{v}_h\|_{0,\Omega} \quad \forall\, \mathbf{v}_h \in V_h.$

iii) $\|\mathtt{h}^{1/2}\boldsymbol{\tau}_h\boldsymbol{\nu}\|_{0,\mathcal{E}_h^\partial} \;\leq\; \widehat{C}_3 \|\boldsymbol{\tau}_h\|_{0,\Omega} \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h.$

Proof. Given $\mathbf{v}_h \in V_h$, we use (3.11) to deduce that

$$\begin{aligned} \|\mathtt{h}^{1/2}\{\!\{\mathbf{v}_h\}\!\}\|^2_{0,\mathcal{E}_h^i} &= \frac{1}{4}\int_{\mathcal{E}_h^i} \mathtt{h}|\mathbf{v}_h^+ + \mathbf{v}_h^-|^2 \;\le\; \frac{1}{2}\int_{\mathcal{E}_h^i} \mathtt{h}\left(|\mathbf{v}_h^+|^2 + |\mathbf{v}_h^-|^2\right) \;\le\; \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_{\partial T} \mathtt{h}|\mathbf{v}_h|^2 \\ &\le\; C_1 \sum_{T\in\mathcal{T}_h} h_T\|\mathbf{v}_h\|^2_{0,\partial T} \;\le\; C_1 C_{\text{inv}} \sum_{T\in\mathcal{T}_h} \|\mathbf{v}_h\|^2_{0,T} \;=\; C_1 C_{\text{inv}}\|\mathbf{v}_h\|^2_{0,\Omega}, \end{aligned}$$

which shows *i)* with $\widehat{C}_1 := (C_1 C_{\text{inv}})^{1/2} > 0$. Next, using that $\mathtt{h} = h_T$ on $\mathcal{E}_h^\partial$, and applying again (3.11), we find that

$$\|\mathtt{h}^{1/2}\mathbf{v}_h\|^2_{0,\mathcal{E}_h^\partial} \;=\; \int_{\mathcal{E}_h^\partial} \mathtt{h}|\mathbf{v}_h|^2 \;\le\; \sum_{T\in\mathcal{T}_h} h_T\|\mathbf{v}_h\|^2_{0,\partial T} \;\le\; C_{\text{inv}}\|\mathbf{v}_h\|^2_{0,\Omega},$$

which proves *ii)* with $\widehat{C}_2 := (C_{\text{inv}})^{1/2}$. Finally, the proof of *iii)* follows from (3.12). □

Using Lemma 3.10, the definition of tensor $\mathbf{S}$ given in (3.5), and the Cauchy-Schwarz inequality, it is easy to check that the operators $\mathcal{B}_h$, $\mathcal{S}_h$ and $\mathcal{C}_h$, and the functionals $\mathcal{F}_h$ and $\mathcal{G}_h$, are all bounded with respect to the corresponding norms. More precisely, the corresponding bounds are established in the following lemma.

Lemma 3.11. *Let $\mathbf{s}_h \in S_h$, $\boldsymbol{\tau}_h \in \Sigma_h$ and $\mathbf{u}_h, \mathbf{v}_h \in V_h$. Then there hold*

$$\begin{aligned} |[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h]| &\le (1+\tau\widehat{C}_1)\,\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\,\|\mathbf{v}_h\|_{0,\Omega} \\ |[\mathcal{S}_h(\mathbf{u}_h),\mathbf{v}_h]| &\le \tau\widehat{C}_1\,\|\mathbf{u}_h\|_{0,\Omega}\,\|\mathbf{v}_h\|_{0,\Omega} \\ |[\mathcal{C}_h(\mathbf{u}_h),\mathbf{v}_h]| &\le 2\tau\widehat{C}_1^2\,\|\mathbf{u}_h\|_{0,\Omega}\,\|\mathbf{v}_h\|_{0,\Omega} \\ |[\mathcal{F}_h,(\mathbf{s}_h,\boldsymbol{\tau}_h)]| &\le \left(\kappa_2 + c_2\widehat{C}_3\right)\mathbb{B}(\mathbf{f},\mathbf{g})\,\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h} \\ |[\mathcal{G}_h,\mathbf{v}_h]| &\le \left(1+\tau h_0\widehat{C}_2\right)\mathbb{B}(\mathbf{f},\mathbf{g})\,\|\mathbf{v}_h\|_{0,\Omega} \end{aligned} \tag{3.13}$$

where

$$\mathbb{B}(\mathbf{f},\mathbf{g}) \;:=\; \|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathtt{h}^{-1/2}\mathbf{g}\|_{0,\mathcal{E}_h^\partial}.$$

Proof. The proof uses Lemma 3.10 and the definitions of each operator and functional. We omit further details and refer to [3, Lemma 4.4]. □

We end this section, by recalling from [18] the following abstract theorem.

Theorem 3.1. *Let X, M be Hilbert spaces and assume that*

i) the operator $\mathcal{A}: X \to X'$ is Lipschitz continuous and strongly monotonic, that is, there exist γ, $\alpha > 0$ such that

$$\|\mathcal{A}(\mathbf{s}_1) - \mathcal{A}(\mathbf{s}_2)\|_{X'} \;\le\; \gamma\,\|\mathbf{s}_1 - \mathbf{s}_2\|_X \quad \forall\,\mathbf{s}_1,\mathbf{s}_2 \in X$$

and

$$[\mathcal{A}(\mathbf{s}_1) - \mathcal{A}(\mathbf{s}_2), \mathbf{s}_1 - \mathbf{s}_2] \;\ge\; \alpha\,\|\mathbf{s}_1 - \mathbf{s}_2\|^2_X \quad \forall\,\mathbf{s}_1,\mathbf{s}_2 \in X;$$

ii) the linear operator $\mathcal{S}$ is positive semidefinite on M, that is

$$[\mathcal{S}(\boldsymbol{\tau}),\boldsymbol{\tau}] \;\ge\; 0 \quad \forall\,\boldsymbol{\tau} \in M;$$

iii) the linear operator $\mathcal{B}$ satisfies an inf-sup condition on $X \times M$, that is, there exists $\beta > 0$ such that

$$\sup_{\substack{\mathbf{s} \in X \\ \mathbf{s} \neq \mathbf{0}}} \frac{[\mathcal{B}(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_X} \;\geq\; \beta \, \|\boldsymbol{\tau}\|_M \quad \forall\, \boldsymbol{\tau} \in M.$$

Then, given $\mathcal{F} \in X'$ and $\mathcal{G} \in M'$, there exists a unique solution $(\mathbf{t}, \boldsymbol{\sigma}) \in X \times M$ of

$$\begin{aligned} [\mathcal{A}(\mathbf{t}), \mathbf{s}] + [\mathcal{B}^*(\boldsymbol{\sigma}), \mathbf{s}] &= [\mathcal{F}, \mathbf{s}] \quad \forall\, \mathbf{s} \in X, \\ [\mathcal{B}(\mathbf{t}), \boldsymbol{\tau}] - [\mathcal{S}(\boldsymbol{\sigma}), \boldsymbol{\tau}] &= [\mathcal{G}, \boldsymbol{\tau}] \quad \forall\, \boldsymbol{\tau} \in M. \end{aligned}$$

In addition, the following estimates hold

$$\begin{aligned} \|\mathbf{t}\|_X &\leq C_1 \Big\{ \|\mathcal{F}\|_{X'} \;+\; \|\mathcal{G}\|_{M'} \;+\; \|\mathcal{A}(\mathbf{0})\|_{X'} \Big\}, \\ \|\boldsymbol{\sigma}\|_M &\leq C_2 \Big\{ \|\mathcal{F}\|_{X'} \;+\; \|\mathcal{G}\|_{M'} \;+\; \|\mathcal{A}(\mathbf{0})\|_{X'} \Big\}, \end{aligned}$$

where

$$C_1 \;:=\; \frac{1}{\alpha} + \frac{\|\mathcal{B}\|}{\alpha} C_2 \quad \textit{and} \quad C_2 \;:=\; \frac{\gamma^2}{\alpha \beta^2} \left(1 + \frac{\|\mathcal{B}\|}{\alpha} \right).$$

Proof. See [18, Lemma 2.1], where it is easy to show the last estimates from expressions (2.8) and (2.9) in [18]. □

3.2 Main result

In order to prove existence and uniqueness of solution of (2.11), we now introduce the nonlinear mapping $\mathbb{T}_h : H_h \times V_h \to H_h \times V_h$ that, given $((\mathbf{r}_h, \boldsymbol{\rho}_h), \mathbf{w}_h) \in H_h \times V_h$, defines $\mathbb{T}_h((\mathbf{r}_h, \boldsymbol{\rho}_h), \mathbf{w}_h) := ((\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{u}_h) \in H_h \times V_h$ as the unique solution of the problem

$$\begin{aligned} [\mathcal{A}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)] \;+\; [\mathcal{B}_h(\mathbf{s}_h, \boldsymbol{\tau}_h), \mathbf{u}_h] &= [\mathcal{F}_h, (\mathbf{s}_h, \boldsymbol{\tau}_h)] \quad \forall\, (\mathbf{s}_h, \boldsymbol{\tau}_h) \in H_h, \\ [\mathcal{B}_h(\mathbf{t}_h, \boldsymbol{\sigma}_h), \mathbf{v}_h] \;-\; [\mathcal{S}_h(\mathbf{u}_h), \mathbf{v}_h] &= [\mathcal{G}_h, \mathbf{v}_h] \;+\; [\mathcal{C}_h(\mathbf{w}_h), \mathbf{v}_h] \quad \forall\, \mathbf{v}_h \in V_h. \end{aligned}$$

Note that actually $\mathbb{T}_h((\mathbf{r}_h, \boldsymbol{\rho}_h), \mathbf{w}_h)$ depends only on the third component $\mathbf{w}_h \in V_h$. In addition, bearing in mind Lemmas 3.5, 3.6, 3.7 and 3.8, it follows from Theorem 3.1 that $\mathbb{T}_h$ is well-defined and there holds

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h)\|_{H_h} \;\leq\; \widehat{C}_a \widetilde{C} \, \mathbb{B}(\mathbf{f}, \mathbf{g}) \;+\; 2 \widehat{C}_1^2 \widehat{C}_a \tau \, \|\mathbf{w}_h\|_{0,\Omega}, \tag{3.14}$$

and

$$\|\mathbf{u}_h\|_{0,\Omega} \;\leq\; \widehat{C}_b \widetilde{C} \, \mathbb{B}(\mathbf{f}, \mathbf{g}) \;+\; 2 \widehat{C}_1^2 \widehat{C}_b \tau \, \|\mathbf{w}_h\|_{0,\Omega}, \tag{3.15}$$

where

$$\begin{aligned} \widetilde{C} &:= 1 + \kappa_2 + \tau h_0 \widehat{C}_2 + c_1^{1/2} \widehat{C}_3 (1 + \tau)^{1/2}, \\ \widehat{C}_a &:= \frac{1}{C_{\mathrm{SM}}} \left(1 + (1 + \tau \widehat{C}_1) \widehat{C}_b \right), \\ \widehat{C}_b &:= \frac{C_{\mathrm{LC}}^2}{C_{\mathrm{SM}} C_{\mathrm{inf}}^2} \left(1 + \frac{1 + \tau \widehat{C}_1}{C_{\mathrm{SM}}} \right), \end{aligned}$$

and the constants $\widehat{C}_1$, $\widehat{C}_2$, and $\widehat{C}_3$ are those from Lemma 3.10. Observe here that the identity $\mathcal{A}_h(\mathbf{0},\mathbf{0}) = (\mathbf{0},\mathbf{0})$ and Lemma 3.11 have been employed to establish the estimates (3.14) and (3.15). Also, we remark that the relevance of the introduction of $\mathbb{T}_h$ has to do with the fact that any eventual solution of (2.11) becomes a fixed point of $\mathbb{T}_h$ and conversely. Moreover, the following lemma establishes that $\mathbb{T}_h$ is indeed a contraction mapping and hence, thanks to the Banach Fixed-Point Theorem, it has a unique fixed point in $H_h \times V_h$.

Lemma 3.12. *Assume that the parameter τ lies in $\big(0, \frac{1}{\theta}\big)$, where*

$$\theta \;:=\; \left(\frac{2\widehat{C}_1^2}{C_{\rm SM}}\right)\left(\frac{C_{\rm LC}}{C_{\rm inf}}\right)\left(1 + \frac{C_{\rm LC}}{C_{\rm inf}}\right) \;>\; 0.$$

Then, $\mathbb{T}_h$ is a contraction.

Proof. Given $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h)$, $((\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h),\widetilde{\mathbf{u}}_h)$, $((\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{w}_h)$, and $((\widetilde{\mathbf{r}}_h,\widetilde{\boldsymbol{\rho}}_h),\widetilde{\mathbf{w}}_h)$ in $H_h \times V_h$ such that

$$\mathbb{T}_h((\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{w}_h) \;=\; ((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \quad \text{and} \quad \mathbb{T}_h((\widetilde{\mathbf{r}}_h,\widetilde{\boldsymbol{\rho}}_h),\widetilde{\mathbf{w}}_h) \;=\; ((\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h),\widetilde{\mathbf{u}}_h),$$

we know from the definition of $\mathbb{T}_h$ that

$$[\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h) - \mathcal{A}_h(\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)] \;+\; [\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{u}_h - \widetilde{\mathbf{u}}_h] \;=\; 0, \tag{3.16a}$$

$$[\mathcal{B}_h(\mathbf{t}_h - \widetilde{\mathbf{t}}_h,\boldsymbol{\sigma}_h - \widetilde{\boldsymbol{\sigma}}_h),\mathbf{v}_h] \;-\; [\mathcal{S}_h(\mathbf{u}_h - \widetilde{\mathbf{u}}_h),\mathbf{v}_h] \;=\; [\mathcal{C}_h(\mathbf{w}_h - \widetilde{\mathbf{w}}_h),\mathbf{v}_h], \tag{3.16b}$$

for all $((\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h) \in H_h \times V_h$. Next, taking $(\mathbf{s}_h,\boldsymbol{\tau}_h) = (\mathbf{t}_h - \widetilde{\mathbf{t}}_h,\boldsymbol{\sigma}_h - \widetilde{\boldsymbol{\sigma}}_h)$ and $\mathbf{v}_h = \mathbf{u}_h - \widetilde{\mathbf{u}}_h$, we obtain from (3.16) that

$$\begin{aligned}&[\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h) - \mathcal{A}_h(\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h),(\mathbf{t}_h,\boldsymbol{\sigma}_h) - (\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h)]\\ &\qquad + [\mathcal{S}_h(\mathbf{u}_h - \widetilde{\mathbf{u}}_h),\mathbf{u}_h - \widetilde{\mathbf{u}}_h] \;=\; -[\mathcal{C}_h(\mathbf{w}_h - \widetilde{\mathbf{w}}_h),\mathbf{u}_h - \widetilde{\mathbf{u}}_h].\end{aligned} \tag{3.17}$$

Then, using the strong monotonicity of $\mathcal{A}_h$, the fact that $\mathcal{S}_h$ is positive semidefinite, and the boundedness of $\mathcal{C}_h$ (cf. (3.13)), we deduce from (3.17) that

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h) - (\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{H_h}^2 \;\le\; \frac{2\tau\widehat{C}_1^2}{C_{\rm SM}}\,\|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{0,\Omega}\,\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,\Omega}. \tag{3.18}$$

On the other hand, employing the inf-sup condition for $\mathcal{B}_h$ (cf. Lemma 3.7), (3.16a), and the Lipschitz-continuity of $\mathcal{A}_h$ (cf. Lemma 3.6), we find that

$$\begin{aligned}\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,\Omega} \;&\le\; \frac{1}{C_{\rm inf}} \sup_{\substack{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h\\ (\mathbf{s}_h,\boldsymbol{\tau}_h)\neq \mathbf{0}}} \frac{|[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{u}_h - \widetilde{\mathbf{u}}_h]|}{\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}},\\ &=\; \frac{1}{C_{\rm inf}} \sup_{\substack{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h\\ (\mathbf{s}_h,\boldsymbol{\tau}_h)\neq \mathbf{0}}} \frac{|-[\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h) - \mathcal{A}_h(\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)]|}{\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}},\\ &\le\; \frac{C_{\rm LC}}{C_{\rm inf}}\,\|(\mathbf{t}_h,\boldsymbol{\sigma}_h) - (\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{H_h},\end{aligned}$$

which, together with (3.18), implies that

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h) - (\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{H_h} \;\le\; \left(\frac{2\tau\widehat{C}_1^2}{C_{\rm SM}}\right)\left(\frac{C_{\rm LC}}{C_{\rm inf}}\right)\|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{0,\Omega}$$

and

$$\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,\Omega} \;\le\; \left(\frac{2\tau\widehat{C}_1^2}{C_{\rm SM}}\right)\left(\frac{C_{\rm LC}}{C_{\rm inf}}\right)^2 \|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{0,\Omega}.$$

In this way, we conclude that

$$\|\mathbb{T}_h((\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{w}_h) - \mathbb{T}_h((\widetilde{\mathbf{r}}_h,\widetilde{\boldsymbol{\rho}}_h),\widetilde{\mathbf{w}}_h)\|_{H_h\times V_h} \;\le\; L\,\|((\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{w}_h) - ((\widetilde{\mathbf{r}}_h,\widetilde{\boldsymbol{\rho}}_h),\widetilde{\mathbf{w}}_h)\|_{H_h\times V_h},$$

with $L := \tau\theta$. Finally, since $C_{\rm inf}$, $C_{\rm LC}$, and $C_{\rm SM}$, are independent of $\tau > 0$, we can choose $\tau \in \big(0, \frac{1}{\theta}\big)$, which insures that $\mathbb{T}_h$ is a contraction and completes the proof. □

Now we are ready to establish the main result of this section.

Theorem 3.2. *Assume that*

$$0 \;<\; \tau \;<\; \min\left\{\frac{1}{\theta},\; \frac{1}{2}\left(\frac{C_{\rm SM}}{(1+C_{\rm SM})\theta + \widehat{C}_1}\right)\right\}.$$

Then, there exists a unique $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \in H_h \times V_h$ *solution of* (2.11)*. Moreover, there holds*

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{\Sigma_h} \;\le\; C_{\rm a}\,\mathbb{B}(\mathbf{f},\mathbf{g}) \quad \textit{and} \quad \|\mathbf{u}_h\|_{0,\Omega} \;\le\; C_{\rm b}\,\mathbb{B}(\mathbf{f},\mathbf{g}),$$

where

$$C_{\rm a} \;:=\; \widehat{C}_a\,(\widetilde{C} + 2\widehat{C}_1^2 C_b\tau) \quad \textit{and} \quad C_{\rm b} \;:=\; 2\widehat{C}_b\widetilde{C}.$$

Proof. The unique solvability of (2.11) follows straightforwardly from its equivalence with the fixed-point equation for $\mathbb{T}_h$, the corresponding Banach Theorem, and the fact that $\mathbb{T}_h$ becomes a contraction when $\tau < \frac{1}{\theta}$ (cf. Lemma 3.12). Then, denoting by $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \in H_h \times V_h$ the unique solution of (2.11), we have from (3.14) and (3.15) that

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h} \;\le\; \widehat{C}_a\widetilde{C}\,\mathbb{B}(\mathbf{f},\mathbf{g}) \;+\; 2\widehat{C}_1^2\widehat{C}_a\tau\,\|\mathbf{u}_h\|_{0,\Omega} \tag{3.19}$$

and

$$\|\mathbf{u}_h\|_{0,\Omega} \;\le\; \widehat{C}_b\widetilde{C}\,\mathbb{B}(\mathbf{f},\mathbf{g}) \;+\; 2\widehat{C}_1^2\widehat{C}_b\tau\,\|\mathbf{u}_h\|_{0,\Omega}. \tag{3.20}$$

It remain to handle the second term on the right-hand side of (3.20). For this purpose, we now note that

$$\begin{aligned}
2\widehat{C}_1^2\widehat{C}_b\tau \;&=\; 2\widehat{C}_1^2\frac{C_{\rm LC}^2}{C_{\rm SM}C_{\rm inf}^2}\left(1 + \frac{1+\tau\widehat{C}_1}{C_{\rm SM}}\right)\tau \\
&=\; \left(\frac{2\widehat{C}_1^2}{C_{\rm SM}}\right)\left(\frac{C_{\rm LC}}{C_{\rm inf}}\right)\left(\frac{C_{\rm LC}}{C_{\rm inf}}\right)\left(1 + \frac{1+\tau\widehat{C}_1}{C_{\rm SM}}\right)\tau \\
&\le\; \theta\left(1 + \frac{1+\tau\widehat{C}_1}{C_{\rm SM}}\right)\tau \;=\; \left(\theta + \frac{\theta + (\theta\tau)\widehat{C}_1}{C_{\rm SM}}\right)\tau
\end{aligned}$$

which, using the assumption on τ, gives

$$2\widehat{C}_1^2\widehat{C}_b\tau \;<\; \left(\theta + \frac{\theta + \widehat{C}_1}{C_{\rm SM}}\right)\tau \;=\; \left(\frac{(1+C_{\rm SM})\theta + \widehat{C}_1}{C_{\rm SM}}\right)\tau \;<\; \frac{1}{2}.$$

In this way, replacing the foregoing inequality back into (3.20), we deduce that

$$\|\mathbf{u}_h\|_{0,\Omega} \;\le\; 2\widehat{C}_b\widetilde{C}\,\mathbb{B}(\mathbf{f},\mathbf{g}) \;=\; C_{\rm b}\,\mathbb{B}(\mathbf{f},\mathbf{g}),$$

which, together with (3.19), yields

$$\|(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h} \;\le\; \left(\widehat{C}_a\widetilde{C} + 2\widehat{C}_1^2\widehat{C}_a C_{\rm b}\tau\right)\mathbb{B}(\mathbf{f},\mathbf{g}) \;=\; C_{\rm a}\,\mathbb{B}(\mathbf{f},\mathbf{g}),$$

thus completing the proof of the theorem. □

4 A-priori error analysis

We now aim to derive the *a priori* error estimates for the augmented HDG scheme (2.11). We begin by remarking that the eventual extension to the present nonlinear case of the projection-based error analysis developed in [11] (see also [13]) does not seem straightforward, precisely because of the nonlinearity, and hence in what follows we adopt a more classical approach. Next, since $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and $\nabla \mathbf{u} = \mathbf{t} \in \mathbb{L}^2(\Omega)$ (cf. (2.4)), we observe that actually $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which guarantees that the jump $[\![\mathbf{u}]\!]$ vanish on any interior face of $\mathcal{T}_h$ and there holds $\{\!\{\mathbf{u}\}\!\} = \mathbf{u}$. In addition, since $\boldsymbol{\sigma} = \boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbb{I} \in \mathbb{L}^2(\Omega)$ and $\mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f}$ in Ω, with $\mathbf{f} \in \mathbf{L}^2(\Omega)$, we conclude that $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$, whence $[\![\boldsymbol{\sigma}]\!] = \mathbf{0}$ on each $F \in \mathcal{E}_h^i$. Then, it is easy to check that $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ satisfies the equations of (2.11), and then we obtain the error equations

$$[\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma}) - \mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)] + [\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{u}-\mathbf{u}_h] \;=\; 0 \quad \forall\, (\mathbf{s}_h,\boldsymbol{\tau}_h) \in H_h, \tag{4.1a}$$

$$[\mathcal{B}_h((\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h)),\mathbf{v}_h] - [\mathcal{S}_h(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h] - [\mathcal{C}_h(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h] \;=\; 0 \quad \forall\, \mathbf{v}_h \in V_h. \tag{4.1b}$$

The following result establishes the Céa estimate for (2.5) and (2.11).

Lemma 4.1. *Assume that*

$$0 \;<\; \tau \;<\; \min\left\{\frac{1}{\theta},\; \frac{1}{2}\left(\frac{C_{\mathrm{SM}}}{(1+C_{\mathrm{SM}})\theta + \widehat{C}_1}\right),\; \frac{1}{\vartheta}\right\},$$

with $\theta > 0$ defined in Lemma 3.12 and

$$\vartheta \;:=\; 2\left(1 + \frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(\frac{C_{\mathrm{LC}}}{C_{\mathrm{inf}}}\right)\left(\frac{\widehat{C}_1 + 2\widehat{C}_2^2}{C_{\mathrm{inf}}}\right) \;>\; 0.$$

Let $((\mathbf{t},\boldsymbol{\sigma}),\mathbf{u}) \in \mathbb{L}^2(\Omega)\times\mathbb{H}(\mathbf{div};\Omega)\times\mathbf{L}^2(\Omega)$ and $((\mathbf{t}_h,\boldsymbol{\sigma}_h),\mathbf{u}_h) \in H_h \times V_h$ be the unique solutions of (2.5) *and* (2.11)*, respectively. Then, there hold the Céa error estimates*

$$\begin{aligned}\|(\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h} \;\le\; & 2\left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(1+\frac{\|\mathcal{B}_h\|}{C_{\mathrm{inf}}}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h}\|(\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\\ &+\left\{\frac{\|\mathcal{B}_h\|}{C_{\mathrm{SM}}} + \left\{1+\left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\frac{\|\mathcal{B}_h\|}{C_{\mathrm{inf}}}\right\}\left(\frac{C_{\mathrm{inf}}}{C_{\mathrm{LC}}}\right)\right\}\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\le\; & 2\left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(\frac{C_{\mathrm{LC}}}{C_{\mathrm{inf}}}\right)\left(1+\frac{\|\mathcal{B}_h\|}{C_{\mathrm{inf}}}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h}\|(\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\\ &+ 2\left\{1+\left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\frac{\|\mathcal{B}_h\|}{C_{\mathrm{inf}}}\right\}\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}.\end{aligned} \tag{4.3}$$

Proof. We proceed as in [31, Proposition 4.1]. In fact, we first set $H_h = \widetilde{H}_h \oplus \widetilde{H}_h^{\perp}$, with $\widetilde{H}_h$ being the kernel of $\mathcal{B}_h$. Hence, given $(\mathbf{s}_h,\boldsymbol{\tau}_h) \in H_h$, we let $(\mathbf{r}_h,\boldsymbol{\rho}_h) \in \widetilde{H}_h^{\perp}$ be the unique solution of

$$[\mathcal{B}_h(\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{v}_h] \;=\; [\mathcal{B}_h((\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s}_h,\boldsymbol{\tau}_h)) - \mathcal{S}_h(\mathbf{u}-\mathbf{u}_h) - \mathcal{C}_h(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h] \quad \forall\, \mathbf{v}_h \in V_h,$$

which there exists thanks to the discrete inf-sup condition and the continuity of $\mathcal{B}_h$. Then, there holds

$$\begin{aligned}C_{\mathrm{inf}}\,\|(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h} \;&\le\; \sup_{\substack{\mathbf{v}_h\in V_h\\ \mathbf{v}_h\neq\mathbf{0}}}\frac{[\mathcal{B}_h(\mathbf{r}_h,\boldsymbol{\rho}_h),\mathbf{v}_h]}{\|\mathbf{v}_h\|_{0,\Omega}}\\ &=\; \sup_{\substack{\mathbf{v}_h\in V_h\\ \mathbf{v}_h\neq\mathbf{0}}}\frac{[\mathcal{B}_h((\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s}_h,\boldsymbol{\tau}_h)) - \mathcal{S}_h(\mathbf{u}-\mathbf{u}_h) - \mathcal{C}_h(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h]}{\|\mathbf{v}_h\|_{0,\Omega}}\\ &\le\; \|\mathcal{B}_h\|\,\|(\mathbf{t},\boldsymbol{\sigma}) - (\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h} + \{\,\|\mathcal{S}_h\| + \|\mathcal{C}_h\|\,\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\end{aligned}$$

that is

$$\|(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h} \;\le\; \frac{\|\mathcal{B}_h\|}{C_{\inf}}\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}+\left\{\frac{\|\mathcal{S}_h\|+\|\mathcal{C}_h\|}{C_{\inf}}\right\}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}. \tag{4.4}$$

Also, note by construction of $(\mathbf{r}_h,\boldsymbol{\rho}_h)\in\widetilde{H}_h^{\perp}$ and (4.1b) that there holds

$$[\mathcal{B}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)),\mathbf{v}_h] \;=\; 0 \quad \forall\ \mathbf{v}_h\in V_h. \tag{4.5}$$

Next, applying the strong monotonicity of $\mathcal{A}_h$ and (4.1a), we get

$$\begin{aligned}
&C_{\mathrm{SM}}\ \|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}^2\\
&\quad\le\ [\mathcal{A}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h))-\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)]\\
&\quad=\ [\mathcal{A}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h))-\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)]\\
&\qquad+\ [\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma})-\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)]\\
&\quad=\ [\mathcal{A}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h))-\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)]\\
&\qquad-\ [\mathcal{B}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)),\mathbf{u}-\mathbf{u}_h].
\end{aligned}$$

In turn, it follows from (4.5) that we can replace $\mathbf{u}_h$ by $\mathbf{v}_h\in V_h$ in the foregoing expression involving $\mathcal{B}_h$, and hence we obtain

$$\begin{aligned}
&C_{\mathrm{SM}}\ \|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}^2\\
&\quad\le\ [\mathcal{A}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h))-\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)]\\
&\qquad-\ [\mathcal{B}_h((\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)),\mathbf{u}-\mathbf{v}_h]\\
&\quad\le\ C_{\mathrm{LC}}\ \|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t},\boldsymbol{\sigma})\|_{H_h}\ \|(\mathbf{s}_h+\mathbf{r}_h,\boldsymbol{\tau}_h+\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}\\
&\qquad+\ \|\mathcal{B}_h\|\ \|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}\ \|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega},
\end{aligned}$$

which yields

$$\|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h} \;\le\; \frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)-(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h}+\frac{\|\mathcal{B}_h\|}{C_{\mathrm{SM}}}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}.$$

Thus, by triangle inequality we deduce that

$$\begin{aligned}
\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}\ &\le\ \|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)-(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h}+\|(\mathbf{s}_h,\boldsymbol{\tau}_h)+(\mathbf{r}_h,\boldsymbol{\rho}_h)-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}\\
&\le\ \left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)-(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h}\ +\ \frac{\|\mathcal{B}_h\|}{C_{\mathrm{SM}}}\ \|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}\\
&\le\ \left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\ +\ \left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\|(\mathbf{r}_h,\boldsymbol{\rho}_h)\|_{H_h}\ +\ \frac{\|\mathcal{B}_h\|}{C_{\mathrm{SM}}}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega},
\end{aligned}$$

which, together with (4.4) and the fact that $(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h$ and $\mathbf{v}_h\in V_h$ are arbitrary, imply

$$\begin{aligned}
\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}\ &\le\ \left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(1+\frac{\|\mathcal{B}_h\|}{C_{\inf}}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h}\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\\
&\quad+\ \frac{\|\mathcal{B}_h\|}{C_{\mathrm{SM}}}\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}+\left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(\frac{\|\mathcal{S}_h\|+\|\mathcal{C}_h\|}{C_{\inf}}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}.
\end{aligned} \tag{4.6}$$

On the other hand, using the inf-sup condition for $\mathcal{B}_h$, (4.1a), and the Lipschitz-continuity of $\mathcal{A}_h$, we find that for each $\mathbf{v}_h \in V_h$ there holds

$$\begin{aligned}
C_{\text{inf}}\,\|\mathbf{v}_h - \mathbf{u}_h\|_{0,\Omega} &\le \sup_{\substack{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h\\(\mathbf{s}_h,\boldsymbol{\tau}_h)\neq\mathbf{0}}} \frac{[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h-\mathbf{u}_h]}{\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}}\\
&= \sup_{\substack{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h\\(\mathbf{s}_h,\boldsymbol{\tau}_h)\neq\mathbf{0}}} \frac{[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h-\mathbf{u}] \;+\; [\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{u}-\mathbf{u}_h]}{\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}}\\
&= \sup_{\substack{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h\\(\mathbf{s}_h,\boldsymbol{\tau}_h)\neq\mathbf{0}}} \frac{[\mathcal{B}_h(\mathbf{s}_h,\boldsymbol{\tau}_h),\mathbf{v}_h-\mathbf{u}] \;-\; [\mathcal{A}_h(\mathbf{t},\boldsymbol{\sigma})-\mathcal{A}_h(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s}_h,\boldsymbol{\tau}_h)]}{\|(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}}\\
&\le \|\mathcal{B}_h\|\;\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega} \;+\; C_{\text{LC}}\,\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h},
\end{aligned}$$

which, together with an application of the triangle inequality, gives

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\le\; \left(1+\frac{\|\mathcal{B}_h\|}{C_{\text{inf}}}\right)\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega} \;+\; \frac{C_{\text{LC}}}{C_{\text{inf}}}\,\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{t}_h,\boldsymbol{\sigma}_h)\|_{H_h}. \tag{4.7}$$

Next, by substituting (4.6) into (4.7), we arrive at

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\le\;& \left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\left(\frac{C_{\text{LC}}}{C_{\text{inf}}}\right)\left(1+\frac{\|\mathcal{B}_h\|}{C_{\text{inf}}}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h}\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\\
&+\;\left\{1+\left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\frac{\|\mathcal{B}_h\|}{C_{\text{inf}}}\right\}\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}\\
&+\;\left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\left(\frac{C_{\text{LC}}}{C_{\text{inf}}}\right)\left(\frac{\|\mathcal{S}_h\|+\|\mathcal{C}_h\|}{C_{\text{inf}}}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}.
\end{aligned}$$

In turn, we know from Lemma 3.11 that $\|\mathcal{S}_h\| \le \tau\widehat{C}_1$ and $\|\mathcal{C}_h\| \le 2\tau\widehat{C}_1^2$, and hence, recalling that $\tau < \frac{1}{\vartheta}$, we deduce that

$$\left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\left(\frac{C_{\text{LC}}}{C_{\text{inf}}}\right)\left(\frac{\|\mathcal{S}_h\|+\|\mathcal{C}_h\|}{C_{\text{inf}}}\right) \;\le\; \left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\left(\frac{C_{\text{LC}}}{C_{\text{inf}}}\right)\left(\frac{\widehat{C}_1+2\widehat{C}_1^2}{C_{\text{inf}}}\right)\tau \;<\; \frac{1}{2},$$

which allows to conclude from the previous inequality that

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\le\;& 2\left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\left(\frac{C_{\text{LC}}}{C_{\text{inf}}}\right)\left(1+\frac{\|\mathcal{B}_h\|}{C_{\text{inf}}}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\tau}_h)\in H_h}\|(\mathbf{t},\boldsymbol{\sigma})-(\mathbf{s}_h,\boldsymbol{\tau}_h)\|_{H_h}\\
&+\,2\left\{1+\left(1+\frac{C_{\text{LC}}}{C_{\text{SM}}}\right)\frac{\|\mathcal{B}_h\|}{C_{\text{inf}}}\right\}\inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}.
\end{aligned}\tag{4.8}$$

Finally, it is easy to see that (4.6) and (4.8) provide (4.2) and (4.3), thus finishing the proof. □

Next, in order to provide the rate of convergence of the discontinuous Galerkin scheme (2.11), we need the approximation properties of the finite element subspaces involved. For this purpose, given $T \in \mathcal{T}_h$, we let $\boldsymbol{\mathcal{P}}_T^k : \mathbb{L}^2(T) \to \mathbb{P}_k(T)$ and $\mathcal{P}_T^{k-1} : \mathbf{L}^2(T) \to \mathbf{P}_{k-1}(T)$ be the $\mathbb{L}^2(T)$ and $\mathbf{L}^2(T)$ – orthogonal projectors, respectively. It is well known (see, e.g. [7, 16]) that for each $\mathbf{s} \in \mathbb{H}^\ell(T)$ and $\mathbf{v} \in \mathbf{H}^{\ell+1}(T)$ there holds

$$\|\mathbf{s}-\boldsymbol{\mathcal{P}}_T^k(\mathbf{s})\|_{0,T} \;\le\; C h_T^{\min\{\ell,k+1\}}|\mathbf{s}|_{\ell,T} \quad \forall\, T\in\mathcal{T}_h, \tag{4.9}$$

and

$$\|\mathbf{v}-\mathcal{P}_T^{k-1}(\mathbf{v})\|_{0,T} \;\le\; C h_T^{\min\{\ell+1,k\}}|\mathbf{v}|_{\ell+1,T} \quad \forall\, T\in\mathcal{T}_h. \tag{4.10}$$

On the other hand, let $\Pi_T^{k-1} : \mathbb{H}^1(T) \to \mathbb{P}_k(T)$ be the Raviart-Thomas interpolation operator (see [2, 16, 29]), which satisfies the approximation property

$$\|\boldsymbol{\tau} - \Pi_T^{k-1}(\boldsymbol{\tau})\|_{\mathbf{div},T} \;\leq\; C h_T^{\min\{\ell,k\}} \Big\{ |\boldsymbol{\tau}|_{\ell,T} \;+\; |\mathbf{div}(\boldsymbol{\tau})|_{\ell,T} \Big\} \quad \forall\, T \in \mathcal{T}_h, \tag{4.11}$$

and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(T)$ such that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^\ell(T)$, with $\ell \geq 1$. Moreover, the interpolation operator Π_T^{k-1} can also be defined as a bounded linear operator from the larger space $\mathbb{H}^\ell(T) \cap \mathbb{H}(\mathbf{div};T)$ into $\mathbb{P}_k(T)$ for all $\ell \in (0,1]$ (see, e.g. [21, Theorem 3.16]). In this case there holds the following interpolation error estimate (see [16, Lemma 3.19])

$$\|\boldsymbol{\tau} - \Pi_T^{k-1}(\boldsymbol{\tau})\|_{0,T} \;\leq\; C h_T^{\ell} \Big\{ |\boldsymbol{\tau}|_{\ell,T} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,T} \Big\} \quad \forall\, T \in \mathcal{T}_h,$$

which, together with (4.11), implies for $\ell > 0$ that

$$\|\boldsymbol{\tau} - \Pi_T^{k-1}(\boldsymbol{\tau})\|_{\mathbf{div},T} \;\leq\; C h_T^{\min\{\ell,k\}} \Big\{ |\boldsymbol{\tau}|_{\ell,T} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,T} \Big\} \quad \forall\, T \in \mathcal{T}_h.$$

On the other hand, observe that, given $Z := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \;:\; \boldsymbol{\tau}|_T \in \mathbb{H}^\ell(T) \quad \forall\, T \in \mathcal{T}_h\}$, we can define $\Pi_{\Sigma_h} : \mathbb{H}(\mathbf{div};\Omega) \cap Z \to \Sigma_h$ by

$$\Pi_{\Sigma_h}(\boldsymbol{\tau})|_T \;:=\; \Pi_T^{k-1}(\boldsymbol{\tau}|_T) \;+\; d\,\mathbb{I} \quad \forall\, T \in \mathcal{T}_h,$$

with $d := -\frac{1}{n|\Omega|} \sum\limits_{T \in \mathcal{T}_h} \int_T \operatorname{tr}\Big(\Pi_T^{k-1}(\boldsymbol{\tau}|_T)\Big) \in R$. Then, it is easy to prove that

$$\|\boldsymbol{\tau} - \Pi_{\Sigma_h}(\boldsymbol{\tau})\|_{\Sigma_h}^2 \;\leq\; \sum_{T \in \mathcal{T}_h} \|\boldsymbol{\tau} - \Pi_T^{k-1}(\boldsymbol{\tau})\|_{\mathbf{div},T}^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) \cap Z,$$

and hence

$$\|\boldsymbol{\tau} - \Pi_{\Sigma_h}(\boldsymbol{\tau})\|_{\Sigma_h} \;\leq\; C \sum_{T \in \mathcal{T}_h} h_T^{\min\{\ell,k\}} \Big\{ |\boldsymbol{\tau}|_{\ell,T} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,T} \Big\}. \tag{4.12}$$

In this way, as a consequence of (4.9), (4.10), (4.12), and the usual interpolation estimates, we find that S_h, Σ_h and V_h satisfy the following approximation properties:

($\mathbf{AP}_h^{\mathbf{t}}$) For each $\ell \geq 0$ and for each $\mathbf{s} \in \mathbb{H}^\ell(\Omega)$ there exists $\mathbf{s}_h \in S_h$ such that

$$\|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \;\leq\; C \sum_{T \in \mathcal{T}_h} h_T^{\min\{\ell,k+1\}} |\mathbf{s}|_{\ell,T}.$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) For each $\ell > 0$ and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^\ell(\Omega)$ there exists $\boldsymbol{\tau}_h \in \Sigma_h$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\Sigma_h} \;\leq\; C \sum_{T \in \mathcal{T}_h} h_T^{\min\{\ell,k\}} \Big\{ |\boldsymbol{\tau}|_{\ell,T} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,T} \Big\}.$$

($\mathbf{AP}_h^{\mathbf{u}}$) For each $\ell \geq 0$ and for each $\mathbf{v} \in \mathbf{H}^\ell(\Omega)$ there exists $\mathbf{v}_h \in V_h$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega} \;\leq\; C \sum_{T \in \mathcal{T}_h} h_T^{\min\{\ell+1,k\}} |\mathbf{v}|_{\ell+1,T}.$$

The following theorem establishes the theoretical rates of convergence of the discrete scheme (2.11), under suitable regularity assumptions on the exact solution.

Theorem 4.1. *Assume the same hypotheses of Lemma 4.1. In addition, suppose that there exists an integer $\ell > 0$ such that $\mathbf{t}|_T \in \mathbb{H}^\ell(T)$, $\boldsymbol{\sigma}|_T \in \mathbb{H}^\ell(T)$, $\mathbf{div}(\boldsymbol{\sigma}|_T) \in \mathbf{H}^\ell(T)$ and $\mathbf{u}|_T \in \mathbf{H}^{\ell+1}(T)$, for all $T \in \mathcal{T}_h$. Then, there exists $C > 0$, independent of h and the polynomial approximation degree k, such that*

$$\begin{aligned} &\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} \;+\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h} \;+\; \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \\ &\quad\le\; C \sum_{T\in\mathcal{T}_h} h_T^{\min\{\ell,k\}} \Big\{ |\mathbf{t}|_{\ell,T} \;+\; |\boldsymbol{\sigma}|_{\ell,T} \;+\; \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,T} \;+\; |\mathbf{u}|_{\ell+1,T} \Big\}. \end{aligned}$$

Proof. It follows from the Céa estimate (cf. Lemma 4.1) and the approximation properties $(\mathbf{AP}_h^{\mathbf{t}})$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ and $(\mathbf{AP}_h^{\mathbf{u}})$. □

Note from the previous theorem and (3.7) that we can also conclude that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \;\le\; C \sum_{T\in\mathcal{T}_h} h_T^{\min\{\ell,k\}} \Big\{ |\mathbf{t}|_{\ell,T} + |\boldsymbol{\sigma}|_{\ell,T} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,T} + |\mathbf{u}|_{\ell+1,T} \Big\}. \tag{4.13}$$

Furthermore, we know from (2.1) that $p = -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma})$, which suggests to define the following postprocessed approximation of the pressure:

$$p_h \;:=\; -\frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\sigma}_h) \quad \text{in} \quad \Omega\,,$$

and therefore

$$\|p - p_h\|_{0,\Omega} \;=\; \frac{1}{n}\|\mathrm{tr}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,\|_{0,\Omega} \;\le\; \frac{1}{n}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}, \tag{4.14}$$

which, thanks to (4.13), gives the *a priori* error estimate for the pressure.

Now, as in [11], we measure the errors of quantities defined on $\partial\mathcal{T}_h$ with the seminorm:

$$\|\boldsymbol{\mu}\|_h \;:=\; \left\{ \sum_{T\in\mathcal{T}_h} h_T \|\boldsymbol{\mu}\|^2_{0,\partial T} \right\}^{1/2},$$

and we let $\Pi_{\mathcal{E}_h} : \mathbf{L}^2(\mathcal{E}_h) \to \mathbf{P}(\mathcal{E}_h)$ be the orthogonal projection onto the space of piecewise polynomials of degree $\le k$ on $\mathcal{E}_h$. Next, we end this section with the *a priori* error estimate for the trace of the velocity unknown, which is established next.

Theorem 4.2. *Assume the same hypotheses of Theorem 4.1. Then, there exists $C > 0$, independent of h and the polynomial approximation degree k, such that*

$$\|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h \;\le\; C \sum_{T\in\mathcal{T}_h} h_T^{\min\{\ell,k\}} \Big\{ |\mathbf{t}|_{\ell,T} + |\boldsymbol{\sigma}|_{\ell,T} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,T} + |\mathbf{u}|_{\ell+1,T} \Big\}.$$

Proof. Since $\Pi_{\mathcal{E}_h}(\mathbf{u}) = \Pi_\Gamma(\mathbf{g}) = \widehat{\mathbf{u}}_h$ on $\mathcal{E}_h^\partial$, we only need to compute the error for each $F \in \mathcal{E}_h^i$. In fact, we have

$$\begin{aligned} \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h^2 \;&=\; \sum_{T\in\mathcal{T}_h} \sum_{F\in\partial T\setminus\Gamma} h_T \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \boldsymbol{\lambda}_h\|^2_{0,F} \\ &\le\; \widetilde{C} \sum_{T\in\mathcal{T}_h} \sum_{F\in\partial T\setminus\Gamma} \mathtt{h} \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \boldsymbol{\lambda}_h\|^2_{0,F} \;=\; 2\widetilde{C} \sum_{F\in\mathcal{E}_h^i} \mathtt{h} \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \boldsymbol{\lambda}_h\|^2_{0,F}, \end{aligned}$$

with $\widetilde{C} \geq 1$ depending only on the shape regularity of the mesh. Then, according to (2.10), (3.6) and the fact that $\{\!\{\mathbf{u}\}\!\} = \mathbf{u}$ and $[\![\boldsymbol{\sigma}]\!] = \mathbf{0}$ on $\mathcal{E}_h^i$, we obtain that

$$\begin{aligned} \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h^2 &\leq 2\widetilde{C} \sum_{F\in\mathcal{E}_h^i} \mathtt{h} \left\| \Pi_{\mathcal{E}_h}(\mathbf{u}) - \{\!\{\mathbf{u}_h\}\!\} + \frac{1}{2}(\tau\mathtt{h})^{-1}[\![\boldsymbol{\sigma}_h]\!] \right\|_{0,F}^2 \\ &\leq C \sum_{F\in\mathcal{E}_h^i} \left\{ \|\mathtt{h}^{1/2}\{\!\{\Pi_{\mathcal{E}_h}(\mathbf{u}) - \mathbf{u}_h\}\!\}\|_{0,F}^2 + \frac{1}{4\tau}\|(\tau\mathtt{h})^{-1/2}[\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]\|_{0,F}^2 \right\} \\ &\leq C \left\{ \|\mathtt{h}^{1/2}\{\!\{\Pi_{\mathcal{E}_h}(\mathbf{u}) - \mathbf{u}_h\}\!\}\|_{0,\mathcal{E}_h^i}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h}^2 \right\}. \end{aligned} \tag{4.15}$$

Next, it is easy to check that $\Pi_{\mathcal{E}_h}(\mathbf{u})|_F = \mathcal{P}_h^k(\mathbf{u})|_F =$ for each $F \in \mathcal{E}_h$, where $\mathcal{P}_h^k : \mathbf{L}^2(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ is the orthogonal projector, which satisfies

$$\|\mathbf{v} - \mathcal{P}_h^k(\mathbf{v})\|_{0,\Omega} \leq C \sum_{T\in\mathcal{T}_h} h_T^{\min\{\ell+1,k+1\}} |\mathbf{v}|_{\ell+1,T} \quad \forall\, \mathbf{v} \in \mathbf{H}^\ell(T), \quad \forall\, T \in \mathcal{T}_h. \tag{4.16}$$

Consequently, using the analogue of the part i) of Lemma 3.10 with $\mathbf{P}_k(\mathcal{T}_h)$ instead of V_h, we deduce from (4.15) that

$$\begin{aligned} \|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h &\leq C \left\{ \|\mathcal{P}_h^k(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h} \right\} \\ &\leq C \left\{ \|\mathbf{u} - \mathcal{P}_h^k(\mathbf{u})\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h} \right\}, \end{aligned}$$

which, together with (4.16) and Theorem 4.1, complete the proof. □

5 Implementation considerations

In this section we describe some general aspects on the computational implementation of the discrete scheme proposed in Section 2. We remark that we refer to the original HDG system (2.9) since, as explained before, the equivalent reduced scheme given by (2.11) was introduced just for sake of the analysis. We begin by considering again problem (2.9) in a single element $T \in \mathcal{T}_h$ with Dirichlet's datum $\mathbf{g} = \mathbf{0}$ (as is usual, the boundary condition can be imposed later), that is

$$\begin{aligned} \int_T \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h - \int_T \mathbf{s}_h : \boldsymbol{\sigma}_h^{\mathrm{d}} &= 0, \\ \int_T \left\{ \mathbf{t}_h - \kappa_1 \boldsymbol{\psi}(\mathbf{t}_h) \right\} : \boldsymbol{\tau}_h^{\mathrm{d}} + \left\{ \kappa_1 \int_T \boldsymbol{\sigma}_h^{\mathrm{d}} : \boldsymbol{\tau}_h^{\mathrm{d}} + \kappa_2 \int_T \mathbf{div}(\boldsymbol{\sigma}_h) \cdot \mathbf{div}(\boldsymbol{\tau}_h) \right\} & \\ + \int_T \mathbf{u}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) - \int_{\partial T} \boldsymbol{\tau}_h \boldsymbol{\nu} \cdot \boldsymbol{\lambda}_h &= -\kappa_2 \int_T \mathbf{f} \cdot \mathbf{div}(\boldsymbol{\tau}_h), \\ - \int_T \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\sigma}_h) + \int_{\partial T} \mathbf{S}\mathbf{u}_h \cdot \mathbf{v}_h - \int_{\partial T} \mathbf{S}\boldsymbol{\lambda}_h \cdot \mathbf{v}_h &= \int_T \mathbf{f} \cdot \mathbf{v}_h, \\ - \int_{\partial T} \boldsymbol{\sigma}_h \boldsymbol{\nu} \cdot \boldsymbol{\mu}_h + \int_{\partial T} \mathbf{S}\mathbf{u}_h \cdot \boldsymbol{\mu}_h - \int_{\partial T} \mathbf{S}\boldsymbol{\lambda}_h \cdot \boldsymbol{\mu}_h &= 0, \end{aligned}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\mu}_h) \in \mathbb{P}_k(T) \times \mathbb{P}_k(T) \times \mathbf{P}_{k-1}(T) \times \mathbf{P}_k(\partial T)$.

Note that, because of the null mean value condition of the trace of $\boldsymbol{\sigma}_h$, that is $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}_h) = 0$, we can not establish the value of $\boldsymbol{\sigma}_h|_T$ only with the information from T (as it is natural in discontinuous Galerkin schemes). For that reason, and in order to rewrite the above local contribution in an equivalent form, we now define the local space

$$\Sigma_{h,0}(T) := \left\{ \boldsymbol{\tau} \in \mathbb{P}_k(T) \ : \ \int_T \mathrm{tr}\,(\boldsymbol{\tau}) = 0 \right\},$$

for which there holds $\mathbb{P}_k(T) = \Sigma_{h,0}(T) \oplus P_0(T)\mathbb{I}$, where $\mathbb{I} \in R^{n\times n}$ is the identity matrix. Next, given $\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \in S_h$, we consider the local decomposition

$$\boldsymbol{\sigma}_h|_T \,=\, \widetilde{\boldsymbol{\sigma}}_h|_T \,+\, \rho_h|_T\mathbb{I} \quad \text{and} \quad \boldsymbol{\tau}_h|_T \,=\, \widetilde{\boldsymbol{\tau}}_h|_T \,+\, \zeta_h|_T\mathbb{I} \quad \forall\, T \in \mathcal{T}_h\,,$$

where $\widetilde{\boldsymbol{\sigma}}_h|_T$, $\widetilde{\boldsymbol{\tau}}_h|_T \in \Sigma_{h,0}(T)$, $\rho_h|_T$, $\zeta_h|_T \in P_0(T)$, and rewrite the above local contribution as

$$\begin{aligned}
\int_T \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h - \int_T \mathbf{s}_h : \widetilde{\boldsymbol{\sigma}}_h^{\mathrm{d}} &= 0,\\
\int_T \Big\{\mathbf{t}_h - \kappa_1\boldsymbol{\psi}(\mathbf{t}_h)\Big\} : \widetilde{\boldsymbol{\tau}}_h^{\mathrm{d}} + \Big\{\kappa_1 \int_T \widetilde{\boldsymbol{\sigma}}_h^{\mathrm{d}} : \widetilde{\boldsymbol{\tau}}_h^{\mathrm{d}} + \kappa_2 \int_T \mathbf{div}(\widetilde{\boldsymbol{\sigma}}_h)\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}}_h)\Big\} & \\
+ \int_T \mathbf{u}_h \cdot \mathbf{div}(\widetilde{\boldsymbol{\tau}}_h) - \int_{\partial T} \widetilde{\boldsymbol{\tau}}_h\boldsymbol{\nu}\cdot\boldsymbol{\lambda}_h &= -\kappa_2 \int_T \mathbf{f}\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}}_h),\\
-\int_T \mathbf{v}_h\cdot\mathbf{div}(\widetilde{\boldsymbol{\sigma}}_h) + \int_{\partial T}\mathbf{S}\mathbf{u}_h\cdot\mathbf{v}_h - \int_{\partial T}\mathbf{S}\boldsymbol{\lambda}_h\cdot\mathbf{v}_h &= \int_T \mathbf{f}\cdot\mathbf{v}_h,\\
-\int_{\partial T}\widetilde{\boldsymbol{\sigma}}_h\boldsymbol{\nu}\cdot\boldsymbol{\mu}_h + \int_{\partial T}\mathbf{S}\mathbf{u}_h\cdot\boldsymbol{\mu}_h - \int_{\partial T}\mathbf{S}\boldsymbol{\lambda}_h\cdot\boldsymbol{\mu}_h - \int_{\partial T}\rho_h\boldsymbol{\mu}_h\cdot\boldsymbol{\nu} &= 0,\\
-\int_{\partial T}\zeta_h\boldsymbol{\lambda}_h\cdot\boldsymbol{\nu} &= 0\,,
\end{aligned}$$

for all $(\mathbf{s}_h, \widetilde{\boldsymbol{\tau}}_h, \mathbf{v}_h, \boldsymbol{\mu}_h, \zeta_h) \in \mathbb{P}_k(T) \times \Sigma_{h,0}(T) \times \mathbf{P}_{k-1}(T) \times \mathbf{P}_k(\partial T) \times P_0(T)$. In addition, it is easy to see that the aforementioned condition on the trace of $\boldsymbol{\sigma}_h$ becomes

$$\sum_{T\in\mathcal{T}_h} \rho_h|_T\,|T| \,=\, 0\,.$$

Then, applying the Newton-Raphson's method to the global nonlinear system, we translate the local contribution for the Newton's linear system in the mth iteration into the form

$$\left(\begin{array}{ccc|cc}
\mathbf{DA}_1(\mathbf{t}_h^m) & \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
-\mathbf{B}^T - \mathbf{DA}_2(\mathbf{t}_h^m) & \mathbf{H} & \mathbf{C} & -\mathbf{E} & \mathbf{0}\\
\mathbf{0} & -\mathbf{C}^T & \mathbf{K} & -\mathbf{F} & \mathbf{0}\\ \hline
\mathbf{0} & -\mathbf{E}^T & \mathbf{F}^T & -\mathbf{D} & \mathbf{G}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^T & 0
\end{array}\right)
\left(\begin{array}{c}
\delta\mathbf{t}_h^m\\ \delta\widetilde{\boldsymbol{\sigma}}_h^m\\ \delta\mathbf{u}_h^m\\ \hline \delta\boldsymbol{\lambda}_h^m\\ \delta\rho_h^m
\end{array}\right)
=
\left(\begin{array}{c}
\mathbf{b}_1^m\\ \mathbf{b}_2^m\\ \mathbf{b}_3^m\\ \hline \mathbf{b}_4^m\\ \mathrm{b}_5^m
\end{array}\right),$$

where $\delta\mathbf{t}_h^m$ corresponds to the mth update for the $\mathbf{t}_h$ variable, that is $\mathbf{t}_h^{m+1} = \mathbf{t}_h^m + \delta\mathbf{t}_h^m$, and similarly for the other variables. The discrete operators $\mathbf{DA}_i(\mathbf{r})$, $i \in \{1, 2\}$, are the respective Gâteaux derivatives, given by

$$[\mathbf{DA}_1(\mathbf{r})\mathbf{t}, \mathbf{s}] \;:=\; \int_T \sum_{i,j,k,l=1}^{n} \frac{\partial}{\partial r_{kl}}\psi_{ij}(\mathbf{r})t_{kl}s_{ij} \;=\; \int_T \frac{\mu'(|\mathbf{r}|)}{|\mathbf{r}|}(\mathbf{r}:\mathbf{t})(\mathbf{r}:\mathbf{s}) \;+\; \int_T \mu(|\mathbf{r}|)\mathbf{t}:\mathbf{s},$$

and

$$[\mathbf{DA}_2(\mathbf{r})\mathbf{t}, \mathbf{s}] \;:=\; \kappa_1[\mathbf{DA}_1(\mathbf{r})\mathbf{t}, \mathbf{s}^{\mathrm{d}}],$$

for all $\mathbf{r}, \mathbf{t}, \mathbf{s} \in \mathbb{L}^2(T)$, with $|\mathbf{r}| = \|\mathbf{r}\|_{R^{n\times n}} \neq 0$. All the above discrete operators can be calculated similarly as in [6]. It is important to note here that the local submatrix

$$\left(\begin{array}{ccc}
\mathbf{DA}_1(\mathbf{t}_h^m) & \mathbf{B} & \mathbf{0}\\
-\mathbf{B}^T - \mathbf{DA}_2(\mathbf{t}_h^m) & \mathbf{H} & \mathbf{C}\\
\mathbf{0} & -\mathbf{C}^T & \mathbf{K}
\end{array}\right) \in R^{(n^2d_q+(n^2d_q-1)+nd_u)\times(n^2d_q+(n^2d_q-1)+nd_u)},$$

with $d_q := \dim P_k(T)$ and $d_u := \dim P_{k-1}(T)$, is invertible when $\mu > 0$ and $|\mathbf{t}_h^m| \neq 0$. Then, as it is usual in the HDG methods, we can obtain the values of $\delta\mathbf{t}_h^m|_T$, $\delta\widetilde{\boldsymbol{\sigma}}_h^m|_T$ and $\delta\mathbf{u}_h^m|_T$ as functions of $\delta\boldsymbol{\lambda}_h^m|_T$ and $\delta\rho_h^m|_T$ (actually, they only depend on $\delta\boldsymbol{\lambda}_h^m|_T$). In other words, we can reduce the stencil of the global linear system on each iteration of the Newton's method.

Finally, we let

$$\begin{aligned} N_{\text{total}} &:= (n^2 d_q + n^2 d_q + n d_u + d_q) \times (\#\text{ of element in } \mathcal{T}_h) \\ &\quad + (n d_l) \times (\#\text{ of faces in } \mathcal{T}_h), \end{aligned}$$

with $d_l := \dim P_k(F)$, $F \in \partial T$, be the total number of degrees of freedom including those for the pressure. In other words, N_{total} is the total number of unknowns defining $\mathbf{t}_h$, $\boldsymbol{\sigma}_h$, $\mathbf{u}_h$, $\boldsymbol{\lambda}_h$ and p_h. On the other hand, we let

$$N_{\text{comp}} := (n d_l) \times (\#\text{ of faces in } \mathcal{T}_h) + (\#\text{ of element in } \mathcal{T}_h)$$

be the number of degrees of freedom effectively employed in the computations, i.e, the total number of unknowns defining $\boldsymbol{\lambda}_h$ and ρ_h.

6 Numerical results

In this section we present several numerical experiments illustrating the performance of the augmented HDG method introduced in Section 2. We set $\tau = 10^{-2}$ for each one of the 4 examples to be reported, which, as shown below, works fine in all the cases. An a priori verification of the hypotheses on τ in Lemma 4.1 would certainly require the explicit knowledge of all the constants involved, which, however, is rarely possible. On the other hand, we take the stabilization parameter $\kappa_1 = \frac{\alpha_0}{\gamma_0^2}$, which obviously satisfies the assumption $\kappa_1 \in \left(0, \frac{2\alpha_0}{\gamma_0^2}\right)$ in Lemma 3.6, and then, as suggested by the value of the strong monotonicity constant C_{SM} at the end of its proof, we simply choose $\kappa_2 = \frac{\kappa_1}{2}$. The corresponding nonlinear algebraic system arising from (2.9) is solved by the Newton method with a tolerance of 10^{-6} and taking as initial iteration the solution of the associated linear Stokes problem (four iterations were required to achieve the given tolerance in each example). Now, according to the definitions given in Section 5, we recall that N_{total} is the total number of degrees of freedom, and N_{comp} is the number of degrees of freedom involved in the implementation of the Newton's method. The numerical results presented below were obtained using a C^{++}code, which was developed following the same techniques from [6]. In turn, the linear systems are solved using the conjugate gradient method with a relative tolerance of 10^{-6}.

In Example 1 we follow [28, 11] and consider the linear Stokes problem given by the flow uncovered by Kovaszany [24]. This means that $\Omega := (-0.5, 1.5) \times (0, 2)$, $\mu = 0.1$, and the data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that the exact solution is given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \left(1 - \exp(\lambda x_1)\cos(2\pi x_2),\ \frac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2)\right), \\ p(\mathbf{x}) &= \frac{1}{2}\exp(2\lambda x_1) - \frac{1}{8\lambda}\left\{\exp(3\lambda) - \exp(-\lambda)\right\}, \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, where $\lambda := \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ and $Re := \mu^{-1} = 10$ is the Reynolds number. It is easy to see in this linear case that $\alpha_0 = \gamma_0 = \mu$. Concerning the triangulations employed in our computations, we first consider seven meshes that are Cartesian refinements of a domain defined in terms of squares, and then we split each square into four congruent triangles.

In Example 2 we deal with the nonlinear version of Example 1. More precisely, we consider instead of $\mu = 0.1$ the kinematic viscosity function $\mu : R^+ \to R^+$ given by the Carreau law, that is $\mu(t) := \mu_0 + \mu_1(1+t^2)^{(\beta-2)/2} \quad \forall\, t \in R^+$, with $\mu_0 = \mu_1 = 0.5$ and $\beta = 1.5$. It is easy to check in this case that the assumptions (2.2) and (2.3) are satisfied with

$$\gamma_0 \;=\; \mu_0 + \mu_1 \left\{ \frac{|\beta-2|}{2} + 1 \right\} \quad \text{and} \quad \alpha_0 \;=\; \mu_0 \,.$$

Then, we let again $\Omega := (-0.5, 1.5) \times (0,2)$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is the same from Example 1. The set of triangulations utilized is also as in Example 1.

Next, in Example 3 we use the same nonlinearity μ from Example 2, consider the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \Big(r^{2/3}\sin(\theta),\; -r^{2/3}\cos(\theta) \Big)\,, \\ p(\mathbf{x}) &= \cos(x_1)\cos(x_2) - \sin^2(1)\,, \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$, where $r := |\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ and $\theta := \arctan\left(\frac{x_2}{x_1}\right)$. We remark that $\nabla\mathbf{u}$ is singular at the origin, and hence lower rates of convergence are expected in our computations. The meshes are generated analogously to the previous examples.

Finally, in Example 4 we consider the three dimensional domain $\Omega := (0,1)^3$, and assume the same kinematic viscosity function μ from Examples 2 and 3. In addition, the data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that the exact solution is given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \Big(x_1(\sin(2\pi x_3) - \sin(2\pi x_2)),\; x_2(\sin(2\pi x_1) - \sin(2\pi x_3)),\; x_3(\sin(2\pi x_2) - \sin(2\pi x_1)) \Big)\,, \\ p(\mathbf{x}) &= x_1 x_2 x_3 \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) \;+\; \frac{1}{8\pi^3}, \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2, x_3)^{\mathrm{t}} \in \Omega$.

It is easy to check that $\mathbf{u}$ is divergence free and $\displaystyle\int_\Omega p = 0$ for each one of the aforedescribed examples.

In Tables 6.1−6.4 we summarize the convergence history of the augmented HDG method (2.9) as applied to Examples 1 and 2 for the polynomial degrees $k \in \{1,2,3,4\}$. We observe there, looking at the experimental rates of convergence, that the orders predicted for each k by Theorems 4.1 and 4.2, and estimates (4.13) and (4.14), are attained in all the unknowns for these smooth examples. Actually, the errors $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ behave exactly as proved, whereas the remaining ones show higher orders of convergence. In particular, $\|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h$ presents a superconvergence phenomenon with two additional powers of h. In addition, it is interesting to notice that these numerical results provide the same rates of convergence obtained for the linear case in [11], and hence they might constitute numerical evidences supporting the conjecture that the a priori error estimates derived in the present paper are not sharp. We plan to address this issue in a separate work. Nevertheless, as already mentioned at the beginning of Section 4, whether the projection-based error analysis developed in [11] will work or not in this nonlinear case is still an open problem.

Furthermore, in Tables 6.5−6.6 we summarize the convergence history of the augmented HDG method (2.9) as applied to Example 3 for the polynomial degrees $k \in \{1,2,3,4\}$. In this case, and because of the singularity at the origin of the exact solution, the theoretical orders of convergence are far to be attained. In fact, similarly as obtained in [6], $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ behaves as $\mathcal{O}(h^{\min\{k,4/3\}})$, whereas $\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} = \mathcal{O}(h^{2/3})$. Also, $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} = \mathcal{O}(h^{2/3})$, $\|\Pi_{\mathcal{E}_h}(\mathbf{u}) - \widehat{\mathbf{u}}_h\|_h = \mathcal{O}(h^{\min\{k,4/3\}})$, and thanks to (4.14), $\|p - p_h\|_{0,\Omega} = \mathcal{O}(h^{2/3})$ as well. Moreover, the behaviour of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Sigma_h}$ in Table 6.5

k	h	N_{total}	N_{comp}	$\Vert\mathbf{t}-\mathbf{t}_h\Vert_{0,\Omega}$ error	order	$\Vert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\Vert_{0,\Omega}$ error	order	$\Vert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\Vert_{\Sigma_h}$ error	order
	0.2000	14080	2881	1.13e-0	−−	2.96e-1	−−	5.08e-0	−−
	0.1333	31620	6421	5.17e-1	1.92	1.37e-1	1.90	3.48e-0	0.93
	0.1000	56160	11361	2.95e-1	1.95	7.81e-2	1.95	2.64e-0	0.96
1	0.0800	87700	17701	1.90e-1	1.97	5.04e-2	1.97	2.12e-0	0.98
	0.0667	126240	25441	1.32e-1	1.98	3.51e-2	1.98	1.77e-0	0.99
	0.0571	171780	34581	9.75e-2	1.98	2.59e-2	1.98	1.52e-0	0.99
	0.0500	224320	45121	7.47e-2	1.99	1.98e-2	1.99	1.33e-0	0.99
	0.2000	27720	4121	8.88e-2	−−	1.88e-2	−−	5.86e-1	−−
	0.1333	62280	9181	2.77e-2	2.88	5.86e-3	2.87	2.71e-1	1.90
	0.1000	110640	16241	1.19e-2	2.92	2.53e-3	2.93	1.55e-1	1.95
2	0.0800	172800	25301	6.20e-3	2.94	1.31e-3	2.95	9.96e-2	1.97
	0.0667	248760	36361	3.62e-3	2.95	7.62e-4	2.96	6.94e-2	1.98
	0.0571	338520	49421	2.29e-3	2.96	4.82e-4	2.97	5.11e-2	1.99
	0.0500	442080	64481	1.54e-3	2.97	3.24e-4	2.98	3.92e-2	1.99
	0.2000	45760	5361	5.59e-3	−−	1.18e-3	−−	4.98e-2	−−
	0.1333	102840	11941	1.15e-3	3.90	2.46e-4	3.87	1.54e-2	2.89
	0.1000	182720	21121	3.69e-4	3.95	7.93e-5	3.93	6.62e-3	2.94
3	0.0800	285400	32901	1.52e-4	3.97	3.28e-5	3.96	3.41e-3	2.96
	0.0667	410880	47281	7.38e-5	3.98	1.59e-5	3.97	1.98e-3	2.98
	0.0571	559160	64261	3.99e-5	3.98	8.62e-6	3.98	1.25e-3	2.98
	0.0500	730240	83841	2.34e-5	3.99	5.06e-6	3.98	8.41e-4	2.99
	0.2000	68200	6601	2.97e-4	−−	6.42e-5	−−	3.35e-3	−−
	0.1333	153300	14701	4.06e-5	4.91	8.92e-6	4.87	6.94e-4	3.88
	0.1000	272400	26001	9.79e-6	4.95	2.16e-6	4.93	2.23e-4	3.94
4	0.0800	425500	40501	3.23e-6	4.96	7.14e-7	4.96	9.22e-5	3.96
	0.0667	612600	58201	1.31e-6	4.97	2.89e-7	4.97	4.47e-5	3.98
	0.0571	833700	79101	6.11e-7	4.93	1.34e-7	4.96	2.42e-5	3.98
	0.0500	1088800	103201	3.16e-7	4.93	6.93e-8	4.96	1.42e-5	3.98

Table 6.1: History of convergence for Example 1 (Part 1).

is explained by the fact that the a priori estimate for $\Vert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\Vert_{\Sigma_h}$ depends on the regularity of $\mathbf{div}(\boldsymbol{\sigma})$, which can be shown to belong precisely to $\mathbf{H}^{-1/3}(\Omega)$. A classical way of circumventing this drawback is the incorporation of an adaptive scheme based on a posteriori error estimates. This issue will also be addressed in a forthcoming paper.

On the other hand, in Tables 6.7−6.8 we present the convergence history of the augmented HDG method (2.9) as applied to Example 4 for the polynomial degrees $k \in \{1, 2\}$. The remarks in this case are exactly the same given above for Examples 1 and 2.

Finally, some components of the approximate and exact solutions for Examples 2, 3, and 4 are displayed in Figures 6.1−6.8. They all correspond to those obtained with the fourth mesh and for the polynomial degree k indicated in each case. Here we use the notations $\mathbf{t}_h = (t_{h,ij})_{i,j=1,n}$, $\boldsymbol{\sigma}_h = (\sigma_{h,ij})_{i,j=1,n}$, and $\mathbf{u}_h = (u_{h,i})_{i=1,n}$.

k	h	N_{total}	N_{comp}	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ error	order	$\|\Pi_{\mathcal{E}_h}(\mathbf{u})-\widehat{\mathbf{u}}_h\|_h$ error	order	$\|p-p_h\|_{0,\Omega}$ error	order
	0.2000	14080	2881	4.75e-1	$--$	7.89e-2	$--$	1.93e-1	$--$
	0.1333	31620	6421	3.17e-1	0.99	2.76e-2	2.59	8.95e-2	1.90
	0.1000	56160	11361	2.38e-1	1.00	1.36e-2	2.47	5.12e-2	1.95
1	0.0800	87700	17701	1.91e-1	1.00	8.00e-3	2.37	3.30e-2	1.97
	0.0667	126240	25441	1.59e-1	1.00	5.27e-3	2.28	2.30e-2	1.98
	0.0571	171780	34581	1.36e-1	1.00	3.74e-3	2.22	1.69e-2	1.98
	0.0500	224320	45121	1.19e-1	1.00	2.80e-3	2.18	1.30e-2	1.99
	0.2000	27720	4121	6.02e-2	$--$	5.88e-3	$--$	1.17e-2	$--$
	0.1333	62280	9181	2.67e-2	2.00	1.26e-3	3.79	3.65e-3	2.87
	0.1000	110640	16241	1.50e-2	2.00	4.16e-4	3.86	1.57e-3	2.93
2	0.0800	172800	25301	9.61e-3	2.00	1.75e-4	3.89	8.14e-4	2.95
	0.0667	248760	36361	6.67e-3	2.00	8.55e-5	3.91	4.74e-4	2.97
	0.0571	338520	49421	4.90e-3	2.00	4.67e-5	3.93	3.00e-4	2.98
	0.0500	442080	64481	3.75e-3	2.00	2.76e-5	3.94	2.01e-4	2.98
	0.2000	45760	5361	5.51e-3	$--$	2.31e-4	$--$	7.37e-4	$--$
	0.1333	102840	11941	1.62e-3	3.01	3.23e-5	4.85	1.54e-4	3.86
	0.1000	182720	21121	6.83e-4	3.01	7.86e-6	4.91	4.97e-5	3.93
3	0.0800	285400	32901	3.49e-4	3.00	2.61e-6	4.94	2.05e-5	3.96
	0.0667	410880	47281	2.02e-4	3.00	1.06e-6	4.96	9.97e-6	3.97
	0.0571	559160	64261	1.27e-4	3.00	4.91e-7	4.97	5.40e-6	3.98
	0.0500	730240	83841	8.52e-5	3.00	2.53e-7	4.97	3.17e-6	3.98
	0.2000	68200	6601	3.96e-4	$--$	8.83e-6	$--$	4.03e-5	$--$
	0.1333	153300	14701	7.75e-5	4.02	8.24e-7	5.85	5.61e-6	4.86
	0.1000	272400	26001	2.44e-5	4.01	1.51e-7	5.91	1.36e-6	4.93
4	0.0800	425500	40501	1.00e-5	4.01	4.01e-8	5.93	4.50e-7	4.95
	0.0667	612600	58201	4.82e-6	4.00	1.36e-8	5.93	1.82e-7	4.97
	0.0571	833700	79101	2.60e-6	4.00	5.42e-9	5.96	8.47e-8	4.97
	0.0500	1088800	103201	1.52e-6	4.00	2.45e-9	5.96	4.36e-8	4.97

Table 6.2: History of convergence for Example 1 (Part 2).

k	h	N_{total}	N_{comp}	$\|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}$ error	order	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$ error	order	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\Sigma_h}$ error	order
1	0.2000	14080	2881	5.46e-1	−−	5.69e-1	−−	1.04e+1	−−
	0.1333	31620	6421	2.45e-1	1.97	2.57e-1	1.96	7.00e-0	0.98
	0.1000	56160	11361	1.39e-1	1.98	1.45e-1	1.98	5.27e-0	0.99
	0.0800	87700	17701	8.90e-2	1.99	9.33e-2	1.99	4.22e-0	0.99
	0.0667	126240	25441	6.19e-2	1.99	6.49e-2	1.99	3.52e-0	0.99
	0.0571	171780	34581	4.56e-2	1.99	4.77e-2	1.99	3.02e-0	1.00
	0.0500	224320	45121	3.49e-2	1.99	3.66e-2	1.99	2.64e-0	1.00
2	0.2000	27720	4121	4.68e-2	−−	3.96e-2	−−	1.31e-0	−−
	0.1333	62280	9181	1.41e-2	2.97	1.19e-2	2.96	5.89e-1	1.97
	0.1000	110640	16241	5.96e-3	2.98	5.04e-3	2.98	3.29e-1	2.02
	0.0800	172800	25301	3.08e-3	2.97	2.61e-3	2.95	2.14e-1	1.93
	0.0667	248760	36361	1.79e-3	2.98	1.52e-3	2.98	1.49e-1	1.99
	0.0571	338520	49421	1.13e-3	2.98	9.57e-4	2.99	1.09e-1	1.99
	0.0500	442080	64481	7.57e-4	2.99	6.42e-4	2.99	8.38e-2	2.00
3	0.2000	45760	5361	3.52e-3	−−	3.19e-3	−−	1.47e-1	−−
	0.1333	102840	11941	7.18e-4	3.92	6.84e-4	3.80	4.87e-2	2.72
	0.1000	182720	21121	2.36e-4	3.87	2.26e-4	3.85	2.28e-2	2.64
	0.0800	285400	32901	9.64e-5	4.01	9.26e-5	4.00	1.10e-2	3.28
	0.0667	410880	47281	4.69e-5	3.95	4.50e-5	3.96	6.41e-3	2.96
	0.0571	559160	64261	2.55e-5	3.96	2.44e-5	3.97	4.06e-3	2.97
	0.0500	730240	83841	1.50e-5	3.95	1.44e-5	3.97	2.74e-3	2.95
4	0.2000	68200	6601	4.03e-4	−−	5.40e-4	−−	3.70e-2	−−
	0.1333	153300	14701	6.70e-5	4.42	8.21e-5	4.65	7.71e-3	3.87
	0.1000	272400	26001	1.62e-5	4.95	1.89e-5	5.11	2.03e-3	4.64
	0.0800	425500	40501	5.87e-6	4.54	7.10e-6	4.38	9.96e-4	3.19
	0.0667	612600	58201	2.41e-6	4.88	2.91e-6	4.89	5.46e-4	3.30
	0.0571	833700	79101	1.13e-6	4.91	1.37e-6	4.92	2.98e-4	3.92
	0.0500	1088800	103201	5.81e-7	5.00	7.03e-7	4.98	1.72e-4	4.13

Table 6.3: History of convergence for Example 2 (Part 1).

k	h	N_{total}	N_{comp}	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ error	order	$\|\Pi_{\mathcal{E}_h}(\mathbf{u})-\widehat{\mathbf{u}}_h\|_h$ error	order	$\|p-p_h\|_{0,\Omega}$ error	order
1	0.2000	14080	2881	4.75e-1	−−	5.28e-2	−−	3.25e-1	−−
	0.1333	31620	6421	3.17e-1	0.99	2.21e-2	2.15	1.47e-1	1.96
	0.1000	56160	11361	2.38e-1	1.00	1.21e-2	2.08	8.34e-2	1.98
	0.0800	87700	17701	1.91e-1	1.00	7.66e-3	2.07	5.35e-2	1.99
	0.0667	126240	25441	1.59e-1	1.00	5.28e-3	2.04	3.72e-2	1.99
	0.0571	171780	34581	1.36e-1	1.00	3.86e-3	2.03	2.74e-2	1.99
	0.0500	224320	45121	1.19e-1	1.00	2.95e-3	2.02	2.10e-2	1.99
2	0.2000	27720	4121	5.91e-2	−−	2.43e-3	−−	1.86e-2	−−
	0.1333	62280	9181	2.62e-2	2.00	5.24e-4	3.78	5.58e-3	2.97
	0.1000	110640	16241	1.47e-2	2.00	1.71e-4	3.90	2.37e-3	2.98
	0.0800	172800	25301	9.44e-3	2.00	7.35e-5	3.77	1.22e-3	2.96
	0.0667	248760	36361	6.55e-3	2.00	3.61e-5	3.90	7.10e-4	2.98
	0.0571	338520	49421	4.81e-3	2.00	1.97e-5	3.92	4.48e-4	2.99
	0.0500	442080	64481	3.69e-3	2.00	1.17e-5	3.93	3.01e-4	2.99
3	0.2000	45760	5361	5.24e-3	−−	1.27e-4	−−	1.32e-3	−−
	0.1333	102840	11941	1.54e-3	3.01	2.01e-5	4.55	2.88e-4	3.76
	0.1000	182720	21121	6.50e-4	3.01	5.88e-6	4.27	9.41e-5	3.89
	0.0800	285400	32901	3.32e-4	3.00	1.74e-6	5.46	3.87e-5	3.98
	0.0667	410880	47281	1.92e-4	3.00	7.18e-7	4.85	1.87e-5	3.98
	0.0571	559160	64261	1.21e-4	3.00	3.39e-7	4.87	1.01e-5	4.00
	0.0500	730240	83841	8.10e-5	3.00	1.78e-7	4.84	5.92e-6	4.01
4	0.2000	68200	6601	3.65e-4	−−	2.01e-5	−−	2.02e-4	−−
	0.1333	153300	14701	7.15e-5	4.02	1.99e-6	5.70	3.02e-5	4.68
	0.1000	272400	26001	2.25e-5	4.01	3.03e-7	6.55	6.88e-6	5.14
	0.0800	425500	40501	9.22e-6	4.01	1.10e-7	4.54	2.62e-6	4.34
	0.0667	612600	58201	4.44e-6	4.01	3.80e-8	5.84	1.07e-6	4.88
	0.0571	833700	79101	2.40e-6	4.00	1.53e-8	5.89	5.05e-7	4.90
	0.0500	1088800	103201	1.40e-6	4.00	6.72e-9	6.17	2.61e-7	4.95

Table 6.4: History of convergence for Example 2 (Part 2).

k	h	N_{total}	N_{comp}	$\Vert\mathbf{t}-\mathbf{t}_h\Vert_{0,\Omega}$ error	order	$\Vert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\Vert_{0,\Omega}$ error	order	$\Vert\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\Vert_{\Sigma_h}$ error	order
1	0.1667	15216	3121	8.56e-2	−−	9.98e-2	−−	7.65e-0	−−
	0.1111	34164	6949	6.60e-2	0.64	7.05e-2	0.86	8.62e-0	-0.30
	0.0833	60672	12289	5.48e-2	0.65	5.52e-2	0.85	9.39e-0	-0.30
	0.0667	94740	19141	4.74e-2	0.65	4.57e-2	0.84	1.00e+1	-0.30
	0.0556	136368	27505	4.21e-2	0.65	3.93e-2	0.83	1.06e+1	-0.30
	0.0455	203632	41009	3.69e-2	0.65	3.34e-2	0.81	1.13e+1	-0.30
	0.0400	262900	52901	3.39e-2	0.65	3.02e-2	0.80	1.17e+1	-0.30
2	0.1667	29952	4465	6.10e-2	−−	5.33e-2	−−	6.84e-0	−−
	0.1111	67284	9937	4.67e-2	0.66	3.92e-2	0.76	7.72e-0	-0.30
	0.0833	119520	17569	3.87e-2	0.66	3.16e-2	0.74	8.41e-0	-0.30
	0.0667	186660	27361	3.34e-2	0.66	2.69e-2	0.73	8.99e-0	-0.30
	0.0556	268704	39313	2.96e-2	0.66	2.36e-2	0.72	9.50e-0	-0.30
	0.0455	401280	58609	2.60e-2	0.66	2.04e-2	0.72	1.01e+1	-0.30
	0.0400	518100	75601	2.39e-2	0.66	1.86e-2	0.72	1.05e+1	-0.30
3	0.1667	49440	5809	4.48e-2	−−	3.65e-2	−−	5.99e-0	−−
	0.1111	111096	12925	3.43e-2	0.66	2.73e-2	0.72	6.76e-0	-0.30
	0.0833	197376	22849	2.84e-2	0.66	2.22e-2	0.71	7.37e-0	-0.30
	0.0667	308280	35581	2.45e-2	0.66	1.90e-2	0.71	7.89e-0	-0.30
	0.0556	443808	51121	2.17e-2	0.66	1.67e-2	0.71	8.33e-0	-0.30
	0.0455	662816	76209	1.91e-2	0.66	1.45e-2	0.70	8.85e-0	-0.30
	0.0400	855800	98301	1.75e-2	0.66	1.32e-2	0.70	9.20e-0	-0.30
4	0.1667	73680	7153	3.40e-2	−−	2.70e-2	−−	5.13e-0	−−
	0.1111	165600	15913	2.60e-2	0.66	2.03e-2	0.71	5.79e-0	-0.30
	0.0833	294240	28129	2.15e-2	0.66	1.66e-2	0.70	6.31e-0	-0.30
	0.0667	459600	43801	1.86e-2	0.66	1.42e-2	0.70	6.75e-0	-0.30
	0.0556	661680	62929	1.65e-2	0.66	1.25e-2	0.70	7.13e-0	-0.30
	0.0455	988240	93809	1.45e-2	0.66	1.08e-2	0.70	7.58e-0	-0.30
	0.0400	1276000	121001	1.33e-2	0.66	9.89e-3	0.70	7.88e-0	-0.30

Table 6.5: History of convergence for Example 3 (Part 1).

k	h	N_{total}	N_{comp}	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ error	order	$\|\Pi_{\mathcal{E}_h}(\mathbf{u})-\widehat{\mathbf{u}}_h\|_h$ error	order	$\|p-p_h\|_{0,\Omega}$ error	order
1	0.1667	15216	3121	6.75e-2	$--$	1.04e-2	$--$	5.65e-2	$--$
	0.1111	34164	6949	4.51e-2	0.99	7.18e-3	0.92	3.81e-2	0.97
	0.0833	60672	12289	3.39e-2	1.00	5.44e-3	0.97	2.88e-2	0.98
	0.0667	94740	19141	2.71e-2	1.00	4.35e-3	1.00	2.32e-2	0.97
	0.0556	136368	27505	2.26e-2	1.00	3.61e-3	1.03	1.95e-2	0.96
	0.0455	203632	41009	1.85e-2	1.00	2.92e-3	1.05	1.61e-2	0.95
	0.0400	262900	52901	1.63e-2	1.00	2.55e-3	1.07	1.43e-2	0.93
2	0.1667	29952	4465	2.51e-3	$--$	5.48e-3	$--$	2.23e-2	$--$
	0.1111	67284	9937	1.36e-3	1.51	3.10e-3	1.41	1.57e-2	0.88
	0.0833	119520	17569	8.84e-4	1.50	2.07e-3	1.40	1.23e-2	0.83
	0.0667	186660	27361	6.35e-4	1.49	1.52e-3	1.38	1.03e-2	0.80
	0.0556	268704	39313	4.85e-4	1.47	1.19e-3	1.37	8.96e-3	0.78
	0.0455	401280	58609	3.62e-4	1.46	9.03e-4	1.36	7.68e-3	0.77
	0.0400	518100	75601	3.00e-4	1.45	7.60e-4	1.35	6.97e-3	0.76
3	0.1667	49440	5809	7.85e-4	$--$	2.36e-3	$--$	1.30e-2	$--$
	0.1111	111096	12925	4.25e-4	1.52	1.29e-3	1.50	9.57e-3	0.76
	0.0833	197376	22849	2.77e-4	1.48	8.45e-4	1.47	7.73e-3	0.74
	0.0667	308280	35581	2.00e-4	1.46	6.12e-4	1.45	6.57e-3	0.73
	0.0556	443808	51121	1.54e-4	1.44	4.71e-4	1.43	5.75e-3	0.73
	0.0455	662816	76209	1.16e-4	1.42	3.55e-4	1.41	4.98e-3	0.72
	0.0400	855800	98301	9.67e-5	1.41	2.97e-4	1.40	4.54e-3	0.72
4	0.1667	73680	7153	3.52e-4	$--$	1.29e-3	$--$	8.93e-3	$--$
	0.1111	165600	15913	1.89e-4	1.53	6.86e-4	1.56	6.66e-3	0.72
	0.0833	294240	28129	1.23e-4	1.50	4.41e-4	1.53	5.42e-3	0.72
	0.0667	459600	43801	8.86e-5	1.47	3.14e-4	1.52	4.62e-3	0.71
	0.0556	661680	62929	6.79e-5	1.46	2.39e-4	1.50	4.06e-3	0.71
	0.0455	988240	93809	5.09e-5	1.44	1.78e-4	1.48	3.52e-3	0.71
	0.0400	1276000	121001	4.24e-5	1.42	1.47e-4	1.47	3.22e-3	0.71

Table 6.6: History of convergence for Example 3 (Part 2).

k	h	N_{total}	N_{comp}	$\|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}$ error	order	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$ error	order	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\Sigma_h}$ error	order
1	0.3464	74100	15601	4.05e-1	−−	4.00e-1	−−	7.19e-0	−−
	0.2474	202272	41749	2.09e-1	1.97	2.11e-1	1.90	5.18e-0	0.98
	0.1925	428652	87481	1.27e-1	1.98	1.30e-1	1.92	4.04e-0	0.98
	0.1732	587400	119401	1.03e-1	2.01	1.06e-1	1.98	3.63e-0	1.01
	0.1332	1287780	259585	6.11e-2	1.98	6.34e-2	1.95	2.81e-0	0.98
	0.1083	2397696	480769	4.05e-2	1.99	4.20e-2	1.98	2.29e-0	0.99
	0.0962	3411720	682345	3.20e-2	2.00	3.32e-2	1.99	2.03e-0	1.00
	0.0912	4011432	801421	2.87e-2	1.99	2.99e-2	1.98	1.92e-0	0.99
2	0.3464	181200	30451	4.14e-2	−−	3.63e-2	−−	1.15e-0	−−
	0.2474	495096	81439	1.58e-2	2.86	1.38e-2	2.87	6.15e-1	1.86
	0.1925	1049760	170587	7.59e-3	2.93	6.64e-3	2.92	3.76e-1	1.96
	0.1732	1438800	232801	5.74e-3	2.65	5.17e-3	2.38	3.26e-1	1.34
	0.1332	3155568	505987	2.59e-3	3.03	2.29e-3	3.10	1.85e-1	2.16
	0.1083	5876736	936961	1.40e-3	2.97	1.22e-3	3.01	1.21e-1	2.07
	0.0962	8363088	1329697	1.00e-3	2.83	9.08e-4	2.54	1.01e-1	1.51
	0.0912	9833640	1561687	8.47e-4	3.09	7.57e-4	3.36	8.86e-2	2.40

Table 6.7: History of convergence for Example 4 (Part 1).

k	h	N_{total}	N_{comp}	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ error	order	$\|\Pi_{\mathcal{E}_h}(\mathbf{u})-\widehat{\mathbf{u}}_h\|_h$ error	order	$\|p-p_h\|_{0,\Omega}$ error	order
1	0.3464	74100	15601	2.63e-1	−−	1.49e-1	−−	1.67e-1	−−
	0.2474	202272	41749	1.89e-1	0.98	7.65e-2	1.99	9.02e-2	1.84
	0.1925	428652	87481	1.47e-1	0.99	4.63e-2	1.99	5.63e-2	1.88
	0.1732	587400	119401	1.33e-1	0.99	3.75e-2	2.01	4.58e-2	1.95
	0.1332	1287780	259585	1.02e-1	1.00	2.22e-2	1.99	2.75e-2	1.94
	0.1083	2397696	480769	8.31e-2	1.00	1.47e-2	2.00	1.83e-2	1.97
	0.0962	3411720	682345	7.39e-2	1.00	1.16e-2	2.00	1.45e-2	1.98
	0.0912	4011432	801421	7.00e-2	1.00	1.04e-2	2.00	1.30e-2	1.98
	0.3464	181200	30451	3.96e-2	−−	5.23e-3	−−	1.17e-2	−−
2	0.2474	495096	81439	2.04e-2	1.97	1.52e-3	3.67	4.11e-3	3.12
	0.1925	1049760	170587	1.24e-2	1.99	5.81e-4	3.83	1.91e-3	3.04
	0.1732	1438800	232801	1.00e-2	1.99	4.19e-4	3.10	1.45e-3	2.64
	0.1332	3155568	505987	5.94e-3	1.99	1.43e-4	4.11	6.29e-4	3.18
	0.1083	5876736	936961	3.92e-3	2.00	6.22e-5	4.00	3.31e-4	3.09
	0.0962	8363088	1329697	3.10e-3	2.00	4.16e-5	3.40	2.42e-4	2.64
	0.0912	9833640	1561687	2.78e-3	2.00	3.28e-5	4.40	2.03e-4	3.31

Table 6.8: History of convergence for Example 4 (Part 2).

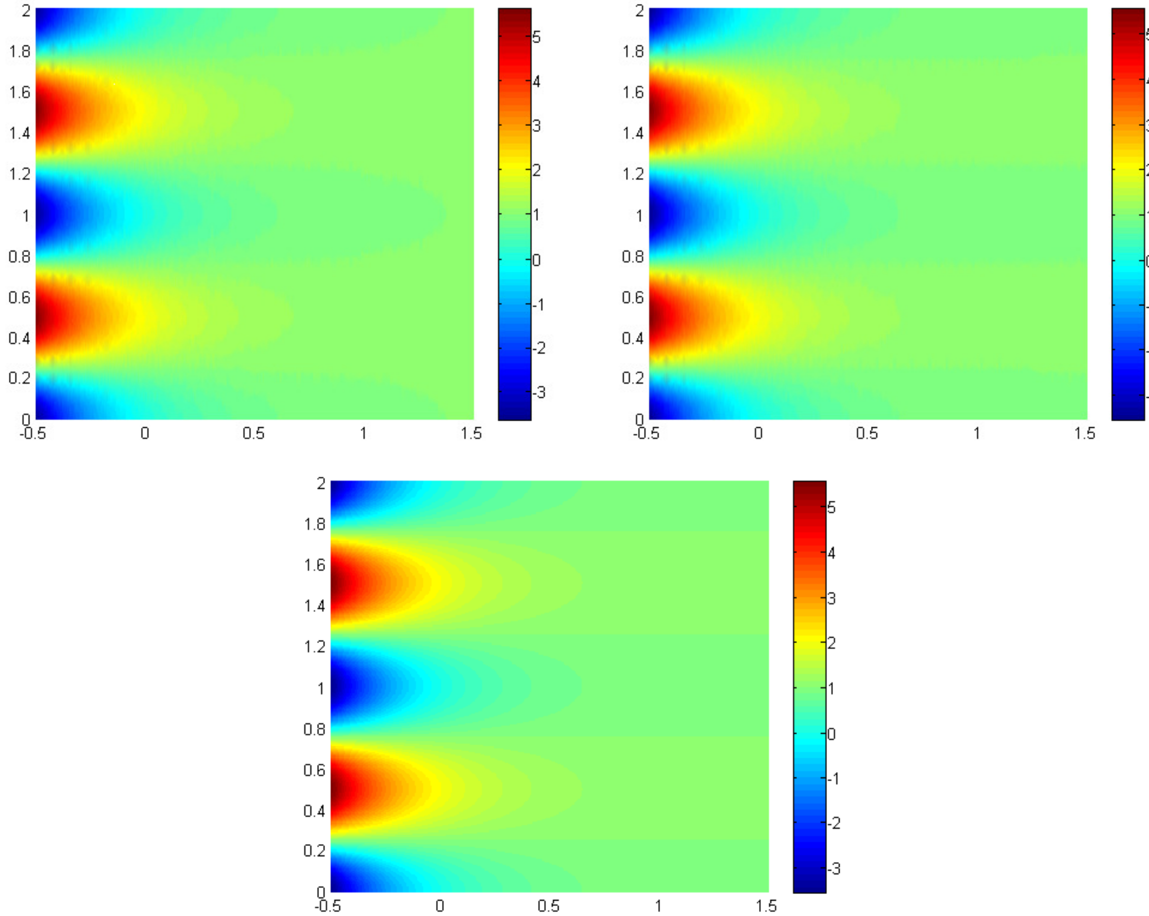

Figure 6.1: Example 2, $u_{h,1}$ for $k = 2$ (top-left), for $k = 3$ (top-right), and its exact value (bottom).

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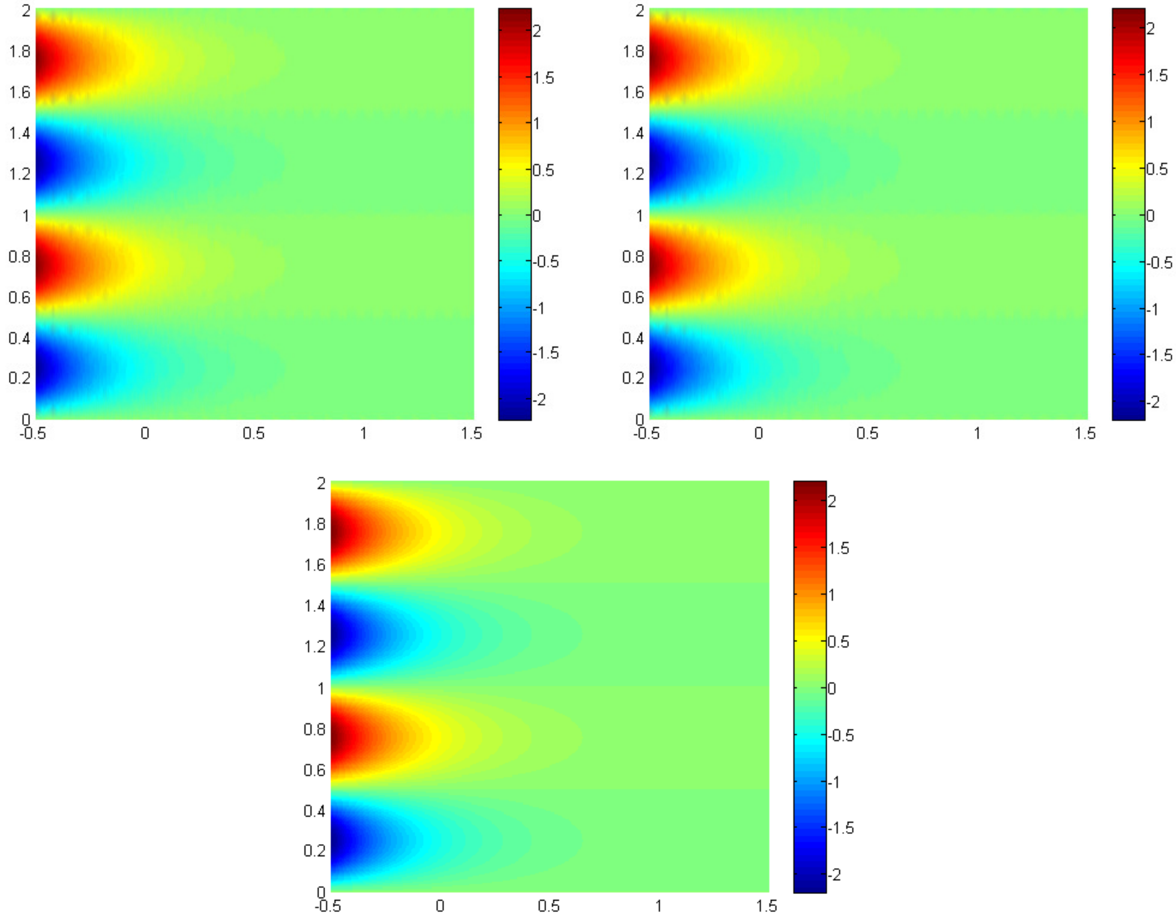

Figure 6.2: Example 2, $u_{h,2}$ for $k = 2$ (top-left), for $k = 3$ (top-right), and its exact value (bottom).

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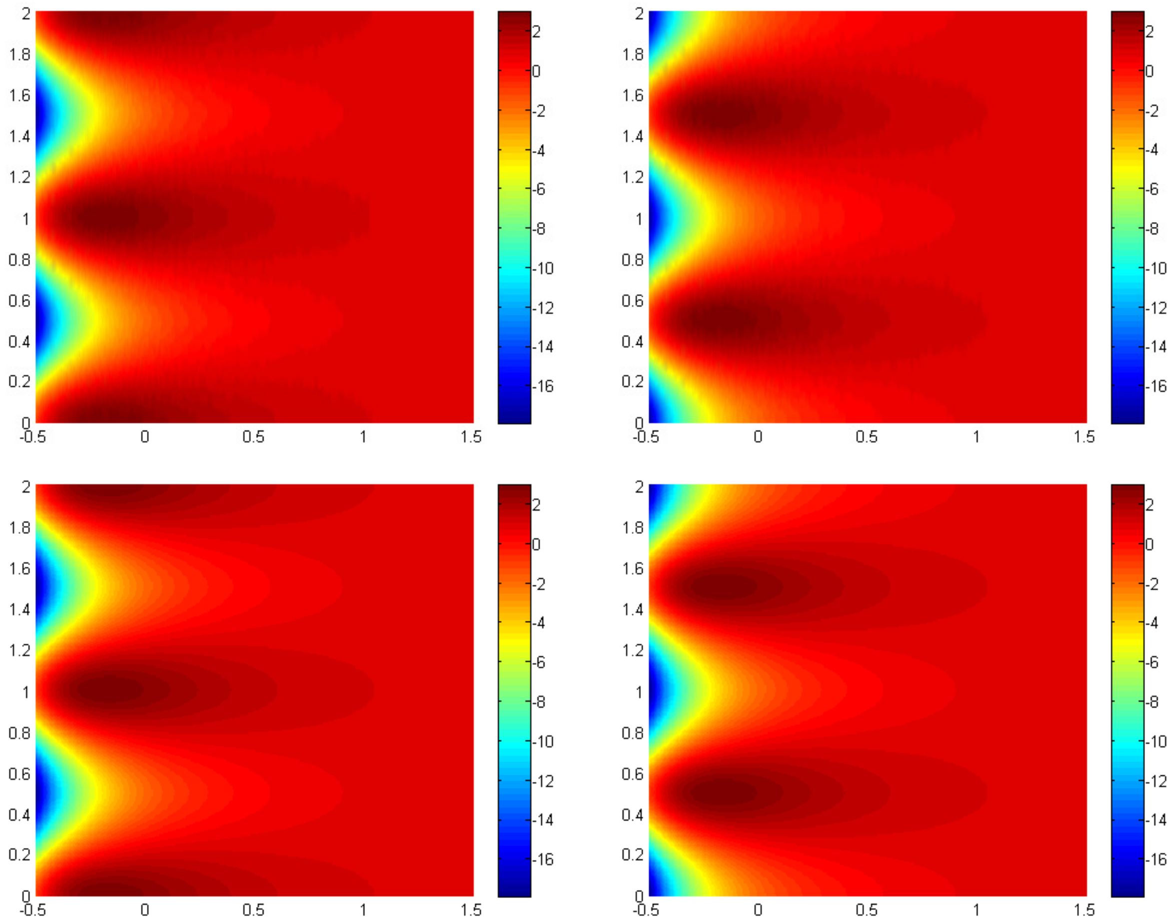

Figure 6.3: Example 2, $\sigma_{h,11}$ (top-left) $\sigma_{h,22}$ (top-right) for $k = 2$, and its exact values (bottom).

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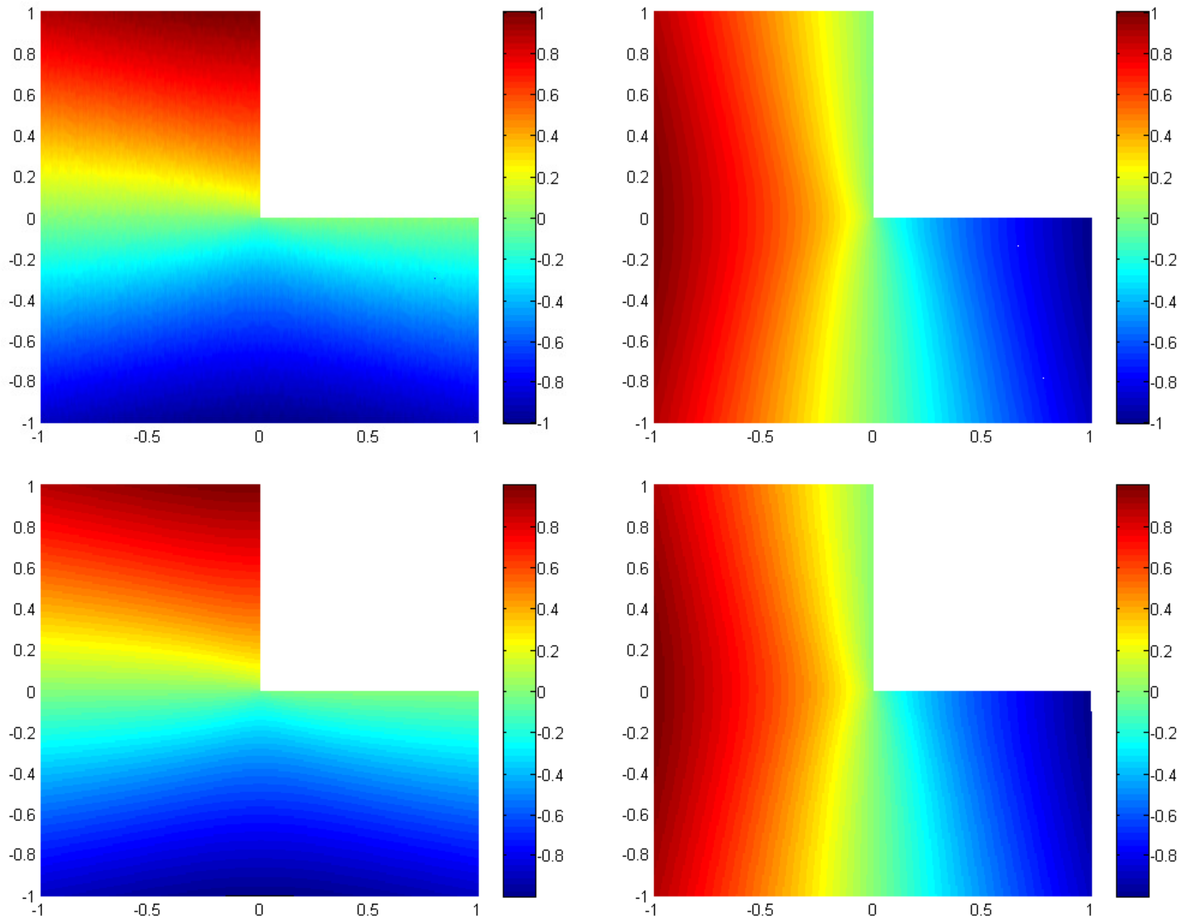

Figure 6.4: Example 3, $u_{h,1}$ (top-left) and $u_{h,2}$ (top-right) for $k = 2$, and its exact values (bottom).

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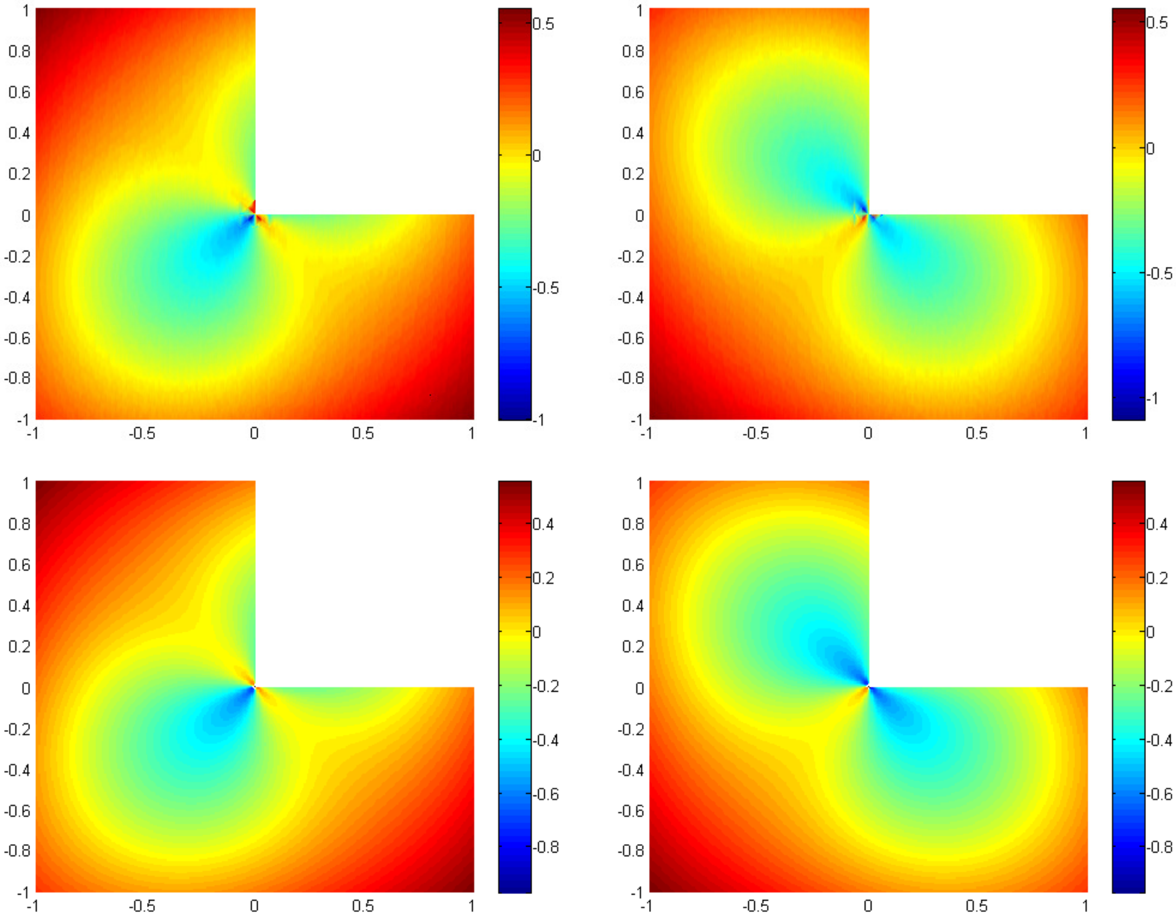

Figure 6.5: Example 3, $\sigma_{h,11}$ (top-left) and $\sigma_{h,22}$ (top-right) for $k = 2$, and its exact values (bottom).

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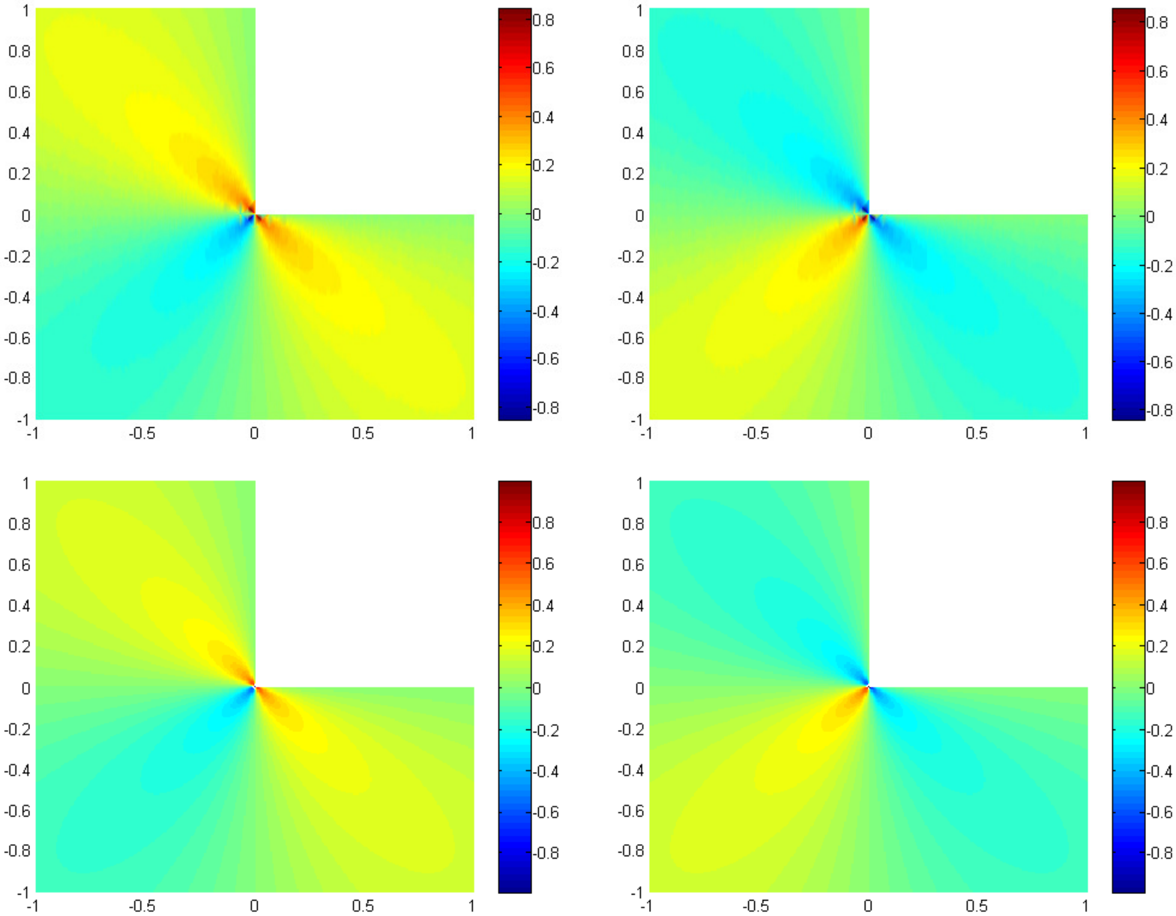

Figure 6.6: Example 3, $t_{h,11}$ (top-left) and $t_{h,22}$ (top-right) for $k = 2$, and its exact values (bottom).

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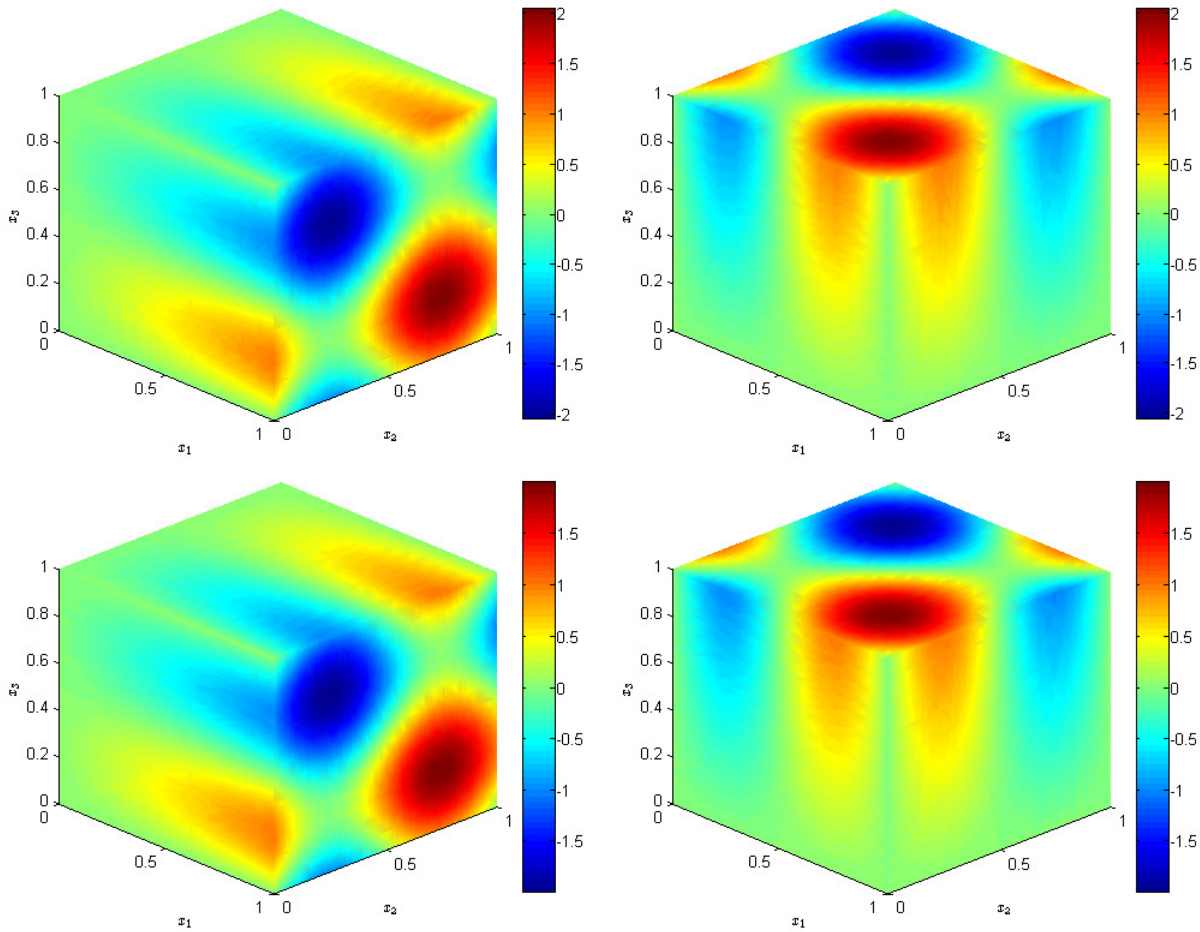

Figure 6.7: Example 4, $u_{h,1}$ (top-left) and $u_{h,3}$ (top-right) for $k = 2$, and its exact values (bottom).

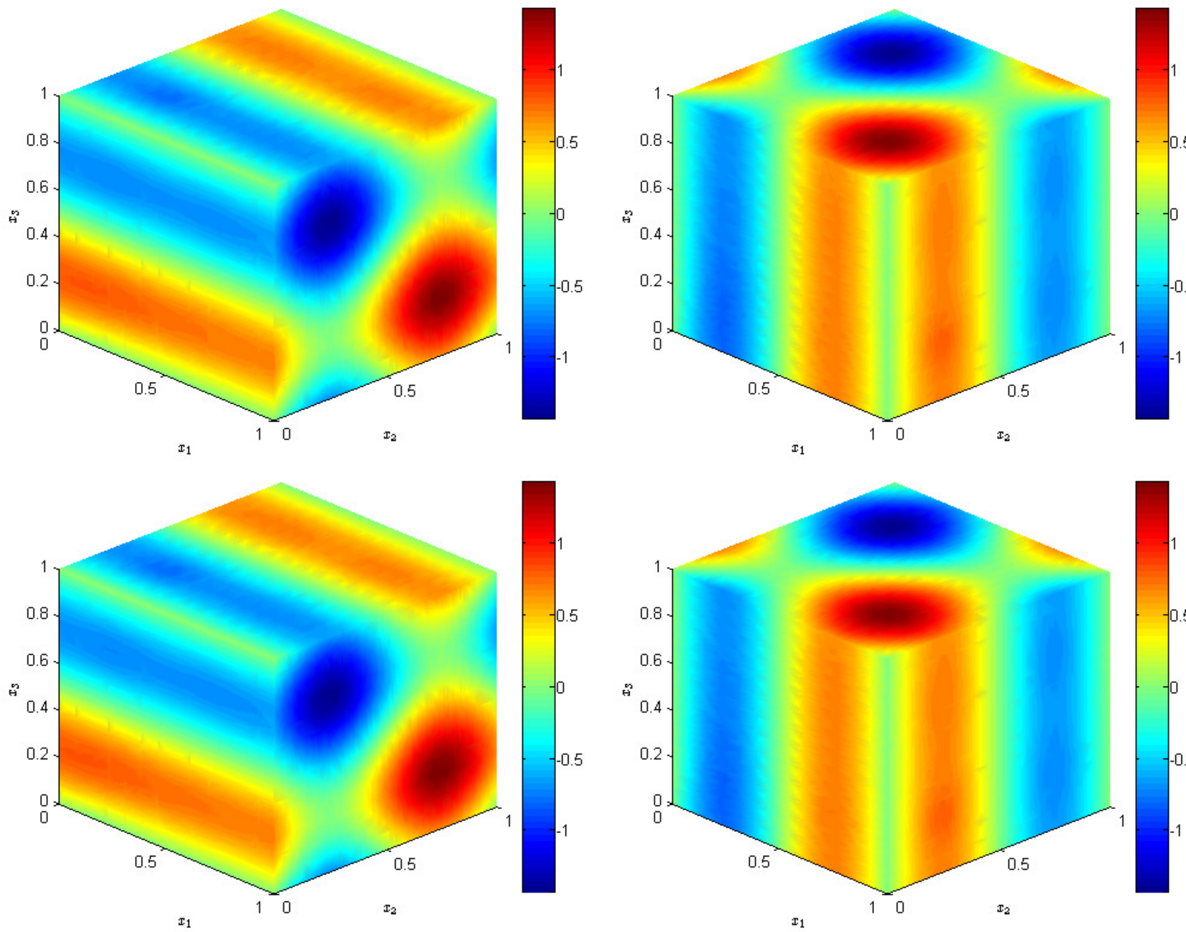

Figure 6.8: Example 4, $\sigma_{h,11}$ (top-left) and $\sigma_{h,33}$ (top-right) for $k = 2$, and its exact values (bottom).

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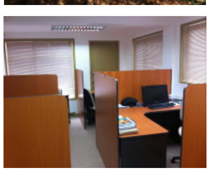

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