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A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem*

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Abstract

In this paper we introduce and analyze a virtual element method (VEM) for a mixed variational formulation of the Stokes problem in which the pseudostress and the velocity are the only unknowns, whereas the pressure is computed via a postprocessing formula. We first recall the corresponding continuous variational formulation, and then, following the basic principles for mixed-VEM, define the virtual finite element subspaces to be employed, introduce the associated interpolation operators, and provide the respective approximation properties. In particular, the latter includes the estimation of the interpolation error for the pseudostress variable measured in the $\mathbb{H}(\mathbf{div})$-norm. We remark that a Bramble-Hilbert type theorem for averaged Taylor polynomials plays a key role in the respective analysis. Next, and in order to define calculable discrete bilinear forms, we propose a new local projector onto a suitable space of polynomials, which takes into account the main features of the continuous solution and allows the explicit integration of the terms involving the deviatoric tensors. The uniform boundedness of the resulting family of local projectors is established and, using the aforementioned compactness theorem, its approximation properties are also derived. In addition, we show that the global discrete bilinear forms satisfy all the hypotheses required by the Babuška-Brezzi theory. In this way, we conclude the well-posedness of the actual Galerkin scheme and derive the associated a priori error estimates for the virtual solution as well as for the fully computable projection of it. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical rates of convergence are presented.

Key words: Stokes equations, virtual element method, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 76D07

1 Introduction

The virtual element method (VEM), which arised as a natural consequence of new developments and interpretations of the mimetic finite difference method (MFDM) (see, e.g. [9]), was first introduced and analyzed in [4] by employing the Poisson problem as a model. The VEM approach can be viewed as an extension of the classical finite element technique to general polygonal and polyhedral meshes, and also as a generalization of the MFDM to arbitrary degrees of accuracy and arbitrary continuity properties. Its basic idea consists of the utilization of one or more virtual discrete spaces defined on

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meshes made of polygonal or polyhedral elements, and the incorporation of approximated bilinear forms that mimic the original ones and that provide still consistence and stability of the resulting discrete scheme. The concept *virtual* when referring to a discrete space means that the corresponding basis functions do not need to be known explicitly in order to implement the method, but only the degrees of freedom defining them uniquely on each element are required. As remarked in [6], the main advantages of VEM, when compared with finite volume methods, MMFD, and related techniques, are given by its solid mathematical ground, the simplicity of the respective coding, and the quality of the numerical results provided. In addition, the computational domain can be decomposed into nonoverlapping convex or nonconvex polygonal elements that can be of very general shape. Further developments of VEM can be found in [5], [12], [1], and [7]. In particular, VEM is utilized in [5] to solve two-dimensional linear elasticity problems, including the compressible and nearly incompressible cases. Moreover, the related application to the linear plate bending problem, in the Kirchhoff-Love formulation, is given in [12]. Also, the eventual incorporation of further global regularity into the discrete solution is investigated in [7]. The main motivation here is the derivation of highly regular methods that could lead to less complicated discretizations of higher-order problems, and also to more direct computations of other variables of physical interest, such as stresses, rotations, and vorticities. Other recent contributions include [6], [23], [33], and [10], which refer, respectively, to practical aspects for the computational implementation of VEM, the application of VEM to three-dimensional linear elasticity problems, the numerical analysis of the two-dimensional Steklov eigenvalue problem by using VEM, and the extension of VEM to the discretization of H(div)-conforming vector fields. Up to the authors' knowledge, [10] is the only work available in the literature that deals with mixed virtual element methods.

According to the above, in the present paper we are interested in continuing the research line drawn by [10], and aim to develop a mixed-VEM for the Stokes problem. More precisely, following previous related contributions on mixed finite element methods in fluid mechanics, we consider here the pseudostress-velocity formulation introduced first in [16], and furtherly developed, among others, in [28] and [29]. Indeed, the derivation of pseudostress-based mixed finite element methods for problems in continuum mechanics has become a very active research area lately, mainly due to the need of finding new ways of circumventing the symmetry requirement of the usual stress-based approach. While the weak imposition of this condition was suggested long before (see, e.g. [2]), the use of the pseudostress has become very popular in recent years, specially in the context of least-squares and augmented methods for incompressible flows, precisely because of the non-necessity of the symmetry condition. As a consequence, two new approaches appeared: the pseudostress-velocity-pressure and pseudostress-velocity formulations (see, e.g. [14], [15], [22], and the references therein). In particular, augmented mixed finite element methods for both pseudostress-based formulations of the stationary Stokes equations are studied in [22]. In addition, the pseudostress-velocity-pressure formulation has also been applied to nonlinear Stokes problems (see, e.g. [21], [27], [32]). Furthermore, the formulation from [16] is modified in [28] by incorporating the pressure into the discrete analysis, thus allowing further flexibility for approximating this unknown. More precisely, it is established there that the corresponding Galerkin scheme only makes sense for pressure finite element subspaces not containing the traces of the pseudostresses subspace. In particular, this is the case when Raviart-Thomas elements of index $k \geq 0$ for the pseudostress, and piecewise discontinuous polynomials of degree k for the velocity and the pressure, are utilized. On the other hand, for recent applications of the pseudostress-based approach in fluid mechanics we refer for instance to [25] and [26], where dual-mixed methods for the linear and nonlinear versions of the two-dimensional Brinkman problem are studied. Actually, the pseudostress is the main unknown of the resulting saddle point problems in [25] and [26], and the velocity and pressure are easily recovered through simple postprocessing formulae. In addition, as it is usual for dual-mixed methods, the Dirichlet boundary condition for the velocity becomes natural in this case, and the Neumann boundary condition, being essential, is imposed weakly through the introduction of the trace of the velocity on that boundary as the associated Lagrange multiplier. Additional contributions on this and related topics include [17], [18], [19], [30], and [34].

The rest of this work is organized as follows. In Section 2 we introduce the boundary value problem of interest, and recall from [28] its pseudostress-velocity mixed formulation and the associated well-posedness result. Then, in Section 3 we follow [10] to introduce the virtual element subspaces that will be employed, and then show the respective unisolvency, define the associated interpolation operators, and provide their approximation properties. Though the proofs of these results are sketched in [10] (see also [8]), for sake of clearness and completeness, in the present paper we try to give as much details as possible in some of them. In particular, a Bramble-Hilbert type theorem for averaged Taylor polynomials (cf. [8], [20]) plays a key role in our analysis. Next, fully calculable discrete bilinear forms are introduced in Section 4 and their boundedness and related properties are established. To this end, a new local projector onto a suitable space of polynomials is proposed here. This operator is somehow suggested by the main features of the continuous solution of the Stokes problem, and it also responds to the need of explicitly integrating the terms of the bilinear form that involves deviatoric tensors. The family of local projectors is shown to be uniformly bounded, and the aforementioned compactness theorem is applied to derive its approximation properties. The actual mixed virtual element method is then introduced and analyzed in Section 5. The classical discrete Babuška-Brezzi theory is applied to deduce the well-posedness of this scheme, and then suitable bounds and identities satisfied by the bilinear forms and the projectors and interpolators involved, allow to derive the a priori error estimates and corresponding rates of convergence for the virtual solution as well as for the projection of it. Finally, several numerical examples showing the good performance of the method, confirming the rates of convergence for regular and singular solutions, and illustrating the accurateness obtained with the approximate solutions, are reported in Section 6 .

We end this section with some notations to be used below. In what follows, $\mathbb{I}$ is the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})\,, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}\,, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\,, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}\,,$$

which corresponds, respectively, to the transpose, the trace, and the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. In addition, we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O} \subset \mathrm{R}^2$ is a domain, $\mathcal{S} \subset \mathrm{R}^2$ is an open or closed Lipschitz curve, and $r \in \mathrm{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2\,, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}\,, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2\,.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding inner products, norms, and semi-norms are denoted, respectively, by $\langle\cdot,\cdot\rangle_{r,\mathcal{O}}$, $\|\cdot\|_{r,\mathcal{O}}$, and $|\cdot|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$), and $\langle\cdot,\cdot\rangle_{r,\mathcal{S}}$, $\|\cdot\|_{r,\mathcal{S}}$, and $|\cdot|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In turn, $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ stands for the usual duality pairing between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. However, when no confusion arises, we write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{\mathcal{S}}$. Furthermore, with div denoting the usual divergence operator, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \big\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}(\mathbf{w}) \in L^2(\mathcal{O}) \big\}\,,$$

is standard in the realm of mixed problems (see [11], [31]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$, where $\mathbf{div}$ stands for the action of div along each row of a tensor. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})$ and also $\boldsymbol{\tau}\mathbf{n} \in \mathbf{H}^{-1/2}(\partial\mathcal{O})$, where $\mathbf{n}$ is the unit outward normal at the boundary $\partial\mathcal{O}$. Finally, we employ $\mathbf{0}$ to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The Stokes problem and its mixed formulation

Let Ω be a bounded and simply connected polygonal domain in R^2 with boundary Γ. Our aim is to find the velocity $\mathbf{u}$, the pseudostress tensor $\boldsymbol{\sigma}$ and the pressure p of a steady flow occupying Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$ and a scalar field p such that

$$\begin{aligned} \boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.1}$$

where μ is the kinematic viscosity. As required by the incompressibility condition, we assume that $\mathbf{g}$ satisfies the compatibility condition $\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0$. In turn, it is easy to see, using that $\mathrm{tr}\,(\nabla\mathbf{u}) = \mathrm{div}\mathbf{u}$ in Ω, that the pair of equations given by

$$\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega \quad \text{and} \quad \mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega\,,$$

is equivalent to

$$\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega \quad \text{and} \quad p + \frac{1}{2}\,\mathrm{tr}\,(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,,$$

and therefore, instead of (2.1), from now on we consider

$$\begin{aligned} \boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ p + \frac{1}{2}\,\mathrm{tr}\,(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.2}$$

Then, proceding as in [28], in particular eliminating the pressure from the third equation of (2.2), we arrive at the following mixed variational formulation: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times Q$ such that

$$\begin{aligned} \mathbf{a}\,(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, \mathbf{b}\,(\boldsymbol{\tau}, \mathbf{u}) \;&=\; \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{g}\rangle \qquad \forall\,\boldsymbol{\tau} \in H\,, \\ \mathbf{b}\,(\boldsymbol{\sigma}, \mathbf{v}) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in Q\,, \end{aligned} \tag{2.3}$$

where

$$H \;:=\; \left\{\boldsymbol{\tau} \in \mathbb{H}\,(\mathbf{div}\,;\Omega) : \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) \;=\; 0\right\}, \qquad Q \;:=\; \mathbf{L}^2\,(\Omega)\,, \tag{2.4}$$

and H is endowed with the usual norm $\|\cdot\|_{\mathbf{div};\Omega}$ of $\mathbb{H}(\mathbf{div};\Omega)$. In turn, $\mathbf{a} : H \times H \to \mathrm{R}$ and $\mathbf{b} : H \times Q \to \mathrm{R}$ are the bounded bilinear forms defined by

$$\mathbf{a}\,(\boldsymbol{\sigma}, \boldsymbol{\tau}) \;:=\; \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\;(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in H \times H\,,$$

and

$$\mathbf{b}\,(\boldsymbol{\tau}, \mathbf{v}) \;:=\; \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\tau}, \mathbf{v}) \in H \times Q\,.$$

The unique solvability of (2.3) is established as follows.

Theorem 2.1 *There exists a unique* $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times Q$ *solution of* (2.3)*. Moreover, there exists a constant* $C > 0$*, depending only on* Ω*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_{H\times Q} \,\le\, C \Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} .$$

Proof. See [28, Theorem 2.1]. $\square$

3 The virtual element subspaces

3.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements. For each $K \in \mathcal{T}_h$ we denote its diameter by h_K, and define, as usual, $h := \max\Big\{ h_K : \quad K \in \mathcal{T}_h \Big\}$. Now, given an integer $k \geq 0$, we let $\mathrm{P}_k(K)$ be the space of polynomials on K of total degree up to k. Then, given an integer $k \geq 1$, we follow [10] and consider the following virtual element subspaces of H and Q, respectively:

$$\begin{gathered} H_h \,:=\, \Big\{ \boldsymbol{\tau} \in H : \quad \boldsymbol{\tau}\,\mathbf{n}\Big|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \in \mathcal{T}_h, \quad \mathbf{div}\,\boldsymbol{\tau}\Big|_K \in \mathbf{P}_{k-1}(K)\,, \\ \mathbf{rot}\,\boldsymbol{\tau}\Big|_K \in \mathbf{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\} , \end{gathered} \tag{3.1}$$

and

$$Q_h \,:=\, \Big\{ \mathbf{v} \in Q : \quad \mathbf{v}\Big|_K \in \mathbf{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\} , \tag{3.2}$$

where

$$\mathbf{rot}\,\boldsymbol{\tau} \,:=\, \begin{pmatrix} \dfrac{\partial \tau_{12}}{\partial x_1} - \dfrac{\partial \tau_{11}}{\partial x_2} \\ \dfrac{\partial \tau_{22}}{\partial x_1} - \dfrac{\partial \tau_{21}}{\partial x_2} \end{pmatrix} \qquad \forall\, \boldsymbol{\tau} \in H\,.$$

Then, the Galerkin scheme associated with (2.3) would read: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ such that

$$\begin{aligned} \mathbf{a}\,(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, \mathbf{b}\,(\boldsymbol{\tau}_h, \mathbf{u}_h) \;&=\; \langle \boldsymbol{\tau}_h \mathbf{n}, \mathbf{g} \rangle \qquad && \forall\, \boldsymbol{\tau}_h \in H_h\,, \\ \mathbf{b}\,(\boldsymbol{\sigma}_h, \mathbf{v}_h) \;&=\; -\int_\Omega \mathbf{f} \cdot \mathbf{v}_h \qquad && \forall\, \mathbf{v}_h \in Q_h\,. \end{aligned} \tag{3.3}$$

Note, according to (3.1) and (3.2), that the right hand sides of (3.3) are calculable explicitly. The same fact is valid for the bilinear form $\mathbf{b}$, which will be re-emphasized at the beginning of Section 4. Nevertheless, we will observe later on that $\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)$ can not be computed explicitly when $\boldsymbol{\sigma}_h$, $\boldsymbol{\tau}_h$ belongs to H_h, and hence a suitable approximation of this bilinear form, namely $\mathbf{a}_h$, will be introduced in that section to redefine (3.3).

3.2 Unisolvency of the virtual element subspaces

In what follows we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}}\,h_K$ and center $\mathbf{x}_B \in K$, that is, for each $x_0 \in B$, all the line segments joining x_0 with any $x \in K$ are contained in K, or, equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in K.

As a consequence of the above hypotheses, one can show that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that the number of edges of each $K \in \mathcal{T}_h$ is bounded above by $N_{\mathcal{T}}$.

Next, in order to choose the degrees of freedom of H_h, given an edge $e \in \mathcal{T}_h$ with medium point x_e and length h_e, and given an integer $\ell \geq 0$, we first introduce the following set of $2(\ell+1)$ normalized monomials on e

$$\mathcal{B}_\ell(e) := \left\{ \left(\left(\frac{x - x_e}{h_e} \right)^j, 0 \right)^{\mathtt{t}} \right\}_{0 \leq j \leq \ell} \bigcup \left\{ \left(0, \left(\frac{x - x_e}{h_e} \right)^j \right)^{\mathtt{t}} \right\}_{0 \leq j \leq \ell}, \tag{3.4}$$

which certainly constitutes a basis of $\mathbf{P}_\ell(e)$. Similarly, given an element $K \in \mathcal{T}_h$ with barycenter $\mathbf{x}_K$, and given an integer $\ell \geq 0$, we define the following set of $(\ell+1)(\ell+2)$ normalized monomials

$$\mathcal{B}_\ell(K) := \left\{ \left(\left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^\alpha, 0 \right)^{\mathtt{t}} \right\}_{0 \leq |\alpha| \leq \ell} \bigcup \left\{ \left(0, \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^\alpha \right)^{\mathtt{t}} \right\}_{0 \leq |\alpha| \leq \ell}, \tag{3.5}$$

which is a basis of $\mathbf{P}_\ell(K)$. Note that (3.5) makes use of the multi-index notation where, given $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathrm{R}^2$ and $\alpha := (\alpha_1, \alpha_2)^{\mathtt{t}}$, with nonnegative integers α_1, α_2, we set $\mathbf{x}^\alpha := x_1^{\alpha_1}\, x_2^{\alpha_2}$ and $|\alpha| := \alpha_1 + \alpha_2$. According to the above and the definition of H_h (cf. (3.1)), we propose the following degrees of freedom for a given $\boldsymbol{\tau} \in H_h$:

$$\begin{array}{llll}
\text{a)} & \displaystyle\int_e \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} & \forall\, \mathbf{q} \in \mathcal{B}_k(e) & \forall \text{ edge } e \in \mathcal{T}_h\,, \\
\text{b)} & \displaystyle\int_K \boldsymbol{\tau} : \nabla \mathbf{q} & \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) \setminus \big\{ (1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}} \big\} & \forall\, K \in \mathcal{T}_h\,, \\
\text{c)} & \displaystyle\int_K \mathbf{q} \cdot \mathbf{rot}\,\boldsymbol{\tau} & \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) & \forall\, K \in \mathcal{T}_h\,.
\end{array} \tag{3.6}$$

We now observe, according to the cardinalities of $\mathcal{B}_k(e)$ and $\mathcal{B}_{k-1}(K)$, that the amount of local degrees of freedom, that is those related to a given $K \in \mathcal{T}_h$, is given by

$$n_k^K := 2(k+1)\, d_K + \big\{ k(k+1) - 2 \big\} + k(k+1) = 2 \Big\{ (k+1)\,(d_K + k) - 1 \Big\},$$

where d_K is the number of edges of K. Moreover, we have the following local unisolvence result.

Lemma 3.1 *Given an integer $k \geq 1$, we define for each $K \in \mathcal{T}_h$ the local space*

$$\begin{gathered} H_h^K := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,; K) \cap \mathbb{H}(\mathbf{rot}\,; K) : \quad \boldsymbol{\tau}\mathbf{n}\Big|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \subseteq \partial K\,, \\ \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{P}_{k-1}(K)\,, \quad \mathbf{rot}\,\boldsymbol{\tau} \in \mathbf{P}_{k-1}(K) \Big\}. \end{gathered} \tag{3.7}$$

Then, the n_k^K local degrees of freedom arising from (3.6) *are unisolvent in H_h^K.*

Proof. Let $\boldsymbol{\tau} \in H_h^K$ such that

$$\begin{aligned}
\int_e \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{q} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,,\\
\int_K \boldsymbol{\tau} : \nabla\mathbf{q} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,,\\
\int_K \mathbf{q}\cdot\mathbf{rot}\,\boldsymbol{\tau} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,.
\end{aligned} \tag{3.8}$$

It follows easily from the definition (3.7) together with the first and third equations of (3.8) that

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial K\,, \quad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad K\,. \tag{3.9}$$

In turn, integrating by parts the second equation in (3.8), we find that

$$0 = \int_K \boldsymbol{\tau} : \nabla\mathbf{q} = -\int_K \mathbf{q}\cdot\mathbf{div}\boldsymbol{\tau} + \int_{\partial K} \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{q} = -\int_K \mathbf{q}\cdot\mathbf{div}\boldsymbol{\tau} \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,,$$

which yields $\mathbf{div}\boldsymbol{\tau} = \mathbf{0}$ in K. Now, since K is simply connected, we know from the second identity in (3.9) and [31, Chapter I, Theorem 2.9] that there exists $\boldsymbol{\phi} \in \mathbf{H}^1(K)$ such that $\boldsymbol{\tau} = \nabla\boldsymbol{\phi}$ in K. In this way, the free divergence property of $\boldsymbol{\tau}$, and the fact that its normal component is the null vector on ∂K, can be rewritten as

$$\Delta\boldsymbol{\phi} = \mathbf{0} \quad \text{in} \quad K\,, \quad \nabla\boldsymbol{\phi}\,\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial K\,.$$

Thus, the classical solvability analysis of this Neumann problem implies that $\boldsymbol{\phi}$ is a constant vector, and hence $\boldsymbol{\tau}$ vanishes in K, which completes the proof. $\square$

3.3 Interpolation on H_h and Q_h

In this section we define suitable interpolation operators on our virtual element subspaces and establish their corresponding approximation properties. To this end, we need some preliminary notations and technical results. For each element $K \in \mathcal{T}_h$ we let $\widetilde{K} := T_K(K)$, where $T_K : \mathrm{R}^2 \longrightarrow \mathrm{R}^2$ is the bijective affine mapping defined by $T_K(\mathbf{x}) := \dfrac{\mathbf{x} - \mathbf{x}_B}{h_K}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$. Note that the diameter $h_{\widetilde{K}}$ of $\widetilde{K}$ is 1, and, according to the assumptions a) and b), it is easy to see that the shortest edge of $\widetilde{K}$ is bigger than $C_{\mathcal{T}}$, and that $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. Recall here that $\mathbf{x}_B$ is the center of the ball B with respect to which K is star-shaped. Then, by connecting each vertex of $\widetilde{K}$ to the center of $\widetilde{B}$, that is to the origin, we generate a partition of $\widetilde{K}$ into $d_{\widetilde{K}}$ triangles $\widetilde{\Delta}_i$, $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $d_{\widetilde{K}} \leq N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\dfrac{\widetilde{h}_i}{\widetilde{\rho}_i} \leq c_{\mathcal{T}}$ $\forall\, i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $\widetilde{h}_i$ is the diameter of $\widetilde{\Delta}_i$ and $\widetilde{\rho}_i$ is the diameter of the largest ball contained in $\widetilde{\Delta}_i$. We also let $\widehat{\Delta}$ be the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$, and for each $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$ we let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\,\mathbf{x}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \widetilde{\Delta}_i$. We remark that the fact that the origin is a vertex of each triangle $\widetilde{\Delta}_i$ allows to choose F_i as indicated.

In what follows, given $\mathbf{v} \in \mathbf{L}^2(K)$, we let $\widetilde{\mathbf{v}} := \mathbf{v} \circ T_K^{-1} \in \mathbf{L}^2(\widetilde{K})$. Then, we have the following result.

Lemma 3.2 *Given an integer $\ell \geq 0$ and an element $K \in \mathcal{T}_h$, we let $\mathcal{P}_\ell^K : \mathbf{L}^2(K) \to \mathbf{P}_\ell(K)$ and $\mathcal{P}_\ell^{\widetilde{K}} : \mathbf{L}^2(\widetilde{K}) \to \mathbf{P}_\ell(\widetilde{K})$ be the corresponding orthogonal projectors. Then $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v} \in \mathbf{L}^2(K)$, and for any pair of nonnegative integers r and s there holds $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(\mathbf{H}^r(\widetilde{K}), \mathbf{H}^s(\widetilde{K}))$, with $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))}$ independent of $\widetilde{K}$, namely depending only on ℓ, s, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$.*

Proof. Denoting $N_\ell := (\ell+1)(\ell+2)$, we let $\left\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_{N_\ell}\right\}$ be a basis of $\mathbf{P}_\ell(K)$, in particular $\mathcal{B}_\ell(K)$ (cf. (3.5)), and observe that $\left\{\widetilde{\boldsymbol{\varphi}}_1, \widetilde{\boldsymbol{\varphi}}_2, \ldots, \widetilde{\boldsymbol{\varphi}}_{N_\ell}\right\}$ becomes a basis of $\mathbf{P}_\ell(\widetilde{K})$. Hence, given $\mathbf{v} \in \mathbf{L}^2(K)$, and bearing in mind that the Jacobian of T_K is h_K^{-2}, we find that for each $j \in \{1, 2, \ldots, N_\ell\}$ there holds

$$\int_{\widetilde{K}} \widetilde{\mathcal{P}_\ell^K(\mathbf{v})} \cdot \widetilde{\boldsymbol{\varphi}}_j \,=\, h_K^{-2} \int_K \mathcal{P}_\ell^K(\mathbf{v}) \cdot \boldsymbol{\varphi}_j \,=\, h_K^{-2} \int_K \mathbf{v} \cdot \boldsymbol{\varphi}_j \,=\, \int_{\widetilde{K}} \widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\varphi}}_j \,=\, \int_{\widetilde{K}} \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}}) \cdot \widetilde{\boldsymbol{\varphi}}_j \,,$$

which shows that $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$. Throughout the rest of the proof we assume for simplicity that $\left\{\widetilde{\boldsymbol{\varphi}}_1, \widetilde{\boldsymbol{\varphi}}_2, \ldots, \widetilde{\boldsymbol{\varphi}}_{N_\ell}\right\}$ is orthonormal, which yields $\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}}) = \sum_{j=1}^{N_\ell} \langle \widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\varphi}}_j \rangle_{0,\widetilde{K}}\, \widetilde{\boldsymbol{\varphi}}_j$. Then, employing the Cauchy-Schwarz inequality, we obtain

$$\|\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})\|_{s,\widetilde{K}} \,\leq\, \left\{\sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}}\right\} \|\widetilde{\mathbf{v}}\|_{0,\widetilde{K}} \,\leq\, \left\{\sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}}\right\} \|\widetilde{\mathbf{v}}\|_{r,\widetilde{K}} \,,$$

which proves that $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(\mathbf{H}^r(\widetilde{K}), \mathbf{H}^s(\widetilde{K}))$, with

$$\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))} \,\leq\, \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}} \,=\, \sum_{j=1}^{N_\ell} \left\{\sum_{i=1}^{d_{\widetilde{K}}} \|\widetilde{\boldsymbol{\varphi}}_j\|^2_{s,\widetilde{\Delta}_i}\right\}^{1/2} \,, \tag{3.10}$$

where the last equality makes use of the aforementioned decomposition of $\widetilde{K}$. We now apply the usual scaling properties connecting the Sobolev integer seminorms in each $\widetilde{\Delta}_i$ with those in $\widehat{\Delta}$. In this way, denoting $\widehat{\boldsymbol{\varphi}}_{j,i} := \widetilde{\boldsymbol{\varphi}}_j|_{\widetilde{\Delta}_i} \circ F_i \in \mathbf{P}_\ell(\widehat{\Delta})$, using the equivalence of norms in $\mathbf{P}_\ell(\widehat{\Delta})$, and noting that $\widetilde{\rho}_i^{-1} \,\leq\, c_{\mathcal{T}}\, \widetilde{h}_i^{-1} \,\leq\, c_{\mathcal{T}}\, C_{\mathcal{T}}^{-1}$, we deduce that for each integer $t \geq 0$ there holds

$$\begin{aligned} |\widetilde{\boldsymbol{\varphi}}_j|_{t,\widetilde{\Delta}_i} \,&\leq\, C_t\, \widehat{h}^t\, \widetilde{\rho}_i^{-t}\, |\det B_i|^{1/2}\, \|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{t,\widehat{\Delta}} \,\leq\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, |\det B_i|^{1/2}\, \widehat{c}\, \|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{0,\widehat{\Delta}} \\ &=\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}\, \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{\Delta}_i} \,\leq\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}\, \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{K}} \,=\, C \,, \end{aligned}$$

where C_t depends on t, whereas $\widehat{c}$ depends on $\mathbf{P}_\ell(\widehat{\Delta})$ and t, and $C \,=\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}$. The foregoing inequality and (3.10) give the announced independence of $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))}$, which ends the proof. $\square$

The next result taken from [8, Lemma 4.3.8] (see also [20]) is required in what follows as well.

Lemma 3.3 *Let $\mathcal{O}$ be a domain of R^2 with diameter 1, such that it is star-shaped with respect to a ball B of radius $> \frac{1}{2}\, \rho_{\mathtt{max}}$, where $\rho_{\mathtt{max}} := \sup\Big\{\rho : \quad \mathcal{O}$ is star-shaped with respect to a ball of radius $\rho\Big\}$. In addition, given an integer $m \geq 1$ and $\mathbf{v} \in \mathbf{H}^m(\mathcal{O})$, we let $\mathbf{T}^m(\mathbf{v}) \in \mathbf{P}_{m-1}(\mathcal{O})$ be the Taylor polynomial of order m of $\mathbf{v}$ averaged over B. Then, there exists $C > 0$, depending only on m and $\rho_{\mathtt{max}}$, such that*

$$|\mathbf{v} - \mathbf{T}^m(\mathbf{v})|_{\ell,\mathcal{O}} \,\leq\, C\, |\mathbf{v}|_{m,\mathcal{O}} \qquad \forall\, \ell \in \{0, 1, \ldots, m\}\,.$$

We now proceed to define our interpolation operators. We begin by letting $\mathcal{P}^h_{k-1} : \mathbf{L}^2(\Omega) \longrightarrow Q_h$ be the orthogonal projector, that is, given $\mathbf{v} \in Q := \mathbf{L}^2(\Omega)$, $\mathcal{P}^h_{k-1}(\mathbf{v})$ is characterized by

$$\int_K \Big(\mathbf{v} \,-\, \mathcal{P}^h_{k-1}(\mathbf{v})\Big)\cdot \mathbf{q} \,=\, 0 \qquad \forall\, K \in \mathcal{T}_h\,, \quad \forall\, \mathbf{q} \in \mathbf{P}_{k-1}(K)\,, \tag{3.11}$$

which means, equivalently, that

$$\mathcal{P}^h_{k-1}(\mathbf{v})\big|_K \,=\, \mathcal{P}^K_{k-1}(\mathbf{v}|_K)\,,$$

where, as indicated in Lemma 3.2, $\mathcal{P}^K_{k-1} : \mathbf{L}^2(K) \to \mathbf{P}_{k-1}(K)$ is the local orthogonal projector. The following lemma establishes the approximation properties of this operator.

Lemma 3.4 *Let k, ℓ and r be integers such that $1 \le r \le k$ and $0 \le \ell \le r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds*

$$|\mathbf{v} \,-\, \mathcal{P}^K_{k-1}(\mathbf{v})|_{\ell,K} \,\le\, C\, h_K^{r-\ell}\, |\mathbf{v}|_{r,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^r(K)\,.$$

Proof. Given integers k, ℓ and r as stated, $K \in \mathcal{T}_h$, and $\mathbf{v} \in \mathbf{H}^r(K)$, we first observe that there hold

$$|\tilde{\mathbf{v}}|_{\ell,\widetilde{K}} \,=\, h_K^{\ell+1}\, |\mathbf{v}|_{\ell,K} \qquad \text{and} \qquad \mathcal{P}^{\widetilde{K}}_{k-1}\big(\widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\big) \,=\, \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\,,$$

where $\widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}}) \in \mathbf{P}_{r-1}(\widetilde{K})$ is the Taylor polynomial of order r of $\tilde{\mathbf{v}}$ averaged over a ball of radius $> \frac{1}{2}\,\widetilde{\rho}_{\mathtt{max}}$, where $\widetilde{\rho}_{\mathtt{max}} := \sup\Big\{\rho : \quad \widetilde{K}$ is star-shaped with respect to a ball of radius $\rho\Big\}$. Recall here that $\widetilde{K}$ has diameter 1 and is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. It follows, using Lemmas 3.2 and 3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
|\mathbf{v} - \mathcal{P}^K_{k-1}(\mathbf{v})|_{\ell,K} \,&=\, h_K^{-\ell-1}\, |\tilde{\mathbf{v}} - \widetilde{\mathcal{P}^K_{k-1}(\mathbf{v})}|_{\ell,\widetilde{K}} \,=\, h_K^{-\ell-1}\, |\tilde{\mathbf{v}} - \mathcal{P}^{\widetilde{K}}_{k-1}(\tilde{\mathbf{v}})|_{\ell,\widetilde{K}} \\
&=\, h_K^{-\ell-1}\, |\big(\mathrm{I} - \mathcal{P}^{\widetilde{K}}_{k-1}\big)\big(\tilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\big)|_{\ell,\widetilde{K}} \,\le\, h_K^{-\ell-1}\, \|\mathrm{I} - \mathcal{P}^{\widetilde{K}}_{k-1}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^\ell(\widetilde{K}))}\, \|\tilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\|_{r,\widetilde{K}} \\
&\le\, C\, h_K^{-\ell-1}\, |\tilde{\mathbf{v}}|_{r,\widetilde{K}} \,=\, C\, h_K^{r-\ell}\, |\mathbf{v}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. $\square$

We now let

$$\widetilde{H} \,:=\, \Big\{\boldsymbol{\tau} \in H : \quad \boldsymbol{\tau}|_K \in \mathbb{L}^s(K) \text{ (for some } s > 2) \text{ and } \mathbf{rot}\,\boldsymbol{\tau}|_K \in \mathbf{L}^1(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \tag{3.12}$$

and introduce an interpolation operator $\Pi^h_k : \widetilde{H} \longrightarrow H_h$. Indeed, given $\boldsymbol{\tau} \in \widetilde{H}$, we let $\Pi^h_k(\boldsymbol{\tau})$ be the unique element in H_h such that

$$\begin{aligned}
0 \,&=\, \int_e \big(\boldsymbol{\tau} - \Pi^h_k(\boldsymbol{\tau})\big)\mathbf{n}\cdot\mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_k(e) && \forall\ \text{edge } e \in \mathcal{T}_h\,, \\
0 \,&=\, \int_K \big(\boldsymbol{\tau} - \Pi^h_k(\boldsymbol{\tau})\big) : \nabla\mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) \setminus \big\{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\big\} && \forall\, K \in \mathcal{T}_h\,, \\
0 \,&=\, \int_K \mathbf{q}\cdot\mathbf{rot}\,\big(\boldsymbol{\tau} - \Pi^h_k(\boldsymbol{\tau})\big) && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) && \forall\, K \in \mathcal{T}_h\,.
\end{aligned} \tag{3.13}$$

Note here that the extra local regularities on $\boldsymbol{\tau}$ and $\mathbf{rot}\,\boldsymbol{\tau}$ allow to define normal traces of $\boldsymbol{\tau}$ on the edges of $\mathcal{T}_h$ and the moments involving $\mathbf{rot}\,\boldsymbol{\tau}$ in each $K \in \mathcal{T}_h$, respectively. In addition, the uniqueness of $\Pi_k^h(\boldsymbol{\tau})$ is guaranteed by Lemma 3.1. Next, we define the local restriction of the interpolation operator as $\Pi_k^K(\boldsymbol{\tau}) := \Pi_k^h(\boldsymbol{\tau})|_K \in H_h^K$. It follows that for each $\mathbf{q} \in \mathbf{P}_{k-1}(K)$ there holds

$$\int_K \mathbf{div}\,(\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})) \cdot \mathbf{q} \,=\, - \int_K (\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})) : \nabla \mathbf{q} + \int_{\partial K} (\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau}))\, \mathbf{n} \cdot \mathbf{q} \,=\, 0\,,$$

which, together with the fact that $\mathbf{div}\,\Pi_k^K(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(K)$, implies that

$$\mathbf{div}\,\Pi_k^K(\boldsymbol{\tau}) \,=\, \mathcal{P}_{k-1}^K(\mathbf{div}\,\boldsymbol{\tau})\,. \tag{3.14}$$

This identity implies the following result.

Lemma 3.5 *Let k, ℓ and r be integers satisfying $1 \le r \le k$ and $0 \le \ell \le r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ and for any $\boldsymbol{\tau}$ verifying additionally that $\mathbf{div}\,\boldsymbol{\tau}|_K \in \mathbf{H}^r(K)$ there holds*

$$|\mathbf{div}\,\boldsymbol{\tau} - \mathbf{div}\,\Pi_k^K(\boldsymbol{\tau})|_{l,K} \,\le\, C\, h_K^{r-l}\, |\mathbf{div}\,\boldsymbol{\tau}|_{r,K}\,. \tag{3.15}$$

Proof. It follows from a straightforward application of Lemma 3.4. □

We now consider $K \in \mathcal{T}_h$ and set the local moments defining H_h^K. Indeed, given $\boldsymbol{\tau}$ as required by (3.13), we define the K-moments:

$$\begin{aligned}
m_{\mathbf{q},e}^{\mathbf{n}}(\boldsymbol{\tau}) \;&:=\; \int_e \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,, \\
m_{\mathbf{q},K}^{\mathbf{div}}(\boldsymbol{\tau}) \;&:=\; \int_K \boldsymbol{\tau} : \nabla \mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) \setminus \big\{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\big\}\,, \\
m_{\mathbf{q},K}^{\mathbf{rot}}(\boldsymbol{\tau}) \;&:=\; \int_K \mathbf{q} \cdot \mathbf{rot}\,(\boldsymbol{\tau}) && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,,
\end{aligned} \tag{3.16}$$

and gather all the above in the set $\Big\{m_{j,K}(\boldsymbol{\tau})\Big\}_{j=1}^{n_k^K}$. Then, we let $\{\boldsymbol{\varphi}_{j,K}\}_{j=1}^{n_k^K}$ be the canonical basis of H_h^K, that is, given $i \in \{1, 2, \ldots, n_k^K\}$, $\boldsymbol{\varphi}_{i,K}$ is the unique element in H_h^K such that

$$m_{j,K}(\boldsymbol{\varphi}_{i,K}) \,=\, \delta_{ij} \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,.$$

It follows easily that

$$\Pi_k^K(\boldsymbol{\tau}) \,:=\, \sum_{j=1}^{n_k^K} m_{j,K}(\boldsymbol{\tau})\, \boldsymbol{\varphi}_{j,K}\,, \tag{3.17}$$

or, equivalently, $\Pi_k^K(\boldsymbol{\tau})$ is the unique element in H_h^K such that

$$m_{j,K}(\Pi_k^K(\boldsymbol{\tau})) \,=\, m_{j,K}(\boldsymbol{\tau}) \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,.$$

The approximation properties of the local operator Π_k^K are provided now (cf. [10, eq.(4.8)]).

Lemma 3.6 *Let k and r be integers such that $k \ge 1$ and $1 \le r \le k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds*

$$\|\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\|_{0,K} \,\le\, C\, h_K^r\, |\boldsymbol{\tau}|_{r,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K)\,. \tag{3.18}$$

As a corollary of Lemmas 3.5 and 3.6 we have the following result.

Lemma 3.7 *Let k and r be integers such that $1 \le r \le k$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds*

$$\|\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\|_{\mathbf{div};K} \le C\, h_K^r \Big\{ |\boldsymbol{\tau}|_{r,K} + |\mathbf{div}\,\boldsymbol{\tau}|_{r,K} \Big\} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K) \quad \textit{with} \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^r(K)\,.$$

Proof. It suffices to apply (3.15) with $\ell = 0$ and then combine it with the estimate provided by Lemma 3.6. □

4 The discrete bilinear forms

The ultimate purpose of this section is to define computable discrete versions $\mathbf{a}_h : H_h \times H_h \longrightarrow \mathrm{R}$ and $\mathbf{b}_h : H_h \times Q_h \longrightarrow \mathrm{R}$ of the bilinear forms $\mathbf{a}$ and $\mathbf{b}$, respectively. To this end, we first observe that, given $(\boldsymbol{\tau}, \mathbf{v}) \in H_h \times Q_h$, the expression

$$\mathbf{b}\,(\boldsymbol{\tau}, \mathbf{v}) := \int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau} = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,,$$

is explicitly calculable since, according to the definitions of H_h and Q_h (cf. (3.1), (3.2)), there holds $\mathbf{v}|_K \in \mathbf{P}_{k-1}(K)$ and $\mathbf{div}\,\boldsymbol{\tau}|_K \in \mathbf{P}_{k-1}(K)$ on each element K, and hence we just set $\mathbf{b}_h = \mathbf{b}$. On the contrary, given $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in H_h$, the expression

$$\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} = \frac{1}{2\mu} \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}}$$

is not explicitly calculable since in general $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}$ are not known on each $K \in \mathcal{T}_h$. In order to overcome this difficulty, we now proceed to introduce suitable spaces on which the elements of H_h will be projected later on, and for which the bilinear form $\mathbf{a}$ is computable. Indeed, let us first consider a particular choice of $\boldsymbol{\tau}$ given by $\boldsymbol{\tau} := \nabla\, \underline{\mathrm{curl}}\, q \in \mathbb{P}_k(K)$ with $q \in \mathrm{P}_{k+2}(K)$, where $\underline{\mathrm{curl}} := \big(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\big)$, and observe that $\mathrm{tr}(\boldsymbol{\tau}) = \mathrm{div}(\underline{\mathrm{curl}}\, q) = 0$, whence $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau}$. It follows that for each $\boldsymbol{\zeta} \in H_h$ there holds

$$\int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} = \int_K \boldsymbol{\zeta} : \boldsymbol{\tau} = \int_K \boldsymbol{\zeta} : \nabla\, \underline{\mathrm{curl}}\, q = -\int_K \underline{\mathrm{curl}}\, q \cdot \mathbf{div}\,\boldsymbol{\zeta} + \int_{\partial K} (\boldsymbol{\zeta}\mathbf{n}) \cdot \underline{\mathrm{curl}}\, q\,, \tag{4.1}$$

which, bearing in mind from Lemma 3.1 that $\mathbf{div}\,\boldsymbol{\zeta}|_K$ and $\boldsymbol{\zeta}\mathbf{n}|_{\partial K}$ are explicitly known, shows that $\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau})$ is in fact calculable in this case. In turn, it is also quite clear that, given $\boldsymbol{\tau} := q\,\mathbb{I} \in \mathbb{P}_k(K)$ with $q \in \mathrm{P}_k(K)$, there holds $\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) = 0$ for all $\boldsymbol{\zeta} \in H_h$. The above suggests to define the subspace of $\mathbb{P}_k(K)$ given by

$$\widehat{H}_k^K := \widehat{H}_{k,\nabla}^K + \widehat{H}_{k,\mathbb{I}}^K\,,$$

where

$$\widehat{H}_{k,\nabla}^K := \Big\{ \nabla\, \underline{\mathrm{curl}}\, q : \quad q \in \mathrm{P}_{k+2}(K) \Big\} \quad \text{and} \quad \widehat{H}_{k,\mathbb{I}}^K := \Big\{ q\,\mathbb{I} : \quad q \in \mathrm{P}_k(K) \Big\}\,.$$

The following lemma establishes the basic properties of the space $\widehat{H}_k^K$.

Lemma 4.1 *There holds $\widehat{H}_k^K := \widehat{H}_{k,\nabla}^K \oplus \widehat{H}_{k,\mathbb{I}}^K$ and $\dim \widehat{H}_k^K = (k+1)\,(k+4)$.*

Proof. Given $\boldsymbol{\tau} \in \widehat{H}^K_{k,\nabla} \cap \widehat{H}^K_{k,\mathbb{I}}$, we have on one hand $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{d}}$, and on the other hand $\boldsymbol{\tau}^{\mathtt{d}} = \mathbf{0}$, so that necessarily $\boldsymbol{\tau} = \mathbf{0}$. This shows the required decomposition and hence

$$\dim \widehat{H}^K_k \,=\, \dim \widehat{H}^K_{k,\nabla} \,+\, \dim \widehat{H}^K_{k,\mathbb{I}} \,=\, \dim \widehat{H}^K_{k,\nabla} \,+\, \dim \mathrm{P}_k(K)\,. \tag{4.2}$$

Alternatively, it is easy to see that $\widehat{H}^K_{k,\nabla}$ and $\widehat{H}^K_{k,\mathbb{I}}$ are orthogonal with respect to the usual inner product of $\mathbb{H}(\mathbf{rot}\,;K)$. In order to determine the remaining dimension in (4.2) we prove now that the set $\Big\{\nabla\,\underline{\mathrm{curl}}\,\mathbf{x}^\alpha : \quad 2 \le |\alpha| \le k+2\Big\}$ is a basis of $\widehat{H}^K_{k,\nabla}$. Indeed, the generation property is quite clear from the fact that $\Big\{\mathbf{x}^\alpha : \quad 0 \le |\alpha| \le k+2\Big\}$ is the canonical basis of $P_{k+2}(K)$ and by observing that $\nabla\,\underline{\mathrm{curl}}\,q = \mathbf{0} \;\;\forall\, q \in \mathrm{P}_1(K)$. Next, we consider scalars $a_\alpha,\; 2 \le |\alpha| \le k+2$, and set $\nabla\,\underline{\mathrm{curl}}\,q = \mathbf{0}$ with $q := \displaystyle\sum_{2\le|\alpha|\le k+2} a_\alpha\,\mathbf{x}^\alpha$. It follows that $\underline{\mathrm{curl}}\,q$ is a constant vector of R^2, that is

$$\begin{aligned}
\frac{\partial q}{\partial x_1} \;&=\; \sum_{1\le|\beta|\le k+1} a_{\beta+(1,0)}\,(\beta_1+1)\,\mathbf{x}^\beta \;=\; \text{constant} \quad \text{in} \quad K\,,\\
\frac{\partial q}{\partial x_2} \;&=\; \sum_{1\le|\beta|\le k+1} a_{\beta+(0,1)}\,(\beta_2+1)\,\mathbf{x}^\beta \;=\; \text{constant} \quad \text{in} \quad K\,,
\end{aligned}$$

which yields $a_{\beta+(1,0)} \,=\, a_{\beta+(0,1)} \,=\, 0$ for all $1 \le |\beta| \le k+1$. In this way, since clearly

$$\big\{a_\alpha :\; 2 \le |\alpha| \le k+2\big\} \;=\; \big\{a_{\beta+(1,0)} :\; 1 \le |\beta| \le k+1\big\} \,\cup\, \big\{a_{\beta+(0,1)} :\; 1 \le |\beta| \le k+1\big\}\,,$$

we deduce that $q = 0$. Having thus identified a basis of $\widehat{H}^K_{k,\nabla}$, whose cardinality is certainly given by $\dim \mathrm{P}_{k+2}(K) - \dim \mathrm{P}_1(K)$, we conclude from (4.2) and the foregoing expression that

$$\dim \widehat{H}^K_k \;=\; \frac{(k+3)(k+4)}{2} \;-\; 3 \;+\; \frac{(k+1)(k+2)}{2} \;=\; (k+1)(k+4)\,,$$

which completes the proof. □

We now introduce a projection operator $\widehat{\Pi}^K_k : \mathbb{H}(\mathbf{div};K) \longrightarrow \widehat{H}^K_k$. To this end, we set for each $K \in \mathcal{T}_h$ the local bilinear form

$$\mathbf{a}^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \;:=\; \frac{1}{2\mu}\int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \in \mathbb{L}^2(K)\,.$$

Then, we define $\widehat{\boldsymbol{\zeta}} \,:=\, \widehat{\Pi}^K_k(\boldsymbol{\zeta}) \in \widehat{H}^K_k$ in terms of the decomposition:

$$\widehat{\boldsymbol{\zeta}} = \widehat{\boldsymbol{\zeta}}_\nabla \,+\, q_{\boldsymbol{\zeta}}\,\mathbb{I} \,+\, c_{\boldsymbol{\zeta}}\,\mathbb{I}\,, \tag{4.3}$$

where the components $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}^K_{k,\nabla}$, $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K) \,:=\, \mathrm{span}\Big\{\mathbf{x}^\alpha : \quad 1 \le |\alpha| \le k\Big\}$, and $c_{\boldsymbol{\zeta}} \,\in\, \mathrm{R}$ are computed according to the following sequentially connected problems:

- Find $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}^K_{k,\nabla}$ such that

$$\mathbf{a}^K(\widehat{\boldsymbol{\zeta}}_\nabla,\boldsymbol{\tau}) \;=\; \mathbf{a}^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \,\in\, \widehat{H}^K_{k,\nabla}\,, \tag{4.4}$$

- Find $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K)$ such that

$$(\mathbf{div}(q_{\boldsymbol{\zeta}}\,\mathbb{I}),\mathbf{div}(q\,\mathbb{I}))_{0,K} \;=\; (\mathbf{div}\big(\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_\nabla\big),\mathbf{div}(q\,\mathbb{I}))_{0,K} \qquad \forall\, q \,\in\, \widehat{\mathrm{P}}_k(K)\,, \tag{4.5}$$

- Find $c_{\boldsymbol{\zeta}} \in \mathrm{R}$ such that:
$$\int_K \operatorname{tr}(\widehat{\boldsymbol{\zeta}}) \;=\; \int_K \operatorname{tr}(\boldsymbol{\zeta})\,. \tag{4.6}$$

We remark that the unique solvability of (4.4) is guaranteed by the identity

$$\mathbf{a}^K(\boldsymbol{\tau},\boldsymbol{\tau}) \;=\; \frac{1}{2\mu}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,K} \;=\; \frac{1}{2\mu}\,\|\boldsymbol{\tau}\|^2_{0,K} \qquad \forall\,\boldsymbol{\tau}\in \widehat{H}^K_{k,\nabla}\,,$$

whereas that of (4.5) follows from the inequality

$$\|\mathbf{div}(q\,\mathbb{I})\|^2_{0,K} \;=\; |q|^2_{1,K} \;>\; 0 \qquad \forall\, q\,\in\,\widehat{\mathrm{P}}_k(K)\setminus\{0\}\,.$$

In this way, having computed $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}^K_{k,\nabla}$ and then $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K)$, we replace them into (4.6), which, using that $\operatorname{tr}(\widehat{\boldsymbol{\zeta}}_\nabla) = 0$, yields

$$c_{\boldsymbol{\zeta}} \;=\; \frac{1}{2|K|}\int_K \Big\{\operatorname{tr}(\boldsymbol{\zeta}) \,-\, 2q_{\boldsymbol{\zeta}}\Big\}\,. \tag{4.7}$$

Let us now check that the right hand sides of (4.4), (4.5), and (4.7) are indeed calculable when $\boldsymbol{\zeta}$ belongs to our local virtual space $H^K_h \,\subseteq\, \mathbb{H}(\mathbf{div};K)$ (cf. (3.7)). Firstly, the fact that $\mathbf{a}^K(\boldsymbol{\zeta},\boldsymbol{\tau})$ can be explicitly computed for $\boldsymbol{\zeta}\in H^K_h$ and $\boldsymbol{\tau}\in\widehat{H}^K_{k,\nabla}$, was already noticed at the beginning of Section 4 (cf. (4.1)). In turn, since $\mathbf{div}\boldsymbol{\zeta} \,\in\, \mathbf{P}_{k-1}(K)$ (cf. (3.7)) and $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \mathbb{P}_k(K)$, it is quite clear that the expression $(\mathbf{div}\big(\boldsymbol{\zeta}-\widehat{\boldsymbol{\zeta}}_\nabla\big),\mathbf{div}(q\,\mathbb{I}))_{0,K}$ is also calculable for each $q\,\in\,\widehat{\mathrm{P}}_k(K)$. Next, for the right hand side of (4.7) we simply observe that

$$\int_K \operatorname{tr}(\boldsymbol{\zeta}) \;=\; \int_K \boldsymbol{\zeta}:\mathbb{I} \;=\; \int_K \boldsymbol{\zeta}:\nabla\mathbf{x} \;=\; -\int_K \mathbf{x}\cdot\mathbf{div}\boldsymbol{\zeta} \;+\; \int_{\partial K}\boldsymbol{\zeta}\,\mathbf{n}\cdot\mathbf{x}\,,$$

which, according to (3.7), is calculable as well. Finally, it is straightforward to check from (4.4) - (4.6) that $\widehat{\Pi}^K_k(\boldsymbol{\zeta}) = \boldsymbol{\zeta} \quad \forall\,\boldsymbol{\zeta}\in\widehat{H}^K_k$, which confirms that $\widehat{\Pi}^K_k$ is in fact a projector. Moreover, the following result establishes the uniform boundedness of the family $\Big\{\widehat{\Pi}^K_k\Big\}_{K\in\mathcal{T}_h} \subseteq \Big\{\mathcal{L}(\mathbb{H}(\mathbf{div};K),\mathbb{L}^2(K))\Big\}_{K\in\mathcal{T}_h}$.

Lemma 4.2 *There exists a constant $C>0$, depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K\in\mathcal{T}_h$ there holds*

$$\|\widehat{\Pi}^K_k(\boldsymbol{\zeta})\|_{0,K} \,\le\, C\Big\{\,\|\boldsymbol{\zeta}\|_{0,K} \,+\, h_K\,\|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K}\,\Big\} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{H}(\mathbf{div};K)\,. \tag{4.8}$$

Proof. Given $\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div};K)$ we utilize again the decomposition (4.3) and set

$$\widehat{\boldsymbol{\zeta}} := \widehat{\Pi}^K_k(\boldsymbol{\zeta}) \;=\; \widehat{\boldsymbol{\zeta}}_\nabla \,+\, q_{\boldsymbol{\zeta}}\,\mathbb{I} \,+\, c_{\boldsymbol{\zeta}}\,\mathbb{I}\,, \tag{4.9}$$

with $\widehat{\boldsymbol{\zeta}}_\nabla\in\widehat{H}^K_{k,\nabla}$, $q_{\boldsymbol{\zeta}}\in\widehat{P}_k(K)$, and $c_{\boldsymbol{\zeta}}\in\mathrm{R}$. Then, it follows straightforwardly from (4.4), (4.5), and (4.7) that

$$\|\widehat{\boldsymbol{\zeta}}_\nabla\|_{0,K} \,\le\, \|\boldsymbol{\zeta}\|_{0,K}\,,\quad |q_{\boldsymbol{\zeta}}|_{1,K} \;=\; \|\mathbf{div}(q_{\boldsymbol{\zeta}}\,\mathbb{I})\|_{0,K} \,\le\, \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K} \,+\, \|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_\nabla)\|_{0,K}\,, \tag{4.10}$$

and

$$\|c_{\boldsymbol{\zeta}}\,\mathbb{I}\|_{0,K} \,\le\, \|\boldsymbol{\zeta}\|_{0,K} \,+\, \sqrt{2}\,\|q_{\boldsymbol{\zeta}}\|_{0,K}\,. \tag{4.11}$$

In what follows we bound $\|q_{\boldsymbol{\zeta}}\|_{0,K}$ and $\|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_{\nabla})\|_{0,K}$ in terms of $|q_{\boldsymbol{\zeta}}|_{1,K}$ and $\|\boldsymbol{\zeta}\|_{0,K}$, respectively. For the first estimate we assume, without loss of generality, that K is star-shaped with respect to a ball B centered at the origin. Otherwise, instead of K we consider the shifted region $\bar{K} := \bar{T}_K(K)$, where $\bar{T}_K(\mathbf{x}) := \mathbf{x} - \mathbf{x}_B \ \ \forall\, \mathbf{x} \in K$, for which there holds $h_K = h_{\bar{K}}$. Then, analogously as described for $\widetilde{K}$ at the beginning of Section 3.3, we now let $\Big\{ \Delta_i : \quad i \in \{1, 2, ..., d_K\} \Big\}$ be the partition of K obtained by connecting each vertex of this element to the origin. In addition, for each $i \in \{1, 2, \dots, d_K\}$ we let h_i and ρ_i be the geometric parameters of Δ_i, and let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\, \mathbf{x} \ \ \forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \Delta_i$. Recall that $\widehat{\Delta}$ is the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$. Hence, we can write

$$\|q_{\boldsymbol{\zeta}}\|^2_{0,K} \;=\; \sum_{i=1}^{d_K} \|q_{\boldsymbol{\zeta}}\|^2_{0,\Delta_i} \;=\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\, \|\widehat{q}_{\boldsymbol{\zeta},i}\|^2_{0,\widehat{\Delta}}\,, \tag{4.12}$$

where $\widehat{q}_{\boldsymbol{\zeta},i} := q_{\boldsymbol{\zeta}}|_{\Delta_i} \circ F_i \in \widehat{P}_k(\widehat{\Delta})$. We emphasize here that the fact that the origin is a vertex of each one of the triangles Δ_i has allowed to choose a linear (not affine) transformation F_i mapping $\widehat{\Delta}$ onto Δ_i, which, given that $q_{\boldsymbol{\zeta}}|_{\Delta_i} \in \widehat{\mathrm{P}}_k(\Delta_i)$, insures that $\widehat{q}_{\boldsymbol{\zeta},i}$ does belong to $\widehat{\mathrm{P}}_k(\widehat{\Delta})$. Moreover, the importance of it lies on the fact that $|\cdot|_{1,\widehat{\Delta}}$ is a norm on $\widehat{\mathrm{P}}_k(\widehat{\Delta})$, and therefore there exists $\widehat{c} > 0$, depending only on k and $\widehat{\Delta}$, such that, in particular, $\|\widehat{q}_{\boldsymbol{\zeta},i}\|^2_{0,\widehat{\Delta}} \le \widehat{c}\, |\widehat{q}_{\boldsymbol{\zeta},i}|^2_{1,\widehat{\Delta}}$. In this way, applying once more the scaling properties between Sobolev seminorms, we obtain from (4.12) that

$$\|q_{\boldsymbol{\zeta}}\|^2_{0,K} \;\le\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\, \widehat{c}\, |\widehat{q}_{\boldsymbol{\zeta},i}|^2_{1,\widehat{\Delta}} \;\le\; \widehat{c} \sum_{i=1}^{d_K} h_i^2\, \widehat{\rho}^{\,-2}\, |q_{\boldsymbol{\zeta},i}|^2_{1,\Delta_i} \;\le\; \widehat{C}\, h_K^2\, |q_{\boldsymbol{\zeta}}|^2_{1,K}\,, \tag{4.13}$$

which, together with the second inequality in (4.10), gives

$$\|q_{\boldsymbol{\zeta}}\|_{0,K} \;\le\; \widehat{C}\, h_K \Big\{ \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K} \,+\, \|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_{\nabla})\|_{0,K} \Big\}\,. \tag{4.14}$$

On the other hand, applying the inverse inequality in each triangle Δ_i, noting from the assumption a) at the beginning of Section 3.2 that $h_i^{-1} \le C_{\mathcal{T}}^{-1}\, h_K^{-1}$, and then using the first estimate in (4.10), we find that

$$\begin{aligned}\|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_{\nabla})\|^2_{0,K} \;&\le\; 2\, |\widehat{\boldsymbol{\zeta}}_{\nabla}|^2_{1,K} \;=\; 2 \sum_{i=1}^{d_K} |\widehat{\boldsymbol{\zeta}}_{\nabla}|^2_{1,\Delta_i} \;\le\; c \sum_{i=1}^{d_K} h_i^{-2}\, \|\widehat{\boldsymbol{\zeta}}_{\nabla}\|^2_{0,\Delta_i} \\ &\le\; c\, C_{\mathcal{T}}^{-2}\, h_K^{-2}\, \|\widehat{\boldsymbol{\zeta}}_{\nabla}\|^2_{0,K} \;\le\; c\, C_{\mathcal{T}}^{-2}\, h_K^{-2}\, \|\boldsymbol{\zeta}\|^2_{0,K}\,,\end{aligned}$$

which, replaced back into (4.14), yields

$$\|q_{\boldsymbol{\zeta}}\|_{0,K} \;\le\; \widehat{C} \Big\{ \|\boldsymbol{\zeta}\|_{0,K} \,+\, h_K\, \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K} \Big\}\,. \tag{4.15}$$

Finally, it is easy to see that (4.9), the first inequality in (4.10), (4.11), and (4.15) imply the required estimate (4.8), thus completing the proof. □

We remark at this point, as indicated in [3], that instead of using $\widehat{\mathrm{P}}_k(K)$ in the decomposition (4.3), one could also employ the space $\mathrm{P}_{k,0}(K) := \Big\{ q \in \mathrm{P}_k(K) : \quad \int_K q = 0 \Big\}$. In this case, the corresponding inequality (4.13) follows from the approximation property given by Lemma 3.4 with $k = 1$, $r = 1$, and $\ell = 0$, and noting that obviously $\mathcal{P}_0^K(q) = 0 \ \ \forall\, q \in \mathrm{P}_{k,0}(K)$. Nevertheless, we prefer to stay with $\widehat{\mathrm{P}}_k(K)$ because of the simplicity of its canonical basis for the implementation of (4.5).

The analogue of Lemma 3.2 for the present operator $\widehat{\Pi}_k^K$ is provided next.

Lemma 4.3 *Given integers $k,\ \ell \geq 1$, and $K \in \mathcal{T}_h$, there holds $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} = \widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})$ for all $\boldsymbol{\zeta} \in \mathbb{H}^k(K)$, and $\widehat{\Pi}_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbb{H}^\ell(\widetilde{K}), \mathbb{L}^2(\widetilde{K})\big)$ with $\|\widehat{\Pi}_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbb{H}^\ell(\widetilde{K}),\mathbb{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$, namely depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$.*

Proof. We first observe that $\boldsymbol{\tau} \in \widehat{H}_k^K$ if and only $\widetilde{\boldsymbol{\tau}} := \boldsymbol{\tau} \circ T_K^{-1} \in \widehat{H}_k^{\widetilde{K}}$. In particular, given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}; K)$, there holds $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} \in \widehat{H}_k^{\widetilde{K}}$, and hence, in order to obtain the required identity, it suffices to show that $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}$ solves the same problem as $\widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})$, namely (4.4) - (4.6) with $K = \widetilde{K}$ and $\boldsymbol{\zeta} = \widetilde{\boldsymbol{\zeta}}$. In fact, setting as before $\widehat{\Pi}_k^K(\boldsymbol{\zeta}) = \widehat{\boldsymbol{\zeta}}_\nabla + q_{\boldsymbol{\zeta}}\,\mathbb{I} + c_{\boldsymbol{\zeta}}\,\mathbb{I}$, where $\widehat{\boldsymbol{\zeta}}_\nabla \in \widehat{H}_{k,\nabla}^K$, $q_{\boldsymbol{\zeta}} \in \widehat{\mathrm{P}}_k(K)$, and $c_{\boldsymbol{\zeta}} \in \mathrm{R}$, we find, according to (4.4), that for each $\boldsymbol{\tau}_\nabla \in \widehat{H}_{k,\nabla}^K$ there holds

$$\mathbf{a}^{\widetilde{K}}(\widetilde{\widehat{\boldsymbol{\zeta}}_\nabla}, \widetilde{\boldsymbol{\tau}_\nabla}) \,=\, h_K^{-2}\,\mathbf{a}^K(\widehat{\boldsymbol{\zeta}}_\nabla, \boldsymbol{\tau}_\nabla) \,=\, h_K^{-2}\,\mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}_\nabla) \,=\, \mathbf{a}^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}_\nabla})\,. \tag{4.16}$$

In turn, for each $q \in \widehat{\mathrm{P}}_k(K)$ we have, in virtue of (4.5), that

$$\begin{aligned}
(\mathbf{div}(\widetilde{q_{\boldsymbol{\zeta}}}\,\mathbb{I}), \mathbf{div}(\widetilde{q}\,\mathbb{I}))_{0,\widetilde{K}} \,&=\, (\mathbf{div}(\widetilde{q_{\boldsymbol{\zeta}}\,\mathbb{I}}), \mathbf{div}(\widetilde{q\,\mathbb{I}}))_{0,\widetilde{K}} \,=\, h_K^{-4}\,(\mathbf{div}(q_{\boldsymbol{\zeta}}\,\mathbb{I}), \mathbf{div}(q\,\mathbb{I}))_{0,K} \\
&=\, h_K^{-4}\,(\mathbf{div}(\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_\nabla), \mathbf{div}(q\,\mathbb{I}))_{0,K} \,=\, (\mathbf{div}(\widetilde{\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_\nabla}), \mathbf{div}(\widetilde{q\,\mathbb{I}}))_{0,\widetilde{K}} \\
&=\, (\mathbf{div}(\widetilde{\boldsymbol{\zeta}} - \widetilde{\widehat{\boldsymbol{\zeta}}_\nabla}), \mathbf{div}(\widetilde{q}\,\mathbb{I}))_{0,\widetilde{K}}\,.
\end{aligned} \tag{4.17}$$

Next, it is easy to see, thanks to (4.6), that

$$\int_{\widetilde{K}} \mathrm{tr}\big(\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}\big) \,=\, \int_{\widetilde{K}} \widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} : \mathbb{I} \,=\, h_K^{-2} \int_K \widehat{\Pi}_k^K(\boldsymbol{\zeta}) : \mathbb{I} = h_K^{-2} \int_K \boldsymbol{\zeta} : \mathbb{I} \,=\, \int_{\widetilde{K}} \widetilde{\boldsymbol{\zeta}} : \mathbb{I} \,=\, \int_{\widetilde{K}} \mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,,$$

which, together with (4.16) and (4.17), confirm that $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}$ does solve the announced problem. Finally, since $h_{\widetilde{K}} = 1$, a direct application of Lemma 4.2 implies the existence of a constant $C > 0$, independent of $\widetilde{K}$, such that

$$\|\widehat{\Pi}_k^{\widetilde{K}}(\boldsymbol{\zeta})\|_{0,\widetilde{K}} \,\leq\, C\,\|\boldsymbol{\zeta}\|_{\mathbf{div};\widetilde{K}} \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}; \widetilde{K})\,,$$

and consequently, for each integer $\ell \geq 1$ there holds

$$\|\widehat{\Pi}_k^{\widetilde{K}}(\boldsymbol{\zeta})\|_{0,\widetilde{K}} \,\leq\, C\,\|\boldsymbol{\zeta}\|_{\ell,\widetilde{K}} \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{H}^\ell(\widetilde{K})\,,$$

which completes the proof. $\square$

Before establishing the next result, we now recall from [28] that if $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times Q$ is the solution of the continuous problem (2.3), then there holds $\boldsymbol{\sigma}^{\mathrm{d}} \,=\, 2\,\mu\,\nabla\mathbf{u} \in \mathbb{L}^2(\Omega)$ and $\mathrm{div}\,\mathbf{u} \,=\, 0$, which formally implies the existence of $w \in \mathrm{H}^2(\Omega)$ such that $\mathbf{u} \,=\, \underline{\mathrm{curl}}\,w$, and hence $\boldsymbol{\sigma}^{\mathrm{d}} \,=\, 2\,\mu\,\nabla\,\underline{\mathrm{curl}}\,w$. These remarks motivate for each integer $r \geq 0$ the introduction of the space

$$\mathbb{H}^r_{\nabla\,\underline{\mathrm{curl}}}(K) \,:=\, \Big\{\boldsymbol{\zeta} \in \mathbb{H}^r(K): \quad \boldsymbol{\zeta}^{\mathrm{d}} \,=\, \nabla\,\underline{\mathrm{curl}}\,w \quad \text{for some} \quad w \in \mathrm{H}^{r+2}(K)\Big\}\,.$$

Then, we have the following projection error for $\widehat{\Pi}_k^K$, which constitutes the analogue of Lemma 3.6.

Lemma 4.4 *Let k and r be integers such that $k \geq 1$ and $1 \leq r \leq k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds*

$$\|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K} \leq C\, h_K^r\, |\boldsymbol{\zeta}|_{r,K} \qquad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^r_{\nabla\, \underline{\mathrm{curl}}}(K)\,.$$

Proof. We proceed similarly as in the proof of Lemma 3.6. In fact, given integers k and r as stated, $K \in \mathcal{T}_h$, and $\boldsymbol{\zeta} \in \mathbb{H}^r_{\nabla\, \underline{\mathrm{curl}}}(K)$, we let $w \in \mathrm{H}^{r+2}(K)$ such that $\boldsymbol{\zeta}^{\mathsf{d}} = \nabla\, \underline{\mathrm{curl}}\, w$, set $\widetilde{w} \in \mathrm{H}^{r+2}(\widetilde{K})$ such that $\widetilde{\boldsymbol{\zeta}}^{\mathsf{d}} = \nabla\, \underline{\mathrm{curl}}\, \widetilde{w}$, denote by $\mathrm{T}^{r+2}(\widetilde{w}) \in \mathrm{P}_{r+1}(\widetilde{K})$ and $\mathrm{T}^r\big(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\big) \in \mathrm{P}_{r-1}(\widetilde{K})$ the averaged Taylor polynomials of order $r+2$ and r of $\widetilde{w}$ and $\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})$, respectively (cf. Lemma 3.3), and observe, since $r+1 \leq k+2$ and $r-1 \leq k$, that

$$\widehat{\Pi}_k^{\widetilde{K}}(\nabla\, \underline{\mathrm{curl}}\, \mathrm{T}^{r+2}(\widetilde{w})) = \nabla\, \underline{\mathrm{curl}}\, \mathrm{T}^{r+2}(\widetilde{w}) \qquad \text{and} \qquad \widehat{\Pi}_k^{\widetilde{K}}\big(\mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I}\big) = \mathrm{T}^r\big(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\big)\,\mathbb{I}\,.$$

It follows, using Lemmas 4.3 and 3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
\|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K} &= h_K^{-1}\, \|\widetilde{\boldsymbol{\zeta}} - \widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}\|_{0,\widetilde{K}} = h_K^{-1}\, \|\widetilde{\boldsymbol{\zeta}} - \widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})\|_{0,\widetilde{K}} \\
&= h_K^{-1} \left\| \big(\mathrm{I} - \widehat{\Pi}_k^{\widetilde{K}}\big) \left(\widetilde{\boldsymbol{\zeta}}^{\mathsf{d}} - \frac{1}{2}\,\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,\mathbb{I} - \nabla\, \underline{\mathrm{curl}}\, \mathrm{T}^{r+2}(\widetilde{w}) + \frac{1}{2}\,\mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I} \right) \right\|_{0,\widetilde{K}} \\
&\leq h_K^{-1}\, \|\mathrm{I} - \widehat{\Pi}_k^{\widetilde{K}}\|_{\mathcal{L}(\mathbb{H}^r(\widetilde{K}),\mathbb{L}^2(\widetilde{K}))} \Big\{ \|\widetilde{\boldsymbol{\zeta}}^{\mathsf{d}} - \nabla\, \underline{\mathrm{curl}}\, \mathrm{T}^{r+2}(\widetilde{w})\|_{r,\widetilde{K}} + \|\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,\mathbb{I} - \mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I}\|_{r,\widetilde{K}} \Big\} \\
&\leq C\, h_K^{-1} \Big\{ \|\widetilde{w} - \mathrm{T}^{r+2}(\widetilde{w})\|_{r+2,\widetilde{K}} + \|\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}) - \mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\|_{r,\widetilde{K}} \Big\} \\
&\leq C\, h_K^{-1} \Big\{ |\widetilde{w}|_{r+2,\widetilde{K}} + |\widetilde{\boldsymbol{\zeta}}|_{r,\widetilde{K}} \Big\} \leq C\, h_K^{-1}\, |\widetilde{\boldsymbol{\zeta}}|_{r,\widetilde{K}} = C\, h_K^r\, |\boldsymbol{\zeta}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. $\square$

We now let $\mathbf{a}_h^K : H_h^K \times H_h^K \longrightarrow \mathrm{R}$ be the local discrete bilinear form given by

$$\mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}), \widehat{\Pi}_k^K(\boldsymbol{\tau})\big) + \mathcal{S}^K\big(\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta}), \boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h^K\,, \tag{4.18}$$

where $\mathcal{S}^K : H_h^K \times H_h^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_k^K \times n_k^K}$ with respect to the basis $\{\boldsymbol{\varphi}_{j,K}\}_{j=1}^{n_k^K}$ of H_k^K (cf. (3.16) - (3.17)), that is

$$\mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{i=1}^{n_k^K} m_{i,K}(\boldsymbol{\zeta})\, m_{i,K}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h^K\,.$$

Next, as suggested by (4.18), we define the global discrete bilinear form $\mathbf{a}_h : H_h \times H_h \longrightarrow \mathrm{R}$

$$\mathbf{a}_h(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{K \in \mathcal{T}_h} \mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h\,. \tag{4.19}$$

The following lemma is a particular case of the inequality given in [10, eq. (5.8)]. In fact, it suffices to take the matrix $\mathbb{K}$ appearing there as the identity $\mathbb{I}$.

Lemma 4.5 *There exist c_0, $c_1 > 0$, depending only on $C_{\mathcal{T}}$, such that*

$$c_0\, \|\boldsymbol{\zeta}\|_{0,K}^2 \leq \mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \leq c_1\, \|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta} \in H_h^K\,. \tag{4.20}$$

As a consequence of the previous lemma and the properties of the projector $\widehat{\Pi}_k^K$, we have the following result.

Lemma 4.6 *For each $K \in \mathcal{T}_h$ there holds*

$$\mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \,=\, \mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta} \in \widehat{H}_k^K, \quad \forall\, \boldsymbol{\tau} \in H_h^K\,, \tag{4.21}$$

and there exist positive constants α_1, α_2, independent of h and K, such that

$$|\mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau})| \le \alpha_1 \left\{ \|\boldsymbol{\zeta}\|_{0,K}\, \|\boldsymbol{\tau}\|_{0,K} + \|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}\, \|\boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau})\|_{0,K} \right\} \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h^K\,, \tag{4.22}$$

and

$$\alpha_2\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,K}^2 \,\le\, \mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \,\le\, \alpha_1 \left\{ \|\boldsymbol{\zeta}\|_{0,K}^2 + \|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}^2 \right\} \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta} \in H_h^K\,. \tag{4.23}$$

Proof. Given $\boldsymbol{\zeta} \in \widehat{H}_k^K$, we certainly have $\boldsymbol{\zeta} \,=\, \widehat{\Pi}_k^K(\boldsymbol{\zeta}) \,:=\, \widehat{\boldsymbol{\zeta}}_\nabla + q_{\boldsymbol{\zeta}}\, \mathbb{I} + c_{\boldsymbol{\zeta}}\, \mathbb{I}$, with $\widehat{\boldsymbol{\zeta}}_\nabla \in \widehat{H}_{k,\nabla}^K$, $q_{\boldsymbol{\zeta}} \in \mathrm{P}_k(K)$, and $c_{\boldsymbol{\zeta}} \in \mathrm{R}$. Hence, using the symmetry of $\mathbf{a}^K$, and bearing in mind problem (4.4), we deduce, starting from (4.18), that given $\boldsymbol{\tau} \in H_h^K$ and denoting the deviatoric tensor of $\widehat{\Pi}_k^K(\boldsymbol{\tau})$ by $\widehat{\boldsymbol{\tau}}_\nabla \in \widehat{H}_{k,\nabla}^K$, there holds

$$\begin{aligned}
\mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \,&=\, \mathbf{a}^K(\widehat{\Pi}_k^K(\boldsymbol{\zeta}), \widehat{\Pi}_k^K(\boldsymbol{\tau})) \,=\, \mathbf{a}^K(\boldsymbol{\zeta}, \widehat{\Pi}_k^K(\boldsymbol{\tau})) \,=\, \mathbf{a}^K(\widehat{\Pi}_k^K(\boldsymbol{\tau}), \boldsymbol{\zeta}) \\
&=\, \mathbf{a}^K(\widehat{\boldsymbol{\tau}}_\nabla, \widehat{\boldsymbol{\zeta}}_\nabla) \,=\, \mathbf{a}^K(\boldsymbol{\tau}, \widehat{\boldsymbol{\zeta}}_\nabla) \,=\, \mathbf{a}^K(\boldsymbol{\tau}, \boldsymbol{\zeta}) \,=\, \mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau})\,,
\end{aligned}$$

which proves (4.21). Next, for the boundedness of $\mathbf{a}_h^K$ we apply the Cauchy-Schwarz inequality, the first estimate in (4.10), and the upper bound in (4.20) (cf. Lemma 4.5), to obtain

$$\begin{aligned}
|\mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau})| \,&\le\, \frac{1}{2\mu}\, \|\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathrm{d}}\|_{0,K} \|\big(\widehat{\Pi}_k^K(\boldsymbol{\tau})\big)^{\mathrm{d}}\|_{0,K} \\
&+ \Big\{ \mathcal{S}^K\big(\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta}), \boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\big) \Big\}^{1/2} \Big\{ S^K\big(\boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau}), \boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau})\big) \Big\}^{1/2} \\
&\le\, \frac{1}{2\mu}\, \|\boldsymbol{\zeta}\|_{0,K}\, \|\boldsymbol{\tau}\|_{0,K} \,+\, c_1\, \|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}\, \|\boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau})\|_{0,K} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h^K\,,
\end{aligned}$$

which gives (4.22) with $\alpha_1 \,:=\, \max\{\frac{1}{2\mu}, c_1\}$. Finally, concerning (4.23), it is clear that the corresponding upper bound follows from (4.22). In turn, applying the lower estimate in (4.20) (cf. Lemma 4.5) we find that

$$\begin{aligned}
\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,K}^2 \,&\le\, 2 \Big\{ \|\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathrm{d}}\|_{0,K}^2 \,+\, \|\big(\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathrm{d}}\|_{0,K}^2 \Big\} \\
&\le\, 4\mu\, \mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}), \widehat{\Pi}_k^K(\boldsymbol{\zeta})\big) \,+\, 2\, \|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}^2 \\
&\le\, 4\mu\, \mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}), \widehat{\Pi}_k^K(\boldsymbol{\zeta})\big) \,+\, \frac{2}{c_0}\, \mathcal{S}^K\big(\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta}), \boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)\,,
\end{aligned}$$

which yields the lower bound in (4.23) with $\alpha_2 \,:=\, \max\{4\mu, \frac{2}{c_0}\}^{-1}$. □

We end this section by observing, as mentioned in [3], that all the tensors in H_h^K vanishing $\mathbf{a}^K(\cdot,\cdot)$ also vanish $\mathbf{a}_h^K(\cdot,\cdot)$, which is important for the stability condition provided by (4.22) and (4.23). In fact, given $\boldsymbol{\tau} \in H_h^K$ such that $0 = \mathbf{a}^K(\boldsymbol{\tau}, \boldsymbol{\tau}) = \frac{1}{2\mu}\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,K}^2$, we have $\boldsymbol{\tau}^{\mathrm{d}} \,=\, \mathbf{0}$, that is $\boldsymbol{\tau} \,=\, \frac{1}{2}\, \mathrm{tr}(\boldsymbol{\tau})\, \mathbb{I}$, which implies, according to the definition of H_h^K (cf. (3.7)), that $\mathbf{div}\, \boldsymbol{\tau} \,=\, \frac{1}{2}\, \nabla\big(\mathrm{tr}(\boldsymbol{\tau})\big) \,\in\, \mathbf{P}_{k-1}(K)$. It follows that $\mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{P}_k(K)$, and hence $\boldsymbol{\tau} \in \widehat{H}_{k,\mathbb{I}}^K \subseteq \widehat{H}_k^K$. In this way, thanks to the consistency condition (4.21), we conclude that $\mathbf{a}_h^K(\boldsymbol{\tau}, \boldsymbol{\tau}) = \mathbf{a}^K(\boldsymbol{\tau}, \boldsymbol{\tau}) = 0$.

5 The mixed virtual element scheme

According to the analysis from the foregoing section, we reformulate the Galerkin scheme associated with (2.3) as: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ such that

$$\begin{aligned} \mathbf{a}_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \langle \boldsymbol{\tau}_h \mathbf{n}, \mathbf{g} \rangle && \forall\, \boldsymbol{\tau}_h \in H_h\,, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}_h && \forall\, \mathbf{v}_h \in Q_h\,. \end{aligned} \tag{5.1}$$

In addition, as suggested by the third equation of (2.2), the postprocessed virtual pressure is defined as follows:

$$p_h = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)\,. \tag{5.2}$$

In what follows we establish the well-posedness of (5.1). We begin the analysis with the following result from [11].

Lemma 5.1 *There exists $c_\Omega > 0$, depending only on Ω, such that*

$$c_\Omega\, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \le \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\zeta} \in H \text{ (cf. (2.4))}\,. \tag{5.3}$$

Proof. See [11, Chapter IV, Proposition 3.1]. □

The ellipticity of $\mathbf{a}_h$ in the discrete kernel of $\mathbf{b}$ is proved next.

Lemma 5.2 *Let $V_h := \Big\{ \boldsymbol{\zeta}_h \in H_h : \quad \mathbf{b}(\boldsymbol{\zeta}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in Q_h \Big\}$. Then, there exists $\alpha > 0$, independent of h, such that*

$$\mathbf{a}_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \ge \alpha\, \|\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega} \qquad \forall\, \boldsymbol{\zeta}_h \in V_h\,. \tag{5.4}$$

Proof. Recalling from (3.1) that for each $\boldsymbol{\zeta}_h \in H_h$ there holds $\mathbf{div}(\boldsymbol{\zeta}_h)|_K \in \mathbf{P}_{k-1}(K)\ \ \forall\, K \in \mathcal{T}_h$, which actually says that $\mathbf{div}(\boldsymbol{\zeta}_h) \in Q_h$, we find that

$$V_h := \Big\{ \boldsymbol{\zeta}_h \in H_h : \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\zeta}_h) = 0 \quad \forall\, \mathbf{v}_h \in Q_h \Big\} = \Big\{ \boldsymbol{\zeta}_h \in H_h : \quad \mathbf{div}(\boldsymbol{\zeta}_h) = 0 \Big\}\,.$$

Hence, according to the definition of $\mathbf{a}_h$ (cf. (4.19)), and applying the lower bound in (4.23), and the estimate (5.3) (cf. Lemma 5.1), we deduce that for each $\boldsymbol{\zeta}_h \in V_h$ there holds

$$\mathbf{a}_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) = \sum_{K\in\mathcal{T}_h} \mathbf{a}_h^K(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \ge \alpha_2 \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,K}^2 = \alpha_2\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,\Omega}^2 \ge \alpha\, \|\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega}^2\,,$$

with $\alpha = c_\Omega\, \alpha_2$, which ends the proof. □

The following lemma provides the discrete inf-sup condition for $\mathbf{b}$.

Lemma 5.3 *Let H_h and Q_h be the virtual subspaces given by (3.1) and (3.2). Then, there exists $\beta > 0$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in H_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega}} \ge \beta\, \|\mathbf{v}_h\|_{0,\Omega} \qquad \forall\, \mathbf{v}_h \in Q_h\,. \tag{5.5}$$

Proof. Since $\mathbf{b}$ satisfies the continuous inf-sup condition, we proceed in the classical way (see, e.g. [24, Section 4.2]) by constructing a corresponding Fortin's operator. In fact, given a convex and bounded domain G containing $\bar{\Omega}$, and given $\boldsymbol{\tau} \in H$ (cf. (2.4)), we let $\mathbf{z} \in \mathbf{H}_0^1(G) \cap \mathbf{H}^2(G)$ be the unique solution of the boundary value problem

$$\Delta \mathbf{z} \,=\, \begin{cases} \operatorname{\mathbf{div}} \boldsymbol{\tau} & \text{in} \quad \Omega\,, \\ 0 & \text{in} \quad G \backslash \bar{\Omega}\,, \end{cases} \quad , \quad \mathbf{z} = 0 \quad \text{on} \quad \partial G\,, \tag{5.6}$$

which, thanks to the corresponding elliptic regularity result, satisfies

$$\|\mathbf{z}\|_{2,\Omega} \,\leq\, C\,\|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega}\,. \tag{5.7}$$

Then, recalling that Π_k^h denotes the interpolation operator mapping $\widetilde{H}$ onto our virtual subspace H_h (cf. (3.12), (3.13)), we now define the operator $\pi_k^h : H \to H_h$ as

$$\pi_k^h(\boldsymbol{\tau}) \,=\, \Pi_k^h(\nabla \mathbf{z}) \,-\, \left\{ \frac{1}{2\,|\Omega|} \int_\Omega \operatorname{tr}\left(\Pi_k^h(\nabla \mathbf{z})\right) \right\} \mathbb{I}\,.$$

It follows, using (3.14) and the fact that Π_k^K and $\mathcal{P}_{k-1}^K$ are the restrictions to $K \in \mathcal{T}_h$ of the operators Π_k^h and $\mathcal{P}_{k-1}^h$, respectively, that

$$\operatorname{\mathbf{div}}\left(\pi_k^h(\boldsymbol{\tau})\right) \,=\, \operatorname{\mathbf{div}}\left(\Pi_k^h(\nabla \mathbf{z})\right) \,=\, \mathcal{P}_{k-1}^h(\operatorname{\mathbf{div}} \nabla \mathbf{z}) \,=\, \mathcal{P}_{k-1}^h\big(\operatorname{\mathbf{div}} \boldsymbol{\tau}\big) \quad \text{in} \quad \Omega\,, \tag{5.8}$$

and hence for each $\mathbf{v}_h \in Q_h$ we obtain

$$\mathbf{b}(\pi_k^h(\boldsymbol{\tau}), \mathbf{v}_h) \,=\, \int_\Omega \mathbf{v}_h \cdot \operatorname{\mathbf{div}}\left(\pi_k^h(\boldsymbol{\tau})\right. \,=\, \int_\Omega \mathbf{v}_h \cdot \mathcal{P}_{k-1}^h\big(\operatorname{\mathbf{div}} \boldsymbol{\tau}\big) \,=\, \int_\Omega \mathbf{v}_h \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} \,=\, \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}_h)\,. \tag{5.9}$$

In turn, using (5.8), (3.18) (with $r = 1$), and (5.7), we find that

$$\begin{aligned} \|\pi_k^h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,&\leq\, \|\Pi_k^h(\nabla \mathbf{z})\|_{0,\Omega} \,+\, \|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \|\nabla \mathbf{z}\|_{0,\Omega} \,+\, \|\nabla \mathbf{z} - \Pi_k^h(\nabla \mathbf{z})\|_{0,\Omega} \,+\, \|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega} \\ &\leq\, \|\nabla \mathbf{z}\|_{0,\Omega} \,+\, c\,h\,\|\nabla \mathbf{z}\|_{1,\Omega} \,+\, \|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \bar{C}\,\|\mathbf{z}\|_{2,\Omega} \,+\, \|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega} \,\leq\, C\,\|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega}\,, \end{aligned}$$

which proves the uniform boundedness of the operators $\{\pi_k^h\}_{h>0}$. This fact and the identity (5.9) confirm that $\{\pi_k^h\}_{h>0}$ constitutes a family of Fortin's operators, which yields (5.5) and ends the proof. $\square$

The unique solvability and stability of the actual Galerkin scheme (5.1) is established now.

Theorem 5.1 *There exists a unique $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ solution of (5.1), and there exists a positive constant C, independent of h, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H \times Q} \,\leq\, C \Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\}\,.$$

Proof. The boundedness of $\mathbf{a}_h : H_h \times H_h \longrightarrow \mathrm{R}$ with respect to the norm $\|\cdot\|_{\mathbf{div};\Omega}$ of $\mathbb{H}(\mathbf{div};\Omega)$ follows easily from (4.22) and (4.8) (cf. Lemma 4.2). In turn, it is quite clear that $\mathbf{b}$ is also bounded. Hence, thanks to Lemmas 5.2 and 5.3, a straightforward application of the Babuška-Brezzi theory completes the proof. $\square$

We now aim to provide the corresponding a priori error estimates. To this end, and just for sake of clearness in what follows, we recall that $\mathcal{P}_{k-1}^h : \mathbf{L}^2(\Omega) \longrightarrow Q_h$ and $\Pi_k^h : \widetilde{H} \longrightarrow H_h$ are the projector and interpolator, respectively, defined by (3.11) and (3.13), whose associated local operators are denoted by $\mathcal{P}_{k-1}^K$ and Π_k^K. In turn, given our local projector $\widehat{\Pi}_k^K$ defined by (4.3) - (4.6), we denote by $\widehat{\Pi}_k^h$ its global counterpart, that is, given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega)$, we let

$$\widehat{\Pi}_k^h(\boldsymbol{\zeta})|_K \;:=\; \widehat{\Pi}_k^K(\boldsymbol{\zeta}|_K) \qquad \forall\, K \in \mathcal{T}_h\,.$$

Then, we have the following main result.

Theorem 5.2 *Let $(\boldsymbol{\sigma},\mathbf{u}) \in H \times Q$ and $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete schemes* (2.3) *and* (5.1)*, respectively, and let $p_h \in L^2(\Omega)$ be the postprocessed virtual pressure defined in* (5.2)*. Then, there exist positive constants C_1, C_2, independent of h, such that*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \;\le\; C_1 \Big\{ \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \Big\}\,, \tag{5.10}$$

and

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \;\le\; C_2 \Big\{ \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\mathbf{u}-\mathcal{P}_{k-1}^h(\mathbf{u})\|_{0,\Omega} \,+\, h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \Big\}\,. \tag{5.11}$$

Proof. We proceed similarly as in [10, Theorem 6.1]. Indeed, we first have, thanks to the triangle inequality, that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} \;\le\; \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\Pi_k^h(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h\|_{0,\Omega}\,, \tag{5.12}$$

whence it just remains to estimate $\boldsymbol{\delta}_h := \Pi_k^h(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h$. We now observe from (3.14) and the second equation of (5.1) that $\mathbf{div}\big(\Pi_k^h(\boldsymbol{\sigma})\big) \,=\, \mathcal{P}_{k-1}^h(\mathbf{div}\boldsymbol{\sigma}) \,=\, \mathcal{P}_{k-1}^h(-\mathbf{f}) \,=\, \mathbf{div}\,\boldsymbol{\sigma}_h$, which says that $\boldsymbol{\delta}_h \in V_h$. It follows from (5.4) (cf. Lemma 5.2), adding and substracting $\widehat{\Pi}_k^h(\boldsymbol{\sigma})$, using the first equations of (5.1) and (2.3), employing the identity (4.21), and applying the boundedness of $\mathbf{a}_h^K$ (cf. (4.22)), $\mathbf{a}^K$ and $\widehat{\Pi}_k^K$ (cf. (4.8)), that

$$\begin{aligned}
\alpha\,\|\boldsymbol{\delta}_h\|^2_{\mathbf{div};\Omega} \;&=\; \alpha\,\|\boldsymbol{\delta}_h\|^2_{0,\Omega} \;\le\; \mathbf{a}_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \;=\; \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}_h(\boldsymbol{\sigma}_h,\boldsymbol{\delta}_h)\\
&=\; \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,+\, \mathbf{a}_h(\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \langle \boldsymbol{\delta}_h\,\mathbf{n},\mathbf{g}\rangle\\
&=\; \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,+\, \mathbf{a}_h(\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\delta}_h)\\
&=\; \sum_{K\in\mathcal{T}_h} \Big\{ \mathbf{a}_h^K(\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}^K(\boldsymbol{\sigma}-\widehat{\Pi}_k^K(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \Big\}\\
&\le\; \alpha_1 \sum_{K\in\mathcal{T}_h} \Big\{ \|\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K(\boldsymbol{\sigma})\|_{0,K} \,+\, \|\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K\{\Pi_k^K(\boldsymbol{\sigma})\}\|_{0,K} \Big\}\,\|\boldsymbol{\delta}_h\|_{0,K}\\
&\quad+\;\; \frac{1}{2\mu} \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\sigma}-\widehat{\Pi}_k^K(\boldsymbol{\sigma})\|_{0,K}\,\|\boldsymbol{\delta}_h\|_{0,K}\,,
\end{aligned}$$

which yields, with $C := \frac{1}{\alpha}\,\max\{\alpha_1,\frac{1}{2\mu}\}$,

$$\|\boldsymbol{\delta}_h\|_{\mathbf{div};\Omega} \;\le\; C\,\Big\{\|\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h\{\Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\Big\}\,. \tag{5.13}$$

Next, adding and substracting $\boldsymbol{\sigma}$, we deduce that

$$\|\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \;\le\; \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,. \tag{5.14}$$

In turn, proceding in the same way and employing the boundedness of $\widehat{\Pi}_k^K$ (cf. (4.8)), we find that

$$\begin{aligned}
\|\Pi_k^h(\boldsymbol{\sigma}) - \widehat{\Pi}_k^h\{\Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \,&\leq\, \|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\widehat{\Pi}_k^h\{\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \\
&\leq\, C\left\{\|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{div}\big(\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\big)\|_{0,\Omega}\right\} \\
&\leq\, C\left\{\,\|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\right\}.
\end{aligned} \tag{5.15}$$

In this way, replacing (5.15) and (5.14) into (5.13), and then the resulting estimate back into (5.12), we conclude the upper bound for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ induced by (5.10). In addition, using from the third equation of (2.2) that $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$, we obtain

$$\|p - p_h\|_{0,\Omega} \,=\, \frac{1}{2}\,\|\mathrm{tr}(\boldsymbol{\sigma}) - \mathrm{tr}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,\leq\, c\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,,$$

which completes the proof of (5.10).

On the other hand, concerning the error $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$, we begin with the triangle inequality again and obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,\leq\, \|\mathbf{u} - \mathcal{P}_{k-1}^h(\mathbf{u})\|_{0,\Omega} \,+\, \|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}\,. \tag{5.16}$$

Next, proceeding as in the proof of Lemma 5.3, taking $\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h \,\in\, Q_h$ instead of $\mathbf{div}(\boldsymbol{\tau})$ in the definition of the auxiliary problem (5.6), we deduce the existence of $\boldsymbol{\sigma}_h^* \in H_h$ such that

$$\mathbf{div}(\boldsymbol{\sigma}_h^*) \,=\, \mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h \quad \text{and} \quad \|\boldsymbol{\sigma}_h^*\|_{\mathbf{div};\Omega} \,\leq\, c\,\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}\,.$$

It follows, employing the first equations of (5.1) and (2.3), and the identity (4.21),that

$$\begin{aligned}
\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}^2 \,&=\, \int_\Omega \big(\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\big)\cdot\mathbf{div}(\boldsymbol{\sigma}_h^*) \,=\, \int_\Omega \big(\mathbf{u} - \mathbf{u}_h\big)\cdot\mathbf{div}(\boldsymbol{\sigma}_h^*) \\
&=\, \mathbf{b}(\boldsymbol{\sigma}_h^*, \mathbf{u}) \,-\, \mathbf{b}(\boldsymbol{\sigma}_h^*, \mathbf{u}_h) \,=\, \mathbf{a}_h(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h^*) \,-\, \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h^*) \\
&=\, \mathbf{a}_h(\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h^*) \,-\, \mathbf{a}(\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h^*)\,,
\end{aligned}$$

which, applying the boundedness of $\mathbf{a}_h^K$ (cf. (4.22)), $\mathbf{a}^K$ and $\widehat{\Pi}_k^K$ (cf. (4.8)), and observing in particular that $\|\boldsymbol{\sigma}_h^* - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h^*)\|_{0,\Omega} \,\leq\, c\,\|\boldsymbol{\sigma}_h^*\|_{\mathbf{div};\Omega} \,\leq\, C\,\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}$, gives

$$\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega} \,\leq\, C\left\{\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\right\}. \tag{5.17}$$

Now, adding and substracting $\boldsymbol{\sigma}$, we readily get

$$\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,\leq\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,. \tag{5.18}$$

Similarly, and utilizing once again the boundedness of $\widehat{\Pi}_k^K$ (cf. (4.8)), we can write

$$\begin{aligned}
\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,&\leq\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\widehat{\Pi}_k^h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \\
&\leq\, C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}\right\}.
\end{aligned} \tag{5.19}$$

Furthermore, since $\mathbf{div}(\boldsymbol{\sigma}_h) \,=\, \mathcal{P}_{k-1}^h(-\mathbf{f})$ and $\mathbf{div}(\boldsymbol{\sigma}) \,=\, -\mathbf{f}$, we have

$$\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \,=\, \|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\,. \tag{5.20}$$

Consequently, replacing (5.19) and (5.18) into (5.17), making use also of (5.20) and the already derived a priori error bound for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$, and then placing the resulting estimate back into (5.16), we arrive at (5.11) and conclude the proof. $\square$

Having established the a priori error estimates for our unknowns, we now provide the corresponding rates of convergence.

Theorem 5.3 *Let $(\boldsymbol{\sigma},\mathbf{u}) \in H \times Q$ and $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete schemes (2.3) and (5.1), respectively, and let $p_h \in L^2(\Omega)$ be the postprocessed virtual pressure defined in (5.2). Assume that for some $r \in [1,k+1]$ and $s \in [1,k]$ there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^r_{\nabla \underline{\mathrm{curl}}}(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{H}^{r-1}(K)$, and $\mathbf{u}|_K \in \mathbf{H}^s(K)$ for each $K \in \mathcal{T}_h$. Then, there exist positive constants $\bar{C}_1$, $\bar{C}_2$, independent of h, such that*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \,\le\, \bar{C}_1\, h^r \left\{ \sum_{K\in\mathcal{T}_h} \Big\{ \|\boldsymbol{\sigma}\|^2_{r,K} \,+\, \|\mathbf{f}\|^2_{r-1,K} \Big\} \right\}^{1/2}, \tag{5.21}$$

and

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \,\le\, \bar{C}_2\, h^r \left\{ \sum_{K\in\mathcal{T}_h} \Big\{ \|\boldsymbol{\sigma}\|^2_{r,K} \,+\, \|\mathbf{f}\|^2_{r-1,K} \Big\} \right\}^{1/2} \,+\, \bar{C}_2\, h^s \left\{ \sum_{K\in\mathcal{T}_h} \|\mathbf{u}\|^2_{s,K} \right\}^{1/2}. \tag{5.22}$$

Proof. The case of integers $r \in [1,k+1]$ and $s \in [1,k]$ follows from straightforward applications of the approximation properties provided by Lemmas 3.6, 4.4, and 3.4, to the terms on the right hand sides of (5.10) and (5.11). In turn, the usual interpolation estimates of Sobolev spaces allow to conclude for the remaining real values of r and s. We omit further details. $\square$

We notice that if the assumed regularities in the foregoing theorem are global, then the estimates (5.21) and (5.22) become, respectively,

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \,\le\, \bar{C}_1\, h^r \Big\{ \|\boldsymbol{\sigma}\|_{r,\Omega} \,+\, \|\mathbf{f}\|_{r-1,\Omega} \Big\},$$

and

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \,\le\, \bar{C}_2\, h^r \Big\{ \|\boldsymbol{\sigma}\|_{r,\Omega} \,+\, \|\mathbf{f}\|_{r-1,\Omega} \Big\} \,+\, \bar{C}_2\, h^s\, |\mathbf{u}|_{s,K}\,.$$

In turn, it is also clear from the range of variability of the integers r and s that the highest possible rate of convergence for $\boldsymbol{\sigma}$ and p is h^{k+1}, whereas that of $\mathbf{u}$ is h^k.

We now introduce the fully computable approximations of $\boldsymbol{\sigma}$ and p given by

$$\widehat{\boldsymbol{\sigma}}_h \,:=\, \widehat{\Pi}^h_k(\boldsymbol{\sigma}_h) \qquad \text{and} \qquad \widehat{p}_h \,=\, -\frac{1}{2}\,\mathrm{tr}\big(\widehat{\boldsymbol{\sigma}}_h\big)\,, \tag{5.23}$$

and establish next the corresponding a priori error estimates.

Theorem 5.4 *There exists a positive constant C_3, independent of h, such that*

$$\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|p-\widehat{p}_h\|_{0,\Omega} \,\le\, C_3 \Big\{ \|\boldsymbol{\sigma}-\Pi^h_k(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}-\widehat{\Pi}^h_k(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{f}-\mathcal{P}^h_{k-1}(\mathbf{f})\|_{0,\Omega} \Big\}. \tag{5.24}$$

Proof. Similarly as at the end of the proof of Theorem 5.2 we have

$$\|p - \widehat{p}_h\|_{0,\Omega} \,=\, \frac{1}{2}\,\|\mathrm{tr}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h)\|_{0,\Omega} \,\leq\, c\,\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,,$$

and then, adding and substracting $\boldsymbol{\sigma}_h$, we get

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,\leq\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega}\,.$$

In this way, utilizing the estimates for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ and $\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega}$ given by (5.10) and (5.19), respectively, and employing (5.20) as well, we arrive at (5.24) and complete the proof. □

We end this section by remarking, according to the upper bounds provided by (5.10) (cf. Theorem 5.2) and (5.24) (cf. Theorem 5.4), that the pairs $(\boldsymbol{\sigma}_h, p_h)$ and $(\widehat{\boldsymbol{\sigma}}_h, \widehat{p}_h)$ share exactly the same rates of convergence given by Theorem 5.3.

6 Numerical results

In this section we present three numerical examples illustrating the good performance of the virtual mixed finite element scheme (5.1), and confirming the rates of convergence predicted by Theorem 5.3. For all the computations we consider the virtual element subspaces H_h and Q_h given by (3.1) and (3.2), with $k = 1$. In turn, for each Example we take kinematic viscosity $\mu = 1$ and assume first decompositions of Ω made of triangles. In addition, in Example 1 we also consider hexagons, whereas Example 2 makes use of general quadrilateral elements, and Example 3 considers distorted squares as well. We begin by introducing additional notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (5.1), that is, $N = \dim H_h + \dim Q_h$. More precisely, according to (3.6) and (3.2), and bearing in mind that $\dim \mathbf{P}_k(e) = 2(k+1)\ \ \forall$ edge $e \in \mathcal{T}_h$, and $\dim \mathbf{P}_{k-1}(K) = k(k+1)\ \forall\, K \in \mathcal{T}_h$, we find that in general

$$N \,=\, 2(k+1) \,\times\, \text{number of edges } e \in \mathcal{T}_h \,+\, \big\{3k(k+1) - 2\big\} \,\times\, \text{number of } K \in \mathcal{T}_h\,,$$

which, in the case $k = 1$, becomes

$$N \,=\, 4 \,\times\, \Big\{\text{number of edges } e \in \mathcal{T}_h \,+\, \text{number of } K \in \mathcal{T}_h\Big\}.$$

Also, the individual errors are defined by

$$\mathtt{e}_0(\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,, \quad \mathtt{e}(p) \,:=\, \|p - \widehat{p}_h\|_{0,\Omega}\,, \quad \text{and} \quad \mathtt{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,,$$

where $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{p}_h$ are computed according to (5.23), and $\mathbf{u}_h$ is provided by (5.1). In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}_0(\boldsymbol{\sigma}) \,:=\, \frac{\log\big(\mathtt{e}_0(\boldsymbol{\sigma})/\mathtt{e}_0'(\boldsymbol{\sigma})\big)}{\log(h/h')}\,, \quad \mathtt{r}(p) \,:=\, \frac{\log\big(\mathtt{e}(p)/\mathtt{e}'(p)\big)}{\log(h/h')}\,, \quad \text{and} \quad \mathtt{r}(\mathbf{u}) \,:=\, \frac{\log\big(\mathtt{e}(\mathbf{u})/\mathtt{e}'(\mathbf{u})\big)}{\log(h/h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the errors for two consecutive meshes with sizes h and h', respectively. The numerical results presented below were obtained using a MATLAB code. The corresponding linear systems were solved using the Conjugate Gradient method as main solver, and applying a stopping criterion determined by a relative tolerance of 10^{-10}. The specific examples to be considered are described next.

In Example 1 we consider $\Omega =]0,1[^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, (\sin(\pi x_1)\,\cos(\pi x_2), -\cos(\pi x_1)\,\sin(\pi x_2))^{\mathrm{t}} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, \frac{1}{x_2^2+1} - \frac{\pi}{4}\,.$$

In Example 2 we consider the L-shaped domain $\Omega :=]-1,1[^2 \backslash [0,1]^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, \frac{\mathbf{x} - (1,1)^{\mathrm{t}}}{(x_1-1)^2 + (x_2-1)^2} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, x_1 + \frac{1}{6}\,.$$

Finally, in Example 3 we consider the same geometry of Example 1, that is $\Omega =]0,1[^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given for each $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, (x_2^2, -x_1^2)^{\mathrm{t}} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, \left(x_1^2 + x_2^2\right)^{1/3} - \int_\Omega \left(x_1^2 + x_2^2\right)^{1/3}.$$

Note in this example that the partial derivatives of p, and hence, in particular $\mathbf{div}\,\boldsymbol{\sigma}$, are singular at the origin. Moreover, because of the power $1/3$, there holds $\boldsymbol{\sigma} \in \mathbb{H}^{5/3-\epsilon}(\Omega)$ and $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^{2/3-\epsilon}(\Omega)$ for each $\epsilon > 0$, which, applying Theorem 5.3 with $r = 5/3 - \epsilon$, should yield a rate of convergence very close to $O(h^{5/3})$ for $\boldsymbol{\sigma}$ and p.

In Tables 6.1 up to 6.4 we summarize the convergence history of the mixed virtual element scheme (5.1) as applied to Examples 1 and 2, for sequences of quasi-uniform refinements of each domain. We notice there that the rates of convergences $O(h^{k+1}) = O(h^2)$ and $O(h^k) = O(h)$ predicted by Theorem 5.3 (when $r = k+1$ and $s = k$) are attained by $(\boldsymbol{\sigma}, p)$ and $\mathbf{u}$, respectively, for triangular as well as for hexagonal and quadrilateral meshes. In turn, in Tables 6.5 and 6.6 we display the corresponding convergence history of Example 3. As predicted in advance, and due to the limited regularity of p and $\boldsymbol{\sigma}$ in this case, we observe that the orders $O(h^{5/3})$ and $O(h)$ are attained by $(\boldsymbol{\sigma}, p)$ and $\mathbf{u}$, respectively. Finally, in order to illustrate the accurateness of the discrete scheme, in Figures 6.1 up to 6.12 we display several components of the approximate and exact solutions for each example.

We end this paper by remarking that the analysis and the numerical examples presented here confirm that the mixed virtual element scheme (5.1) is a quite valid and attractive alternative to solve the Stokes problem. For further details, in particular those concerning the computational implementation of (5.1), we refer to [13]. Future developments in the direction of this work should consider related models such as linear elasticity and Stokes-Darcy coupling, as well as nonlinear problems and a posteriori error analysis.

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N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
352	0.354	$1.028E-01$	–	$1.212E-01$	–	$1.826E-01$	–
1344	0.177	$2.654E-02$	1.953	$3.036E-02$	1.998	$9.225E-02$	0.985
5248	0.088	$6.712E-03$	1.984	$7.660E-03$	1.986	$4.624E-02$	0.996
20736	0.044	$1.688E-03$	1.992	$1.931E-03$	1.988	$2.314E-02$	0.999
82432	0.022	$4.233E-04$	1.995	$4.852E-04$	1.992	$1.157E-02$	1.000
328704	0.011	$1.060E-04$	1.998	$1.217E-04$	1.996	$5.785E-03$	1.000

Table 6.1: Example 1, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
1508	0.139	$3.512E-02$	–	$1.462E-02$	–	$9.630E-02$	–
6228	0.065	$7.984E-03$	1.965	$3.020E-03$	2.093	$4.619E-02$	0.975
13156	0.044	$3.646E-03$	2.025	$1.282E-03$	2.213	$3.154E-02$	0.986
22644	0.034	$2.081E-03$	2.019	$7.049E-04$	2.152	$2.395E-02$	0.991
36324	0.026	$1.296E-03$	1.968	$4.297E-04$	2.058	$1.886E-02$	0.993
51252	0.022	$9.105E-04$	2.020	$2.966E-04$	2.120	$1.585E-02$	0.994
68740	0.019	$6.746E-04$	2.018	$2.171E-04$	2.100	$1.367E-02$	0.995
91380	0.017	$5.083E-04$	1.968	$1.619E-04$	2.041	$1.185E-02$	0.996

Table 6.2: Example 1, quasi-uniform refinement with hexagons.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
272	0.707	$3.866E-02$	–	$3.770E-02$	–	$1.131E-01$	–
1024	0.354	$9.502E-03$	2.025	$7.461E-03$	2.337	$5.747E-02$	0.976
3968	0.177	$2.507E-03$	1.922	$1.851E-03$	2.011	$2.888E-02$	0.993
15616	0.088	$6.419E-04$	1.966	$4.549E-04$	2.025	$1.446E-02$	0.998
61952	0.044	$1.625E-04$	1.982	$1.141E-04$	1.996	$7.231E-03$	1.000
246784	0.022	$4.100E-05$	1.987	$2.867E-05$	1.992	$3.616E-03$	1.000
985088	0.011	$1.029E-05$	1.995	$7.179E-06$	1.998	$1.808E-03$	1.000

Table 6.3: Example 2, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
176	0.800	$6.059E-02$	–	$3.400E-02$	–	$1.352E-01$	–
640	0.431	$1.736E-02$	2.020	$8.099E-03$	2.319	$7.081E-02$	1.045
2432	0.215	$4.347E-03$	1.989	$1.352E-03$	2.571	$3.584E-02$	0.978
9472	0.110	$1.097E-03$	2.050	$3.182E-04$	2.154	$1.796E-02$	1.028
37376	0.055	$2.746E-04$	1.990	$6.277E-05$	2.332	$8.997E-03$	0.993
148480	0.028	$6.873E-05$	2.066	$1.328E-05$	2.317	$4.504E-03$	1.032
591872	0.014	$1.718E-05$	2.015	$2.987E-06$	2.168	$2.253E-03$	1.007
2363392	0.007	$4.295E-06$	2.002	$7.091E-07$	2.077	$1.127E-03$	1.001

Table 6.4: Example 2, quasi-uniform refinement with quadrilaterals.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
352	0.354	$3.626E-03$	$-$	$4.324E-03$	$-$	$9.552E-02$	$-$
1344	0.177	$1.215E-03$	1.578	$1.406E-03$	1.621	$4.802E-02$	0.992
5248	0.088	$3.963E-04$	1.616	$4.522E-04$	1.636	$2.405E-02$	0.998
20736	0.044	$1.275E-04$	1.636	$1.444E-04$	1.647	$1.203E-02$	1.000
82432	0.022	$4.070E-05$	1.648	$4.591E-05$	1.654	$6.014E-03$	1.000
328704	0.011	$1.293E-05$	1.655	$1.454E-05$	1.658	$3.007E-03$	1.000
1312768	0.006	$4.093E-06$	1.659	$4.598E-06$	1.661	$1.504E-03$	1.000

Table 6.5: Example 3, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
64	0.707	$3.088E-02$	$-$	$4.188E-02$	$-$	$2.285E-01$	$-$
224	0.495	$1.211E-02$	2.624	$1.601E-02$	2.696	$1.267E-01$	1.654
832	0.277	$4.451E-03$	1.722	$5.858E-03$	1.729	$6.757E-02$	1.081
3200	0.143	$1.337E-03$	1.812	$1.797E-03$	1.780	$3.447E-02$	1.014
12544	0.072	$3.985E-04$	1.765	$5.423E-04$	1.747	$1.732E-02$	1.003
49664	0.036	$1.214E-04$	1.720	$1.659E-04$	1.714	$8.673E-03$	1.001
197632	0.018	$3.744E-05$	1.698	$5.125E-05$	1.695	$4.338E-03$	1.000
788480	0.009	$1.164E-05$	1.686	$1.594E-05$	1.685	$2.169E-03$	1.000

Table 6.6: Example 3, quasi-uniform refinement with distorted squares.

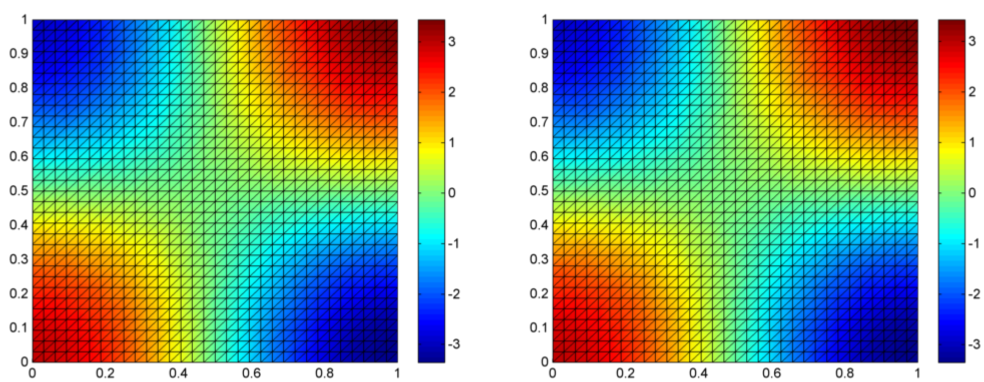

Figure 6.1: Example 1, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 20736$).

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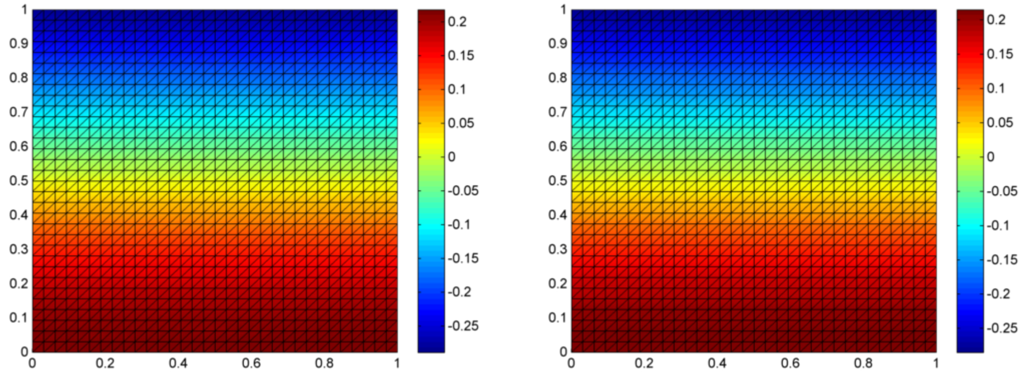

Figure 6.2: Example 1, $\widehat{p}_h$ and p for a mesh with triangles ($N = 20736$).

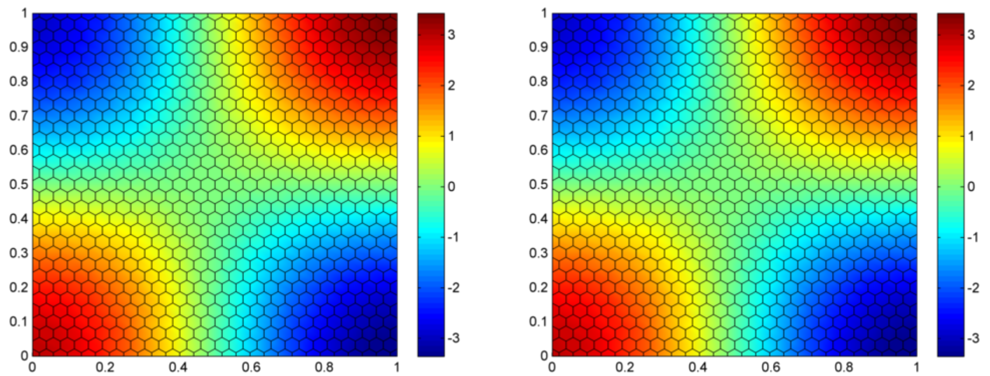

Figure 6.3: Example 1, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with hexagons ($N = 13156$).

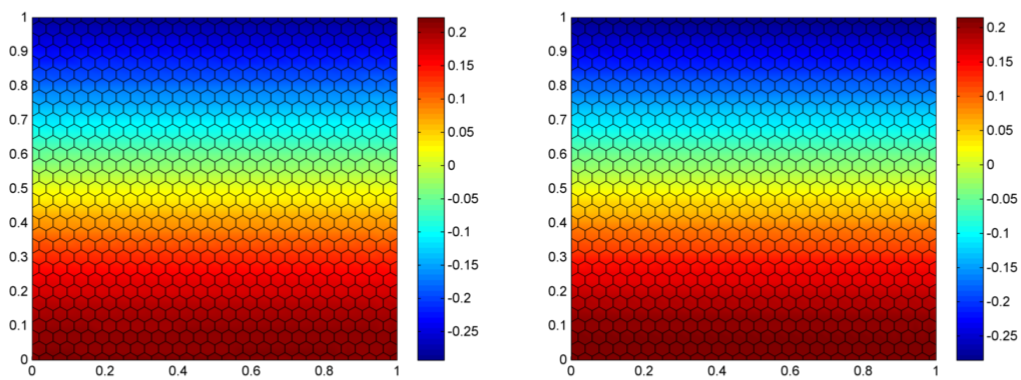

Figure 6.4: Example 1, $\widehat{p}_h$ and p for a mesh with hexagons ($N = 13156$).

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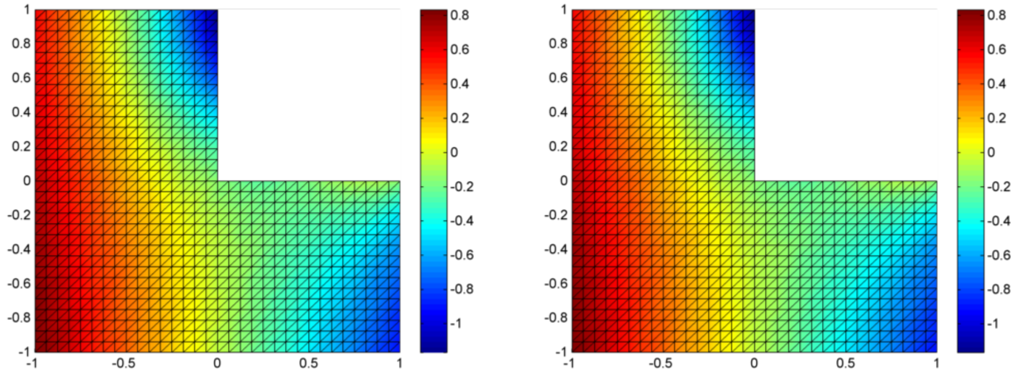

Figure 6.5: Example 2, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 15616$).

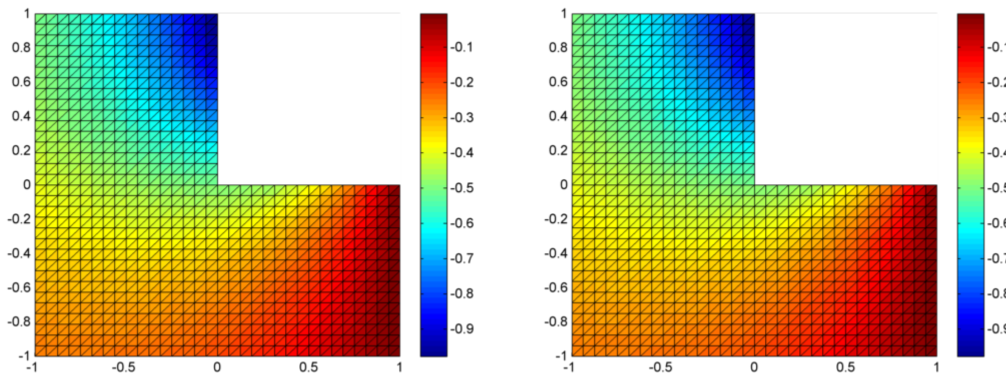

Figure 6.6: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 15616$).

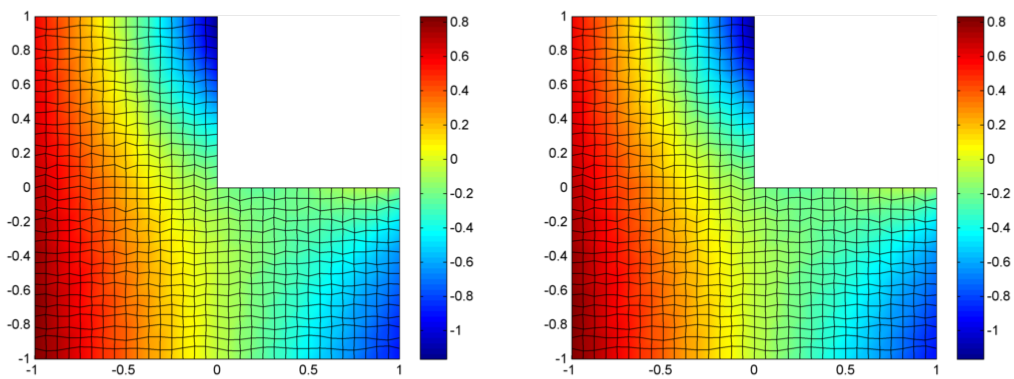

Figure 6.7: Example 2, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with quadrilaterals ($N = 9472$).

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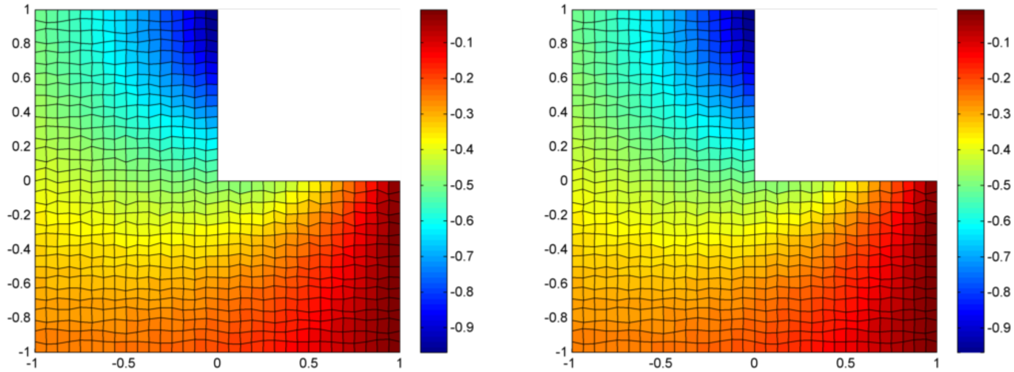

Figure 6.8: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with quadrilaterals ($N = 9472$).

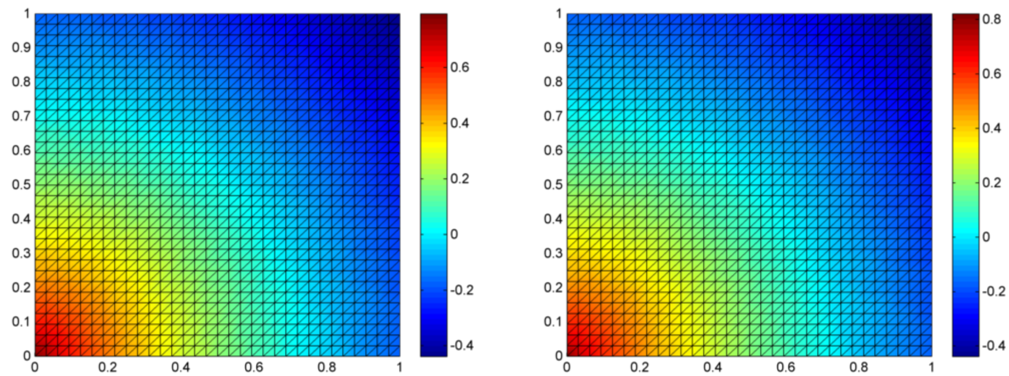

Figure 6.9: Example 3, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 20736$).

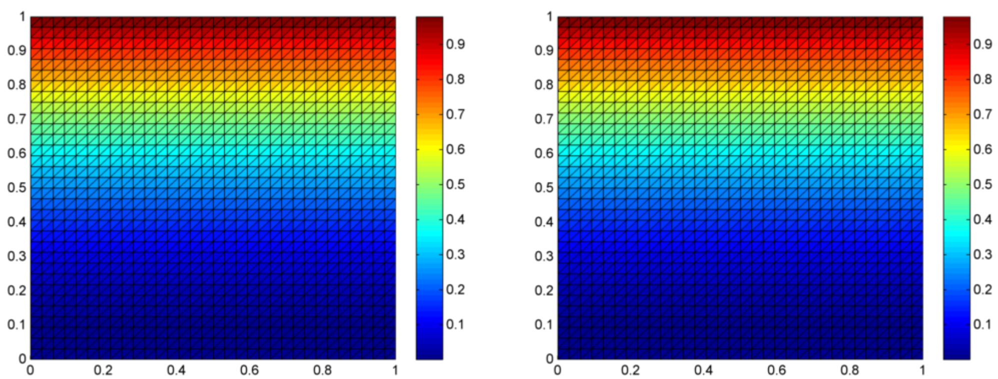

Figure 6.10: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 20736$).

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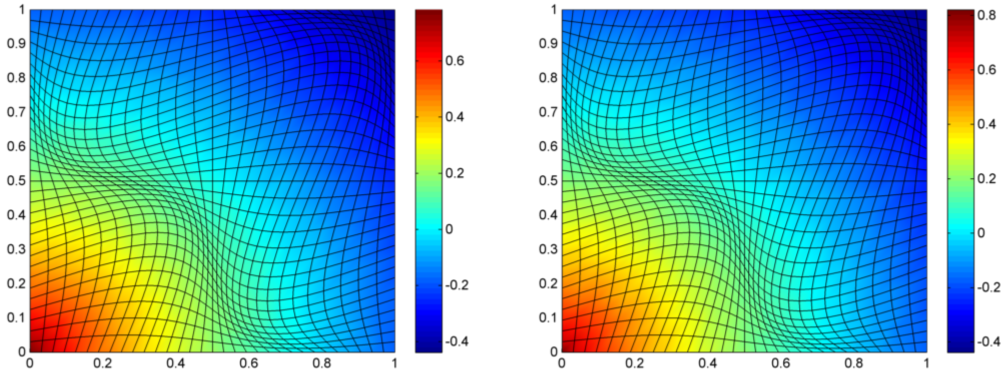

Figure 6.11: Example 3, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with distorted squares ($N = 12544$).

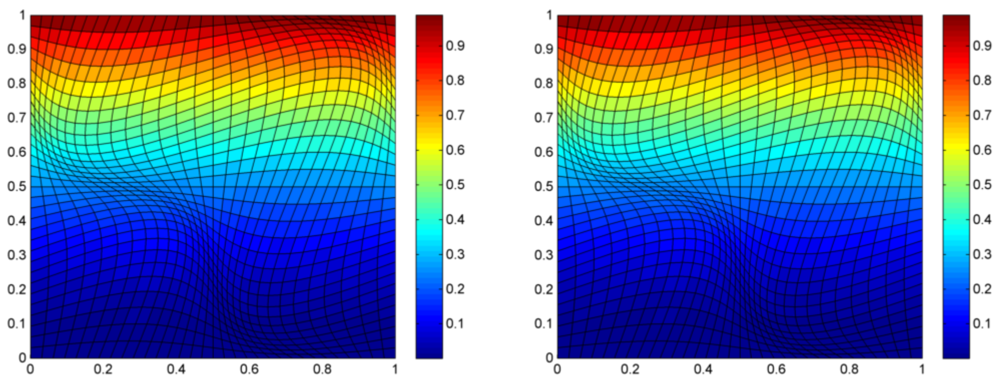

Figure 6.12: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with distorted squares ($N = 12544$).

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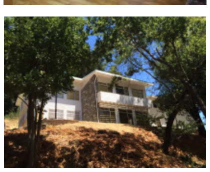

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