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Marco Discacciati, Ricardo Oyarzúa

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A conforming mixed finite element method for the Navier-Stokes/Darcy coupled problem *

Marco Discacciati[†] and Ricardo Oyarzúa[‡]

Abstract

In this paper we develop the a priori analysis of a mixed finite element method for the coupling of fluid flow with porous media flow. Flows are governed by the Navier-Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We consider the standard mixed formulation in the Navier-Stokes domain and the dual-mixed one in the Darcy region, which yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. The finite element subspaces defining the discrete formulation employ Bernardi-Raugel and Raviart-Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange multiplier. We show stability, convergence, and a priori error estimates for the associated Galerkin scheme. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical rates of convergence are reported.

Keywords: mixed finite element, Navier-Stokes equation, Darcy equation

Mathematics Subject Classifications (1991): 65N15, 65N30, 76D05, 76S05

1 Introduction

The devising of suitable numerical methods for the coupling of fluid flow with porous media flow has become a very active research area during the last decades, mostly due to the relevance of this physical process for a variety of phenomena in medicine (filtration process of blood through vessel walls), geoscience (flow of a river and its riverbed) and industry (oil extraction process), to name a few.

One of the most studied models for this type of phenomena is the Stokes-Darcy coupled system, which consists in a set of equations where the Stokes model (for the free fluid flow) is coupled with the Darcy model (for the fluid flow in the porous medium) through a set of interface conditions, namely, continuity of the normal velocities (mass conservation), balance of normal forces, and the Beavers-Joseph-Saffman law. So far, several numerical methods have been

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[†]Laboratori de Càlcul Numèric (LaCàN), Departmento de Matemática Aplicada III, Universitat Politècnica de Catalunya, E-08034, Barcelona, Spain, email: marco.discacciati@upc.edu

[‡]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile email: royarzua@ubiobio.cl

developed to approximate the solution of the Stokes-Darcy coupled problem (see for instance [9, 10, 12, 13, 14, 16, 19, 20, 21, 25, 29, 30, 32, 33, 34, 4]), most of them based on appropriate combinations of stable elements for both media. In this direction, the first theoretical results go back to [32] and [14]. In [14] the authors introduce an iterative subdomain method employing the standard velocity-pressure formulation for the Stokes equation and the primal one in the Darcy domain, whereas in [32] the authors apply the velocity-pressure formulation in the free fluid domain and the dual-mixed velocity-pressure formulation in the porous medium, yielding the introduction of the trace of the porous medium pressure on the interface as an additional unknown. Next, in [19] and [21] new mixed finite element discretizations of the variational formulation from [32] have been introduced and analyzed. The stability of a specific Galerkin method, employing the Bernardi-Raugel and Raviart-Thomas elements for the free fluid and the fluid in the porous medium, respectively, is the main result in [19]. The results from [19] are improved in [21] where it is shown that the use of any pair of stable Stokes and Darcy elements implies the well-posedness of the Stokes-Darcy Galerkin scheme. In particular, this includes Hood-Taylor, Bernardi-Raugel, and MINI element for the Stokes region, and Raviart-Thomas of any order for the Darcy domain. The analysis in [21] hinges on the fact that the linear operator defining the continuous variational formulation is given by a compact perturbation of an invertible linear mapping.

The purpose of the present work is to contribute to the development of new numerical methods for the coupling of fluid flow with porous media flow by extending the approach in [19] to the Navier-Stokes/Darcy coupled problem. Up to the authors' knowledge, the first works in developing numerical methods for the Navier-Stokes/Darcy coupled problem are [24] and [3]. In [24] the authors introduce and analyze a DG discretization for the nonlinear coupled problem considering the usual nonsymmetric interior penalty Galerkin (NIPG), symmetric interior penalty Galerkin (SIPG), and incomplete interior penalty Galerkin (IIPG) bilinear forms for the discretization of the Laplacian in both media and the upwind Lesaint-Raviart discretization of the convective term in the free fluid domain. In [3] the authors extend the approach in [14] (see also [12]) and introduce an iterative subdomain method employing the velocity-pressure formulation for the Navier-Stokes equation and the primal one for the Darcy equation. They approximate the coupled nonlinear problem using conformal finite elements in both media and study the convergence properties Newton-like iterative methods for solving this problem. We point out that, differently from [32] and [19], the approach adopted in [24] and [3] avoids the introduction of Lagrange multipliers to impose the continuity of the normal velocity of the fluid through the interface. Indeed, this condition is imposed weakly using the primal formulation of the Darcy problem for the sole pressure unknown.

According to the above discussion, in this paper we extend the analysis developed in [19] and study a conforming mixed finite element method for the Navier-Stokes/Darcy coupled problem. We consider the standard velocity-pressure formulation for the Navier-Stokes equation and the dual-mixed formulation for the Darcy equation, which yields the velocity and the pressure of the fluid in both media as the main unknowns of the coupled system. Since one of the interface conditions becomes essential, we proceed similarly to [32] and [19] and incorporate the trace of the porous medium pressure as an additional unknown. To analyze the continuous problem we linearize the coupled system by considering the Oseen linearization in the free fluid domain and apply the classical Babuška-Brezzi theory and Banach's fixed point theorem to establish the well-posedness of the nonlinear coupled problem. Using similar arguments we prove the well-posedness of the discrete problem for a specific choice of discrete spaces, namely, Bernardi-

Raugel elements for the velocity in the free fluid region, Raviart-Thomas elements of lowest order for the filtration velocity in the porous media, piecewise constants with null mean value for the pressures, and continuous piecewise linear elements for the Lagrange multiplier on the interface. It is important to remark that the interpolation properties of the Raviart-Thomas and Bernardi-Raugel operators, mainly those holding on the edges of the triangulations (see Eqs. (3.11), (4.2), and (4.7) in [19]), play a key role in the proof of one of the required discrete inf-sup conditions. We point out here that, nevertheless, this approach can not be extended to arbitrary finite element subspaces as in [21] since the operator defining the continuous variational formulation is nonlinear, and then, the classical result on projection methods for Fredholm operators of index zero is not applicable.

The rest of this paper is organized as follows. In Section 2 we introduce the continuous coupled problem, its weak formulation, the corresponding variational system and we prove its well-posedness. In Section 3 we define the Galerkin scheme, we prove its well-posedness and derive the corresponding Cea's estimate and rate of convergence. Finally, several numerical examples illustrating the performance of the method and confirming the theoretical order of convergence are reported in Section 4.

2 The continuous problem

2.1 Statement of the model problem

For simplicity of exposition we set the problem in $\mathbb{R}^2$. However, our study can be extended to the 3D case with few modifications that we will point out in the paper.

In order to describe the geometry, we let Ω_S and Ω_D be two bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \emptyset$ and $\Omega_S \cap \Omega_D = \emptyset$. Then, let $\Gamma_S := \partial\Omega_S \backslash \bar{\Sigma}$, $\Gamma_D := \partial\Omega_D \backslash \bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_S \cup \Sigma \cup \Omega_D$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Figure 2.1).

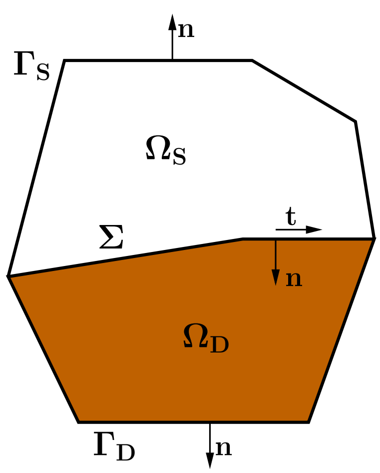

Figure 2.1: Domains for the 2D Navier-Stokes/Darcy model

The free/porous-medium flow problem can be modelled by coupling the Navier-Stokes and the Darcy equations. More precisely, in the free fluid domain Ω_S, the motion of the fluid can be described by the incompressible Navier-Stokes equations:

$$\begin{aligned} \boldsymbol{\sigma}_S &= -p_S \mathbf{I} + 2\mu\, \mathbf{e}(\mathbf{u}_S) \quad \text{in } \Omega_S, \\ -\mathbf{div}\, \boldsymbol{\sigma}_S + \rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S &= \mathbf{f}_S \quad \text{in } \Omega_S, \\ \operatorname{div} \mathbf{u}_S &= 0 \quad \text{in } \Omega_S, \end{aligned} \tag{2.1}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, ρ is its density, $\mathbf{u}_S$ is the fluid velocity, p_S the pressure, $\boldsymbol{\sigma}_S$ is the Cauchy stress tensor, $\mathbf{I}$ is the 2×2 identity matrix, $\mathbf{f}_S$ is a given external force, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}$ is the strain tensor:

$$\mathbf{e}(\mathbf{u}_S) := \frac{1}{2}\Big(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^t\Big),$$

where the superscript ^{t} denotes transposition.

In the porous medium Ω_D we consider the following Darcy model:

$$\begin{aligned} \mathbf{K}^{-1}\mathbf{u}_D &= -\nabla p_D + \mathbf{f}_D \quad \text{in } \Omega_D, \\ \operatorname{div} \mathbf{u}_D &= 0 \quad \text{in } \Omega_D, \end{aligned} \tag{2.2}$$

where $\mathbf{u}_D$ is the Darcy velocity (specific discharge), p_D is the pressure, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_D)$ is a symmetric and uniformly positive definite tensor in Ω_D representing the intrinsic permeability $\boldsymbol{\kappa}$ of the porous medium divided by the dynamic viscosity μ of the fluid. Throughout the paper we assume that there exists $C > 0$ such that

$$\boldsymbol{\xi} \cdot \mathbf{K}(x)\, \boldsymbol{\xi} \geq C\, \|\boldsymbol{\xi}\|^2,$$

for almost all $x \in \Omega_D$, and for all $\boldsymbol{\xi} \in \mathbb{R}^2$. Finally, $\mathbf{f}_D$ is a given external force that accounts for gravity, i.e. $\mathbf{f}_D = \rho\mathbf{g}$ where ρ is the density of the fluid and $\mathbf{g}$ is the gravity acceleration.

The transmission conditions that couple the Navier-Stokes and the Darcy models through the interface Σ are given by

$$\begin{aligned} \mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad \text{on } \Sigma, \\ \boldsymbol{\sigma}_S\, \mathbf{n} + \frac{\alpha_d \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}}\, (\mathbf{u}_S \cdot \mathbf{t})\, \mathbf{t} &= -p_D\, \mathbf{n} \quad \text{on } \Sigma, \end{aligned} \tag{2.3}$$

where α_d is a dimensionless constant which depends only on the geometrical characteristics of the porous medium.

The first condition in (2.3) is a consequence of the incompressibility of the fluid and of the conservation of mass across Σ. The second transmission condition on Σ can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_S\, \mathbf{n}) \cdot \mathbf{n} = -p_D \quad \text{and} \quad (\boldsymbol{\sigma}_S\, \mathbf{n}) \cdot \mathbf{t} = -\frac{\alpha_d \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}}\, (\mathbf{u}_S \cdot \mathbf{t}) \quad \text{on } \Sigma. \tag{2.4}$$

The first equation in (2.4) corresponds to the balance of normal forces [13, 24, 32], whereas the second one is known as the Beavers-Joseph-Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_D \cdot \mathbf{t}$ is negligible). We refer to [5, 28, 37] for further details on this interface condition.

To complete the definition of the Navier-Stokes/Darcy problem, suitable boundary conditions must be imposed. For simplicity in our analysis we consider

$$\mathbf{u}_\mathrm{S} = \mathbf{0} \text{ on } \Gamma_\mathrm{S} \quad \text{and} \quad \mathbf{u}_\mathrm{D}\cdot\mathbf{n} = 0 \text{ on } \Gamma_\mathrm{D}. \tag{2.5}$$

Other possible choices are discussed e.g. in [12].

2.2 The variational formulation

Let us first introduce some notation. Given $\star \in \{\mathrm{S},\mathrm{D}\}$, we denote

$$(u,v)_\star := \int_{\Omega_\star} u\,v, \qquad (\mathbf{u},\mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}\,,$$

where, given two arbitrary tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, $\boldsymbol{\sigma} : \boldsymbol{\tau} = \mathrm{tr}(\boldsymbol{\sigma}^t\boldsymbol{\tau}) = \sum_{i,j=1}^2 \sigma_{ij}\tau_{ij}$.

We use the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain and $r\in\mathbb{R}$, we define $\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2$. For $r=0$ we write $\mathbf{L}^2(\mathcal{O})$ and $L^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$ and $H^0(\Gamma)$, respectively, where Γ is a closed Lipschitz curve. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$ and $\mathbf{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{\mathbf{w}\in\mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\;\mathbf{w}\in L^2(\mathcal{O})\right\},$$

with norm $\|\cdot\|_{\mathrm{div}\,,\mathcal{O}}$, is standard in the realm of mixed problems (see, e.g. [8]).

On the other hand, the symbol for the $L^2(\Gamma)$ inner product

$$\langle \xi,\lambda\rangle_\Gamma := \int_\Gamma \xi\,\lambda \quad \forall\,\xi,\,\lambda\in L^2(\Gamma),$$

will also be employed for their respective extension as the duality product $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$. In addition, given two Hilbert spaces H_1 and H_2, the product space $H_1\times H_2$ will be endowed with the norm $\|\cdot\|_{H_1\times H_2} = \|\cdot\|_{H_1} + \|\cdot\|_{H_2}$. Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^2$, $\mathbb{P}_k(S)$ stands for the space of polynomials defined on S of degree $\le k$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

The unknowns in the variational formulation of the Navier-Stokes/Darcy problem and the corresponding spaces will be:

$$\mathbf{u}_\mathrm{S}\in\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}), \quad p_\mathrm{S}\in L^2(\Omega_\mathrm{S}), \quad \mathbf{u}_\mathrm{D}\in\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D}), \quad p_\mathrm{D}\in L^2(\Omega_\mathrm{D}),$$

where

$$\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) := \left\{\mathbf{v}\in\mathbf{H}^1(\Omega_\mathrm{S}) : \quad \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{S}\right\},$$

$$\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D}) := \{\mathbf{v}\in\mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_\mathrm{D}\}.$$

In addition, we need to define a further unknown on the coupling boundary:

$$\lambda := p_\mathrm{D} \in H^{1/2}(\Sigma). \tag{2.6}$$

Note that, in principle, the space for p_D does not allow enough regularity for the trace λ to exist. However, the solution of (2.2) has the pressure in $H^1(\Omega_\mathrm{D})$.

Next, for the derivation of the weak formulation of (2.1)-(2.3), (2.5) we write $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, we define the space

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \quad \int_\Omega q = 0 \right\},$$

and we group the unknowns and spaces as follows:

$$\begin{aligned}
&\mathbf{u} := (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}; \Omega_{\mathrm{D}}) , \\
&(p, \lambda) \in \mathbf{Q} := L_0^2(\Omega) \times H^{1/2}(\Sigma) ,
\end{aligned}$$

where $p := p_{\mathrm{S}} \chi_{\Omega_{\mathrm{S}}} + p_{\mathrm{D}} \chi_{\Omega_{\mathrm{D}}}$, with $\chi_{\Omega_\star}$ being the characteristic function:

$$\chi_{\Omega_\star} := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \backslash \overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{D}, \mathrm{S}\}$.

Hence, we proceed as in [32] to find the mixed variational formulation: Find $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned}
\mathbf{a}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p, \lambda)) &= \mathbf{f}(\mathbf{v}) && \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H} , \\
\mathbf{b}(\mathbf{u}, (q, \xi)) &= 0 && \forall\, (q, \xi) \in \mathbf{Q} ,
\end{aligned} \tag{2.7}$$

where $\mathbf{a} : \mathbf{H} \times (\mathbf{H} \times \mathbf{H}) \to \mathbb{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ are the forms defined by

$$\begin{aligned}
\mathbf{a}(\mathbf{w}; \mathbf{u}, \mathbf{v}) &:= A_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) + O_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}}; \mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) + A_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}}), \\
\mathbf{b}(\mathbf{v}, (q, \xi)) &:= -(q, \mathrm{div}\, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (q\, \mathrm{div}\, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{S}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_\Sigma ,
\end{aligned}$$

with

$$\begin{aligned}
&A_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) := 2\,\mu\, (\mathbf{e}(\mathbf{u}_{\mathrm{S}}), \mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}} + \frac{\alpha_d \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}} \langle \mathbf{u}_{\mathrm{S}} \cdot \mathbf{t}, \mathbf{v}_{\mathrm{S}} \cdot \mathbf{t} \rangle_\Sigma, \\
&O_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}}; \mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) := \rho\, ((\mathbf{w}_{\mathrm{S}} \cdot \nabla) \mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}, \\
&A_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}}) := (\mathbf{K}^{-1} \mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}},
\end{aligned}$$

and $\mathbf{f}(\mathbf{v})$ is the linear functional $\mathbf{f} : \mathbf{H} \to \mathbb{R}$ defined as

$$\mathbf{f}(\mathbf{v}) = (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \qquad \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H}.$$

Let us observe that the forms A_{S}, A_{D}, O_{S} and $\mathbf{b}$ are continuous: there exist positive constants C_{S}, C_{D}, C_O and $C_{\mathbf{b}}$, such that

$$\begin{aligned}
&|A_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})| \le C_{\mathrm{S}} \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}, \\
&|O_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}}; \mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})| \le C_O \rho \|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}, \\
&|A_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})| \le C_{\mathrm{D}} \|\mathbf{u}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}} \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}, \\
&|\mathbf{b}(\mathbf{v}, (q, \xi))| \le C_{\mathbf{b}} \|\mathbf{v}\|_{\mathbf{H}} \|(q, \xi)\|_{\mathbf{Q}}.
\end{aligned} \tag{2.8}$$

In addition, the continuity of the functional $\mathbf{f}$ is straightforward:

$$|\mathbf{f}(\mathbf{v})| \leq (\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}})\|\mathbf{v}\|_{\mathbf{H}}. \tag{2.9}$$

We end this section by observing that owing to the well known Korn and Poincaré inequalities (see, e.g. [15]) and the fact that $\mathbf{K}^{-1}$ is symmetric and positive definite, we easily obtain that there exist constants α_{S}, $\alpha_{\mathrm{D}} > 0$, depending only on Ω_{S} and the tensor $\mathbf{K}$, respectively, such that

$$A_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) \geq 2\,\mu\,\alpha_{\mathrm{S}}\,\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}^2 \quad \text{and} \quad A_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}}) \geq \alpha_{\mathrm{D}}\|\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}^2, \tag{2.10}$$

for all $\mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H}$.

2.3 The Oseen-Darcy coupled problem

In this section we study the well-posedness of the linearized version of problem (2.7): Given $\mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}}) \in \mathbf{H}$, with $\operatorname{div}\mathbf{w}_{\mathrm{S}} = 0$ in Ω_{S}, find $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\mathbf{w}; \mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p, \lambda)) &= \mathbf{f}(\mathbf{v}) \qquad && \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H}\,, \\ \mathbf{b}(\mathbf{u}, (q, \xi)) &= 0 && \forall\, (q, \xi) \in \mathbf{Q}\,, \end{aligned} \tag{2.11}$$

which corresponds to the variational formulation of the Oseen-Darcy coupled problem. Having studied the well-posedness of problem (2.11), in what follows we will be able to reformulate (2.7) as an equivalent fixed-point problem, and as a result, to apply the Classical Banach's fixed point theorem to prove the solvability of (2.7).

In the forthcoming analysis we will make use of the classical Babuška-Brezzi theory [7]. To do this, we first introduce some merely technical results and further notations.

2.3.1 Preliminaries

Given $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\operatorname{div}; \Omega_{\mathrm{D}})$, the boundary condition $\mathbf{v}_{\mathrm{D}} \cdot \mathbf{n} = 0$ on Γ_{D} means

$$\langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, E_{0,\mathrm{D}}(\zeta) \rangle_{\partial\Omega_{\mathrm{D}}} = 0 \qquad \forall\, \zeta \in H_{00}^{1/2}(\Gamma_{\mathrm{D}}), \tag{2.12}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega_{\mathrm{D}}}$ stands for the usual duality pairing between $H^{-1/2}(\partial\Omega_{\mathrm{D}})$ and $H^{1/2}(\partial\Omega_{\mathrm{D}})$ with respect to the $L^2(\partial\Omega_{\mathrm{D}})$-inner product, $E_{0,\mathrm{D}} : H^{1/2}(\Gamma_{\mathrm{D}}) \to L^2(\partial\Omega_{\mathrm{D}})$ is the extension operator defined by

$$E_{0,\mathrm{D}}(\zeta) := \begin{cases} \zeta & \text{on} \quad \Gamma_{\mathrm{D}} \\ 0 & \text{on} \quad \Sigma \end{cases} \qquad \forall\, \zeta \in H^{1/2}(\Gamma_{\mathrm{D}})\,,$$

and

$$H_{00}^{1/2}(\Gamma_{\mathrm{D}}) = \left\{ \zeta \in H^{1/2}(\Gamma_{\mathrm{D}}) : \quad E_{0,\mathrm{D}}(\zeta) \in H^{1/2}(\partial\Omega_{\mathrm{D}}) \right\},$$

endowed with the norm $\|\zeta\|_{1/2,00,\Gamma_{\mathrm{D}}} := \|E_{0,\mathrm{D}}(\zeta)\|_{1/2,\partial\Omega_{\mathrm{D}}}$.

As a consequence, it is not difficult to prove (see e.g. Section 2 in [16]) that the restriction of $\mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}$ to Σ can be identified with an element of $H^{-1/2}(\Sigma)$, namely

$$\langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_{\Sigma} := \langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, E_{\mathrm{D}}(\xi) \rangle_{\partial\Omega_{\mathrm{D}}} \qquad \forall\, \xi \in H^{1/2}(\Sigma)\,, \tag{2.13}$$

where $E_{\mathrm{D}} : H^{1/2}(\Sigma) \to H^{1/2}(\partial\Omega_{\mathrm{D}})$ is any bounded extension operator. In particular, given $\xi \in H^{1/2}(\Sigma)$, one could define $E_{\mathrm{D}}(\xi) := z|_{\partial\Omega_{\mathrm{D}}}$, where $z \in H^1(\Omega_{\mathrm{D}})$ is the unique solution of the boundary value problem: $\Delta z = 0$ in Ω_{D}, $z = \xi$ on Σ, $\nabla z \cdot \mathbf{n} = 0$ on Γ_{D}. In addition, one can show (see [16, Lemma 2.2]) that for all $\psi \in H^{1/2}(\partial\Omega_{\mathrm{D}})$, there exist unique elements $\psi_\Sigma \in H^{1/2}(\Sigma)$ and $\psi_{\Gamma_{\mathrm{D}}} \in H^{1/2}_{00}(\Gamma_{\mathrm{D}})$ such that

$$\psi = E_{\mathrm{D}}(\psi_\Sigma) + E_{0,\mathrm{D}}(\psi_{\Gamma_{\mathrm{D}}}), \tag{2.14}$$

and

$$\begin{aligned} C_1 \left(\|\psi_\Sigma\|_{1/2,\Sigma} + \|\psi_{\Gamma_{\mathrm{D}}}\|_{1/2,00,\Gamma_{\mathrm{D}}} \right) &\leq \|\psi\|_{1/2,\partial\Omega_{\mathrm{D}}} \\ &\leq C_2 \left(\|\psi_\Sigma\|_{1/2,\Sigma} + \|\psi_{\Gamma_{\mathrm{D}}}\|_{1/2,00,\Gamma_{\mathrm{D}}} \right). \end{aligned} \tag{2.15}$$

Finally we observe that, since $H^{1/2}(\partial\Omega_{\mathrm{S}})$ is continuously embedded into $L^4(\partial\Omega_{\mathrm{S}})$ and the trace operator is continuous, the following inequality holds:

$$\|\mathbf{v}_{\mathrm{S}}\|_{4,\Sigma} = \|\mathbf{v}_{\mathrm{S}}\|_{4,\partial\Omega_{\mathrm{S}}} \leq C_{sob}\|\mathbf{v}_{\mathrm{S}}\|_{1/2,\partial\Omega_{\mathrm{S}}} \leq C_{sob}C_{trace}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}, \tag{2.16}$$

for all $\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$.

2.3.2 Well-posedness of the Oseen-Darcy problem

We begin by proving the continuous inf-sup condition for $\mathbf{b}$. For its proof we proceed as in [19, Lemma 2.1] and in [22, Lemma 3.3].

Lemma 2.1 *There exists $\beta > 0$ such that*

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\, \|(q,\xi)\|_{\mathbf{Q}} \qquad \forall\, (q,\xi) \in \mathbf{Q}. \tag{2.17}$$

Proof. Let $(q,\xi) \in \mathbf{Q}$. Since $q \in L^2_0(\Omega)$, it is well known (see, e.g. Corollary 2.4 in Chapter I of [23]) that there exists $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ such that $\operatorname{div}\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \leq c\,\|q\|_{0,\Omega}$. Setting $\hat{\mathbf{v}} = (\hat{\mathbf{v}}_{\mathrm{S}}, \hat{\mathbf{v}}_{\mathrm{D}})$ with $\hat{\mathbf{v}}_\star = \mathbf{z}|_{\Omega_\star}$ for $\star \in \{\mathrm{S},\mathrm{D}\}$, we find that $\hat{\mathbf{v}}_{\mathrm{S}}\cdot\mathbf{n} = \hat{\mathbf{v}}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ and $\|\hat{\mathbf{v}}\|_{\mathbf{H}} \leq c\,\|q\|_{0,\Omega}$, and hence

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \frac{|\,\mathbf{b}(\hat{\mathbf{v}},(q,\xi))\,|}{\|\hat{\mathbf{v}}\|_{\mathbf{H}}} = \frac{\|q\|^2_{0,\Omega}}{\|\hat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\,\|q\|_{0,\Omega}. \tag{2.18}$$

On the other hand, given $\phi \in H^{-1/2}(\Sigma)$, we define $\eta \in H^{-1/2}(\partial\Omega_{\mathrm{D}})$ as

$$\langle \eta, \mu\rangle_{\partial\Omega_{\mathrm{D}}} := \langle \phi, \mu_\Sigma\rangle_\Sigma \qquad \forall\, \mu \in H^{1/2}(\partial\Omega_{\mathrm{D}}), \tag{2.19}$$

where μ_Σ is given by the decomposition (2.14). It is not difficult to see that

$$\langle \eta, E_{0,\mathrm{D}}(\rho)\rangle_{\partial\Omega_{\mathrm{D}}} = 0 \qquad \forall\, \rho \in H^{1/2}_{00}(\Gamma_{\mathrm{D}}), \tag{2.20}$$

$$\langle \eta, E_{\mathrm{D}}(\xi)\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \phi, \xi\rangle_\Sigma\,, \tag{2.21}$$

and

$$\|\eta\|_{-1/2,\partial\Omega_{\mathrm{D}}} \leq C\,\|\phi\|_{-1/2,\Sigma}. \tag{2.22}$$

We set $\tilde{\mathbf{v}}_{\mathrm{D}} := \nabla z$ in Ω_{D}, with $z \in H^1(\Omega_{\mathrm{D}})$ being the unique solution of the boundary value problem:

$$-\Delta z = -\frac{1}{|\Omega_{\mathrm{D}}|}\langle \eta, 1\rangle_{\partial\Omega_{\mathrm{D}}} \text{ in } \Omega_{\mathrm{D}}, \quad \nabla z \cdot \mathbf{n} = \eta \text{ on } \partial\Omega_{\mathrm{D}}, \quad \int_{\Omega_{\mathrm{D}}} z = 0.$$

Observe that $\operatorname{div} \tilde{\mathbf{v}}_{\mathrm{D}} = \frac{1}{|\Omega_{\mathrm{D}}|}\langle \eta, 1\rangle_{\partial\Omega_{\mathrm{D}}} \in \mathbb{P}_0(\Omega_{\mathrm{D}})$, $\tilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n} = \eta$ on $\partial\Omega_{\mathrm{D}}$, and $\|\tilde{\mathbf{v}}_{\mathrm{D}}\|_{\operatorname{div},\Omega_{\mathrm{D}}} \leq C\,\|\eta\|_{-1/2,\partial\Omega_{\mathrm{D}}} \leq C\,\|\phi\|_{-1/2,\Sigma}$. In addition, owing to (2.13), (2.20) and (2.21), we find that

$$\langle \tilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, \xi\rangle_{\Sigma} = \langle \tilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, E_{\mathrm{D}}(\xi)\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \eta, E_{\mathrm{D}}(\xi)\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \phi, \xi\rangle_{\Sigma},$$

and

$$\langle \tilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, E_{0,\mathrm{D}}(\rho)\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \eta, E_{0,\mathrm{D}}(\rho)\rangle_{\partial\Omega_{\mathrm{D}}} = 0 \qquad \forall \rho \in H^{1/2}_{00}(\Gamma_{\mathrm{D}}).$$

The latter means that $\tilde{\mathbf{v}}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\operatorname{div};\Omega_{\mathrm{D}})$. In this way, defining $\tilde{\mathbf{v}} := (\mathbf{0}, \tilde{\mathbf{v}}_{\mathrm{D}}) \in \mathbf{H}$, we obtain

$$\begin{aligned}\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} &\geq \frac{|\,\mathbf{b}(\tilde{\mathbf{v}},(q,\xi))\,|}{\|\tilde{\mathbf{v}}\|_{\mathbf{H}}} = \frac{|\,\langle \phi, \xi\rangle_{\Sigma} + \frac{1}{|\Omega_{\mathrm{D}}|}\langle \eta, 1\rangle_{\partial\Omega_{\mathrm{D}}} \int_{\Omega_{\mathrm{D}}} q\,|}{\|\tilde{\mathbf{v}}_{\mathrm{D}}\|_{\operatorname{div},\Omega_{\mathrm{D}}}}\\ &\geq c_2\,\frac{|\,\langle \phi, \xi\rangle_{\Sigma}\,|}{\|\phi\|_{-1/2,\Sigma}} - c_3\,\|q\|_{0,\Omega},\end{aligned}$$

and using that $\phi \in H^{-1/2}(\Gamma_2)$ is arbitrary, we get

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq c_2\,\|\xi\|_{1/2,\Sigma} - c_3\,\|q\|_{0,\Omega}. \tag{2.23}$$

Finally, after a simple computation we deduce that (2.18) and (2.23) imply (2.17) with β depending on c_1, c_2, and c_3. □

Now, let us consider the subspace

$$\mathbf{V} := \{\,\mathbf{v}\in\mathbf{H} : \quad \mathbf{b}(\mathbf{v},(q,\xi)) = 0 \quad \forall\,(q,\xi)\in\mathbf{Q}\,\}, \tag{2.24}$$

which corresponds to the kernel of the bilinear form $\mathbf{b}$. According to the definition of $\mathbf{b}$, we observe that $\mathbf{v} = (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{V}$, if and only if,

$$(q, \operatorname{div}\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (q, \operatorname{div}\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} = 0 \qquad \forall\, q \in L^2_0(\Omega)$$

and

$$\langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_{\Sigma} = 0 \qquad \forall\,\xi\in H^{1/2}(\Sigma).$$

Then, noting that $L^2(\Omega) = L^2_0(\Omega)\oplus\mathbb{R}$, and taking $\xi\in\mathbb{R}$ in the latter equation, we deduce that

$$(q, \operatorname{div}\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (q, \operatorname{div}\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} = 0 \qquad \forall\, q \in L^2(\Omega),$$

which implies

$$\operatorname{div}\mathbf{v}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \quad \operatorname{div}\mathbf{v}_{\mathrm{D}} = 0 \quad \text{in} \quad \Omega_{\mathrm{D}}. \tag{2.25}$$

Next, we establish the ellipticity of $\mathbf{a}(\mathbf{w};\cdot,\cdot)$ on $\mathbf{V}$ for a suitable $\mathbf{w}\in\mathbf{H}$.

Lemma 2.2 *Let* $\mathbf{w} := (\mathbf{w}_\mathrm{S}, \mathbf{w}_\mathrm{D}) \in \mathbf{H}$*, such that* $\operatorname{div} \mathbf{w}_\mathrm{S} = 0$ *in* Ω_S *and*

$$\|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq \frac{2\mu\alpha_\mathrm{S}}{\rho C_{trace}^3 C_{sob}^2}. \tag{2.26}$$

Then, there exists $\alpha > 0$*, such that*

$$\mathbf{a}(\mathbf{w}; \mathbf{v}, \mathbf{v}) \;\geq\; \alpha\,\|\mathbf{v}\|_{\mathbf{H}}^2 \qquad \forall\,\mathbf{v} \in \mathbf{V}. \tag{2.27}$$

Proof. Let $\mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{H}$ satisfying (2.26) and $\operatorname{div} \mathbf{w}_\mathrm{S} = 0$ in Ω_S. Integrating by parts, it is easy to see that

$$O_\mathrm{S}(\mathbf{w}_\mathrm{S}; \mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{S}) = \frac{\rho}{2}\int_\Sigma (\mathbf{w}_\mathrm{S}\cdot\mathbf{n})|\mathbf{v}_\mathrm{S}|^2 \geq -\frac{\rho}{2}\left|\int_\Sigma (\mathbf{w}_\mathrm{S}\cdot\mathbf{n})|\mathbf{v}_\mathrm{S}|^2\right|. \tag{2.28}$$

In turn, from (2.16), the continuity of the trace operator, and the Hölder inequality, we obtain

$$\left|\int_\Sigma (\mathbf{w}_\mathrm{S}\cdot\mathbf{n})|\mathbf{v}_\mathrm{S}|^2\right| \leq \|\mathbf{w}_\mathrm{S}\|_{0,\Sigma}\|\mathbf{v}_\mathrm{S}\|_{L^4(\Sigma)}^2 \leq C_{trace}^3 C_{sob}^2 \|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}^2. \tag{2.29}$$

Therefore, combining (2.10), (2.28), (2.29), (2.26), and the fact that $\operatorname{div} \mathbf{v}_\mathrm{D} = 0$ in Ω_D, we obtain

$$\mathbf{a}(\mathbf{w}_\mathrm{S}; \mathbf{v}, \mathbf{v}) \geq \mu\alpha_\mathrm{S}\,\|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}^2 \;+\; \alpha_\mathrm{D}\|\mathbf{v}_\mathrm{D}\|_{\operatorname{div},\Omega_\mathrm{D}}^2, \tag{2.30}$$

which yields the result setting $\alpha = \frac{1}{2}\min(\mu\alpha_\mathrm{S}, \alpha_\mathrm{D})$. □

We are now in position of establishing the well-posedness of (2.11).

Theorem 2.1 *Let* $\mathbf{w} := (\mathbf{w}_\mathrm{S}, \mathbf{w}_\mathrm{D}) \in \mathbf{H}$*, such that* $\operatorname{div} \mathbf{w}_\mathrm{S} = 0$ *in* Ω_S *and*

$$\|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq \frac{2\mu\alpha_\mathrm{S}}{\rho C_{trace}^3 C_{sob}^2},$$

and let $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$ *and* $\mathbf{f}_\mathrm{D} \in \mathbf{L}^2(\Omega_\mathrm{D})$*. Then, there exists a unique* $(\mathbf{u}, (\lambda, p)) \in \mathbf{H} \times \mathbf{Q}$ *solution of* (2.11)*. Moreover, there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\mathbf{u}, (\lambda, p))\|_{\mathbf{H}\times\mathbf{Q}} \;\leq\; C\,(\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}). \tag{2.31}$$

Proof. From Lemmas 2.1 and 2.2, and from a direct application of the classical Babuška-Brezzi theory, it follows that problem (2.11) is well posed and the estimate (2.31) holds. □

2.4 Analysis of the continuous Navier-Stokes/Darcy problem

In this section we analyze the well-posedness of problem (2.7). To that end, we now introduce the reduced version of problem (2.7) on the kernel of $\mathbf{V}$ (see (2.24)), which consists in finding $\mathbf{u} := (\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}) \in \mathbf{V}$ such that

$$\mathbf{a}(\mathbf{u}; \mathbf{u}, \mathbf{v}) \;=\; \mathbf{f}(\mathbf{v}), \quad \forall\,\mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{V}. \tag{2.32}$$

The following equivalence property is standard (see [23]).

Lemma 2.3 *If* $(\mathbf{u}, (p, \lambda)) = ((\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}), (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *is a solution of* (2.7)*, then* $\mathbf{u} \in \mathbf{V}$ *is also a solution of* (2.32)*. Conversely, if* $\mathbf{u} = (\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}) \in \mathbf{V}$ *is a solution of* (2.32)*, then there exists a unique* $(p, \lambda) \in \mathbf{Q}$*, such that* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *is a solution of* (2.7)*.*

Remark 2.1 *We recall that the existence of* (p,λ) *in Lemma 2.3 is guaranteed thanks to the inf-sup condition* (2.17)*.*

In this way, thanks to Lemma 2.3, it suffices to prove that (2.32) is well posed. To this aim, we introduce the set

$$\begin{aligned}\mathbf{X} := \{\mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{V} \ : \\ \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,,\Omega_{\mathrm{D}}} \,\leq\, \alpha^{-1}(\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}})\}\,,\end{aligned} \tag{2.33}$$

and the mapping

$$\mathbb{T} : \mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}}) \,\in\, \mathbf{X} \,\to\, \mathbf{u} := (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \,\in\, \mathbf{X},$$

as the solution of the linearized version of problem (2.32): Find $\mathbf{u} \in \mathbf{V}$ such that

$$\mathbf{a}(\mathbf{w}; \mathbf{u}, \mathbf{v}) \,=\, \mathbf{f}(\mathbf{v}), \quad \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{V}. \tag{2.34}$$

Notice that (2.34) is nothing but the reduced version of the well posed problem (2.11), which owing to the inf-sup condition (2.17), is equivalent to (2.34). According to this, we deduce that $\mathbb{T}$ is well defined. Alternatively, owing to Lemma 2.2, it is clear that $\mathbf{a}$ is elliptic on $\mathbf{X}$, which readily implies that problem (2.34) is well posed as well, or equivalently, $\mathbb{T}$ is well defined. In addition, assuming that

$$\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \leq \alpha \frac{2\mu\alpha_{\mathrm{S}}}{\rho C_{trace}^3 C_{sob}^2}, \tag{2.35}$$

it is not difficult to see that $\mathbb{T}$ maps $\mathbf{X}$ into $\mathbf{X}$. In fact, given $\mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}})$ in $\mathbf{X}$, from (2.35) we obtain that $\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \frac{2\mu\alpha_{\mathrm{S}}}{\rho C_{trace}^3 C_{sob}^2}$. Therefore, owing to Lemma 2.2 we easily obtain

$$\begin{aligned}\|\mathbb{T}(\mathbf{w})\|_{\mathbf{H}}^2 &= \|\mathbf{u}\|_{\mathbf{H}}^2 \,\leq\, \alpha^{-1}\mathbf{a}(\mathbf{w};\, \mathbf{u}, \mathbf{u}) = \alpha^{-1}\mathbf{f}(\mathbf{u}) \\ &\leq \alpha^{-1}(\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}})\|\mathbf{u}\|_{\mathbf{H}},\end{aligned}$$

which implies that $\mathbb{T}(\mathbf{w})$ is in $\mathbf{X}$.

In the next lemma we prove that $\mathbb{T}$ has a unique fixed point. To do that, we make use of Banach's fixed point theorem in the following form: *Let* X *be a Banach space, and let* T *a mapping of* X *into itself. Assume that there exists* $0 < r < 1$*, such that* $\|T(u) - T(v)\|_X \leq r\|u - v\|_X$ *for all* $u, v \in X$*, that is,* T *is a contraction mapping. Then there exists a unique* $u \in X$ *such that* $T(u) = u$*.*

Lemma 2.4 *Let* $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ *and* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ *such that*

$$\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} < \gamma, \tag{2.36}$$

with

$$\gamma := \frac{\alpha}{\rho} \min\left(\frac{2\mu\alpha_{\mathrm{S}}}{C_{trace}^3 C_{sob}^2}\,,\, \frac{\alpha}{C_O}\right)$$

Then, $\mathbb{T}$ *has a unique fixed point.*

Proof. According to Banach's fixed point theorem, it is suffices to prove that $\mathbb{T}$ is a contraction mapping. To do that, we let $\mathbf{w} := (\mathbf{w}_{\mathrm{S}}, \mathbf{w}_{\mathrm{D}}), \tilde{\mathbf{w}} := (\tilde{\mathbf{w}}_{\mathrm{S}}, \tilde{\mathbf{w}}_{\mathrm{D}})$ in $\mathbf{X}$ and $\mathbf{u} := \mathbb{T}(\mathbf{w})$, $\tilde{\mathbf{u}} := \mathbb{T}(\tilde{\mathbf{w}})$, and notice that, owing to assumption (2.36), there holds $\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \frac{2\mu\alpha_{\mathrm{S}}}{\rho C_{trace}^3 C_{sob}^2}$.

Now, from the definition of $\mathbb{T}$ we observe that

$$\mathbf{a}(\mathbf{w},\mathbf{u},\mathbf{v}) - \mathbf{a}(\tilde{\mathbf{w}},\tilde{\mathbf{u}},\mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in \mathbf{X},$$

which implies

$$\begin{aligned} A_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}} - \tilde{\mathbf{u}}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) + A_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}} - \tilde{\mathbf{u}}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}}) & \\ + O_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}}; \mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) - O_{\mathrm{S}}(\tilde{\mathbf{w}}_{\mathrm{S}}; \tilde{\mathbf{u}}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}) = 0 & \qquad \forall\, \mathbf{v} \in \mathbf{X}. \end{aligned} \tag{2.37}$$

In particular, for $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$, we utilize (2.30) and add and subtract suitable terms, to obtain

$$\alpha \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{H}}^2 \le \mathbf{a}(\mathbf{w}, \mathbf{u} - \tilde{\mathbf{u}}, \mathbf{u} - \tilde{\mathbf{u}}) = -O_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}} - \tilde{\mathbf{w}}_{\mathrm{S}}; \tilde{\mathbf{u}}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}} - \tilde{\mathbf{u}}_{\mathrm{S}}),$$

which together with the continuity of O_{S} implies

$$\alpha \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{H}}^2 \le C_O \rho \|\mathbf{w}_{\mathrm{S}} - \tilde{\mathbf{w}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\tilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S}} - \tilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}.$$

But, since $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_{\mathrm{S}}, \tilde{\mathbf{u}}_{\mathrm{D}}) \in \mathbf{X}$, we easily get

$$\begin{aligned} \alpha \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{H}}^2 & \le \frac{C_O \rho}{\alpha} \|\mathbf{w}_{\mathrm{S}} - \tilde{\mathbf{w}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S}} - \tilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} (\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}) \\ & \le \frac{C_O \rho}{\alpha} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathbf{H}} \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{H}} (\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}), \end{aligned}$$

that is

$$\|\mathbb{T}(\mathbf{w}) - \mathbb{T}(\tilde{\mathbf{w}})\|_{\mathbf{H}} \le \frac{C_O \rho}{\alpha^2} (\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}) \|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathbf{H}}.$$

The latter estimate and assumption (2.36) imply that $\mathbb{T}$ is a contraction in $\mathbf{X}$, which concludes the proof. □

Now, we are in position of establishing the main result of this section, namely, the well-posedness of problem (2.7).

Theorem 2.2 *Let* $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ *and* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ *and assume that* (2.36) *holds. Then, there exists a unique* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$*, solution of* (2.7)*. In addition, there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\mathbf{u}, (p, \lambda))\|_{\mathbf{H} \times \mathbf{Q}} \le C(\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}). \tag{2.38}$$

Proof. Let $\mathbf{u} \in \mathbf{X}$ be the unique fixed point of $\mathbb{T}$. Then, according to the definition of $\mathbb{T}$, we easily obtain that $\mathbf{u}$ is the unique solution of (2.32). In turn, applying Lemma 2.3 we deduce the existence of $(p, \lambda) \in \mathbf{Q}$, such that $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (2.7).

Next, owing to the inf-sup condition (2.17), the first equation of (2.7) and the continuity of A_{S}, A_{D} and O_{S}, we have

$$\begin{aligned} \beta\ \|(p, \lambda)\|_{\mathbf{Q}} \le & \sup_{\substack{\mathbf{v} \in \mathbf{H} \\ \mathbf{v} \neq \mathbf{0}}} \frac{\mathbf{b}(\mathbf{v}, (p, \lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} = \sup_{\substack{\mathbf{v} \in \mathbf{H} \\ \mathbf{v} \neq \mathbf{0}}} \frac{\mathbf{f}(\mathbf{v}) - \mathbf{a}(\mathbf{u}; \mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}} \\ \le & \ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \\ & \qquad + C_{\mathrm{S}} \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + C_O \rho \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}^2 + C_{\mathrm{D}} \|\mathbf{u}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}. \end{aligned}$$

Then, from the latter estimate, recalling that $\mathbf{u} \in \mathbf{X}$ and using assumption (2.36), we obtain the estimate (2.38), which concludes the proof. □

3 The discrete problem

Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D} formed by shape-regular triangles of diameter h_T and denote by h_{S} and h_{D} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Hereafter $h := \max\{h_{\mathrm{S}}, h_{\mathrm{D}}\}$.

For each $T \in \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of the lowest order [35]:

$$\mathrm{RT}_0(T) := \operatorname{span}\{(1,0),\,(0,1),\,(x_1,x_2)\}\,.$$

In addition, for each $T \in \mathcal{T}_h^{\mathrm{S}}$ we denote by $\mathrm{BR}(T)$ the local Bernardi-Raugel space (see [6, 23]):

$$\mathrm{BR}(T) := [\mathbb{P}_1(T)]^2 \oplus \operatorname{span}\{\,\eta_2\,\eta_3\,\mathbf{n}_1, \eta_1\,\eta_3\,\mathbf{n}_2, \eta_1\,\eta_2\,\mathbf{n}_3\,\}\,,$$

where $\{\eta_1,\eta_2,\eta_3\}$ are the baricentric coordinates of T, and $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. Hence, we define the following finite element subspaces:

$$\begin{aligned}
\mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \{\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) : \mathbf{v}|_T \in \mathrm{BR}(T), \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\},\\
\mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \{\mathbf{v} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}}) : \mathbf{v}|_T \in \mathrm{RT}_0(T), \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\},\\
L_h(\Omega) &:= \{q \in L^2(\Omega) \,:\, q|_T \in \mathbb{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\}.
\end{aligned}$$

The finite element subspaces for the velocities and pressure are, respectively,

$$\begin{aligned}
\mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) &:= \mathbf{H}_h(\Omega_{\mathrm{S}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}),\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}),\\
L_{h,0}(\Omega) &:= L_h(\Omega) \cap L^2_0(\Omega).
\end{aligned}$$

Next, to introduce the finite element subspace of $H^{1/2}(\Sigma)$, we denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$) and we assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (i.e., the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. If the number of edges of Σ_h is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. Then, we define the following finite element subspace for $\lambda \in H^{1/2}(\Sigma)$

$$\Lambda_h(\Sigma) = \{\,\xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \in \Sigma_{2h}\,\}\,.$$

In this way, grouping the unknowns and spaces as follows:

$$\begin{aligned}
&\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}) \in \mathbf{H}_h := \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}),\\
&(p_h, \lambda_h) \in \mathbf{Q}_h := L_{h,0}(\Omega) \times \Lambda_h(\Sigma),
\end{aligned}$$

where $p_h := p_{h,\mathrm{S}}\chi_{\Omega_{\mathrm{S}}} + p_{h,\mathrm{D}}\chi_{\Omega_{\mathrm{D}}}$, the Galerkin approximation of (2.7) reads: Find $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned}
\mathbf{a}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p_h, \lambda_h)) &= \mathbf{f}(\mathbf{v}) \qquad \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_h\,,\\
\mathbf{b}(\mathbf{u}_h, (q, \xi)) &= 0 \qquad \forall\, (q, \xi) \in \mathbf{Q}_h\,.
\end{aligned} \tag{3.1}$$

Here, $\mathbf{a}_h : \mathbf{H}_h \times (\mathbf{H}_h \times \mathbf{H}_h) \to \mathbb{R}$ is the discrete version of $\mathbf{a}$ defined by

$$\mathbf{a}_h(\mathbf{w};\mathbf{u},\mathbf{v}) \;:=\; A_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}) + O^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}};\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}) + A_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}),$$

where O^h_{S} is the well-known skew-symmetric convection form (see [38]):

$$O^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}};\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}) := \rho((\mathbf{w}_{\mathrm{S}}\cdot\nabla)\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + \frac{\rho}{2}(\mathrm{div}\,\mathbf{w}_{\mathrm{S}}\,\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}},$$

for all $\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\mathbf{w}_{\mathrm{S}} \in \mathbf{H}_h(\Omega_{\mathrm{S}})$. Observe that integrating by parts, there holds

$$O^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}};\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}) = \frac{\rho}{2}\int_{\Sigma}(\mathbf{w}_{\mathrm{S}}\cdot\mathbf{n})|\mathbf{v}_{\mathrm{S}}|^2 \qquad \forall\, \mathbf{w},\mathbf{v} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}). \tag{3.2}$$

Moreover, owing to standard Sobolev inequalities, it is easy to see that there exists $\tilde{C}_O$, independent of h, such that, for all $\mathbf{w}_{\mathrm{S}},\, \mathbf{u}_{\mathrm{S}},\, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$,

$$|O^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}};\mathbf{u}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})| \;\le\; \tilde{C}_O\rho\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}. \tag{3.3}$$

In what follows, we proceed similarly to the continuous case to prove that problem (3.1) is well posed. We start by proving the solvability of the discrete version of (2.11).

3.1 The discrete Oseen-Darcy coupled problem

In this section we will apply the classical Babuška-Brezzi theory to prove the well-posedness of the problem: Given $\mathbf{w}_h := (\mathbf{w}_{\mathrm{S},h},\mathbf{w}_{\mathrm{D},h}) \in \mathbf{H}_h$ and $\mathbf{f} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, find $(\mathbf{u}_h,(p_h,\lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{a}_h(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}) \,+\, \mathbf{b}(\mathbf{v},(p_h,\lambda_h)) \;&=\; \mathbf{f}(\mathbf{v}) \qquad && \forall\,\mathbf{v} := (\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_h\,,\\ \mathbf{b}(\mathbf{u}_h,(q,\xi)) \;&=\; 0 \qquad && \forall\,(q,\xi) \,\in\, \mathbf{Q}_h\,. \end{aligned} \tag{3.4}$$

To do that, we first need to introduce some notations and previous results.

3.1.1 Preliminaries

Let $\Pi_{\mathrm{S}} \,:\, \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \to \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ be the Bernardi-Raugel interpolation operator (see [6, 23]), which is linear and bounded with respect to the $\mathbf{H}^1(\Omega_{\mathrm{S}})$-norm. We remark that, given $\mathbf{v} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, there holds

$$\int_e \Pi_{\mathrm{S}}(\mathbf{v})\cdot\mathbf{n} \;=\; \int_e \mathbf{v}\cdot\mathbf{n} \qquad \text{for each edge } e \text{ of } \mathcal{T}^{\mathrm{S}}_h\,, \tag{3.5}$$

and hence

$$\int_{\Omega_{\mathrm{S}}} q\,\mathrm{div}\,\Pi_{\mathrm{S}}(\mathbf{v}) \;=\; \int_{\Omega_{\mathrm{S}}} q\,\mathrm{div}\,\mathbf{v} \qquad \forall\, q \,\in\, L_h(\Omega)\,. \tag{3.6}$$

Equivalently, if $\mathcal{Q}_{\mathrm{S}}$ denotes the $L^2(\Omega_{\mathrm{S}})$-orthogonal projection onto the restriction of $L_h(\Omega)$ to Ω_{S}, then the relation (3.6) can be written as

$$\mathcal{Q}_{\mathrm{S}}(\mathrm{div}\,(\Pi_{\mathrm{S}}(\mathbf{v}))) \;=\; \mathcal{Q}_{\mathrm{S}}(\mathrm{div}\,\mathbf{v}) \qquad \forall\,\mathbf{v} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\,. \tag{3.7}$$

Now, let $\Pi_{\rm D} : \mathbf{H}^1(\Omega_{\rm D}) \to \mathbf{H}_h(\Omega_{\rm D})$ be the Raviart-Thomas interpolation operator, which owing to [1, Theorem 3.1], can also be defined from the larger space $\mathbf{H}^\delta(\Omega_{\rm D}) \cap \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})$ onto $\mathbf{H}_h(\Omega_{\rm D})$ for all $\delta \in (0,1)$. In addition, as established by [27, Theorem 3.16], for all $\boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega_{\rm D}) \cap \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})$, there holds

$$\|\boldsymbol{\tau} - \Pi_{\rm D}(\boldsymbol{\tau})\|_{0,\Omega_{\rm D}} \,\le\, C\, h_2^\delta \Big\{ |\boldsymbol{\tau}|_{\delta,\Omega_{\rm D}} \,+\, \|\mathrm{div}\,\boldsymbol{\tau}\|_{0,\Omega_{\rm D}} \Big\}. \tag{3.8}$$

We also recall that, given $\boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega_{\rm D}) \cap \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})$, there holds

$$\int_e \Pi_{\rm D}(\boldsymbol{\tau}) \cdot \mathbf{n} \,=\, \int_e \boldsymbol{\tau} \cdot \mathbf{n} \qquad \text{for each edge } e \text{ of } \mathcal{T}_h^{\rm D}\,, \tag{3.9}$$

and hence

$$\int_{\Omega_{\rm D}} q \,\mathrm{div}\,(\Pi_{\rm D}(\boldsymbol{\tau})) \,=\, \int_{\Omega_{\rm D}} q \,\mathrm{div}\,\boldsymbol{\tau} \qquad \forall\, q \,\in\, L_h(\Omega)\,. \tag{3.10}$$

Equivalently, if $\mathcal{Q}_{\rm D}$ denotes the $L^2(\Omega_{\rm D})$-orthogonal projection onto the restriction of $L_h(\Omega)$ to $\Omega_{\rm D}$, then the relation (3.10) can be written as

$$\mathrm{div}\,(\Pi_{\rm D}(\boldsymbol{\tau})) \,=\, \mathcal{Q}_{\rm D}(\mathrm{div}\,(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega_{\rm D}) \cap \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})\,. \tag{3.11}$$

Let us now observe that the set of discrete normal traces on Σ of $\mathbf{H}_h(\Omega_{\rm D})$ is given by

$$\Phi_h(\Sigma) \,:=\, \Big\{ \phi_h : \Sigma \to \mathbb{R} : \quad \phi_h|_e \,\in\, \mathbb{P}_0(e) \qquad \forall \text{ edge } e \,\in\, \Sigma_h \Big\}. \tag{3.12}$$

In [34, Theorem A.1] it has been proved that there exists a discrete lifting

$$\mathbf{L}_h \,: \Phi_h(\Sigma) \to \mathbf{H}_{h,\Gamma_{\rm D}}(\Omega_{\rm D}), \tag{3.13}$$

such that, for all $\phi_h \in \Phi_h(\Sigma)$,

$$\|\mathbf{L}_h(\phi_h)\|_{\mathrm{div}\,;\Omega_{\rm D}} \le c_\star \|\phi_h\|_{-1/2,\Sigma} \quad \text{and} \quad \mathbf{L}_h(\phi_h) \cdot \mathbf{n} = \phi_h \quad \text{on} \quad \Sigma. \tag{3.14}$$

In addition, in [20, Lemma 5.2] it has been proved that there exists $\widehat{\beta}_\Sigma > 0$, independent of h, such that the pair of subspaces $(\Phi_h(\Sigma), \Lambda_h(\Sigma))$ satisfies the discrete inf-sup condition:

$$\sup_{\substack{\phi_h \,\in\, \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\ge\, \widehat{\beta}_\Sigma \,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h(\Sigma)\,. \tag{3.15}$$

3.1.2 The discrete inf-sup condition

In what follows we prove that the bilinear form $\mathbf{b}$ satisfies the corresponding discrete inf-sup condition. We start by establishing the following two previous lemmas.

Lemma 3.1 *There exists $\tilde{C}_1 > 0$, independent of h, such that*

$$\sup_{\substack{\mathbf{v} \in \mathbf{H}_h\\ \mathbf{v} \neq \mathbf{0}}} \frac{\mathbf{b}(\mathbf{v}, (q_h, \xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}} \,\ge\, \tilde{C}_1 \,\|\xi_h\|_{1/2,\Sigma} \,-\, \|q_h\|_{0,\Omega}, \tag{3.16}$$

$\forall\, (q_h, \xi_h) \,\in\, \mathbf{Q}_h$.

Proof. Let $(q_h, \xi_h) \in \mathbf{Q}_h$. Given $\phi_h \in \Phi_h(\Sigma)$, we define $\bar{\mathbf{v}}_{h,\mathrm{D}} = \mathbf{L}_h(\phi_h)$, with $\mathbf{L}_h$ being the lifting defined in (3.13). From (3.14), it follows that

$$\|\bar{\mathbf{v}}_{h,\mathrm{D}}\|_{\mathrm{div}\,;\Omega_\mathrm{D}} \le c_\star \|\phi_h\|_{-1/2,\Sigma} \quad \text{and} \quad \bar{\mathbf{v}}_{h,\mathrm{D}} \cdot \mathbf{n} = \phi_h \quad \text{on} \quad \Sigma.$$

Hence, defining $\bar{\mathbf{v}}_h := (\mathbf{0}, \bar{\mathbf{v}}_{h,\mathrm{D}}) \in \mathbf{H}_h$, we deduce that

$$\begin{aligned}
\sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q_h,\xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}} \ge \ & \frac{\mathbf{b}(\bar{\mathbf{v}}_h,(q_h,\xi_h))}{\|\bar{\mathbf{v}}_h\|_{\mathbf{H}}} = \frac{\left|\,\langle \bar{\mathbf{v}}_{h,\mathrm{D}}\cdot\mathbf{n}, \xi_h\rangle_\Sigma \,-\, (\mathrm{div}\,\bar{\mathbf{v}}_{h,\mathrm{D}}, q_h)_\mathrm{D}\,\right|}{\|\bar{\mathbf{v}}_{h,\mathrm{D}}\|_{\mathrm{div}\,;\Omega_\mathrm{D}}} \\
\ge \ & \frac{|\,\langle \phi_h, \xi_h\rangle_\Sigma\,|}{\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_\mathrm{D}}} - \|q_h\|_{0,\Omega_\mathrm{D}} \\
\ge \ & \frac{1}{c_\star}\,\frac{|\,\langle \phi_h, \xi_h\rangle_\Sigma\,|}{\|\phi_h\|_{-1/2,\Sigma}} \,-\, \|q_h\|_{0,\Omega}\,,
\end{aligned}$$

which, noting that ϕ_h is arbitrary in $\Phi_h(\Sigma)$, yields

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q_h,\xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}} \ \ge\ \frac{1}{c_\star} \sup_{\substack{\phi_h\in\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,-\, \|q_h\|_{0,\Omega}\,.$$

This inequality and (3.15) imply the result and complete the proof. ☐

Lemma 3.2 *There exist positive constants $\tilde{C}_2$ and $\tilde{C}_3$, independent of h, such that for all $(q_h, \xi_h) \in \mathbf{Q}_h$, there holds*

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q_h,\xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}} \ \ge\ \tilde{C}_2\,\|q_h\|_{0,\Omega} \,-\, \tilde{C}_3\, h_\mathrm{D}^{1/2}\,\|\xi_h\|_{1/2,\Sigma}\,. \tag{3.17}$$

Proof. Let $(q_h, \xi_h) \in \mathbf{Q}_h$. Since $q_h \in L^2_0(\Omega)$ there exists $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ such that

$$\mathrm{div}\,\mathbf{z} = -q \quad \text{in} \quad \Omega \qquad \text{and} \qquad \|\mathbf{z}\|_{1,\Omega} \,\le\, c\,\|q\|_{0,\Omega}\,. \tag{3.18}$$

Let $\mathbf{z}_\star := \mathbf{z}|_{\Omega_\star}$ for $\star \in \{\mathrm{S}, \mathrm{D}\}$. From (3.5), (3.9), and the fact that $\mathbf{z}_\mathrm{S} = \mathbf{z}_\mathrm{D}$ on Σ, it follows that

$$\int_\Sigma (\Pi_\mathrm{S}(\mathbf{z}_\mathrm{S}) - \Pi_\mathrm{D}(\mathbf{z}_\mathrm{D}))\cdot\mathbf{n} = 0\,. \tag{3.19}$$

Let us now define $\chi_h \in L^2(\partial\Omega_\mathrm{D}) \subseteq H^{-1/2}(\partial\Omega_\mathrm{D})$ as

$$\chi_h := \begin{cases} (\Pi_\mathrm{S}(\mathbf{z}_\mathrm{S}) - \Pi_\mathrm{D}(\mathbf{z}_\mathrm{D}))\cdot\mathbf{n} & \text{on} \quad \Sigma, \\ 0 & \text{on} \quad \Gamma_\mathrm{D}, \end{cases}$$

which clearly satisfies

$$\langle \chi_h, E_\mathrm{D}(\xi_h)\rangle_{\partial\Omega_\mathrm{D}} \ =\ \langle (\Pi_\mathrm{S}(\mathbf{z}_\mathrm{S}) - \Pi_\mathrm{D}(\mathbf{z}_\mathrm{D}))\cdot\mathbf{n}, \xi_h\rangle_\Sigma\ ,$$

$$\langle \chi_h, \psi\rangle_{\partial\Omega_\mathrm{D}} \ =\ \langle (\Pi_\mathrm{S}(\mathbf{z}_\mathrm{S}) - \Pi_\mathrm{D}(\mathbf{z}_\mathrm{D}))\cdot\mathbf{n}, \psi_\Sigma\rangle_\Sigma \qquad \forall\,\psi \in H^{1/2}(\partial\Omega_\mathrm{D}),$$

with ψ_Σ being the element in $H^{1/2}(\Sigma)$ satisfying (2.14), and

$$\|\chi_h\|_{-1/2,\partial\Omega_D} \,\le\, C\,\|(\Pi_S(\mathbf{z}_S) - \Pi_D(\mathbf{z}_D))\cdot\mathbf{n}\|_{-1/2,\Sigma}\,. \tag{3.20}$$

In addition, from (3.19), and the definition of χ_h, it is not difficult to see that

$$\langle\chi_h, 1\rangle_{\partial\Omega_D} \,=\, \langle\Pi_S(\mathbf{z}_S) - \Pi_D(\mathbf{z}_D), 1\rangle_\Sigma = 0\,.$$

In this way, we let $\varphi \in H^1(\Omega_D)$ be the unique weak solution of the problem:

$$-\,\Delta\varphi \,=\, 0 \ \text{ in } \ \Omega_D\,, \qquad \frac{\partial\varphi}{\partial\mathbf{n}} \,=\, \chi_h \ \text{ on } \ \partial\Omega_D\,, \qquad \int_{\Omega_D}\varphi \,=\, 0\,, \tag{3.21}$$

and define

$$\mathbf{w}_{h,S} \,:=\, \Pi_S(\mathbf{z}_S) \qquad \text{and} \qquad \mathbf{w}_{h,D} \,:=\, \Pi_D(\mathbf{z}_D) \,+\, \Pi_D(\nabla\varphi)\,. \tag{3.22}$$

Recalling that $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$, we observe that $\mathbf{z}_D \in \mathbf{H}^1_{\Gamma_D}(\Omega_D)$, and then

$$\int_e \mathbf{w}_{h,D}\cdot\mathbf{n} = \int_e(\,\Pi_D(\mathbf{z}_D) \,+\, \Pi_D(\nabla\varphi))\cdot\mathbf{n} \,=\, \int_e(\mathbf{z}_D\cdot\mathbf{n} + \chi_h) = 0$$

for all edge e on Γ_D, which implies that $\mathbf{w}_{h,D} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D)$. Then, we proceed analogously to the proof of [19, Lemma 4.2], set $\mathbf{w}_h := (\mathbf{w}_{h,S}, \mathbf{w}_{h,D}) \in \mathbf{H}_h$, and use the properties of the interpolation operators in Section 3.1.1, to find that

$$\|\mathbf{w}_h\|_{\mathbf{H}} \,\le\, C\,\|q_h\|_{0,\Omega}\,, \tag{3.23}$$

$$\int_{\Omega_S} q_h \,\mathrm{div}\,\mathbf{w}_{h,S} + \int_{\Omega_D} q_h \,\mathrm{div}\,\mathbf{w}_{h,D} \,=\, -\|q_h\|^2_{0,\Omega}\,, \tag{3.24}$$

and

$$|\,\langle\mathbf{w}_{h,S}\cdot\mathbf{n} - \mathbf{w}_{h,D}\cdot\mathbf{n}, \xi_h\rangle_\Sigma\,| \,\le\, C\,h_D^{1/2}\,\|\xi_h\|_{1/2,\Sigma}\,\|q_h\|_{0,\Omega}\,, \tag{3.25}$$

from which

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \ge \frac{|\mathbf{b}(\mathbf{w}_h,(q_h,\xi_h))|}{\|\mathbf{w}_h\|_{\mathbf{H}}} \ge \tilde{C}_2\,\|q_h\|_{0,\Omega} \,-\, \tilde{C}_3\,h_D^{1/2}\,\|\xi_h\|_{1/2,\Sigma},$$

which completes the proof. □

Now, we are in position of establishing the discrete inf-sup condition of $\mathbf{b}$.

Lemma 3.3 *Assume that*

$$h_D \le \left(\frac{\tilde{C}_1\tilde{C}_2}{2\tilde{C}_3}\right)^2, \tag{3.26}$$

where $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$, are the constants in Lemmas 3.1 and 3.2. Then there exists $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq \mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q_h,\xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}} \,\ge\, \tilde{\beta}\,\|(q_h,\xi_h)\|_{\mathbf{Q}}\,. \tag{3.27}$$

Proof. Setting

$$S := \sup_{\substack{\mathbf{v}\in\mathbf{H}_h \\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\mathbf{v},(q_h,\xi_h))}{\|\mathbf{v}\|_{\mathbf{H}}},$$

from Lemmas 3.1 and 3.2, we obtain

$$\frac{\tilde{C}_2}{2} S \geq \frac{\tilde{C}_2\tilde{C}_1}{2} \|\xi_h\|_{1/2,\Sigma} - \frac{\tilde{C}_2}{2}\|q_h\|_{0,\Omega}$$

and

$$S \geq \tilde{C}_2 \|q_h\|_{0,\Omega} - \tilde{C}_3\, h_{\mathrm{D}}^{1/2} \|\xi_h\|_{1/2,\Sigma},$$

which combined yield

$$\left(1+\frac{\tilde{C}_2}{2}\right) S \geq \frac{\tilde{C}_2}{2}\|q_h\|_{0,\Omega} + \left(\frac{\tilde{C}_2\tilde{C}_1}{2} - \tilde{C}_3\, h_{\mathrm{D}}^{1/2}\right) \|\xi_h\|_{1/2,\Sigma}.$$

Together to (3.26), this implies the result. □

Remark 3.1 *Observe that the existence of a stable lifting* $\mathbf{L}_h$ *satisfying* (3.14) *and the inf-sup condition* (3.15) *play an important role in the proof of the discrete inf-sup condition* (3.16). *In particular, as established in Section 3.1.1, the existence of a stable lifting* $\mathbf{L}_h$*, satisfying* (3.14)*, has been proved in [34, Theorem A.1] for the 2D case, where the only restriction on the grid is shape regularity (previously in [20], a similar result was proved under a quasi-uniformity condition on the mesh near the interface* Σ*). However, the 3D analogue of [34, Theorem A.1], being an open problem, cannot be employed, and in order to prove the 3D version of the inf-sup condition* (3.15) *we need to define the discrete subspace* Λ_h *on a suitable independent triangulation of* Σ*. Indeed, defining an independent triangulation* $\Sigma_{\tilde{h}}$ *of the interface* Σ *formed by triangles of diameter* $\tilde{h}_K$*, setting* $\tilde{h}_\Sigma := \max\{\tilde{h}_K : K \in \Sigma_{\tilde{h}}\}$*, and defining the set of normal traces of* $\mathbf{H}_h(\Omega_{\mathrm{D}})$ *as in* (3.12) *(considering triangles instead of edges), with* $h_\Sigma := \max\{h_K : K \in \Sigma_h\}$*, it can be proved (see e.g. the second part of the proof of [18, Lemma 7.5]) that there exists* $C_0 \in (0,1)$ *such that, for each pair* $(h_\Sigma, \tilde{h}_\Sigma)$ *verifying* $h_\Sigma \leq C_0\tilde{h}_\Sigma$*, the 3D version of* (3.15) *is satisfied. Furthermore, in order to construct the stable discrete lifting of the normal traces of* $\mathbf{H}_h(\Omega_{\mathrm{D}})$*, we need to employ some inverse inequalities on* Σ*, which require quasi-uniform meshes in a neighbourhood of the interface.*

3.1.3 Well-posedness of the discrete Oseen-Darcy problem

Now we prove the well-posedness of problem (3.4). We begin by establishing the ellipticity of $\mathbf{a}_h(\mathbf{w},\cdot,\cdot)$ on the discrete kernel of $\mathbf{b}$:

$$\mathbf{V}_h := \{\mathbf{v} := (\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_h : \mathbf{b}(\mathbf{v},(q_h,\xi_h)) = 0 \quad \forall\,(q_h,\xi_h)\in\mathbf{Q}_h\},$$

for a suitable $\mathbf{w}\in\mathbf{H}_h$.

Observe that, similarly to the continuous case, $\mathbf{v}\in\mathbf{V}_h$, if and only if,

$$\int_{\Omega_{\mathrm{S}}} q_h \operatorname{div}\mathbf{v}_{\mathrm{S}} + \int_{\Omega_{\mathrm{D}}} q_h \operatorname{div}\mathbf{v}_{\mathrm{D}} = 0, \quad \forall\, q_h \in L_{h,0}(\Omega),$$

and

$$\langle \mathbf{v}_\mathrm{S}\cdot\mathbf{n} - \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, \xi_h\rangle_\Sigma = 0, \quad \forall\, \xi_h \in \Lambda_h(\Sigma),$$

which, in particular imply that

$$\mathrm{div}\,\mathbf{v}_\mathrm{D} \;=\; 0 \quad \text{in} \quad \Omega_\mathrm{D}. \tag{3.28}$$

Remark 3.2 *We recall here that if* $\mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D})\ \mathbf{V}_h$*, then* $\mathbf{v}_\mathrm{S}$ *is not necessarily divergence-free. This fact motivates the utilization of the skew-symmetric convective form* O^h_S*.*

In the next lemma we establish the ellipticity of $\mathbf{a}_h$ on $\mathbf{V}_h$.

Lemma 3.4 *Let* $\mathbf{w} := (\mathbf{w}_\mathrm{S}, \mathbf{w}_\mathrm{D}) \in \mathbf{H}_h$*, such that*

$$\|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \le \frac{2\mu\alpha_\mathrm{S}}{\rho C^3_{trace} C^2_{sob}}. \tag{3.29}$$

Then, there holds

$$\mathbf{a}_h(\mathbf{w}; \mathbf{v}, \mathbf{v}) \;\ge\; \alpha\,\|\mathbf{v}\|^2_\mathbf{H} \qquad \forall\,\mathbf{v}\,\in\,\mathbf{V}_h, \tag{3.30}$$

with $\alpha = \frac{1}{2}\min\{\mu\alpha_\mathrm{S}, \alpha_\mathrm{D}\}$*.*

Proof. Let $\mathbf{w} := (\mathbf{w}_\mathrm{S}, \mathbf{w}_\mathrm{D}) \in \mathbf{H}_h$ such that (3.29) holds. First, from identity (3.2), for all $\mathbf{v}_\mathrm{S} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, we obtain

$$O^h_\mathrm{S}(\mathbf{w}_\mathrm{S}; \mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{S}) = \frac{1}{2}\int_\Sigma (\mathbf{w}_\mathrm{S}\cdot\mathbf{n})|\mathbf{v}_\mathrm{S}|^2 \ge -\frac{1}{2}\left|\int_\Sigma (\mathbf{w}_\mathrm{S}\cdot\mathbf{n})|\mathbf{v}_\mathrm{S}|^2\right|.$$

Then, the result follows analogously to the proof of Lemma 2.2. We omit further details. □

We are now in a position of establishing the well-posedness of (3.4).

Theorem 3.1 *Let* $\mathbf{w} := (\mathbf{w}_\mathrm{S}, \mathbf{w}_\mathrm{D}) \in \mathbf{H}_h$*, such that*

$$\|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \le \frac{2\mu\alpha_\mathrm{S}}{\rho C^3_{trace} C^2_{sob}}.$$

and assume that (3.26) *holds. Then, for each* $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$ *and* $\mathbf{f}_\mathrm{D} \in \mathbf{L}^2(\Omega_\mathrm{D})$*, there exists a unique* $(\mathbf{u}_h, (\lambda_h, p_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *solution of* (3.4)*. Moreover, there exists a constant* $\tilde{C} > 0$*, independent of the solution, such that*

$$\|(\mathbf{u}_h, (\lambda_h, p_h))\|_{\mathbf{H}\times\mathbf{Q}} \;\le\; \tilde{C}\,(\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}). \tag{3.31}$$

Proof. It follows from Lemmas 3.3 and 3.4, and a straightforward application of the classical Babuška-Brezzi theory. □

3.2 Well-posedness of the discrete Navier-Stokes/Darcy problem

In this section, we proceed analogously to Section 2.4 and prove the well-posedness of problem (3.1) by means of Banach's fixed point theorem.

3.2.1 The discrete contraction mapping

Let us consider the reduced version of problem (2.7): Find $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}) \in \mathbf{V}_h$ such that

$$\mathbf{a}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) = \mathbf{f}(\mathbf{v}), \quad \forall\, \mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{V}_h. \tag{3.32}$$

Since the discrete inf-sup condition holds (see Lemma 3.3), it is easy to see that problems (3.1) and (3.32) are equivalent. In fact, we have the following standard result (see [23]).

Lemma 3.5 *Assume that* (3.26) *holds. If* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *is a solution of* (3.1)*, then* $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}) \in \mathbf{V}_h$ *is also a solution of* (3.32)*. Conversely, if* $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}})$ *is a solution of* (3.32)*, then there exists a unique* $(p_h, \lambda_h) \in \mathbf{Q}_h$*, such that* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *is a solution of* (3.1)*.*

Now, similarly to the analysis of the continuous problem, we assume that

$$\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \leq \alpha \frac{2\mu\alpha_\mathrm{S}}{\rho C_{trace}^3 C_{sob}^2}, \tag{3.33}$$

and define the set

$$\mathbf{X}_h := \Big\{ \mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{V}_h \,:\, \|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + \|\mathbf{v}_\mathrm{D}\|_{\mathrm{div},\Omega_\mathrm{D}} \,\leq\, \alpha^{-1}(\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}) \Big\}, \tag{3.34}$$

and the mapping

$$\mathbb{T}_h : \mathbf{w}_h := (\mathbf{w}_{h,\mathrm{S}}, \mathbf{w}_{h,\mathrm{D}}) \,\in\, \mathbf{X}_h \,\to\, \mathbf{u}_h := (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}) \,\in\, \mathbf{X}_h,$$

where $\mathbf{u}_h$ is the unique element in $\mathbf{X}_h$, such that

$$\mathbf{a}(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}) = \mathbf{f}(\mathbf{v}), \quad \forall\, \mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{X}_h. \tag{3.35}$$

Proceeding as in Section 2.4, we easily obtain that owing to assumption (3.33), the mapping $\mathbb{T}_h$ is well defined and maps $\mathbf{X}_h$ into $\mathbf{X}_h$.

The following result establishes that $\mathbb{T}_h$ has a unique fixed point in $\mathbf{X}_h$. Its proof follows analogously to the proof of Lemma 2.4 by means of Banach's fixed point theorem. We omit further details.

Lemma 3.6 *Let* $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$ *and* $\mathbf{f}_\mathrm{D} \in \mathbf{L}^2(\Omega)$*, such that*

$$\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} < \tilde{\gamma}, \tag{3.36}$$

with

$$\tilde{\gamma} := \frac{\alpha}{\rho} \min\left(\frac{2\mu\alpha_\mathrm{S}}{C_{trace}^3 C_{sob}^2}, \frac{\alpha}{\tilde{C}_O} \right)$$

Then, $\mathbb{T}_h$ *has a unique fixed point in* $\mathbf{X}_h$*.*

3.2.2 The main result

Now, we are in position of establishing the main result of this section, namely, existence and uniqueness of solution of problem (3.1).

Theorem 3.2 *Let* $\mathbf{f}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S})$ *and* $\mathbf{f}_{\rm D} \in \mathbf{L}^2(\Omega)$. *Assume that* (3.26) *and* (3.36) *hold. Then, there exists a unique* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, *solution of* (3.1). *In addition, there exists a constant* $\tilde{C} > 0$, *independent of the solution, such that*

$$\|(\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq \tilde{C}(\|\mathbf{f}_{\rm S}\|_{0,\Omega_{\rm S}} + \|\mathbf{f}_{\rm D}\|_{0,\Omega_{\rm D}}). \tag{3.37}$$

Proof. The proof follows analogously to the proof of Theorem 2.2. In fact, we first observe that $\mathbf{u}_h$ is the unique solution of (3.32) if and only if $\mathbb{T}_h(\mathbf{u}_h) = \mathbf{u}_h$, that is $\mathbf{u}_h$ is the unique fixed point of $\mathbb{T}_h$. Then, according to the equivalence established in Lemma 3.5, and owing to Lemma 3.6, it follows that problem (3.1) is well posed. In turn, as in the proof of Theorem 2.2, the continuous dependence result (3.37) easily follows from the inf-sup condition (3.27), the fact that the solution $\mathbf{u}_h$ is in $\mathbf{X}_h$, and the continuity of $A_{\rm S}$, $A_{\rm D}$ and $O^h_{\rm S}$. □

3.3 Convergence of the Galerkin scheme

Our next goal is to provide the corresponding Cea's estimate and rate of convergence of the Galerkin scheme (3.1). To this end and in order to simplify the subsequent analysis, we write $\mathbf{e}_{\mathbf{u}_{\rm S}} = \mathbf{u}_{\rm S} - \mathbf{u}_{h,{\rm S}}$, $\mathbf{e}_{\mathbf{u}_{\rm D}} = \mathbf{u}_{\rm D} - \mathbf{u}_{h,{\rm D}}$, $\mathrm{e}_p = p - p_h$, and $\mathrm{e}_\lambda = \lambda - \lambda_h$, where $(\mathbf{u}, (p, \lambda)) := ((\mathbf{u}_{\rm S}, \mathbf{u}_{\rm D}), (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ and $(\mathbf{u}_h, (p_h, \lambda_h)) := ((\mathbf{u}_{h,{\rm S}}, \mathbf{u}_{h,{\rm D}}), (p, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ are the unique solutions of (2.7) and (3.1), respectively.

On the other hand, since the exact solution $\mathbf{u}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$ satisfies $\operatorname{div} \mathbf{u}_{\rm S} = 0$ in $\Omega_{\rm S}$, we have

$$O^h_{\rm S}(\mathbf{u}_{\rm S}, \mathbf{u}_{\rm S}, \mathbf{v}_{h,{\rm S}}) = O_{\rm S}(\mathbf{u}_{\rm S}, \mathbf{u}_{\rm S}, \mathbf{v}_{h,{\rm S}}) \qquad \forall\, \mathbf{v}_{h,{\rm S}} \in \mathbf{H}_{h,\Gamma_{\rm S}}(\Omega_{\rm S}).$$

Owing to this identity, the following Galerkin orthogonality property holds:

$$\begin{aligned} A_{\rm S}(\mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{v}_{\rm S}) + A_{\rm D}(\mathbf{e}_{\mathbf{u}_{\rm D}}, \mathbf{v}_{\rm D}) + O^h_{\rm S}(\mathbf{u}_{\rm S}, \mathbf{u}_{\rm S}, \mathbf{v}_{\rm S}) & \\ -O^h_{\rm S}(\mathbf{u}_{h,{\rm S}}, \mathbf{u}_{h,{\rm S}}, \mathbf{v}_{\rm S}) + \mathbf{b}(\mathbf{v}, (\mathrm{e}_p, \mathrm{e}_\lambda)) &= 0 \\ \mathbf{b}((\mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{e}_{\mathbf{u}_{\rm D}}), (q, \xi)) &= 0 \end{aligned} \tag{3.38}$$

for all $\mathbf{v} := (\mathbf{v}_{\rm S}, \mathbf{v}_{\rm D}) \in \mathbf{H}_h$, and $(q, \xi) \in \mathbf{Q}_h$.

The following theorem provides the corresponding Cea's estimate.

Theorem 3.3 *Let* $\mathbf{f}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S})$ *and* $\mathbf{f}_{\rm D} \in \mathbf{L}^2(\Omega_{\rm D})$ *such that*

$$\|\mathbf{f}_{\rm S}\|_{0,\Omega_{\rm S}} + \|\mathbf{f}_{\rm D}\|_{0,\Omega_{\rm D}} \leq \frac{1}{2} \min\{\gamma, \tilde{\gamma}\}, \tag{3.39}$$

where γ *and* $\tilde{\gamma}$ *are the constants in Lemmas 2.4 and 3.6, respectively. Assume that* (3.26) *holds. Let* $(\mathbf{u}, (p, \lambda)) := ((\mathbf{u}_{\rm S}, \mathbf{u}_{\rm D}), (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, (p_h, \lambda_h)) := ((\mathbf{u}_{h,{\rm S}}, \mathbf{u}_{h,{\rm D}}), (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.7) *and* (3.1), *respectively. Then there exists* $C > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} &\|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\quad \leq C \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} + \inf_{(q_h, \xi_h) \in \mathbf{Q}_h} \|(p, \lambda) - (q_h, \xi_h)\|_{\mathbf{Q}} \right\}. \end{aligned} \tag{3.40}$$

Proof. Given $\bar{\mathbf{v}} = (\bar{\mathbf{v}}_{h,\mathrm{S}}, \bar{\mathbf{v}}_{h,\mathrm{D}}) \in \mathbf{V}_h$ and $(\bar{q}_h, \bar{\xi}_h) \in \mathbf{Q}_h$, as usual we decompose these errors into

$$\mathbf{e}_{\mathbf{u}_\mathrm{S}} = \boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}} + \boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \quad \mathbf{e}_{\mathbf{u}_\mathrm{D}} = \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}} + \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}, \quad \mathrm{e}_p = \delta_p + \eta_p, \quad \mathrm{e}_\lambda = \delta_\lambda + \eta_\lambda, \tag{3.41}$$

where

$$\begin{aligned} \boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}} &= \mathbf{u}_\mathrm{S} - \bar{\mathbf{v}}_{h,\mathrm{S}}, \quad \boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}} = \bar{\mathbf{v}}_{h,\mathrm{S}} - \mathbf{u}_{h,\mathrm{S}}, \\ \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}} &= \mathbf{u}_\mathrm{D} - \bar{\mathbf{v}}_{h,\mathrm{D}}, \quad \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}} = \bar{\mathbf{v}}_{h,\mathrm{D}} - \mathbf{u}_{h,\mathrm{D}}, \\ \delta_p &= p - \bar{q}_h, \quad \eta_p = \bar{q}_h - p_h, \quad \delta_\lambda = \lambda - \bar{\xi}_h, \quad \eta_\lambda = \bar{\xi}_h - \lambda_h. \end{aligned} \tag{3.42}$$

Now, we recall that owing to assumption (3.39), it follows that $\mathbf{u} = (\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}) \in \mathbf{X}$ and $\mathbf{u}_h = (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}) \in \mathbf{X}_h$ (cf. (2.33) and (3.34)), which implies

$$\begin{aligned} \|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} &\le \alpha^{-1}(\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}), \\ \|\mathbf{u}_{h,\mathrm{S}}\|_{1,\Omega_\mathrm{S}} &\le \alpha^{-1}(\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}), \end{aligned} \tag{3.43}$$

and

$$\mathbf{u} \in \mathbf{V}, \quad \mathbf{u}_h \in \mathbf{V}_h. \tag{3.44}$$

In particular, from (3.44) we have

$$(\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}) \in \mathbf{V}_h. \tag{3.45}$$

According to the above, and noting that for all $\mathbf{v}_\mathrm{S} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, there holds

$$\begin{aligned} O_\mathrm{S}^h(\mathbf{u}_\mathrm{S}; \mathbf{u}_\mathrm{S}, \mathbf{v}_\mathrm{S}) &- O_\mathrm{S}^h(\mathbf{u}_{h,\mathrm{S}}; \mathbf{u}_{h,\mathrm{S}}, \mathbf{v}_\mathrm{S}) \\ &= O_\mathrm{S}^h(\mathbf{e}_{\mathbf{u}_\mathrm{S}}; \mathbf{u}_\mathrm{S}, \mathbf{v}_\mathrm{S}) + O_\mathrm{S}^h(\mathbf{u}_{h,\mathrm{S}}; \mathbf{e}_{\mathbf{u}_\mathrm{S}}, \mathbf{v}_\mathrm{S}) \\ &= O_\mathrm{S}^h(\mathbf{u}_{h,\mathrm{S}}; \boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \mathbf{v}_\mathrm{S}) + R, \end{aligned} \tag{3.46}$$

with

$$R = O_\mathrm{S}^h(\mathbf{u}_{h,\mathrm{S}}; \boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}, \mathbf{v}_\mathrm{S}) + O_\mathrm{S}^h(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}; \mathbf{u}_\mathrm{S}, \mathbf{v}_\mathrm{S}) + O_\mathrm{S}^h(\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}; \mathbf{u}_\mathrm{S}, \mathbf{v}_\mathrm{S}),$$

we add and subtract suitable terms in the first equation of (3.38) with $\mathbf{v} = (\boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})$, and observe that $\mathbf{b}((\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), (\eta_p, \eta_\lambda)) = 0$, to obtain

$$\begin{aligned} &\mathbf{a}_h(\mathbf{u}_h; (\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), (\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})) = \\ &\qquad - A_\mathrm{S}(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}) - A_\mathrm{D}(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}) - R - \mathbf{b}((\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), (\delta_p, \delta_\lambda)). \end{aligned}$$

Hence, proceeding analogously to the proof of Lemma 2.2, and using the continuity of A_S, A_D, O_S^h and $\mathbf{b}$, we obtain

$$\begin{aligned} \mu\alpha_\mathrm{S}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}}^2 &+ \alpha_\mathrm{D}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}\|_{\mathrm{div},\Omega_\mathrm{D}}^2 \\ &\le C_\mathrm{S}\|\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}} + C_\mathrm{D}\|\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}\|_{\mathrm{div},\Omega_\mathrm{D}}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}\|_{\mathrm{div},\Omega_\mathrm{D}} \\ &\quad + \tilde{C}_O(\|\mathbf{u}_{h,\mathrm{S}}\|_{1,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}})\|\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}} \\ &\quad + \tilde{C}_O\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}}^2 + C_\mathbf{b}\|(\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})\|_\mathbf{H}\|(\delta_p, \delta_\lambda)\|_\mathbf{Q}, \end{aligned}$$

which together with (3.43) and assumption (3.39), implies that there exists $C > 0$, independent of h, such that

$$\|(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\eta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} \leq C\big\{\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\delta_p, \delta_\lambda)\|_{\mathbf{Q}}\big\}. \tag{3.47}$$

In this way, from (3.41), (3.47) and the triangle inequality, we obtain

$$\begin{aligned} \|(\mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{e}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} &\leq \|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\eta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} \\ &\leq \tilde{C}\big\{\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\delta_p, \delta_\lambda)\|_{\mathbf{Q}}\big\}. \end{aligned} \tag{3.48}$$

Now, to estimate e_p and e_λ we observe that from the discrete inf-sup condition (3.27), the first equation of (3.38), and the first equation of (3.46), there holds

$$\begin{aligned} \tilde{\beta}\|(\eta_p, \eta_\lambda)\|_{\mathbf{Q}} &\leq \sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v}, (\eta_p, \eta_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} = \sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq 0}} \frac{\mathbf{b}(\mathbf{v}_h, (\mathrm{e}_p, \mathrm{e}_\lambda)) - \mathbf{b}(\mathbf{v}, (\delta_p, \delta_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \\ = \sup_{\substack{\mathbf{v}\in\mathbf{H}_h\\ \mathbf{v}\neq 0}} -\Bigg(&\frac{A_{\rm S}(\mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{v}_{\rm S}) + A_{\rm D}(\mathbf{e}_{\mathbf{u}_{\rm D}}, \mathbf{v}_{\rm D}) + O_{\rm S}^h(\mathbf{e}_{\mathbf{u}_{\rm S}}; \mathbf{u}_{\rm S}, \mathbf{v}_{\rm S})}{\|\mathbf{v}\|_{\mathbf{H}}} \\ &+ \frac{O_{\rm S}^h(\mathbf{u}_{h,{\rm S}}; \mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{v}_{\rm S}) + \mathbf{b}(\mathbf{v}, (\delta_p, \delta_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}}\Bigg). \end{aligned}$$

Then, owing to the continuity of $A_{\rm S}$, $A_{\rm D}$, $O_{\rm S}^h$, $\mathbf{b}$, inequalities (3.48) and (3.43), and assumption (3.39), we obtain

$$\|(\eta_p, \eta_\lambda)\|_{\mathbf{Q}} \leq c\big\{\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\delta_p, \delta_\lambda)\|_{\mathbf{Q}}\big\},$$

which together to the triangle inequality implies

$$\begin{aligned} \|(\mathrm{e}_p, \mathrm{e}_\lambda)\|_{\mathbf{Q}} &\leq \|(\eta_p, \eta_\lambda)\|_{\mathbf{Q}} + \|(\delta_p, \delta_\lambda)\|_{\mathbf{Q}} \\ &\leq \tilde{c}\big\{\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}, \boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\delta_p, \delta_\lambda)\|_{\mathbf{Q}}\big\}, \end{aligned} \tag{3.49}$$

with $\tilde{c} > 0$ independent of h.

Therefore, recalling that $\bar{\mathbf{v}}_h \in \mathbf{V}_h$ and $(\bar{q}_h, \bar{\lambda}_h) \in \mathbf{Q}_h$ are arbitrary, from (3.48) and (3.49) we obtain

$$\begin{aligned} &\|((\mathbf{e}_{\mathbf{u}_{\rm S}}, \mathbf{e}_{\mathbf{u}_{\rm D}}), (\mathrm{e}_p, \mathrm{e}_\lambda))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\quad \leq C\Big\{\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} + \inf_{(q_h,\xi_h)\in\mathbf{Q}_h} \|(p, \lambda) - (q_h, \xi_h)\|_{\mathbf{Q}}\Big\}. \end{aligned}$$

We conclude the proof by recalling that the discrete inf-sup condition (3.27), and a classical result on mixed methods (see, for instance [17, Theorem 2.6]) ensure the existence of a constant $c > 0$, independent of h, such that

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} \leq c \inf_{\mathbf{v}_h\in\mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}}.$$

□

Now, in order to provide the theoretical rate of convergence of the Galerkin scheme (3.1), we recall the approximation properties of the subspaces involved (see, e.g. [2, 6, 8, 17, 36]). Note that each one of them is named after the unknown to which it is applied later on.

$(\mathbf{AP}_h^{\mathbf{u}_{\rm S}})$ For each $\mathbf{v}_{\rm S} \in \mathbf{H}^2(\Omega_{\rm S})$, there holds

$$\|\mathbf{v}_{\rm S} - \Pi_{\rm S}(\mathbf{v}_{\rm S})\|_{1,\Omega_{\rm S}} \leq C\,h\,\|\mathbf{v}_{\rm S}\|_{2,\Omega_{\rm S}}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_{\rm D}})$ For each $\mathbf{v}_{\rm D} \in \mathbf{H}^1(\Omega_{\rm D})$ with $\mathrm{div}\,\mathbf{v}_{\rm D} \in H^1(\Omega_{\rm D})$, there holds

$$\|\mathbf{v}_{\rm D} - \Pi_{\rm D}(\mathbf{v}_{\rm D})\|_{\mathrm{div}\,;\Omega_{\rm D}} \leq C\,h\Big\{\|\mathbf{v}_{\rm D}\|_{1,\Omega_{\rm D}} + \|\mathrm{div}\,\mathbf{v}_{\rm D}\|_{1,\Omega_{\rm D}}\Big\}\,.$$

$(\mathbf{AP}_h^{p_h})$ For each $q \in H^1(\Omega) \cap L_0^2(\Omega)$, there exists $q_h \in L_{h,0}(\Omega)$ such that

$$\|q - q_h\|_{0,\Omega} \leq C\,h\,\|q\|_{1,\Omega}\,.$$

$(\mathbf{AP}_h^{\lambda})$ For each $\xi \in H^{3/2}(\Sigma)$, there exists $\xi_h \in \Lambda_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \leq C\,h\,\|\xi\|_{3/2,\Sigma}\,.$$

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (3.1), under suitable regularity assumptions on the exact solution.

Theorem 3.4 *Let* $\mathbf{f}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S})$ *and* $\mathbf{f}_{\rm D} \in \mathbf{L}^2(\Omega_{\rm D})$*, such that* (3.39) *holds. Assume that* (3.26) *holds. Let* $(\mathbf{u},(p,\lambda)) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\mathbf{u}_h,(p_h,\lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.7) *and* (3.1)*, respectively, and assume that* $\mathbf{u}_{\rm S} \in \mathbf{H}^2(\Omega_{\rm S})$, $\mathbf{u}_{\rm D} \in \mathbf{H}^1(\Omega_{\rm D})$, $\mathrm{div}\,\mathbf{u}_{\rm D} \in H^1(\Omega_{\rm D})$*,* $p \in H^1(\Omega)$*, and* $\lambda \in H^{3/2}(\Sigma)$*. Then there exists* $C > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned}\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq C\,h\Big\{&\|\mathbf{u}_{\rm S}\|_{2,\Omega_{\rm S}} + \|\mathbf{u}_{\rm D}\|_{1,\Omega_{\rm D}}\\ &+ \|\mathrm{div}\,\mathbf{u}_{\rm D}\|_{1,\Omega_{\rm D}} + \|p\|_{1,\Omega} + \|\lambda\|_{3/2,\Sigma}\Big\}\,.\end{aligned} \tag{3.50}$$

Proof. It is suffices to apply Theorem 3.3 and the approximation properties of the discrete subespaces. We omit further details. □

4 Numerical results

In this section we present some examples illustrating the performance of our mixed finite element scheme (3.1) on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a *FreeFem++* code [26], in conjunction with the direct linear solver *UMFPACK* [11].

In order to solve the nonlinear problem, we propose the Newton-type strategy: Given $\mathbf{u}^0 = (\mathbf{u}_{\rm S}^0,\mathbf{u}_{\rm D}^0) \in \mathbf{H}_h$, $p^0 \in L_{h,0}(\Omega)$ and $\lambda^0 \in \Lambda_h(\Sigma)$, for $m \geq 1$, find $\mathbf{u}^m = (\mathbf{u}_{\rm S}^m,\mathbf{u}_{\rm D}^m) \in \mathbf{H}_h$, $p^m \in L_{h,0}(\Omega)$ and $\lambda^m \in \Lambda_h(\Sigma)$, such that

$$\begin{aligned}&A_{\rm S}(\mathbf{u}_{\rm S}^m,\mathbf{v}_{\rm S}) + O_{\rm S}^h(\mathbf{u}_{\rm S}^{m-1};\mathbf{u}_{\rm S}^m,\mathbf{v}_{\rm S}) + O_{\rm S}^h(\mathbf{u}_{\rm S}^m;\mathbf{u}_{\rm S}^{m-1},\mathbf{v}_{\rm S})\\ &\qquad + A_{\rm D}(\mathbf{u}_{\rm D}^m,\mathbf{v}_{\rm D}) + \mathbf{b}(\mathbf{v},(p^m,\lambda^m)) = O_{\rm S}^h(\mathbf{u}_{\rm S}^{m-1};\mathbf{u}_{\rm S}^{m-1},\mathbf{v}_{\rm S}) + \mathbf{f}(\mathbf{v})\\ &\mathbf{b}(\mathbf{u}^m,(q,\xi)) = 0,\end{aligned} \tag{4.1}$$

for all $\mathbf{v} = (\mathbf{v}_{\rm S},\mathbf{v}_{\rm D}) \in \mathbf{H}_h$, $q \in L_{h,0}(\Omega)$ and $\xi \in \Lambda_h(\Sigma)$.

In all the numerical experiments below, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is,

$$\frac{\|\mathbf{Coeff}^{m+1}-\mathbf{Coeff}^{m}\|_{l^2}}{\|\mathbf{Coeff}^{m+1}\|_{l^2}}\leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$, and tol is a fixed tolerance chosen as $tol = 1e-06$. For each example shown below we simply take $\mathbf{u}^0=\mathbf{0}$ and $(p^0,\lambda^0)=\mathbf{0}$ as initial guess.

We now introduce some additional notations. We denote by $h_\Sigma := \max\{h_e : e\in\Sigma_{2h}\}$. As in Section 3.3, the individual errors are denoted by $\mathbf{e}_{\mathbf{u}_{\mathrm{S}}}=\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{h,\mathrm{S}}$, $\mathbf{e}_{\mathbf{u}_{\mathrm{D}}}=\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{h,\mathrm{D}}$, $\mathrm{e}_{p_{\mathrm{S}}}=p_{\mathrm{S}}-p_{h,\mathrm{S}}$, $\mathrm{e}_{p_{\mathrm{D}}}=p_{\mathrm{D}}-p_{h,\mathrm{D}}$ and $\mathrm{e}_\lambda=\lambda-\lambda_h$. Also, we let $\mathbf{r}_{\mathbf{u}_{\mathrm{S}}}$, $\mathbf{r}_{\mathbf{u}_{\mathrm{D}}}$, $\mathrm{r}_{p_{\mathrm{S}}}$, $\mathrm{r}_{p_{\mathrm{D}}}$ and r_λ be the experimental rates of convergence given by

$$\mathbf{r}_{\mathbf{u}_{\mathrm{S}}}:=\frac{\log(\mathbf{e}_{\mathbf{u}_{\mathrm{S}}}/\mathbf{e}'_{\mathbf{u}_{\mathrm{S}}})}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})},\quad \mathbf{r}_{\mathbf{u}_{\mathrm{D}}}:=\frac{\log(\mathbf{e}_{\mathbf{u}_{\mathrm{D}}}/\mathbf{e}'_{\mathbf{u}_{\mathrm{D}}})}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})},$$
$$\mathrm{r}_{p_{\mathrm{S}}}:=\frac{\log(\mathrm{e}_{p_{\mathrm{S}}}/\mathrm{e}'_{p_{\mathrm{S}}})}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})},\quad \mathrm{r}_{p_{\mathrm{D}}}:=\frac{\log(\mathrm{e}_{p_{\mathrm{D}}}/\mathrm{e}'_{p_{\mathrm{D}}})}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})}\mathrm{r}_\lambda:=\frac{\log(\mathrm{e}_\lambda/\mathrm{e}'_\lambda)}{\log(h_\Sigma/h'_\Sigma)},$$

where $h_\star$ and $h'_\star$ ($\star\in\{\mathrm{S},\mathrm{D},\Sigma\}$) denote two consecutive mesh sizes with their respective errors $\mathbf{e}$, $\mathbf{e}'$ (or e, e$'$).

For each example below we consider the parameters $\alpha_d=1$, $\rho=1$, $\kappa=1$ and $\mathbf{K}=\mathbf{I}$.

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega=\Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$, with $\Omega_{\mathrm{S}}:=(-1/2,1/2)\times(0,1/2)$ and $\Omega_{\mathrm{D}}:=(-1/2,1/2)\times(-1/2,0)$. We consider the viscosity $\mu=1$ and the terms on the right-hand side are adjusted so that the exact solution is given by the functions

$$\mathbf{u}_{\mathrm{S}}(x_1,x_2)=\begin{pmatrix}16x_2\cos(\pi x_1)^2(x_2^2-1/4)\\[2ex] 8\pi\cos(\pi x_1)\sin(\pi x_1)(x_2^2-1/4)^2\end{pmatrix}\quad\text{in}\quad\Omega_{\mathrm{S}},$$

$$\mathbf{u}_{\mathrm{D}}(x_1,x_2)=\begin{pmatrix}-2x_2\cos(\pi x_1)^2\\[2ex] -2\pi\cos(\pi x_1)\sin(\pi x_1)(x_2^2-1/4)\end{pmatrix}\quad\text{in}\quad\Omega_{\mathrm{D}},$$

$$p_\star(x_1,x_2)=e^{x_2}\sin(x_1)\quad\text{in}\quad\Omega_\star,$$

with $\star\in\{\mathrm{S},\mathrm{D}\}$. Notice that, $\mathbf{u}_{\mathrm{S}}=\mathbf{u}_{\mathrm{D}}$ on Σ.

In Table 4.1 we summarize the convergence history for a sequence of quasi-uniform triangulations. We observe that the rate of convergence $O(h)$ predicted by Theorem 3.4 is attained in all the cases. Next, in Figure 4.1 we display (to the left) the vector field of the approximate velocity $\mathbf{u}_h$ and the magnitude of the error $|\mathbf{u}-\mathbf{u}_h|$ (to the right) with $N=443758$. Notice that our method preserves the direction of the velocities on Σ as expected. Also observe that the maximum value of the error in Ω_{S} is of the order of $3e-05$ whereas in Ω_{D} is of the order of $3e-03$. In addition, in Figure 4.2 we display (to the left) the approximate pressure and the magnitude of the error $|p-p_h|$ (to the right) with $N=443758$. Notice that the maximum value of $|p - p_h|$ in Ω_S is of the order of $3e-02$, whereas in Ω_D is of the order of $2e-05$. As noted from Figures 4.1 and 4.2 (to the right), the approximation is not very accurate in those regions of high gradients. Nevertheless, this aspect could be easily fixed by applying an adaptive algorithm based on suitable a posteriori error estimates.

Table 4.1: EXAMPLE 1: Degrees of freedom N, mesh sizes $h_\star$, errors, and rates of convergence for the mixed approximation of the Navier-Stokes/Darcy problem with $\mu = 1$.

N	h_S	$\mathbf{e}_{\mathbf{u}_S}$	$\mathbf{r}_{\mathbf{u}_S}$	e_{p_S}	r_{p_S}
491	0.1875	0.3844	–	0.1617	–
1824	0.1085	0.1799	1.3878	0.0699	1.5331
7099	0.0500	0.0916	0.8713	0.0341	0.9253
27986	0.0274	0.0450	1.1861	0.0168	1.1847
111931	0.0131	0.0228	0.9135	0.0078	1.0343
443758	0.0071	0.0113	1.1499	0.0039	1.1414

N	h_D	$\mathbf{e}_{\mathbf{u}_D}$	$\mathbf{r}_{\mathbf{u}_D}$	e_{p_D}	r_{p_D}
491	0.2001	0.0847	–	0.0154	–
1824	0.0938	0.0433	0.8844	0.0077	0.9224
7099	0.0494	0.0211	1.1227	0.0038	1.0751
27986	0.0262	0.0107	1.0747	0.0019	1.0779
111931	0.0141	0.0053	1.1257	0.0009	1.1283
443758	0.0070	0.0027	0.9796	0.0004	0.9868

N	h_Σ	e_λ	r_λ
491	0.1250	0.0304	–
1824	0.0625	0.0114	1.4132
7099	0.0312	0.0050	1.1924
27986	0.0156	0.0027	0.8994
111931	0.0078	0.0013	1.0430
443758	0.0039	0.0006	0.9898

In our second example we focus on the performance of the iterative method (4.1) with respect to the viscosity μ. To do this we consider the domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, with $\Omega_S := (-1/2, 3/2) \times (0, 1/2)$ and $\Omega_D := (-1/2, 3/2) \times (-1/2, 0)$. Then, the terms on the right-hand

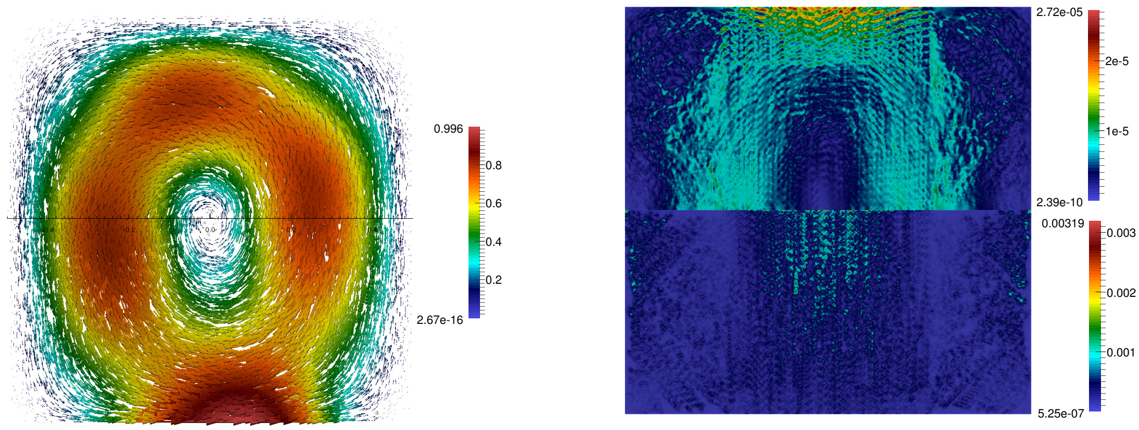

Figure 4.1: Example 1: Velocity vector fields $\mathbf{u}_h$ (left) and $|\mathbf{u} - \mathbf{u}_h|$ (right) with $N = 443758$.

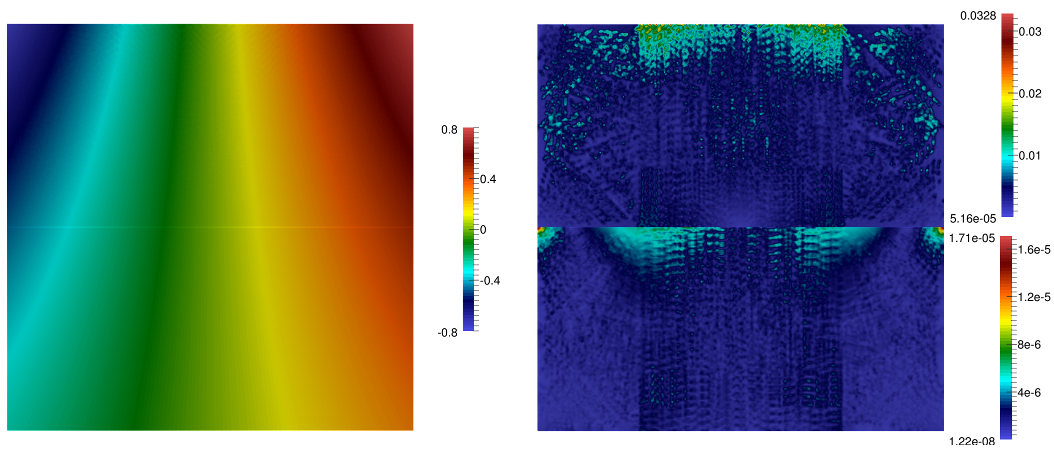

Figure 4.2: Example 1: p_h (left) and $|p - p_h|$ (right) with $N = 443758$.

side are adjusted so that the exact solution is given by the functions:

$$\mathbf{u}_{\mathrm{S}}(x_1, x_2) \;=\; \begin{pmatrix} 1 - e^{\gamma x_1}\cos(2\pi x_2) \\[2ex] \frac{\gamma}{2\pi} e^{\gamma x_1} \sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{S}}\,,$$

$$p_{\mathrm{S}}(x_1, x_2) \;=\; -\frac{1}{2} e^{2\gamma x_1} + c \quad \text{in} \quad \Omega_{\mathrm{S}},$$

$$\mathbf{u}_{\mathrm{D}}(x_1, x_2) \;=\; \begin{pmatrix} (x_1 + 0.5)(x_1 - 1.5) \\[2ex] -(x_2 + 2)(2x_1 - 1.0) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{D}}\,,$$

$$p_{\mathrm{D}}(x_1, x_2) = (x_1 - 0.5)^3 (x_2 + 1) \quad \text{in} \quad \Omega_{\mathrm{D}}$$

where

$$\gamma := \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-2} + 16\pi^2}}.$$

and the constant c is such that $\int_\Omega p = 0$. Notice that $(\mathbf{u}_{\mathrm{S}}, p_{\mathrm{S}})$ is the well known analytical solution for the Navier-Stokes problem obtained by Kovasznay in [31], which presents a boundary layer at $\{-1/2\} \times (0, 2)$.

In Table 4.2 we show the behaviour of the iterative method (4.1) as a function of the viscosity μ, considering different mesh sizes $h := \max\{h_{\mathrm{S}}, h_{\mathrm{D}}\}$, and a tolerance $tol = 1e{-}06$. Here we observe that the smaller the parameter μ the higher the number of iterations as it occurs also in the Newton method for the sole Navier-Stokes equations. Numerical experiments for smaller values of μ are not reported since the iterative methods need too many iterations to converge (more than 100). Next, the numerical results in Table 4.3 show the convergence history for a sequence of quasi-uniform triangulations, considering the viscosity $\mu = 0.1$. We see there that the rate of convergence $O(h)$ provided by Theorem 3.4 is attained by the unknowns.

Table 4.2: Example 2: Convergence behavior of the iterative method (4.1) with respect to the parameter μ.

μ	$h = 0.4129$	$h = 0.1955$	$h = 0.1084$	$h = 0.0517$	$h = 0.0320$
1	5	5	5	5	5
0.1	6	6	6	6	6
0.01	9	7	7	7	7

References

[1] A. Agouzal and J.-M. Thomas. An extension theorem for equilibrium finite elements spaces. *Japan J. Indust. Appl. Math.*, 13(2):257–266, 1996.

[2] A.K. Aziz and I. Babuška. Survey lectures on the mathematical foundations of the finite element method. In *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. B. Kellogg.

[3] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier-Stokes/Darcy coupling. *Numer. Math.*, 115(2):195–227, 2010.

[4] S. Badia and R. Codina. Stokes, Maxwell and Darcy: a single finite element approximation for three model problems. *Appl. Numer. Math.*, 62(4):246–263, 2012.

[5] G.S. Beavers and D.D. Joseph. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.*, 30:197–207, 1967.

[6] C. Bernardi and G. Raugel. Analysis of some finite elements for the Stokes problem. *Math. Comp.*, 44(169):71–79, 1985.

Table 4.3: EXAMPLE 2: Degrees of freedom N, mesh sizes $h_\star$, errors, and rates of convergence for the mixed approximation of the Navier-Stokes/Darcy problem with $\mu = 0.1$.

N	h_{S}	$\mathbf{e}_{\mathbf{u}_{\mathrm{S}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{S}}}$	$\mathrm{e}_{p_{\mathrm{S}}}$	$\mathrm{r}_{p_{\mathrm{S}}}$
1017	0.3802	6.1181	–	1.3192	–
3763	0.1955	3.0741	1.0349	0.6799	0.9967
14725	0.1084	1.5002	1.2171	0.3251	1.2518
59191	0.0506	0.7505	0.9080	0.1611	0.9205
235996	0.0316	0.3712	1.4936	0.0778	1.5436

N	h_{D}	$\mathbf{e}_{\mathbf{u}_{\mathrm{D}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{D}}}$	$\mathrm{e}_{p_{\mathrm{D}}}$	$\mathrm{r}_{p_{\mathrm{D}}}$
1017	0.4129	0.4374	–	0.1071	–
3763	0.1955	0.2043	1.0182	0.0452	1.1538
14725	0.0998	0.1003	1.0585	0.0230	1.0063
59191	0.0517	0.0492	1.0821	0.0110	1.1218
235996	0.0320	0.0246	1.4383	0.0054	1.4664

N	h_Σ	e_λ	r_λ
1017	0.1250	0.3677	–
3763	0.0625	0.1378	1.4158
14725	0.0312	0.0555	1.3121
59191	0.0156	0.0220	1.3325
235996	0.0078	0.0099	1.1617

[7] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle, sér. Rouge*, 8:129–151, 1974.

[8] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Method.* Springer, New York, 1991.

[9] Y. Cao, M. Gunzburger, X. He, and X. Wang. Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition. *Numer. Math.*, 117(4):601–629, 2011.

[10] W. Chen, M. Gunzburger, F. Hua, and X. Wang. A parallel Robin-Robin domain decomposition method for the Stokes-Darcy system. *SIAM J. Numer. Anal.*, 49(3):1064–1084, 2011.

[11] T.A. Davis. Algorithm 832: UMFPACK V4.3–an unsymmetric-pattern multifrontal method. *ACM Transactions on Mathematical Software*, 30(2):196–199, 2004.

[12] M. Discacciati. *Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows.* PhD thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2004.

[13] M. Discacciati. Coupling free and porous-media flows: models and numerical approximation. In *Simulation of Flow in Porous Media. Applications in Energy and Environment*,

volume 12 of *Radon Series on Computational and Applied Mathematics*, pages 107–138. De Gruyter, Berlin and Boston, 2013.

[14] M. Discacciati, E. Miglio, and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57–74, 2002.

[15] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer, New York, 2004.

[16] J. Galvis and M. Sarkis. Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations. *Electron. Trans. Numer. Anal.*, 26:350–384, 2007.

[17] G.N. Gatica. *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*. Springer Briefs in Mathematics. Springer, 2014.

[18] G.N. Gatica, G.C. Hsiao, and S. Meddahi. A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body. *SIAM J. Numer. Anal.*, 48(4):1338–1368, 2010.

[19] G.N. Gatica, S. Meddahi, and R. Oyarzúa. A conforming mixed finite-element method for the coupling of fluid flow with porous media flow. *IMA J. Numer. Anal.*, 29(1):86–108, 2009.

[20] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas. Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem. *Math. Comp.*, 80(276):1911–1948, 2011.

[21] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas. Convergence of a family of Galerkin discretizations for the Stokes-Darcy coupled problem. *Numer. Methods Partial Differential Equations*, 27(3):721–748, 2011.

[22] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas. A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow. *IMA J. Numer. Anal.*, 32(3):845–887, 2012.

[23] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Springer, Berlin, 1986.

[24] V. Girault and B. Rivière. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM J. Numer. Anal.*, 47:2052–2089, 2009.

[25] V. Girault, D. Vassilev, and I. Yotov. Mortar multiscale finite element methods for Stokes-Darcy flows. *Numer. Math.*, 127(1):93–165, 2014.

[26] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[27] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.

[28] W. Jäger and A. Mikelić. On the interface boundary condition of Beavers, Joseph and Saffman. *SIAM J. Appl. Math.*, 60(4):1111–1127, 2000.

[29] G. Kanschat and B. Rivière. A strongly conservative finite element method for the coupling of Stokes and Darcy flow. *J. Comput. Phys.*, 229(17):5933–5943, 2010.

[30] T. Karper, K.-A. Mardal, and R. Winther. Unified finite element discretizations of coupled Darcy-Stokes flow. *Numer. Methods Partial Differential Equations*, 25(2):311–326, 2009.

[31] L. I. G. Kovasznay. Laminar flow behind a two-dimensional grid. *Proceedings of the Cambridge Philosophical Society.*, 44:58–62, 1948.

[32] W.L. Layton, F. Schieweck, and I. Yotov. Coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 40:2195–2218, 2003.

[33] K. Lipnikov, D. Vassilev, and I. Yotov. Discontinuous Galerkin and mimetic finite difference methods for coupled Stokes-Darcy flows on polygonal and polyhedral grids. *Numer. Math.*, 126(2):321–360, 2014.

[34] A. Márquez, S. Meddahi, and F.-J. Sayas. Strong coupling of finite element methods for the Stokes-Darcy problem. *IMA J. Numer. Anal.*, 2014.

[35] P.-A. Raviart and J.M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical Aspects of Finite Element Methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.

[36] J.E. Roberts and J.-M. Thomas. Mixed and Hybrid Methods. In *Handbook of Numerical Analysis, Vol. II*, pages 523–639. North-Holland, Amsterdam, 1991.

[37] P.G. Saffman. On the boundary condition at the interface of a porous medium. *Stud. Appl. Math.*, 1:93–101, 1971.

[38] R. Temam. *Navier-Stokes Equations. Theory and Numerical Analysis.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.

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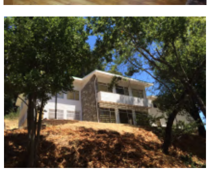
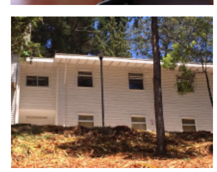

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