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EXPONENTIAL STABILITY TO TIMOSHENKO SYSTEM WITH SHEAR BOUNDARY DISSIPATION

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Abstract. In this paper we consider a Timoshenko model with boundary dissipation over the shear force effective in only one side. We prove that to get the exponential stability of the related contraction semigroup, the equality of the wave propagations is not enough, it is necessary additional conditions over the coefficient of the system.

1. Introduction

In this work, we study the stabilization of a Timoshenko model system which arises in the theory of the transverse vibration of a beam. This system is given by

$$\begin{aligned} &\rho_1\,\varphi_{tt}-\kappa\,(\varphi_x+\psi)_x=0 && \text{in } (0,\ell)\times(0,\infty),\\ &\rho_2\,\psi_{tt}-b\,\psi_{xx}+\kappa\,(\varphi_x+\psi)=0 && \text{in } (0,\ell)\times(0,\infty), \end{aligned} \tag{1.1}$$

where t denotes the time variable, x the space variable along the beam of length ℓ in its equilibrium configuration; $S=\kappa(\varphi_x+\psi)$ and $M=b\psi_x$ denote the shear force and the bending moment, respectively. We denote by $\varphi=\varphi(x,t)$ the transversal displacement(vertical deflection) and $\psi=\psi(x,t)$ is the rotation angle of the filament. Here $\rho_1=\rho A$, $\rho_2=\rho I$, $\kappa=KAG$, $b=EI$, where ρ denotes the density, A is the cross-sectional area, I is the area moment of inertia, E is the modulus of elasticity, K is the shear factor and G is the shear modulus.

Here from on we consider the system (1.1) with the initial conditions

$$\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\quad \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x)\ \text{ in } (0,\ell), \tag{1.2}$$

and the boundary conditions

$$\begin{aligned} &\varphi(\ell,t)=0, && \psi(0,t)=0 && \text{in } (0,\infty),\\ &\kappa\,\varphi_x(0,t)=\gamma\,\varphi_t(0,t), && \psi_x(\ell,t)=0, && \text{in } (0,\infty), \end{aligned} \tag{1.3}$$

with $\gamma>0$.

Several authors studied the Timoshenko model with different mechanism of dissipation and the most of them considered this mechanism effective only on the bending moment. We mention here a few of them. For Timoshenko model with frictional damping we can refer to, e.g., [1, 22]. There are also several works analyzing Timoshenko models with thermal dissipation, depending on the Fourier law (see, e.g., [2]) or the Maxwell-Cattaneo law (see, e.g., [10, 21]) or the Pikpin and Gurtin constitutive law (see, e.g. [8]) for the heat flux or with thermoelasticity of type III (see, e.g., [15, 16]). In case of the Timoshenko model with dissipative memory effect, we recall [5, 9, 14]. In all this cases, except for the Timoshenko model with Maxwell-Cattaneo law (see [8, 10]), the condition necessary and sufficient for a corresponding semigroup associated to system be exponentially stable is that the wave speeds of the system are equals, that is,

$$\frac{\kappa}{\rho_1}=\frac{b}{\rho_2}. \tag{1.4}$$

For dissipation boundary to model Timoshenko we have the pioneer work due to Kim and Renardy [11]. They showed that the model with the dissipation effective in the shear force and

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the bending moment is exponentially stable. This result has been extended in [24] for a more general dissipative boundary condition.

Ammar-Khodja et al. in [6] proved that the exponential stability holds "up to a finite dimensional space of initial data", provided that the dissipative mechanism is effective in both sides of the boundary of the bending moment (see [7, Theorem 2]). This means that the authors do not consider dissipative mechanism over the shear force, they only consider over the bending moment, but they claim that is possible that there exist some initial data, in a finite dimensional space, for which the exponentially stability does not holds.

Concerning of the Timoshenko system with only one boundary dissipation mechanism the situation is completely different. That is to say, in this case the equality of the speed waves is not sufficient to establish exponential stability. In fact, Bassam et al. [7] showed the polynomial stability for the Timoshenko system with boundary dissipation mechanism only on one side of the bending moment, provided the coefficients of the system satisfy some conditions.

The main result of this paper is to prove that when the length ℓ of the interval is small the corresponding semigroup associated to system (1.1)-(1.3) is exponential stability if only if the condition (1.4) holds and

$$\frac{(j_1^2-j_2^2)^2}{j_1^2+j_2^2}\neq\frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2, \tag{1.5}$$

for all natural odd numbers $j_1,\ j_2,\ j_1\neq j_2$.

The remaining part of this paper is organized as follows. In the next section 2, we show that the model is well posed. In section 3 we show that there exists the condition that ensure the strong stability of the semigroup associated to system (1.1)-(1.3) and in section 4 we prove that it is exponential stable when (1.4)-(1.5) hold. Our main tools are recent results due to Prüss [20] as well as spectral arguments. Finally, in section 5 we calculate numerically some large eigenvalues near the imaginary axis using the Chebyshev-tau method [18], and we present some numerical results illustrating the asymptotic behavior of the energy based on Finite Differences of second order and the $\beta-$Newmark Method [12].

Throughout this paper, C is a generic constant, not necessarily the same at each occasion (it will change line to line), which depends in an increasing way on the indicated quantities.

2. Existence and uniqueness

In this section we will show the well-posedness of the problem (1.1)–(1.3) using the semigroup techniques (see [19]).

Firstly, we introduce the notation. Given a Banach space X, let $\|\cdot\|_X$ be the usual norm defined on X. In particular, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm defined on $L^2(0,\ell)$, respectively. Before stating the existence and the uniqueness result of problem (1.1)–(1.3), we first set-up the following short-hand notation for function space

$$\begin{aligned} H^1_\ell(0,\ell) &= \left\{u\in H^1(0,\ell):\quad u(\ell)=0\right\},\\ H^1_*(0,\ell) &= \left\{u\in H^1(0,\ell):\quad u(0)=0\right\}.\end{aligned}$$

The phase space of our problem is the Hilbert space

$$\mathcal{H}=H^1_l(0,\ell)\times L^2(0,\ell)\times H^1_*(0,\ell)\times L^2(0,\ell),$$

provided by inner product

$$\begin{aligned}\langle(\varphi_1,\Phi_1,\psi_1,\Psi_1),(\varphi_2,\Phi_2,\psi_2,\Psi_2)\rangle_{\mathcal{H}} &= \kappa\int_0^\ell(\varphi_{1x}+\psi_1)\overline{(\varphi_{2x}+\psi_2)}dx\ +\ b\int_0^\ell\psi_{1x}\overline{\psi_{2x}}dx\\ &\quad+\rho_1\int_0^\ell\Phi_1\overline{\Phi_2}dx\ +\ \rho_2\int_0^\ell\Psi_1\overline{\Psi_2}dx\end{aligned}$$

and normed by

$$\|(\varphi, \Phi, \psi, \Psi)\|_{\mathcal{H}}^2 = \kappa\|\varphi_x + \psi\|^2 + \rho_1\|\Phi\|^2 + b\|\psi_x\|^2 + \rho_2\|\Psi\|^2.$$

Putting $\Phi = \varphi_t$ and $\Psi = \psi_t$ and introducing the state vector

$$U(t) = (\varphi(t), \Phi(t), \psi(t), \Psi(t))^\top,$$

the system (1.1)–(1.3) can be written as a Cauchy problem in $\mathcal{H}$ of the form

$$\begin{aligned} &\frac{d}{dt}U(t) = \mathbb{A}\, U(t), \\ &U(0) = U_0, \end{aligned} \tag{2.1}$$

where $U_0 = (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1)^\top$ and $\mathbb{A} : \mathcal{D}(\mathbb{A}) \to \mathcal{H}$ is the operator linear given by

$$\mathbb{A} = \begin{bmatrix} 0 & Id & 0 & 0 \\ \dfrac{\kappa}{\rho_1}(.)_{xx} & 0 & \dfrac{\kappa}{\rho_1}(.)_x & 0 \\ 0 & 0 & 0 & Id \\ -\dfrac{\kappa}{\rho_2}(.)_x & 0 & \dfrac{b}{\rho_2}(.)_{xx} - \dfrac{\kappa}{\rho_2}Id & 0 \end{bmatrix}$$

while domain $\mathcal{D}(\mathbb{A})$ is the subspace

$$\mathcal{D}(\mathbb{A}) = \left\{U \in \mathcal{H} : \varphi, \psi \in H^2(0,\ell), \Phi \in H^1_\ell(0,\ell), \Psi \in H^1_*(0,\ell), \kappa\varphi_x(0) - \gamma\Phi(0) = 0, \psi_x(\ell) = 0\right\}.$$

It is not difficult to show that the operator $\mathbb{A}$ is dissipative, that is,

$$\mathrm{Re}\langle \mathbb{A}U, U\rangle_{\mathcal{H}} = -\gamma|\Phi(0)|^2 \le 0, \quad \forall\, U \in \mathcal{D}(\mathbb{A}), \tag{2.2}$$

and that 0 belongs to the resolvent set $\rho(\mathbb{A})$. Consequently, it follows from the Lumer Phillips Theorem that the operator $\mathbb{A}$ generates a C_0-semigroup of contractions

$$S_{\mathbb{A}}(t) = e^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}$$

on the space $\mathbb{H}$. Hence

Proposition 2.1. *For any $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)^\top \in \mathcal{H}$ the problem* (2.1) *has a unique weak solution $U(t) = S_{\mathbb{A}}(t)\, U_0 = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t))^\top$ in $U \in C([0,\infty) :\ \mathcal{H})$. Moreover, if $U_0 \in \mathcal{D}(\mathbb{A}^n)$ then*

$$U \in C^{n-k}([0,\infty) :\ [\mathcal{D}(\mathbb{A}^k))]) \cap C^n([0,\, \infty) :\ \mathcal{H}),\ k = 1,\ \dots,\ n.$$

3. Strong stability

In this section we will prove the uniform stability of the semigroup $\{S_{\mathbb{A}}(t)\}_{t\ge 0}$. Therefore we will need to study the resolvent equation $(i\lambda\,\mathbb{I} - \mathbb{A})U = F$, namely

$$i\lambda\varphi - \Phi = f_1 \quad \text{in } H^1_\ell(0,\ell), \tag{3.1}$$

$$i\lambda\rho_1\,\Phi - \kappa\,(\varphi_x + \psi)_x = f_2 \quad \text{in } L^2(0,\ell), \tag{3.2}$$

$$i\lambda\psi - \Psi = f_3 \quad \text{in } H^1_*(0,\ell), \tag{3.3}$$

$$i\lambda\rho_2\,\Psi - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = f_4 \quad \text{in } L^2(0,\ell), \tag{3.4}$$

where $F = (f_1, f_2, f_3, f_4)^\top \in \mathcal{H}$, $U = (\varphi, \Phi, \psi, \Psi) \in \mathcal{D}(\mathbb{A})$ and $\lambda \in \mathbb{R}$. Taking the real part of the inner product in $\mathcal{H}$ of $\mathbb{A}U$ with $U \in \mathcal{D}(\mathbb{A})$ we obtain

$$|\mathrm{Re}\langle \mathbb{A}U, U\rangle_{\mathcal{H}}| \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{3.5}$$

and using (2.2) we get

$$|\Phi(0)|^2 \leq \frac{1}{\gamma}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.6}$$

By (3.1) we conclude that

$$|\varphi(0)|^2 \leq \frac{C}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{\lambda^2}\|F\|^2_{\mathcal{H}} \tag{3.7}$$

for a positive constant C. Since $\kappa\varphi(0) = \gamma\Phi(0)$, it follows that

$$|\varphi_x(0)|^2 \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{3.8}$$

for a positive constant C.

Lemma 3.1. *The imaginary axis $i\mathbb{R}$ is contained in resolvent set $\rho(\mathbb{A})$ if only if*

$$\frac{(\rho_1 b j_1^2 - \rho_2\kappa j_2^2)(\rho_2\kappa j_1^2 - \rho_1 b j_2^2)}{(j_1^2 + j_2^2)} \neq \rho_1\kappa(\rho_1 b + \rho_2\kappa)\left(\frac{2\ell}{\pi}\right)^2 \tag{3.9}$$

for all natural odd numbers $j_1,\ j_2,\ j_1 \neq j_2$.

Proof. Since $\mathcal{D}(\mathbb{A})$ has compact immersion over the phase space $\mathcal{H}$, the bounded linear operator $\mathbb{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is compact and then the spectrum $\sigma(\mathbb{A})$ has only eigenvalues. We will prove that $\mathbb{A}$ does not have pure imaginary eigenvalues.

By contradiction argument, let $0 \neq U = (\varphi, \Phi, \psi, \Psi) \in \mathcal{D}(\mathbb{A})$ and $0 \neq \lambda \in \mathbb{R}$ such that

$$\mathbb{A}\, U = i\lambda U,$$

manely,

$$i\lambda\varphi - \Phi = 0 \quad \text{in } \ H^1_\ell(0,\ell), \tag{3.10}$$

$$i\lambda\rho_1\,\Phi - \kappa\,(\varphi_x + \psi)_x = 0 \quad \text{in } \ L^2(0,\ell), \tag{3.11}$$

$$i\lambda\psi - \Psi = 0 \quad \text{in } \ H^1_*(0,\ell), \tag{3.12}$$

$$i\lambda\rho_2\,\Psi - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = 0 \quad \text{in } \ L^2(0,\ell), \tag{3.13}$$

Our goal is to find a contradiction by proving that $U = 0$. Since

$$\langle \mathbb{A}U, U\rangle = i\lambda\|U\|^2_{\mathcal{H}}$$

it follows that $\mathrm{Re}\langle \mathbb{A}U, U\rangle = 0$, and from (2.2) we have

$$\Phi(0) = 0.$$

Hence, from (3.10) and by the condition $\kappa\varphi_x(0) = \gamma\Phi(0)$ we obtain

$$\varphi(0) = 0, \quad \varphi_x(0) = 0, \quad \text{and} \quad \Phi_x(0) = 0.$$

Moreover, as $\psi(0) = 0$ and $\varphi_x(0) = 0$ we can conclude by (3.12)-(3.13) that $\psi_{xx}(0) = 0$. Therefore, by (3.11) we have that

$$\varphi_{xxx}(0) = 0.$$

Finally, since that $\psi_x(\ell) = 0$ it follows by (3.11) that

$$\varphi_{xx}(\ell) = 0.$$

From equations (3.10) and (3.11) we have

$$b\psi_{xxx} = -\lambda^2\frac{\rho_1 b}{\kappa}\varphi_{xx} \ - \ b\varphi_{xxxx}$$

and from (3.12) and (3.13) we obtain

$$b\psi_{xxx} = (\kappa - \lambda^2\rho_2)\psi_x \ + \ \kappa\varphi_{xx}.$$

Thus, we get ψ_x in function of φ_{xx} and φ_{xxxx}. Substituting ψ_x into (3.10) we get that φ is a solution of the ODE

$$\kappa b\,\varphi_{xxxx} + (\rho_1 b + \rho_2\kappa)\,\lambda^2\varphi_{xx} + \left(\rho_1\rho_2\lambda^4 - \rho_1\kappa\lambda^2\right)\varphi = 0, \tag{3.14}$$

whose general solutions depends on the roots of the polynomial

$$p(r) = \kappa b\,r^4 + (\rho_1 b + \rho_2\kappa)\,\lambda^2 r^2 + \left(\rho_1\rho_2\lambda^4 - \rho_1\kappa\lambda^2\right), \tag{3.15}$$

which are given by

$$r = \pm\sqrt{-\frac{1}{2}\left(\frac{\rho_1}{\kappa} + \frac{\rho_2}{b}\right)\lambda^2 \pm \frac{1}{2}\sqrt{\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right)^2\lambda^4 + 4\frac{\rho_1}{b}\lambda^2}}.$$

In case of

$$\rho_1\rho_2\lambda^4 - \rho_1\kappa\lambda^2 = 0 \tag{3.16}$$

i.e,

$$\lambda^2 = \frac{\kappa}{\rho_2},$$

we have that $r = 0$ is a double root of (3.15). Denoting by $r_1 = i\alpha$, $r_2 = -i\alpha$, $r_3 = r_4 = 0$ the roots of (3.15), the general solution of (3.14) must be of the form

$$\varphi(x) = c_1 e^{i\alpha x} + c_2 e^{-i\alpha x} + c_3 + c_4 x.$$

Due to the boundary condition $\varphi_{xxx}(0) = 0$ we have that $c_1 = c_2$. Therefore we can write

$$\varphi(x) = 2c_1\cos(\alpha x) + c_3 + c_4 x.$$

Since $\varphi_x(0) = 0$ we have that $c_4 = 0$. Using that $\varphi(0) = 0$ we get that $c_3 = -2c_1$ and thus

$$\varphi(x) = 2c_1(\cos(\alpha x) - 1).$$

Finally, using $\varphi(\ell) = \varphi_{xx}(\ell) = 0$, we obtain

$$2c_1(\cos(\alpha\ell) - 1) = 0 \quad \text{and} \quad 2c_1\alpha^2\cos(\alpha\ell) = 0.$$

Therefore we obtain that $c_1 = 0$. This implies that $\varphi = 0$. Therefore, using (3.11) we get that $\psi = 0$. So we arrive at contradiction. We conclude that the equality (3.16) does not hold.

Now we can suppose that the polynomial $p(r)$, given in (3.15), has two imaginary roots and two real roots, then its roots are of the type $r_1 = i\alpha_1$, $r_2 = -i\alpha_1$, $r_3 = \alpha_2$, $r_4 = -\alpha_2$ with $|\alpha_1| \neq |\alpha_2|$. Therefore the solution of the ODE (3.14) can be written as

$$\varphi(x) = c_1 e^{i\alpha_1 x} + c_2 e^{-i\alpha_1 x} + c_3 e^{\alpha_2 x} + c_4 e^{-\alpha_2 x}.$$

Using that $\varphi_x(0) = \varphi_{xxx}(0) = 0$ we obtain that

$$i\alpha_1(c_1 - c_2) + \alpha_2(c_3 - c_4) = 0, \quad -i\alpha_1^3(c_1 - c_2) + \alpha_2^3(c_3 - c_4) = 0,$$

and we conclude that

$$c_1 = c_2 \quad \text{and} \quad c_3 = c_4.$$

Since $\varphi(0) = 0$ we have $c_1 = -c_3$ and , hence the function φ must be of the form

$$\varphi(x) = 2c_1[\cos(\alpha_1 x) - \cosh(\alpha_2 x)].$$

Proceeding as before, if $c_1 \neq 0$, using that $\varphi(\ell) = \varphi_{xx}(\ell) = 0$ we obtain

$$\begin{cases} \cos(\alpha_1\ell) \;-\; \cosh(\alpha_2\ell) = 0, \\ \alpha_1^2\cos(\alpha_1\ell) \;-\; \alpha_2^2\cosh(\alpha_2\ell) = 0. \end{cases}$$

Since $|\alpha_1| \neq |\alpha_2|$, it follows that $\cos(\alpha_1\ell) = 0$ and $\cosh(\alpha_2\ell) = 0$, which is not possible. Therefore, $\varphi = 0$ and then $\psi = 0$. Our conclusion follows.

Now, we suppose that all the four roots of the polynomial (3.15) are imaginary and different, that is, $r_1 = i\alpha_1$, $r_2 = -i\alpha_1$, $r_3 = i\alpha_2$, $r_4 = -i\alpha_2$ with $0 < \alpha_2 < \alpha_1$. Therefore the solution can be written as

$$\varphi(x) = c_1 e^{i\alpha_1 x} + c_2 e^{-i\alpha_1 x} + c_3 e^{i\alpha_2 x} + c_4 e^{-i\alpha_2 x}.$$

Using that $\varphi_x(0) = \varphi_{xxx}(0) = 0$ we get

$$\alpha_1(c_1 - c_2) + \alpha_2(c_3 - c_4) = 0, \quad \alpha_1^3(c_1 - c_2) + \alpha_2^3(c_3 - c_4) = 0.$$

Since $\alpha_1 \neq \alpha_2$ we conclude that $c_1 - c_2 = c_3 - c_4 = 0$. Moreover using that $\varphi(0) = 0$ we conclude that $c_1 = -c_3$. Hence, the function φ must be of the form

$$\varphi(x) = 2c_1[\cos(\alpha_1 x) - \cos(\alpha_2 x)].$$

If $c_1 \neq 0$, since $\varphi(\ell) = \varphi_{xx}(\ell) = 0$ we obtain

$$\begin{cases} \cos(\alpha_1 \ell) - \cos(\alpha_2 \ell) = 0, \\ \alpha_1^2 \cos(\alpha_1 \ell) - \alpha_2^2 \cos(\alpha_2 \ell) = 0, \end{cases}$$

and as $\alpha_1 \neq \alpha_2$ we conclude that

$$\cos(\alpha_1 \ell) = 0 \qquad \text{and} \qquad \cos(\alpha_2 \ell) = 0.$$

Therefore, there are non negative integer numbers m, n, $m \neq n$, such that

$$\alpha_1 = \frac{(1+2m)\pi}{2\ell}, \qquad \alpha_2 = \frac{(1+2n)\pi}{2\ell},$$

i.e.

$$\alpha_1 = \frac{j_1 \pi}{2\ell}, \qquad \alpha_2 = \frac{j_2 \pi}{2\ell}, \;\; j_1,\; j_2 \in \mathbb{N},\; j_1 \neq j_2, \tag{3.17}$$

with $j_1,\; j_2$ being odd numbers. Note that

$$\alpha_1^2 + \alpha_2^2 = \left(\frac{\rho_1 b + \rho_2 \kappa}{\kappa b} \right) \lambda^2 \tag{3.18}$$

and

$$\kappa b\, \alpha_1^4 - (\rho_1 b + \rho_2 \kappa)\, \lambda^2 \alpha_1^2 + \left(\rho_1 \rho_2 \lambda^4 - \rho_1 \kappa \lambda^2 \right) = 0. \tag{3.19}$$

Substitution of α_1 given by (3.17) and λ given by (3.18) in (3.19) yields

$$\kappa b \left(\frac{\pi}{2\ell} \right)^4 j_1^4 - b\kappa (j_1^2 + j_2^2) j_1^2 \left(\frac{\pi}{2\ell} \right)^4 + \frac{\rho_1 \rho_2 \kappa^2 b^2}{(\rho_1 b + \rho_2 \kappa)^2} (j_1^2 + j_2^2)^2 \left(\frac{\pi}{2\ell} \right)^4$$
$$- \frac{\rho_1 \kappa^2 b}{(\rho_1 b + \rho_2 \kappa)} (j_1^2 + j_2^2) \left(\frac{\pi}{2\ell} \right)^2 = 0,$$

i.e.,

$$-b\kappa j_2^2 j_1^2 + \frac{\rho_1 \rho_2 \kappa^2 b^2}{(\rho_1 b + \rho_2 \kappa)^2} (j_1^2 + j_2^2)^2 - \frac{\rho_1 \kappa^2 b}{(\rho_1 b + \rho_2 \kappa)} (j_1^2 + j_2^2) \left(\frac{2\ell}{\pi} \right)^2 = 0.$$

Thus we get

$$\frac{(\rho_1 b\, j_1^2 - \rho_2 \kappa\, j_2^2)(\rho_2 \kappa\, j_1^2 - \rho_1 b\, j_2^2)}{(j_1^2 + j_2^2)} = \rho_1 \kappa (\rho_1 b + \rho_2 \kappa) \left(\frac{2\ell}{\pi} \right)^2$$

for $j_1,\; j_2$ being natural odd numbers, $j_1 \neq j_2$, which is not possible due to hypothesis (3.9). Then the unique solution of the equation (3.14) must be $\varphi = 0$. In that case we conclude that $\psi = 0$. So our conclusion follows.

4. Exponential Stability

The main aim of this section is to prove the exponential stability of the corresponding semigroup associated to system (1.1)–(1.3). Our main tool is the well known result (see [20]):

Theorem 4.1. *Let $\{S_{\mathbb{B}}(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions on Hilbert space $\mathbb{H}$ with infinitesimal generator $\mathbb{B}$. Then $S(t)$ is exponentially stable if and only if*

(a) $i\mathbb{R} \subset \rho(\mathbb{B})$,

(b) $\overline{\lim\limits_{|\lambda|\to\infty}} ||(i\lambda I - \mathbb{B})^{-1}||_{\mathcal{L}(\mathbb{H})} < \infty.$

Here and in what follows we assume that (3.9) holds. Given $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, let $U = (\varphi, \Phi, \psi, \Psi)$ be the unique solution of the resolvent equation $(i\lambda\,\mathbb{I} - \mathbb{A})U = F$, namely

$$i\lambda\varphi - \Phi = f_1 \quad \text{in} \quad H^1_\ell(0,\ell), \tag{4.1}$$

$$i\lambda\rho_1\,\Phi - \kappa\,(\varphi_x + \psi)_x = \rho_1 f_2 \quad \text{in} \quad L^2(0,\ell), \tag{4.2}$$

$$i\lambda\psi - \Psi = f_3 \quad \text{in} \quad H^1_*(0,\ell), \tag{4.3}$$

$$i\lambda\rho_2\,\Psi - b\,\psi_{xx} + \kappa\,(\varphi_x + \psi) = \rho_2 f_4 \quad \text{in} \quad L^2(0,\ell). \tag{4.4}$$

Let us introduce the following notation

$$\begin{aligned}
&\mathcal{I}_\psi(\alpha) = \rho_2|\Psi(\alpha)|^2 + b|\psi_x(\alpha)|^2,\\
&\mathcal{I}_\varphi(\alpha) = \rho_1|\Phi(\alpha)|^2 + \kappa|\varphi_x(\alpha) + \psi(\alpha)|^2,\\
&\mathcal{I}(\alpha) = \mathcal{I}_\varphi(\alpha) + \mathcal{I}_\psi(\alpha),\\
&\mathcal{I} = \int_0^\ell (\mathcal{I}_\psi(s) + \mathcal{I}_\varphi(s))ds.
\end{aligned}$$

Lemma 4.2. *For any $q \in H^1(0,\ell)$ we have that*

$$\int_0^\ell q'(s)\mathcal{I}_\varphi(s)\,ds = q(\alpha)\,\mathcal{I}_\varphi(\alpha)|_0^\ell \;+\; 2\rho_1\,\mathrm{Re}\int_0^\ell q(x)\Phi\overline{\Psi}\,dx \;+\; R_1, \tag{4.5}$$

$$\int_0^\ell q'(s)\mathcal{I}_\psi(s)\,ds = q(\alpha)\,\mathcal{I}_\psi(\alpha)|_0^\ell \;-\; 2\kappa\,\mathrm{Re}\int_0^\ell q(\varphi_x + \psi)\overline{\psi}_x\,dx \;+\; R_2, \tag{4.6}$$

where the term R_j verifies

$$|R_j| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},\; j = 1, 2,$$

for a positive constant C.

Proof. To get (4.5), let us multiply equation (4.2) by $q(\overline{\varphi_x + \psi})$. Integrating on $(0,\ell)$ we obtain

$$i\lambda\rho_1\int_0^\ell \Phi q\overline{(\varphi_x + \psi)}dx - \kappa\int_0^\ell (\varphi_x + \psi)_x\,q\,\overline{(\varphi_x + \psi)}dx = \rho_1\int_0^\ell f_2\,q\,\overline{(\varphi_x + \psi)}dx,$$

$$-\,\rho_1\int_0^\ell \Phi q\overline{(i\lambda\varphi_x + i\lambda\psi)}dx - \kappa\int_0^\ell (\varphi_x + \psi)_x\,q\,\overline{(\varphi_x + \psi)}dx = \rho_1\int_0^\ell f_2\,q\,\overline{(\varphi_x + \psi)}dx.$$

Therefore, taking the real part we get

$$\begin{aligned}
-\frac{\rho_1}{2}\int_0^\ell q(s)\frac{d}{dx}|\Phi|^2dx - \frac{\kappa}{2}\int_0^\ell q\,\frac{d}{dx}|\varphi_x + \psi|^2dx &= \rho_1\,\mathrm{Re}\int_0^\ell q\Phi\overline{\Psi}dx\\
&\quad\underbrace{+\,\rho_1\,\mathrm{Re}\int_0^\ell q\Phi\overline{(f_{1x} + f_3)}dx + \rho_1\,\mathrm{Re}\int_0^\ell q\,f_2\,\overline{(\varphi_x + \psi)}dx}_{:=\frac{R_1}{2}}\,.
\end{aligned}$$

Performing an integration by parts we arrive at

$$\int_0^\ell q'(s)[\rho_1|\Phi(s)|^2+\kappa|\varphi_x(s)+\psi(s)|^2]\,ds = q\mathcal{I}_\varphi|_0^\ell + 2\rho_1\operatorname{Re}\int_0^\ell q\Phi\overline{\Psi}\,dx + R_1. \tag{4.7}$$

Similarly, multiplying the equation (4.4) by $q\overline{\psi}_x$ and integrating on $(0,\ell)$ we obtain

$$i\lambda\rho_2\int_0^\ell \Psi q\overline{\psi}_x dx - b\int_0^\ell q\psi_{xx}\,\overline{\psi_x}dx + \kappa\int_0^\ell (\varphi_x+\psi)\,q\,\overline{\psi}_x dx = \rho_2\int_0^\ell f_4\,q\,\overline{\psi}_x dx$$

and taking the real part we get

$$-\rho_2\mathrm{Re}\int_0^\ell q\Psi\overline{\Psi}_x dx - b\mathrm{Re}\int_0^\ell \psi_{xx}\,q\,\overline{\psi}_x dx = -\kappa\operatorname{Re}\int_0^\ell q(\varphi_x+\psi)\overline{\psi}_x\,dx$$
$$+\underbrace{\rho_2\operatorname{Re}\int_0^\ell \Psi\,q\,\overline{f_3}_x dx + \rho_2\operatorname{Re}\int_0^\ell f_4\,q\,\overline{\psi}_x dx}_{:=\frac{R_2}{2}}.$$

Therefore

$$-\frac{\rho_2}{2}\int_0^\ell q(s)\frac{d}{dx}|\Psi|^2 ds - \frac{b}{2}\int_0^\ell q\frac{d}{dx}|\psi_x|^2 ds = -\kappa\operatorname{Re}\int_0^\ell q(\varphi_x+\psi)\overline{\psi}_x dx + \frac{R_2}{2}.$$

Integrating by part implies

$$\int_0^\ell q'(s)[\rho_2|\Psi(s)|^2+b|\psi_x(s)|^2]\,ds = q\mathcal{I}_\psi|_0^\ell - 2\kappa\int_0^\ell q(\varphi_x+\psi)\overline{\psi_x}\,dx + R_2. \tag{4.8}$$

Our conclusion follows.

Remark 4.3. Multiplying equation (4.2) by $q\overline{\varphi_x}$, performing integration by parts and proceeding as made in the last Lemma we get

$$\int_0^\ell q'[\rho_1|\Phi|^2+\kappa|\varphi_x|^2]dx = q(\alpha)\mathcal{I}_\varphi(\alpha)|_0^\ell + 2\kappa\mathrm{Re}\int_0^\ell q\,\varphi_x\overline{\psi_x}\,dx + R_1', \tag{4.9}$$

whit $|R_1'|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$, for a positive constant C. Summing (4.9) and (4.6) we obtain the following observability result

$$q(\alpha)\mathcal{I}_\varphi(\alpha)|_0^\ell + q(\alpha)\mathcal{I}_\psi(\alpha)|_0^\ell = \int_0^\ell q'[\rho_1|\Phi|^2+\rho_2|\Psi|^2+\kappa|\varphi_x|^2+b|\psi_x|^2]dx + R_2'$$

with

$$|R_2'|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|}\|U\|^2.$$

Lemma 4.4. *We have that*

$$b\,\operatorname{Re}\varphi_x(0)\overline{\psi_x(0)} - b\,\chi\,\operatorname{Re}\int_0^\ell \Phi_x\overline{\Psi}dx + \kappa\int_0^\ell|\varphi_x+\psi|^2dx = b\int_0^\ell|\psi_x|^2dx + R_5 \tag{4.10}$$

where $R_5 = bR_3 + bR_4$,

$$\chi = \frac{\rho_2}{b} - \frac{\rho_1}{\kappa} \tag{4.11}$$

and

$$|R_3|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \qquad \textit{and} \qquad |R_4|\leq \frac{C}{|\lambda|}\|U\|_{\mathcal{H}}^2.$$

Proof. Multiplying equation (4.2) by $\overline{\psi_x}$ and (4.4) by $\overline{\varphi}_x$, integrating on $(0,\ell)$ we obtain the equations

$$
\begin{aligned}
&-\rho_1\int_0^\ell \Phi\overline{\Psi}_x dx-\kappa\int_0^\ell(\varphi_x+\psi)_x\overline{\psi}_x dx=\rho_1\int_0^\ell \Phi\,\overline{f_3}_x dx+\rho_1\int_0^\ell f_2\,\overline{\psi}_x dx,\\
&-\rho_2\int_0^\ell \Psi\overline{\Phi}_x dx-b\int_0^\ell \psi_{xx}\,\overline{(\varphi_x+\psi)}dx+\kappa\int_0^\ell(\varphi_x+\psi)\overline{(\varphi_x+\psi)}dx=-b\int_0^\ell \psi_{xx}\,\overline{\psi}dx\\
&\qquad+\kappa\int_0^\ell(\varphi_x+\psi)\overline{\psi}dx+\rho_2\int_0^\ell f_4\overline{\varphi}_x\,dx+\rho_2\int_0^\ell \Psi\overline{f_1}_x dx.
\end{aligned}
$$

Taking the real part of each one of the above equations and summing the results we obtain

$$
\begin{aligned}
&\Big(\frac{\rho_1}{\kappa}-\frac{\rho_2}{b}\Big)\mathrm{Re}\int_0^\ell \Phi_x\overline{\Psi}dx-\mathrm{Re}\int_0^\ell\frac{d}{dx}[(\varphi_x+\psi)\overline{\psi}_x]dx+\frac{\kappa}{b}\int_0^\ell|\varphi_x+\varphi|^2dx=-\mathrm{Re}\int_0^\ell\overline{\psi_{xx}}\,\psi dx\\
&\qquad+\underbrace{\mathrm{Re}\Big(\frac{\rho_1}{\kappa}\int_0^\ell\Phi\,\overline{f_3}_x dx+\frac{\rho_1}{\kappa}\int_0^\ell f_2\,\overline{\psi}_x dx+\frac{\rho_2}{b}\int_0^\ell\overline{f_4}\varphi_x\,dx+\frac{\rho_2}{b}\int_0^\ell\overline{\Psi}f_{1x}dx\Big)}_{:=R_3}\\
&\qquad+\underbrace{\mathrm{Re}\,\frac{\kappa}{b}\int_0^\ell\overline{(\varphi_x+\psi)}\psi dx}_{:=R_4}.
\end{aligned}
$$

Due to boundary conditions $\Phi(l)=\Psi(0)=\psi(0)=\psi_x(l)=0$, after that we perform some algebraic manipulations we obtain (4.10).

Lemma 4.5. *We have that*

$$
\rho_2\int_0^\ell|\Psi|^2dx=\kappa\int_0^\ell|\varphi_x+\psi|^2dx-\,b\mathrm{Re}\;\psi_x(0)\overline{\varphi_x(0)}-b\,\chi\,\mathrm{Re}\;\int_0^\ell\Psi\overline{\Phi_x}dx+R_6,\tag{4.12}
$$

where χ is given in (4.11) *and $|R_6|\le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$, for a positive constant C.*

Proof. Multiplying equation (4.4) by $\overline{\varphi_x+\psi}$ and integrating on $(0,\ell)$ we obtain

$$
\begin{aligned}
\kappa\int_0^\ell|\varphi_x+\psi|^2dx&=\rho_2\int_0^\ell\Psi\overline{(\Phi_x+\Psi)}dx+b\psi_x\overline{(\varphi_x+\psi)}\;|_0^\ell-b\int_0^\ell\psi_x\overline{(\varphi_x+\psi)_x}dx\\
&\quad+\rho_2\int_0^\ell f_4\overline{(\varphi_x+\psi)}dx+\rho_2\int_0^\ell\Psi\overline{(f_{1x}+\Psi)}dx\\
&=\rho_2\int_0^\ell|\Psi|^2dx+\rho_2\int_0^\ell\Psi\overline{\Phi_x}dx+b\psi_x\overline{(\varphi_x+\psi)}\;|_0^\ell+\frac{b\rho_1}{\kappa}\int_0^\ell\Psi_x\overline{\Phi}dx\\
&\quad+\rho_2\int_0^\ell f_4\overline{(\varphi_x+\psi)}dx+\frac{b\rho_1}{\kappa}\int_0^\ell\psi_x\overline{f_2}dx+\rho_2\int_0^\ell\Psi\overline{(f_{1x}+\Psi)}dx.
\end{aligned}
$$

Therefore we have that

$$
\begin{aligned}
\kappa\int_0^\ell|\varphi_x+\psi|^2dx&=\rho_2\int_0^\ell|\Psi|^2dx+b\mathrm{Re}\;\psi_x\overline{(\varphi_x+\psi)}\;|_0^\ell+\frac{b\rho_1}{\kappa}\mathrm{Re}\;\Psi\overline{\Phi}\;|_0^\ell+b\,\chi\mathrm{Re}\;\int_0^\ell\Psi\overline{\Phi_x}dx\\
&\quad+\underbrace{\rho_2\mathrm{Re}\;\int_0^\ell f_4\overline{(\varphi_x+\psi)}dx+\frac{b\rho_1}{\kappa}\mathrm{Re}\;\int_0^\ell\psi_x\overline{f_2}dx+\rho_2\int_0^\ell\Psi\overline{(f_{1x}+\Psi)}dx}_{:=-R_6}\\
&=\rho_2\int_0^\ell|\Psi|^2dx+b\mathrm{Re}\;\psi_x(0)\overline{\varphi_x(0)}+b\,\chi\mathrm{Re}\;\int_0^\ell\Psi\overline{\Phi_x}dx-R_6
\end{aligned}
$$

where

$$|R_6| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Lemma 4.6. *There exists a positive constant C such that*

$$\kappa\int_0^\ell |\varphi_x+\psi|^2dx = \rho_1\int_0^\ell |\Phi|^2dx + \kappa\,\mathrm{Re}\,\varphi_x(0)\overline{\varphi(0)} \;+\; R_7 \;+\; R_8, \tag{4.13}$$

where

$$|R_7| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \qquad \text{and} \qquad |R_8| \leq \frac{C}{|\lambda|}\|U\|^2_{\mathcal{H}}.$$

Proof. Multiplying the equation (4.2) by $\overline{\varphi}$, integrating on $(0,\ell)$ and integrating by parts we get

$$-\rho_1\int_0^\ell \Phi\overline{i\lambda\varphi}dx + k\varphi_x(0)\overline{\varphi(0)} + k\int_0^\ell |\varphi_x+\psi|^2dx = \rho_1\int_0^\ell f_2\overline{\varphi}dx + k\int_0^\ell (\varphi_x+\psi)\overline{\psi}dx.$$

Since $i\lambda\varphi = \phi + f_1$ we have

$$\begin{aligned} k\int_0^\ell |\varphi_x+\psi|^2dx \;&= \rho_1\int_0^\ell |\Phi|^2dx - k\mathrm{Re}\varphi_x(0)\overline{\varphi(0)} + \underbrace{\rho_1\mathrm{Re}\int_0^\ell f_2\overline{\varphi}dx + \rho_1\mathrm{Re}\int_0^\ell \Phi\overline{f_1}dx}_{:=R_7} \\ &\quad + \underbrace{k\mathrm{Re}\int_0^\ell (\varphi_x+\psi)\overline{\psi}dx}_{:=R_8}. \end{aligned}$$

Our results follows.

Lemma 4.7. *Suppose $0<\ell<\sqrt{\frac{\rho_2}{2\rho_1}}$. Moreover, assume that $\chi=0$ and*

$$\frac{(j_1^2-j_2^2)^2}{j_1^2+j_2^2} \neq \frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2,$$

for all natural odd numbers $j_1,\ j_2,\ j_1\neq j_2$. Then there exists a positive constant C such that

$$\|(i\lambda\,\mathbb{I}-\mathbb{A})\|_{\mathcal{L}(\mathcal{H})} \leq C$$

for $|\lambda|$ large enough.

Proof. If we consider in (4.5) the function $q(x)=x-\ell,\ x\in(0,\ell)$ then we get

$$\int_0^\ell \mathcal{I}_\varphi(x)dx = \ell k|\varphi_x(0)+\psi(0)|^2 + \rho_1\ell|\Phi(0)|^2 + 2\rho_1\,\mathrm{Re}\int_0^\ell (x-\ell)\Phi\overline{\Psi}\,dx + R_1.$$

Applying the Young inequality and using estimates (3.6), (3.8) we obtain

$$\begin{aligned} \int_0^\ell \mathcal{I}_\varphi(x)dx \;&\leq\; 2\rho_1\ell\int_0^\ell |\Phi||\Psi|dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \\ &\leq\; \frac{\rho_1}{2}\int_0^\ell |\Phi|^2dx + 2\ell^2\rho_1\int_0^\ell |\Psi|^2dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned}$$

Therefore

$$\int_0^\ell \mathcal{I}_\varphi(x)dx \;\leq\; \frac{4\ell^2\rho_1}{\rho_2}\int_0^\ell \rho_2|\Psi|^2dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.14}$$

By equations (4.12) and (4.13) we obtain

$$\begin{aligned}\int_0^\ell \rho_2|\Psi|^2dx &= \frac{1}{2}\int_0^\ell \kappa|\varphi_x+\psi|^2dx + \frac{1}{2}\int_0^\ell \kappa|\varphi_x+\psi|^2dx - b\mathrm{Re}\psi_x(0)\overline{\varphi_x(0)} + R_6\\ &= \frac{1}{2}\int_0^\ell [\rho_1|\Phi|^2+\kappa|\varphi_x+\psi|^2]dx + \kappa\mathrm{Re}\varphi_x(0)\overline{\varphi(0)} - b\mathrm{Re}\psi_x(0)\overline{\varphi_x(0)} + \sum_{j=6}^{8} R_j.\end{aligned}$$

It results by equation (4.14) and estimates (3.7) and (3.8) that

$$\begin{aligned}\int_0^\ell \rho_2|\Psi|^2dx \;&\le\; \frac{2\ell^2\rho_1}{\rho_2}\int_0^\ell \rho_2|\Psi|^2dx + C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + \frac{C}{|\lambda|^2}\|U\|_{\mathcal H}\|F\|_{\mathcal H}\\ &\quad + C|\psi_x(0)||\varphi_x(0)| + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal H} + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H}.\end{aligned}$$

Thus

$$\begin{aligned}\left(1-\frac{2\rho_1\ell^2}{\rho_2}\right)\int_0^\ell \rho_2|\Psi|^2dx \;&\le\; C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + C|\psi_x(0)||\varphi_x(0)|\\ &\quad + \frac{C}{|\lambda|^2}\|U\|_{\mathcal H}\|F\|_{\mathcal H} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal H} + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H}\end{aligned} \tag{4.15}$$

for a positive constant C. From the equation (4.10) and from the estimates (4.14) and (4.15) we obtain

$$\int_0^\ell |\psi_x|^2dx \le C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + C|\psi_x(0)||\varphi_x(0)| + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H} + C\|F\|^2_{\mathcal H}$$

for $|\lambda|>1$. Therefore, we have

$$\int_0^\ell \mathcal{I}_\psi(x)dx \le C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + C|\psi_x(0)||\varphi_x(0)| + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H} + C\|F\|^2_{\mathcal H} \tag{4.16}$$

for a positive constant C and $|\lambda|>1$. By (4.14), (4.15) and (4.16) we obtain

$$\mathcal{I} \;\le\; C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + C|\psi_x(0)||\varphi_x(0)| + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H} + C\|F\|^2_{\mathcal H}, \tag{4.17}$$

for a positive constant C and $|\lambda|>1$. In Lemma 4.2, taking $q(x)=x-l$ and summing the equations (4.5) and (4.6) we get

$$\begin{aligned}\ell(\rho_1|\Phi(0)|^2+\kappa|\varphi_x(0)|^2+b|\psi_x(0)|^2) \;&\le\; \mathcal{I} + 2\rho_1\ell\int_0^\ell |\Phi||\Psi|dx + 2\kappa\ell\int_0^\ell |\varphi_x+\psi||\psi_x|dx\\ &\quad + C\|U\|_{\mathcal H}\|F\|_{\mathcal H}.\end{aligned}$$

Therefore using (3.8), (4.17) and the Young inequality we obtain

$$\ell(\rho_1|\Phi(0)|^2+\kappa|\varphi_x(0)|^2+b|\psi_x(0)|^2) \;\le\; C\|U\|_{\mathcal H}\|F\|_{\mathcal H} + \frac{b\ell}{2}|\psi_x(0)|^2 + \frac{C}{|\lambda|}\|U\|^2_{\mathcal H} + C\|F\|^2_{\mathcal H}.$$

Finally, it follows from last estimate and from (4.17) that

$$\left(1-\frac{C}{|\lambda|}\right)\|U\|^2_{\mathcal H} \le C\|F\|^2_{\mathcal H} \tag{4.18}$$

for $|\lambda|>1$ and a positive constant C. Our result follows.

Theorem 4.8. *Assume that the conditions of the Lemma 4.7 hold. Then semigroup* $\{S_{\mathbb{A}}(t)\}_{t\ge 0}$ *is exponentially stable.*

Proof. The result follows from Lemmas 3.1, 4.7 and Theorem 4.1 and due to continuity of the operator $(\lambda I-\mathbb{A})^{-1}$ on $\mathbb{C}$.

5. Numerical Examples

In this section, we present some numerical results illustrating the asymptotic behavior of the energy and importance of the different conditions and assumptions for the exponential decay.

5.1. **Numerical study of the spectrum.** We present numerical results on the linear stability of our system. We use normal mode analysis and set

$$\phi(x,t) = e^{\lambda t} P(x), \qquad \psi(x,t) = e^{\lambda t} Q(x).$$

Thus, (1.1), (1.3) become the following eigenvalues problem:

$$\kappa(P_x + Q)_x = \lambda^2 P \tag{5.1}$$

$$bQ_{xx} - \kappa(P_x + Q) = \lambda^2 Q \tag{5.2}$$

$$\kappa P_x(0) = \lambda\gamma P(0) \tag{5.3}$$

$$P(\ell) = Q(0) = Q(\ell) = 0 \tag{5.4}$$

We discretize $P(x)$ and $Q(x)$ by the Chebyshev-tau method [18] (see also [11]). This is a spectral method where the expansion functions are the Chebyshev polynomials $T_n(z)$ defined by $T_n(\cos\theta) = \cos n\theta$ when $z = \cos\theta$. This method approximates discrete eigenvalues belonging to C^∞ eigenfunctions with infinite-order accuracy. We rescale the spatial variable to $z = 2x/\ell - 1$, so that $-1 < z < 1$. We set

$$P(z) = \sum_{n=0}^{N} p_n T_n(z), \text{ and } Q(z) = \sum_{n=0}^{N} q_n T_n(z) \tag{5.5}$$

and substitute into (5.1)-(5.4). The four boundary conditions are imposed as part of the conditions determining the coefficients p_n and q_n. The $(4N+4)\times(4N+4)$ matrix equation is given by

$$\begin{pmatrix} \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{K}_1 & \mathbf{O} & \mathbf{K}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\ \mathbf{K}_4 & \mathbf{O} & \mathbf{K}_3 & \mathbf{O} \end{pmatrix} \begin{pmatrix} p_n \\ r_n \\ q_n \\ z_n \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{0,0} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{0,0} \end{pmatrix} \begin{pmatrix} p_n \\ r_n \\ q_n \\ z_n \end{pmatrix} \tag{5.6}$$

where $\mathbf{K}_i$, with $i = 1, \ldots, 4$ are the matrices applied to the coefficients of the series (5.5), $\mathbf{O}$ are the null matrix, $\mathbf{I}$ the identity and $\mathbf{I}_{0,0} = \mathrm{diag}\{1, \ldots, 1, 0, 0\}$.

5.2. **Some examples for different cases.** In this subsection we show some numerical examples of the spectrum for different cases. Here we assume that $N = 500$. In Figure 1, we observe the spectrum for different sizes of L, with the coefficients $\rho_1 = \rho_2 = \kappa = b = 1$. That is, we consider here the case $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$. In the same Figure 1 we note that for $\ell = 0.25$, $\ell = 0.5$, $\ell = 1.0$ and even $\ell = 2.0$, the spectrum is well separated from the imaginary axis, as is to be expected when exponential decay of energy. Recall that Theorem 4.8 requires the assumption $0 < \ell < \sqrt{\frac{\rho_2}{2\rho_1}}$, that in this example holds for $\ell < 1/\sqrt{2}$. However, for the case $\ell = 5$, it is numerically observed that the spectrum approaches the imaginary axis with a pair of eigenvalues that apparently are on the imaginary axis. In this example there is not exponential decay since besides not verified hypothesis of Theorem 4.8. Further observed that in all cases of this Figure 1, $\frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2} \neq \frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2$.

Furthermore, we observe in Figure 2, several spectral examples with parameters such that $\frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2 = \frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2}$, or, when this condition is verified asymptotically, i.e. for a sequence of parameters converging to this condition. The graph in Figure 2(A) shows a set of three spectra, in which the fixed parameters are $\ell = 0.25$, $\rho_2 = \kappa = 1.0$, and $\rho_1 = \kappa = \frac{1}{2}\frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2}\left(\frac{\pi}{2\ell}\right)^2$, for three pairs of odd indices (j_1, j_2). We observe in the three cases for this graph $((j_1 = 3, j_2 = 1), (j_1 =$

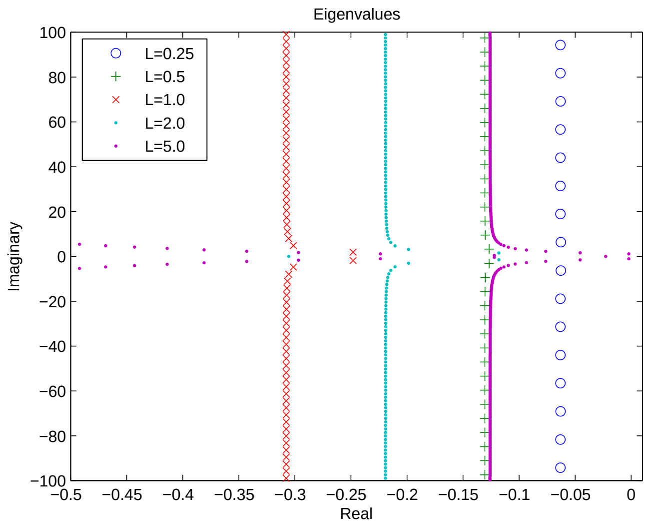
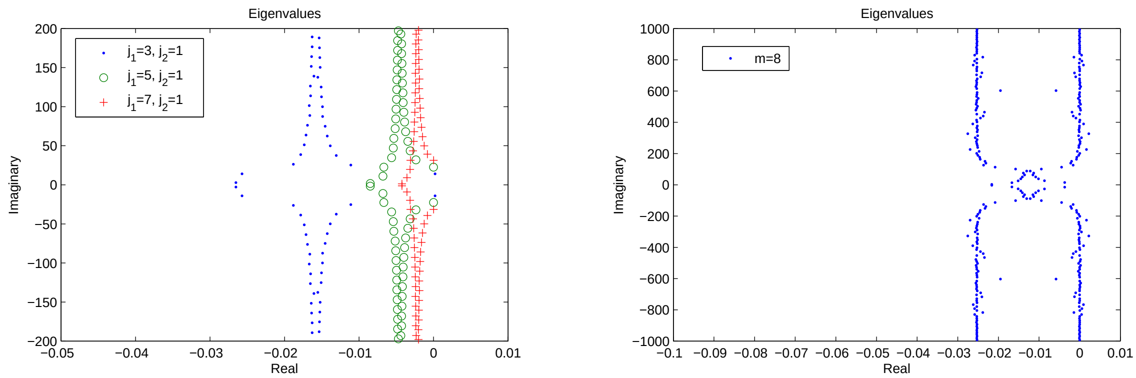


FIGURE 1. Exponential Decay: case $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ and $\frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2} \neq \frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2$. Eigenvalues for different sizes of L.



(A) Eigenvalues for different values of j_1 and j_2 with $\rho_2 = b = 1$ and $\rho_1 = \kappa = \frac{(j_1^2 - j_2^2)^2}{2(j_1^2 + j_2^2)}\left(\frac{\pi}{2\ell}\right)^2$.

(B) Eigenvalues for the case $\rho_2 = b = 1$ and $\rho_1 = \kappa = \frac{m}{2}\left(\frac{\pi}{2\ell}\right)^2$ with $m = 8$.

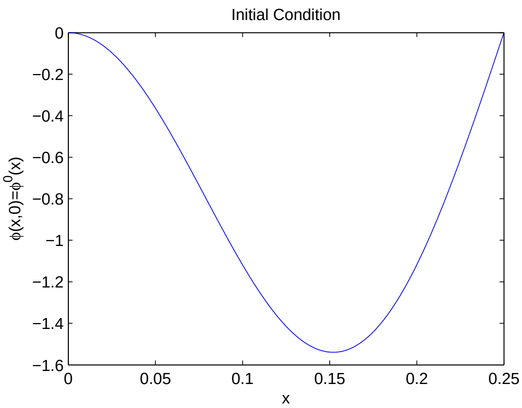
FIGURE 2. Spectra for parameters around condition $\frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2 = \frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2}$, when it is verified, or when an infinite number of cases approaching it.

$5, j_2 = 1)$ and $(j_1 = 7, j_2 = 1)$), the three pair of eigenvalues on the imaginary axis, or at least very close to this (away by some numerical inaccuracy). This eigenvalues are $\lambda_{3,1} \approx \pm i14.0496$, $\lambda_{5,1} \approx \pm i22.6543$ and $\lambda_{7,1} \approx \pm i31.4159$, respectively.

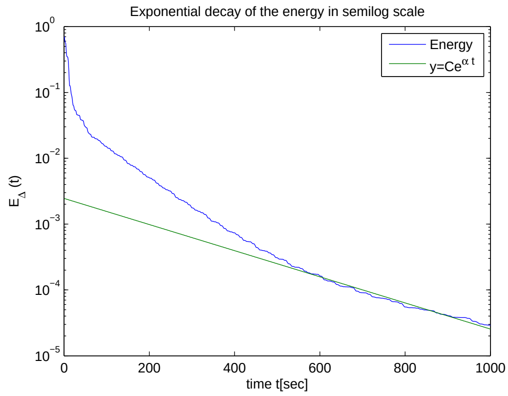
On the other hand, the graph in Figure 2(B) shows the spectrum with the parameters $\ell = 0.25$, $\rho_2 = \kappa = 1.0$, and $\rho_1 = \kappa = \frac{m}{2}\left(\frac{\pi}{2\ell}\right)^2$, with $m = 8$. This case is special, and corresponds to the accumulation point of the sequence of consecutive indexes odd (j_1, j_2), with $j_2 = j_1 + 2$. That is

$$\lim_{j_2 = j_1 + 2 \to \infty} \frac{(j_1^2 - j_2^2)^2}{j_1^2 + j_2^2} = 8.$$

Here, there are countless pairs of indices close to the condition, but that equality is not verified for any of them, then in theory should be exponential decay, and the values themselves distanced from the imaginary axis. However, numerically and as shown in Figure 2(B), a large number of eigenvalues pasted to the imaginary axis is observed.



(A) $\varphi(x,0)=\cos(\frac{3\pi}{2\ell}x)-\cos(\frac{\pi}{2\ell}x)$



(B) Exponential decay of the Energy.

FIGURE 3. Exponential decay of the Energy for $L=0.25$, with parameters such that $\frac{\kappa}{\rho_1}=\frac{b}{\rho_2}$ and $\frac{(j_1^2-j_2^2)^2}{j_1^2+j_2^2}\neq\frac{2\kappa}{b}\left(\frac{2\ell}{\pi}\right)^2$. Comparison with $y=Ce^{\alpha t}$ with $C=0.0025$ and $\alpha=-0.0046$.

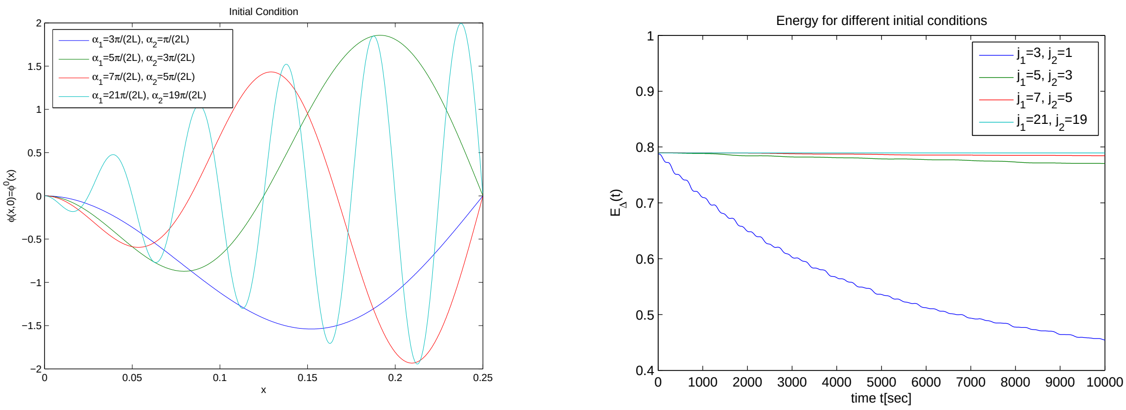
5.3. **Numerical study of the energy decay.** We study numerically here, the decay of Energy. For this, we use Finite Difference (of second order in space and time) which are more stable than the Chebyshev-tau method for solving evolution equations. Furthermore, the method of $\beta-$Newmark is a second order method preserving the discrete energy always when the discrete system of equations of motion is symmetric (i.e. matrices associated to the system should be symmetric).

We consider J an integer non-negative and $h=\ell/(J+1)$ an spatial subdivision of the interval $(0,\ell)$ given by $0=x_0<x_1<\ldots<x_J<x_{J+1}=\ell$, with $x_j=jh$ each node of the mesh. We use $\varphi_j(t)$, $\psi_j(t)$, for all $j=1,2,\ldots,J$ and $t>0$ to denote the approximate values of $\varphi(jh,t)$ and $\psi(jh,t)$, respectively. In addition, we denote the discrete operator $\Delta_h\vartheta_j=\frac{\vartheta_{j+1}-2\vartheta_j+\vartheta_{j-1}}{h^2}$ We assume the following finite difference scheme applied to system (1.1)-(1.3)

$$\begin{aligned}
\rho_1\frac{\varphi''_{j+1}+2\varphi''_j+\varphi''_{j-1}}{4}-\kappa\Delta_h\varphi_j-\kappa\frac{\psi_{j+1}-\psi_{j-1}}{2h} &= 0, && (5.7)\\
\rho_2\frac{\psi''_{j+1}+2\psi''_j+\psi''_{j-1}}{2}-b\Delta_h\psi_j+\kappa\frac{\varphi_{j+1}-\varphi_{j-1}}{2h}+\kappa\frac{\psi_{j+1}+2\psi_j+\psi_{j-1}}{4} &= 0, && (5.8)\\
j &= 1,\ldots,J,\\
\varphi_J=\psi_0=\psi_{J+1}-\psi_J &= 0, && (5.9)\\
\kappa\frac{\varphi_1-\varphi_0}{h} &= \gamma\varphi'_0 && (5.10)\\
\varphi_j(0)=\varphi_j^0,\ \varphi'_j(0)=\varphi_j^1,\ \psi_j(0)=\psi_j^0,\ \psi'_j(0)=\psi_j^1,\qquad j &= 1,\ldots,J. && (5.11)
\end{aligned}$$

The discrete Energy of (5.7)-(5.11) is given by

$$E_\Delta(t)=\rho_1\frac{h}{2}\sum_{j=0}^{J}|\varphi'_j|^2+\rho_2\frac{h}{2}\sum_{j=0}^{J}|\psi'_j|^2+\frac{h}{2}\left[b\left|\frac{\psi_{j+1}-\psi_j}{h}\right|^2+\kappa\left|\frac{\varphi_{j+1}-\varphi_j}{h}+\frac{\psi_{j+1}+\psi_j}{2}\right|^2\right] \tag{5.12}$$



(A) $\varphi(x,0) = \cos(\alpha_1 x) - \cos(\alpha_2 x)$ (B) Exponential decay of the Energy.

FIGURE 4. Energy behaviour for the case $\rho_2 = b = 1$ and $\rho_1 = \kappa = \frac{m}{2}\left(\frac{\pi}{2\ell}\right)^2$ with $m = 8$, and different initial conditions.

5.4. **Equation of motion and time discretization.** The system (5.7)-(5.10) can be rewritten as

$$\mathbf{M}\begin{bmatrix} \ddot{\varphi}_h \\ \ddot{\psi}_h \end{bmatrix} + \mathbf{C}\begin{bmatrix} \dot{\varphi}_h \\ \dot{\psi}_h \end{bmatrix} + \mathbf{K}\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} = \mathbf{0}, \tag{5.13}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system in $\mathcal{M}_{2J}(\mathbb{R})$, and $\varphi_h = (\varphi_1, \ldots, \varphi_J)^\top$, $\psi_h = (\psi_1, \ldots, \psi_J)^\top \in \mathbb{R}^J$.

The Newmark algorithm [17] is based on a set of two relations expressing the forward displacement $[\varphi_h^{n+1}, \psi_h^{n+1}]^\top$ and velocity $[\Phi_h^{n+1}, \Psi_h^{n+1}]^\top = [\dot{\varphi}_h^{n+1}, \dot{\psi}_h^{n+1}]^\top$. The method consists in updating the displacement, velocity and acceleration vectors from current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$,

$$\Phi_h^{n+1} = \Phi_h^n + (1-\gamma)\delta t\, \dot{\Phi}_h^n + \gamma\delta t\, \dot{\Phi}_h^{n+1} \tag{5.14}$$

$$\varphi_h^{n+1} = \varphi_h^n + \left(\frac{1}{2} - \beta\right)\delta t^2\, \dot{\Phi}_h^n + \beta\delta t^2\, \dot{\Phi}_h^{n+1} \tag{5.15}$$

$$\Psi_h^{n+1} = \Psi_h^n + (1-\gamma)\delta t\, \dot{\Psi}_h^n + \gamma\delta t\, \dot{\Psi}_h^{n+1} \tag{5.16}$$

$$\psi_h^{n+1} = \psi_h^n + \left(\frac{1}{2} - \beta\right)\delta t^2\, \dot{\Psi}_h^n + \beta\delta t^2\, \dot{\Psi}_h^{n+1}, \tag{5.17}$$

where β and γ are parameters of the methods that will be fixed later. Replacing (5.14)-(5.17) in the equation of motion (5.13), we obtain

$$\begin{aligned}(\mathbf{M} + \gamma\delta t\,\mathbf{C} + \beta\delta t^2\mathbf{K})\begin{bmatrix} \dot{\Phi}_h^{n+1} \\ \dot{\Psi}_h^{n+1} \end{bmatrix} &= -\mathbf{C}\left(\begin{bmatrix} \Phi_h^n \\ \Psi_h^n \end{bmatrix} + (1-\gamma)\delta t\begin{bmatrix} \dot{\Phi}_h^n \\ \dot{\Psi}_h^n \end{bmatrix}\right) \\ &\quad -\mathbf{K}\left(\begin{bmatrix} \varphi_h^n \\ \psi_h^n \end{bmatrix} + \delta t\begin{bmatrix} \Phi_h^n \\ \Psi_h^n \end{bmatrix} + \left(\frac{1}{2} - \beta\right)\delta t^2\begin{bmatrix} \dot{\Phi}_h^n \\ \dot{\Psi}_h^n \end{bmatrix}\right).\end{aligned} \tag{5.18}$$

The acceleration $[\dot{\Phi}_h^{n+1}, \dot{\Psi}_h^{n+1}]^\top$ is computed from (5.18), and the velocities $[\Phi_h^{n+1}, \Psi_h^{n+1}]^\top$ are obtained from (5.14) and (5.16), respectively. Finally, displacement $[\varphi_h^{n+1}, \psi_h^{n+1}]^\top$ follow from (5.15) and (5.17), by simple matrix operations. Thus, the fully discrete energy of the system (5.14)-(5.18) is given by

$$\mathcal{E}_h^n := \frac{1}{2}\left[\Phi_h^\top, \Psi_h^\top\right]\mathbf{M}\begin{bmatrix} \Phi_h \\ \Psi_h \end{bmatrix} + \frac{1}{2}\left[\varphi_h^\top, \psi_h^\top\right]\mathbf{K}\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} \tag{5.19}$$

which is an approximation of energy for the continuous case. The increment of this energy can be expressed in terms of mean values and increments of the displacement and velocity. Then, we choose $\gamma = \frac{1}{2}$ and $\beta = \frac{\gamma}{2}$, reducing the above expression to

$$\mathcal{E}_\delta^{n+1} - \mathcal{E}_\delta^n \quad = \quad -\frac{1}{2}\left\{ \left[\Delta\varphi_h^\top, \Delta\psi_h^\top\right] \mathbf{C} \left[\begin{array}{c} \Delta\varphi_h \\ \Delta\psi_h \end{array}\right] + \delta t \left[\Phi_h^{n+\frac{1}{2},\top}, \Psi_h^{n+\frac{1}{2},\top}\right] \mathbf{C} \left[\begin{array}{c} \Phi_h^{n+\frac{1}{2}} \\ \Psi_h^{n+\frac{1}{2}} \end{array}\right] \right\} \leqslant 0.$$

With this, the fully discrete Energy obtained by the $\beta-$Newmark method is decreasing and we expect that its asymptotic behavior be a reflection of the continuous case (see [12] and also [3,4]).

5.5. **Numerical examples.** We make simulations with parameters $\ell = 0.25$, $\rho_2 = b = \gamma = 1$, and the initial condition:

$$\varphi(x, 0) = \cos(\alpha_1 x) - \cos(\alpha_2 x), \qquad x \in (0, \ell), \tag{5.20}$$

and $\Phi(x, 0) = \psi(x, 0) = \Psi(x, 0) = 0$.

In Figure 3, the exponential decay of energy is observed in scale semi-log from the initial condition (5.20) with $\alpha_1 = \frac{3\pi}{2\ell}$, $\alpha_2 = \frac{\pi}{2\ell}$, $\ell = 0.25$ and $\rho_1 = \kappa = \rho_2 = b = \gamma = 1$. This is compared to an exponential curve whose rate is given by $\alpha = -0.0046$, and is in line with the separation between the spectrum and the imaginary axis seen in Figure 1.

Figure 4 shows the evolution in time of the energies simulated using parameters that give the spectrum shown in the graph of Figure 2(B). In this graph certain eigenvalues stick to the imaginary axis, and therefore it is not expected to have any decay of the energy, or at least numerically decay has been extremely slow. That is what actually is observed in Figure 4(B), and specifically this happens for hight frequencies of the initial condition in terms of cosine functions, i.e. when we come to the eigenfunctions of the eigenvalues closer to the imaginary axis. In particular, for the higher frequencies pair $(j_1, j_2) = (21;\ 19)$, the energy remains practically constant as shown in Figure 4(B).

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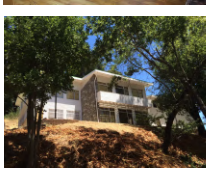

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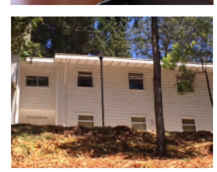

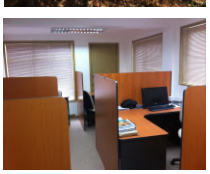

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