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**Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters**

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# Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters *

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**Abstract**

In this paper we analyze a conforming finite element method for the numerical simulation of non-isothermal incompressible fluid flows subject to a heat source modelled by a generalized Boussinesq problem with temperature-dependent parameters. We consider the standard velocity-pressure formulation for the fluid flow equations which is coupled with a primal-mixed scheme for the convection-diffusion equation modelling the temperature. In this way, the unknowns of the resulting formulation are given by the velocity, the pressure, the temperature, and the normal derivative of the latter on the boundary. Hence, assuming standard hypotheses on the discrete spaces, we prove existence and stability of solutions of the associated Galerkin scheme, and derive the corresponding Cea's estimate for small and smooth solutions. In particular, any pair of stable Stokes elements, such as Hood-Taylor elements, for the fluid flow variables, continuous piecewise polynomials of degree $\leq k+1$ for the temperature, and piecewise polynomials of degree $\leq k$ for the boundary unknown become feasible choices of finite element subspaces. Finally, we derive optimal a priori error estimates, and provide several numerical results illustrating the performance of the conforming method and confirming the theoretical rates of convergence.

**Key words**: generalized Boussinesq problem, conforming finite element method, Hood-Taylor, temperature-dependent parameters, primal-mixed formulation

**Mathematics Subject Classifications (2000)**: 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

## 1 Introduction

This paper is concerned with the numerical approximation of the stationary generalized Boussinesq problem:

$$\begin{aligned} -\mathbf{div}\,(\nu(\theta)\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{g}\theta &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \Gamma, \\ -\operatorname{div}(\kappa(\theta)\nabla\theta) + \mathbf{u}\cdot\nabla\theta &= 0 \quad \text{in} \quad \Omega, \\ \theta &= \theta_D \quad \text{on} \quad \Gamma. \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a polyhedral domain, $\mathbf{u}$ is the fluid velocity, $p$ is the fluid pressure, $\theta$ is the fluid temperature, $\theta_D$ is a given non-vanishing boundary temperature and $\mathbf{g}$ is a given external force per unit mass, usually acting in the opposite direction to gravity. The functions $\nu(\cdot)$ and $\kappa(\cdot)$ are the fluid viscosity and the thermal conductivity, respectively, which are assumed to be Lipschitz continuous and bounded from above and from below: there exist constants $\nu_{\text{lip}}$, $\kappa_{\text{lip}} > 0$ and $\nu_1$, $\nu_2$, $\kappa_1$, $\kappa_2 > 0$, such that

$$|\nu(\theta_1) - \nu(\theta_2)| \leq \nu_{\text{lip}}|\theta_1 - \theta_2|, \qquad |\kappa(\theta_1) - \kappa(\theta_2)| \leq \kappa_{\text{lip}}|\theta_1 - \theta_2|, \tag{1.2}$$

for all values of $\theta_1, \theta_2$, and

$$0 < \nu_1 \leq \nu(\theta) \leq \nu_2, \qquad 0 < \kappa_1 \leq \kappa(\theta) \leq \kappa_2, \tag{1.3}$$

for all values of $\theta$. In particular, we are interested in approximating the unknowns of the system (1.1) by using conforming finite elements.

The study of numerical methods for approximating the solution of incompressible non-isothermal fluid flows, modelled by the Bousinnesq equations and its generalizations, has significantly increased in recent years (see e.g. [1, 2, 7, 9, 10, 14, 15, 23, 24, 26], and the references therein). This fact has been motivated by the diverse applications of this model in industry (fume cupboard ventilation, heat exchangers, cooling of electronic equipments, cooling of nuclear reactors, etc.) and in geophysics or oceanography (climate predictions, oceanic flows, etc.), to name a few. In particular, the work [26] studies a finite element method for time-dependent non-isothermal incompressible fluid flow problems. Here, the governing equations are discretized by the backward Euler method in time and conforming finite elements in space. On each time step, in [26] the convective and diffusive terms are linearized by considering the solution computed in the previous step, and as a result, there is no need of analyzing the corresponding steady state nonlinear problem. More recently, in [24] has been proposed and analyzed a new mixed finite element method with exactly divergence-free velocities for the numerical simulation of the generalized Boussinesq problem (1.1). The method proposed in [24] is based on using divergence-conforming elements of order $k$ for the velocities, discontinuous elements of order $k-1$ for the pressure, and standard continuous elements of order $k$ for the temperature. The $H^1$-conformity of the velocities is enforced by a discontinuous Galerkin approach. Similarly to [22], in [24] it is shown existence and stability of discrete solutions by assuming that there exists a small enough (in the $L^3$-norm) discrete lifting of the temperature boundary data into the computational domain. The latter is a delicate issue since constructing such a lifting satisfying the required smallness assumption may be difficult.

In this paper we propose and analyze conforming finite element discretizations for the numerical approximation of the non-isothermal fluid flow problem (1.1). We employ the standard velocity-pressure formulation for the Navier-Stokes equations, which is coupled with a primal-mixed scheme for the convection-diffusion equation modelling the temperature, and introduce a new primal-mixed variational formulation for the generalized Boussinesq equations (1.1), which yields the velocity, the pressure, the temperature, and the normal derivative of the latter on the boundary as the main unknowns of the resulting formulation. We point out that the utilization of the primal-mixed scheme for the temperature equation, as has been recently applied in [9], allows us to avoid the necessity of assuming a smallness assumption on the discrete lifting of the temperature boundary data, as it is required in [24]. Then, similarly to [24], we introduce an equivalent fixed-point setting and apply the classical Brower's Theorem, combined with the generalized Lax-Milgram Theorem and the Babuška-Brezzi theory to prove solvability and stability of the corresponding discrete problem under a sufficiently small data assumption. In particular, any pair of stable Stokes elements for the velocity and pressure, such as the Hood-Taylor elements, combined with continuous piecewise polynomials of degree $\leq k+1$ for the temperature, and piecewise polynomials of degree $\leq k$ for the boundary unknown become feasible choices of finite element subspaces, yielding optimal convergence. The rest of the paper is organized as follows. In Section 2 we derive the primal-mixed variational formulation, and analyze existence and stability of solution under a smallness assumption on the data. Next, in Section 3 we introduce the Galerkin scheme, and derive general hypotheses on the finite element subspaces ensuring existence of discrete solutions and the corresponding Cea's estimate. In addition, suitable choices of finite element subspaces satisfying these assumptions are introduced in Section 4. Finally, in Section 5 we provide several numerical results illustrating the performance of the primal-mixed finite element method and confirming the theoretical rates of convergence.

Throughout the rest of the paper, we utilize the standard terminology for Sobolev spaces, norms and seminorms, employ $\mathbf{0}$ to denote a generic null vector and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

# 2 Continuous problem

## 2.1 Preliminaries

Let us first introduce some notations and previous results that we will serve for the forthcoming analysis. Let $\mathcal{O}$ be a domain in $\mathbb{R}^n$, $n \in \{2,3\}$ with Lipschitz boundary $\partial\mathcal{O}$. For $r \geq 0$ and $p \in [1,\infty]$, we denote by $\mathrm{L}^p(\mathcal{O})$ and $\mathrm{W}^{r,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{W}^{r,p}(\mathcal{O})}$, respectively. Note that $\mathrm{W}^{0,p}(\mathcal{O}) = \mathrm{L}^p(\mathcal{O})$. If $p = 2$, we write $\mathrm{H}^r(\mathcal{O})$ instead of $\mathrm{W}^{r,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{r,\mathcal{O}}$, respectively. For $r \geq 0$, we write $|\cdot|_{r,\mathcal{O}}$ for the seminorm. $\mathrm{H}^1_0(\mathcal{O})$ is the space of functions in $\mathrm{H}^1(\mathcal{O})$ with vanishing trace on $\partial\mathcal{O}$, and $\mathrm{L}^p_0(\mathcal{O})$ is the space of $\mathrm{L}^2(\mathcal{O})$ functions with vanishing mean value over $\mathcal{O}$. Spaces of vector-valued functions are denoted in bold face. For example, $\mathbf{H}^r(\mathcal{O}) = [\mathrm{H}^r(\mathcal{O})]^n$, for $r \geq 0$. For simplicity, we also write $\|\cdot\|_{0,\mathcal{O}}$ and $|\cdot|_{r,\mathcal{O}}$ for the corresponding norms and seminorms on these spaces. In the subsequent analysis, we denote by $C_\infty > 0$ the embedding constant such that

$$\|\mathbf{u}\|_{1,\mathcal{O}} \leq C_\infty \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{O})} \quad \text{and} \quad \|\theta\|_{1,\mathcal{O}} \leq C_\infty \|\theta\|_{\mathrm{W}^{1,\infty}(\mathcal{O})} \tag{2.1}$$

for all $\mathbf{u} \in \mathbf{W}^{1,\infty}(\mathcal{O})$ and $\theta \in \mathrm{W}^{1,\infty}(\mathcal{O})$.

Now, let $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ be the trace operator satisfying $\gamma_0(\varphi) := \varphi\big|_\Gamma \ \forall\, \varphi \in \mathrm{H}^1(\Omega)$. Then, we define the trace space as:

$$\mathrm{H}^{1/2}(\Gamma) := \gamma_0(\mathrm{H}^1(\Omega)),$$

endowed with the norm

$$\|\psi\|_{1/2,\Gamma} := \inf\left\{\|w\|_{1,\Omega} : w \in \mathrm{H}^1(\Omega) \text{ such that } \gamma_0(w) = \psi\right\}.$$

It readily follows that

$$\|\gamma_0(w)\|_{1/2,\Gamma} \leq \|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega). \tag{2.2}$$

As usual, the dual space of $\mathrm{H}^{1/2}(\Gamma)$ is denoted by $\mathrm{H}^{-1/2}(\Gamma)$. In addition, we use $\langle \cdot, \cdot \rangle_\Gamma$ to denote the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product. In addition, it is easy to see that there holds:

$$\mathrm{H}^1(\Omega) \,=\, \mathrm{H}^1_0(\Omega) \oplus [\mathrm{H}^1_0(\Omega)]^\perp,$$

where $\mathrm{H}^1_0(\Omega)$ is the null space of $\gamma_0$ (see, e.g. [17, Theorem 1.3-1]). Then, defining the linear bounded bijection $\tilde{\gamma_0} := \gamma_0\big|_{[\mathrm{H}^1_0(\Omega)]^\perp}$, it follows that there exists $C_{\mathrm{lift}} > 0$, such that

$$\|\tilde{\gamma_0}^{-1}(\psi)\|_{1,\Omega} \,\leq\, C_{\mathrm{lift}} \|\psi\|_{1/2,\Gamma} \quad \forall\, \psi \in \mathrm{H}^{1/2}(\Gamma). \tag{2.3}$$

## 2.2 The weak formulation

The unknowns in the weak formulation will be the velocity $\mathbf{u}$, the pressure $p$ and the temperature $\theta$. The corresponding spaces will be

$$\mathbf{u} \in \mathbf{H}^1_0(\Omega), \quad p \in L^2_0(\Omega) \quad \text{and} \quad \theta \in \mathrm{H}^1(\Omega).$$

In addition, we will need to define the following unknown on the boundary:

$$\lambda := -\kappa(\theta)\nabla\theta \cdot \mathbf{n} \,\in\, \mathrm{H}^{-1/2}(\Gamma),$$

where $\mathbf{n}$ is the unit normal vector field on $\Gamma$ which points outwards from $\Omega$. We point out that the introduction of this new unknown is due to the fact that we are not imposing the Dirichlet boundary condition $\theta = \theta_D$ on $\Gamma$ in the space where the unknown $\theta$ is looked for (see, e.g., [16, Chapter 2]).

Now, in order to obtain the weak formulation of (1.1) we test the first and fourth equations of (1.1) with arbitrary test functions $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ and $\psi \in \mathrm{H}^1(\Omega)$, respectively, integrate by parts and utilize the Dirichlet boundary conditions $\mathbf{u} = \mathbf{0}$ on $\Gamma$ and the definition of the new unknown $\lambda$, to obtain the variational equations

$$\int_\Omega \nu(\theta)\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} - \int_\Omega p \,\mathrm{div}\,\mathbf{v} - \int_\Omega \mathbf{g}\theta\cdot\mathbf{v} \,=\, 0 \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{2.4}$$

$$\int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\psi + \int_\Omega (\mathbf{u}\cdot\nabla\theta)\psi + \langle\lambda,\psi\rangle_\Gamma \,=\, 0 \quad \forall\psi \in \mathrm{H}^1(\Omega). \tag{2.5}$$

In addition, the incompressibility condition $\mathrm{div}\,\mathbf{u} = 0 \in \Omega$, as well as the Dirichlet boundary condition $\theta = \theta_D$ on $\Gamma$, are imposed weakly as follows:

$$\int_\Omega q \,\mathrm{div}\,\mathbf{u} \,=\, 0 \qquad \forall q \in \mathrm{L}^2_0(\Omega) \tag{2.6}$$

and

$$\langle\xi,\theta\rangle_\Gamma \,=\, \langle\xi,\theta_D\rangle_\Gamma \quad \forall\xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{2.7}$$

According to the above, the weak formulation of (1.1) reduces to: Find $((\mathbf{u},p),(\theta,\lambda)) \in (\mathbf{H}^1_0(\Omega)\times \mathrm{L}^2_0(\Omega)) \times (\mathrm{H}^1(\Omega)\times \mathrm{H}^{-1/2}(\Gamma))$, such that

$$\begin{aligned}
\int_\Omega \nu(\theta)\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} - \int_\Omega p\,\mathrm{div}\,\mathbf{v} - \int_\Omega \theta\mathbf{g}\cdot\mathbf{v} &= 0\\
\int_\Omega q\,\mathrm{div}\,\mathbf{u} &= 0\\
\int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\psi + \int_\Omega (\mathbf{u}\cdot\nabla\theta)\psi + \langle\lambda,\psi\rangle_\Gamma &= 0\\
\langle\xi,\theta\rangle_\Gamma &= \langle\xi,\theta_D\rangle_\Gamma,
\end{aligned} \tag{2.8}$$

for all $((\mathbf{v},q),(\psi,\xi)) \in (\mathbf{H}^1_0(\Omega)\times \mathrm{L}^2_0(\Omega)) \times (\mathrm{H}^1(\Omega)\times \mathrm{H}^{-1/2}(\Gamma))$.

### 2.3 Analysis of the continuous problem

Now, we stablish the main aspects of the continuous problem, namely, existence, uniqueness and stability. In what follows we basically adapt the results provided in [22] and [24] to our primal-mixed variational problem. We start by identifying the forms appearing in the weak formulation (2.8):

$$\begin{aligned}
A_{\mathrm{F}}(\phi;\mathbf{u},\mathbf{v}) &= \int_\Omega \nu(\phi)\,\nabla\mathbf{u}:\nabla\mathbf{v}, & O_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v}) &= \int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v},\\
A_{\mathrm{T}}(\varphi;\theta,\psi) &= \int_\Omega \kappa(\varphi)\nabla\theta\cdot\nabla\psi, & O_{\mathrm{T}}(\mathbf{v};\theta,\psi) &= \int_\Omega (\mathbf{v}\cdot\nabla\theta)\psi,\\
B_{\mathrm{F}}(\mathbf{v},q) &= \int_\Omega q\,\mathrm{div}\,\mathbf{v}, & D_{\mathrm{F}}(\theta,\mathbf{v}) &= \int_\Omega \theta\mathbf{g}\cdot\mathbf{v},\\
B_{\mathrm{T}}(\theta,\xi) &= \langle\xi,\theta\rangle_\Gamma.
\end{aligned} \tag{2.9}$$

In this way, the variational problem (2.8) can be rewritten as follows: find $(\mathbf{u},p)\in \mathbf{H}^1_0(\Omega)\times\mathrm{L}^2_0(\Omega)$ and $(\theta,\lambda)\in \mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Omega)$, such that

$$\begin{aligned}
A_{\mathrm{F}}(\theta;\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}) - B_{\mathrm{F}}(\mathbf{v},p) - D_{\mathrm{F}}(\theta,\mathbf{v}) &= 0,\\
B_{\mathrm{F}}(\mathbf{u},q) &= 0,\\
A_{\mathrm{T}}(\theta;\theta,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta,\psi) + B_{\mathrm{T}}(\psi,\lambda) &= 0,\\
B_{\mathrm{T}}(\theta,\xi) &= \langle\xi,\theta_D\rangle_\Gamma,
\end{aligned} \tag{2.10}$$

for all $(\mathbf{v},q)\in \mathbf{H}^1_0(\Omega)\times\mathrm{L}^2_0(\Omega)$ and $(\psi,\xi)\in \times\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$.

#### 2.3.1 Stability properties

Let us now discuss the stability properties of the forms in (2.9). We begin by observing that, according to hypothesis (1.3), the forms $A_{\mathrm{F}}$ and $A_{\mathrm{T}}$ are bounded in the last two components:

$$|A_{\mathrm{F}}(\cdot;\mathbf{u},\mathbf{v})| \le \nu_2\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \quad \text{and} \quad |A_{\mathrm{T}}(\cdot;\theta,\psi)| \le \kappa_2\|\theta\|_{1,\Omega}\|\psi\|_{1,\Omega}, \tag{2.11}$$

for all $\mathbf{u},\mathbf{v}\in\mathbf{H}^1_0(\Omega)$ and $\theta,\psi\in\mathrm{H}^1(\Omega)$. Moreover, from the Lipschitz continuity of $\nu$ and $\kappa$ (cf. (1.2)), and the Hölder's inequality, it readily follows that for all $\theta_1,\theta_2\in\mathrm{H}^1(\Omega)$, $\mathbf{u}\in\mathbf{W}^{1,\infty}(\Omega)$

and $\theta \in \mathrm{W}^{1,\infty}(\Omega)$, there hold

$$|A_{\mathrm{F}}(\theta_1;\mathbf{u},\mathbf{v}) - A_{\mathrm{F}}(\theta_2;\mathbf{u},\mathbf{v})| \leq \nu_{\mathrm{lip}}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)}\|\theta_1-\theta_2\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \qquad \mathbf{v}\in\mathbf{H}^1(\Omega), \tag{2.12}$$

$$|A_{\mathrm{T}}(\theta_1;\theta,\psi) - A_{\mathrm{T}}(\theta_2;\theta,\psi)| \leq \kappa_{\mathrm{lip}}\|\theta\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\theta_1-\theta_2\|_{1,\Omega}\|\psi\|_{1,\Omega}, \qquad \psi\in\mathrm{H}^1(\Omega). \tag{2.13}$$

Now, owing to standard Sobolev embeddings (cf. [27]) and the Hölder's inequality, it is not difficult to see that the convective terms $O_{\mathrm{F}}$ and $O_{\mathrm{T}}$ are bounded:

$$|O_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v})| \leq C_{O_{\mathrm{F}}}\|\mathbf{w}\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \quad \text{and} \quad |O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \leq C_{O_{\mathrm{T}}}\|\mathbf{w}\|_{1,\Omega}\|\theta\|_{1,\Omega}\|\psi\|_{1,\Omega}, \tag{2.14}$$

for all $\mathbf{w},\mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega)$ and $\theta,\psi\in\mathrm{H}^1(\Omega)$. Similarly, we have

$$|D_{\mathrm{F}}(\theta,\mathbf{v})| \leq C_D\|\mathbf{g}\|_{0,\Omega}\|\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \qquad \theta\in\mathrm{H}^1(\Omega),\ \mathbf{v}\in\mathbf{H}^1(\Omega). \tag{2.15}$$

In turn, owing to the Hölder's inequality and to property (2.2), we easily obtain

$$|B_{\mathrm{F}}(\mathbf{v},q)| \leq C_{B_{\mathrm{F}}}\|\mathbf{v}\|_{1,\Omega}\|q\|_{0,\Omega} \quad \text{and} \quad |B_{\mathrm{T}}(\psi,\xi)| \leq \|\psi\|_{1,\Omega}\|\xi\|_{-1/2,\Gamma}, \tag{2.16}$$

for all $(\mathbf{v},q)\in\mathbf{H}_0^1(\Omega)\times q\in L_0^2(\Omega)$ and $(\psi,\xi)\in\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$.

Next, we observe that the form $A_{\mathrm{F}}$ is elliptic on $\mathbf{H}_0^1(\Omega)$ in the last two components. In fact, owing to the Poincaré inequality and to (1.3), it follows that there exists $\alpha_{\mathrm{F}}>0$, such that

$$A_{\mathrm{F}}(\cdot;\mathbf{v},\mathbf{v}) \geq \alpha_{\mathrm{F}}\|\mathbf{v}\|_{1,\Omega}^2, \quad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega). \tag{2.17}$$

Analogously, we can obtain that $A_{\mathrm{T}}$ is elliptic on $\mathrm{H}_0^1(\Omega)$ in the last two components, that is, there exists $\alpha_{\mathrm{T}}>0$, such that

$$A_{\mathrm{T}}(\cdot;\psi,\psi) \geq \alpha_{\mathrm{T}}\|\psi\|_{1,\Omega}^2, \quad \forall\psi\in\mathrm{H}_0^1(\Omega). \tag{2.18}$$

On the other hand, by integrating by parts, it readily follows that

$$\begin{aligned} O_{\mathrm{F}}(\mathbf{w};\mathbf{v},\mathbf{v}) &= 0, \qquad \mathbf{w}\in\mathbf{X}, \quad \mathbf{v}\in\mathbf{H}^1(\Omega),\\ O_{\mathrm{T}}(\mathbf{w};\psi,\psi) &= 0, \qquad \mathbf{w}\in\mathbf{X}, \quad \psi\in\mathrm{H}^1(\Omega), \end{aligned} \tag{2.19}$$

where $\mathbf{X}$ is the kernel of the bilinear form $B_{\mathrm{F}}$, that is

$$\mathbf{X} := \left\{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\ :\ B_{\mathrm{F}}(\mathbf{v},q)=0 \quad \forall\,q\in\mathrm{L}_0^2(\Omega)\right\} = \left\{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\ :\ \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega\right\}.$$

We conclude this section by noting that the bilinear forms $B_{\mathrm{F}}$ and $B_{\mathrm{T}}$ satisfy the following inf-sup conditions:

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{B_{\mathrm{F}}(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}} \geq \beta_{\mathrm{F}}\|q\|_{0,\Omega}, \qquad \forall q\in\mathrm{L}_0^2(\Omega), \tag{2.20}$$

$$\sup_{\substack{\psi\in\mathrm{H}^1(\Omega)\\ \psi\neq0}} \frac{B_{\mathrm{T}}(\psi,\xi)}{\|\psi\|_{1,\Omega}} \geq \beta_{\mathrm{T}}\|\xi\|_{-1/2,\Gamma}, \qquad \forall\xi\in\mathrm{H}^{-1/2}(\Gamma), \tag{2.21}$$

where $\beta_{\mathrm{F}}$ and $\beta_{\mathrm{T}}$ are positive constants depending on $|\Omega|$. The proof of (2.20) and (2.21) can be found in [17, Theorem 3.7] and [16, Section 2.4.4], respectively.

### 2.3.2 Existence and uniqueness of solution

Now, we study the existence and uniqueness of solution of problem (2.10). To that end, it is enough to study the reduced version of problem (2.10): Find $\mathbf{u} \in \mathbf{X}$ and $\theta \in \mathrm{H}^1(\Omega)$, such that $\theta|_\Gamma = \theta_D$ and

$$\begin{aligned} A_{\mathrm{F}}(\theta;\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}) - D_{\mathrm{F}}(\theta,\mathbf{v}) &= 0,\\ A_{\mathrm{T}}(\theta;\theta,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta,\psi) &= 0, \end{aligned} \tag{2.22}$$

for all $\mathbf{v} \in \mathbf{X}$ and $\psi \in \times\mathrm{H}^1_0(\Omega)$.

In fact, the next lemma establishes the equivalence between problems (2.10) and (2.22).

**Lemma 2.1** *If $((\mathbf{u},p),(\theta,\lambda)) \in (\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega))\times(\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma))$ is a solution of (2.10), then $\mathbf{u}\in\mathbf{X}$ and $(\mathbf{u},\theta)\in\mathbf{X}\times\mathrm{H}^1(\Omega)$ is a solution of* (2.22)*. Conversely, if $(\mathbf{u},\theta)\in\mathbf{X}\times\mathrm{H}^1(\Omega)$ is a solution of* (2.22)*, then there exists $p\in\mathrm{L}^2_0(\Omega)$ and $\lambda\in\mathrm{H}^{-1/2}(\Gamma)$ such that $((\mathbf{u},p),(\theta,\lambda))\in(\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega))\times(\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma))$ is a solution of (2.10).*

*Proof.* Let $((\mathbf{u},p),(\theta,\lambda)) \in (\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega))\times(\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma))$ be a solution of (2.10). Then it readily follows that $\mathbf{u}\in\mathbf{X}$, $\theta|_\Gamma=\theta_D$, and therefore $(\mathbf{u},\theta)\in\mathbf{X}\times\mathrm{H}^1(\Omega)$ is also a solution of (2.22).

Conversely, let $(\mathbf{u},\theta)\in\mathbf{X}\times\mathrm{H}^1(\Omega)$ be a solution of (2.22). Then, owing to the inf-sup condition (2.20), we easily obtain that there exists $p\in L^2_0(\Omega)$, such that

$$B_{\mathrm{F}}(\mathbf{v},p) \;=\; A_{\mathrm{F}}(\theta;\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}) - D_{\mathrm{F}}(\theta,\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega).$$

Similarly, since the bilinear form $B_{\mathrm{T}}$ satisfies the inf-sup condition (2.21), we obtain that there exists $\lambda\in\mathrm{H}^{-1/2}(\Gamma)$, such that

$$B_{\mathrm{T}}(\psi,\lambda) \;=\; -A_{\mathrm{T}}(\theta;\theta,\psi) - O_{\mathrm{T}}(\mathbf{u};\theta,\psi), \quad \forall\,\psi\in\mathrm{H}^1(\Omega).$$

Finally, since $\mathbf{u}\in\mathbf{X}$ and $\theta|_\Gamma=\theta_D$, then

$$B_{\mathrm{F}}(\mathbf{u},q) \;=0 \quad \text{and} \quad B_{\mathrm{T}}(\theta,\xi) \;=\; \langle\xi,\theta_D\rangle_\Gamma,$$

for all $q\in L^2_0(\Omega)$ and $\xi\in\mathrm{H}^{-1/2}(\Gamma)$, which concludes the proof. □

The next result provides the existence of solution of problem (2.22). Its proof can be found in [22, Theorem 2.1]

**Theorem 2.2** *Given $\theta_D\in\mathrm{H}^{1/2}(\Gamma)$ and $\mathbf{g}\in\mathbf{L}^2(\Omega)$, there exists $(\mathbf{u},\theta)\in\mathbf{X}\times\mathrm{H}^1(\Omega)$ solution to problem* (2.22)*.*

Next, we establish the corresponding uniqueness result of problem (2.22). Its proof can be found in [24, Theorem 2.3] and is based on an additional smallness assumption on the solution.

**Theorem 2.3** *Let $(\mathbf{u},\theta)\in[\mathbf{X}\cap\mathbf{W}^{1,\infty}(\Omega)]\times\mathrm{W}^{1,\infty}(\Omega)$ be a solution to problem* (2.22)*, and assume that there exists $M>0$, such that*

$$\max\{\|\mathbf{g}\|_{0,\Omega},\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)},\|\theta\|_{\mathrm{W}^{1,\infty}(\Omega)}\} \le M < \min\left\{\frac{\alpha_{\mathrm{F}}}{C_{O_{\mathrm{F}}}C_\infty+K}\,,\,\frac{\alpha_{\mathrm{T}}}{\kappa_{\mathrm{lip}}+K}\right\}, \tag{2.23}$$

*with $K=(C_{D_{\mathrm{F}}}+\nu_{\mathrm{lip}}+C_{O_{\mathrm{T}}}C_\infty)/2$. Then, the solution of problem* (2.22) *is unique.*

We end this section by observing that, owing to the equivalence of problems (2.10) and(2.22) (Lemma 2.1), and the existence of solution of problem (2.22) (Theorem 2.2), we can conclude that problem (2.10) admits a solution, which is unique if we assume further that (2.23) holds.

#### 2.3.3 Stability of solution

Now, we establish the corresponding stability result of problem (2.10). We begin by observing that, analogously to the analysis presented in Section 2.3.2, and according to Lemma 2.1, it suffices to analyze the stability of solution of problem (2.22).

Let us now point out that this result has been already established in [22, Section 3] for the reduced problem (2.22), assuming that there exists a sufficiently small, in the $L^3$-norm, lifting of the temperature boundary datum $\theta_D$ into the computational domain. More precisely, we have the following result.

**Theorem 2.4** *Given* $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$*, let* $(\mathbf{u}, \theta) \in \mathbf{X} \times \mathrm{H}^1(\Omega)$ *be a solution of problem* (2.22)*. Given* $\epsilon > 0$*, assume that there exists* $\theta_1 \in \mathrm{H}^1(\Omega)$*, such that* $\theta_1|_\Gamma = \theta_D$ *and* $\|\theta_1\|_{L^3(\Omega)} < \epsilon$*. Then there exist positive constants* $c_1$ *and* $c_2$*, independent of the solution, such that*

$$\|\mathbf{u}\|_{1,\Omega} \leq c_1 \|\theta_1\|_{1,\Omega} \quad \textit{and} \quad \|\theta\|_{1,\Omega} \leq c_2 \|\theta_1\|_{1,\Omega}.$$

At this point, we observe that the assumption $\|\theta_1\|_{L^3(\Omega)} < \epsilon$ may be a delicate matter when solving the equations (1.1) numerically. In fact, as it is done for the continuous case, to ensure the stability of a finite element solution of problem (1.1), one needs to be able to find a small enough, in the $L^3$-norm, discrete lifting $\theta_{1,h}$ of a suitable approximation of the datum $\theta_D$ on the boundary $\Gamma$. One option (see [24, Section 4.2] for other options) is to approximate $\theta_D$ by using the Lagrange interpolant (assuming more regularity for $\theta_D$), and then define $\theta_{1,h}$ as the resulting approximation of $\theta_D$ on the boundary, and zero in all the internals nodes. Hence, in this case $\|\theta_{1,h}\|_{L^3(\Omega)}$ is small enough as the discretization parameter $h$ tends to zero. However, $\|\theta_{1,h}\|_{1,\Omega}$ blows up as $h$ tends to zero, making the numerical method unstable. This issue motivates the utilization of the primal-mixed formulation for the heat equation since, in this case, the datum $\theta_D$ appears on the right-hand side of the resulting scheme, avoiding the necessity of approximating $\theta_D$ on the boundary. In this way, as we will see later, the stability result for the corresponding finite element discretization can be obtained assuming now that the original datum $\theta_D$ is small enough in the $H^{1/2}$ norm.

According to the above, we now provide the following alternative proof for the stability of solution of problem (2.22).

**Lemma 2.5** *Given* $\theta_D \in H^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$*, let* $(\mathbf{u}, \theta) \in \mathbf{X} \times \mathrm{H}^1(\Omega)$ *be a solution of problem* (2.22)*. Assume that*

$$C_{stab} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \leq \frac{1}{2}, \tag{2.24}$$

*with*

$$C_{stab} := \frac{C_{O_\mathrm{T}} C_{D_\mathrm{F}} C_{\mathrm{lift}}}{\alpha_\mathrm{T} \alpha_\mathrm{F}}. \tag{2.25}$$

*Then, there exist constants* $C_\mathbf{u}$ *and* $C_\theta$ *depending only on* $\|\mathbf{g}\|_{0,\Omega}$ *and the stability constants, such that*

$$\|\mathbf{u}\|_{1,\Omega} \leq C_\mathbf{u} \|\theta_D\|_{1/2,\Gamma} \quad \textit{and} \quad \|\theta\|_{1,\Omega} \leq C_\theta \|\theta_D\|_{1/2,\Gamma}. \tag{2.26}$$

*(Explicit expressions for* $C_\mathbf{u}$ *and* $C_\theta$ *can be found in* (2.31) *and* (2.32)*, respectively.)*

*Proof.* Let $\theta_1 = \tilde{\gamma_0}^{-1}(\theta_D)$, with $\tilde{\gamma_0}$ being the operator defined in Section 2.1, and let $\theta_0 = \theta - \theta_1$. It follows that $\theta_0 \in \mathrm{H}^1_0(\Omega)$ and $\theta_1\big|_\Gamma = \theta_D$. In addition, from (2.3), we have

$$\|\theta_1\|_{1,\Omega} \leq C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma}. \tag{2.27}$$

Then, owing to (2.19), from (2.22) with $(\mathbf{v}, \psi) = (\mathbf{u}, \theta_0) \in \mathbf{X} \times \mathrm{H}_0^1(\Omega)$ we easily obtain the identities

$$A_{\mathrm{F}}(\theta; \mathbf{u}, \mathbf{u}) = D_{\mathrm{F}}(\theta, \mathbf{u}) \quad \text{and} \quad A_{\mathrm{T}}(\theta; \theta_0, \theta_0) = -A_{\mathrm{T}}(\theta; \theta_1, \theta_0) - O_{\mathrm{T}}(\mathbf{u}; \theta_1, \theta_0),$$

which together with (2.11), (2.14), (2.15), (2.17) and (2.18), implies

$$\|\mathbf{u}\|_{1,\Omega} \;\leq\; \alpha_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}} \|\mathbf{g}\|_{0,\Omega} \|\theta\|_{1,\Omega}, \tag{2.28}$$

and

$$\|\theta_0\|_{1,\Omega} \;\leq\; \alpha_{\mathrm{T}}^{-1} \kappa_2 \|\theta_1\|_{1,\Omega} + \alpha_{\mathrm{T}}^{-1} C_{O_{\mathrm{T}}} \|\mathbf{u}\|_{1,\Omega} \|\theta_1\|_{1,\Omega}.$$

In particular, according to (2.27), the latter can be rewritten as

$$\|\theta_0\|_{1,\Omega} \;\leq\; \alpha_{\mathrm{T}}^{-1} \kappa_2 C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} + \alpha_{\mathrm{T}}^{-1} C_{O_{\mathrm{T}}} C_{\mathrm{lift}} \|\mathbf{u}\|_{1,\Omega} \|\theta_D\|_{1/2,\Gamma},$$

which together with the identity $\theta = \theta_0 + \theta_1$ and the triangle inequality yields

$$\begin{aligned} \|\theta\|_{1,\Omega} \;&\leq\; \|\theta_0\|_{1,\Omega} + \|\theta_1\|_{1,\Omega} \\ &\leq\; \|\theta_0\|_{1,\Omega} + C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} \\ &\leq\; (\kappa_2 + \alpha_{\mathrm{T}})\, \alpha_{\mathrm{T}}^{-1} C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} + \alpha_{\mathrm{T}}^{-1} C_{O_{\mathrm{T}}} C_{\mathrm{lift}} \|\mathbf{u}\|_{1,\Omega} \|\theta_D\|_{1/2,\Gamma}. \end{aligned} \tag{2.29}$$

Hence, replacing (2.29) into (2.28), we obtain

$$\begin{aligned} &\left(1 - \alpha_{\mathrm{T}}^{-1} \alpha_{\mathrm{F}}^{-1} C_{O_{\mathrm{T}}} C_{D_{\mathrm{F}}} C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} \|\mathbf{g}\|_{0,\Omega}\right) \|\mathbf{u}\|_{1,\Omega} \\ &\qquad\qquad\qquad \leq (\kappa_2 + \alpha_{\mathrm{T}})\, \alpha_{\mathrm{T}}^{-1} \alpha_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}} C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} \|\mathbf{g}\|_{0,\Omega}, \end{aligned}$$

which combined with assumption (2.24), implies

$$\|\mathbf{u}\|_{1,\Omega} \;\leq\; C_{\mathbf{u}} \|\theta_D\|_{1/2,\Gamma}, \tag{2.30}$$

with

$$C_{\mathbf{u}} \;=\; 2\, (\kappa_2 + \alpha_{\mathrm{T}})\, \alpha_{\mathrm{T}}^{-1} \alpha_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}} C_{\mathrm{lift}} \|\mathbf{g}\|_{0,\Omega}. \tag{2.31}$$

Similarly, replacing (2.30) into (2.29), we obtain

$$\|\theta\|_{1,\Omega} \;\leq\; \left(1 + 2\alpha_{\mathrm{T}}^{-1} \alpha_{\mathrm{F}}^{-1} C_{O_{\mathrm{T}}} C_{D_{\mathrm{F}}} C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma} \|\mathbf{g}\|_{0,\Omega}\right) (\kappa_2 + \alpha_{\mathrm{T}})\, \alpha_{\mathrm{T}}^{-1} C_{\mathrm{lift}} \|\theta_D\|_{1/2,\Gamma},$$

which combined with assumption (2.24) yields

$$\|\theta\|_{1,\Omega} \;\leq\; C_\theta \|\theta_D\|_{1/2,\Gamma},$$

with

$$C_\theta = 2\, (\kappa_2 + \alpha_{\mathrm{T}})\, \alpha_{\mathrm{T}}^{-1} C_{\mathrm{lift}}. \tag{2.32}$$

□

As a consequence of Lemma 2.5 we can obtain the stability of solution of problem (2.10).

**Theorem 2.6** *Given* $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$, *let* $((\mathbf{u},p),(\theta,\lambda)) \in (\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)) \times (\mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma))$ *be a solution of problem* (2.10). *Assume that* $\theta_D$ *and* $\mathbf{g}$ *satisfy* (2.24), *that is*

$$C_{stab}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} \leq \frac{1}{2},$$

*with* $C_{stab}$ *given by* (2.25). *Then,*

$$\|\mathbf{u}\|_{1,\Omega} \leq C_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma} \quad \textit{and} \quad \|\theta\|_{1,\Omega} \leq C_\theta\|\theta_D\|_{1/2,\Gamma}.$$

*with* $C_{\mathbf{u}}$ *and* $C_\theta$ *given by* (2.31) *and* (2.32), *respectively. In addition, there exist positive constants* $C_p$ *and* $C_\lambda$, *depending only on* $\|\mathbf{g}\|_{0,\Omega}$ *and the stability constants, such that*

$$\|p\|_{0,\Omega} \leq C_p\|\theta_D\|_{1/2,\Gamma} \quad \textit{and} \quad \|\lambda\|_{-1/2,\Gamma} \leq C_\lambda\|\theta_D\|_{1/2,\Gamma}. \tag{2.33}$$

*(Explicit expressions for* $C_p$ *and* $C_\lambda$ *can be found in (2.34) and (2.35), respectively).*

*Proof.* Since $((\mathbf{u},p),(\theta,\lambda))$ is a solution to (2.10) owing to Lemma 2.1 and Lemma 2.5, it readily follows that $\mathbf{u}$ and $\theta$ satisfy the estimates (2.26). In this way, it suffices to verify the estimates in (2.33) to conclude the proof.

First, from inf-sup condition (2.20), the first equation in (2.10) and the continuity properties (2.11), (2.14) and (2.15), we obtain

$$\begin{aligned}
\beta_{\mathrm{F}}\|p\|_{0,\Omega} &\leq \sup_{\substack{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{B_{\mathrm{F}}(\mathbf{v},p)}{\|\mathbf{v}\|_{1,\Omega}} \\
&= \sup_{\substack{\mathbf{v}\in\mathbf{H}^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \left\{\frac{A_{\mathrm{F}}(\theta;\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}) - D_{\mathrm{F}}(\theta,\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}}\right\} \\
&\leq \|\mathbf{u}\|_{1,\Omega}(\nu_2 + C_{O_{\mathrm{F}}}\|\mathbf{u}\|_{1,\Omega}) + C_{D_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\theta\|_{1,\Omega}.
\end{aligned}$$

which together with (2.26) and (2.24), implies

$$\|p\|_{0,\Omega} \leq C_p\|\theta_D\|_{1/2,\Gamma},$$

with $C_p$ is defined as

$$C_p = \beta_{\mathrm{F}}^{-1}\left[C_{\mathbf{u}}\left(\nu_2 + \frac{C_{O_{\mathrm{F}}}C_{\mathbf{u}}\alpha_{\mathrm{T}}\alpha_{\mathrm{F}}}{2C_{O_{\mathrm{T}}}C_{D_{\mathrm{F}}}C_{\mathrm{lift}}}\right) + C_{D_{\mathrm{F}}}C_\theta\right]\|\mathbf{g}\|_{0,\Omega}. \tag{2.34}$$

Similarly, from the inf-sup condition (2.21), the third equation of (2.10) , and the continuity of $A_{\mathrm{T}}$ and $O_{\mathrm{T}}$ in (2.11) and (2.14), respectively, we have

$$\begin{aligned}
\beta_{\mathrm{T}}\|\lambda\|_{-1/2,\Gamma} &\leq \sup_{\substack{\psi\in\mathrm{H}^1(\Omega)\\ \psi\neq 0}} \frac{B_{\mathrm{T}}(\psi,\lambda)}{\|\psi\|_{1,\Omega}} \\
&= \sup_{\substack{\psi\in\mathrm{H}^1(\Omega)\\ \psi\neq 0}} \left\{\frac{-A_{\mathrm{T}}(\theta;\theta,\psi) - O_{\mathrm{T}}(\mathbf{u};\theta,\psi)}{\|\psi\|_{1,\Omega}}\right\} \\
&\leq (\kappa_2 + C_{O_{\mathrm{T}}}\|\mathbf{u}\|_{1,\Omega})\,\|\theta\|_{1,\Omega},
\end{aligned}$$

which together with (2.24) and (2.26), yields

$$\|\lambda\|_{-1/2,\Gamma} \leq C_\lambda \|\theta_D\|_{1/2,\Gamma},$$

with

$$C_\lambda = \beta_{\mathrm{T}}^{-1} C_\theta \left( \kappa_2 + \frac{C_{O_{\mathrm{T}}} C_{\mathbf{u}} \alpha_{\mathrm{T}} \alpha_{\mathrm{F}}}{2 C_{O_{\mathrm{T}}} C_{D_{\mathrm{F}}} C_{\mathrm{lift}}} \right). \tag{2.35}$$

□

# 3 The Galerkin scheme

In this section we introduce the Galerkin scheme of the variational problem (2.10), analyze its solvability and provide the corresponding Cea's estimate.

## 3.1 Preliminaries

We begin by taking arbitrary finite dimensional subspaces

$$\mathbf{H}_h \subseteq \mathbf{H}_0^1(\Omega), \quad \mathrm{Q}_h \subseteq \mathrm{L}_0^2(\Omega), \quad \Psi_h \subseteq \mathrm{H}^1(\Omega) \quad \text{and} \quad \Lambda_h \subseteq \mathrm{H}^{-1/2}(\Gamma).$$

Hereafter, $h$ stands for the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles $K$ (when $n = 2$) or tetrahedra $K$, (when $n = 3$) of diameter $h_K$, that is $h := \{h_K : K \in \mathcal{T}_h\}$.

As we will see in Section 4, the pair $(\mathbf{H}_h, \mathrm{Q}_h)$ will be chosen as a pair of stable finite element subspaces for the Stokes problem which, as it is well known, may not produce divergence-free velocities, and as a consequence, at the discrete level properties (2.19) may not be necessarily satisfied. According to this, in what follows we will consider discrete versions of the convective terms $O_{\mathrm{F}}$ and $O_{\mathrm{T}}$, denoted respectively by $O_{\mathrm{F}}^h$ and $O_{\mathrm{T}}^h$, both of them linear on each variable and satisfying

$$|O_{\mathrm{F}}^h(\mathbf{w};\mathbf{u},\mathbf{v})| \leq \widehat{C}_{O_{\mathrm{F}}} \|\mathbf{w}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \qquad \mathbf{w},\mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega), \tag{3.1}$$

$$|O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi)| \leq \widehat{C}_{O_{\mathrm{T}}} \|\mathbf{w}\|_{1,\Omega} \|\theta\|_{1,\Omega} \|\psi\|_{1,\Omega}, \qquad \mathbf{w} \in \mathbf{H}^1(\Omega),\ \theta,\psi \in \mathrm{H}^1(\Omega), \tag{3.2}$$

with $\widehat{C}_{O_{\mathrm{F}}}$, and $\widehat{C}_{O_{\mathrm{T}}}$ being positive constants independent of the parameter of discretization $h$. Concrete examples for these discrete convective terms will be provided in Section 4.

In this way, considering the forms $A_{\mathrm{F}}$, $A_{\mathrm{T}}$, $B_{\mathrm{F}}$ ,$B_{\mathrm{T}}$ and $D_{\mathrm{F}}$, introduced in Section 2.3 , the Galerkin scheme associated to (2.10) reads: Find $((\mathbf{u}_h, p_h), (\theta_h, \lambda_h)) \in (\mathbf{H}_h \times \mathrm{Q}_h) \times (\Psi_h \times \Lambda_h)$, such that

$$\begin{aligned} A_{\mathrm{F}}(\theta_h;\mathbf{u}_h,\mathbf{v}_h) + O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) - B_{\mathrm{F}}(\mathbf{v}_h,p_h) - D_{\mathrm{F}}(\theta_h,\mathbf{v}_h) &= 0, \\ B_{\mathrm{F}}(\mathbf{u}_h,q_h) &= 0, \\ A_{\mathrm{T}}(\theta_h;\theta_h,\psi_h) + O_{\mathrm{T}}^h(\mathbf{u}_h;\theta_h,\psi_h) + B_{\mathrm{T}}(\psi_h,\lambda_h) &= 0, \\ B_{\mathrm{T}}(\theta_h,\xi_h) &= \langle \xi_h, \theta_D \rangle_\Gamma, \end{aligned} \tag{3.3}$$

for all $((\mathbf{v}_h, q_h), (\psi_h, \xi_h)) \in (\mathbf{H}_h \times \mathrm{Q}_h) \times (\Psi_h \times \Lambda_h)$.

In what follows we derive general hypotheses on the spaces $\mathbf{H}_h$, $\mathrm{Q}_h$, $\Psi_h$ and $\Lambda_h$ that will allow us to show in Section 3.2 below the solvability of (3.3). Our approach consists of adapting to the present discrete case the arguments employed in [24, Section 4]. We begin by introducing the following two hypotheses providing sufficient conditions (Babuška-Brezzi conditions) to ensure the well-posedness of Oseen-type problems (see [16, Section 2.5]):

(**H.0**) There exists $\widehat{\alpha}_{\rm F} > 0$, independent of the discretization parameter $h$, such that for given $\mathbf{w}_h \in \mathbf{X}_h$ and $\varphi_h \in \Psi_h$, there holds

$$\sup_{\substack{\mathbf{z}_h \in \mathbf{X}_h \\ \mathbf{z}_h \neq 0}} \frac{A_{\rm F}(\varphi_h; \mathbf{v}_h, \mathbf{z}_h) + O_{\rm T}^h(\mathbf{w}_h; \mathbf{v}_h, \mathbf{z}_h)}{\|\mathbf{z}_h\|_{1,\Omega}} \geq \widehat{\alpha}_{\rm F} \|\mathbf{v}_h\|_{1,\Omega} \quad \forall\, \mathbf{v}_h \,\in\, \mathbf{X}_h, \tag{3.4}$$

where $\mathbf{X}_h$ is the Kernel of the bilinear form $B_{\rm F}$, that is

$$\mathbf{X}_h \,:=\, \{\mathbf{v}_h \in \mathbf{H}_h \,:\, B_{\rm F}(\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in {\rm Q}_h\}\,.$$

(**H.1**) There exists $\widehat{\beta}_{\rm F} > 0$, independent of the discretization parameter $h$, such that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq 0}} \frac{B_{\rm F}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widehat{\beta}_F \|q_h\|_{0,\Omega} \qquad \forall q_h \in {\rm Q}_h. \tag{3.5}$$

In fact, owing to hypotheses (**H.0**) and (**H.1**) we easily obtain the solvability of the problem: Given $\mathbf{w}_h \in \mathbf{H}_h$ and $\varphi_h \in \Psi_h$, find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times {\rm Q}_h$, such that

$$\begin{aligned} A_{\rm F}(\varphi_h; \mathbf{u}_h, \mathbf{v}_h) + O_{\rm F}^h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) - B_{\rm F}(\mathbf{v}_h, p_h) &= D_{\rm F}(\varphi_h, \mathbf{v}_h) \\ B_{\rm F}(\mathbf{u}_h, q_h) &= 0, \end{aligned} \tag{3.6}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{H}_h \times {\rm Q}_h)$.

Similarly, due to the mixed structure of the convection-diffusion equation modelling the temperature, we will require the following hypotheses:

(**H.2**) There exists $\widehat{\beta}_{\rm T} > 0$, independent of the discretization parameter $h$, such that

$$\sup_{\substack{\psi_h \in \Psi_h \\ \psi_h \neq 0}} \frac{B_{\rm T}(\psi_h, \xi_h)}{\|\psi_h\|_{1,\Omega}} \geq \widehat{\beta}_T \|\xi_h\|_{-1/2,\Gamma} \quad \forall \xi_h \in \Lambda_h. \tag{3.7}$$

(**H.3**) There exists $\widehat{\alpha}_{\rm T} > 0$, independent of the discretization parameter $h$, such that for given $\varphi_h \in \Psi_h$ and $\mathbf{w}_h \in \mathbf{X}_h$, there holds

$$\sup_{\substack{\phi_h \in \Psi_{h,0} \\ \phi_h \neq 0}} \frac{A_{\rm T}(\varphi_h; \psi_h, \phi_h) + O_{\rm T}^h(\mathbf{w}_h; \psi_h, \phi_h)}{\|\phi_h\|_{1,\Omega}} \geq \widehat{\alpha}_{\rm T} \|\psi_h\|_{1,\Omega} \quad \forall\, \psi_h \,\in\, \Psi_{h,0}, \tag{3.8}$$

where $\Psi_{h,0}$ is the Kernel of the bilinear form $B_{\rm T}$, that is

$$\Psi_{h,0} := \{\psi_h \in \Psi_h \,:\, B_{\rm T}(\psi_h, \xi_h) = 0, \quad \forall\, \xi_h \in \Lambda_h\}. \tag{3.9}$$

Finally, in order to achieve the derivation of the Cea's estimate, we will assume that the discrete convective terms $O_{\rm F}^h$ and $O_{\rm T}^h$ are consistent with the continuous ones in the following sense:

(**H.4**) Given $\mathbf{w} \in \mathbf{X}$, there hold

$$O_{\rm F}^h(\mathbf{w}; \mathbf{u}, \mathbf{v}) = O_{\rm F}(\mathbf{w}; \mathbf{u}, \mathbf{v}) \quad \text{and} \quad O_{\rm T}^h(\mathbf{w}; \theta, \psi) = O_{\rm T}(\mathbf{w}; \theta, \psi), \tag{3.10}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\theta$, $\psi \in {\rm H}^1(\Omega)$.

### 3.2 Existence and stability of solution

In what follows we establish existence and stability of solution of problem (3.3). To do that, we first observe that owing to hypothesis (**H.1**), it readily follows that (3.3) is equivalent to the reduced problem: Find $\mathbf{u}_h \in \mathbf{X}_h$ and $(\theta_h, \lambda_h) \in \Psi_h \times \Lambda_h$, such that

$$\begin{aligned} A_{\mathrm{F}}(\theta_h; \mathbf{u}_h, \mathbf{v}_h) + O^h_{\mathrm{F}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - D_{\mathrm{F}}(\theta_h, \mathbf{v}_h) &= 0, \\ A_{\mathrm{T}}(\theta_h; \theta_h, \psi_h) + O^h_{\mathrm{T}}(\mathbf{u}_h; \theta_h, \psi_h) + B_{\mathrm{T}}(\psi_h, \lambda_h) &= 0, \\ B_{\mathrm{T}}(\theta_h, \xi_h) &= \langle \xi_h, \theta_D \rangle_\Gamma, \end{aligned} \tag{3.11}$$

for all $\mathbf{v}_h \in \mathbf{X}_h$ and $(\psi_h, \xi_h) \in \Psi_h \times \Lambda_h$. In fact, we have the following result:

**Lemma 3.1** *If* $((\mathbf{u}_h, p_h), (\theta_h, \lambda_h)) \in (\mathbf{H}_h \times \mathrm{Q}_h) \times (\Psi_h \times \Lambda_h)$ *is a solution of* (3.3)*, then* $\mathbf{u}_h \in \mathbf{X}_h$ *and* $(\mathbf{u}_h, (\theta_h, \lambda_h)) \in \mathbf{X}_h \times (\Psi_h, \Lambda_h)$ *is also a solution of* (3.11)*. Conversely, if* $(\mathbf{u}_h, (\theta_h, \lambda_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$ *is a solution of* (3.11)*, then there exists a discrete pressure* $p_h \in \mathrm{Q}_h$ *such that* $((\mathbf{u}_h, p_h), (\theta_h, \lambda_h)) \in (\mathbf{H}_h \times \mathrm{Q}_h) \times \Psi_h \times \Lambda_h)$ *is also a solution of* (3.3)*.*

According to the above, in the sequel we analyze problem (3.11).

#### 3.2.1 Two technical results

Before proving the main results of this section, we first provide two technical results that will serve for the forthcoming analysis.

**Lemma 3.2** *Let* $\mathbf{g} \in \mathbf{L}^2(\Omega)$*,* $\mathbf{w}_h \in \mathbf{X}_h$ *and* $\varphi_h \in \Psi_h$*. Assume that* (**H.0**) *holds. Then there exists a unique* $\mathbf{u}_h \in \mathbf{X}_h$*, such that*

$$A_{\mathrm{F}}(\varphi_h; \mathbf{u}_h, \mathbf{v}_h) + O^h_F(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) = D_{\mathrm{F}}(\varphi_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h. \tag{3.12}$$

*Moreover,*

$$\|\mathbf{u}_h\|_{1,\Omega} \le \widehat{\alpha}_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}} \|\mathbf{g}\|_{0,\Omega} \|\varphi_h\|_{1,\Omega}. \tag{3.13}$$

*Proof.* Given $\mathbf{w}_h \in \mathbf{X}_h$ and $\varphi_h \in \Psi_h$, from (**H.0**) (cf. (3.4)), (2.11) and (3.2), we easily obtain that $A_{\mathrm{F}}(\varphi_h, \cdot, \cdot) + O^h_F(\mathbf{w}_h; \cdot, \cdot)$ induces a continuous and bijective linear operator from $\mathbf{X}_h$ to $\mathbf{X}_h$, which implies the existence and uniqueness of solution of (3.12). In addition, from (2.15), (3.4) and (3.12), it readily follows that

$$\begin{aligned} \widehat{\alpha}_{\mathrm{F}} \|\mathbf{u}_h\|_{1,\Omega} &\le \sup_{\substack{\mathbf{z}_h \in \mathbf{X}_h \\ \mathbf{z}_h \neq 0}} \frac{A_{\mathrm{F}}(\varphi_h; \mathbf{u}_h, \mathbf{z}_h) + O^h_{\mathrm{T}}(\mathbf{w}_h; \mathbf{u}_h, \mathbf{z}_h)}{\|\mathbf{z}_h\|_{1,\Omega}} \\ &= \sup_{\substack{\mathbf{z}_h \in \mathbf{X}_h \\ \mathbf{z}_h \neq 0}} \frac{D_{\mathrm{F}}(\varphi_h, \mathbf{z}_h)}{\|\mathbf{z}_h\|_{1,\Omega}} \\ &\le C_{D_{\mathrm{F}}} \|\mathbf{g}\|_{0,\Omega} \|\varphi_h\|_{1,\Omega}, \end{aligned}$$

which implies (3.13) and concludes the proof. □

**Lemma 3.3** *Let* $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$*,* $\mathbf{g} \in \mathbf{L}^2(\Omega)$*,* $\mathbf{w}_h \in \mathbf{X}_h$ *and* $\varphi_h \in \Psi_h$*. Assume that* (**H.2**) *and* (**H.3**) *hold. Then, there exists a unique* $(\theta_h, \lambda_h) \in \Psi_h \times \Lambda_h$*, such that*

$$\begin{aligned} A_{\mathrm{T}}(\varphi_h; \theta_h, \psi_h) + O^h_{\mathrm{T}}(\mathbf{w}_h; \theta_h, \psi_h) + B_{\mathrm{T}}(\psi_h, \lambda_h) &= 0, \\ B_{\mathrm{T}}(\theta_h, \xi_h) &= \langle \xi_h, \theta_D \rangle_\Gamma, \end{aligned} \tag{3.14}$$

*for all* $(\psi_h, \xi_h) \in \Psi_h \times \Lambda_h$. *Moreover,*

$$\|\theta_h\|_{1,\Omega} \leq \widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{\beta}_{\mathrm{T}}^{-1}\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega} + \widehat{\alpha}_{\mathrm{T}}\right)\|\theta_D\|_{1/2,\Gamma} \tag{3.15}$$

*and*

$$\|\lambda_h\|_{-1/2,\Gamma} \leq \widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{\beta}_{\mathrm{T}}^{-2}(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega})\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega} + \widehat{\alpha}_{\mathrm{T}}\right)\|\theta_D\|_{1/2,\Gamma}. \tag{3.16}$$

*Proof.* Since hypotheses (**H.2**) and (**H.3**) hold, a straightforward application of the classical Babuška-Brezzi theory implies that problem (3.14) is well posed. Now, let $\theta_{h,0} \in \Psi_{h,0}$ and $\theta_{h,1} \in \Psi_{h,0}^{\perp}$, such that $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\Psi_{h,0}^{\perp}$ being the orthogonal complement of $\Psi_{h,0}$. Then, owing the inf-sup condition (3.7) and the second equation of (3.14), a classical result in functional analysis (see, e.g., [16, Section 2.2]) implies that

$$\|\theta_{h,1}\|_{1,\Omega} \;\leq\; \widehat{\beta}_{\mathrm{T}}^{-1} \sup_{\substack{\xi_h\in\Lambda_h\\ \xi_h\neq 0}} \frac{B_{\mathrm{T}}(\theta_{h,1},\xi_h)}{\|\xi_h\|_{-1/2,\Gamma}} \;=\; \widehat{\beta}_{\mathrm{T}}^{-1} \sup_{\substack{\xi_h\in\Lambda_h\\ \xi_h\neq 0}} \frac{\langle \xi_h, \theta_D\rangle_\Gamma}{\|\xi_h\|_{-1/2,\Gamma}} \;\leq\; \widehat{\beta}_{\mathrm{T}}^{-1}\|\theta_D\|_{1/2,\Gamma}. \tag{3.17}$$

Then, from the inf-sup condition (3.8), the first equation of (3.14) and the continuity of $A_{\mathrm{T}}$ and $O_{\mathrm{T}}^h$ in (2.11) and (3.2), respectively, we obtain

$$\begin{aligned}
\widehat{\alpha}_T\|\theta_{h,0}\|_{1,\Omega} &\leq \sup_{\substack{\phi_h\in\Psi_{h,0}\\ \phi_h\neq 0}} \frac{A_T(\varphi_h;\theta_{h,0},\phi_h) + O_{\mathrm{T}}^h(\mathbf{w}_h;\theta_{h,0},\phi_h)}{\|\phi_h\|_{1,\Omega}}\\
&= \sup_{\substack{\phi_h\in\Psi_{h,0}\\ \phi_h\neq 0}} \frac{-A_{\mathrm{T}}(\varphi_h;\theta_{h,1},\phi_h) - O_{\mathrm{T}}^h(\mathbf{w}_h;\theta_{h,1},\phi_h)}{\|\phi_h\|_{1,\Omega}}\\
&\leq (\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega})\|\theta_{h,1}\|_{1,\Omega},
\end{aligned}$$

which together with (3.17), implies

$$\|\theta_{h,0}\|_{1,\Omega} \leq \widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{\beta}_{\mathrm{T}}^{-1}\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega}\right)\|\theta_D\|_{1/2,\Gamma}.$$

Hence, owing the triangle inequality and (3.17), we easily obtain

$$\begin{aligned}
\|\theta_h\|_{1,\Omega} &\leq \|\theta_{h,0}\|_{1,\Omega} + \|\theta_{h,1}\|_{1,\Omega}\\
&\leq \|\theta_{h,0}\|_{1,\Omega} + \widehat{\beta}_{\mathrm{T}}^{-1}\|\theta_D\|_{1/2,\Gamma}\\
&\leq \widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{\beta}_{\mathrm{T}}^{-1}\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega} + \widehat{\alpha}_{\mathrm{T}}\right)\|\theta_D\|_{1/2,\Gamma}.
\end{aligned} \tag{3.18}$$

Finally, for (3.16) we utilize the inf-sup condition (3.7), the first equation in (3.14), and the continuity of $A_{\mathrm{T}}$ and $O_{\mathrm{T}}^h$ in (2.11) and (3.2), respectively, to obtain

$$\begin{aligned}
\widehat{\beta}_{\mathrm{T}}\|\lambda_h\|_{-1/2,\Gamma} &\leq \sup_{\substack{\psi_h\in\Psi_h\\ \psi_h\neq 0}} \frac{B_{\mathrm{T}}(\psi_h,\lambda_h)}{\|\psi_h\|_{1,\Omega}}\\
&= \sup_{\substack{\psi_h\in\Psi_h\\ \psi_h\neq 0}} \left\{\frac{-A_{\mathrm{T}}(\varphi_h;\theta_h,\psi_h) - O_{\mathrm{T}}^h(\mathbf{w}_h;\theta_h,\psi_h)}{\|\psi_h\|_{1,\Omega}}\right\}\\
&\leq (\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega})\|\theta_h\|_{1,\Omega},
\end{aligned}$$

which together with (3.18) concludes the proof. □

#### 3.2.2 Existence and stability of discrete solutions

In this section we establish the existence and stability of discrete solutions of problem (3.3). To do that, we proceed as in [24, Section 4.3] and first analyze the existence and stability of the reduced problem (3.11) by making use of the classical Brouwer's fixed point theorem in the following form (see e.g., [6]): *Let $\mathcal{K}$ be a non-empty compact convex subset of a finite dimensional normed space, and let $\mathcal{L}$ be a continuous mapping of $\mathcal{K}$ into itself. Then $\mathcal{L}$ has a fixed point in $\mathcal{K}$.*

In order to fit our analysis into the framework of the Brouwer's fixed point theorem, we first let $\widehat{C}_{\mathbf{u}}$, $\widehat{C}_\theta$ and $\widehat{C}_\lambda$ be positive constants (to be specified later), independent of $h$, and introduce the finite dimensional, convex and compact set:

$$\mathcal{K} := \left\{ \begin{array}{c} (\mathbf{v}_h, (\psi_h, \xi_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h) : \ \|\mathbf{v}_h\|_{1,\Omega} \le \widehat{C}_{\mathbf{u}} \|\theta_D\|_{1/2,\Gamma}, \\ \|\psi_h\|_{1,\Omega} \le \widehat{C}_\theta \|\theta_D\|_{1/2,\Gamma} \quad \text{and} \quad \|\xi_h\|_{-1/2,\Gamma} \le \widehat{C}_\lambda \|\theta_D\|_{1/2,\Gamma} \end{array} \right\},$$

In addition, we define the mapping

$$\mathcal{L} : \mathcal{K} \to \mathcal{K}, \quad (\mathbf{w}_h, (\varphi_h, \chi_h)) \mapsto \mathcal{L}(\mathbf{w}_h, (\varphi_h, \chi_h)) = (\mathbf{u}_h, (\theta_h, \lambda_h)), \tag{3.19}$$

where $\mathbf{u}_h \in \mathbf{X}_h$ and $(\theta_h, \lambda_h) \in \Psi_h \times \Lambda_h$ are the unique solutions of problems (3.12) and (3.14), respectively. It is clear that $(\mathbf{u}_h, (\theta_h, \lambda_h))$ is a solution to (3.11), if and only if,

$$\mathcal{L}(\mathbf{u}_h, (\theta_h, \lambda_h)) = (\mathbf{u}_h, (\theta_h, \lambda_h)),$$

that is, $(\mathbf{u}_h, (\theta_h, \lambda_h))$ is a fixed point to $\mathcal{L}$. In turn, it is not difficult to see that, by assuming a smallness assumption on the data, and for specific constants $\widehat{C}_{\mathbf{u}}$, $\widehat{C}_\theta$ and $\widehat{C}_\lambda$, the mapping $\mathcal{L}$ is well defined and maps $\mathcal{K}$ into itself. In fact, we have the following lemma.

**Lemma 3.4** *Let $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$ and $\mathbf{g} \in \mathbf{L}^2(\Omega)$. Assume that hypotheses* (**H.0**), (**H.2**) *and* (**H.3**) *hold. Then, given $(\mathbf{w}_h, (\varphi_h, \chi_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$, there exists a unique $(\mathbf{u}_h, (\theta_h, \lambda_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$, such that $\mathcal{L}(\mathbf{w}_h, (\varphi_h, \chi_h)) = (\mathbf{u}_h, (\theta_h, \lambda_h))$. In addition, under the further assumption on the data*

$$\widehat{C}_{stab} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \le \frac{1}{2}, \tag{3.20}$$

*with*

$$\widehat{C}_{stab} := \frac{\widehat{C}_{O_{\mathrm{T}}} C_{D_{\mathrm{F}}}}{\widehat{\alpha}_{\mathrm{F}} \widehat{\alpha}_{\mathrm{T}} \widehat{\beta}_{\mathrm{T}}}, \tag{3.21}$$

*and setting the constants $\widehat{C}_{\mathbf{u}}$, $\widehat{C}_\theta$ and $\widehat{C}_\lambda$ as*

$$\widehat{C}_{\mathbf{u}} = \frac{2 C_{D_{\mathrm{F}}} (\kappa_2 + \widehat{\alpha}_{\mathrm{T}}) \|\mathbf{g}\|_{0,\Omega}}{\widehat{\alpha}_{\mathrm{F}} \widehat{\alpha}_{\mathrm{T}} \widehat{\beta}_{\mathrm{T}}}, \quad \widehat{C}_\theta = \frac{2(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})}{\widehat{\alpha}_{\mathrm{T}} \widehat{\beta}_{\mathrm{T}}} \quad \text{and} \quad \widehat{C}_\lambda = \frac{2(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})(2\kappa_2 + \widehat{\alpha}_{\mathrm{T}})}{\widehat{\alpha}_{\mathrm{T}} \widehat{\beta}_{\mathrm{T}}^2}, \tag{3.22}$$

*the mapping $\mathcal{L}$ maps $\mathcal{K}$ into itself.*

*Proof.* Let $(\mathbf{w}_h, (\varphi_h, \chi_h)) \in \mathcal{K}$. The existence and uniqueness of $(\mathbf{u}_h, (\theta_h, \lambda_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$ such that $\mathcal{L}(\mathbf{w}_h, (\varphi_h, \chi_h)) = (\mathbf{u}_h, (\theta_h, \lambda_h))$ is guaranteed by Lemmata 3.2 and 3.3. Therefore, it remains to prove that, under hypothesis (3.20), $(\mathbf{u}_h, (\theta_h, \lambda_h))$ belongs to $\mathcal{K}$. To do that, we need to find suitable constants $\widehat{C}_{\mathbf{u}}$, $\widehat{C}_\theta$ and $\widehat{C}_\lambda$ for which the inequalities $\|\mathbf{u}_h\|_{1,\Omega} \le \widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma}$, $\|\theta_h\|_{1,\Omega} \le \widehat{C}_\theta\|\theta_D\|_{1/2,\Gamma}$ and $\|\lambda_h\|_{-1/2,\Gamma} \le \widehat{C}_\lambda\|\theta_D\|_{1/2,\Gamma}$ hold.

First, we recall that, since $(\mathbf{w}_h, \varphi_h, \chi_h) \in \mathcal{K}$, then

$$\|\mathbf{w}_h\|_{1,\Omega} \le \widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma}, \quad \|\varphi_h\|_{1,\Omega} \le \widehat{C}_\theta\|\theta_D\|_{1/2,\Gamma} \quad \text{and} \quad \|\chi_h\|_{-1/2,\Gamma} \le \widehat{C}_\lambda\|\theta_D\|_{1/2,\Gamma}. \tag{3.23}$$

Now, since $\mathbf{u}_h \in \mathbf{X}_h$ and $(\theta_h, \lambda_h) \in \Psi_h \times \Lambda_h$ are the unique solutions of problems (3.12) and (3.14), respectively, from (3.13) and (3.15) and (3.23), we obtain

$$\|\mathbf{u}_h\|_{1,\Omega} \le \widehat{\alpha}_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}} \widehat{C}_\theta \|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma},$$

and

$$\|\theta_h\|_{1,\Omega} \le \widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{\beta}_{\mathrm{T}}^{-1}\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma} + \widehat{\alpha}_{\mathrm{T}}\right)\|\theta_D\|_{1/2,\Gamma}. \tag{3.24}$$

Then, setting $\widehat{C}_{\mathbf{u}} = \alpha_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}}\widehat{C}_\theta\|\mathbf{g}\|_{0,\Omega}$, from (3.24) and (3.20), we obtain

$$\begin{aligned}\|\theta_h\|_{1,\Omega} &\le \frac{\widehat{C}_{O_{\mathrm{T}}}\widehat{C}_{D_{\mathrm{F}}}\widehat{C}_\theta}{\widehat{\alpha}_{\mathrm{F}}\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|^2_{1/2,\Gamma} + \frac{\kappa_2 + \widehat{\alpha}_{\mathrm{T}}}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}\|\theta_D\|_{1/2,\Gamma} \\ &= \left(\widehat{C}_\theta\widehat{C}_{stab}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} + \frac{\kappa_2 + \widehat{\alpha}_{\mathrm{T}}}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}\right)\|\theta_D\|_{1/2,\Gamma} \\ &\le \left(\frac{\widehat{C}_\theta}{2} + \frac{\kappa_2 + \widehat{\alpha}_{\mathrm{T}}}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}\right)\|\theta_D\|_{1/2,\Gamma}.\end{aligned} \tag{3.25}$$

In this way, we set $\widehat{C}_\theta = \frac{\widehat{C}_\theta}{2} + \frac{\kappa_2 + \widehat{\alpha}_{\mathrm{T}}}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}$, and find that

$$\widehat{C}_\theta = \frac{2(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}, \tag{3.26}$$

which according to the above, implies that

$$\widehat{C}_{\mathbf{u}} = \widehat{\alpha}_{\mathrm{F}}^{-1} C_{D_{\mathrm{F}}}\widehat{C}_\theta\|\mathbf{g}\|_{0,\Omega} = \frac{2C_{D_{\mathrm{F}}}(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})\|\mathbf{g}\|_{0,\Omega}}{\widehat{\alpha}_{\mathrm{F}}\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}. \tag{3.27}$$

Finally, from (3.20) and (3.27), we easily obtain that

$$\widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega} \le \widehat{C}_{O_{\mathrm{T}}}\widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma} = \frac{2\widehat{C}_{O_{\mathrm{T}}}C_{D_{\mathrm{F}}}(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})}{\widehat{\alpha}_{\mathrm{F}}\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}_{\mathrm{T}}}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} \le \kappa_2 + \widehat{\alpha}_{\mathrm{T}},$$

which together to (3.16), implies

$$\begin{aligned}\|\lambda_h\|_{-1/2,\Gamma} &\le \frac{(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega})}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}^2_{\mathrm{T}}}\left(\kappa_2 + \widehat{C}_{O_{\mathrm{T}}}\|\mathbf{w}_h\|_{1,\Omega} + \widehat{\alpha}_{\mathrm{T}}\right)\|\theta_D\|_{1/2,\Gamma} \\ &\le \widehat{C}_\lambda\|\theta_D\|_{1/2,\Gamma},\end{aligned}$$

with

$$\widehat{C}_\lambda = \frac{2(\kappa_2 + \widehat{\alpha}_{\mathrm{T}})(2\kappa_2 + \widehat{\alpha}_{\mathrm{T}})}{\widehat{\alpha}_{\mathrm{T}}\widehat{\beta}^2_{\mathrm{T}}}.$$

In this way, if $(\mathbf{w}_h,(\varphi_h,\chi_h))\in\mathcal{K}$, with $\mathcal{K}$ defined with the constants in (3.22), then $(\mathbf{u}_h,(\theta_h,\lambda_h))\in\mathcal{K}$, which concludes the proof. $\square$

We are now in position of establishing the existence and stability of solutions of problem (3.11).

**Theorem 3.5** *Let $\theta_D\in \mathrm{H}^{1/2}(\Gamma)$ and $\mathbf{g}\in\mathbf{L}^2(\Omega)$. Assume that hypotheses (**H.0**), (**H.2**) and (**H.3**) hold. Assume further that*

$$\widehat{C}_{stab}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma}\ \le\frac{1}{2},$$

*with $\widehat{C}_{stab}$ being the constant in (3.21). Then, problem (3.11) admits a solution $(\mathbf{u}_h,(\theta_h,\lambda_h))\in\mathbf{X}_h\times(\Psi_h\times\Lambda_h)$, satisfying*

$$\|\mathbf{u}_h\|_{1,\Omega}\le\widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma},\quad \|\theta_h\|_{1,\Omega}\le\widehat{C}_{\theta}\|\theta_D\|_{1/2,\Gamma}\quad\textit{and}\quad \|\lambda_h\|_{-1/2,\Gamma}\le\widehat{C}_{\lambda}\|\theta_D\|_{1/2,\Gamma},\qquad(3.28)$$

*with $\widehat{C}_{\mathbf{u}}$, $\widehat{C}_{\theta}$ and $\widehat{C}_{\lambda}$ being the constants in (3.22).*

*Proof.* As mentioned before, according to the definition of $\mathcal{L}$ and the Brouwer's fixed point theorem, it suffices to prove that $\mathcal{L}$ is continuous operator. To do that, we let $(\mathbf{w},(\varphi,\chi))\in\mathcal{K}$ and a sequence $\{(\mathbf{w}_m,(\varphi_m,\chi_m))\}_{m\in\mathbb{N}}\subset\mathcal{K}$, satisfying

$$\|\mathbf{w}_m-\mathbf{w}\|_{1,\Omega}\overset{m\to\infty}{\longrightarrow}0,\quad \|\varphi_m-\varphi\|_{1,\Omega}\overset{m\to\infty}{\longrightarrow}0,\quad\text{and}\quad \|\chi_m-\chi\|_{-1/2,\Gamma}\overset{m\to\infty}{\longrightarrow}0.\qquad(3.29)$$

Then, setting $(\mathbf{u},(\theta,\lambda))=\mathcal{L}(\mathbf{w},(\varphi,\chi))$ and $(\mathbf{u}_m,(\theta_m,\lambda_m))=\mathcal{L}(\mathbf{w}_m,(\varphi_m,\chi_m))$, we proceed similarly to the proof of [24, Theorem 4.1] to prove that

$$\|\mathbf{u}_m-\mathbf{u}\|_{1,\Omega}\overset{m\to\infty}{\longrightarrow}0,\qquad \|\theta_m-\theta\|_{1,\Omega}\overset{m\to\infty}{\longrightarrow}0,\quad\text{and}\quad \|\lambda_m-\lambda\|_{-1/2,\Gamma}\overset{m\to\infty}{\longrightarrow}0.\qquad(3.30)$$

In fact, from the definition of $\mathcal{L}$ in (3.19), and from (3.12) and (3.14), we first observe that there hold

$$\begin{aligned} A_{\mathrm{F}}(\varphi_m;\mathbf{u}_m,\mathbf{v})+O^h_{\mathrm{F}}(\mathbf{w}_m;\mathbf{u}_m,\mathbf{v})\ &=\ D_{\mathrm{F}}(\varphi_m,\mathbf{v})\\ A_{\mathrm{T}}(\varphi_m;\theta_m,\psi)+O^h_{\mathrm{T}}(\mathbf{w}_m;\theta_m,\psi)+B_{\mathrm{T}}(\psi,\lambda_m)\ &=\ 0\\ B_{\mathrm{T}}(\theta_m,\xi)\ &=\ \langle\xi,\theta_D\rangle_\Gamma, \end{aligned}$$

for all $(\mathbf{v},(\psi,\xi))\in\mathbf{X}_h\times(\Psi_h\times\Lambda_h)$, and

$$\begin{aligned} A_{\mathrm{F}}(\varphi;\mathbf{u},\mathbf{v})+O^h_{F}(\mathbf{w};\mathbf{u},\mathbf{v})\ &=\ D_{\mathrm{F}}(\varphi,\mathbf{v})\\ A_{\mathrm{T}}(\varphi;\theta,\psi)+O^h_{\mathrm{T}}(\mathbf{w};\theta,\psi)+B_{\mathrm{T}}(\psi,\lambda)\ &=\ 0\\ B_{\mathrm{T}}(\theta,\xi)\ &=\ \langle\xi,\theta_D\rangle_\Gamma, \end{aligned}$$

for all $(\mathbf{v},(\psi,\xi))\in\mathbf{X}_h\times(\Psi_h\times\Lambda_h)$. Then, subtracting the two systems from each other we easily obtain

$$[A_{\mathrm{F}}(\varphi_m;\mathbf{u}_m,\mathbf{v})-A_{\mathrm{F}}(\varphi;\mathbf{u},\mathbf{v})]+[O^h_{\mathrm{F}}(\mathbf{w}_m;\mathbf{u}_m,\mathbf{v})-O^h_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v})]-D_{\mathrm{F}}(\varphi_m-\varphi,\mathbf{v})=0\qquad(3.31)$$

$$[A_{\mathrm{T}}(\varphi_m;\theta_m,\psi)-A_{\mathrm{T}}(\varphi;\theta,\psi)]+[O^h_{\mathrm{T}}(\mathbf{w}_m;\theta_m,\psi)-O^h_{\mathrm{T}}(\mathbf{w};\theta,\psi)]+B_{\mathrm{T}}(\psi,\lambda-\lambda_m)=0\qquad(3.32)$$

$$B_{\mathrm{T}}(\theta-\theta_m,\xi)=0,\qquad(3.33)$$

for all $(\mathbf{v},(\psi,\xi)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$. In particular, from (3.31), we add and subtract suitable terms, to obtain

$$
\begin{aligned}
A_{\rm F}(\varphi_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v}) + O^h_{\rm F}(\mathbf{w}_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v}) = \quad & -[A_{\rm F}(\varphi;\mathbf{u},\mathbf{v}) - A_{\rm F}(\varphi_m;\mathbf{u},\mathbf{v})] \\
& -O^h_{\rm F}(\mathbf{w}-\mathbf{w}_m;\mathbf{u},\mathbf{v})) - D_{\rm F}(\varphi_m-\varphi,\mathbf{v}).
\end{aligned}
$$

Then, owing to (**H.0**) (cf. (3.4)), (2.12), (2.15) and (3.1), from the previous identity, it follows that

$$
\begin{aligned}
\widehat{\alpha}_{\rm F}\|\mathbf{u}-\mathbf{u}_m\|_{1,\Omega} &\le \sup_{\substack{\mathbf{v}\in\mathbf{X}_h\\ \mathbf{v}\neq 0}} \frac{A_{\rm F}(\varphi_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v}) + O^h_{\rm F}(\mathbf{w}_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}} \\
&= \sup_{\substack{\mathbf{v}\in\mathbf{X}_h\\ \mathbf{v}\neq 0}} \frac{-[A_{\rm F}(\varphi;\mathbf{u},\mathbf{v}) - A_{\rm F}(\varphi_m;\mathbf{u},\mathbf{v})] - O^h_{\rm F}(\mathbf{w}-\mathbf{w}_m;\mathbf{u},\mathbf{v})) - D_{\rm F}(\varphi_m-\varphi,\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}} \\
&\le \nu_{\rm lip}\|\varphi-\varphi_m\|_{1,\Omega}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} + \widehat{C}_{O_{\rm F}}\|\mathbf{w}-\mathbf{w}_m\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} + C_{D_{\rm F}}\|\mathbf{g}\|_{0,\Omega}\|\varphi_m-\varphi\|_{1,\Omega},
\end{aligned}
$$

which together with (3.29), yields

$$
\lim_{m\to\infty}\|\mathbf{u}-\mathbf{u}_m\|_{1,\Omega} = 0. \tag{3.34}
$$

Similarly to the above, we now add and subtract suitable terms in (3.32), to obtain

$$
\begin{aligned}
A_{\rm T}(\varphi_m;\theta-\theta_m,\psi) + O^h_{\rm T}(\mathbf{w}_m;\theta-\theta_m,\psi) \quad = \quad & -[A_{\rm T}(\varphi;\theta,\psi) - A_{\rm T}(\varphi_m;\theta,\psi)] \\
& -O^h_{\rm T}(\mathbf{w}-\mathbf{w}_m;\theta,\psi) + B_{\rm T}(\psi,\lambda-\lambda_m),
\end{aligned} \tag{3.35}
$$

for all $\psi \in \Psi_h$. Hence, noting that $\theta-\theta_m \in \Psi_{h,0}$ (cf. (3.9)), we apply the inf-sup condition (3.8) to the left-hand side of (3.35) and utilize the estimates (2.13) and (3.2), to get

$$
\begin{aligned}
\widehat{\alpha}_{\rm T}\|\theta-\theta_m\|_{1,\Omega} \quad &\le \sup_{\substack{\psi\in\Psi_{h,0}\\ \psi\neq 0}} \frac{A_T(\varphi_m;\theta-\theta_m,\psi) + O^h_{\rm T}(\mathbf{w}_m;\theta-\theta_m,\psi)}{\|\psi\|_{1,\Omega}} \\
&= \sup_{\substack{\psi\in\Psi_{h,0}\\ \psi\neq 0}} \frac{-[A_{\rm T}(\varphi;\theta,\psi) - A_{\rm T}(\varphi_m;\theta,\psi)] - O^h_{\rm T}(\mathbf{w}-\mathbf{w}_m;\theta,\psi)}{\|\psi\|_{1,\Omega}} \\
&\le \kappa_{\rm lip}\|\varphi-\varphi_m\|_{1,\Omega}\|\theta\|_{{\rm W}^{1,\infty}(\Omega)} + \widehat{C}_{O_{\rm T}}\|\mathbf{w}-\mathbf{w}_m\|_{1,\Omega}\|\theta\|_{1,\Omega},
\end{aligned}
$$

which together with (3.29), yields

$$
\lim_{m\to\infty}\|\theta-\theta_m\|_{1,\Omega} = 0. \tag{3.36}
$$

Finally, from the inf-sup condition of $B_{\rm T}$ (cf. (3.7)), the identity (3.35), and the estimates (2.11), (2.13) and (3.2), it follows that

$$
\begin{aligned}
\widehat{\beta}_{\rm T}\|\lambda-\lambda_m\|_{-1/2,\Gamma} \le \quad & \sup_{\substack{\psi\in\Psi_h\\ \psi\neq 0}} \frac{B_{\rm T}(\psi,\lambda-\lambda_m)}{\|\psi\|_{1,\Omega}}, \\
\le \quad & \kappa_2\|\theta-\theta_m\|_{1,\Omega} + \widehat{C}_{O_{\rm T}}\|\mathbf{w}_m\|_{1,\Omega}\|\theta-\theta_m\|_{1,\Omega} \\
& + \kappa_{\rm lip}\|\theta\|_{{\rm W}^{1,\infty}(\Omega)}\|\varphi-\varphi_m\|_{1,\Omega} + \widehat{C}_{O_{\rm T}}\|\mathbf{w}-\mathbf{w}_m\|_{1,\Omega}\|\theta\|_{1,\Omega}
\end{aligned}
$$

which together with (3.34), (3.36), and the fact that $\{\mathbf{w}_m\}_{m\in\mathbb{N}}$ and $\{\varphi_m\}_{m\in\mathbb{N}}$ are convergent sequences, leads us to

$$\lim_{m\to\infty} \|\lambda - \lambda_m\|_{-1/2,\Gamma} = 0, \tag{3.37}$$

which concludes the proof. □

We end this section with the main result of this section, namely, existence and stability of solutions of problem (3.3).

**Theorem 3.6** *Let $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$ and $\mathbf{g} \in \mathbf{L}^2(\Omega)$. Assume that hypotheses* (**H.0**)*,* (**H.1**)*,* (**H.2**) *and* (**H.3**) *hold. Assume further that*

$$\widehat{C}_{stab}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} \;\leq \frac{1}{2},$$

*with $\widehat{C}_{stab}$ being the constant in* (3.21)*. Then, problem* (3.3) *admits a solution $((\mathbf{u}_h, p_h), (\theta_h, \lambda_h)) \in (\mathbf{H}_h \times \mathrm{Q}_h) \times (\Psi_h \times \Lambda_h)$, with $\mathbf{u}_h$, $\theta_h$ and $\lambda_h$ satisfying the stability estimates* (3.28)*. In addition, there exists a positive constant $\widehat{C}_p$, depending only on $\|\mathbf{g}\|_{0,\Omega}$ and the stability constants, such that*

$$\|p_h\|_{0,\Omega} \;\leq\; \widehat{C}_p\|\theta_D\|_{1/2,\Gamma}. \tag{3.38}$$

*Proof.* It is clear that the existence of solutions of problem (3.3) follows from Lemma 3.1 and Theorem 3.5. In addition, owing to Theorem 3.5 the estimates (3.28) hold. Then, it suffices to prove the estimate (3.38) to conclude the proof. To do that, we first observe that from the inf-sup condition (3.5), and the first equation of (3.3), we have

$$\begin{aligned} \widehat{\beta}_{\mathrm{F}}\|p_h\|_{0,\Omega} \;&\leq\; \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{B_{\mathrm{F}}(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \\ &=\; \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \left\{\frac{A_{\mathrm{F}}(\theta_h;\mathbf{u}_h,\mathbf{v}_h) + O^h_{\mathrm{F}}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) - D_{\mathrm{F}}(\theta_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}}\right\}, \end{aligned}$$

which together with the continuity of $A_F$, $O^h_{\mathrm{F}}$ and $D_{\mathrm{F}}$ in (2.11), (3.1) and (2.15), respectively, implies

$$\|p_h\|_{0,\Omega} \;\;\leq\;\; \widehat{\beta}_{\mathrm{F}}^{-1}(\nu_2 + \widehat{C}_{O_{\mathrm{F}}}\|\mathbf{u}_h\|_{1,\Omega})\|\mathbf{u}_h\|_{1,\Omega} + \widehat{\beta}_{\mathrm{F}}^{-1}C_{D_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\theta_h\|_{1,\Omega}. \tag{3.39}$$

Hence, using the estimates (3.28), from the previous inequality, we obtain

$$\|p_h\|_{0,\Omega} \leq \widehat{\beta}_{\mathrm{F}}^{-1}\left(\widehat{C}_{\mathbf{u}}(\nu_2 + \widehat{C}_{O_{\mathrm{F}}}\widehat{C}_{\mathbf{u}}\|\theta_D\|_{1/2,\Gamma}) + C_{D_{\mathrm{F}}}\widehat{C}_{\theta}\|\mathbf{g}\|_{0,\Omega}\right)\|\theta_D\|_{1/2,\Gamma}.$$

Hence, assumption (3.20) and the previous inequality yield the result. □

## 3.3 Céa's estimate

Now, we derive the corresponding Céa's estimate of our Galerkin scheme (3.3). To that end, we first provide some previous results that will serve for the forthcoming analysis. We begin by defining the set

$$\Psi_{h,D} := \{\varphi_h \in \Psi_h \,:\, B_{\mathrm{T}}(\phi_h, \xi_h) = \langle\xi_h, \theta_D\rangle_\Gamma \quad \forall \xi \in \Lambda_h\}\,, \tag{3.40}$$

which, according to hypothesis (**H.3**) is a non-empty space. In addition, it is not difficult to see that the following inequality holds (see for instance [16, Theorem 2.6]):

$$\inf_{\zeta_h \in \Psi_{h,D}} \|\theta - \zeta_h\|_{1,\Omega} \leq \hat{c}_1 \inf_{\phi_h \in \Psi_h} \|\theta - \phi_h\|_{1,\Omega}, \tag{3.41}$$

with $\hat{c}_1 > 0$ independent of $h$. Similarly, owing to hypothesis (**H.1**), it is possible to prove that there exists $\hat{c}_2 > 0$ independent of $h$, such that

$$\inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} \ \leq \ \hat{c}_2 \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega}. \tag{3.42}$$

In order to simplify the subsequent analysis, we write $\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h$, $\mathrm{e}_p = p - p_h$, $\mathrm{e}_\theta = \theta - \theta_h$ and $\mathrm{e}_\lambda = \lambda - \lambda_h$. As usual, for given $\widehat{\mathbf{v}}_h \in \mathbf{X}_h$, $\widehat{q}_h \in \mathrm{Q}_h$, $\widehat{\psi}_h \in \Psi_{h,D}$ and $\widehat{\xi}_h \in \Lambda_h$, we shall decompose these errors into

$$\mathbf{e}_{\mathbf{u}} = \mathbf{r}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, \quad \mathrm{e}_p = \mathrm{r}_p + \chi_p, \quad \mathrm{e}_\theta = \mathrm{r}_\theta + \chi_\theta \quad \text{and} \quad \mathrm{e}_\lambda = \mathrm{r}_\lambda + \chi_\lambda,$$

with

$$\begin{aligned} &\mathbf{r}_{\mathbf{u}} := \mathbf{u} - \widehat{\mathbf{v}}_h \in \mathbf{X}, && \boldsymbol{\chi}_{\mathbf{u}} := \widehat{\mathbf{v}}_h - \mathbf{u}_h \in \mathbf{X}_h, \\ &\mathrm{r}_p := p - \widehat{q}_h \in \mathrm{Q}, && \chi_p := \widehat{q}_h - p_h \in \mathrm{Q}_h, \\ &\mathrm{r}_\theta := \theta - \widehat{\psi}_h \in \mathrm{H}^1(\Omega), && \chi_\theta := \widehat{\psi}_h - \theta_h \in \Psi_{h,0}, \\ &\mathrm{r}_\lambda := \lambda - \widehat{\xi}_h \in \mathrm{H}^{-1/2}(\Gamma), && \chi_\lambda := \widehat{\xi}_h - \lambda_h \in \Lambda_h. \end{aligned} \tag{3.43}$$

We first provide the estimates for $\mathbf{u}$, $\theta$ and $\lambda$.

**Theorem 3.7** *Assume that* (**H.0**), (**H.2**), (**H.3**) *and* (**H.4**) *hold. Assume further that*

$$\max\{C_{stab}, \widehat{C}_{stab}\} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \ \leq \ \frac{1}{2}, \tag{3.44}$$

*with $C_{stab}$ and $\widehat{C}_{stab}$ given by* (2.25) *and* (3.21), *respectively. Let $((\mathbf{u}, p), (\theta, \lambda)) \in (\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)) \times (\mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma))$ and $(\mathbf{u}_h, (\theta_h, \lambda_h)) \in \mathbf{X}_h \times (\Psi_h \times \Lambda_h)$ be solutions of the continuous and discrete problems* (2.10) *and* (3.11), *respectively, and assume that*

$$\max\{\|\mathbf{g}\|_{0,\Omega}, \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}}, \|\theta\|_{\mathrm{W}^{1,\infty}(\Omega)}\} \ \leq \ \widehat{M} \ := \ \min\Big\{M, \ \widetilde{M}\Big\}, \tag{3.45}$$

*with $M$ and $\widetilde{M}$ sufficiently small as specified in* (2.23) *and* (3.52) *below, respectively. Then, there exist positive constants $\mathcal{C}_1^{cea}$ and $\mathcal{C}_2^{cea}$ independent of the discretization parameter $h$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\theta - \theta_h\|_{1,\Omega} \leq \mathcal{C}_1^{cea} \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \inf_{\psi \in \Psi_h} \|\theta - \psi_h\|_{1,\Omega} \right\}, \tag{3.46}$$

$$\|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq \mathcal{C}_2^{cea} \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \inf_{\psi \in \Psi_h} \|\theta - \psi_h\|_{1,\Omega} + \inf_{\xi_h \in \Lambda_h} \|\lambda - \xi_h\|_{-1/2,\Gamma} \right\}. \tag{3.47}$$

*Proof.* Let $\widehat{\mathbf{v}}_h \in \mathbf{X}_h$, $\widehat{\psi}_h \in \Psi_{h,D}$ and $\widehat{\xi}_h \in \Lambda_h$, and define $\mathbf{r_u}$, $\boldsymbol{\chi}_\mathbf{u}$, $\mathrm{r}_\theta, \chi_\theta$, $\mathrm{r}_\lambda$ and $\chi_\lambda$ as in (3.43).

First, since $((\mathbf{u},p),(\theta,\lambda)) \in (\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega))\times(\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma))$ is the solution of (2.10), for a given $(\mathbf{v}_h,\psi_h,\xi_h)\in\mathbf{X}_h\times\Psi_h\times\Lambda_h$, it readily follows that

$$\begin{aligned} A_\mathrm{F}(\theta;\mathbf{u},\mathbf{v}_h)+O_\mathrm{F}(\mathbf{u};\mathbf{u},\mathbf{v}_h)-D_\mathrm{F}(\theta,\mathbf{v}_h) &= 0,\\ A_\mathrm{T}(\theta;\theta,\psi_h)+O_\mathrm{T}(\mathbf{u};\theta,\psi_h)+B_\mathrm{T}(\psi_h,\lambda) &= 0,\\ B_\mathrm{T}(\theta,\xi_h) &= \langle\xi_h,\theta_D\rangle_\Gamma, \end{aligned} \tag{3.48}$$

Then, we subtract equations (3.48) and (3.11) and utilize hypothesis (**H.4**) (cf. (3.10)), to obtain

$$\begin{aligned} [A_\mathrm{F}(\theta;\mathbf{u},\mathbf{v}_h)-A_\mathrm{F}(\theta_h;\mathbf{u}_h,\mathbf{v}_h)]+[O_\mathrm{F}^h(\mathbf{u};\mathbf{u},\mathbf{v}_h)-O_\mathrm{F}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)]-D_\mathrm{F}(\mathrm{e}_\theta,\mathbf{v}_h) &= 0,\\ [A_\mathrm{T}(\theta;\theta,\psi_h)-A_\mathrm{T}(\theta_h;\theta_h,\psi_h)]+[O_\mathrm{T}^h(\mathbf{u};\theta,\psi_h)-O_\mathrm{T}^h(\mathbf{u}_h;\theta_h,\psi_h)]+B_\mathrm{T}(\psi_h,\mathrm{e}_\lambda) &= 0,\\ B_\mathrm{T}(\mathrm{e}_\theta,\xi_h) &= 0, \end{aligned} \tag{3.49}$$

for all $(\mathbf{v}_h,\psi_h,\xi_h)\in\mathbf{X}_h\times\Psi_h\times\Lambda_h$. In particular, using the first equation of (3.49), we add and subtract suitable terms, to obtain

$$\begin{aligned} A_\mathrm{F}(\theta_h;\boldsymbol{\chi}_\mathbf{u},\mathbf{v}_h)+O_\mathrm{F}^h(\mathbf{u}_h;\boldsymbol{\chi}_\mathbf{u},\mathbf{v}_h) = \;& -[A_\mathrm{F}(\theta_h;\mathbf{r_u},\mathbf{v}_h)+O_\mathrm{F}^h(\mathbf{u}_h;\mathbf{r_u},\mathbf{v}_h)]\\ & -[A_\mathrm{F}(\theta;\mathbf{u},\mathbf{v}_h)-A_\mathrm{F}(\theta_h;\mathbf{u},\mathbf{v}_h)]\\ & -O_\mathrm{F}^h(\mathbf{e_u};\mathbf{u},\mathbf{v}_h))+D_\mathrm{F}(\mathrm{e}_\theta,\mathbf{v}_h), \end{aligned}$$

for all $\mathbf{v}_h\in\mathbf{X}_h$, which together with hypothesis (**H.0**), the fact that $\boldsymbol{\chi}_\mathbf{u}\in\mathbf{X}_h$, and the estimates (2.11), (2.12), (2.15), (3.1), implies

$$\begin{aligned} \widehat{\alpha}_\mathrm{F}\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\Omega} \leq\; & \sup_{\substack{\mathbf{v}_h\in\mathbf{X}_h\\ \mathbf{v}_h\neq0}} \frac{A_\mathrm{F}(\theta_h;\boldsymbol{\chi}_\mathbf{u},\mathbf{v}_h)+O_\mathrm{T}^h(\mathbf{u}_h;\boldsymbol{\chi}_\mathbf{u},\mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}}\\ \leq\; & \nu_2\|\mathbf{r_u}\|_{1,\Omega}+\widehat{C}_{O_\mathrm{F}}\|\mathbf{u}_h\|_{1,\Omega}\|\mathbf{r_u}\|_{1,\Omega}+\nu_\mathrm{lip}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)}\|\mathrm{e}_\theta\|_{1,\Omega}\\ & +\widehat{C}_{O_\mathrm{F}}\|\mathbf{e_u}\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}+C_{D_\mathrm{F}}\|\mathbf{g}\|_{0,\Omega}\|\mathrm{e}_\theta\|_{1,\Omega}. \end{aligned} \tag{3.50}$$

Similarly, adding and subtracting suitable terms in the second equation of (3.49), we get

$$\begin{aligned} A_\mathrm{T}(\theta_h;\chi_\theta,\psi_h)+O_\mathrm{T}^h(\mathbf{u}_h;\chi_\theta,\psi_h) = \;& -[A_\mathrm{T}(\theta_h;\mathrm{r}_\theta,\psi_h)+O_\mathrm{T}^h(\mathbf{u}_h;\mathrm{r}_\theta,\psi_h)]\\ & -[A_\mathrm{T}(\theta;\theta,\psi_h)-A_\mathrm{T}(\theta_h;\theta,\psi_h)]\\ & -O_\mathrm{T}^h(\mathbf{e_u};\theta,\psi_h)-B_\mathrm{T}(\psi_h,\mathrm{e}_\lambda), \end{aligned}$$

which together with hypothesis (**H.3**) (cf. (3.8)), the estimates (2.11), (2.13) and (3.2), and the fact that $\chi_\theta\in\Psi_{h,0}$, yields

$$\begin{aligned} \widehat{\alpha}_\mathrm{T}\|\chi_\theta\|_{1,\Omega} \leq\; & \sup_{\substack{\psi_h\in\Psi_{h,0}\\ \mathbf{v}_h\neq0}} \frac{A_\mathrm{T}(\theta_h;\chi_\theta,\psi_h)+O_\mathrm{T}^h(\mathbf{u}_h;\chi_\theta,\psi_h)}{\|\psi_h\|_{1,\Omega}}\\ \leq\; & \kappa_2\|\mathrm{r}_\theta\|_{1,\Omega}+\widehat{C}_{O_\mathrm{T}}\|\mathbf{u}_h\|_{1,\Omega}\|\mathrm{r}_\theta\|_{1,\Omega}+\kappa_\mathrm{lip}\|\theta\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\mathrm{e}_\theta\|_{1,\Omega}+\widehat{C}_{O_\mathrm{T}}\|\mathbf{e_u}\|_{1,\Omega}\|\theta\|_{1,\Omega}. \end{aligned} \tag{3.51}$$

Then, using the continuous dependence result $\|\mathbf{u}_h\|_{1,\Omega} \le \widehat{C}_{\mathbf{u}} \|\theta_D\|_{1/2,\Gamma}$, the triangle inequality for $\|\mathbf{e_u}\|_{1,\Omega}$ and $\|\mathrm{e}_\theta\|_{1,\Omega}$, hypothesis (3.45), and the fact that $\|\mathbf{u}\|_{1,\Omega} \le C_\infty \|\mathbf{u}\|_{\mathbf{W}^1(\Omega)}$ and $\|\theta\|_{1,\Omega} \le C_\infty \|\theta\|_{W^1(\Omega)}$, from (3.50) and (3.51), we obtain

$$\begin{aligned} \widehat{\alpha}_{\mathrm{F}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\Omega} + \widehat{\alpha}_{\mathrm{T}} \|\chi_\theta\|_{1,\Omega} \le \quad & C(\|\mathbf{r_u}\|_{1,\Omega} \, + \, \|\mathrm{r}_\theta\|_{1,\Omega}) + C_\infty (\widehat{C}_{O_\mathrm{F}} + \widehat{C}_{O_\mathrm{T}}) \widetilde{M} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\Omega} \\ & + (\nu_{\mathrm{lip}} + \kappa_{\mathrm{lip}} + C_{D_\mathrm{F}}) \widetilde{M} \|\chi_\theta\|_{1,\Omega}, \end{aligned}$$

with $C > 0$, depending on $C_\infty$, $\|\theta_D\|_{1/2,\Gamma}$, and on the stability constants. Then, choosing $\widetilde{M}$, such that

$$\widetilde{M} \le \frac{1}{2} \min \left\{ \frac{\widehat{\alpha}_{\mathrm{F}}}{C_\infty (\widehat{C}_{O_\mathrm{F}} + \widehat{C}_{O_\mathrm{T}})}, \frac{\widehat{\alpha}_{\mathrm{T}}}{\nu_{\mathrm{lip}} + \kappa_{\mathrm{lip}} + C_{D_\mathrm{F}}} \right\}, \tag{3.52}$$

it follows that

$$\frac{\widehat{\alpha}_{\mathrm{F}}}{2} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\Omega} + \frac{\widehat{\alpha}_{\mathrm{T}}}{2} \|\chi_\theta\|_{1,\Omega} \le C(\|\mathbf{r_u}\|_{1,\Omega} \, + \, \|\mathrm{r}_\theta\|_{1,\Omega})$$

which together with the triangle inequality, yields

$$\begin{aligned} \|\mathbf{e_u}\|_{1,\Omega} + \|\mathrm{e}_\theta\|_{1,\Omega} \le \quad & \|\mathbf{r_u}\|_{1,\Omega} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\Omega} + \|\mathrm{r}_\theta\|_{1,\Omega} + \|\chi_\theta\|_{1,\Omega} \\ \le \quad & \tilde{C} \left\{ \|\mathbf{r_u}\|_{1,\Omega} + \|\mathrm{r}_\theta\|_{1,\Omega} \right\} \\ = \quad & \tilde{C} \left\{ \|\mathbf{u} - \widehat{\mathbf{v}}_h\|_{1,\Omega} + \|\theta - \widehat{\psi}_h\|_{1,\Omega} \right\}, \end{aligned}$$

for all $\widehat{\mathbf{v}}_h \in \mathbf{X}_h$ and $\widehat{\psi}_h \in \Psi_{h,D}$, which implies

$$\|\mathbf{e_u}\|_{1,\Omega} + \|\mathrm{e}_\theta\|_{1,\Omega} \le \tilde{C} \left\{ \inf_{\widehat{\mathbf{v}}_h \in \mathbf{X}_h} \|\mathbf{u} - \widehat{\mathbf{v}}_h\|_{1,\Omega} + \inf_{\widehat{\psi}_h \in \Psi_{h,D}} \|\theta - \widehat{\psi}_h\|_{1,\Omega} \right\}, \tag{3.53}$$

with $\tilde{C} > 0$ depending on $\|\theta_D\|_{1/2,\Gamma}$ and the stability constants. In this way, the estimate (3.46) follows from (3.53) and the properties (3.41) and (3.42).

Now, we prove estimate (3.47). To do that we first observe that, adding and subtracting suitable terms in the second equation of (3.49), there holds

$$\begin{aligned} B_{\mathrm{T}}(\psi_h, \chi_\lambda) = \quad & -[A_{\mathrm{T}}(\theta_h; \mathrm{e}_\theta, \psi_h) + O_{\mathrm{T}}^h(\mathbf{u}_h; \mathrm{e}_\theta, \psi_h)] - [A_{\mathrm{T}}(\theta; \theta, \psi_h) - A_{\mathrm{T}}(\theta_h; \theta, \psi_h)] \\ & - O_{\mathrm{T}}^h(\mathbf{e_u}; \theta, \psi_h) - B_{\mathrm{T}}(\psi_h, \mathrm{r}_\lambda), \end{aligned}$$

which together with (**H.2**) (cf. (3.7)), and the estimates (2.11), (2.13), (2.16) and (3.2), implies

$$\begin{aligned} \|\chi_\lambda\|_{-1/2,\Gamma} \le \quad & \widehat{\beta}_{\mathrm{T}}^{-1} \sup_{\substack{\psi_h \in \Psi_h \\ \psi_h \neq 0}} \left\{ \frac{|A_{\mathrm{T}}(\theta_h; e_\theta, \psi_h) + O_{\mathrm{T}}^h(\mathbf{u}_h; e_\theta, \psi_h) + O_{\mathrm{T}}^h(\mathbf{e_u}; \theta, \psi_h)|}{\|\psi_h\|_{1,\Omega}} \right\} \\ & + \widehat{\beta}_{\mathrm{T}}^{-1} \sup_{\substack{\psi_h \in \Psi_h \\ \psi_h \neq 0}} \left\{ \frac{|A_{\mathrm{T}}(\theta; \theta, \psi_h) - A_{\mathrm{T}}(\theta_h; \theta, \psi_h) + B_{\mathrm{T}}(\psi_h, \mathrm{r}_\lambda)|}{\|\psi_h\|_{1,\Omega}} \right\} \\ \le \quad & \widehat{\beta}_{\mathrm{T}}^{-1} \big( \kappa_2 \|\mathrm{e}_\theta\|_{1,\Omega} + \widehat{C}_{O_\mathrm{T}} \|\mathbf{u}_h\|_{1,\Omega} \|\mathrm{e}_\theta\|_{1,\Omega} + \widehat{C}_{O_\mathrm{T}} \|\mathbf{e_u}\|_{1,\Omega} \|\theta\|_{1,\Omega} \\ & + \kappa_2 \|\theta\|_{\mathrm{W}^{1,\infty}(\Omega)} \|\mathrm{e}_\theta\|_{1,\Omega} + \|\mathrm{r}_\lambda\|_{-1/2,\Gamma} \big). \end{aligned}$$

Hence, recalling that $\|\theta\|_{1,\Omega} \leq C_\infty \|\theta\|_{W^{1,\infty}(\Omega)}$ and $\|\mathbf{u}_h\|_{1,\Omega} \leq \widehat{C}_{\mathbf{u}} \|\theta_D\|_{1/2,\Gamma}$, from (3.44), we obtain that there exists $c > 0$, independent of $h$, such that

$$\|\chi_\lambda\|_{-1/2,\Gamma} \leq c\,(\|\mathrm{e}_\theta\|_{1,\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega}) + \widehat{\beta}_{\mathrm{T}}^{-1} \|\mathrm{r}_\lambda\|_{-1/2,\Gamma} \tag{3.54}$$

Therefore, the triangle inequality and the previous estimate imply that

$$\begin{aligned} \|\mathrm{e}_\lambda\|_{-1/2,\Gamma} &\leq \|r_\lambda\|_{-1/2,\Gamma} + \|\chi_\lambda\|_{-1/2,\Gamma} \\ &\leq c\,(\|\mathrm{e}_\theta\|_{1,\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega}) + \left(1 + \widehat{\beta}_{\mathrm{T}}^{-1}\right) \|r_\lambda\|_{-1/2,\Gamma} \end{aligned}$$

which together with (3.46) yields (3.47) and completes the proof. □

We end this section by providing the corresponding estimate for the pressure.

**Corollary 3.8** *Assume that hypotheses of Theorem 3.7 hold. Assume further that* (**H.1**) *holds. Let $p \in L^2_0(\Omega)$ and $p_h \in \mathrm{Q}_h$, such that $((\mathbf{u}, p), (\theta, \lambda))$ and $((\mathbf{u}_h, p_h), (\theta_h, \lambda_h))$ are solutions to* (2.10) *and* (3.3)*, respectively. Then, there exits a positive constant $\mathcal{C}_3^{cea}$, independent the parameter discretization $h$, such that*

$$\|p - p_h\|_{0,\Omega} \leq \mathcal{C}_3^{cea} \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \inf_{\psi \in \Psi_h} \|\theta - \psi_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \right\}. \tag{3.55}$$

*Proof.* Let $\widehat{p}_h \in Q_h$ and define $\mathrm{r}_p \in L^2_0(\Omega)$ and $\chi_p \in \mathrm{Q}_h$ as in (3.43). Similarly to the proof of Theorem 3.7 we subtract the first equations of (2.10) and (3.3) and add and subtract suitable terms to obtain

$$\begin{aligned} B_{\mathrm{F}}(\mathbf{v}_h, \chi_p) = \quad & A_{\mathrm{F}}(\theta_h; \mathbf{e}_{\mathbf{u}}, \mathbf{v}_h) + O_{\mathrm{F}}^h(\mathbf{u}_h; \mathbf{e}_{\mathbf{u}}, \mathbf{v}_h) + [A_{\mathrm{F}}(\theta; \mathbf{u}, \mathbf{v}_h) - A_{\mathrm{F}}(\theta_h; \mathbf{u}, \mathbf{v}_h)] \\ & + O_{\mathrm{F}}^h(\mathbf{e}_{\mathbf{u}}; \mathbf{u}, \mathbf{v}_h) - D_{\mathrm{F}}(\mathrm{e}_\theta, \mathbf{v}_h) - B_{\mathrm{F}}(\mathbf{v}_h, \mathrm{r}_p), \end{aligned}$$

for all $\mathbf{v}_h \in \mathbf{H}_h$. Then, from (**H.1**) (cf. (3.5)), it readily follows that

$$\begin{aligned} \widehat{\beta}_F \|\chi_p\|_{0,\Omega} \leq & \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \left\{ \frac{|A_{\mathrm{F}}(\theta_h; \mathbf{e}_{\mathbf{u}}, \mathbf{v}_h) + O_{\mathrm{F}}^h(\mathbf{u}_h; \mathbf{e}_{\mathbf{u}}, \mathbf{v}_h) + A_{\mathrm{F}}(\theta; \mathbf{u}, \mathbf{v}_h) - A_{\mathrm{F}}(\theta_h; \mathbf{u}, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} \right\} \\ & + \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \left\{ \frac{|O_{\mathrm{F}}^h(\mathbf{e}_{\mathbf{u}}; \mathbf{u}, \mathbf{v}_h) - D_{\mathrm{F}}(\mathrm{e}_\theta, \mathbf{v}_h) + B_{\mathrm{F}}(\mathbf{v}_h, \mathrm{r}_p)|}{\|\mathbf{v}_h\|_{1,\Omega}} \right\}, \end{aligned}$$

which together with (2.11), (2.12), (2.15), (2.16) and (3.1), yields

$$\begin{aligned} \widehat{\beta}_F \|\chi_p\|_{0,\Omega} \leq \quad & \nu_2 \|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega} + \widehat{C}_{O_{\mathrm{F}}} \|\mathbf{u}_h\|_{1,\Omega} \|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega} + \nu_{\mathrm{lip}} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} \|\mathrm{e}_\theta\|_{1,\Omega} \\ & + \widehat{C}_{O_{\mathrm{F}}} \|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} + C_{D_{\mathrm{F}}} \|\mathbf{g}\|_{0,\Omega} \|\mathrm{e}_\theta\|_{1,\Omega} + C_{B_{\mathrm{F}}} \|\mathrm{r}_p\|_{0,\Omega}. \end{aligned}$$

Then, using that $\|\mathbf{u}\|_{1,\Omega} \leq C_\infty \|\mathbf{u}\|_{W^{1,\infty}(\Omega)}$ and $\|\mathbf{u}_h\|_{1,\Omega} \leq \widehat{C}_{\mathbf{u}} \|\theta_D\|_{1/2,\Gamma}$, from (3.44), we obtain that there exists $c > 0$, independent of $h$, such that

$$\|\chi_p\|_{0,\Omega} \leq c\,(\|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega} + \|e_\theta\|_{1,\Omega}) + \widehat{\beta}_{\mathrm{F}}^{-1} C_{B_{\mathrm{F}}} \|\mathrm{r}_p\|_{0,\Omega},$$

which together with the triangle inequality implies

$$\|\mathrm{e}_p\|_{0,\Omega} \leq c\,(\|\mathbf{e}_{\mathbf{u}}\|_{1,\Omega} + \|e_\theta\|_{1,\Omega}) + (\widehat{\beta}_{\mathrm{F}}^{-1} C_{B_{\mathrm{F}}} + 1) \|p - \widehat{p}_h\|_{0,\Omega},$$

for all $\widehat{p}_h \in \mathrm{Q}_h$. Then the result readily follows from the previous estimate and (3.46). □

## 4 Concrete finite element discretizations

In this section we provide concrete examples of finite elements subspaces and convective forms satisfying hypotheses (**H.0**)–(**H.4**).

First, for the discretization of the convective forms we adopt the well known skew-symmetric forms (see [27]), given by

$$\begin{aligned} O^h_{\mathrm{F}}(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}_h) \,&=\, \int_\Omega [(\mathbf{w}_h\cdot\nabla)\mathbf{u}_h]\cdot\mathbf{v}_h + \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{w}_h)\mathbf{u}_h\cdot\mathbf{v}_h \qquad \mathbf{w}_h,\,\mathbf{u}_h,\,\mathbf{v}_h\in\mathbf{H}_h,\\ O^h_{\mathrm{T}}(\mathbf{w}_h;\theta_h,\psi_h) \,&=\, \int_\Omega (\mathbf{w}_h\cdot\nabla\theta_h)\psi_h + \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{w}_h)\theta_h\psi_h \qquad \theta_h\;\psi_h\in\Psi_h, \end{aligned} \tag{4.1}$$

which clearly satisfy (3.1), (3.2) and hypothesis (**H.4**) (for further choices of convective forms, see, for instance, [21]). In addition, by using integration by parts, it readily follows that

$$\begin{aligned} O^h_{\mathrm{F}}(\mathbf{w}_h;\mathbf{v}_h,\mathbf{v}_h) \,&=\, 0, \quad \mathbf{w}_h,\;\mathbf{v}_h\in\mathbf{H}_h,\\ O^h_{\mathrm{T}}(\mathbf{w}_h;\psi_h,\psi_h) \,&=\, 0, \quad \mathbf{w}_h\in\mathbf{H}_h, \quad \psi_h\in\Psi_h. \end{aligned} \tag{4.2}$$

Next, among all the choices of finite element subspaces that can be utilized to approximate the solution of (2.8), in what follows we introduce two particular examples. To do that, given an integer $k\geq 0$ and a set $S$ of $\mathbb{R}^n$, in the sequel we denote by $\mathrm{P}_k(S)$ the space of polynomial functions on $S$ of degree $\leq k$.

### 4.1 Hood-Taylor + Lagrange + piecewise polynomials

Let $k\geq 0$ an integer. Then, the well known Hood-Taylor element (see, e.g. [17]) consists of the pair $(\mathbf{H}_h,\mathrm{Q}_h)$, where

$$\mathbf{H}_h \,:=\, \big\{\mathbf{v}_h\in[C(\overline{\Omega})]^n \,:\, \mathbf{v}_h\big|_K\in[\mathrm{P}_{k+2}(K)]^n \quad \forall K\in\mathcal{T}_h, \quad \mathbf{v}_h \,=\, 0 \text{ on } \Gamma\big\}$$

and

$$\mathrm{Q}_h \,:=\, \big\{q_h\in C(\overline{\Omega}) \,:\, q_h\big|_K\in\mathrm{P}_{k+1}(K) \quad \forall K\in\mathcal{T}_h\big\}\,.$$

It is well known that this pair satisfy hypothesis (**H.1**) (see, for instance [3, 4, 17]). In turn, given an integer $l\geq 0$, to approximate the temperature $\theta$ we can simply choose the discrete space

$$\Psi_h \,:=\, \big\{\psi_h\in C(\overline{\Omega}) \,:\, \psi_h\big|_K\in\mathrm{P}_{l+1}(K) \quad \forall K\in\mathcal{T}_h\big\}\,. \tag{4.3}$$

Now, to approximate the Lagrange multiplier $\lambda$, for technical reasons that can be found in [16, Section 4.3], we first let $\{\overline{\Gamma}_1,\overline{\Gamma}_2,\ldots,\overline{\Gamma}_m\}$ be an independent triangulation of $\Gamma$ (made of triangles in $\mathbb{R}^3$ or straight segments in $\mathbb{R}^2$), and define $\tilde{h}:=\max\{|\Gamma_j| : j\in\{1,\ldots,m\}\}$. Then, we define the finite element subspace to approximate the Lagrange multiplier $\lambda$ as

$$\Lambda_{\tilde{h}} \,:=\, \Big\{\xi_{\tilde{h}}\in\mathrm{L}^2(\Gamma) \,:\, \xi_{\tilde{h}}\big|_{\overline{\Gamma}_j}\in\mathrm{P}_l(\overline{\Gamma}_j) \quad \forall j\in\{1,\ldots,m\}\Big\}\,.$$

The following lemma establishes that the pair $(\Psi_h,\Lambda_h)$ satisfies hypotheses (**H.2**).

**Lemma 4.1** *There exist $C_0 > 0$ and $\widehat{\beta}_{\mathrm{T}} > 0$, independent of $h$ and $\tilde{h}$, such that for each $h \leq C_0\tilde{h}$, there holds*

$$\sup_{\substack{\psi_h \in \Psi_h \\ \psi_h \neq 0}} \frac{B_{\mathrm{T}}(\psi_h, \xi_{\tilde{h}})}{\|\psi_h\|_{1,\Omega}} \geq \widehat{\beta}_{\mathrm{T}} \|\xi_{\tilde{h}}\|_{-1/2,\Gamma} \quad \forall \xi_{\tilde{h}} \in \Lambda_{\tilde{h}}. \tag{4.4}$$

*Proof.* It follows basically using the same arguments employed in [16, Lemma 4.7], where the approximating spaces for $\theta$ and $\lambda$ are defined as above but with $l = 0$. In fact, it suffices to replace the orthogonal projector from $\mathrm{H}^1(\Omega)$ onto the continuous piecewise polynomials of degree $\leq 1$ (employed there), by the one onto the continuous piecewise polynomials of degree $\leq l+1$ (required here). We omit Further details. □

Finally, in order to complete the analysis of this section, it remains to verify that the finite elements spaces and the convective forms described above satisfy hypotheses (**H.0**) and (**H.3**). This result is established in the following lemma.

**Lemma 4.2** *There exist positive constants $\widehat{\alpha}_{\mathrm{F}}$ and $\widehat{\alpha}_{\mathrm{T}}$, independent of $h$ and $\tilde{h}$, such that the inf-sup conditions* (3.4) *and* (3.8) *hold.*

*Proof.* We start by verifying the inf-sup condition (3.4). To do this, we first recall that the discrete kernel of $B_{\mathrm{F}}$ is given by

$$\mathbf{X}_h := \{\mathbf{v}_h \in \mathbf{H}_h \,:\, B_{\mathrm{F}}(\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in \mathrm{Q}_h\}.$$

Now, given $\mathbf{w}_h$, $\mathbf{v}_h \in \mathbf{X}_h$ and $\varphi_h \in \Psi_h$, from (2.17) and (4.2), it readily follows that

$$A_{\mathrm{F}}(\varphi_h; \mathbf{v}_h, \mathbf{v}_h) + O_{\mathrm{T}}^h(\mathbf{w}_h; \mathbf{v}_h, \mathbf{v}_h) \geq \alpha_{\mathrm{F}} \|\mathbf{v}_h\|_{1,\Omega}^2,$$

which clearly implies (3.4) with $\widehat{\alpha}_{\mathrm{F}} = \alpha_{\mathrm{F}}$.

Next, for (3.8) we recall that the discrete kernel of $B_{\mathrm{T}}$ is given by

$$\Psi_{h,0} := \{\psi_h \in \Psi_h \,:\, B_{\mathrm{T}}(\psi_h, \xi_{\tilde{h}}) = 0, \quad \forall\, \xi_{\tilde{h}} \in \Lambda_{\tilde{h}}\}.$$

In particular, $\xi_{\tilde{h}} \equiv 1$ belongs to $\Lambda_{\tilde{h}}$, and hence $\Psi_{h,0}$ is contained in the space

$$V := \left\{\psi \in \mathrm{H}^1(\Omega) : \quad \int_\Gamma \psi = 0\right\},$$

where, thanks to the generalized Poincaré inequality (cf. [19, Theorem 5.11.2]), $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ become equivalent on $V$, which in particular implies that there exists $C_P > 0$, depending only on $\Omega$, such that $|\psi|_{1,\Omega}^2 \geq C_P\|\psi\|_{1,\Omega}^2$, for all $\psi \in V$. Then, given $\mathbf{w}_h \in \mathbf{X}_h$ and $\varphi_h \in \Psi_h$, from (4.2) and (1.3), it follows that

$$A_{\mathrm{T}}(\varphi_h; \psi_h, \psi_h) + O_{\mathrm{T}}^h(\mathbf{w}_h; \psi_h, \psi_h) \geq \kappa_1 |\psi_h|_{1,\Omega}^2 \geq \kappa_1 C_P \|\psi_h\|_{1,\Omega}^2 \quad \forall\, \psi_h \in \Psi_{h,0},$$

which clearly implies (3.8) and completes the proof. □

Let us now recall the approximation properties of the subspaces specified above.

($\mathbf{AP}_h^{\mathbf{u}}$) There exits $C > 0$, independent of $h$, such that for all $\mathbf{u} \in \mathbf{H}^{k+3}(\Omega)$, there holds

$$\inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \leq C h^{k+2} \|\mathbf{u}\|_{k+3,\Omega}.$$

($\mathbf{AP}_h^p$) There exits $C > 0$, independent of $h$, such that for all $p \in \mathrm{H}^{k+2}(\Omega)$, there holds

$$\inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \leq C h^{k+2} \|p\|_{k+2,\Omega}.$$

($\mathbf{AP}_h^\theta$) There exits $C > 0$, independent of $h$, such that for all $\theta \in \mathrm{H}^{l+2}(\Omega)$, there holds

$$\inf_{\psi_h \in \Psi_h} \|\theta - \psi_h\|_{1,\Omega} \leq C h^{l+1} \|\theta\|_{l+2,\Omega}.$$

($\mathbf{AP}_{\tilde{h}}^\lambda$) There exits $C > 0$, independent of $\tilde{h}$, such that for all $\lambda \in \mathrm{H}^{l+\frac{1}{2}}(\Gamma)$, there holds

$$\inf_{\xi_h \in \Lambda_{\tilde{h}}} \|\lambda - \xi_{\tilde{h}}\|_{-1/2,\Gamma} \leq C \tilde{h}^{l+1} \|\lambda\|_{l+\frac{1}{2},\Gamma}.$$

Owing to these approximation properties, we now can establish the theoretical rate of convergence of our method.

**Theorem 4.3** *Assume that hypotheses of Theorem 3.7 and Corollary 3.8 hold. Given* $k,\, l \geq 0$, *assume further that* $\mathbf{u} \in \mathbf{H}^{k+3}(\Omega)$, $p \in \mathrm{H}^{k+2}(\Omega)$, $\theta \in \mathrm{H}^{l+1}(\Omega)$ *and* $\lambda \in \mathrm{H}^{l+1/2}(\Gamma)$. *Then, there exist* $C_1$, $C_2$, $C_3$, $> 0$, *independent of* $h$ *and* $\tilde{h}$, *such that for all* $h \leq C_0 \tilde{h}$, *there holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|\theta - \theta_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq C_1 h^{k+2} \big\{ \|\mathbf{u}\|_{k+3,\Omega} + \|p\|_{k+2,\Omega} \big\}$$
$$+ C_2 h^{l+1} \|\theta\|_{l+2,\Omega} + C_3 \tilde{h}^{l+1} \|\lambda\|_{l+\frac{1}{2},\Gamma}. \tag{4.5}$$

*Proof.* It follows from the Céa estimates (3.46), (3.47) and (3.55), and the approximation properties ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^p$) ,($\mathbf{AP}_h^\theta$) and ($\mathbf{AP}_{\tilde{h}}^\lambda$).

□

## 4.2 MINI-element + Lagrange + piecewise polynomials

In what follows, for the sake of simplicity we restrict ourselves to the 2D case. Then, for each $T \in \mathcal{T}_h$, we let $\mathrm{P}_{1,b}(T)$ be the space (see, e.g. [17])

$$\mathrm{P}_{1,b}(T) := [\mathrm{P}_1(T) \oplus \mathrm{span}\{b_T\}]^2,$$

where $b_T := \varphi_1\varphi_2\varphi_3$ is a $P_3$ bubble function in $T$, and $\varphi_1$, $\varphi_2$ , $\varphi_3$ are the barycentric coordinates of $T$. Then, the MINI-element (see, e.g. [17]) is the pair $(\mathbf{H}_h, \mathrm{Q}_h)$, where

$$\mathbf{H}_h := \left\{ \mathbf{v}_h \in [C(\overline{\Omega})]^2 \,:\, \mathbf{v}_h\big|_K \in \mathrm{P}_{1,b}(T) \quad \forall K \in \mathcal{T}_h, \quad \mathbf{v}_h = 0 \text{ on } \Gamma \right\}$$

and

$$\mathrm{Q}_h := \left\{ q_h \in C(\overline{\Omega}) \,:\, q_h\big|_K \in \mathrm{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

It is well known that this pair satisfy hypothesis (**H.1**) (see, for instance [13, 17]). Now, to approximate $\theta$ and $\lambda$, we proceed as in Section 4.1 and simply choose the discrete spaces

$$\Psi_h := \left\{ \psi_h \in C(\overline{\Omega}) \,:\, \psi_h\big|_K \in \mathrm{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\},$$

and

$$\Lambda_{\tilde{h}} := \left\{ \xi_{\tilde{h}} \in \mathrm{L}^2(\Gamma) \,:\, \xi_{\tilde{h}}|_{\overline{\Gamma}_j} \in \mathrm{P}_0(\overline{\Gamma}_j) \quad \forall j \in \{1, \ldots, m\} \right\},$$

where $\{\overline{\Gamma}_1, \overline{\Gamma}_2, \ldots, \overline{\Gamma}_m\}$ and $\tilde{h}$ are defined as in Section 4.1. Let us observe that, by using the same arguments employed in Section 4.1, it follows that hypotheses (**H.0**), (**H.2**) and (**H.3**) hold.

We now state the approximation properties of these subspaces and the rate of convergence of our method.

($\widehat{\mathbf{AP}}_h^{\mathbf{u}}$) There exits $C > 0$, independent of $h$, such that for all $\mathbf{u} \in \mathbf{H}^2(\Omega)$, there holds

$$\inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \leq Ch\|\mathbf{u}\|_{2,\Omega}.$$

($\widehat{\mathbf{AP}}_h^{p}$) There exits $C > 0$, independent of $h$, such that for all $p \in \mathrm{H}^1(\Omega)$, there holds

$$\inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \leq Ch\|p\|_{1,\Omega}.$$

($\widehat{\mathbf{AP}}_h^{\theta}$) There exits $C > 0$, independent of $h$, such that for all $\theta \in \mathrm{H}^2(\Omega)$, there holds

$$\inf_{\psi_h \in \Psi_h} \|\theta - \psi_h\|_{1,\Omega} \leq Ch\|\theta\|_{2,\Omega}.$$

($\widehat{\mathbf{AP}}_{\tilde{h}}^{\lambda}$) There exits $C > 0$, independent of $\tilde{h}$, such that for all $\lambda \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, there holds

$$\inf_{\xi_h \in \Lambda_{\tilde{h}}} \|\lambda - \xi_{\tilde{h}}\|_{-1/2,\Gamma} \leq C\tilde{h}\|\lambda\|_{1/2,\Gamma}.$$

**Theorem 4.4** *Assume that hypotheses of Theorem 3.7 and Corollary 3.8 hold. Assume further that $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $p \in \mathrm{H}^1(\Omega)$, $\theta \in \mathrm{H}^2(\Omega)$ and $\lambda \in \mathrm{H}^{1/2}(\Gamma)$. Then, there exist $C_1$, $C_2$, $C_3$, $> 0$, independent of $h$ and $\tilde{h}$, such that for all $h \leq C_0\tilde{h}$, there holds*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|\theta - \theta_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq C_1 h\big\{\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\big\} \\ &+ C_2 h\|\theta\|_{2,\Omega} + C_3\tilde{h}\|\lambda\|_{\frac{1}{2},\Gamma}. \end{aligned} \tag{4.6}$$

*Proof.* It follows from the Céa's estimates (3.46), (3.47) and (3.55), and the approximation properties ($\widehat{\mathbf{AP}}_h^{\mathbf{u}}$), ($\widehat{\mathbf{AP}}_h^{p}$) ,($\widehat{\mathbf{AP}}_h^{\theta}$) and ($\widehat{\mathbf{AP}}_{\tilde{h}}^{\lambda}$). □

# 5 Numerical results

In this section we present some numerical results illustrating the performance of our mixed finite element scheme (3.3) on unstructured meshes under a set of uniform refinements of the corresponding domain and considering the finite element spaces introduced in Section 4. Our implementation is based on a *FreeFem++* code (cf. [18]), in conjunction with the direct linear solver *UMFPACK* (cf.[12]).

In order to solve the nonlinear problem, we use the fixed point strategy suggested by the operator introduced in Section 3.2.2:

Given $(\mathbf{u}^0, p^0, \theta^0, \lambda^0) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h \times \Lambda_{\tilde{h}}$, for $m \geq 1$, find $(\mathbf{u}^m, p^m, \theta^m, \lambda^m) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h \times \Lambda_{\tilde{h}}$ such that

$$
\begin{aligned}
A_{\mathrm{F}}(\theta^{m-1};\mathbf{u}^m,\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}^{m-1};\mathbf{u}^m,\mathbf{v}) - B_{\mathrm{F}}(\mathbf{v},p^m) &= D_{\mathrm{F}}(\theta^{m-1},\mathbf{v}),\\
B_{\mathrm{F}}(\mathbf{u}^m,q) &= 0,\\
A_{\mathrm{T}}(\theta^{m-1};\theta^m,\psi) + O_{\mathrm{T}}^h(\mathbf{u}^{m-1};\theta^m,\psi) + B_{\mathrm{T}}(\psi,\lambda^m) &= 0,\\
B_{\mathrm{T}}(\theta^m,\xi) &= \langle \xi, \theta_D\rangle_\Gamma,
\end{aligned}
\tag{5.1}
$$

for all $(\mathbf{v}, q, \psi, \xi) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h \times \Lambda_h$.

The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is,

$$
\frac{\|\mathbf{Coeff}^{m+1} - \mathbf{Coeff}^m\|_{l^2}}{\|\mathbf{Coeff}^{m+1}\|_{l^2}} \leq tol,
$$

where $\|\cdot\|_{l^2}$ is the standard $l^2$-norm in $\mathbb{R}^N$, with $N$ denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$, $\mathrm{Q}_h$, $\Psi_h$ and $\Lambda_{\tilde{h}}$, and $tol$ is a fixed tolerance chosen as $tol = 1e-08$. For each example shown below we simply take $\mathbf{u}_h^0 = \mathbf{0}$ and $\theta_h^0 = 0$ as initials guess. In what follows, for practical purposes, the restriction on the meshsizes established in Lemma 4.1 is verified in an heuristic sense only. More precisely, since the constant $C_0$ is actually unknown, we simply assume $C_0 = 1/2$ and consider a partition of $\Gamma$ with a meshsize $\widetilde{h}$ given approximately by the double of $h$. The numerical results below confirm the suitability of this choice.

We now introduce some additional notations. The individual errors are denoted by

$$
\mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega},
$$

$$
\mathrm{e}(\theta) := \|\theta - \theta_h\|_{1,\Omega}, \quad \mathrm{e}(\lambda) := \|\lambda - \lambda_{\tilde{h}}\|_{0,\Gamma}.
$$

Also, we let $r(\mathbf{u})$, $r(p)$, $r(\theta)$ and $r(\lambda)$ be the experimental rates of convergence given by

$$
r(\mathbf{u}) := \frac{\log(\mathrm{e}(\mathbf{u})/\mathrm{e}'(\mathbf{u}))}{\log(h/h')}, \qquad r(p) := \frac{\log(\mathrm{e}(p)/\mathrm{e}'(p))}{\log(h/h')}
$$

$$
r(\theta) := \frac{\log(\mathrm{e}(\theta)/\mathrm{e}'(\theta))}{\log(h/h')}, \qquad r(\lambda) := \frac{\log(\mathrm{e}(\lambda)/\mathrm{e}'(\lambda))}{\log(\tilde{h}/\tilde{h}')},
$$

where $h$ and $h'$ (resp. $\tilde{h}$ and $\tilde{h}'$) denote two consecutive mesh sizes with their respective errors $\mathrm{e}$ and $\mathrm{e}'$.

For all the examples below, we consider the domain $\Omega := (0,1)^2$ and choose the temperature–dependent parameters as

$$
\nu(\theta) := e^{-\theta}, \qquad \kappa(\theta) := e^{\theta}.
$$

In our first example, we take $\mathbf{g} = (0,1)^t$ and adequately manufacture the data so that the exact solution is given by the smooth functions

$$
\begin{aligned}
\mathbf{u}(x_1,x_2) &:= \begin{pmatrix} 2x_1^2 x_2(2x_2-1)(x_2-1)(x_1-1)^2 \\ -2x_1x_2^2(x_2-1)^2(2x_1-1)(x_1-1) \end{pmatrix}\\
p(x_1,x_2) &:= e^{x_2}(x_1-0.5)^3\\
\theta(x_1,x_2) &:= x_1^2 + x_2^4
\end{aligned}
$$

| $N$ | $h$ | $\mathrm{e}(\mathbf{u})$ | $r(\mathbf{u})$ | $\mathrm{e}(p)$ | $r(p)$ | $\mathrm{e}(\theta)$ | $r(\theta)$ |
|---|---|---|---|---|---|---|---|
| 1,116 | 0.1964 | 2.9619E-03 | – | 1.9480E-03 | – | 1.1890E-02 | – |
| 4,237 | 0.0997 | 7.1929E-04 | 2.0869 | 4.5509E-04 | 2.1440 | 2.4792E-03 | 2.3117 |
| 16,354 | 0.0487 | 1.9507E-04 | 1.8213 | 1.2350E-04 | 1.8204 | 5.3607E-04 | 2.1374 |
| 64,443 | 0.0250 | 4.9249E-05 | 2.0604 | 2.8953E-05 | 2.1713 | 1.2415E-04 | 2.1896 |
| 256,158 | 0.0136 | 1.2075E-05 | 2.3082 | 6.9087E-06 | 2.3527 | 2.9349E-05 | 2.3680 |
| 1,008,151 | 0.0072 | 2.7675E-06 | 2.3171 | 1.6736E-06 | 2.2301 | 6.9932E-06 | 2.2560 |

| $N$ | $\tilde{h}$ | $\mathrm{e}(\lambda)$ | $r(\lambda)$ | iterations |
|---|---|---|---|---|
| 1,116 | 0.2500 | 2.6590E-01 | - | 13 |
| 4,237 | 0.1250 | 7.3446E-02 | 1.8562 | 13 |
| 16,354 | 0.0625 | 1.8858E-02 | 1.9615 | 13 |
| 64,443 | 0.0312 | 4.7216E-03 | 1.9979 | 13 |
| 256,158 | 0.0156 | 1.1776E-03 | 2.0034 | 13 |
| 1,008,151 | 0.0078 | 2.9387E-04 | 2.0026 | 13 |

Table 5.1: Example 1: Degrees of freedom, meshsizes, errors, rates of convergence and iterations for the $[\mathrm{P}_2]^2 - \mathrm{P}_1 - \mathrm{P}_2 - \mathrm{P}_1$ approximation of the generalized Boussinesq problem.

In Table 5.1 and 5.2 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the subspaces provided in Section 4.1 with $k = 0$ and $l = 1$ ($[\mathrm{P}_2]^2 - \mathrm{P}_1 - \mathrm{P}_2 - \mathrm{P}_1$) and the subspaces in Section 4.2 ($\mathrm{P}_{1,b} - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$), respectively. In Table 5.1 we observe that the rate of convergence $O(h^2)$ predicted by Theorem 4.3 is attained by all the unknowns. Similarly, in Table 5.2 we can observe a first order convergence for all fields, confirming the expected results from Theorem 4.4. Next, in Figure 5.1 we display (to the left) the approximate vector field, the approximate pressure, and the approximate temperature, respectively, and we compare them with their corresponding exact counterparts (to the right). All the figures were built using the $[\mathrm{P}_2]^2 - \mathrm{P}_1 - \mathrm{P}_2 - \mathrm{P}_1$ approximation with $N = 1,008,151$ degrees of freedom. In all the cases we observe that the finite element subspaces employed provide very accurate approximations to the unknowns.

In our second example we illustrate a more realistic situation in which the exact solution is unknown. Here, we consider the external force $\mathbf{g} = (0, 1)^t$ and the boundary data

$$\theta_D(x_1, x_2) = e^{x_1 x_2} \quad \text{on} \quad \Gamma.$$

Notice, that $\theta_D$ attains its maximum value at $(x_1, x_2) = (1, 1)$, whereas $\theta_D = 1$ on $\{0\} \times (0, 1)$ and $(0, 1) \times \{0\}$. In Table 5.3 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering a $\mathrm{P}_{1,b} - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximation. There, the errors and experimental rates of convergence are computed by considering the discrete solution obtained with a finer mesh ($N = 2,508,849$) as the exact solution. We observe that the rate of convergence $O(h)$ predicted by Theorem 4.4 is attained by all the fields. Next, in Figure 5.2 we display the approximate velocity magnitude and velocity vector filed (top left and top right, respectively) together with the approximate pressure and temperature (bottom left and bottom right, respectively). All the figures were obtained with $N = 620,009$ degrees of freedom. We can observe that the discrete temperature preserve the prescribed boundary condition.

| $N$ | $h$ | e($\mathbf{u}$) | $r(\mathbf{u})$ | e($p$) | $r(p)$ | e($\theta$) | $r(\theta)$ |
|---|---|---|---|---|---|---|---|
| 705 | 0.1964 | 1.6109E-02 | – | 6.0464E-03 | – | 1.8407E-01 | – |
| 2,585 | 0.0997 | 7.8785E-03 | 1.0547 | 2.4493E-03 | 1.3324 | 8.8390E-02 | 1.0817 |
| 10,017 | 0.0487 | 3.8720E-03 | 0.9915 | 9.6648E-04 | 1.2979 | 4.3696E-02 | 0.9833 |
| 39,561 | 0.0250 | 1.9484E-03 | 1.0280 | 4.2686E-04 | 1.2232 | 2.2112E-02 | 1.0196 |
| 157,441 | 0.0136 | 9.6101E-04 | 1.1605 | 1.8493E-04 | 1.3735 | 1.0996E-02 | 1.1470 |
| 620,009 | 0.0072 | 4.6761E-04 | 1.1330 | 8.2510E-05 | 1.2694 | 5.4859E-03 | 1.0937 |

| $N$ | $\tilde{h}$ | e($\lambda$) | $r(\lambda)$ | iterations |
|---|---|---|---|---|
| 705 | 0.2500 | 2.1729E-00 | - | 13 |
| 2,585 | 0.1250 | 1.1262E-00 | 0.9482 | 13 |
| 10,017 | 0.0625 | 5.5209E-01 | 1.0284 | 13 |
| 39,561 | 0.0312 | 2.7302E-01 | 1.0159 | 13 |
| 157,441 | 0.0156 | 1.3455E-01 | 1.0209 | 13 |
| 620,009 | 0.0078 | 6.6790E-02 | 1.0104 | 13 |

Table 5.2: EXAMPLE 1: Degrees of freedom, meshsizes, errors, rates of convergence and iterations for the $\mathrm{P}_{1,b} - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximation of the generalized Boussinesq problem.

| $N$ | $h$ | e($\mathbf{u}$) | $r(\mathbf{u})$ | e($p$) | $r(p)$ | e($\theta$) | $r(\theta)$ |
|---|---|---|---|---|---|---|---|
| 705 | 0.1964 | 2.3553E-02 | – | 2.6231E-03 | – | 2.0396E-01 | – |
| 2,585 | 0.0997 | 1.1439E-02 | 1.0649 | 1.3508E-03 | 0.9786 | 9.0159E-02 | 1.2037 |
| 10,017 | 0.0487 | 6.0892E-03 | 0.8800 | 5.9926E-04 | 1.1344 | 4.1347E-02 | 1.0881 |
| 39,561 | 0.0250 | 3.0282E-03 | 1.0456 | 2.2325E-04 | 1.4780 | 2.0379E-02 | 1.0590 |
| 157,441 | 0.0136 | 1.4881E-03 | 1.1666 | 9.3239E-05 | 1.4336 | 1.0046E-02 | 1.1614 |
| 620,009 | 0.0072 | 7.8121E-04 | 1.0136 | 4.8769E-05 | 1.0193 | 5.1740E-03 | 1.0437 |

| $N$ | $\tilde{h}$ | e($\lambda$) | $r(\lambda)$ | iterations |
|---|---|---|---|---|
| 705 | 0.2500 | 7.5565E-00 | - | 11 |
| 2,585 | 0.1250 | 4.8078E-00 | 0.6524 | 11 |
| 10,017 | 0.0625 | 2.7053E-00 | 0.8296 | 11 |
| 39,561 | 0.0312 | 1.4305E-00 | 0.9193 | 10 |
| 157,441 | 0.0156 | 7.2451E-01 | 0.9814 | 10 |
| 620,009 | 0.0078 | 3.4733E-01 | 1.0607 | 10 |

Table 5.3: EXAMPLE 2: Degrees of freedom, meshsizes, errors, rates of convergence and iterations for the $\mathrm{P}_{1,b} - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximation of the generalized Boussinesq problem.

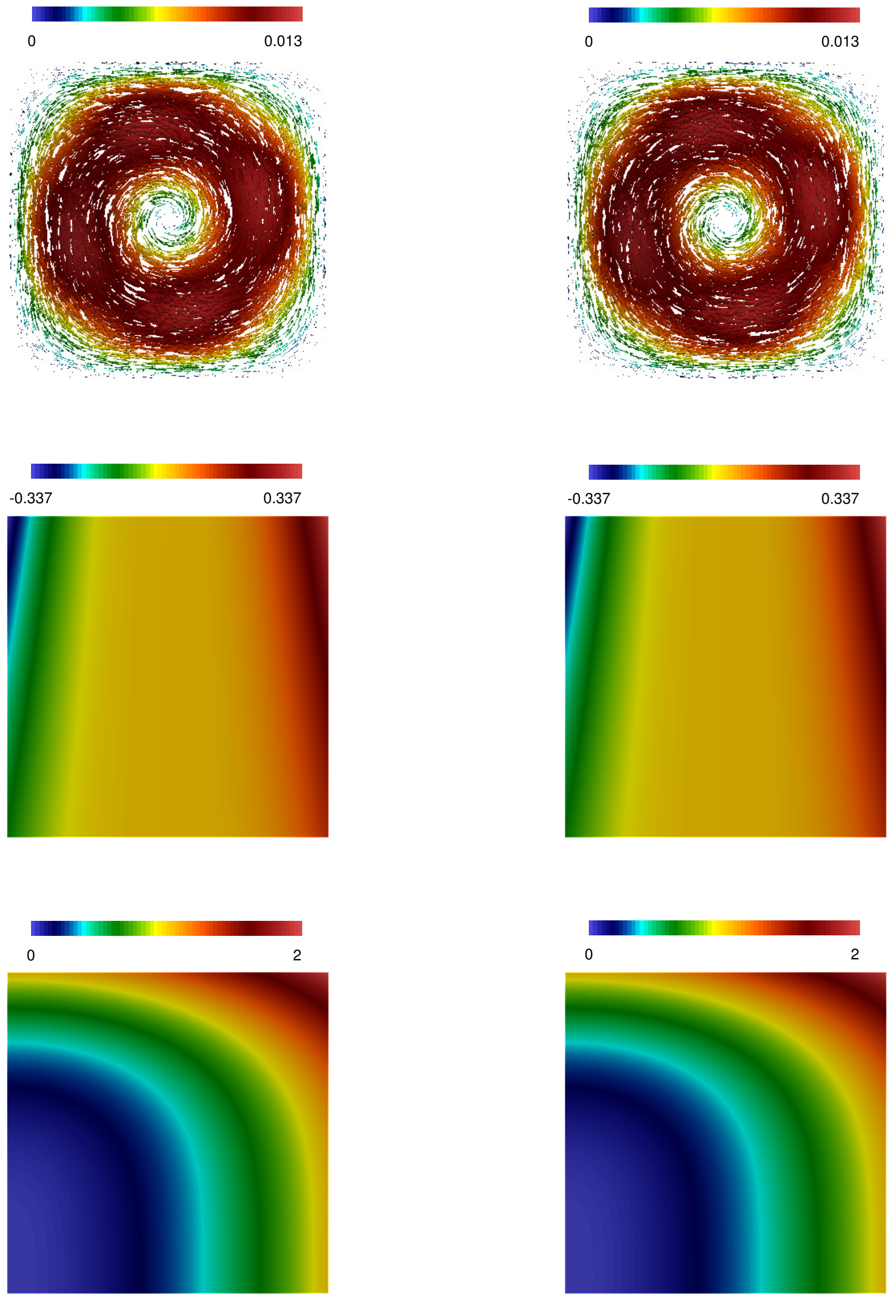


Figure 5.1: Example 1: Approximate velocity vector field (top left), exact velocity vector field (top right) $p_h$ (center left), $p$ (center right), $\theta_h$ (bottom left) and $\theta$ (bottom right) with $N = 1,008,151$.

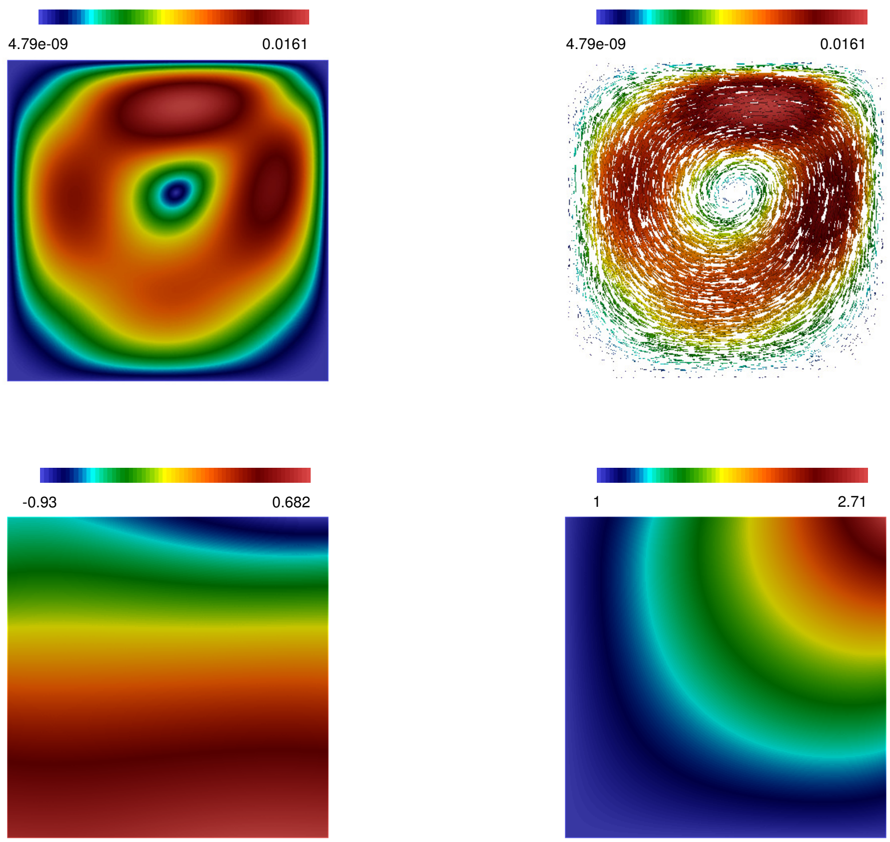


Figure 5.2: Example 2: Approximate velocity magnitude (top left), approximate velocity vector field (top left), exact velocity vector field (top right) $p_h$ (bottom left), $\theta_h$ (bottom left) and $\theta$ (bottom right) with $N = 620,009$.

## References


[1] K. Allali, *A priori and a posteriori error estimates for Boussinesq equations*, International Journal of Numerical Analysis and Modeling, vol. 2, no. 2, pp. 179–196, (2005).

[2] C. Bernardi, B. Métivet, B. Pernaud-Thomas, *Couplage des équations de Navier-Stokes et de la chaleur: le modèle et son approximation par éléments finis*. ESAIM: Mathematical Modelling and Numerical Analysis, vol. 29, no. 7, pp. 871-921, (1995).

[3] D. Boffi, *Stability of Higher Order Triangular Hood-Taylor Methods for Stationary Stokes Equations*. Mathematical Models and Methods in Applied Sciences, vol. 2, 4 , pp. 223–235, (1994).

[4] D. Boffi, *Three-dimensional finite element methods for the Stokes problem*. SIAM Journal on Numerical Analysis, vol. 34, 664-670, (1997).

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, (1991).

[6] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York Dordrecht Heidelberg London, (2011).

[7] J. Boland and W. Layton, *Error analysis for finite element methods for steady natural convection problems*. Numerical Functional Analysis and Optimization, vol. 11, no. 5–6, pp. 449–483, (1990).

[8] P.G. Ciarlet, *The finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, New York, Oxford, (1978).

[9] E. Colmenares, G.N. Gatica and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem*. Numerical Methods for Partial Differential Equations, DOI:10.1002/num.22001, (2015).

[10] A. Çibik, K. Songül *A projection-based stabilized finite element method for steady-state natural convection problem*, Journal of Mathematical Analysis and Applications, vol. 381, pp. 469–484, (2011).

[11] C. Cox, H. Lee, and D. Szurley, *Finite element approximation of the non-isothermal Stokes-Oldroyd equations*. Internat. J. Numer. Anal. Model, vol. 1, 1-18, (2007).

[12] T. A. Davis, *Algorithm 832: UMFPACK V4.3-an unsymmetric-pattern multifrontal method*. ACM Transactions on Mathematical Software, vol. 30, pp. 196-199, (2004).

[13] A. Ern and J.-L.Guermond, *Applied Mathematical Sciences*, Springer-Verlag, New York, 2004., vol. 159, (2004).

[14] M. Farhloul, S. Nicaise and L. Paquet, *A mixed Formulation of Boussinesq equations: Analysis of nonsingular solutions*. Mathematics of Computation, vol. 69, 231, pp. 965–986, (2000).

[15] M. Farhloul, S. Nicaise and L. Paquet, *A refined mixed finite element method for the Boussinesq equations in polygonal domains*, IMA Journal of Numerical Analysis, vol. 21, no. 2, pp. 525–551, (2001).

[16] G.N. Gatica: *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications* Springer Briefs in Mathematics, Springer, Cham Heidelberg New York Dordrecht London, (2014).

[17] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier–Stokes Equations*. Lecture Notes in Mathematics, Berlin Springer Verlag, vol. 749, (1979).

[18] F. Hecht, *New development in FreeFem++*. Journal of Numerical Mathematics, vol. 20, 3-4, pp. 251-265, (2012).

[19] A. Kufner, O. John, S. Fučík, *Function Spaces.Monographs and Textbooks on Mechanics of Solids and Fluids*. Mechanics: Analysis, pp. xv+454. Noordhoff, Leyden; Academia, Prague (1977)

[20] A. Kufner, J. Oldrich and F. Svatopluk, *Function spaces*. vol. 3, Springer Science & Business Media, (1977).

[21] W. Layton, C. Manica, M. Neda, M. Olshanskii and L.G. Rebholz, *On the accuracy of the rotation form in simulations of the Navier-Stokes equations*. Journal of Computational Physics, vol 228, 9, pp. 3433-3447, (2009).

[22] S.A. Lorca and J.L. Boldrini, *Stationary solutions for generalized Boussinesq models. J. Differential Equations*, vol. 134, pp. 389-406, (1996).

[23] G. Lube, T. Knopp, G. Rapin, R. Gritzki and M. Rösler Stabilized finite element methods to predict ventilation efficiency and thermal comfort in buildings International Journal for Numerical Methods in Fluids, vol. 57, no. 9, pp. 1269–1290, (2008).

[24] R. Oyarzúa, T. Qin and D. Schötzau, *An exactly divergence-free finite element method for a generalized Boussinesq problem*. IMA Journal of Numerical Analysis, vol. 34, 3, pp. 1104–1135, (2014).

[25] P.A. Raviart and J-M. Thomas, *Introduction á L'Analyse Numérique des Equations aux Dérivées Partielles*. Masson, Paris, (1983).

[26] M, Tabata, D. Tagami, *Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients*. Numerische Mathematik, vol. 100, no. 2, 351–372, (2005).

[27] R. Temam *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

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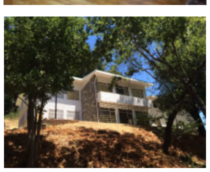

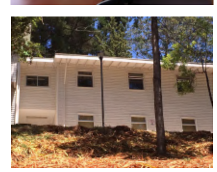

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