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Finite element analysis of a pressure-stress formulation for the time-domain fluid-structure interaction problem*

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Abstract

We present a convergence analysis for the space discretization of a time-dependent system of partial differential equations modeling an elasto-acoustic interaction problem. We use the Arnold-Falk-Winther mixed finite element method with weak symmetry in the solid and the usual Lagrange finite element method in the acoustic medium. The error analysis of the resulting global semi-discrete scheme relies essentially on the mapping properties of an adequate projector. We show that the method is stable uniformly with respect to the space discretization parameter and the Poisson modulus and we prove asymptotic error estimates.

Mathematics Subject Classification. 65N30, 65M12, 65M15, 74H15

Keywords. mixed finite elements, fluid-solid interaction, error estimates

1 Introduction

In this paper, we aim to compute the vibrations of an elastic structure enclosing in its interior an inviscid compressible fluid. The model problem consists in a scalar-valued equation describing the propagation of acoustic pressure waves and a vector-valued equation modeling the propagation of elastic waves. The two systems are coupled through adequate transmission conditions on the contact boundary. Traditionally, a displacement formulation in the solid is combined with a formulation using either the acoustic pressure (as in [14]) or the fluid displacement (as in [7]) as main variables in the fluid domain. The displacement-pressure formulation studied in [14] leads to a non-symmetric weak formulation involving time derivatives on boundary terms. The displacement-displacement formulation introduced in [7] is symmetric but it is not adapted to deal with nearly incompressible elastic materials.

More recently, dual-mixed formulations have been considered in the solid for the static elastoacoustic source problem (see, e.g., [19] and [20]). This approach may be considered as the dual procedure to the one proposed in [7]. In such a case, the Cauchy stress tensor is used as a main variable in the solid structure, in combination with the pressure in the fluid domain. The resulting formulation is

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symmetric and delivers direct finite element approximations of the stresses. In addition, it has been shown that an approximation scheme based on the Lagrange and Arnold-Falk-Winther (AFW) [4] finite elements in the fluid and solid domains, respectively, provides a stable Galerkin method in the nearly incompressible case. Moreover, it has been proved in [22] that the former mixed finite element method provides a spectrally correct approximation of the corresponding eigenproblem. In the present paper, we use the same Galerkin method for a space discretization of the elastoacoustic problem in the time domain transient problem and conclude that it provides the same convergence and stability performances shown in [22] and [20] for the spectral and the static source problems, respectively. More precisely, we prove the stability of the AFW/Lagrange finite element scheme when the Lamé coefficient λ tends to infinity and when the mesh size h goes to 0, and then we establish asymptotic error estimates.

The paper is organized as follows. We begin by introducing in Section 2 some basic notations and properties needed in the forthcoming analysis. In Section 3 we introduce a mixed formulation of the time-domain elastoacoustic problem and prove its well-posedness. We also provide conditions on the initial data that permits to express the displacement field explicitly in terms of the stress tensor. In Section 4 we introduce a space-discretization of the problem based on the AFW and Lagrange finite elements. Then, in Section 5 we prepare the convergence analysis of this conforming Galerkin scheme by introducing an adequate projector and studying its mapping properties and those of its discrete counterpart. Finally, the convergence analysis of the numerical scheme is performed in Section 6.

2 Notations and preliminary results

We denote by $\boldsymbol{I}$ the identity matrix of $\mathbb{R}^{n\times n}$ $(n = 2, 3)$, and $\mathbf{0}$ represents the null vector in $\mathbb{R}^n$ or the null tensor in $\mathbb{R}^{n\times n}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{n} \tau_{ij}\sigma_{ij}$. We now let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^n$ $(n = 2, 3)$, with boundary $\partial\Omega$, and denote by $\mathcal{D}(\Omega)$ the space of indefinitely differentiable functions with compact support in Ω. For $s \in \mathbb{R}$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ or $[\mathrm{H}^s(\Omega)]^{n\times n}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also denote by $(\cdot,\cdot)_{0,\Omega}$ the inner product in $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\Omega)^n$ or $\mathrm{L}^2(\Omega)^{n\times n}$. We introduce the Hilbert space

$$\mathbf{H}(\mathbf{div},\Omega) := \left\{\boldsymbol{\tau} \in [\mathrm{L}^2(\Omega)]^{n\times n}; \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{L}^2(\Omega)^n\right\},$$

whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$. In addition, given $p \in [1, +\infty]$ and a separable Hilbert space V with norm $\|\cdot\|_V$, we let $\mathrm{L}^p(V)$ be the space of classes of functions $f : (0,T) \to V$ that are Böchner-measurable and such that $\|f\|_{\mathrm{L}^p(V)} < \infty$, with

$$\|f\|^p_{\mathrm{L}^p(V)} := \int_0^T \|f(t)\|^p_V \,\mathrm{d}t \quad (1 \le p < \infty), \quad \|f\|_{\mathrm{L}^\infty(V)} := \operatorname*{ess\,sup}_{[0,T]} \|f(t)\|_V\,.$$

For any $k \in \mathbb{N}$, we consider the space $\mathcal{C}^k(V)$ of all functions f with (strong) derivatives $\dfrac{\mathrm{d}^j f}{\mathrm{d}t^j}$ in $\mathcal{C}^0(V)$ for all $1 \le j \le k$, where $\mathcal{C}^0(V)$ stands for the Banach space consisting of all continuous functions $f : [0,T] \to V$. In what follows, we will use also denote $\dot{f} := \dfrac{\mathrm{d}f}{\mathrm{d}t}$ and $\ddot{f} := \dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2}$ the first and second derivatives with respect to the variable t. Furthermore, we will use the Sobolev space

$$\mathrm{W}^{1,p}(V) := \left\{f :\ \exists g \in \mathrm{L}^p(V) \text{ and } \exists f_0 \in V \text{ such that } f(t) = f_0 + \int_0^t g(s)\,\mathrm{d}s \quad \forall t \in [0,T]\right\}.$$

The space $\mathrm{W}^{k,p}(V)$ is defined recursively for all $k \in \mathbb{N}$.

Finally, we need to recall a classical result that will be recurrently used in the following and that concerns the well-posedness of a variational problem defined in terms of a bilinear form satisfying the inf-sup condition. Indeed, given two Hilbert spaces $\big(S, \langle\cdot,\cdot\rangle_S\big)$ and $\big(Q, \langle\cdot,\cdot\rangle_Q\big)$ and a bounded bilinear form $\mathcal{A} : S \times Q \to \mathbb{R}$, we let $\mathbf{A} : S \to Q$ be the bounded linear operator induced by $\mathcal{A}$, that is $\langle \mathbf{A}(s), q\rangle_Q = \mathcal{A}(s,q) \quad \forall\,(s,q) \in S \times Q$, and introduce the null space

$$N(\mathbf{A}) := \Big\{ s \in S : \ \mathbf{A}(s) = 0 \Big\} \ = \ \Big\{ s \in S : \ \mathcal{A}(s,q) = 0 \quad \forall\, q \in Q \Big\}$$

and its polar

$$N(\mathbf{A})^\circ \ := \ \Big\{ \chi \in S'; \quad \chi(s) = 0 \quad \forall\, s \in N(\mathbf{A}) \Big\}\,.$$

In addition, we let $\mathcal{R}_S : S' \to S$ be the corresponding Riesz operator. Then, we have the following theorem (cf. [10]).

Theorem 2.1. *Assume that there exists $\kappa > 0$ such that*

$$\|\mathbf{A}^*(q)\| \ := \ \sup_{0 \neq s \in S} \frac{\mathcal{A}(s,q)}{\|s\|_S} \geq \kappa\, \|q\|_Q \qquad \forall\, q \in Q\,. \tag{2.1}$$

Then, for each $\ell \in N(\mathbf{A})^\circ$ there exists a unique $q \in Q$ such that $\mathbf{A}^*(q) = \mathcal{R}_S(\ell)$, that is

$$\mathcal{A}(s,q) = \ell(s) \quad \forall\, s \in S\,.$$

Proof. It suffices to see that (2.1) establishes, equivalently, that $\mathbf{A}^*$ is injective and has closed range $R(\mathbf{A}^*)$, whence $R(\mathbf{A}^*) \ = \ N(\mathbf{A})^\perp \ = \ \mathcal{R}_S\big(N(\mathbf{A})^\circ\big)$. □

Throughout this paper we use C (with or without subscripts) to denote generic constants independent of the parameters indicated at each instance. We point out that these constants may take different values at different places.

3 The model problem

We aim to compute the linear oscillations of a structure $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{F}}$ consisting of a solid body, represented by a polyhedral Lipschitz domain Ω_{S}, and a cavity Ω_{F} completely filled with an homogeneous, inviscid and compressible fluid, see Figure 3.1. The fluid-structure interface is given by $\Sigma := \partial\Omega_{\mathrm{F}}$ and the external boundary $\Gamma := \partial\Omega$ of the solid consists of a part $\Gamma_{\mathrm{D}} \neq \emptyset$ where the structure is fixed and a part Γ_{N} on which it is free from tractions. We impose on Σ the orientation given by the unit normal vector $\boldsymbol{n}$ pointing outward to Ω_{F}. The outward unit normal vector to Γ is also denoted by $\boldsymbol{n}$, as shown in Figure 3.1. We assume that the fluid-structure system is subject to a volume load $\boldsymbol{f} : (0,T] \times \Omega_{\mathrm{S}} \to \mathbb{R}^n$ acting on the solid. We can combine the constitutive law

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega_{\mathrm{S}}, \tag{3.1}$$

and the equation of motion

$$\rho_{\mathrm{S}}\ddot{\boldsymbol{u}} = \mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{3.2}$$

to eliminate either the displacement field $\boldsymbol{u}$ in the solid or the Cauchy stress tensor $\boldsymbol{\sigma}$ from the global formulation of the fluid-structure problem. Here, $\rho_{\mathrm{S}} > 0$ is a constant representing the solid density,

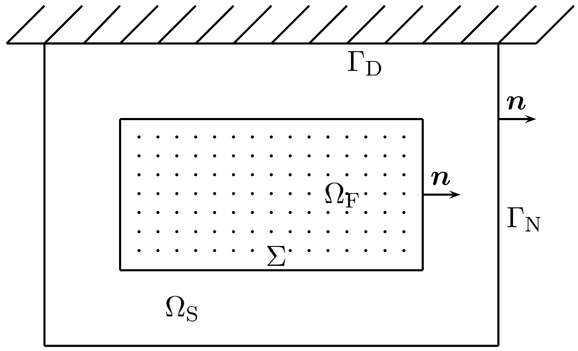

Figure 3.1: Fluid and solid domains

$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathtt{t}}\Big\}$ is the linearized strain tensor, and $\mathcal{C}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the Hooke operator, which is given in terms of the Lamé coefficients λ and μ by

$$\mathcal{C}\boldsymbol{\tau} := \lambda\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}^{n\times n}\,.$$

In what follows we eliminate the displacement $\boldsymbol{u}$ and maintain the stress tensor $\boldsymbol{\sigma}$ as a main variable, which leads to the following dual mixed formulation in the solid,

$$\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}} - \rho_S{}^{-1}\boldsymbol{\varepsilon}\,(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}) = \mathbf{0} \qquad \text{in } \Omega_S \times (0,T], \tag{3.3}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}} \qquad \text{in } \Omega_S \times (0,T], \tag{3.4}$$

$$\rho_S{}^{-1}(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}) = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T], \tag{3.5}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_N \times (0,T] \tag{3.6}$$

$$\boldsymbol{\sigma}\boldsymbol{n} + p\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Sigma \times (0,T]. \tag{3.7}$$

We notice that the transmission condition (3.7) represents an equilibrium of forces on the contact boundary Σ where the fluid pressure p is acting here as a prescribed normal stress. The model problem is described in the fluid domain Ω_F in terms of the pressure,

$$c^{-2}\ddot{p} - \Delta p = 0 \qquad \text{in } \Omega_F \times (0,T], \tag{3.8}$$

$$\frac{\partial p}{\partial \boldsymbol{n}} + \frac{\rho_F}{\rho_S}(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Sigma \times (0,T]. \tag{3.9}$$

Here, $c > 0$ is the acoustic speed and ρ_F stands for the (constant) fluid density. Equation (3.9) corresponds to the so-called wall slipping condition, which expresses the matching of the normal components of the fluid and solid displacements on the transmission boundary Σ. Summing up, our model problem is given by the system (3.3)-(3.9) and the initial conditions

$$(\boldsymbol{\sigma}(0), p(0)) = (\boldsymbol{\sigma}_0, p_0) \quad \text{and} \quad (\dot{\boldsymbol{\sigma}}(0), \dot{p}(0)) = (\boldsymbol{\sigma}_1, p_1). \tag{3.10}$$

Now, we consider the orthogonal decomposition $[\mathrm{L}^2(\Omega_S)]^{n\times n} = [\mathrm{L}^2(\Omega_S)]^{n\times n}_{\mathrm{sym}} \oplus [\mathrm{L}^2(\Omega_S)]^{n\times n}_{\mathrm{skew}}$, where

$$[\mathrm{L}^2(\Omega_S)]^{n\times n}_{\mathrm{sym}} := \Big\{\boldsymbol{\tau} \in [\mathrm{L}^2(\Omega_S)]^{n\times n};\quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}}\Big\}$$

and

$$[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}} := \left\{\boldsymbol{\tau} \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}; \quad \boldsymbol{\tau} = -\boldsymbol{\tau}^{\mathtt{t}}\right\},$$

and introduce the closed subspaces of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}})$ given by

$$\boldsymbol{\mathcal{W}} := \Big\{\boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}); \quad \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}}\Big\}$$

and its symmetric counterpart

$$\boldsymbol{\mathcal{W}}^{\text{sym}} := \boldsymbol{\mathcal{W}} \cap [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{sym}}.$$

On the other hand, since the equation (3.7) is an essential transmission condition that must be explicitly satisfied by the solution pair $(\boldsymbol{\sigma}, p)$, we need to consider the energy space

$$\mathbb{X} := \Big\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_{\mathrm{F}}); \quad \boldsymbol{\tau}\boldsymbol{n} + q\boldsymbol{n} = 0 \qquad \text{on } \Sigma\Big\},$$

which is a closed subspace of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ when endowed with the Hilbertian norm

$$\|(\boldsymbol{\tau}, q)\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|q\|^2_{1,\Omega_{\mathrm{F}}}.$$

We notice that the density of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ in $\mathbb{H} := [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n} \times \mathrm{L}^2(\Omega_{\mathrm{F}})$ proves that the space $\mathbb{X}^{\text{sym}} := \left\{(\boldsymbol{\tau}, q) \in \mathbb{X}; \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}}\right\}$ is also densely embedded in $\mathbb{H}^{\text{sym}} := [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{F}})$. We may then construct the dual $(\mathbb{X}^{\text{sym}})'$ of $\mathbb{X}^{\text{sym}}$ pivotal to $\mathbb{H}^{\text{sym}}$, in such a way that the identification

$$\Big\langle(\boldsymbol{f}, g), (\boldsymbol{\tau}, q)\Big\rangle_{(\mathbb{X}^{\text{sym}})', \mathbb{X}^{\text{sym}}} = \Big((\boldsymbol{f}, g), (\boldsymbol{\tau}, q)\Big)_{\mathbb{H}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\text{sym}}$$

holds true for all $(\boldsymbol{f}, g) \in \mathbb{H}^{\text{sym}} \hookrightarrow (\mathbb{X}^{\text{sym}})'$. Here, $\langle\cdot,\cdot\rangle_{(\mathbb{X}^{\text{sym}})', \mathbb{X}^{\text{sym}}}$ represents the duality pairing between $(\mathbb{X}^{\text{sym}})'$ and $\mathbb{X}^{\text{sym}}$ and $(\cdot,\cdot)_{\mathbb{H}}$ is the natural inner product in $\mathbb{H}$ whose norm is given by

$$\|(\boldsymbol{\tau}, q)\|^2_0 := \|\boldsymbol{\tau}\|^2_{0,\Omega_{\mathrm{S}}} + \|q\|^2_{0,\Omega_{\mathrm{F}}}.$$

Next, given $\boldsymbol{f} \in \mathrm{L}^1(\mathrm{L}^2(\Omega_{\mathrm{S}})^n)$, $(\boldsymbol{\sigma}_0, p_0) \in \mathbb{X}^{\text{sym}}$ and $(\boldsymbol{\sigma}_1, p_1) \in \mathbb{H}^{\text{sym}}$, it is straightforward to show that the variational formulation of (3.3)-(3.10) is given by:

Find $(\boldsymbol{\sigma}, p) \in \mathrm{L}^\infty(\mathbb{X}^{\text{sym}}) \cap \mathrm{W}^{1,\infty}(\mathbb{H}^{\text{sym}})$ such that

$$\Big((\ddot{\boldsymbol{\sigma}}, \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}, p)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\text{sym}} \tag{3.11}$$

$$\big(\boldsymbol{\sigma}(0), p(0)\big) = (\boldsymbol{\sigma}_0, p_0), \qquad \big(\dot{\boldsymbol{\sigma}}(0), \dot{p}(0)\big) = (\boldsymbol{\sigma}_1, p_1),$$

where

$$\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} := (\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \frac{1}{\rho_{\mathrm{F}} c^2}(p, q)_{0,\Omega_{\mathrm{F}}} \tag{3.12}$$

and

$$A\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big) := \rho_{\mathrm{S}}^{-1}(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \rho_{\mathrm{F}}^{-1}(\nabla p, \nabla q)_{0,\Omega_{\mathrm{F}}}.$$

In the forthcoming analysis, we need to keep track of the parameter λ. For this reason, it is important to notice that

$$\begin{aligned}\|(\boldsymbol{\tau}, q)\|^2_{0,\mathcal{C}} := \Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} &= \frac{1}{2\mu}\left\|\boldsymbol{\tau}^{\mathtt{D}}\right\|^2_{0,\Omega_{\mathrm{S}}} + \frac{1}{n(n\lambda + 2\mu)}\|\mathrm{tr}(\boldsymbol{\tau})\|^2_{0,\Omega_{\mathrm{S}}} + \frac{1}{\rho_{\mathrm{F}} c^2}\|q\|^2_{0,\Omega_{\mathrm{F}}} \\ &\le \max\left\{\frac{1}{2\mu}, \frac{1}{\rho_{\mathrm{F}} c^2}\right\}\|(\boldsymbol{\tau}, q)\|^2_0 \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{H}.\end{aligned} \tag{3.13}$$

In addition, the following result proves that $\|(\boldsymbol{\tau}, q)\|^2_{0,\mathcal{C}} + A\Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big)$ is a Hilbertian norm on $\mathbb{X}$ that is equivalent to the $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$-norm uniformly in λ.

Lemma 3.1. *There exists a constant* $\alpha > 0$*, independent of* λ*, such that*

$$\alpha \, \|(\boldsymbol{\tau}, q)\|^2 \leq \|(\boldsymbol{\tau}, q)\|^2_{0,\mathcal{C}} + A\Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big) \leq C \, \|(\boldsymbol{\tau}, q)\|^2 \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}. \tag{3.14}$$

with $C = \max\left\{\frac{1}{2\mu}, \frac{1}{\rho_{\mathrm{S}}}, \frac{1}{\rho_{\mathrm{F}}}, \frac{1}{\rho_{\mathrm{F}} c^2}\right\}$.

Proof. See [22, Lemma 2.1]. □

Theorem 3.1. *Assume that* $\boldsymbol{f} \in \mathrm{W}^{1,1}(\mathrm{L}^2(\Omega_{\mathrm{S}})^n)$*. Then, problem* (3.11) *admits a unique solution* $(\boldsymbol{\sigma}, p) \in \mathrm{L}^\infty(\mathbb{X}^{sym}) \cap \mathrm{W}^{1,\infty}(\mathbb{H}^{sym})$*. Moreover, there exists a constant* $C > 0$*, independent of* λ*, such that*

$$\begin{aligned} &\operatorname*{ess\,sup}_{[0,T]} \|(\boldsymbol{\sigma}, p)(t)\| + \operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol{\sigma}}, \dot{p})(t)\|_{0,\mathcal{C}} \\ &\quad \leq C \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_0, p_0)\| + \|(\boldsymbol{\sigma}_1, p_1)\|_0 \Big\}. \end{aligned} \tag{3.15}$$

Proof. We only provide the main ideas of the proof, which makes use of the classical Galerkin procedure (cf. [13, 23]). More precisely, following the same steps adopted in [15, Lemma 3.2], we first consider a family of finite dimensional subspaces $\{\mathbb{X}^{\mathrm{sym}}_n\}_{n\in\mathbb{N}}$ of $\mathbb{X}^{\mathrm{sym}}$ such that

$$\lim_{n\to\infty} \inf_{(\boldsymbol{\tau}_n, q_n)\in\mathbb{X}^{\mathrm{sym}}_n} \|(\boldsymbol{\tau}, q) - (\boldsymbol{\tau}_n, q_n)\| = 0 \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\mathrm{sym}}.$$

Next, we denote by $(\boldsymbol{\sigma}_{0,n}, p_{0,n})$ the $(\mathbb{X}^{\mathrm{sym}}, \|\cdot\|)$-orthogonal projection of $(\boldsymbol{\sigma}_0, p_0)$ onto $\mathbb{X}^{\mathrm{sym}}_n$ and by $(\boldsymbol{\sigma}_{1,n}, p_{1,n})$ the $(\mathbb{H}^{\mathrm{sym}}, \|\cdot\|_0)$-orthogonal projection of $(\boldsymbol{\sigma}_1, p_1)$ onto $\mathbb{X}^{\mathrm{sym}}_n$. Then, it is easy to show, by using the classical ODE theory, that the problem:

Find $(\boldsymbol{\sigma}_n, p_n) \in \mathcal{C}^1(\mathbb{X}^{\mathrm{sym}})$ such that

$$\begin{aligned} &\Big((\ddot{\boldsymbol{\sigma}}_n, \ddot{p}_n)(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}_n, p_n)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\mathrm{sym}}, \\ &\big(\boldsymbol{\sigma}_n(0), p_n(0)\big) = (\boldsymbol{\sigma}_{0,n}, p_{0,n}), \qquad \big(\dot{\boldsymbol{\sigma}}_n(0), \dot{p}_n(0)\big) = (\boldsymbol{\sigma}_{1,n}, p_{1,n}), \end{aligned} \tag{3.16}$$

admits a unique solution. Furthermore, since $(\cdot,\cdot)_{\mathcal{C}}$ and $A(\cdot,\cdot)$ are symmetric bilinear forms, taking formally $(\boldsymbol{\tau}, q) = (\dot{\boldsymbol{\sigma}}_n(t), \dot{p}_n(t))$ in (3.16) gives

$$\dot{\mathcal{E}}\big((\boldsymbol{\sigma}_n, p_n)\big)(t) \;=\; -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\, \dot{\boldsymbol{\sigma}}_n(t)\big)_{0,\Omega_{\mathrm{S}}}, \tag{3.17}$$

where the energy functional $\mathcal{E}$ is defined by

$$\mathcal{E}\big((\boldsymbol{\tau}, q)\big)(t) := \frac{1}{2}\Big((\dot{\boldsymbol{\tau}}, \dot{q})(t), (\dot{\boldsymbol{\tau}}, \dot{q})(t)\Big)_{\mathcal{C}} + \frac{1}{2} A\Big((\boldsymbol{\tau}, q)(t), (\boldsymbol{\tau}, q)(t)\Big) \quad \forall\, (\boldsymbol{\tau}, q) \in W^{1,\infty}(\mathbb{X}). \tag{3.18}$$

In this way, integrating (3.17) on $[0,t]$ and using the time regularity assumption on $\boldsymbol{f}$ to perform an integration by parts, we find that

$$\begin{aligned} \mathcal{E}\big((\boldsymbol{\sigma}_n, p_n)\big)(t) \;=\; \mathcal{E}\big((\boldsymbol{\sigma}_n, p_n)\big)(0) \,+\, \int_0^t \big(\dot{\boldsymbol{f}}(s), \mathbf{div}\, \boldsymbol{\sigma}_n(s)\big)_{0,\Omega_{\mathrm{S}}} \mathrm{d}s \\ -\, \big(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\sigma}_n(t)\big)_{0,\Omega_{\mathrm{S}}} \,+\, \big(\boldsymbol{f}(0), \mathbf{div}\, \boldsymbol{\sigma}_{0,n}\big)_{0,\Omega_{\mathrm{S}}}, \end{aligned}$$

from which, employing the Cauchy-Schwarz inequality, the Sobolev embedding $\mathrm{W}^{1,1}(\mathrm{L}^2(\Omega)^n) \hookrightarrow \mathcal{C}^0(\mathrm{L}^2(\Omega)^n)$ (see [24, Lemma 7.1]), and the continuous dependence result for (3.16), we deduce that

$$\operatorname*{ess\,sup}_{[0,T]} \mathcal{E}\big((\boldsymbol{\sigma}_n, p_n)\big)^{1/2}(t) \leq C_1 \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_{0,n}, p_{0,n})\| + \|(\boldsymbol{\sigma}_{1,n}, p_{1,n})\|_0 \Big\}. \tag{3.19}$$

It follows now easily from the last estimate and (3.18) that

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol\sigma}_n,\dot p_n)(t)\|_{0,\mathcal{C}} + \operatorname*{ess\,sup}_{[0,T]}\Big(\|(\boldsymbol\sigma_n,p_n)(t)\|_{0,\mathcal{C}}^2 + A\Big((\boldsymbol\sigma_n,p_n)(t),(\boldsymbol\sigma_n,p_n)(t)\Big)\Big)^{1/2} \\
&\qquad \leq\, C_2\left\{\|\boldsymbol f\|_{\mathrm{W}^{1,1}(\boldsymbol L^2(\Omega_{\mathrm S}))} + \|(\boldsymbol\sigma_{0,n},p_{0,n})\| + \|(\boldsymbol\sigma_{1,n},p_{1,n})\|_0\right\}.
\end{aligned}
\tag{3.20}
$$

Finally, using (3.14) and the fact that $\|(\boldsymbol\sigma_{0,n},p_{0,n})\|$ and $\|(\boldsymbol\sigma_{1,n},p_{1,n})\|_0$ are bounded by $\|(\boldsymbol\sigma_0,p_0)\|$ and $\|(\boldsymbol\sigma_1,p_1)\|_0$, respectively, yield

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{[0,T]} \|(\boldsymbol\sigma_n,p_n)(t)\| + \operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol\sigma}_n,\dot p_n)(t)\|_{0,\mathcal{C}} \\
&\qquad \leq\, C_3\left\{\|\boldsymbol f\|_{\mathrm{W}^{1,1}(\boldsymbol L^2(\Omega_{\mathrm S}))} + \|(\boldsymbol\sigma_0,p_0)\| + \|(\boldsymbol\sigma_1,p_1)\|_0\right\}.
\end{aligned}
\tag{3.21}
$$

It is clear from (3.21) that $(\dot{\boldsymbol\sigma}_n,\dot p_n)_n$ and $(\boldsymbol\sigma_n,p_n)_n$ are uniformly bounded in the spaces $\mathrm{L}^\infty(\mathbb{H}^{\mathrm{sym}})$ and $\mathrm{L}^\infty(\mathbb{X}^{\mathrm{sym}})$, respectively, and hence, a classical procedure (cf. [15, Lemma 3.2]) shows that the sequence $\Big\{(\boldsymbol\sigma_n,p_n)\Big\}_{n\in\mathbb N}$ converges to a solution $(\boldsymbol\sigma,p)\in \mathrm{L}^\infty(\mathbb{X}^{\mathrm{sym}})\cap \mathrm{W}^{1,\infty}(\mathbb{H}^{\mathrm{sym}})$ of (3.11). Finally, taking the limit in (3.21) we arrive at the required estimate (3.15), whereas the uniqueness of solution follows from a standard procedure (cf. [13, 23] or [15, Lemma 3.3]). □

At this point we remark that, following [23, Section 11.2.4], it can be shown that the solution $(\boldsymbol\sigma,p)$ to problem (3.11) is actually in $\mathcal{C}^0(\mathbb{X}^{\mathrm{sym}})\cap\mathcal{C}^1(\mathbb{H}^{\mathrm{sym}})$. In turn, it is important to notice that the kernel of the seminorm $A\Big((\boldsymbol\tau,q),(\boldsymbol\tau,q)\Big)^{1/2}$ is given by

$$
\mathbb{K} := \Big\{(\boldsymbol\tau,\xi)\in\mathbb{X}_c^{\mathrm{sym}};\quad \mathbf{div}\,\boldsymbol\tau = 0\Big\},
$$

where $\mathbb{X}_c^{\mathrm{sym}} := \mathbb{X}_c\cap\mathbb{X}^{\mathrm{sym}}$ with $\mathbb{X}_c := \{(\boldsymbol\tau,\xi)\in\mathbb{X};\quad \xi=\text{constant}\}$. Finally, the orthogonal of $\mathbb K$ in $\mathbb{X}^{\mathrm{sym}}$ with respect to the inner product $(\cdot,\cdot)_{\mathcal C}$ is denoted

$$
\mathbb{K}^\perp := \Big\{(\boldsymbol\sigma,p)\in\mathbb{X}^{\mathrm{sym}};\quad \Big((\boldsymbol\sigma,p),(\boldsymbol\tau,\xi)\Big)_{\mathcal C} = 0\quad \forall(\boldsymbol\tau,\xi)\in\mathbb K\Big\}.
$$

On the other hand, the existence of a constant $\beta_0>0$ such that

$$
\sup_{\substack{\boldsymbol\tau\in\boldsymbol{\mathcal W}\\ \boldsymbol\tau\boldsymbol n\,=\,0\ \text{on}\ \Sigma}} \frac{(\boldsymbol s,\boldsymbol\tau)_{0,\Omega_{\mathrm S}} + (\boldsymbol v,\mathbf{div}\,\boldsymbol\tau)_{0,\Omega_{\mathrm S}}}{\|\boldsymbol\tau\|_{\mathrm H(\mathbf{div},\Omega_{\mathrm S})}} \,\geq\, \beta_0\left\{\|\boldsymbol v\|_{0,\Omega_{\mathrm S}} + \|\boldsymbol s\|_{0,\Omega_{\mathrm S}}\right\}
\tag{3.22}
$$

for all $(\boldsymbol v,\boldsymbol s)\in \mathrm{L}^2(\Omega_{\mathrm S})^n\times[\mathrm{L}^2(\Omega_{\mathrm S})]^{n\times n}_{\mathrm{skew}}$, constitutes a crucial inf-sup condition in the analysis of the mixed formulation of the elastostatic problem with reduced symmetry (cf. [4, 8]). Indeed, as we show next, it plays an essential role in the recovery of the displacement field $\boldsymbol u$ from $(\boldsymbol\sigma,p)$. More precisely, we now introduce the linear operator $\mathbf D:\mathbb K^\perp\to \mathrm{L}^2(\Omega_{\mathrm S})^n\times[\mathrm{L}^2(\Omega_{\mathrm S})]^{n\times n}_{\mathrm{skew}}$ defined, for any $(\boldsymbol\sigma,p)\in\mathbb K^\perp$, by the unique solution $(\boldsymbol u,\boldsymbol r):=\mathbf D(\boldsymbol\sigma,p)\in \mathrm{L}^2(\Omega_{\mathrm S})^n\times[\mathrm{L}^2(\Omega_{\mathrm S})]^{n\times n}_{\mathrm{skew}}$ of

$$
(\mathbf{div}\,\boldsymbol\tau,\boldsymbol u)_{0,\Omega_{\mathrm S}} + (\boldsymbol r,\boldsymbol\tau)_{0,\Omega_{\mathrm S}} = -\Big((\boldsymbol\sigma,p),(\boldsymbol\tau,\xi)\Big)_{\mathcal C}\quad \forall(\boldsymbol\tau,\xi)\in\mathbb X_c.
\tag{3.23}
$$

The operator $\mathbf D$ is well-defined by virtue of Theorem 2.1 and (3.22). In fact, the functional on the right hand side of (3.23), that is $\mathbb X_c\ni(\boldsymbol\tau,\xi)\mapsto -\Big((\boldsymbol\sigma,p),(\boldsymbol\tau,\xi)\Big)_{\mathcal C}$, belongs to the polar of $\mathbb K$ in $(\mathbb X_c)'$, and the inf-sup condition

$$
\sup_{(\boldsymbol\tau,\xi)\in\mathbb X_c} \frac{(\boldsymbol s,\boldsymbol\tau)_{0,\Omega_{\mathrm S}} + (\mathbf{div}\,\boldsymbol\tau,\boldsymbol v)_{0,\Omega_{\mathrm S}}}{\|(\boldsymbol\tau,\xi)\|} \,\geq\, \beta_0\left\{\|\boldsymbol v\|_{0,\Omega_{\mathrm S}} + \|\boldsymbol s\|_{0,\Omega_{\mathrm S}}\right\}
\tag{3.24}
$$

for all $(\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$, is a direct consequence of (3.22). Further properties concerning the range of $\mathbf{D}$ in $\mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$ are provided by the following Lemma.

Lemma 3.2. *Given $(\boldsymbol{\sigma}, p) \in \mathbb{X}^{sym}$, the following two statements are equivalent:*

i) $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$

ii) *There exists a unique $\boldsymbol{u} \in \mathrm{H}^1(\Omega_\mathrm{S})^n$ with $\boldsymbol{u}|_{\Gamma_\mathrm{D}} = \mathbf{0}$, such that $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$,*

$$\int_\Sigma \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\rho_\mathrm{F} c^2}\int_{\Omega_\mathrm{F}} p = 0, \tag{3.25}$$

and $\mathbf{D}(\boldsymbol{\sigma}, p) = (\boldsymbol{u}, \boldsymbol{r})$, where $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2$.

Proof. Given $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$, we first let $(\boldsymbol{u}, \boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma}, p)$ according to (3.23). Then, for each $\boldsymbol{\varphi} \in [\mathcal{D}(\Omega_\mathrm{S})]^{n\times n}$ we have that $(\boldsymbol{\tau}, 0) \in \mathbb{X}_c$, which replaced into (3.23) yields $\nabla\boldsymbol{u} = \mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r} \in \mathrm{L}^2(\Omega_\mathrm{S})^{n\times n}$. From this identity and the fact that $\mathcal{C}^{-1}\boldsymbol{\sigma}$ is symmetric (because $\boldsymbol{\sigma}$ is), we readily deduce that there hold $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{C}^{-1}\boldsymbol{\sigma}$ and $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2$. In turn, testing now (3.23) with $(\boldsymbol{\tau}, 0) \in \mathbb{X}_c$ and $(\boldsymbol{\tau}, 1) \in \mathbb{X}_c^{\mathrm{sym}}$, and integrating by parts in both cases, we obtain the boundary condition $\boldsymbol{u}|_{\Gamma_\mathrm{D}} = \mathbf{0}$ and (3.25), respectively. Conversely, given $(\boldsymbol{\sigma}, p) \in \mathbb{X}^{\mathrm{sym}}$ such that ii) holds true, we set the tensor $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2 \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$ and observe that $\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r} - \nabla\boldsymbol{u} = \mathbf{0}$. Hence, given $(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c$, we test the foregoing equation with $\boldsymbol{\tau}$, integrate by parts in Ω_S, and use (3.25), to find

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u})_{0,\Omega_\mathrm{S}} + (\boldsymbol{r}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} = -(\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} - \frac{1}{\rho_\mathrm{F} c^2}(p, \xi)_{0,\Omega_\mathrm{F}}\,. \tag{3.26}$$

Finally, since the left hand side of (3.26) vanishes for $(\boldsymbol{\tau}, \xi) \in \mathbb{K}$, we conclude from there that $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$ and $(\boldsymbol{u}, \boldsymbol{r}) = \mathbf{D}(\boldsymbol{\sigma}, p)$. □

The following result establishes the relation between the solution $(\boldsymbol{\sigma}, p)$ of (3.11) and the solution of the displacement-pressure formulation of the fluid-structure interaction problem.

Theorem 3.2. *Assume that the initial data of problem* (3.11) *are such that $(\boldsymbol{\sigma}_0, p_0)$, $(\boldsymbol{\sigma}_1, p_1) \in \mathbb{K}^\perp$, and let $(\boldsymbol{u}_0, \boldsymbol{r}_0) := \mathbf{D}(\boldsymbol{\sigma}_0, p_0)$ and $(\boldsymbol{u}_1, \boldsymbol{r}_1) := \mathbf{D}(\boldsymbol{\sigma}_1, p_1)$. If $(\boldsymbol{\sigma}, p)$ is the solution of* (3.11) *then the pair $(\boldsymbol{u}, p)$, with*

$$\boldsymbol{u}(t) \;:=\; \int_0^t \Big\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z)\Big)\,dz\Big\}\,ds \;+\; \boldsymbol{u}_0 \;+\; t\boldsymbol{u}_1, \tag{3.27}$$

solves the displacement-pressure formulation of the fluid-structure interaction problem,

$$\rho_\mathrm{S}\ddot{\boldsymbol{u}} - \mathbf{div}\,\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega_\mathrm{S}\times(0,T] \tag{3.28}$$

$$c^{-2}\ddot{p} - \Delta p = 0 \qquad \text{in } \Omega_\mathrm{F}\times(0,T] \tag{3.29}$$

$$\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} + p\boldsymbol{n} = \mathbf{0} \qquad \text{in } \Sigma\times(0,T] \tag{3.30}$$

$$\frac{\partial p}{\partial\boldsymbol{n}} + \rho_\mathrm{F}\ddot{\boldsymbol{u}} = 0 \qquad \text{in } \Sigma\times(0,T] \tag{3.31}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on } \Gamma_\mathrm{D}\times(0,T], \tag{3.32}$$

$$\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_\mathrm{N}\times(0,T] \tag{3.33}$$

subject to the initial conditions $(\boldsymbol{u}(0), p(0)) = (\boldsymbol{u}_0, p_0)$ and $(\dot{\boldsymbol{u}}(0), \dot{p}(0)) = (\boldsymbol{u}_1, p_1)$.

Proof. Integrating the first equation of (3.11) twice with respect to time we deduce that

$$\begin{aligned}\Big((\boldsymbol{\sigma}(t),p(t)),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} &= \Big((\boldsymbol{\sigma}_0,p_0),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} + t\Big((\boldsymbol{\sigma}_1,p_1),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} \\ &- \int_0^t \left(\int_0^s A\Big((\boldsymbol{\sigma},p)(z),(\boldsymbol{\tau},q)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\,\mathrm{d}z\right)\mathrm{d}s\,,\end{aligned} \tag{3.34}$$

for all $(\boldsymbol{\tau},q)\in\mathbb{X}^{\mathrm{sym}}$. It follows that $(\boldsymbol{\sigma}(t),p(t))\in\mathbb{K}^{\perp}$ for all $t\in[0,T]$, and hence Lemma 3.2 ensures the existence of a unique pair $(\boldsymbol{u}(t),\boldsymbol{r}(t))=\mathbf{D}(\boldsymbol{\sigma}(t),p(t))\in\mathrm{L}^2(\Omega_{\mathrm{S}})^n\times[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$ satisfying $\boldsymbol{u}(t)\in\mathrm{H}^1(\Omega_{\mathrm{S}})^n$ $\boldsymbol{\sigma}(t)=\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})(t)$, $\boldsymbol{u}|_{\Gamma_{\mathrm{D}}}=\mathbf{0}$ and

$$(\boldsymbol{r}(t),\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}(t))_{0,\Omega_{\mathrm{S}}} = -\Big((\boldsymbol{\sigma}(t),p(t)),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} \quad \forall(\boldsymbol{\tau},\xi)\in\mathbb{X}_c. \tag{3.35}$$

On the other hand, we readily obtain from (3.22) the inf-sup condition

$$\sup_{(\boldsymbol{\tau},q)\in\mathbb{X}} \frac{(\boldsymbol{s},\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}}}{\|(\boldsymbol{\tau},q)\|} \geq \beta_0\,\|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \quad \forall\boldsymbol{s}\in[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}.$$

In this way, applying Theorem 2.1, we deduce from (3.34) the existence of a unique $\bar{\boldsymbol{r}}(t)\in[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$ satisfying

$$\begin{aligned}(\bar{\boldsymbol{r}}(t),\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} &= -\Big((\boldsymbol{\sigma}(t),p(t)),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} + \Big((\boldsymbol{\sigma}_0,p_0),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} + t\Big((\boldsymbol{\sigma}_1,p_1),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} \\ &- \int_0^t\left(\int_0^s A\Big((\boldsymbol{\sigma},p)(z),(\boldsymbol{\tau},q)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\,\mathrm{d}z\right)\mathrm{d}s\,,\end{aligned} \tag{3.36}$$

for all $(\boldsymbol{\tau},q)\in\mathbb{X}$. Then, replacing (3.35) in (3.36), yields

$$\begin{aligned}(\bar{\boldsymbol{r}}(t),\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} &= (\boldsymbol{r}(t),\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}(t))_{0,\Omega_{\mathrm{S}}} + \Big((\boldsymbol{\sigma}_0,p_0),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} \\ &+ t\Big((\boldsymbol{\sigma}_1,p_1),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} - \int_0^t\left(\int_0^s A\Big((\boldsymbol{\sigma},p)(z),(\boldsymbol{\tau},\xi)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\,\mathrm{d}z\right)\mathrm{d}s \\ &= \big(\boldsymbol{r}(t)-\boldsymbol{r}_0-t\boldsymbol{r}_1,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} + \big(\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}(t)-\boldsymbol{u}_0-t\boldsymbol{u}_1\big)_{0,\Omega_{\mathrm{S}}} \\ &- \int_0^t\left(\int_0^s \big(\rho_{\mathrm{S}}^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}(z)+\rho_{\mathrm{S}}^{-1}\boldsymbol{f}(z),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\,\mathrm{d}z\right)\mathrm{d}s\end{aligned} \tag{3.37}$$

for all $(\boldsymbol{\tau},\xi)\in\mathbb{X}_c$, from which it follows that

$$\begin{aligned}&\left(\boldsymbol{u}(t) - \int_0^t\left\{\int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z)+\boldsymbol{f}(z)\Big)\,\mathrm{d}z\right\}\mathrm{d}s \,-\, \boldsymbol{u}_0 \,-\, t\boldsymbol{u}_1,\,\mathbf{div}\,\boldsymbol{\tau}\right)_{0,\Omega_{\mathrm{S}}} \\ &\quad + \quad \big(\boldsymbol{r}(t)-\boldsymbol{r}_0-t\boldsymbol{r}_1-\bar{\boldsymbol{r}}(t),\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} = 0 \qquad \forall\boldsymbol{\tau}\in\mathbb{X}_c\,.\end{aligned}$$

Thus, the foregoing equation and the inf-sup condition (3.24) imply (3.27) and

$$\bar{\boldsymbol{r}}(t) = \boldsymbol{r}(t)-\boldsymbol{r}_0\,-\,t\,\boldsymbol{r}_1 \qquad \forall\,t\in[0,T]\,. \tag{3.38}$$

Finally, differentiating (3.27) twice with respect to time we obtain the motion equation (3.28), whereas substituting (3.28) back into (3.9) yields (3.31), which completes the proof. □

We end this section remarking that, after differentiating (3.36) twice with respect to time and using (3.38), we find that

$$\Big((\ddot{\boldsymbol{\sigma}},\ddot{p})(t),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} + \big(\ddot{\boldsymbol{r}}(t),\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} + A\Big((\boldsymbol{\sigma},p)(t),(\boldsymbol{\tau},q)\Big) = -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} \quad \forall(\boldsymbol{\tau},q)\in\mathbb{X}\,. \tag{3.39}$$

This identity is employed later on in Section 6.

4 The discrete problem

We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega} = \bar{\Omega}_{\rm S} \cup \bar{\Omega}_{\rm F}$, into triangles/tetrahedra K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. In what follows, we assume that each triange/tetrahedron of $\mathcal{T}_h$ is contained either in $\bar{\Omega}_{\rm S}$ or in $\bar{\Omega}_{\rm F}$, and denote

$$\mathcal{T}_h^{\rm S} := \left\{K \in \mathcal{T}_h; \quad K \subset \bar{\Omega}_{\rm S}\right\} \quad \text{and} \quad \mathcal{T}_h^{\rm F} := \left\{K \in \mathcal{T}_h; \quad K \subset \bar{\Omega}_{\rm F}\right\}.$$

Moreover, we let Σ_h be the triangulation induced by $\mathcal{T}_h$ on Σ. Next, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^d$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D. The space of piecewise polynomial functions of degree at most m associated with $\mathcal{T}_h^*$, $* \in \{\rm S, F\}$, is denoted by

$$\mathcal{P}_m(\mathcal{T}_h^*) := \left\{v \in L^2(\Omega_*); \quad v|_K \in \mathcal{P}_m(K), \quad \forall K \in \mathcal{T}_h^*\right\}.$$

Similarly, $\mathcal{P}_m(\Sigma_h) := \left\{\phi \in L^2(\Sigma); \quad \phi|_T \in \mathcal{P}_m(T), \quad \forall T \in \Sigma_h\right\}$. In addition, for $k \geq 1$, the finite element spaces

$$\boldsymbol{\mathcal{W}}_h := \mathcal{P}_k(\mathcal{T}_h^{\rm S})^{n\times n} \cap \boldsymbol{\mathcal{W}}, \quad \boldsymbol{\mathcal{Q}}_h := \mathcal{P}_{k-1}(\mathcal{T}_h^{\rm S})^{n\times n} \cap [\mathrm{L}^2(\Omega_{\rm S})]^{n\times n}_{\rm skew}, \quad \text{and} \quad \mathcal{U}_h := \mathcal{P}_{k-1}(\mathcal{T}_h^{\rm S})^n,$$

correspond to the $k^{\rm th}$-order element of the Arnold-Falk-Winther (AFW) family introduced for the mixed formulation of elastostatic problem with reduced symmetry. It is shown in [3, Theorem 11.9] that the discrete inf-sup condition

$$\sup_{\substack{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h \\ \boldsymbol{\tau n}\,=\,0 \text{ on } \Sigma}} \frac{(\boldsymbol{\tau}, \boldsymbol{s})_{0,\Omega_{\rm S}} + (\mathbf{div}\,\tau, \boldsymbol{v})_{0,\Omega_{\rm S}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta_0^*\Big\{ \|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega} \Big\}, \quad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \tag{4.1}$$

holds true for a constant $\beta_0^* > 0$ independent of h. It is important to notice that the weakly symmetric version

$$\boldsymbol{\mathcal{W}}_h^{\rm sym} = \left\{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h; \quad \int_{\Omega_{\rm S}} \boldsymbol{\tau}_h : \boldsymbol{s} = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\right\}$$

of $\boldsymbol{\mathcal{W}}_h$ is not a subspace of $\boldsymbol{\mathcal{W}}^{\rm sym}$. Moreover, it is generally not possible to construct a basis for the finite element space $\boldsymbol{\mathcal{W}}_h^{\rm sym}$. Hence, in all what follows, we implicitly assume that a Lagrange multiplier is needed in order to deal, from the practical point of view, with the weak symmetry constraint defining $\boldsymbol{\mathcal{W}}_h^{\rm sym}$. We deliberately have chosen here to hide this additional variable (which is none other than the discrete counterpart of the rotation $\boldsymbol{r}$) for economy in notations.

We approximate the pressure in the usual Lagrange finite element space $V_h := \mathcal{P}_k(\mathcal{T}_h^{\rm F}) \cap \mathrm{H}^1(\Omega_{\rm F})$. We recall some well-known approximation properties of the finite element spaces introduced above. Given $s > 0$, it is well-known that the usual $k^{\rm th}$-order Brezzi-Douglas-Marini (BDM) interpolation operator (see [10]) $\boldsymbol{\Pi}_h : [\mathrm{H}^s(\Omega_{\rm S})]^{n\times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$ satisfies for $0 < s \leq 1/2$ the error estimate

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega_{\rm S}} \leq Ch^s\Big\{ \|\boldsymbol{\tau}\|_{s,\Omega_{\rm S}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega_{\rm S}} \Big\} \qquad \forall \boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\rm S})]^{n\times n} \cap \boldsymbol{\mathcal{W}}. \tag{4.2}$$

For more regular functions $\boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\rm S})]^{n\times n}$ with $s > 1/2$, it holds

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega_{\rm S}} \leq Ch^{\min\{s,k+1\}}\,\|\boldsymbol{\tau}\|_{s,\Omega_{\rm S}}, \qquad \forall \boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\rm S})]^{n\times n}. \tag{4.3}$$

Moreover, we have the commuting diagram properties

$$\mathbf{div}(\boldsymbol{\Pi}_h\boldsymbol{\tau}) = U_h(\mathbf{div}\,\boldsymbol{\tau}) \quad \text{and} \quad (\boldsymbol{\Pi}_h\boldsymbol{\sigma})\boldsymbol{n} = \boldsymbol{\pi}_h(\boldsymbol{\sigma n}) \tag{4.4}$$

for all $\boldsymbol{\tau} \in \mathrm{H}^s(\Omega_\mathrm{S})^{n\times n} \cap \mathbf{H}(\mathbf{div}, \Omega_\mathrm{S})$, $s > 0$, where $U_h : \mathrm{L}^2(\Omega_\mathrm{S})^n \to \mathcal{U}_h$ is the $\mathrm{L}^2(\Omega_\mathrm{S})^n$-orthogonal projector, π_h is the $\mathrm{L}^2(\Sigma)$-orthogonal projector onto $\mathcal{P}_k(\Sigma_h)$, and $\boldsymbol{\pi}_h$ is the vectorial version of π_h. In addition, we denote by $\boldsymbol{R}_h : [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $[\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}$-norm, and let $\Pi_h : \mathrm{H}^1(\Omega_\mathrm{F}) \to V_h$ be the operator that, given $p \in \mathrm{H}^1(\Omega_\mathrm{F})$, is uniquely characterized by

$$(\nabla \Pi_h p, \nabla q)_{0,\Omega_\mathrm{F}} = (\nabla p, \nabla q)_{0,\Omega_\mathrm{F}} \quad \forall q \in V_h \quad \text{and} \quad \int_{\Omega_\mathrm{F}} \Pi_h p = 0\,. \tag{4.5}$$

Then, there hold

$$\|\boldsymbol{r} - \boldsymbol{R}_h \boldsymbol{r}\|_{0,\Omega_\mathrm{S}} \le C h^{\min\{s,k\}} \|\boldsymbol{r}\|_{s,\Omega_\mathrm{S}} \qquad \forall \boldsymbol{r} \in [\mathrm{H}^s(\Omega_\mathrm{S})]^{n\times n} \cap [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}, \tag{4.6}$$

$$\|\boldsymbol{v} - U_h \boldsymbol{v}\|_{0,\Omega_\mathrm{S}} \le C h^{\min\{s,k\}} \|\boldsymbol{v}\|_{s,\Omega_\mathrm{S}} \qquad \forall \boldsymbol{v} \in \mathrm{H}^s(\Omega_\mathrm{S})^n, \tag{4.7}$$

$$|p - \Pi_h p|_{1,\Omega_\mathrm{F}} \le C h^{\min\{s,k\}} \|p\|_{1+s,\Omega_\mathrm{F}} \qquad \forall p \in \mathrm{H}^{1+s}(\Omega_\mathrm{F}), \tag{4.8}$$

$$\|\boldsymbol{\varphi} - \pi_h \boldsymbol{\varphi}\|_{0,\Sigma} \le C h^{\min\{s,k+1\}} \Big(\sum_{e\in\Sigma_h} \|\boldsymbol{\varphi}\|^2_{s,e}\Big)^{1/2} \qquad \forall \boldsymbol{\varphi} \in \prod_{e\in\Sigma_h} \mathrm{H}^s(e)^n\,. \tag{4.9}$$

Furthermore, we introduce the discrete energy space

$$\mathbb{X}_h := \{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}}_h \times V_h; \quad \boldsymbol{\tau n} + p\boldsymbol{n} = 0 \qquad \text{on } \Sigma\}\,,$$

and its subspace $\mathbb{X}_{h,c} = \{(\boldsymbol{\tau}, \xi) \in \mathbb{X}_h; \quad \xi = \text{constant}\}$. We also consider their weakly symmetric versions

$$\mathbb{X}_h^{\mathrm{sym}} := \left\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}} \times V_h; \quad \boldsymbol{\tau n} + p\boldsymbol{n} = 0 \qquad \text{on } \Sigma\right\},$$

and $\mathbb{X}_{h,c}^{\mathrm{sym}} := \mathbb{X}_{h,c} \cap \mathbb{X}_h^{\mathrm{sym}}$, respectively. The kernel $\mathbb{K}_h$ of the bilinear form A in $\mathbb{X}_h^{\mathrm{sym}}$ is given by

$$\mathbb{K}_h := \left\{(\boldsymbol{\tau}, \xi) \in \mathbb{X}_{h,c}^{\mathrm{sym}}; \quad \mathbf{div}\, \boldsymbol{\tau} = 0\right\}\,.$$

In turn, we set

$$\mathbb{K}_h^\perp := \left\{(\boldsymbol{\sigma}_h, p_h) \in \mathbb{X}_h^{\mathrm{sym}}; \quad \Big((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} = 0 \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{K}_h\right\}\,,$$

and notice that, in general, neither $\mathbb{K}_h \subseteq \mathbb{K}$ nor $\mathbb{K}_h^\perp \subseteq \mathbb{K}^\perp$.

According to the above discussions and notations, we consider in what follows the following semi-discrete Galerkin discretization of (3.11):

$$\begin{aligned}
&\text{Find } (\boldsymbol{\sigma}_h, p_h) \in \mathcal{C}^1(\mathbb{X}_h^{\mathrm{sym}}) \text{ such that} \\
&\Big((\ddot{\boldsymbol{\sigma}}_h, \ddot{p}_h)(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}_h, p_h)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_\mathrm{S}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\mathrm{sym}} \\
&(\boldsymbol{\sigma}_h(0), p_h(0)) = (\boldsymbol{\sigma}_{0,h}, p_{0,h}), \qquad (\dot{\boldsymbol{\sigma}}_h(0), \dot{p}_h(0)) = (\boldsymbol{\sigma}_{1,h}, p_{1,h}),
\end{aligned} \tag{4.10}$$

where the discrete initial data $(\boldsymbol{\sigma}_{0,h}, p_{0,h}) \in \mathbb{K}_h^\perp$ and $(\boldsymbol{\sigma}_{1,h}, p_{1,h}) \in \mathbb{K}_h^\perp$ are given approximations of $(\boldsymbol{\sigma}_0, p_0)$ and $(\boldsymbol{\sigma}_1, p_1)$, respectively.

We end this section by remarking that, exactly as in [15, Section 5.3], [16], and the proof of Theorem 3.1, the well-posedness of (4.10) also follows from classical ODE theory. We omit further details and refer to those works or to Theorem 3.1. In turn, similarly as in [15, Section 5], the corresponding convergence analysis is carried out later on in Section 6 by applying the properties of the continuous and discrete versions of the auxiliary operator to be introduced in the following section.

5 An auxiliary operator

As already announced, and in order to facilitate the convergence analysis of the Galerkin scheme (4.10), in this section we first introduce a suitable auxiliary operator and a discrete approximation of it, and then we derive the corresponding error estimate between them.

5.1 The continuous version

In what follows we define an operator $\Xi : \mathbb{X} \to \mathbb{X}^{\mathrm{sym}}$ whose restriction to $\mathbb{X}^{\mathrm{sym}}$ coincides with the $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal projection of $\mathbb{X}^{\mathrm{sym}}$ onto $\mathbb{K}^{\perp}$. More precisely, given $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, we first let

$$\bar{p} = p - \frac{1}{\Omega_{\mathrm{F}}} \int_{\Omega_{\mathrm{F}}} p \quad \text{in} \quad \Omega_{\mathrm{F}}\,, \tag{5.1}$$

and then define $\Xi(\boldsymbol{\sigma}, p) := (\boldsymbol{\sigma}^*, p^*)$, where

$$p^* \;:=\; \bar{p} \;-\; \frac{\rho_{\mathrm{F}} c^2}{|\Omega_{\mathrm{F}}|} \int_{\Sigma} \boldsymbol{u}^* \cdot \boldsymbol{n} \quad \text{in} \quad \Omega_{\mathrm{F}} \tag{5.2}$$

and the pair $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*)$ is characterized by the set of equations,

$$\begin{gathered} \mathcal{C}^{-1}\boldsymbol{\sigma}^* = \boldsymbol{\varepsilon}(\boldsymbol{u}^*) \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \boldsymbol{\sigma}^* = (\boldsymbol{\sigma}^*)^{\mathtt{t}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \mathbf{div}\,\boldsymbol{\sigma}^* = \mathbf{div}\,\boldsymbol{\sigma} \quad \text{in} \quad \Omega_{\mathrm{S}} \\ \boldsymbol{\sigma}^*\boldsymbol{n} = -p^*\boldsymbol{n} \quad \text{on} \quad \Sigma\,, \quad \boldsymbol{\sigma}^*\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{N}}\,, \quad \boldsymbol{u}^* = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{D}}\,. \end{gathered} \tag{5.3}$$

Note from (5.1) and (5.2) that there holds

$$\int_{\Sigma} \boldsymbol{u}^* \cdot \boldsymbol{n} + \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} p^* \;=\; 0\,. \tag{5.4}$$

Actually, the constant value given by the second term on the right hand side of (5.2) has been chosen so that (5.4) holds. Then, motivated by the Neumann boundary condition on Σ, we now consider the spaces

$$\boldsymbol{\mathcal{Y}} \;:=\; \big\{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \quad \boldsymbol{\tau}\boldsymbol{n} \in \mathrm{L}^2(\Sigma)^n\big\} \quad \text{and} \quad \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \;:=\; \boldsymbol{\mathcal{Y}} \cap \boldsymbol{\mathcal{W}}^{\mathrm{sym}}\,,$$

both endowed with the graph norm

$$\|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{Y}}} := \|\boldsymbol{\tau}\|^2_{\mathbf{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|\boldsymbol{\tau}\boldsymbol{n}\|^2_{0,\Sigma}\,. \tag{5.5}$$

Hence, with these notations at hand, and realizing that the auxiliary unknown $\boldsymbol{\psi}^* := \boldsymbol{u}^*|_{\Sigma}$ becomes the Lagrange multiplier corresponding to the weak imposition of the aforementioned condition on Σ, we arrive at the following dual-mixed variational formulation of problem (5.3)

Find $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \mathrm{L}^2(\Sigma)^n$ such that

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}^*, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \\ (\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} && \forall \boldsymbol{v} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n, \\ (\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} - \frac{c^2\rho_{\mathrm{F}}}{|\Omega_{\mathrm{F}}|} \left\{\int_{\Sigma} \boldsymbol{\psi}^* \cdot \boldsymbol{n}\right\} \left\{\int_{\Sigma} \boldsymbol{\varphi} \cdot \boldsymbol{n}\right\} &= -(\bar{p}, \boldsymbol{\varphi} \cdot \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n\,. \end{aligned} \tag{5.6}$$

A further simplification is obtained by taking $\boldsymbol{\varphi} = \boldsymbol{n}$ in the last equation of (5.6), which yields

$$\int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n} = \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|} \left\{ (\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_\Sigma \bar{p} \right\} . \tag{5.7}$$

As a consequence of the foregoing identity, and defining the space

$$\Psi := \left\{ \boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n ; \quad \int_\Sigma \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \right\} ,$$

we can reformulate (5.6) as follows:

Find $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \operatorname{\mathbf{div}} \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau} \boldsymbol{n})_{0,\Sigma} &= -\frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \int_\Sigma \bar{p}\, (\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} , \\ (\operatorname{\mathbf{div}} \boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} &= (\operatorname{\mathbf{div}} \boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} && \forall \boldsymbol{v} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n , \\ (\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\bar{p}, \boldsymbol{\varphi} \cdot \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\varphi} \in \Psi , \end{aligned} \tag{5.8}$$

where

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) := (\mathcal{C}^{-1} \boldsymbol{\sigma}^*, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} (\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} (\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} .$$

More precisely, the following lemma establishes the equivalence between (5.6) and (5.8).

Lemma 5.1. *Let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ be a solution of (5.6), and let*

$$\boldsymbol{\psi}_0^* := \boldsymbol{\psi}^* - \frac{1}{|\Sigma|} \left\{ \int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n} \right\} \boldsymbol{n} . \tag{5.9}$$

Then $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ is a solution of (5.8). Conversely, let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ be a solution of (5.8) and let

$$\boldsymbol{\psi}^* := \boldsymbol{\psi}_0^* + \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \left\{ (\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_\Sigma \bar{p} \right\} \boldsymbol{n} . \tag{5.10}$$

Then $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ is solution of (5.6).

Proof. Let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \mathrm{L}^2(\Sigma)^n$ be a solution of (5.6), and define $\boldsymbol{\psi}_0^*$ by (5.9), which belongs to Ψ. Then, replacing $\boldsymbol{\psi}^*$ by $\boldsymbol{\psi}_0^* + \frac{1}{|\Sigma|} \left\{ \int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n} \right\} \boldsymbol{n}$ in the first equation of (5.6) and using (5.7), we deduce the first equation of (5.8). In turn, testing the third equation of (5.6) with $\boldsymbol{\varphi} \in \Psi$, we obtain the third equation of (5.8), and hence $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ becomes a solution of (5.8). Conversely, let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$ be a solution of (5.8), and define $\boldsymbol{\psi}^*$ as indicated in (5.10) (which is suggested by (5.7) and (5.9)). Then, replacing the resulting expression for $\boldsymbol{\psi}_0^*$ in the first equation of (5.8), we arrive at the first equation of (5.6). On the other hand, given $\boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n$, we certainly have that $\boldsymbol{\varphi}_0 := \boldsymbol{\varphi} - \frac{1}{|\Sigma|} (\boldsymbol{\varphi}, \boldsymbol{n})_{0,\Sigma}\, \boldsymbol{n}$ belongs to Ψ. Thus, employing $\boldsymbol{\varphi}_0$ in the third equation of (5.8), and using from (5.10) that there holds (5.7), we obtain the third equation of (5.6), from which we conclude that $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ is solution of (5.6). □

Next, in order to prove that problem (5.8) (or equivalently (5.6)) is well-posed, we need to establish the following inf-sup condition.

Lemma 5.2. *There exists a constant* $\beta_1 >$ *such that*

$$S(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \,:=\, \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{Y}}} \frac{(\boldsymbol{s},\boldsymbol{\tau})_{0,\Omega_{\rm S}} + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\rm S}} + (\boldsymbol{\varphi},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \geq \beta_1\Big\{\, \|\boldsymbol{s}\|_{0,\Omega_{\rm S}} + \|\boldsymbol{v}\|_{0,\Omega_{\rm S}} + \|\boldsymbol{\varphi}\|_{0,\Sigma}\Big\} \tag{5.11}$$

for all $(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \in [\mathrm{L}^2(\Omega_{\rm S})]^{n\times n}_{skew} \times \mathrm{L}^2(\Omega_{\rm S})^n \times \Psi$.

Proof. Given $(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \in [\mathrm{L}^2(\Omega_{\rm S})]^{n\times n}_{\rm skew} \times \mathrm{L}^2(\Omega_{\rm S})^n \times \Psi$, we first observe, thanks to (3.22), that there holds

$$S(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \,\geq\, \sup_{\substack{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}} \\ \boldsymbol{\tau}\boldsymbol{n}\,=\,0 \text{ on } \Sigma}} \frac{(\boldsymbol{s},\boldsymbol{\tau})_{0,\Omega_{\rm S}} + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\rm S}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_{\rm S})}} \,\geq\, C_0 \left\{\, \|\boldsymbol{s}\|_{0,\Omega_{\rm S}} + \|\boldsymbol{v}\|_{0,\Omega_{\rm S}} \,\right\}. \tag{5.12}$$

Next, we let $\boldsymbol{w} \in \mathrm{H}^1(\Omega_{\rm S})^n$ be the unique solution of the (vectorial) Laplace problem

$$\begin{array}{rcll} \mathbf{div}(\nabla\boldsymbol{w}) &=& \mathbf{0} & \text{in } \Omega_{\rm S}\,, \\ \boldsymbol{w} &=& \mathbf{0} & \text{on } \Gamma_{\rm D}\,, \\ (\nabla\boldsymbol{w})\boldsymbol{n} &=& \mathbf{0} & \text{on } \Gamma_{\rm N}\,, \\ (\nabla\boldsymbol{w})\boldsymbol{n} &=& \boldsymbol{\varphi} & \text{on } \Sigma\,, \end{array} \tag{5.13}$$

and define $\bar{\boldsymbol{\sigma}} := \nabla\boldsymbol{w} \in \boldsymbol{\mathcal{Y}}$. It is clear from (5.13) and its associated continuous dependence result that $\mathbf{div}\,\bar{\boldsymbol{\sigma}} = \mathbf{0}$ in $\Omega_{\rm S}$, $\bar{\boldsymbol{\sigma}}\boldsymbol{n} = \boldsymbol{\varphi}$ on Σ, and that there exists $C_1 > 0$, independent of $\boldsymbol{\varphi}$, such that

$$\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}} \leq C_1\, \|\boldsymbol{\varphi}\|_{0,\Sigma}\,.$$

It follows that

$$\begin{aligned} S(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \,&\geq\, \frac{(\boldsymbol{s},\bar{\boldsymbol{\sigma}})_{0,\Omega_{\rm S}} + (\boldsymbol{v},\mathbf{div}\,\bar{\boldsymbol{\sigma}})_{0,\Omega_{\rm S}} + (\boldsymbol{\varphi},\bar{\boldsymbol{\sigma}}\boldsymbol{n})_{0,\Sigma}}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \,=\, \frac{(\boldsymbol{s},\bar{\boldsymbol{\sigma}})_{0,\Omega_{\rm S}} + \|\boldsymbol{\varphi}\|^2_{0,\Sigma}}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \\ &\geq\, \frac{\|\boldsymbol{\varphi}\|^2_{0,\Sigma}}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} - \|\boldsymbol{s}\|_{0,\Omega_{\rm S}} \,\geq\, \frac{1}{C_1}\, \|\boldsymbol{\varphi}\|_{0,\Sigma} - \|\boldsymbol{s}\|_{0,\Omega_{\rm S}}\,. \end{aligned} \tag{5.14}$$

In this way, the inf-sup condition (5.11) is now obtained by multiplying (5.14) by $\frac{C_0}{2}$ and adding the resulting estimate to (5.12). □

We are now in a position to show that (5.8) is well posed.

Lemma 5.3. *There exists a unique* $(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\psi_0^*) \in \boldsymbol{\mathcal{Y}}^{sym} \times \mathrm{L}^2(\Omega_{\rm S})^n \times \Psi$ *solution of* (5.8), *and there exists* $C > 0$, *independent of* λ *and the given* $(\boldsymbol{\sigma},p) \in \mathbb{X}$, *such that*

$$\|(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\psi_0^*)\| \,\leq\, C\, \|(\boldsymbol{\sigma},p)\|\,. \tag{5.15}$$

Proof. We begin by introducing

$$\mathbf{K} := \Big\{\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\rm sym}; \quad (\boldsymbol{v},\mathbf{div},\boldsymbol{\tau})_{0,\Omega_{\rm S}} + (\boldsymbol{\varphi},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall(\boldsymbol{v},\boldsymbol{\varphi}) \in \mathrm{L}^2(\Omega_{\rm S})^n \times \Psi\Big\}\,,$$

which reduces to $\mathbf{K} = \Big\{\boldsymbol{\tau}; \quad (\boldsymbol{\tau},\xi) \in \mathbb{K}\Big\}$. Then, using the estimate (3.14) and the fact that $\boldsymbol{\tau}\boldsymbol{n} = -\xi\boldsymbol{n}$ on Σ, we find that

$$\begin{aligned} a(\boldsymbol{\tau},\boldsymbol{\tau}) &= (\mathcal{C}^{-1}\boldsymbol{\tau},\boldsymbol{\tau})_{0,\Omega_{\rm S}} + \frac{|\Omega_{\rm F}|}{\rho_{\rm F} c^2}\xi^2 = \Big((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)\Big) \\ &\geq\, \alpha\, \|(\boldsymbol{\tau},\xi)\|^2 \,=\, \alpha\left(\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\rm S})} + \xi^2|\Omega_{\rm F}|\right) \,\geq\, \alpha\min\left\{1, \frac{|\Omega_{\rm F}|}{|\Sigma|}\right\} \|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{Y}}} \end{aligned} \tag{5.16}$$

for all $\boldsymbol{\tau} \in \mathbf{K}$. In this way, thanks to the ellipticity property (5.16) and the inf-sup condition (5.11), a straightforward application of the well-known Babuška-Brezzi theory implies the well-posedness of the saddle point problem (5.8) and the continuous dependence estimate (5.15). $\square$

As a consequence of the foregoing lemma, the auxiliary operator Ξ given originally by (5.1), (5.2), and (5.3), is now well defined. Actually, due to the equivalence between (5.6) and (5.8), and the identity (5.7), we can redefine Ξ as

$$\Xi(\boldsymbol{\sigma}, p) := (\boldsymbol{\sigma}^*, p^*) \qquad \forall\, (\boldsymbol{\sigma}, p) \in \mathbb{X}\,, \tag{5.17}$$

where

$$p^* := \bar{p} - \frac{1}{|\Sigma|}\left\{(\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_\Sigma \bar{p}\right\}, \tag{5.18}$$

$\bar{p}$ is given by (5.1), and $\boldsymbol{\sigma}^*$ is the first component of $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\text{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$, the unique solution of (5.8). Moreover, we have the following result.

Lemma 5.4. *There exists a constant $C > 0$, independent of λ, such that*

$$\|\Xi(\boldsymbol{\sigma}, p)\| \le C\, \|(\boldsymbol{\sigma}, p)\| \quad \forall (\boldsymbol{\sigma}, p) \in \mathbb{X}\,.$$

In addition, the operator $\widetilde{\Xi} := \Xi|_{\mathbb{X}^{sym}}$ is the $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal projection in $\mathbb{X}^{sym}$ onto $\mathbb{K}^\perp$.

Proof. The uniform boundedness of Ξ with respect to λ follows directly from the definition of this operator and (5.15). Furthermore, it is straightforward to see that $\widetilde{\Xi}^2 = \widetilde{\Xi}$, and hence we have the stable and direct splitting,

$$\mathbb{X}^{\text{sym}} = \widetilde{\Xi}(\mathbb{X}) + N(\widetilde{\Xi})\,. \tag{5.19}$$

On the other hand, it is clear that $N(\widetilde{\Xi}) = \mathbb{K}$, and the first equation of (5.8) shows that, for any $(\boldsymbol{\sigma}^*, p^*) = \widetilde{\Xi}(\boldsymbol{\sigma}, p)$, with $(\boldsymbol{\sigma}, p)$ arbitrary in $\mathbb{X}$, there holds

$$\Big(\widetilde{\Xi}(\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} = 0 \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{K}\,,$$

which proves that the decomposition (5.19) is $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal and $\widetilde{\Xi}(\mathbb{X}) = \mathbb{K}^\perp$. $\square$

We now notice from (5.11), taking in particular $(\boldsymbol{v}, \boldsymbol{\varphi}) = (\mathbf{0}, \mathbf{0})$, that there also holds

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}} \frac{(\boldsymbol{r}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \ge \beta_1\, \|\boldsymbol{r}\|_{0,\Omega_{\mathrm{S}}} \quad \forall\, \boldsymbol{r} \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}}\,.$$

Hence, bearing in mind that the first equation of (5.8) can be rewritten as

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \mathbf{div}\, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} + \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \int_\Sigma \bar{p}(\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\text{sym}}\,,$$

and that $\boldsymbol{\mathcal{Y}}^{\text{sym}}$ is the kernel of the operator induced by the bilinear form $\mathcal{A} : \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}} \to \mathbb{R}$ defined by $\mathcal{A}(\boldsymbol{\tau}, \boldsymbol{r}) = (\boldsymbol{r}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}}$ $\forall\, (\boldsymbol{\tau}, \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}}$, we can apply Theorem 2.1 to conclude that there exists a unique $\boldsymbol{r}^* \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}}$ such that

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \mathbf{div}\, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} + \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \int_\Sigma \bar{p}(\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = -(\boldsymbol{r}^*, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}\,. \tag{5.20}$$

The following result provides an explicit connection between the operators $\mathbf{D}$ and Ξ.

Lemma 5.5. *Given* $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, *we let* $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}} \times \mathrm{L}^2(\Omega_\mathrm{S})^n \times \mathrm{L}^2(\Sigma)^n$ *and* $\boldsymbol{r}^* \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{skew}$ *be the unique solutions of* (5.6) *and* (5.20)*, respectively, and set* $(\boldsymbol{\sigma}^*, p^*) := \Xi(\boldsymbol{\sigma}, p)$ *(*cf. (5.17)*). Then there hold*

$$(\boldsymbol{u}^*, \boldsymbol{r}^*) = \mathbf{D}(\boldsymbol{\sigma}^*, p^*) \qquad \textit{and} \qquad \boldsymbol{\psi}^* = \boldsymbol{u}^*|_\Sigma\,.$$

Proof. Given $(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c$, that is $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$, $\xi \in \mathbb{R}$, and $\boldsymbol{\tau}\boldsymbol{n} = -\xi\boldsymbol{n}$ on Σ, it is clear that $\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}$. Then, recalling the definition of $\boldsymbol{\psi}_0^*$ (cf. (5.9)), we deduce from (5.20) that

$$(\boldsymbol{r}^*, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}^*)_{0,\Omega_\mathrm{S}} = -(\mathcal{C}^{-1}\boldsymbol{\sigma}^*, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} - (\boldsymbol{\psi}^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} \quad \forall\,(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c\,.$$

Next, from the definition of p^* (cf. (5.18)), the identity (5.7), and the fact that $\boldsymbol{\tau}\boldsymbol{n} = -\xi\boldsymbol{n}$ on Σ, we find after minor algebraic manipulations that

$$(\boldsymbol{r}^*, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}^*)_{0,\Omega_\mathrm{S}} = -\Big((\boldsymbol{\sigma}^*, p^*), (\boldsymbol{\tau}, \xi)\Big)_\mathcal{C} \quad \forall\,(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c\,,$$

which, according to the characterization (3.23) of the operator $\mathbf{D}$, yields $(\boldsymbol{u}^*, \boldsymbol{r}^*) = \mathbf{D}(\boldsymbol{\sigma}^*, p^*)$. Finally, the interpretation $\boldsymbol{\psi}^* = \boldsymbol{u}^*|_\Sigma$ is obtained by integrating by parts in the first equation of (5.6). □

5.2 The discrete version

We now aim to define a discrete version of the operator Ξ. To this end, we introduce the subspace of Ψ given by $\Psi_h := \mathcal{P}_k(\Sigma_h)^n \cap \Psi$, recall the definition of the operator Π_h (cf. (4.5)), and consider the following Galerkin approximation of problem (5.8):

Find $(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}} \times \mathcal{U}_h \times \Psi_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h^*, \boldsymbol{\tau}) + (\boldsymbol{u}_h^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\psi}_{0,h}^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= -\frac{|\Omega_\mathrm{F}|}{\rho_\mathrm{F} c^2 |\Sigma|^2} \int_\Sigma \Pi_h p (\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}}\,,\\ (\mathbf{div}\,\boldsymbol{\sigma}_h^*, \boldsymbol{v})_{0,\Omega_\mathrm{S}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_\mathrm{S}} && \forall \boldsymbol{v} \in \mathcal{U}_h\,,\\ (\boldsymbol{\sigma}_h^*\boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\Pi_h p, \boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\varphi} \in \Psi_h\,. \end{aligned} \tag{5.21}$$

We begin the analysis of (5.21) with the following discrete inf-sup condition.

Lemma 5.6. *There exists a constant* $\beta_1^* > 0$*, independent of* h*, such that*

$$S_h(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) := \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\varphi}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \geq \beta_1^* \Big\{ \|\boldsymbol{s}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{v}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{\varphi}\|_{0,\Sigma} \Big\} \tag{5.22}$$

for all $(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$.

Proof. We proceed very similarly to the proof of Lemma 5.2. In fact, given $(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$, we first realize, thanks to (4.1), that there exists a constant $\beta_0^* > 0$, independent of h, such that

$$S_h(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \geq \sup_{\substack{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h \\ \boldsymbol{\tau}\boldsymbol{n} = 0 \text{ on } \Sigma}} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_\mathrm{S})}} \geq \beta_0^* \Big\{ \|\boldsymbol{s}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{v}\|_{0,\Omega_\mathrm{S}} \Big\}. \tag{5.23}$$

Now, we consider again the solution $\boldsymbol{w}$ of problem (5.13), but with a Neumann data $\boldsymbol{\varphi} \in \Psi_h \in \mathrm{L}^2(\Sigma)^n$. Then, classical regularity results for the Poisson problem in polyhedral (polygonal) domains (cf. [12]) ensure the existence of $\varepsilon \in (0,1)$, depending on the geometry of Ω_S, such that $\boldsymbol{w} \in \mathrm{H}^{1+\varepsilon}(\Omega_\mathrm{S})^n$ and

$$\|\boldsymbol{w}\|_{1+\varepsilon,\Omega_\mathrm{S}} \leq C_1 \|\boldsymbol{\varphi}\|_{0,\Sigma}\,. \tag{5.24}$$

It follows that $\bar{\boldsymbol{\sigma}} := \nabla \boldsymbol{w}$ belongs to $\boldsymbol{\mathcal{Y}} \cap \mathrm{H}^{\varepsilon}(\Omega_{\mathrm{S}})^n$, and hence $\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}$ is meaningful. In addition, by virtue of (4.4) we have that $\mathbf{div}\,\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}} = \mathbf{0}$ in Ω_{S}, $(\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})\boldsymbol{n} = \mathbf{0}$ on Γ_{N}, and $(\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})\boldsymbol{n} = \boldsymbol{\varphi}$ on Σ, whereas the approximation property of $\boldsymbol{\Pi}_h$ (cf. (4.2)) yields the existence of a constant $C_2 > 0$, independent of h, such that

$$\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} = \|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{0,\Omega_{\mathrm{S}}} \le C_2 \, \|\bar{\boldsymbol{\sigma}}\|_{\varepsilon,\Omega_{\mathrm{S}}} \,.$$

Combining the foregoing inequality with (5.24) gives

$$\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}} \le C_3 \, \|\boldsymbol{\varphi}\|_{0,\Sigma} \,,$$

and therefore, noting that $\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}} \,\in\, \boldsymbol{\mathcal{W}}_h$, we find that

$$\begin{aligned} S_h(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \,&\ge\, \frac{(\boldsymbol{s},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \\ = \frac{(\boldsymbol{s},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \,&\ge\, \frac{\|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \,\ge\, \frac{1}{C_3}\,\|\boldsymbol{\varphi}\|_{0,\Sigma} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \,. \end{aligned} \tag{5.25}$$

Finally, an adequate combination of (5.23) and (5.25) implies (5.22) and finishes the proof. □

The well-posedness of (5.21) is provided next.

Lemma 5.7. *There exists a unique* $(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h^{sym} \times \mathcal{U}_h \times \Psi_h$ *solution of* (5.21), *and there exists* $C > 0$, *independent of* λ, h, *and the given* $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, *such that*

$$\|(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*))\| \,\le\, C\,\|(\boldsymbol{\sigma}, p)\|\,. \tag{5.26}$$

Proof. Proceeding analogously as in the proof of Lemma 5.4, we begin by introducing the discrete null space

$$\mathbf{K}_h := \Big\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}}; \quad (\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall (\boldsymbol{v},\boldsymbol{\varphi}) \in \mathcal{U}_h \times \Psi_h \Big\}\,,$$

which becomes $\mathbf{K}_h = \Big\{ \boldsymbol{\tau}; \quad (\boldsymbol{\tau}, \xi) \in \mathbb{K}_h \Big\}$. Then, it is readily seen that (3.14) implies the $\boldsymbol{\mathcal{Y}}$-ellipticity of $a(\cdot,\cdot)$ on $\mathbf{K}_h$, with a constant independent of h and λ. In this way, thanks to this result and the inf-sup condition (5.6), a direct application of the discrete Babuška-Brezzi theory completes the proof of the lemma. □

As a consequence of the foregoing lemma, we now introduce the operator $\Xi_h : \mathbb{X} \to \mathbb{X}_h^{\mathrm{sym}}$, which is defined as the discrete analogue of (5.17), that is

$$\Xi_h(\boldsymbol{\sigma}, p) \,:=\, (\boldsymbol{\sigma}_h^*, p_h^*) \qquad \forall\,(\boldsymbol{\sigma}, p) \,\in\, \mathbb{X}\,, \tag{5.27}$$

where

$$p_h^* := \Pi_h p - \frac{1}{|\Sigma|}\left\{ (\boldsymbol{\sigma}_h^* \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_{\Sigma} \Pi_h p \right\}\,, \tag{5.28}$$

and $\boldsymbol{\sigma}_h^*$ is the first component of the solution $(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}} \times \mathcal{U}_h \times \Psi_h$ of problem (5.21). In addition, there exists $C > 0$, independent of λ, h, and the given $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, such that

$$\|\Xi_h(\boldsymbol{\sigma}, p)\| \,\le\, C\,\|(\boldsymbol{\sigma}, p)\| \qquad \forall\,(\boldsymbol{\sigma}, p) \,\in\, \mathbb{X}\,. \tag{5.29}$$

In the following section we deal with the a priori error estimate for $\Xi - \Xi_h$.

5.3 The associated error estimate

We begin by remarking that our analysis up to now has not required any formulation involving explicitly the rotation $\boldsymbol{r}^*$ nor its corresponding discrete version $\boldsymbol{r}_h^*$. Actually, the main novelty of our approach has been precisely the fact that these complementary unknowns have remained some how hiden. Nevertheless, for the derivation of the aforementioned error estimate, we now need to introduce the extended versions of (5.8) and (5.21), which are given, respectively, as follows:

Find $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{r}^*, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= -\frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \int_\Sigma \bar{p} (\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}\,, \\
(\boldsymbol{\sigma}^*, \boldsymbol{s})_{0,\Omega_{\mathrm{S}}} &= 0\,, \\
(\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}}\,, \\
(\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\bar{p}, \boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma}\,,
\end{aligned}
\tag{5.30}
$$

for all $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$, and

Find $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}_h^*, \boldsymbol{\tau}) + (\boldsymbol{r}_h^*, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{u}_h^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\psi}_{0,h}^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= -\frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}} c^2 |\Sigma|^2} \int_\Sigma \Pi_h p (\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}\,, \\
(\boldsymbol{\sigma}_h^*, \boldsymbol{s})_{0,\Omega_{\mathrm{S}}} &= 0\,, \\
(\mathbf{div}\,\boldsymbol{\sigma}_h^*, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}}\,, \\
(\boldsymbol{\sigma}_h^* \boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\Pi_h p, \boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma} h\,,
\end{aligned}
\tag{5.31}
$$

for all $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$.

Then, we have the following result.

Lemma 5.8. *There exists a constant $C > 0$, independent of h and λ, such that*

$$
\begin{aligned}
\|\Xi(\boldsymbol{\sigma}, p) - \Xi_h(\boldsymbol{\sigma}, p)\| \le C \Big\{ &\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}} \\
+ &\inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} + \inf_{\boldsymbol{\varphi}_h \in \Psi_h} \|\boldsymbol{\psi}^* - \boldsymbol{\varphi}_h\|_{0,\Sigma} + |p - \Pi_h p|_{1,\Omega_{\mathrm{F}}} \Big\}\,.
\end{aligned}
\tag{5.32}
$$

Proof. A straightforward application of the first Strang lemma to the formulations (5.30) and (5.31) yields the existence of a constant $C_1 > 0$, independent of h and λ, such that

$$
\begin{aligned}
\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}^* - \boldsymbol{\psi}_h^*\|_{0,\Sigma} \le C_1 \Bigg\{ &\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} \\
+ \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} &+ \inf_{\boldsymbol{\varphi}_h \in \Psi_h} \|\boldsymbol{\psi}^* - \boldsymbol{\varphi}_h\|_{0,\Sigma} \\
+ \sup_{\boldsymbol{\varphi} \in \Psi_h} \frac{\int_\Sigma (\bar{p} - \Pi_h p) \boldsymbol{\varphi}\cdot\boldsymbol{n}}{\|\boldsymbol{\varphi}\|_{0,\Sigma}} + \sup_{\boldsymbol{\tau} \in \mathbf{K}_h} &\frac{\int_\Sigma (\bar{p} - \Pi_h p)(\boldsymbol{n}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \Bigg\}\,.
\end{aligned}
\tag{5.33}
$$

Then, employing the Cauchy-Schwarz inequality and the trace theorem, we find that

$$\sup_{\boldsymbol{\varphi}\in\Psi_h} \frac{\displaystyle\int_\Sigma (p-\Pi_h p)\boldsymbol{\varphi}\cdot\boldsymbol{n}}{\|\boldsymbol{\varphi}\|_{0,\Sigma}} \;\le\; C_2\,|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}$$

and

$$\sup_{\boldsymbol{\tau}\in\mathbf{K}_h} \frac{\displaystyle\int_\Sigma (\bar{p}-\Pi_h p)(\boldsymbol{n},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \;\le\; C_3\,|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\,,$$

whereas the definitions of p^*, p_h^*, and $\|\cdot\|_{\boldsymbol{\mathcal{Y}}}$ (cf. (5.18), (5.28), and (5.5)) yield

$$\|p^*-p_h^*\|_{1,\Omega_{\mathrm{F}}} \;\le\; C_4\Big\{|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}} + \|\boldsymbol{\sigma}^*-\boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}}\Big\}.$$

In this way, bounding $\|\boldsymbol{\sigma}^*-\boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}}$ according to (5.33), and combining this estimate with the last three inequalities, we arrive at (5.32) and finish the proof. □

We end this section with the following corollary of the a priori error estimate provided by Lemma 5.8.

Lemma 5.9. *Assume that $(\boldsymbol{\sigma},p)\in\mathbb{K}^\perp$ with $\boldsymbol{\sigma}\in[\mathrm{H}^\epsilon(\Omega_{\mathrm{S}})]^{n\times n}$ for some $\varepsilon>0$, and let $(\boldsymbol{u},\boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma},p)$ and $\boldsymbol{\psi} := \boldsymbol{u}|_\Sigma$. Then, there exists a constant $C>0$, independent of h and λ, such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},p)-\Xi_h(\boldsymbol{\sigma},p)\| \;\le\; C\Big\{ &\|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|p\boldsymbol{n}-\boldsymbol{\pi}_h(p\boldsymbol{n})\|_{0,\Sigma} + \|\boldsymbol{r}-\mathbf{R}_h\boldsymbol{r}\|_{0,\Omega_{\mathrm{S}}}\\ + \quad &\|\boldsymbol{u}-U_h\boldsymbol{u}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}-\pi_h\boldsymbol{\varphi}\|_{0,\Sigma} + |p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\Big\}.\end{aligned}$$

Proof. We first recall from Lemma 5.4 that $\Xi(\boldsymbol{\sigma},p)=(\boldsymbol{\sigma},p)$ for all $(\boldsymbol{\sigma},p)\in\mathbb{K}^\perp$. Then, it follows from Lemma 5.5 that $(\boldsymbol{u}^*,\boldsymbol{r}^*) := \mathbf{D}(\boldsymbol{\sigma},p) = (\boldsymbol{u},\boldsymbol{r})$ and $\boldsymbol{\psi}^* = \boldsymbol{u}|_\Sigma = \boldsymbol{\psi}$, which combined with (5.32), gives

$$\begin{aligned}\|(\boldsymbol{\sigma},p)-\Xi_h(\boldsymbol{\sigma},p)\| \;\le\; C\Big\{ &\inf_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} + \inf_{\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\|\boldsymbol{r}-\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}\\ + \quad &\inf_{\boldsymbol{v}_h\in\mathcal{U}_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} + \inf_{\boldsymbol{\varphi}_h\in\Psi_h}\|\boldsymbol{\psi}-\boldsymbol{\varphi}_h\|_{0,\Sigma} + |p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\Big\}.\end{aligned} \tag{5.34}$$

Next, since $\boldsymbol{\sigma}\in\boldsymbol{\mathcal{W}}\cap[\mathrm{H}^\varepsilon(\Omega_{\mathrm{S}})]^{n\times n}$ with $\varepsilon>0$, we can employ the BDM-interpolation operator $\boldsymbol{\Pi}_h$ (cf. Section 4) to obtain

$$\begin{aligned}\inf_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|^2_{\boldsymbol{\mathcal{Y}}} \;\le\; \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|^2_{\boldsymbol{\mathcal{Y}}} \;&=\; \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|\boldsymbol{\sigma}\boldsymbol{n}-\boldsymbol{\pi}_h(\boldsymbol{\sigma}\boldsymbol{n})\|^2_{0,\Sigma}\\ &\le\; \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|p\boldsymbol{n}-\boldsymbol{\pi}_h(p\boldsymbol{n})\|^2_{0,\Sigma}\,,\end{aligned} \tag{5.35}$$

which, together with (5.34) and the definitions of the projectors $\mathbf{R}_h$ and U_h (cf. Section 4), implies the required estimate and ends the proof. □

6 Analysis of the semi-discrete problem

From now on, we assume that the discrete Galerkin problem (4.10) is supplied with the initial data

$$(\boldsymbol{\sigma}_{0,h},p_{0,h}) := \Xi_h(\boldsymbol{\sigma}_0,p_0) \quad\text{and}\quad (\boldsymbol{\sigma}_{1,h},p_{1,h}) := \Xi_h(\boldsymbol{\sigma}_1,p_1).$$

Then, we introduce

$$\boldsymbol{e}_{\sigma,h}(t) = \boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}_h(t) \quad \text{and} \quad e_{p,h}(t) = p_h^*(t) - p_h(t),$$

where $(\boldsymbol{\sigma}_h^*(t), p_h^*(t)) = \Xi_h(\boldsymbol{\sigma}(t), p(t))$, and notice that $\boldsymbol{e}_{\sigma,h}(0) = \dot{\boldsymbol{e}}_{\sigma,h}(0) = \mathbf{0}$ and $e_{p,h}(0) = \dot{e}_{p,h}(0) = 0$.

The following lemma establishes an a priori estimate for the error between the solution $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0(\mathbb{X}^{\text{sym}}) \cap \mathcal{C}^1(\mathbb{H}^{\text{sym}})$ of the continuous problem (3.11) and its semi-discrete approximation given by the solution $(\boldsymbol{\sigma}_h, p_h) \in \mathcal{C}^1(\mathbb{X}_h^{\text{sym}})$ of the Galerkin scheme (4.10).

Lemma 6.1. *Assume that $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{X} \cap \mathbb{H}^\epsilon(\Omega))$ for some $\epsilon > 0$, and that $p \in \mathcal{C}^2(\mathrm{H}^1(\Omega_\mathrm{F}))$. Then, there exists a constant $C > 0$, independent of λ and h, such that*

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma}, p)(t) - (\boldsymbol{\sigma}_h, p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}}, \dot{p})(t) - (\dot{\boldsymbol{\sigma}}_h, \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \leq C \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbf{H}(\mathbf{div},\Omega_\mathrm{S}))} + \|\boldsymbol{r} - \mathbf{R}_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{skew})} + \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_\mathrm{S})^n)} \qquad (6.1) \\
&\quad + \quad \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_\mathrm{F})^n)} + \|\boldsymbol{\psi} - \pi_h \boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma)^n)} + \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma))} \Big\}.
\end{aligned}$$

where $(\boldsymbol{u}, \boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma}, p)$ and $\boldsymbol{\psi} := \boldsymbol{u}|_\Sigma$.

Proof. Let us first notice that the fact that $(\boldsymbol{\sigma}(t), p(t)) \in \mathbb{K}^\perp$ for all $t \in [0, T]$ guarantes that

$$(\boldsymbol{\sigma}^*(t), p^*(t)) := \Xi(\boldsymbol{\sigma}(t), p(t)) = (\boldsymbol{\sigma}(t), p(t)) \qquad \forall\, t \in [0, T]. \tag{6.2}$$

Moreover, because of the regularity assumptions, it holds that

$$\Big(\frac{\mathrm{d}^i \boldsymbol{\sigma}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i p^*}{\mathrm{d}t^i}(t)\Big) := \frac{\mathrm{d}^i \Xi(\boldsymbol{\sigma}(t), p(t))}{\mathrm{d}t^i} = \Xi(\frac{\mathrm{d}^i \boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i p}{\mathrm{d}t^i}(t)) \qquad \forall\, i \in \{1, 2\}, \quad \forall\, t \in [0, T], \tag{6.3}$$

and hence, by virtue of Lemma 5.9 and (6.3), there exists $C_1 > 0$, independent of h and λ, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} = \|(\boldsymbol{\sigma}, p) - \Xi_h(\boldsymbol{\sigma}, p)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} \\
&\quad \leq C_1 \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbf{H}(\mathbf{div},\Omega_\mathrm{S}))} + \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma))} + \|\boldsymbol{r} - \mathbf{R}_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\text{skew}})} \\
&\quad + \quad \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_\mathrm{S})^n)} + \|\boldsymbol{\psi} - \pi_h \boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma)^n)} + \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_\mathrm{F})^n)} \Big\},
\end{aligned} \tag{6.4}$$

with $(\boldsymbol{u}(t), \boldsymbol{r}(t)) := \mathbf{D}(\boldsymbol{\sigma}(t), \boldsymbol{u}(t))$ and $\boldsymbol{\psi}(t) := \boldsymbol{u}(t)|_\Sigma$. Now, adding and substracting $(\ddot{\boldsymbol{\sigma}}, \ddot{p})$, and then using the identity (3.39) and the first equation of (4.10), we obtain the error equation,

$$\begin{aligned}
&\Big((\ddot{\boldsymbol{e}}_{\sigma,h}(t), \ddot{e}_{p,h}(t)), (\boldsymbol{\tau}, q)\Big)_\mathcal{C} + A\Big((\boldsymbol{e}_{\sigma,h}(t), e_{p,h}(t)), (\boldsymbol{\tau}, q)\Big) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_\mathcal{C} - (\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + A\Big((\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}, p_h^* - p)(t), (\boldsymbol{\tau}, q)\Big) \qquad (6.5) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}^*, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_\mathcal{C} - (\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + A\Big((\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}, p_h^* - p)(t), (\boldsymbol{\tau}, q)\Big)
\end{aligned}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\text{sym}}$, where the first expression of the last equality makes use of the fact that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$ (cf. (6.2)). In addition, by virtue of the inclusion $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) \subset \mathcal{U}_h$, it turns out that

$$\big(\mathbf{div}(\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}^*)(t), \mathbf{div}\, \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} = 0 \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\text{sym}}. \tag{6.6}$$

Furthermore, according to the definitions of p^* and p_h^* (cf. (5.18) and (5.28)), there holds

$$(\nabla(p_h^* - p^*)(t), \nabla q)_{0,\Omega_\mathrm{F}} = (\nabla(p - \Pi_h p)(t), \nabla q)_{0,\Omega_\mathrm{F}} = 0 \qquad \forall q \in V_h, \tag{6.7}$$

and it is straightforward that

$$(\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} = (\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\mathrm{sym}}. \tag{6.8}$$

Next, rewriting (6.5) by taking into account (6.2) and (6.6)-(6.8), we deduce that

$$\begin{aligned}
&\Big((\ddot{\boldsymbol{e}}_{\sigma,h}(t), \ddot{e}_{p,h}(t)), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{e}_{\sigma,h}(t), e_{p,h}(t)), (\boldsymbol{\tau}, q)\Big) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} - \big(\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t), \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\mathrm{sym}}.
\end{aligned}$$

Moreover, choosing $(\boldsymbol{\tau}, q) = (\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)$ in the foregoing identity, recalling the definition of the energy functional $\mathcal{E}$ (cf. (3.18)), and applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
\dot{\mathcal{E}}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t) \;&\le\; \|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_{0,\mathcal{C}} \, \|(\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)\|_{0,\mathcal{C}} \\
&+ \Big(\mathcal{C}\big(\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\big), \ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\Big)_{0,\Omega_\mathrm{S}}^{1/2} \Big(\mathcal{C}^{-1} \dot{\boldsymbol{e}}_{\sigma,h}, \dot{\boldsymbol{e}}_{\sigma,h}\Big)_{0,\Omega_\mathrm{S}}^{1/2} \\
&\le\; \Big(\|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_{0,\mathcal{C}} + \sqrt{2\mu}\, \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\|_{0,\Omega_\mathrm{S}} \Big) \|(\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)\|_{0,\mathcal{C}}
\end{aligned}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\mathrm{sym}}$, where we used that $\mathcal{C}\boldsymbol{s} = 2\mu \boldsymbol{s}$ for all $\boldsymbol{s} \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$. Hence, by virtue of the estimate (3.13), there exists $C_2 > 0$, independent of h and λ, such that

$$\frac{\dot{\mathcal{E}}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t)}{2\sqrt{\mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t)}} \;\le\; C_2 \left\{ \|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_0 + \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\|_{0,\Omega_\mathrm{S}} \right\},$$

and integrating in time yields

$$\max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) \le C_2 \int_0^T \Big\{ \|(\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h - \ddot{p})(t)\|_0 + \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\|_{0,\Omega_\mathrm{S}} \Big\}\, \mathrm{d}t. \tag{6.9}$$

On the other hand, according to the definition of $\mathcal{E}$ (cf. (3.18)), it holds

$$\max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)\big)^{1/2}(t) \;\le\; C_3 \left\{ \max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) + \max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\big)^{1/2}(t) \right\}, \tag{6.10}$$

and taking into account (3.14), we deduce that

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \le\; C_4 \Big\{ \max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) + \max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\big)^{1/2}(t) \Big\}.
\end{aligned} \tag{6.11}$$

Then, combining (6.9) and (6.11) gives

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \le\; C_5 \Big\{ \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} + \max_{[0,T]} \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\|_{0,\Omega_\mathrm{S}} \Big\},
\end{aligned} \tag{6.12}$$

where C_3, C_4, and hence $C_5 > 0$, are all constants independent of h and λ. Finally, it is readily seen that the required estimate (6.1) follows from (6.4) and (6.12). □

As a straightforward consequence of Lemma 6.1 we have the following theorem establishing the rates of convergence of our semi-discrete scheme (4.10).

Theorem 6.1. *Assume that the solution* $(\boldsymbol{u}, p)$ *to* (3.28)-(3.33) *satisfies the regularity assumptions* $(\boldsymbol{u}, p) \in \mathcal{C}^2\big(\mathrm{H}^{1+s}(\Omega_{\mathrm{S}})^n \times \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})\big)$ *and* $\mathbf{div}\,\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{C}^2(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)$, *for some* $s > 1/2$. *Then, there exists a constant* $C > 0$, *independent of* h *and* λ, *such that*

$$
\begin{aligned}
\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \;\le\; C\, h^{\min\{k,s\}} \Big\{ & \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})} \\
+ \quad \|\mathbf{div}\,\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)} + \|\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})} + \|\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)} & \\
+ \quad \|p\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^{1+s}(\Omega_{\mathrm{F}}))} + \Big(\sum_{e\in\Sigma_h} \|\boldsymbol{\psi}\|^2_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(e)^n)}\Big)^{1/2} + \Big(\sum_{e\in\Sigma_h} \|p\|^2_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(e))}\Big)^{1/2} \Big\}. &
\end{aligned}
$$

Proof. It is clear from the hypotheses that $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{C}^2([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathcal{C}^2(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)$, $\boldsymbol{r} = (\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}})/2 \in \mathcal{C}^2([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})$, $\boldsymbol{\psi} = \boldsymbol{u}|_\Sigma \in \mathrm{H}^{s+1/2}(\Sigma)^n$, and $p|_\Sigma \in \mathrm{H}^{s+1/2}(\Sigma)$. Hence, the result follows directly from (6.1) by using the approximation properties given by (4.3), (4.4) and (4.6)-(4.9). □

In addition to the above, and inspired from (3.2), we also propose the following explicit expression for the semi-discrete displacement field:

$$
\boldsymbol{u}_h(t) = \int_0^t \Big\{ \int_0^s \rho_{\mathrm{S}}^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}_h(z) + \mathbf{U}_h \boldsymbol{f}(z) \Big)\,\mathrm{d}z \Big\}\,\mathrm{d}s + \boldsymbol{u}_{0,h} + t\boldsymbol{u}_{1,h}\,, \tag{6.13}
$$

where $\boldsymbol{u}_{0,h}$ and $\boldsymbol{u}_{1,h}$ are obtained by solving (5.21) with data $(\mathbf{div}\,\boldsymbol{\sigma}_0, p_0)$ and $(\mathbf{div}\,\boldsymbol{\sigma}_1, p_1)$, respectively. It is then clear that, under the regularity conditions of Theorem 6.1, there holds

$$
\max_{[0,T]} \|(\boldsymbol{u} - \boldsymbol{u}_h)(t)\|_{0,\Omega_{\mathrm{S}}} = O(h^{\min\{k,s\}})\,.
$$

We end this paper by remarking that in a forthcoming work we show that the analysis given in [15, Section 6] for the dual-mixed formulation of the elastodynamic equations, can be adapted to deal with the time discretization, based on the Newmark trapezoidal rule, of our present problem (4.10). Numerical tests illustrating the good performance of this scheme are in progress.

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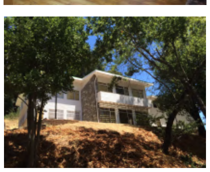

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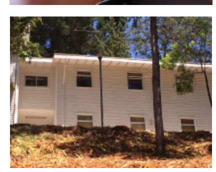
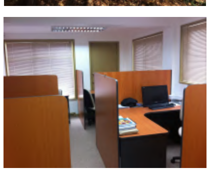

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