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**Error in Sobolev norms of orthogonal projection onto polynomials in the unit ball**

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PREPRINT 2016-20

SERIE DE PRE-PUBLICACIONES

# ERROR IN SOBOLEV NORMS OF ORTHOGONAL PROJECTION ONTO POLYNOMIALS IN THE UNIT BALL

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Abstract. We study approximation properties of weighted $\mathrm{L}^2$-orthogonal projectors onto spaces of polynomials of bounded degree in the Euclidean unit ball, where the weight is of the generalized Gegenbauer form $x \mapsto (1 - \|x\|^2)^\alpha$, $\alpha > -1$. Said properties are measured in Sobolev-type norms in which the same weighted $\mathrm{L}^2$ norm is used to control all involved weak derivatives. The method of proof does not rely on any particular basis of orthogonal polynomials, which allows for a streamlined and dimension-independent exposition.

## 1. Introduction

It has been known since the early eighties [3] that the orthogonal projector $S_N$ mapping $\mathrm{L}^2(-1,1)$ onto the space of univariate polynomials of degree less than or equal to $N$ (equivalently, $S_N$ is the operation consisting in truncating the Fourier–Legendre series of its argument at degree $N$) satisfies the bound

$$(1) \qquad (\forall\, u \in \mathrm{H}^l(-1,1)) \qquad \|u - S_N(u)\|_{\mathrm{H}^1(-1,1)} \leq C N^{3/2-l} \|u\|_{\mathrm{H}^l(-1,1)},$$

where $C > 0$ depends only on $l$ and $\mathrm{H}^1(-1,1)$ and $\mathrm{H}^l(-1,1)$ denote standard Sobolev spaces (see [4, Ch. 5] for a detailed proof of (1) and its Chebyshev weight and periodic unweighted analogues and [10] for its general Gegenbauer weight analogue). Recently [9] this result was extended to the unit disk for Gegenbauer-type weights.

The purpose of this work is proving a weighted analogue of (1) in the case of the unit ball of any dimension. Our main result (Theorem 3.8) is

$$(2) \qquad (\forall\, u \in \mathrm{H}^l_w(B^d)) \qquad \|u - S_N(u)\|_{\mathrm{H}^r_w(B^d)} \leq C N^{-1/2+2r-l} \|u\|_{\mathrm{H}^l_w(B^d)},$$

where $B^d$ is the unit ball of $\mathbb{R}^d$, $\mathrm{L}^2_w(B^d) = w^{-1/2}\,\mathrm{L}^2(B^d)$ with its natural norm, $\mathrm{H}^l_w(B^d)$ and $\mathrm{H}^r_w(B^d)$ are corresponding weighted Sobolev spaces, $S_N$ is the $\mathrm{L}^2_w(B^d)$-orthogonal projector onto $\Pi^d_N$, the space of $d$-variate polynomials of degree less than or equal to $N$, and $C$ depends only on the integers $1 \leq r \leq l$ and the weight $w$, which in turn is of the Gegenbauer form $x \mapsto (1 - \|x\|^2)^\alpha$ with $\alpha > -1$.

Our main result has applications in the analysis of polynomial interpolation operators, themselves important in the analysis of spectral methods (cf. [3] and [4, Ch. 5]) and in the characterization of approximability spaces relevant to the analysis of nonlinear iterative methods for the numerical solution of high-dimensional PDE (cf. [8, Ch. 4]).

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2010 *Mathematics Subject Classification.* 41A25, 41A10, 46E35.

*Key words and phrases.* Orthogonal projection, weighted Sobolev space, unit ball, orthogonal polynomials.

The author was supported by the CONICYT grant FONDECYT-1130923 and by the Universidad de Concepción VRID-Enlace grant 215.013.040-1.0.

We emphasize that the case $r = 0$ is explicitly excluded from consideration in (2), for in that case the provably optimal power on $N$ is $-l$ (cf. Lemma 2.2 below) and thus this case does not follow the pattern in (2). We also note that if $2r \geq l + 1/2$ in (2), $S_N(u)$ need not converge to $u$ in $\mathrm{H}^r_w(B^d)$ as $N$ tends to infinity. We further remark that (2) is not a best or quasi-best approximation result (for those see [4, Ch. 5], [10], [13, § 4] and [5, § 5]), because in general the orthogonal projection of $\mathrm{H}^r_w(B^d)$ onto $\Pi^d_N$ need not coincide with the restriction of $S_N$ to $\mathrm{H}^r_w(B^d)$.

In every proof of a particular instance of (2) that we are aware of, an important role was played by spectral differentiation formulas, which connect the orthogonal expansion coefficients of a functions and one of its derivatives; e.g., [4, Eq. (2.3.18)]

$$(\forall\, k \in \{0, 1, 2, \dots\}) \quad \hat{u}^{(1)}_k = (2k + 1) \sum_{q=0}^{\infty} \hat{u}_{k+1+2q},$$

where $u = \sum_{k=0}^{\infty} \hat{u}_k\, L_k$ and $u' = \sum_{k=0}^{\infty} \hat{u}^{(1)}_k\, L_k$ are the orthogonal expansions of $u \in \mathrm{H}^1(-1, 1)$ and its weak derivative with respect to the basis $(L_k)_{k=0}^{\infty}$ of Legendre polynomials. See [4, Eq. (2.4.22)]—the '+' sign there is a typo—, [10, Eq. (2.13)] and [9, Lem. 3.4] for spectral differentiation formulas for Chebyshev, Gegenbauer and Zernike orthogonal polynomial expansions. Whereas in one and two dimensions those 'named' bases of orthogonal polynomials are known to satisfy a wealth of simple identities so as to make spectral differentiation formulas simple to derive, that might not be the case for known explicit orthogonal polynomial bases $\mathrm{L}^2_w(B^d)$ with $d \geq 3$ (cf. the example bases in [7, § 5.2]).

In this work we introduce a streamlined technique to prove (2) which circumvents the need for spectral differentiation formulas and actually dispenses with the usage of bases of orthogonal polynomials altogether, focusing instead on orthogonal polynomial spaces; that is, spaces of polynomials of a certain degree orthogonal to all polynomials of lower degree (cf. (3) and the opening remarks of [7, Ch. 3]). In this way we can settle our main result seamlessly for any dimension.

The outline of this article is as follows. After introducing some general notation in subsection 1.1 we introduce the relevant weighted $\mathrm{L}^2$ and Sobolev spaces, the associated orthogonal polynomial spaces and some known properties of their members and their associated projectors in section 2. The core of this work is in section 3, in which we prove auxiliary results concerning orthogonal polynomial spaces and their projectors, bound a differentiation-projection commutator, prove our main result in Theorem 3.8 and an interpolation corollary and wrap up with some general remarks.

1.1. **General notation.** We denote by $\mathbb{N}$ the set of strictly positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote its Euclidean norm by $\|x\|$. Members of $[\mathbb{N}_0]^d$ will be called multi-indices and for every multi-index $\gamma \in [\mathbb{N}_0]^d$, point $x \in \mathbb{R}^d$ and (strongly or weakly) differentiable enough complex-valued function $f$ defined on some open set of $\mathbb{R}^d$ we shall write $|\gamma| = \sum_{i=1}^{d} \gamma_i$, $x^\gamma = \prod_{i=1}^{d} x_i^{\gamma_i}$ and $\partial_\gamma f = \partial^{|\gamma|} f / (\partial x_1^{\gamma_1} \cdots \partial x_d^{\gamma_d})$. As already mentioned, $B^d$ denotes the unit ball of $\mathbb{R}^d$ and $\Pi^d_n$ denotes the space of complex $d$-variate polynomials of total degree less than or equal to $n$; we adopt the convention $\Pi^d_n = \{0\}$ if $n < 0$.

## 2. Orthogonal polynomials and weighted Sobolev spaces

Given $d \in \mathbb{N} = \{1, 2, \dots\}$ and $\alpha > -1$ let $W_\alpha$ denote the weight function $B^d \ni x \mapsto (1 - \|x\|^2)^\alpha \in \mathbb{R}$. Note that we omit $d$ from the notation of $W_\alpha$ (and of $\mathrm{L}^2_\alpha$, $\mathrm{H}^m_\alpha$, $\mathrm{proj}^\alpha_k$, etc. below); we do this in order to avoid cluttering and because all of our arguments are dimension-independent. Let $\mathrm{L}^2_\alpha$ denote the vector space of (equivalence classes of) measurable functions $f \colon B^d \to \mathbb{C}$ such that $W_\alpha^{1/2} f \in \mathrm{L}^2(B^d)$. Equipped with its natural inner product $\langle f, g\rangle_\alpha := \int_\Omega f(x)\,\overline{g}(x)\,W_\alpha(x)\,\mathrm{d}x$, $\mathrm{L}^2_\alpha$ is a Hilbert space. In the notation of (2), $\mathrm{L}^2_\alpha = \mathrm{L}^2_w(B^d)$ with $w = W_\alpha$. The restriction on $\alpha$ ensures that $\mathrm{C}(\overline{B^d})$ is contained in $\mathrm{L}^2_\alpha$. If $-1 < \alpha \le \alpha'$ then $W_{\alpha'} \le W_\alpha$, whence $\mathrm{L}^2_\alpha$ is continuously embedded into $\mathrm{L}^2_{\alpha'}$ with the injection operator norm bounded by 1.

Let us define the space of orthogonal polynomials of degree $k$ with respect to the weight $W_\alpha$ (cf. [7, Def. 3.1.1])

$$\mathcal{V}^\alpha_k := \left\{p \in \Pi^d_k \mid (\forall\, q \in \Pi^d_{k-1})\ \langle p, q\rangle_\alpha = 0\right\}. \tag{3}$$

We note that the convention adopted for the $\Pi^d_k$ implies that $\mathcal{V}^\alpha_k = \{0\}$ for $k < 0$. As $W_\alpha$ is centrally symmetric, it transpires from [7, Th. 3.3.11] that for all $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ there holds the following parity relation:

$$(\forall\, p_k \in \mathcal{V}^\alpha_k)\ (\forall\, x \in B^d) \quad p_k(-x) = (-1)^k p_k(x). \tag{4}$$

From [7, Th. 3.2.18],

$$(\forall\, n \in \mathbb{N}_0)\ \Pi^d_n = \bigoplus_{k=0}^{n} \mathcal{V}^\alpha_k \qquad \text{and} \qquad \mathrm{L}^2_\alpha = \bigoplus_{k=0}^{\infty} \mathcal{V}^\alpha_k. \tag{5}$$

Let $\mathrm{proj}^\alpha_k$ denote the orthogonal projection from $\mathrm{L}^2_\alpha$ onto $\mathcal{V}^\alpha_k$ and let $S^\alpha_n$ denote the orthogonal projection from $\mathrm{L}^2_\alpha$ onto $\Pi^d_n$. From (5),

$$(\forall\, n \in \mathbb{N}_0) \quad S^\alpha_n = \sum_{k=0}^{n} \mathrm{proj}^\alpha_k\,. \tag{6}$$

We will denote the entrywise application of $S^\alpha_n$ to $\mathrm{L}^2_\alpha$-valued vectors and higher-order tensors by $S^\alpha_n$ as well (cf. Corollary 3.7 below).

From [7, Eq. (5.2.3) and Th. 8.1.3] and straightforward algebraic manipulation it is readily computed that the members of $\mathcal{V}^\alpha_k$ are eigenfunctions of the second order differential operator $p \mapsto -W_\alpha^{-1}\mathrm{div}\,(W_{\alpha+1}\nabla p) - \sum_{1\le i<j\le d} D^2_{i,j} p$, where $D_{i,j}$ denotes the first order angular differential operator $x_i\partial_j - x_j\partial_i$ [6, § 1.8], with associated eigenvalue $k(k + d + 2\alpha)$. By integration by parts the following integral form follows:

$$(\forall\, p_k \in \mathcal{V}^\alpha_k)\ \left(\forall\, q \in \mathrm{C}^1(\overline{B^d})\right)$$
$$\langle \nabla p_k, \nabla q\rangle_{[\mathrm{L}^2_{\alpha+1}]^d} + \sum_{1\le i<j\le d} \langle D_{i,j} p_k, D_{i,j} q\rangle_\alpha = k(k + d + 2\alpha)\langle p_k, q\rangle_\alpha. \tag{7}$$

For $m \in \mathbb{N}$ we introduce the weighted Sobolev space

$$\mathrm{H}^m_\alpha := \left\{u \in \mathrm{L}^2_\alpha \mid \|u\|^2_{\mathrm{H}^m_\alpha} := \sum_{k=0}^{m} |u|^2_{\mathrm{H}^k_\alpha} < \infty\right\}, \tag{8a}$$

where the seminorms $|\cdot|_{\mathrm{H}^k_\alpha}$ are defined by

(8b) $$|u|^2_{\mathrm{H}^k_\alpha} := \|\nabla_k u\|^2_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} = \sum_{|\gamma|=k} \binom{k}{\gamma} \|\partial_\gamma u\|^2_\alpha \,,$$

and in turn $\binom{k}{\gamma} = k!/(\gamma_1!\cdots\gamma_d!)$ is the number of times the multi-index $\gamma$ of order $k$ appears in the $k$-dimensional array-valued $\nabla_k u$. The seminorm (8b) is of course equivalent to the common choice $u \mapsto \left(\sum_{|\gamma|=k} \|\partial_\gamma u\|^2_\alpha\right)^{1/2}$. In the notation of (2), $\mathrm{H}^m_\alpha = \mathrm{H}^m_w(B^d)$ with $w = W_\alpha$. Given $m \in \mathbb{N}_0$ and $\theta \in (0,1)$ we define $\mathrm{H}^{m+\theta}_\alpha$ by complex interpolation [1, ¶7.51–52]:

(9) $$\mathrm{H}^{m+\theta}_\alpha := \left[\mathrm{H}^m_\alpha, \mathrm{H}^{m+1}_\alpha\right]_\theta .$$

**Lemma 2.1.** *Let $d \in \mathbb{N}$, $\alpha > -1$ and $m \in \mathbb{N}_0$. Then, $\mathrm{C}^\infty(\overline{B^d})$ is dense in $\mathrm{H}^m_\alpha$.*

*Proof.* This follows from [12, Rem. 11.12.(iii)] upon the realization that $W_\alpha$ is bounded from above and below by positive multiples of $\operatorname{dist}(\cdot, \partial B^d)$. □

We cite from [9, Cor. 2.7 and Lem. 2.11] the following $\mathrm{L}^2_\alpha$ bound on the $S^\alpha_n$ projection error and an inverse or Markov-type inequality:

**Lemma 2.2.** *For all $\alpha > -1$, $d \in \mathbb{N}$ and $l \in \mathbb{N}_0$ there exists a positive constant $C = C(\alpha, d, l)$ such that*

$$(\forall\, n \in \mathbb{N}_0)\ (\forall\, u \in \mathrm{H}^l_\alpha) \quad \|u - S^\alpha_n(u)\|_\alpha \le C(n+1)^{-l} \|u\|_{\mathrm{H}^l_\alpha} .$$

**Lemma 2.3.** *For $\alpha > -1$ and $d \in \mathbb{N}$ there exists a positive constant $C = C(\alpha, d) > 0$ such that*

$$(\forall\, n \in \mathbb{N}_0)\ (\forall\, p_n \in \Pi^d_n) \quad \|\nabla p_n\|_{[\mathrm{L}^2_\alpha]^d} \le C n^2 \|p_n\|_{\mathrm{L}^2_\alpha} .$$

## 3. Main result

The following proposition collects results concerning relations between spaces of orthogonal polynomials and their associated projectors not involving differentiation.

**Proposition 3.1.** *Let $\alpha > -1$ and $d \in \mathbb{N}$.*

(i) *Let $p_k \in \mathcal{V}^{\alpha+1}_k$. Then, $(1 - \|\cdot\|^2) p_k \in \mathcal{V}^\alpha_k \oplus \mathcal{V}^\alpha_{k+2}$.*
(ii) *Let $q_k \in \mathcal{V}^\alpha_k$. Then, $q_k = \mathrm{proj}^{\alpha+1}_{k-2}(q_k) + \mathrm{proj}^{\alpha+1}_k(q_k)$.*
(iii) *Let $u \in \mathrm{L}^2_\alpha$. Then, $\mathrm{proj}^{\alpha+1}_k(u) = \mathrm{proj}^{\alpha+1}_k\left(\mathrm{proj}^\alpha_k(u) + \mathrm{proj}^\alpha_{k+2}(u)\right)$.*
(iv) *Let $u \in \mathrm{L}^2_\alpha$. Then,*
$$\mathrm{proj}^{\alpha+1}_k(u) = \mathrm{proj}^\alpha_k(u) + \mathrm{proj}^{\alpha+1}_k \circ \mathrm{proj}^\alpha_{k+2}(u) - \mathrm{proj}^{\alpha+1}_{k-2} \circ \mathrm{proj}^\alpha_k(u).$$

*Proof.* Given $q \in \Pi^d_{k-1}$, $\langle (1 - \|\cdot\|^2) p_k, q\rangle_\alpha = \langle p_k, q\rangle_{\alpha+1} = 0$ by definition (3). Also, by the parity relation (4), $(1 - \|\cdot\|^2) p_k \perp_\alpha \mathcal{V}^\alpha_{k+1}$. Therefore part (i) stems from the first equality in (5). An analogous argument accounts for part (ii). Part (iii) comes from the fact that given $p_k \in \mathcal{V}^{\alpha+1}_k$,

$$\begin{aligned}\langle \mathrm{proj}^{\alpha+1}_k(u), p_k\rangle_{\alpha+1} &= \langle u, p_k\rangle_{\alpha+1} = \langle u, (1 - \|\cdot\|^2) p_k\rangle_\alpha \\ &\overset{\text{(i)}}{=} \langle \mathrm{proj}^\alpha_k(u) + \mathrm{proj}^\alpha_{k+2}(u), (1 - \|\cdot\|^2) p_k\rangle_\alpha = \langle \mathrm{proj}^\alpha_k(u) + \mathrm{proj}^\alpha_{k+2}(u), p_k\rangle_{\alpha+1}.\end{aligned}$$

Part (iv) is obtained from adding and subtracting $\mathrm{proj}^{\alpha+1}_{k-2}(\mathrm{proj}^\alpha_k(u))$ to the right hand side of part (iii) and using part (ii). □

We will now present another collection of results, this time involving differentiation. To this end we introduce the first order differentiation operator $d_j^\alpha$, $\alpha > -1$ and $j \in \{1, \ldots, d\}$, by

$$d_j^\alpha q(x) := -W_\alpha(x)^{-1} \frac{\partial}{\partial x_j} (W_{\alpha+1}(x)\, q(x)) = -(1 - \|x\|^2)\, \partial_j q(x) + 2(\alpha+1)\, x_j\, q(x).$$

**Proposition 3.2.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $j \in \{1, \ldots, d\}$.*

(i) $d_j^\alpha$ *maps* $\Pi_k^d$ *into* $\Pi_{k+1}^d$.
(ii) *Given* $p, q \in \mathrm{C}^1(\overline{B^d})$, $\langle \partial_j p, q\rangle_{\alpha+1} = \langle p, d_j^\alpha q\rangle_\alpha$.
(iii) *Let* $r_k \in \mathcal{V}_k^{\alpha+1}$. *Then,* $d_j^\alpha(r_k) \in \mathcal{V}_{k+1}^\alpha$.
(iv) *Let* $p_k \in \mathcal{V}_k^\alpha$. *Then,* $\partial_j p_k \in \mathcal{V}_{k-1}^{\alpha+1}$.
(v) *Let* $u \in \mathrm{C}^1(\overline{B^d})$. *Then,* $\partial_j \operatorname{proj}_k^\alpha(u) = \operatorname{proj}_{k-1}^{\alpha+1}(\partial_j u)$.

*Proof.* Part (i) is straightforward. Part (ii) is obtained by integration by parts and noticing that no boundary term appears on account of $(1 - \|\cdot\|^2)^{\alpha+1}$ vanishing on the boundary of $B^d$.

Given $r_k \in \mathcal{V}_k^{\alpha+1}$, by part (i), $d_j^\alpha(r_k) \in \Pi_{k+1}^d$, and, on account of part (ii), it is $\mathrm{L}^2_\alpha$-orthogonal to $\Pi_k^d$, whence part (iii). An analogous argument accounts for part (iv).

Given $u \in \mathrm{C}^1(\overline{B^d})$, by part (iv), $\partial_j \operatorname{proj}_k^\alpha(u) \in \mathcal{V}_{k-1}^{\alpha+1}$. Part (v) then comes about from the fact that for all $r \in \mathcal{V}_{k-1}^{\alpha+1}$,

$$\langle \partial_j \operatorname{proj}_k^\alpha(u), r\rangle_{\alpha+1} \overset{\text{(ii)}}{=} \langle \operatorname{proj}_k^\alpha(u), d_j^\alpha r\rangle_\alpha \overset{\text{(iii)}}{=} \langle u, d_j^\alpha r\rangle_\alpha \overset{\text{(ii)}}{=} \langle \partial_j u, r\rangle_{\alpha+1}.$$

□

*Remark* 3.3 (Shift operators). Part (iii) of Proposition 3.2 means that $d_j^\alpha$ is a *backward shift/degree raising* operator in the sense of [11]. Similarly, by part (iv), $\partial_j$ is a *forward shift/degree lowering* operator (see also (11) below).

*Remark* 3.4 (Relations with identities satisfied by bases). In the one-dimensional case ($d = 1$), $\mathcal{V}_k^\alpha = \operatorname{span}(\{P_k^{(\alpha,\alpha)}\})$, where the $P_k^{(\alpha,\alpha)}$ are Jacobi polynomials [14, Ch. 4]. Then, from the 'id-shift' identity (a combination of (6.4.21) and (6.4.23) of [2]; it must be slightly modified if $\alpha = -1/2$ and $k = 0$)

(10)
$$P_k^{(\alpha,\alpha)} = \frac{(k+2\alpha+1)(k+2\alpha+2)}{(2k+2\alpha+1)(2k+2\alpha+2)} P_k^{(\alpha+1,\alpha+1)} - \frac{k+\alpha}{2(2k+2\alpha+1)} P_{k-2}^{(\alpha+1,\alpha+1)},$$

it is possible to furnish alternative proofs of parts (ii) and (iii) of Proposition 3.1 and hence of its part (iv). In that rough sense Proposition 3.1 corresponds to (10). Similarly [14, Eq. (4.21.7)],

$$P_k^{(\alpha,\alpha)\prime} = \frac{k+2\alpha+1}{2} P_{k-1}^{(\alpha+1,\alpha+1)}, \tag{11}$$

allows for proving part (v) of Proposition 3.2 and so, again in a rough sense, Proposition 3.2 corresponds to (11). Using (10) and explicit formulas for the norms of Jacobi polynomials (cf. [14, Eq. (4.3.3)]) it is possible to reconstruct Proposition 3.5, although the necessary computations are not short.

In the two-dimensional case, $\mathcal{V}_k^\alpha = \operatorname{span}(\{P_{m,n}^{(\alpha)} \mid m+n=k\})$, where each $P_{m,n}^{(\alpha)}$ is a Zernike polynomial [15]. Then, the identities (10) and (11) find appropriate analogues in [9, Eq. (3.12)] and [15, Eq. (5.3)], respectively.

Inasmuch as it allows for quantifying a 'wrong' ($\mathrm{L}^2_\alpha$) norm of a member of a space of orthogonal polynomials ($\mathcal{V}_k^{\alpha+1}$), the following result is distantly related to [8, Eq. (4.43)] and [9, Prop. 3.12] in the $d = 1$ and $d = 2$ cases, respectively.

**Proposition 3.5.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, for all $p, q \in \mathcal{V}_k^{\alpha+1}$,*

$$\langle p, q\rangle_\alpha = \left(\frac{k + d/2}{\alpha + 1} + 1\right) \langle p, q\rangle_{\alpha+1}.$$

*Proof.* We start with the observation that if $s$ is a homogeneous polynomial of degree $k$—that is, of the form $s(x) = \sum_{|\gamma|=n} c_\gamma x^\gamma$—, it satisfies $x \cdot \nabla s(x) = k\, s(x)$, which also goes on to show that the $x \cdot \nabla$ operator exactly preserves the degree of any $d$-variate polynomial.

Let $p, q \in \mathcal{V}_k^{\alpha+1}$. As every member of $\mathcal{V}_k^{\alpha+1}$ is a linear combination of homogeneous polynomials of degree ranging from 0 to $k$, there exists a homogeneous polynomial $s_p$ of degree $k$ such that $p - s_p \in \Pi^d_{k-1}$ and hence $x \cdot \nabla p - x \cdot \nabla s_p \in \Pi^d_{k-1}$. Thus,

$$\langle x \cdot \nabla p, q\rangle_{\alpha+1} = \langle x \cdot \nabla s_p, q\rangle_{\alpha+1} = k\langle s_p, q\rangle_{\alpha+1} = k\langle p, q\rangle_{\alpha+1}. \tag{12}$$

Using the facts that $\nabla(1 - \|x\|^2)^{\alpha+1} = -2(\alpha + 1)(1 - \|x\|^2)^\alpha x$, $\operatorname{div}(x) = d$, integration by parts and (12), which of course is still valid if the roles of $p$ and $q$ are interchanged,

$$\begin{aligned} 2(\alpha+1) \int_{B^d} p(x)\overline{q(x)}\, \|x\|^2 (1-\|x\|^2)^\alpha \,\mathrm{d}x &= \int_{B^d} \operatorname{div}\left(p(x)\overline{q(x)}x\right) (1-\|x\|^2)^{\alpha+1} \,\mathrm{d}x \\ &= (\langle x \cdot \nabla p, q\rangle_{\alpha+1} + \langle p, x \cdot \nabla q\rangle_{\alpha+1} + d\langle p, q\rangle_{\alpha+1}) = (2k + d)\langle p, q\rangle_{\alpha+1}. \end{aligned}$$

The desired result then follows from the fact that $(1 - \|x\|^2)^\alpha = \|x\|^2 (1 - \|x\|^2)^\alpha + (1 - \|x\|^2)^{\alpha+1}$. □

**Lemma 3.6.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $l \in \mathbb{N}$. Then, there exists $C = C(\alpha, d, l) > 0$ such that for all $u \in \mathrm{H}^l_\alpha$, $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, d\}$,*

$$\|\partial_j S_n^\alpha(u) - S_n^\alpha(\partial_j u)\|_\alpha \le C(n + 1)^{3/2-l} \|\partial_j u\|_{\mathrm{H}^{l-1}_\alpha}.$$

*Proof.* Let us first assume that $u \in \mathrm{C}^\infty(\overline{B^d})$. Combining part (iv) of Proposition 3.1 and part (v) of Proposition 3.2, we obtain

$$\partial_j \operatorname{proj}^\alpha_{k+1}(u) - \operatorname{proj}^\alpha_k(\partial_j u) = \operatorname{proj}^{\alpha+1}_k \circ \operatorname{proj}^\alpha_{k+2}(\partial_j u) - \operatorname{proj}^{\alpha+1}_{k-2} \circ \operatorname{proj}^\alpha_k(\partial_j u). \tag{13}$$

Using (6) to express $S_n^\alpha$ in terms of the $\operatorname{proj}^\alpha_k$, using (13), noticing that a telescoping sum results and using part (ii) of Proposition 3.1 to expand an appearance of $\operatorname{proj}^\alpha_n(\partial_j u) \in \mathcal{V}_n^\alpha$,

$$\begin{aligned} \partial_j S_n^\alpha(u) - S_n^\alpha(\partial_j u) &= \sum_{k=0}^n \partial_j \operatorname{proj}^\alpha_k(u) - \sum_{k=0}^n \operatorname{proj}^\alpha_k(\partial_j u) \\ &= \sum_{k=0}^{n-1} \left(\partial_j \operatorname{proj}^\alpha_{k+1}(u) - \operatorname{proj}^\alpha_k(\partial_j u)\right) - \operatorname{proj}^\alpha_n(\partial_j u) \\ &= \operatorname{proj}^{\alpha+1}_{n-2} \circ \operatorname{proj}^\alpha_n(\partial_j u) + \operatorname{proj}^{\alpha+1}_{n-1} \circ \operatorname{proj}^\alpha_{n+1}(\partial_j u) - \operatorname{proj}^\alpha_n(\partial_j u) \\ &= \operatorname{proj}^{\alpha+1}_{n-1} \circ \operatorname{proj}^\alpha_{n+1}(\partial_j u) - \operatorname{proj}^{\alpha+1}_n \circ \operatorname{proj}^\alpha_n(\partial_j u). \end{aligned} \tag{14}$$

Now, by Proposition 3.5, the fact that $\|\mathrm{proj}_{n-1}^{\alpha+1}\|_{\mathcal{L}(\mathrm{L}^2_{\alpha+1})} \leq 1$ and the fact that $\|\cdot\|_{\alpha+1} \leq \|\cdot\|_{\alpha}$ in $\mathrm{L}^2_\alpha$ we have that for all $n \geq 1$,

$$\left\|\mathrm{proj}_{n-1}^{\alpha+1} \circ \mathrm{proj}_{n+1}^{\alpha}(\partial_j u)\right\|_{\alpha}^2 \leq \frac{n + d/2 + \alpha}{\alpha + 1} \left\|\mathrm{proj}_{n+1}^{\alpha}(\partial_j u)\right\|_{\alpha}^2 . \tag{15}$$

Of course, if $n = 0$, our conventions imply that $\left\|\mathrm{proj}_{n-1}^{\alpha+1} \circ \mathrm{proj}_{n+1}^{\alpha}(\partial_j u)\right\|_{\alpha}^2 = 0$. Analogous arguments show that for all $n \in \mathbb{N}_0$,

$$\left\|\mathrm{proj}_{n}^{\alpha+1} \circ \mathrm{proj}_{n}^{\alpha}(\partial_j u)\right\|_{\alpha}^2 \leq \frac{n + 1 + d/2 + \alpha}{\alpha + 1} \left\|\mathrm{proj}_{n}^{\alpha}(\partial_j u)\right\|_{\alpha}^2 . \tag{16}$$

Taking the squared $\mathrm{L}^2_\alpha$ norm of both ends of (14), exploiting the $\mathrm{L}^2_\alpha$ orthogonality of $\mathcal{V}_{n-1}^{\alpha+1}$ and $\mathcal{V}_{n}^{\alpha+1}$ (a consequence of the parity relation (4)) and the bounds (15) and (16) we observe that

$$\|\partial_j S_n^\alpha(u) - S_n^\alpha(\partial_j u)\|_\alpha^2 \leq \frac{n + 1 + d/2 + \alpha}{\alpha + 1} \left\|\partial_j u - S_{n+2}^\alpha(\partial_j u)\right\|_\alpha^2 .$$

As $\partial_j u \in \mathrm{H}^{l-1}_\alpha$, we can appeal to Lemma 2.2 to obtain the desired result for $u \in \mathrm{C}^\infty(\overline{B^d})$ after realizing that there exists a constant $\tilde{C}$ depending only on $\alpha$, $d$ and $l$ such that $\frac{n+1+d/2+\alpha}{\alpha+1}((n+3)^{-(l-1)})^2 \leq \tilde{C}(n+1)^{3-2l}$ for all $n \in \mathbb{N}_0$. The general result then follows via the density result in Lemma 2.1. ☐

**Corollary 3.7.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $r, l \in \mathbb{N}$ with $r \leq l$. Then, there exists $C = C(\alpha, d, l, r) > 0$ such that for all $u \in \mathrm{H}^l_\alpha$ and $n \in \mathbb{N}_0$,*

$$\|\nabla_r S_n^\alpha(u) - S_n^\alpha(\nabla_r u)\|_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} \leq C(n+1)^{2r-1/2-l} \|u\|_{\mathrm{H}^l_\alpha} .$$

*Proof.* Let us first note that iterating Lemma 2.3 we find that for all $r \in \mathbb{N}$ there exists $C > 0$ depending on $\alpha$, $d$ and $r$ such that

$$(\forall\, n \in \mathbb{N}_0)\ (\forall\, p \in \Pi_n^d) \quad |p|_{\mathrm{H}^r_\alpha} \leq C n^{2r} \|p\|_{\mathrm{L}^2_\alpha} . \tag{17}$$

We will now operate by induction on $r$. Taking the square root of the sum with respect to $j$ of the square of both sides of the inequality in Lemma 3.6 the case $r = 1$ follows almost immediately. Let us suppose now that our desired result holds for some $r \in \{1, \ldots, l\}$ and that $r + 1 \leq l$. Then, for all $j \in \{1, \ldots, d\}$, by the triangle inequality,

$$\begin{aligned} &\|\nabla_r \partial_j S_n^\alpha(u) - S_n^\alpha(\nabla_r \partial_j u)\|_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} \\ &\qquad \leq |\partial_j S_n^\alpha(u) - S_n^\alpha(\partial_j u)|_{\mathrm{H}^r_\alpha} + \|\nabla_r S_n^\alpha(\partial_j u) - S_n^\alpha(\nabla_r \partial_j u)\|_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} . \end{aligned}$$

By (17) and Lemma 3.6 the first term is bounded by an appropriate constant times $n^{2r}(n+1)^{3/2-l} \|\partial_j u\|_{\mathrm{H}^{l-1}_\alpha}$. By the induction hypothesis and the fact that $\partial_j u \in \mathrm{H}^{l-1}_\alpha$ the second term is bounded by an appropriate constant times $(n+1)^{2r-1/2-(l-1)} \|\partial_j u\|_{\mathrm{H}^{l-1}_\alpha}$. Then the desired result in the $r+1$ case follows from summing up with respect to $j$ and standard inequalities connecting vector 1- and 2-norms. ☐

We are now in a position to state our main result in Theorem 3.8 and the interpolation Corollary 3.9. As they are almost completely analogous to Theorem 3.9 and Corollary 3.10 of [9] we only sketch their proofs here.

**Theorem 3.8.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $r, l \in \mathbb{N} = \{1, 2, \dots\}$ with $r \leq l$. Then, there exists $C = C(\alpha, d, l, r) > 0$ such that for all $u \in \mathrm{H}^l_\alpha$ and $n \in \mathbb{N}_0$,*

$$\|u - S^\alpha_n(u)\|_{\mathrm{H}^r_\alpha} \leq C n^{2r-1/2-l} \|u\|_{\mathrm{H}^l_\alpha} . \tag{18}$$

*Proof.* For every $k \in \{1, \dots, r\}$,

$$|u - S^\alpha_n(u)|^2_{\mathrm{H}^k_\alpha} \leq 2 \|\nabla_k u - S^\alpha_n(\nabla_k u)\|^2_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} + 2 \|S^\alpha_n(\nabla_k u) - \nabla_k S^\alpha_n(u)\|^2_{[\mathrm{L}^2_\alpha]^{d\times\cdots\times d}} .$$

We bound the first term using Lemma 2.2 and the second using Corollary 3.7 and the desired result follows upon summing up with respect to $k$ and taking the square root. □

**Corollary 3.9.** *Let $\alpha > -1$, $d \in \mathbb{N}$ and $r, l \geq 0$ with $r \leq l$. Then, there exists $C = C(\alpha, d, l, r) > 0$ such that for all $u \in \mathrm{H}^l_\alpha$ and $n \in \mathbb{N}_0$,*

$$\|u - S^\alpha_n(u)\|_{\mathrm{H}^r_\alpha} \leq C n^{e(l,r)} \|u\|_{\mathrm{H}^l_\alpha} \quad \text{where} \quad e(l,r) = \begin{cases} 3/2\, r - l & \text{if } 0 \leq r \leq 1, \\ 2\, r - 1/2 - l & \text{if } r \geq 1. \end{cases}$$

*Proof.* The desired bound on the operator norm of $T^\alpha_{n,l,r} \colon \mathrm{H}^l_\alpha \to \mathrm{H}^r_\alpha$ defined by $T^\alpha_{n,l,r} := I - S^\alpha_n$ (with $I$ being the identity operator) holds when $r$ and $l$ are integers from Lemma 2.2 in the $r = 0$ case and Theorem 3.8 in the $r \in \mathbb{N}$ case. The non-integer cases then follow by using the exact interpolation and reiteration theorems. □

*Remark* 3.10 (Real interpolation). Just as it was remarked upon in the $d = 2$ case in [9], essentially the same argument used in Corollary 3.9 would work if we used real instead of complex interpolation to define the weighted Sobolev spaces with non-integer differentiation parameter in (9).

*Remark* 3.11 (On the optimality of the main result). There are four parameters in our main result, Theorem 3.8: The dimension $d \in \mathbb{N}$, the weight parameter $\alpha \in (-1, \infty)$, the regularity parameter for the function being approximated $l \in \mathbb{N}$ and the regularity parameter in which to measure the residual $r \in \{1, \dots, l\}$. We are aware of optimality proofs in the cases $(d, \alpha, l, r) = (1, -1/2, 1, 1)$ [3, pp. 76, 78], $(d, \alpha, r, r) = (1, 0, 1, 1)$ [4, p. 285], $(d, \alpha, l, r) \in \{2\} \times (-1, \infty) \times \mathbb{N} \times \{1\}$ [9, Th. 3.13] (the latter can be adapted to $(d, \alpha, l, r) \in \{1\} \times (-1, \infty) \times \mathbb{N} \times \{1\}$). All those proofs exploit a number of simple identities satisfied by particular bases of orthogonal polynomials. Notice also that all those parameter regimes have $r = 1$ (arguably the most important $r$ in Theorem 3.8 because of its connection with the analysis of weak forms of second order PDE). In [9] numerical experiments were used to support the conjecture that Theorem 3.8 is also true for $(d, \alpha, l, r) \in \{2\} \times (-1, \infty) \times \{(l, r) \in \mathbb{N} \times \mathbb{N} \mid r \leq l\}$. For general $d$ we do not know of bases of $\mathcal{V}^\alpha_k$ satisfying identities (particularly regarding differentiation) simple enough so as to enable us to completely extend the optimality proofs mentioned above. Nevertheless, always in the $r = 1$ case, we managed to generalize the techniques used in [9] for $(\alpha, l)$ in a certain proper subset of its natural range $(-1, \infty) \times \mathbb{N}$. The arguments behind such a partial result being rather involved and out of character with the rest of this work, we decided against including them here.

*Remark* 3.12. We expect our main sequence of results in this section 3 to extend to a wider class of of reflection invariant weights. If we focused on Gegenbauer-type weights it was mostly on account of their importance in applications and the ready availability of Lemma 2.1, Lemma 2.2 and Lemma 2.3.

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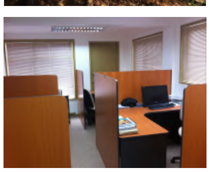

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