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Sergio Caucao, Gabriel N. Gatica,
Ricardo Oyarzúa

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A posteriori error analysis of a fully-mixed formulation for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity*

Sergio Caucao[†] Gabriel N. Gatica[‡] Ricardo Oyarzúa[§]

Abstract

In this paper we consider an augmented fully-mixed variational formulation that has been recently proposed for the coupling of the Navier–Stokes equations (with nonlinear viscosity) and the linear Darcy model, and derive a reliable and efficient residual-based *a posteriori* error estimator for the associated mixed finite element scheme. The finite element subspaces employed are piecewise constants, Raviart–Thomas elements of lowest order, continuous piecewise linear elements, and piecewise constants for the strain, Cauchy stress, velocity, and vorticity in the fluid, respectively, whereas Raviart–Thomas elements of lowest order for the velocity, piecewise constants for the pressure, and continuous piecewise linear elements for the traces, are considered in the porous medium. The proof of reliability of the estimator relies on a global inf-sup condition, suitable Helmholtz decompositions in the fluid and the porous medium, the local approximation properties of the Clément and Raviart–Thomas operators, and a smallness assumption on the data. In turn, inverse inequalities, the localization technique based on bubble functions, and known results from previous works, are the main tools yielding the efficiency estimate. Finally, several numerical results confirming the properties of the estimator and illustrating the performance of the associated adaptive algorithm are reported.

Key words: Navier–Stokes problem, Darcy problem, stress-velocity formulation, mixed finite element methods, efficiency, reliability, a posteriori error analysis.

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

Over the last decades, a wide range of numerical methods capturing the behaviour of a free fluid flow interacting with a porous medium have been proposed. The reason of such an interest by the numerical analysis community relies on the fact that, in industry, engineering sciences and several other disciplines, several interesting phenomena can be described under the framework of this kind of interaction problems (groundwater flows in karst aquifers, petroleum extraction, filtration of blood through

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[†]CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: scaucao@ci2ma.udec.cl.

[‡]CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

[§]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^2MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl.

arterial vessel walls, etc.). One of the most popular models utilized to describe the aforementioned interaction is the Navier-Stokes/Darcy (or Stokes/Darcy) model, which consists in a set of differential equations where the Navier-Stokes (or Stokes) problem is coupled with the Darcy model through a set of coupling equations acting on a common interface given by mass conservation, balance of normal forces, and the so called Beavers-Joseph-Saffman condition. In [3, 12, 13, 15, 21, 22, 23, 7, 35, 36, 38], and in the references therein, we can find a large list of contributions devoted to numerically approximate the solution of this interaction problem, including primal and mixed conforming formulations, as well as nonconforming methods.

In the recent work [12], it has been introduced and analyzed a new augmented-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity. The formulation there considers dual-mixed formulations in both domains, and in order to deal with the nonlinear viscosity, the strain tensor and the vorticity are introduced as auxiliary unknowns. In turn, since the transmission conditions become essential, they are imposed weakly, which yields the introduction of the traces of the porous media pressure and the fluid velocity as associated Lagrange multipliers. Furthermore, since the convective term in the fluid forces the velocity to live in a smaller space than usual, similarly to [8] and [9], the variational formulation is augmented with suitable Galerkin type terms arising from the constitutive and equilibrium equations of the Navier–Stokes model, as well as from the relations defining the strain and vorticity tensors. The resulting augmented variational system of equations is then suitably ordered so that it exhibits a twofold saddle point structure, which is similar to the one analyzed in [31] for the Stokes–Darcy coupled problem with nonlinear viscosity. The formulation is then written equivalently as a fixed point equation, and the well-known Schauder and Banach theorems, as well as the abstract theory developed in [31], which is based on classical results on bijective monotone operators, are applied to prove the unique solvability of the continuous and discrete systems. A feasible choice of finite element subspaces for the formulation introduced in [12] is given by piecewise constants, Raviart–Thomas spaces of lowest order, continuous piecewise linear elements, and piecewise constants for the strain, Cauchy stress, velocity, and vorticity in the fluid, respectively, whereas Raviart–Thomas spaces of lowest order and piecewise constants for the velocity and pressure, together with continuous piecewise linear elements for the Lagrange multipliers, can be utilized in the Darcy region. Optimal a priori error estimates were also derived.

Now, it is well known that under the eventual presence of singularities, as well as when dealing with nonlinear problems, as in the present case, most of the standard Galerkin procedures such as finite element and mixed finite element methods inevitably lose accuracy, and hence one usually tries to recover it by applying an adaptive algorithm based on a posteriori error estimates. In this direction, and particularly for the coupling of fluid flows with porous media flows, we refer to [2, 7, 14, 18, 19, 37, 38, 42, 44, 45, 49] where different contributions addressing this interesting issue, most of them devoted to the Stokes-Darcy coupled problem, can be found. Up to the authors' knowledge, the first work dealing with adaptive algorithms for the Navier-Stokes/Darcy coupling is [42], where an a posteriori error estimator for a discontinuous Galerkin approximation of this coupled problem with constant parameters is proposed.

According to the above discussion, and in order to complement the study started in [12] for the Navier-Stokes/Darcy equations with variable viscosity, in this paper we proceed similarly to [37, 38] and [7], and develop an a posteriori error analysis for the finite element method studied in [12]. More precisely, assuming a smallness condition on the data, we derive a reliable and efficient residual-based a posteriori error estimator for the three dimensional version of the augmented-mixed method introduced in [12]. The global inf-sup condition, a suitable Helmholtz decomposition recently provided in [27], and the approximation properties of the Clemént and Raviart-Thomas operators, among others, are the main tools yielding the reliability. In turn, the efficiency estimate is consequence of standard arguments such as inverse inequalities, the localization technique based on bubble functions, and other known results to be specified later on in Section 3.4. The rest of this work is organized as follows. In Section 2 we recall from [12, Section 2] the model problem and its continuous and discrete augmented fully-mixed variational formulations. In Section 3, we derive the a posteriori error estimator. The reliability analysis is carried out in Section 3.3, whereas in Section 3.4 we provide the efficiency analysis. Finally, some numerical results confirming the reliability and efficiency of the a posteriori error estimator and showing the good performance of the associated adaptive algorithm for the fully-mixed finite element method, are presented in Section 4.

We end this section by introducing some definitions and fixing some notations. Given the vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, with $n \in \{2,3\}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor field $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} := (\zeta_{ij})_{i,j=1,n}$, we define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Additionally, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, Γ is an open or closed Lipschitz curve (respectively surface in $\mathbb{R}^3$), and $s \in \mathbb{R}$, we define

$$\mathbf{H}^s(\mathcal{O}) := [\mathrm{H}^s(\mathcal{O})]^n, \quad \mathbb{H}^s(\mathcal{O}) := [\mathrm{H}^s(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^s(\Gamma) := [\mathrm{H}^s(\Gamma)]^n.$$

However, when $s = 0$ we usually write $\mathbf{L}^2(\mathcal{O}), \mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O}), \mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$ for $\mathrm{H}^s(\mathcal{O})$, $\mathbf{H}^s(\mathcal{O})$ and $\mathbb{H}^s(\mathcal{O})$, and $\|\cdot\|_{s,\Gamma}$ for $\mathrm{H}^s(\Gamma)$ and $\mathbf{H}^s(\Gamma)$. For $s \geq 0$, we write $|\cdot|_{s,\mathcal{O}}$ for the $\mathbf{H}^s$-seminorm. In addition, we recall that

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) :\ \operatorname{div}\mathbf{w} \in L^2(\mathcal{O})\right\},$$

is a standard Hilbert space in the realm of mixed problems (see, e.g. [5, 26, 41]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\operatorname{div};\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div};\mathcal{O})$. The norms of $\mathbf{H}(\operatorname{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\operatorname{div},\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div},\mathcal{O}}$, respectively. On the other hand, the following symbols for the $\mathrm{L}^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda\rangle_\Gamma := \int_\Gamma \xi\lambda \quad \forall \xi,\lambda \in \mathrm{L}^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda}\rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall \boldsymbol{\xi},\boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)$, respectively. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^n$, $P_k(S)$ denotes the space of polynomial functions defined on S of degree $\leq k$. In addition, we set $\mathbf{P}_k(S) := [P_k(S)]^n$ and $\mathbb{P}_k(S) := [P_k(S)]^{n\times n}$. Finally, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The Navier–Stokes/Darcy coupled problem

In this section we recall from [12] the Navier–Stokes/Darcy model, its fully-mixed variational formulation, the associated Galerkin scheme, and the main results concerning the corresponding solvability analysis.

2.1 The model problem

In order to describe the geometry under consideration we let Ω_{S} and Ω_{D} be bounded and simply connected open polyhedral domains in $\mathbb{R}^n$, such that $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$ and $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Fig. 2.1 below) and by $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$. The problem we are interested in consists of the movement of an incompressible quasi-Newtonian viscous fluid occupying Ω_{S} which flows towards and from a porous medium Ω_{D} through Σ, where Ω_{D} is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In Ω_{S}, the governing equations are those of the Navier–Stokes problem with constant density and variable viscosity, which are written in the following nonstandard stress-velocity-pressure formulation:

$$\begin{gathered}\boldsymbol{\sigma}_{\mathrm{S}} = \mu(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbb{I} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \operatorname{div} \mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \\ -\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}},\end{gathered} \tag{2.1}$$

where $\boldsymbol{\sigma}_{\mathrm{S}}$ is the nonlinear stress tensor, $\mathbf{u}_{\mathrm{S}}$ is the velocity, p_{S} is the pressure, $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is the nonlinear kinematic viscosity, $\mathbf{e}(\mathbf{u}_{\mathrm{S}}) := \frac{1}{2}\Big\{\nabla\mathbf{u}_{\mathrm{S}} + (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\Big\}$ is the strain tensor (or symmetric part of the velocity gradient) and $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ is a known volume force.

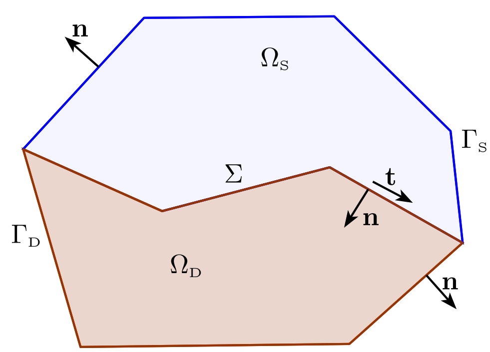

Figure 2.1: Sketch of a 2D geometry of our Navier–Stokes/Darcy model

Furthermore, we assume that μ is of class $\mathcal{C}^1$, and that there exist constants $\mu_1, \mu_2 > 0$, such that

$$\mu_1 \leq \mu(s) \leq \mu_2 \quad \text{and} \quad \mu_1 \leq \mu(s) + s\mu'(s) \leq \mu_2 \qquad \forall s \geq 0, \tag{2.2}$$

which, according to the results provided in [40, Theorem 3.8], implies Lipschitz continuity of the nonlinear operator induced by μ. This fact will be used later on in Sections 3.3 and 3.4. In addition, it is easy to see that the forthcoming analysis also applies to the slightly more general case of a viscosity function acting on $\Omega \times \mathbb{R}^+$, that is $\mu : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. Some examples of nonlinear μ are the following:

$$\mu(s) := 2 + \frac{1}{1+s} \quad \text{and} \quad \mu(s) := \alpha_0 + \alpha_1(1+s^2)^{(\beta-2)/2}, \tag{2.3}$$

where $\alpha_0, \alpha_1 > 0$ and $\beta \in [1,2]$. The first example is basically academic but the second one corresponds to a particular case of the well-known Carreau law in fluid mechanics. It is easy to see that they both satisfy (2.2) with $(\mu_1, \mu_2) = (2,3)$ and $(\mu_1, \mu_2) = (\alpha_0, \alpha_0 + \alpha_1)$, respectively.

Next, we adopt the approach from [12] (see also [33, 34]) and introduce the additional unknowns $\mathbf{t}_{\mathrm{S}} := \mathbf{e}(\mathbf{u}_{\mathrm{S}})$ and $\boldsymbol{\rho}_{\mathrm{S}} := \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\right)$, where $\boldsymbol{\rho}_{\mathrm{S}}$ is the vorticity (or skew-symmetric part of the velocity gradient). In this way, we observe that the equations in (2.1) can be rewritten equivalently as

$$
\begin{gathered}
\mathbf{t}_{\mathrm{S}} = \nabla\mathbf{u}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} = \mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}} - (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \\
-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad p_{\mathrm{S}} = -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})) \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}.
\end{gathered} \tag{2.4}
$$

Note that the fourth equation in (2.4) allows us to eliminate the pressure p_{S} from the system and compute it as a simple post-process of the solution.

On the other hand, in Ω_{D} we consider the linearized Darcy model with homogeneous Neumann boundary condition on Γ_{D}:

$$
\mathbf{u}_{\mathrm{D}} = -\mathbf{K}\nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathrm{div}\,\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}, \tag{2.5}
$$

where $\mathbf{u}_{\mathrm{D}}$ and p_{D} denote the velocity and pressure, respectively, $f_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$ is a source term satisfying $\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0$, and $\mathbf{K} \in [\mathrm{L}^\infty(\Omega_{\mathrm{D}})]^{n\times n}$ is a positive definite symmetric tensor describing the permeability of Ω_{D} divided by a constant approximation of the viscosity.

Finally, the transmission conditions are given by

$$
\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} \;=\; \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n} + \sum_{l=1}^{n-1} \omega_l^{-1}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t}_l)\mathbf{t}_l \;=\; -p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.6}
$$

where $\{\omega_1, \ldots, \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.6) corresponds to mass conservation on Σ, whereas the second one establishes the balance of normal forces and a Beavers–Joseph–Saffman law.

2.2 The fully-mixed variational formulation

In this section we introduce the weak formulation derived in [12, Section 2.2] for the coupled problem given by (2.4), (2.5), and (2.6). To this end, let us first introduce further notations and definitions. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, $u, v \in \mathrm{L}^2(\Omega_\star)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$, we set

$$
(u, v)_\star := \int_{\Omega_\star} uv, \quad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.
$$

In addition, we let $\mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}})$ and $\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})$ be the subspaces of symmetric and skew-symmetric tensors of $\mathbb{L}^2(\Omega_{\mathrm{S}})$, respectively, that is

$$
\begin{aligned}
\mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}}) \quad &:= \left\{ \mathbf{r}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \quad \mathbf{r}_{\mathrm{S}}^{\mathrm{t}} = \mathbf{r}_{\mathrm{S}} \right\}, \\
\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \quad &:= \left\{ \boldsymbol{\eta}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \quad \boldsymbol{\eta}_{\mathrm{S}}^{\mathrm{t}} = -\boldsymbol{\eta}_{\mathrm{S}} \right\}.
\end{aligned}
$$

Furthermore, we define the spaces

$$
\begin{aligned}
\mathbf{H}_0(\mathrm{div}\,;\Omega_{\mathrm{D}}) \quad &:= \left\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \right\}, \\
\mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}}) \quad &:= \left\{ \mathbf{r}_{\mathrm{S}} \in \mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}}) : \quad \mathrm{tr}\,\mathbf{r}_{\mathrm{S}} = 0 \right\}, \\
\mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \quad &:= \left\{ v_{\mathrm{S}} \in \mathrm{H}^1(\Omega_{\mathrm{S}}) : \quad v_{\mathrm{S}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \right\}, \qquad \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \;:=\; [\mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})]^n,
\end{aligned}
$$

and the space of traces

$$\mathrm{H}_{00}^{1/2}(\Sigma) \,:=\, \Big\{ v|_\Sigma : \quad v \in \mathrm{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \Big\}\,, \qquad \mathbf{H}_{00}^{1/2}(\Sigma) \,:=\, [\mathrm{H}_{00}^{1/2}(\Sigma)]^n\,.$$

Equivalently, if $E_{0,\mathrm{S}} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega_\mathrm{S})$ is the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) := \begin{cases} \psi & \text{on } \Sigma \\ 0 & \text{on } \Gamma_\mathrm{S} \end{cases} \qquad \forall \psi \in \mathrm{H}^{1/2}(\Sigma)\,,$$

we have that

$$\mathrm{H}_{00}^{1/2}(\Sigma) \,=\, \Big\{ \psi \in \mathrm{H}^{1/2}(\Sigma) : \quad E_{0,\mathrm{S}}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_\mathrm{S}) \Big\}\,,$$

which is endowed with the norm $\|\psi\|_{1/2,00,\Sigma} := \|E_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_\mathrm{S}}$. In addition, $\|\cdot\|_{1/2,00,\Sigma}$ also stands for the corresponding product norm of $\mathbf{H}_{00}^{1/2}(\Sigma)$. In turn, $\mathrm{H}_{00}^{-1/2}(\Sigma)$ and $\mathbf{H}_{00}^{-1/2}(\Sigma)$ are the dual spaces of $\mathrm{H}_{00}^{1/2}(\Sigma)$ and $\mathbf{H}_{00}^{1/2}(\Sigma)$, respectively, with norms denoted in both cases by $\|\cdot\|_{-1/2,00,\Sigma}$.

Now, in order to deduce our variational system we need to add two auxiliary unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_\mathrm{S}|_\Sigma \in \mathbf{H}_{00}^{1/2}(\Sigma) \quad \text{and} \quad \lambda := p_\mathrm{D}|_\Sigma \in \mathrm{H}^{1/2}(\Sigma)\,.$$

In this way, our variational system will be written in terms of the unknowns $\underline{\mathbf{t}} := (\mathbf{t}_\mathrm{S}, \boldsymbol{\sigma}_\mathrm{S}, \mathbf{u}_\mathrm{S}, \boldsymbol{\rho}_\mathrm{S}, \mathbf{u}_\mathrm{D})$, $\underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda)$ and p_D. Let us recall from [12, Section 2.3] that, given any constant $c \in \mathbb{R}$, the vector defined by $((\mathbf{t}_\mathrm{S}, \boldsymbol{\sigma}_\mathrm{S} - c\mathbb{I}, \mathbf{u}_\mathrm{S}, \boldsymbol{\rho}_\mathrm{S}, \mathbf{u}_\mathrm{D}), (\boldsymbol{\varphi}, \lambda + c), p_\mathrm{D} + c)$ also becomes a solution of the problem defined below. Hence, in order to ensure uniqueness of solution, we will require the Darcy pressure p_D to live in $\mathrm{L}^2_0(\Omega_\mathrm{D})$, where

$$\mathrm{L}^2_0(\Omega_\mathrm{D}) := \Big\{ q \in \mathrm{L}^2(\Omega_\mathrm{D}) : \quad (q,1)_\mathrm{D} = 0 \Big\}\,.$$

Then, defining the spaces

$$\mathbf{X} := \mathbb{L}^2_{\mathrm{tr}}(\Omega_\mathrm{S}) \times \mathbb{H}(\mathbf{div}\,;\Omega_\mathrm{S}) \times \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) \times \mathbf{H}_0(\mathrm{div}\,;\Omega_\mathrm{D}),$$

$$\mathbf{M} := \mathbf{H}_{00}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma), \quad \mathbb{X} := \mathbf{X} \times \mathbf{M}, \quad \text{and} \quad \mathbb{M} := \mathrm{L}^2_0(\Omega_\mathrm{D}),$$

with $\mathbf{X}$, $\mathbf{M}$, $\mathbb{X}$ and $\mathbb{X} \times \mathbb{M}$ endowed with the product norms

$$\|\underline{\mathbf{r}}\|_\mathbf{X} := \|\mathbf{r}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{\tau}_\mathrm{S}\|_{\mathbf{div},\Omega_\mathrm{S}} + \|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + \|\boldsymbol{\eta}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{v}_\mathrm{D}\|_{\mathrm{div},\Omega_\mathrm{D}},$$

$$\|\underline{\boldsymbol{\psi}}\|_\mathbf{M} := \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} + \|\xi\|_{1/2,\Sigma}, \quad \|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_\mathbb{X} := \|\underline{\mathbf{r}}\|_\mathbf{X} + \|\underline{\boldsymbol{\psi}}\|_\mathbf{M},$$

$$\|((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), q_\mathrm{D})\|_{\mathbb{X}\times\mathbb{M}} := \|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_\mathbb{X} + \|q_\mathrm{D}\|_{0,\Omega},$$

as explained in [12, Section 2.2], we arrive at the following modified variational formulation for (2.4), (2.5), and (2.6): Find $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_\mathrm{D}) \in \mathbb{X} \times \mathbb{M}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_\mathrm{S})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), p_\mathrm{D}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall\, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X}, \\ [\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), q_\mathrm{D}] &= [\mathbf{G}, q_\mathrm{D}] \qquad \forall\, q_\mathrm{D} \in \mathbb{M}, \end{aligned} \tag{2.7}$$

where, given $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, the operator $\mathbf{A}(\mathbf{z}_\mathrm{S}) : \mathbb{X} \to \mathbb{X}'$ is defined by

$$[\mathbf{A}(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := [a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}), \underline{\mathbf{r}}] + [b(\underline{\mathbf{t}}), \underline{\boldsymbol{\psi}}] + [b(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}}] - [c(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}], \tag{2.8}$$

with

$$
\begin{aligned}
[a(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}] &:= [a_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}] + [a_2(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}],\\
[a_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}] &:= (\mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}},\mathbf{r}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{r}_{\mathrm{S}},\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}} + (\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}} + \kappa_1(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}}\\
&\quad + \kappa_2(\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\\
&\quad + (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\rho}_{\mathrm{S}})_{\mathrm{S}} - (\boldsymbol{\sigma}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} + \kappa_3(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \mathbf{t}_{\mathrm{S}},\mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}}\\
&\quad + \kappa_4\left(\boldsymbol{\rho}_{\mathrm{S}} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}),\boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}} + (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}},\\
[a_2(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}] &:= ((\mathbf{z}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}},\kappa_1\boldsymbol{\tau}^{\mathrm{d}} - \mathbf{r}_{\mathrm{S}})_{\mathrm{S}},\\
[b(\underline{\mathbf{r}}),\underline{\boldsymbol{\psi}}] &:= \langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} - \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma}\,,\\
[c(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}] &:= \langle\boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma} - \langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma} + \sum_{l=1}^{n-1}\omega_l^{-1}\,\langle\boldsymbol{\varphi}\cdot\mathbf{t}_l,\boldsymbol{\psi}\cdot\mathbf{t}_l\rangle_{\Sigma}\,,
\end{aligned}
\tag{2.9}
$$

whereas the operator $\mathbf{B}:\mathbb{X}\to\mathbb{M}'$ and the functionals $\mathbf{F}:\mathbb{X}\to\mathbb{R}$ and $\mathbf{G}:\mathbb{M}\to\mathbb{R}$ are given by

$$[\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),q_{\mathrm{D}}] := -(\mathrm{div}\,\mathbf{v}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}, \tag{2.10}$$

and

$$[\mathbf{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] := -\kappa_2(\mathbf{f}_{\mathrm{S}},\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \quad\text{and}\quad [\mathbf{G},q_{\mathrm{D}}] := -(f_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}. \tag{2.11}$$

In all the foregoing terms, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators and κ_i, $i\in\{1,\ldots,4\}$, are positive parameters to be specified below in Theorem 2.1.

Furthermore, we notice from (2.9) that, owing to the Cauchy–Schwarz and Hölder's inequalities, and the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega_{\mathrm{S}})$ into $\mathbf{L}^4(\Omega_{\mathrm{S}})$ (see e.g. [1, Theorem 6.3] or [46, Theorem 1.3.5]), there holds

$$|[a_2(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}]| \le c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2}\|\mathbf{z}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\underline{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\,\underline{\mathbf{t}},\underline{\mathbf{r}}\in\mathbf{X}, \tag{2.12}$$

where $c_2(\Omega_{\mathrm{S}}) := \|\mathbf{i}_c\|^2$. Additionally, we observe that (2.7) is equivalent to the variational formulation defined in [12, Section 2.2], in which $\boldsymbol{\sigma}_{\mathrm{S}}$ is decomposed as $\boldsymbol{\sigma}_{\mathrm{S}} = \boldsymbol{\sigma} + l\mathbb{I}$, with $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$ and $l\in\mathbb{R}$, where

$$\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}}) := \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\Omega_{\mathrm{S}}): \quad (\mathrm{tr}\,\boldsymbol{\tau},1)_{\mathrm{S}} = 0\Big\}.$$

The following result taken from [12] establishes the well-posedness of (2.7).

Theorem 2.1 *Assume that*

$$\kappa_1\in\left(0,\frac{2\delta_1\mu_1}{L_\mu}\right), \quad \kappa_2>0\,, \quad \kappa_3\in\left(0,2\delta_2\left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1}\right)\right), \quad \textit{and} \quad \kappa_4\in\left(0,2\delta_3 C_{\mathrm{Ko}}\kappa_3\left(1-\frac{\delta_2}{2}\right)\right),$$

with $L_\mu := \max\{\mu_2, 2\mu_2-\mu_1\}$, C_{Ko} *the Korn's constant given by* [12, eq. (3.10)], $\delta_1\in\left(0,\dfrac{2}{L_\mu}\right)$, $\delta_2\in(0,2)$, *and* $\delta_3\in(0,2)$. *In addition, given* $r\in(0,r_0)$, *with*

$$r_0 := \frac{\alpha_0(\Omega)}{2c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2}}\,, \tag{2.13}$$

where $c_2(\Omega_S)$ is the constant in (2.12) *and $\alpha_0(\Omega)$ is the strong monotonicity constant of the nonlinear operator a (see* [12, eq. (3.16)]*), we let $W_r := \Big\{ \mathbf{z}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S) : \quad \|\mathbf{z}_S\|_{1,\Omega_S} \le r \Big\}$, and assume that the data $\mathbf{f}_S$ and f_D satisfy*

$$c_{\mathbf{T}} \Big\{ \|\mathbf{f}_S\|_{0,\Omega_S} + \|f_D\|_{0,\Omega_D} \Big\} \le r\,, \tag{2.14}$$

where $c_{\mathbf{T}}$ is the positive constant, independent of the data, provided by [12, Lemma 3.6]. *Then, the augmented fully-mixed formulation* (2.7) *has a unique solution $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_D) \in \mathbb{X} \times \mathbb{M}$ with $\mathbf{u}_S \in W_r$, which satisfies*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_D)\|_{\mathbb{X}\times\mathbb{M}} \le c_{\mathbf{T}} \Big\{ \|\mathbf{f}_S\|_{0,\Omega_S} + \|f_D\|_{0,\Omega_D} \Big\}\,. \tag{2.15}$$

Proof. See [12, Theorem 3.11] for details. $\square$

2.3 The fully-mixed finite element method

Here, for clarity of exposition of the a posteriori error estimator to be defined next in Section 3, we restrict ourselves to the particular case provided in [12, Section 6.2] with $k = 0$ and introduce a Galerkin scheme for the 3D version of (2.7). To that end we let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D, which are formed by shape-regular tetrahedra T of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$. Then, for each $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ we set the local Raviart–Thomas space of lowest order,

$$\mathrm{RT}_0(T) := \mathbf{P}_0(T) + P_0(T)\mathbf{x}\,,$$

where $\mathbf{x}$ is a generic vector in $\mathbb{R}^3$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$), which is formed by triangles e of diameter h_e, and set $h_\Sigma := \max\{h_e : e \in \Sigma_h\}$. Furthermore, we introduce the following discrete subspaces

$$\begin{aligned}
\mathrm{L}^2_h(\Omega_\star) &:= \big\{ q_h \in \mathrm{L}^2(\Omega_\star) : \, q_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h^\star \big\}\,, \quad \star \in \{S, D\},\\
\mathbf{H}_h(\Omega_\star) &:= \{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \, \tau_h|_T \in \mathrm{RT}_0(T) \quad \forall T \in \mathcal{T}_h^\star \}\,, \quad \star \in \{S, D\},\\
\mathbf{H}^1_h(\Omega_S) &:= \big\{ \mathbf{v}_h \in [\mathcal{C}(\overline{\Omega}_S)]^3 : \, \mathbf{v}_h|_T \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_h^S \big\}\,,\\
\mathbb{L}^2_{\mathrm{tr}\,,h}(\Omega_S) &:= \big\{ \mathbf{r}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) : \, \mathbf{r}_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h^S \big\}\,,\\
\mathbb{L}^2_{\mathrm{skew},h}(\Omega_S) &:= \big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_S) : \, \boldsymbol{\eta}_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h^S \big\}\,.
\end{aligned}$$

In turn, in order to define the discrete spaces for the unknowns on the interface Σ, we introduce an independent triangulation $\widehat{\Sigma}_h$ of Σ, by triangles $\widehat{e}$ of diameter $h_{\widehat{e}}$, and define the associated meshsize $h_{\widehat{\Sigma}} := \max\{h_{\widehat{e}} : \widehat{e} \in \widehat{\Sigma}_h\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$\begin{aligned}
\Lambda^S_h(\Sigma) &:= \Big\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_{\widehat{e}} \in P_1(\widehat{e}) \quad \forall \widehat{e} \in \widehat{\Sigma}_h, \quad \psi_h = 0 \quad \text{on} \quad \partial\Sigma \Big\}\,,\\
\Lambda^D_h(\Sigma) &:= \Big\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_{\widehat{e}} \in P_1(\widehat{e}) \quad \forall \widehat{e} \in \widehat{\Sigma}_h \Big\}\,.
\end{aligned} \tag{2.16}$$

Employing the above notations, we set

$$\begin{aligned}
\mathbb{H}_h(\Omega_S) &:= \big\{ \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}\,;\Omega_S) : \, \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall\, \mathbf{c} \in \mathbb{R}^3 \big\}\,,\\
\mathbb{H}_{h,0}(\Omega_S) &:= \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_S),\\
\mathbf{H}^1_{h,\Gamma_S}(\Omega_S) &:= \mathbf{H}^1_h(\Omega_S) \cap \mathbf{H}^1_{\Gamma_S}(\Omega_S),
\end{aligned}$$

$$
\begin{aligned}
\mathbf{H}_{h,0}(\Omega_\mathrm{D}) &:= \mathbf{H}_h(\Omega_\mathrm{D}) \cap \mathbf{H}_0(\mathrm{div}\,;\Omega_\mathrm{D}),\\
L^2_{h,0}(\Omega_\mathrm{D}) &:= L^2_h(\Omega_\mathrm{D}) \cap \mathrm{L}^2_0(\Omega_\mathrm{D}),\\
\boldsymbol{\Lambda}^\mathrm{S}_h(\Sigma) &:= [\Lambda^\mathrm{S}_h(\Sigma)]^3.
\end{aligned}
$$

Then, defining the global spaces, unknowns, and test functions as follows

$$
\mathbf{X}_h := \mathbb{L}^2_{\mathrm{tr},h}(\Omega_\mathrm{S}) \times \mathbb{H}_{h,0}(\Omega_\mathrm{S}) \times \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew},h}(\Omega_\mathrm{S}) \times \mathbf{H}_{h,0}(\Omega_\mathrm{D}),
$$

$$
\mathbf{M}_h := \boldsymbol{\Lambda}^\mathrm{S}_h(\Sigma) \times \Lambda^\mathrm{D}_h(\Sigma), \quad \mathbb{X}_h := \mathbf{X}_h \times \mathbf{M}_h, \quad \mathbb{M}_h := L^2_{h,0}(\Omega_\mathrm{D}),
$$

$$
\underline{\mathbf{t}}_h := (\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\rho}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{M}_h,
$$

$$
\underline{\mathbf{r}}_h := (\mathbf{r}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{M}_h,
$$

$$
p_{\mathrm{D},h} \in \mathbb{M}_h, \quad \text{and} \quad q_{\mathrm{D},h} \in \mathbb{M}_h,
$$

the Galerkin scheme for problem (2.7) reads: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h}) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$
\begin{aligned}
{[\mathbf{A}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), p_{\mathrm{D},h}]} &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h,\\
{[\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), q_{\mathrm{D},h}]} &= [\mathbf{G}, q_{\mathrm{D},h}] \qquad \forall\, q_{\mathrm{D},h} \in \mathbb{M}_h.
\end{aligned}
\tag{2.17}
$$

The following theorem, also taken from [12], provides the well-posedness of (2.17), the associated Céa estimate, and the corresponding theoretical rate of convergence.

Theorem 2.2 *Assume that the conditions on $\kappa_i, i \in \{1, \ldots, 4\}$, required by Theorem 2.1 hold. In addition, given $r \in (0, r_0)$, with r_0 defined by (2.13), we let*

$$
W^h_r := \left\{ \mathbf{z}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \|\mathbf{z}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \le r \right\},
$$

and assume that the data $\mathbf{f}_\mathrm{S}$ and f_D satisfy

$$
\tilde{c}_\mathbf{T} \big\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \big\} \le r,
\tag{2.18}
$$

where $\tilde{c}_\mathbf{T}$ is the positive constant, independent of the data, provided by [12, Lemma 4.2]*. Then there exists a constant $C_0 > 0$ such that, whenever $h_\Sigma \le C_0 h_{\widehat{\Sigma}}$, there exists a unique $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h}) \in \mathbb{X}_h \times \mathbb{M}_h$ solution to problem (2.17) with $\mathbf{u}_{\mathrm{S},h} \in W^h_r$. In addition, there holds*

$$
\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h})\|_{\mathbb{X}\times\mathbb{M}} \le \tilde{c}_\mathbf{T} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\},
\tag{2.19}
$$

and there exists $C_1 > 0$, independent of h, h_Σ, and $h_{\widehat{\Sigma}}$, such that

$$
\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h})\|_{\mathbb{X}\times\mathbb{M}} \le C_1 \mathrm{dist}\big(((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_\mathrm{D}), \mathbb{X}_h \times \mathbb{M}_h \big).
$$

Assume further that there exists $\delta > 0$ such that $\mathbf{t}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$, $\boldsymbol{\sigma}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$, $\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{S} \in \mathbf{H}^\delta(\Omega_\mathrm{S})$, $\mathbf{u}_\mathrm{S} \in \mathbf{H}^{1+\delta}(\Omega_\mathrm{S})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\boldsymbol{\rho}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$, $\mathbf{u}_\mathrm{D} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$, and $\mathrm{div}\,\mathbf{u}_\mathrm{D} \in \mathrm{H}^\delta(\Omega_\mathrm{D})$. Then $p_\mathrm{D} \in \mathrm{H}^{1+\delta}(\Omega_\mathrm{D})$, $\lambda \in \mathrm{H}^{1/2+\delta}(\Sigma)$, and there exists $C_2 > 0$, independent of h, h_Σ, and $h_{\widehat{\Sigma}}$, such that

$$
\begin{aligned}
\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_\mathrm{D}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h})\|_{\mathbb{X}\times\mathbb{M}} \le\ & C_2\, h^\delta \Big\{ \|\mathbf{t}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{S}\|_{1+\delta,\Omega_\mathrm{S}}\\
& + \|\boldsymbol{\rho}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|p_\mathrm{D}\|_{1+\delta,\Omega_\mathrm{D}} \Big\}.
\end{aligned}
$$

Proof. We refer the reader to [12, Theorems 4.3, 5.4, and 6.2] for details. $\square$

We end this section by pointing out that the assumption $h_\Sigma \leq C_0 h_{\widehat{\Sigma}}$ required in Theorem 2.2 is needed to prove the discrete inf-sup condition for the bilinear form b (cf. (2.9)). We omit further details about this issue and refer the reader to [30, Lemma 7.5] for more details.

3 A residual-based a posteriori error estimator

In this section we derive a reliable and efficient residual-based *a posteriori* error estimator for the three dimensional Galerkin scheme (2.17). The corresponding a posteriori error analysis for the 2D case should be quite straightforward. We remark in advance that most of the proofs here make extensive use of estimates already available in the literature. In particular, we apply results from [27, 25, 34, 37, 39], among others.

3.1 Preliminaries

We begin by introducing further notations and definitions. First, given $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, we let $\mathcal{E}(T)$ be the set of faces of T, and denote by $\mathcal{E}_h$ the set of all faces of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{D}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_\star) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_\star\}$, $\mathcal{E}_h(\Omega_\star) := \{e \in \mathcal{E}_h : e \subseteq \Omega_\star\}$, for $\star \in \{\mathrm{S}, \mathrm{D}\}$, and the faces of $\mathcal{E}_h(\Sigma)$ are exactly those forming the previously defined partition Σ_h, that is $\mathcal{E}_h(\Sigma) := \{e \in \mathcal{E}_h : e \subseteq \Sigma\}$. Also, for each $e \in \mathcal{E}_h(\Omega_\star)$ we fix a unit normal $\mathbf{n}_e$, and then, given $\mathbf{v} = (v_1, v_2, v_3)^{\mathrm{t}} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\tau} := (\tau_{ij})_{3\times 3} \in \mathbb{L}^2(\Omega)$ such that $\mathbf{v}|_T \in \mathbf{C}(T)$ and $\boldsymbol{\tau}|_T \in \mathbb{C}(T)$ on each $T \in \mathcal{T}_h$, we let $[\![\mathbf{v}\times\mathbf{n}_e]\!]$ and $[\![\boldsymbol{\tau}\times\mathbf{n}_e]\!]$ be the corresponding jumps of the tangential traces across e. In other words, $[\![\mathbf{v}\times\mathbf{n}_e]\!] := (\mathbf{v}|_T - \mathbf{v}|_{T'})|_e \times \mathbf{n}_e$ and $[\![\boldsymbol{\tau}\times\mathbf{n}_e]\!] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e \times \mathbf{n}_e$, respectively, where T and T' are the elements of $\mathcal{T}_h^\star$ having e as a common face and

$$\boldsymbol{\tau} \times \mathbf{n}_e := \begin{pmatrix} (\tau_{11}, \tau_{12}, \tau_{13}) \times \mathbf{n}_e \\ (\tau_{21}, \tau_{22}, \tau_{23}) \times \mathbf{n}_e \\ (\tau_{31}, \tau_{32}, \tau_{33}) \times \mathbf{n}_e \end{pmatrix}.$$

From now on, when no confusion arises, we simple write $\mathbf{n}$ instead of $\mathbf{n}_e$. In the sequel we will also make use of the following differential operators:

$$\mathrm{curl}\,(\mathbf{v}) = \nabla \times \mathbf{v} := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$$

and

$$\underline{\mathbf{curl}}\,(\boldsymbol{\tau}) := \begin{pmatrix} \mathrm{curl}\,(\tau_{11}, \tau_{12}, \tau_{13}) \\ \mathrm{curl}\,(\tau_{21}, \tau_{22}, \tau_{23}) \\ \mathrm{curl}\,(\tau_{31}, \tau_{32}, \tau_{33}) \end{pmatrix}.$$

In turn, the tangential curl operator $\mathbf{curl}_{\mathbf{s}} : \mathrm{H}^{1/2}(\Sigma) \to \mathcal{L}(\mathrm{H}^{-1/2}(\Sigma))$, with $\mathcal{L}(\mathrm{H}^{-1/2}(\Sigma))$ denoting the tangential vector fields of order $-1/2$, will also be needed. This operator which can be defined by $\mathbf{curl}_{\mathbf{s}}(\xi) = \nabla\xi \times \mathbf{n}$ for any sufficiently smooth function ξ, is linear and continuous (see [6, Propositions 3.4 and 3.6] for details). A tensor version of $\mathbf{curl}_{\mathbf{s}}$, say $\underline{\mathbf{curl}}_{\mathbf{s}} : \mathbf{H}^{1/2}(\Sigma) \to \mathcal{L}(\mathbf{H}^{-1/2}(\Sigma))$, which is defined component-wise by $\mathbf{curl}_{\mathbf{s}}$, will be also utilized.

Let us now recall the main properties of the Raviart-Thomas interpolator of lowest order (see [5, 26, 41]) and the Clément operator onto the space of continuous piecewise linear functions [17]. We begin with the aforementioned Raviart–Thomas operator $\Pi_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h(\Omega_\star)$ (recall the definition of $\mathbf{H}_h(\Omega_\star)$ in Section 2.3), $\star \in \{\mathrm{S}, \mathrm{D}\}$, which is characterized by the identity

$$\int_e \Pi_h^\star \mathbf{v} \cdot \mathbf{n} = \int_e \mathbf{v} \cdot \mathbf{n} \quad \forall \text{ face } e \text{ of } \mathcal{T}_h^\star. \tag{3.1}$$

As a consequence of (3.1), there holds

$$\operatorname{div}(\Pi_h^\star \mathbf{v}) = \mathcal{P}_h^\star(\operatorname{div} \mathbf{v}), \tag{3.2}$$

where $\mathcal{P}_h^\star$ is the $\mathrm{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant functions on $\Omega_\star$. A tensor version of $\Pi_h^\star$, say $\mathbf{\Pi}_h^\star : \mathbb{H}^1(\Omega_\star) \to \mathbb{H}_h(\Omega_\star)$, which is defined row-wise by $\Pi_h^\star$, and a vector version of $\mathcal{P}_h^\star$, say $\mathbf{P}_h^\star$, which is the $\mathbf{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant vectors on $\Omega_\star$, might also be required. The local approximation properties of $\Pi_h^\star$ (and hence of $\mathbf{\Pi}_h^\star$) are established in the following lemma. For the corresponding proof we refer to [5] (see also [26]).

Lemma 3.1 *For each $\star \in \{\mathrm{S}, \mathrm{D}\}$ there exist constants $c_1, c_2 > 0$, independent of h, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$ there hold*

$$\|\mathbf{v} - \Pi_h^\star \mathbf{v}\|_{0,T} \le c_1 h_T \|\mathbf{v}\|_{1,T} \quad \forall T \in \mathcal{T}_h^\star,$$

and

$$\|\mathbf{v} \cdot \mathbf{n} - \Pi_h^\star \mathbf{v} \cdot \mathbf{n}\|_{0,e} \le c_2 h_e^{1/2} \|\mathbf{v}\|_{1,T_e} \quad \forall \text{ face } e \text{ of } \mathcal{T}_h^\star,$$

where T_e is a tetrahedron of $\mathcal{T}_h^\star$ containing e on its boundary.

In turn, the Clément operator $I_h^\star : \mathrm{H}^1(\Omega_\star) \to \mathrm{H}_h^1(\Omega_\star)$, with

$$\mathrm{H}_h^1(\Omega_\star) := \Big\{ v \in \mathcal{C}(\overline{\Omega}_\star) : \ v|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h^\star \Big\},$$

approximates optimally non-smooth functions by continuous piecewise linear functions. The local approximation properties of this operator are established in the following lemma (see [17]).

Lemma 3.2 *For each $\star \in \{\mathrm{S}, \mathrm{D}\}$ there exist constants $c_3, c_4, c_5 > 0$, independent of h, such that for all $v \in \mathrm{H}^1(\Omega_\star)$ there holds*

$$\|v - I_h^\star v\|_{0,T} \le c_3 h_T \|v\|_{1,\Delta_\star(T)} \quad \forall T \in \mathcal{T}_h^\star,$$

and

$$\|v - I_h^\star v\|_{0,e} \le c_4 h_e^{1/2} \|v\|_{1,\Delta_\star(e)} \quad \forall e \in \mathcal{E}_h,$$

where

$$\Delta_\star(T) := \cup \Big\{ T' \in \mathcal{T}_h^\star : \ T' \cap T \neq \emptyset \Big\} \quad \text{and} \quad \Delta_\star(e) := \cup \Big\{ T' \in \mathcal{T}_h^\star : \ T' \cap e \neq \emptyset \Big\}.$$

In what follows, a vector version of $I_h^\star$, say $\mathbf{I}_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h^1(\Omega_\star)$, which is defined component-wise by $I_h^\star$, will be needed as well.

For the forthcoming analysis we will also utilize a couple of results providing stable Helmholtz decompositions for $\mathbb{H}(\mathbf{div}; \Omega_\mathrm{S})$ and $\mathbf{H}_0(\operatorname{div}; \Omega_\mathrm{D})$. In this regard, we remark in advance that the decomposition for $\mathbf{H}_0(\operatorname{div}; \Omega_\mathrm{D})$ will require the boundary Γ_D to lie in a "*convex part*" of Ω_D, which means that there exists a convex domain containing Ω_D, and whose boundary contains Γ_D. More precisely, we have the following lemma.

Lemma 3.3

a) *For each* $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}\,;\Omega_{\mathrm{S}})$ *there exist* $\boldsymbol{\eta} \in \mathbf{H}^2(\Omega_{\mathrm{S}})$ *and* $\boldsymbol{\chi} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$ *such that*

$$\boldsymbol{\tau}_{\mathrm{S}} = \nabla\boldsymbol{\eta} + \underline{\mathbf{curl}}\,\boldsymbol{\chi} \quad in \quad \Omega_{\mathrm{S}} \quad and \quad \|\boldsymbol{\eta}\|_{2,\Omega_{\mathrm{S}}} + \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \le C_{\mathrm{S}}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}, \tag{3.3}$$

where C_{S} *is a positive constant independent of all the foregoing variables.*

b) *Assume that there exists a convex domain* Ξ *such that* $\Omega_{\mathrm{D}} \subseteq \Xi$ *and* $\Gamma_{\mathrm{D}} \subseteq \partial\Xi$*. Then, given* $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_0(\mathrm{div}\,;\Omega_{\mathrm{D}})$ *there exist* $w \in \mathrm{H}^2(\Omega_{\mathrm{D}})$ *and* $\boldsymbol{\beta} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$ *such that*

$$\mathbf{v}_{\mathrm{D}} = \nabla w + \mathrm{curl}\,\boldsymbol{\beta} \quad in \quad \Omega_{\mathrm{D}} \quad and \quad \|w\|_{2,\Omega_{\mathrm{D}}} + \|\boldsymbol{\beta}\|_{1,\Omega_{\mathrm{D}}} \le C_{\mathrm{D}}\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}, \tag{3.4}$$

where C_{D} *is a positive constant independent of all the foregoing variables, and*

$$\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) := \Big\{\boldsymbol{\beta} \in \mathbf{H}^1(\Omega_{\mathrm{D}}) : \quad \boldsymbol{\beta}|_{\Gamma_{\mathrm{D}}} \in \mathbf{P}_0(\Gamma_{\mathrm{D}})\Big\}.$$

Proof. See [27, Theorems 3.1 and 3.2]. □

We end this section with a lemma providing estimates in terms of local quantities for the $\mathrm{H}^{-1/2}_{00}(\Sigma)$ and $\mathrm{H}^{-1/2}(\Sigma)$ norms of functions in particular subspaces of $\mathrm{L}^2(\Sigma)$ and $\mathrm{H}^{-1/2}(\Sigma)\cap\mathrm{L}^2(\Sigma)$, respectively. More precisely, having in mind the definitions of $\Lambda^{\mathrm{S}}_h(\Sigma)$ and $\Lambda^{\mathrm{D}}_h(\Sigma)$ (cf. (2.16)), which are subspaces of $\mathrm{H}^{1/2}_{00}(\Sigma)$ and $\mathrm{H}^{1/2}(\Sigma)$, respectively, we introduce the orthogonal-type spaces

$$\Lambda^{\mathrm{S},\perp}_h(\Sigma) := \Big\{\lambda \in \mathrm{L}^2(\Sigma) : \quad \langle\lambda,\psi_h\rangle_\Sigma = 0 \quad \forall\,\psi_h \in \Lambda^{\mathrm{S}}_h(\Sigma)\Big\} \tag{3.5}$$

and

$$\Lambda^{\mathrm{D},\perp}_h(\Sigma) := \Big\{\lambda \in \mathrm{H}^{-1/2}(\Sigma)\cap\mathrm{L}^2(\Sigma) : \quad \langle\lambda,\xi_h\rangle_\Sigma = 0 \quad \forall\,\xi_h \in \Lambda^{\mathrm{D}}_h(\Sigma)\Big\}. \tag{3.6}$$

Then, the announced lemma is stated as follows.

Lemma 3.4 *Assume that for each* $e \in \Sigma_h$ *there exists* $\widehat{e} \in \widehat{\Sigma}_h$ *such that* $e \subseteq \widehat{e}$ *and* $h_{\widehat{e}} \le C_1\,h_e$*, with a constant* $C_1 > 0$ *independent of* h_Σ *and* $h_{\widehat{\Sigma}}$*. Then, there exists* $C > 0$*, independent of the aforementioned meshsizes, such that*

$$\|\lambda\|^2_{-1/2,00,\Sigma} \le C \sum_{e\in\Sigma_h} h_e\,\|\lambda\|^2_{0,e} \qquad \forall\,\lambda \in \Lambda^{\mathrm{S},\perp}_h(\Sigma)\,, \tag{3.7}$$

and

$$\|\lambda\|^2_{-1/2,\Sigma} \le C \sum_{e\in\Sigma_h} h_e\,\|\lambda\|^2_{0,e} \qquad \forall\,\lambda \in \Lambda^{\mathrm{D},\perp}_h(\Sigma)\,. \tag{3.8}$$

Proof. Given $\lambda \in \Lambda^{\mathrm{S},\perp}_h(\Sigma)$, we first observe that $\lambda \in \mathrm{H}^{-1/2}_{00}(\Sigma)$ and that

$$\|\lambda\|_{-1/2,00,\Sigma} = \sup_{\substack{\xi\in\mathrm{H}^{1/2}_{00}(\Sigma)\\ \xi\neq0}} \frac{\langle\lambda,\xi\rangle_\Sigma}{\|\xi\|_{1/2,00,\Sigma}} \le \sup_{\substack{v\in\mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\\ v\neq0}} \frac{\langle\lambda,v\rangle_\Sigma}{\|v\|_{1,\Omega_{\mathrm{S}}}}\,. \tag{3.9}$$

Next, we let $\widehat{\mathcal{T}}^{\mathrm{S}}_h$ be a regular triangulation of the domain Ω_{S} which coincides with $\widehat{\Sigma}_h$ on Σ, and let

$$\widehat{I}_h : \mathrm{H}^1(\Omega_{\mathrm{S}}) \to \widehat{Y}_h := \Big\{v \in \mathcal{C}(\overline{\Omega}_{\mathrm{S}}) : \, v|_T \in P_1(T) \quad \forall\,T \in \widehat{\mathcal{T}}^{\mathrm{S}}_h\Big\}$$

be the usual Clément operator (see Section 3.1). Then, since $\widehat{I}_h(v)|_\Sigma \in \Lambda_h^{\mathrm{S}}(\Sigma) \quad \forall\, v \in \mathrm{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, it follows from (3.5), (3.9), and the Cauchy-Schwarz inequality, that

$$\|\lambda\|_{-1/2,00,\Sigma} \,\le\, \sup_{\substack{v\in \mathrm{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})\\ v\neq 0}} \frac{\langle \lambda, v-\widehat{I}_h(v)\rangle_\Sigma}{\|v\|_{1,\Omega_\mathrm{S}}} \,\le\, \sup_{\substack{v\in \mathrm{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})\\ v\neq 0}} \frac{\displaystyle\sum_{e\in\Sigma_h} \|\lambda\|_{0,e}\|v-\widehat{I}_h(v)\|_{0,\widehat{e}}}{\|v\|_{1,\Omega_\mathrm{S}}}\,, \tag{3.10}$$

where we also use that $\|v-\widehat{I}_h(v)\|_{0,e} \,\le\, \|v-\widehat{I}_h(v)\|_{0,\widehat{e}}$. In turn, applying the second approximation property from Lemma 3.2, the estimate $h_{\widehat{e}} \le C_1\, h_e$, and the fact that the number of triangles of the macro-elements $\Delta(\widehat{e})$ are uniformly bounded, we find that

$$\begin{aligned}
&\sum_{e\in\Sigma_h} \|\lambda\|_{0,e}\|v-\widehat{I}_h(v)\|_{0,\widehat{e}} \,\le\, \sum_{e\in\Sigma_h} h_{\widehat{e}}^{1/2}\, \|\lambda\|_{0,e}\, \|v\|_{1,\Delta(\widehat{e})}\\
&\quad \le\, \left\{\sum_{e\in\Sigma_h} h_{\widehat{e}}\, \|\lambda\|_{0,e}^2\right\}^{1/2} \left\{\sum_{e\in\Sigma_h} \|v\|^2_{1,\Delta(\widehat{e})}\right\}^{1/2} \,\le\, C\, \left\{\sum_{e\in\Sigma_h} h_e\, \|\lambda\|_{0,e}^2\right\}^{1/2} \|v\|_{1,\Omega_\mathrm{S}}\,,
\end{aligned}$$

which, replaced back into (3.10), gives (3.7). The proof of (3.8), being similar to that of (3.7), is omitted. $\square$

3.2 The main result

In what follows we assume that the hypotheses of Theorem 2.1, Theorem 2.2, and Lemma 3.4, hold and let $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_\mathrm{D}) \in \mathbb{X}\times\mathbb{M}$ and $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h}) \in \mathbb{X}_h\times\mathbb{M}_h$ be the unique solutions of problems (2.7) and (2.17), respectively. Then, our global *a posteriori* error estimator is defined by:

$$\Theta := \left\{\sum_{T\in\mathcal{T}_h^\mathrm{S}} \Theta^2_{\mathrm{S},T} + \sum_{T\in\mathcal{T}_h^\mathrm{D}} \Theta^2_{\mathrm{D},T}\right\}^{1/2}\,, \tag{3.11}$$

where the local error indicators $\Theta^2_{\mathrm{S},T}$ (with $T\in\mathcal{T}_h^\mathrm{S}$) and $\Theta^2_{\mathrm{D},T}$ (with $T\in\mathcal{T}_h^\mathrm{D}$) are given by

$$\begin{aligned}
\Theta^2_{\mathrm{S},T} &:= \|\mathbf{f}_\mathrm{S} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} + \big\|\mathbf{f}_\mathrm{S} - \mathbf{P}_h^\mathrm{S}(\mathbf{f}_\mathrm{S})\big\|_{0,T} + \left\|\boldsymbol{\rho}_{\mathrm{S},h} - \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S},h} - (\nabla\mathbf{u}_{\mathrm{S},h})^\mathrm{t}\right)\right\|^2_{0,T}\\
&+ \|\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) - \mathbf{t}_{\mathrm{S},h}\|^2_{0,T} \,+\, \big\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}^\mathrm{t}_{\mathrm{S},h}\big\|^2_{0,T} + \left\|\boldsymbol{\sigma}^\mathrm{d}_{\mathrm{S},h} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^\mathrm{d}\right\|^2_{0,T}\\
&+ h_T^2\, \big\|\nabla\mathbf{u}_{\mathrm{S},h} - \big(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}\big)\big\|^2_{0,T} + h_T^2\, \big\|\underline{\mathbf{curl}}\,(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h})\big\|^2_{0,T}\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{S})} h_e\, \big\|[\![\big(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}\big)\times\mathbf{n}]\!]\big\|^2_{0,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_\mathrm{S})} h_e\, \big\|\big(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}\big)\times\mathbf{n}\big\|^2_{0,e}\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \Big\{h_e\, \big\|\big(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}\big)\times\mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h\big\|^2_{0,e} + h_e\, \|\boldsymbol{\varphi}_h + \mathbf{u}_{\mathrm{S},h}\|^2_{0,e}\Big\}\\
&+ \sum_{e\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \sum_{l=1}^2 \omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l + \lambda_h\mathbf{n}\right\|^2_{0,e},
\end{aligned} \tag{3.12}$$

and

$$\Theta_{\mathrm{D},T}^2 := \|f_\mathrm{D} - \operatorname{div}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\left\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right\|_{0,T}^2 + h_T^2\left\|\operatorname{curl}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\right\|_{0,T}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{D})} h_e\left\|[\![\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\times\mathbf{n}]\!]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_\mathrm{D})} h_e\left\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\times\mathbf{n}\right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{h_e\left\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\times\mathbf{n} + \mathbf{curl}_\mathbf{s}\lambda_h\right\|_{0,e}^2 + h_e\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 + h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2\right\}. \tag{3.13}$$

The main goal of the present Section 3 is to establish, under suitable assumptions, the existence of positive constants $C_{\mathtt{rel}}$ and $C_{\mathtt{eff}}$, independent of the meshsizes and the continuous and discrete solutions, such that

$$C_{\mathtt{eff}}\,\Theta + \text{h.o.t.} \le \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \le C_{\mathtt{rel}}\,\Theta, \tag{3.14}$$

where h.o.t. stands, eventually, for one or several terms of higher order.

The upper and lower bounds in (3.14), which are known as the reliability and efficiency of Θ, are derived below in Sections 3.3 and 3.4, respectively.

3.3 Reliability of Θ

Proceeding analogously to [12, Section 5.2], we first let $\mathbf{P} : \mathbb{X}\times\mathbb{M} \to (\mathbb{X}\times\mathbb{M})' := \mathbb{X}'\times\mathbb{M}'$ and $\mathbf{P}_h : \mathbb{X}_h\times\mathbb{M}_h \to (\mathbb{X}_h\times\mathbb{M}_h)' := \mathbb{X}_h'\times\mathbb{M}_h'$ be the nonlinear operators suggested by the left hand sides of (2.7) and (2.17) with the given velocity solutions $\mathbf{u}_\mathrm{S}\in W_r$ and $\mathbf{u}_{\mathrm{S},h}\in W_r^h$, that is

$$\begin{aligned}[\mathbf{P}(\vec{\mathbf{s}}),\vec{\mathbf{r}}] &:= [(a_1 + a_2(\mathbf{u}_\mathrm{S}))(\underline{\mathbf{s}}),\underline{\mathbf{r}}] + [b(\underline{\mathbf{s}}),\underline{\boldsymbol{\psi}}] + [b(\underline{\mathbf{r}}),\underline{\boldsymbol{\phi}}] - [c(\underline{\boldsymbol{\phi}}),\underline{\boldsymbol{\psi}}] \\ &\quad + [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}), r_\mathrm{D}] + [\mathbf{B}(\underline{\mathbf{s}},\underline{\boldsymbol{\phi}}), q_\mathrm{D}],\end{aligned} \tag{3.15}$$

for all $\vec{\mathbf{s}} = ((\underline{\mathbf{s}},\underline{\boldsymbol{\phi}}), r_\mathrm{D})$, $\vec{\mathbf{r}} = ((\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}), q_\mathrm{D}) \in \mathbb{X}\times\mathbb{M}$, and

$$\begin{aligned}[\mathbf{P}_h(\vec{\mathbf{s}}_h),\vec{\mathbf{r}}_h] &:= [(a_1 + a_2(\mathbf{u}_{\mathrm{S},h}))(\underline{\mathbf{s}}_h),\underline{\mathbf{r}}_h] + [b(\underline{\mathbf{s}}_h),\underline{\boldsymbol{\psi}}_h] + [b(\underline{\mathbf{r}}_h),\underline{\boldsymbol{\phi}}_h] - [c(\underline{\boldsymbol{\phi}}_h),\underline{\boldsymbol{\psi}}_h] \\ &\quad + [\mathbf{B}(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h), r_{\mathrm{D},h}] + [\mathbf{B}(\underline{\mathbf{s}}_h,\underline{\boldsymbol{\phi}}_h), q_{\mathrm{D},h}],\end{aligned} \tag{3.16}$$

for all $\vec{\mathbf{s}}_h = ((\underline{\mathbf{s}}_h,\underline{\boldsymbol{\phi}}_h), r_{\mathrm{D},h})$, $\vec{\mathbf{r}}_h = ((\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h), q_{\mathrm{D},h}) \in \mathbb{X}_h\times\mathbb{M}_h$. Then, setting $\mathcal{F} := (\mathbf{F},\mathbf{G}) \in \mathbb{X}'\times\mathbb{M}'$, it is clear from (2.7) and (2.17) that $\mathbf{P}$ and $\mathbf{P}_h$ satisfy

$$[\mathbf{P}(\vec{\mathbf{t}}),\vec{\mathbf{r}}] = [\mathcal{F},\vec{\mathbf{r}}] \qquad \forall\,\vec{\mathbf{r}}\in\mathbb{X}\times\mathbb{M} \tag{3.17}$$

and

$$[\mathbf{P}_h(\vec{\mathbf{t}}_h),\vec{\mathbf{r}}_h] = [\mathcal{F},\vec{\mathbf{r}}_h] \qquad \forall\,\vec{\mathbf{r}}_h\in\mathbb{X}_h\times\mathbb{M}_h, \tag{3.18}$$

respectively. In addition, since μ is assumed to be of class C^1 (cf. (2.2)), we find, as explained in [12, Section 5.2], that a_1 (cf. (2.9)) has hemi-continuous first order Gâteaux derivative $\mathcal{D}a_1 : \mathbf{X} \to \mathcal{L}(\mathbf{X},\mathbf{X}')$. In this way, the Gâteaux derivative of $\mathbf{P}$ at $\vec{\mathbf{s}}$ is obtained by replacing $[a_1(\cdot),\cdot]$ in (3.15) by $\mathcal{D}a_1(\vec{\mathbf{s}})(\cdot,\cdot)$ (see [12, Lemma 5.3] for details), that is

$$\begin{aligned}\mathcal{D}\mathbf{P}(\vec{\mathbf{s}})(\vec{\mathbf{t}},\vec{\mathbf{r}}) &:= \mathcal{D}a_1(\underline{\mathbf{s}})(\underline{\mathbf{t}},\underline{\mathbf{r}}) + [a_2(\mathbf{u}_\mathrm{S})(\underline{\mathbf{t}}),\underline{\mathbf{r}}] + [b(\underline{\mathbf{t}}),\underline{\boldsymbol{\psi}}] + [b(\underline{\mathbf{r}}),\underline{\boldsymbol{\varphi}}] - [c(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}] \\ &\quad + [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}), p_\mathrm{D}] + [\mathbf{B}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}), q_\mathrm{D}],\end{aligned}$$

for all $\vec{\mathbf{t}} = ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_{\mathrm{D}})$, $\vec{\mathbf{r}} = ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), q_{\mathrm{D}}) \in \mathbb{X} \times \mathbb{M}$, which, according to [12, Lemma 5.3], becomes a uniformly bounded (with respect to $\vec{\mathbf{s}}$) bilinear form on $(\mathbb{X} \times \mathbb{M}) \times (\mathbb{X} \times \mathbb{M})$. Moreover, thanks to the assumptions on κ_i, $i \in \{1, \ldots, 4\}$, required by Theorem 2.1, recalling that c is positive-semidefinite, employing the continuous version of [12, Theorem 5.2], and proceeding again as in [12, Section 5.2], we deduce the existence of a positive constant $C_{\mathbf{P}}$, independent of $\vec{\mathbf{s}}$ and the continuous and discrete solutions, such that the following global inf-sup condition holds

$$C_{\mathbf{P}} \, \|\vec{\boldsymbol{\zeta}}\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \sup_{\substack{\vec{\mathbf{r}}\in\mathbb{X}\times\mathbb{M} \\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{\mathcal{D}\mathbf{P}(\vec{\mathbf{s}})(\vec{\boldsymbol{\zeta}}, \vec{\mathbf{r}})}{\|\vec{\mathbf{r}}\|_{\mathbb{X}\times\mathbb{M}}} \qquad \forall\, \vec{\boldsymbol{\zeta}} \in \mathbb{X} \times \mathbb{M}\,. \tag{3.19}$$

We are now in position of establishing the following preliminary a posteriori error estimate.

Theorem 3.5 *Given $r \in (0, r_0)$, with r_0 defined by* (2.13)*, assume that the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} satisfy*

$$\tilde{c}_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\} \,\leq\, \frac{C_{\mathbf{P}}\, r}{\alpha_0(\Omega)}\,, \tag{3.20}$$

where $\tilde{c}_{\mathbf{T}}$ and $\alpha_0(\Omega)$ are the positive constants, independent of the data, provided by [12, Lemma 4.2 and eq. (3.16)]*, and $C_{\mathbf{P}}$ is given above in* (3.19)*. Then, there holds*

$$\|\vec{\mathbf{t}} \,-\, \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \frac{2}{C_{\mathbf{P}}} \, \|\mathrm{R}\|_{\left(\mathbb{X}\times\mathbb{M}\right)'}\,, \tag{3.21}$$

where $\mathrm{R} : \mathbb{X} \times \mathbb{M} \to \mathbb{R}$ is the residual functional given by $\mathrm{R}(\vec{\mathbf{r}}) \,:=\, [\mathcal{F} - \mathbf{P}_h(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}] \quad \forall\, \vec{\mathbf{r}} \in \mathbb{X} \times \mathbb{M}$*, which satisfies*

$$\mathrm{R}(\vec{\mathbf{r}}_h) \,=\, 0 \qquad \forall\, \vec{\mathbf{r}}_h \,\in\, \mathbb{X}_h \times \mathbb{M}_h\,. \tag{3.22}$$

Proof. Since $\vec{\mathbf{t}}$ and $\vec{\mathbf{t}}_h$ belong to $\mathbb{X}\times\mathbb{M}$, a straightforward application of the mean value theorem yields the existence of a convex combination of $\vec{\mathbf{t}}$ and $\vec{\mathbf{t}}_h$, say $\vec{\mathbf{s}}_h \in \mathbb{X} \times \mathbb{M}$, such that (see for instance the proof of [39, Lemma 3.5])

$$\mathcal{D}\mathbf{P}(\vec{\mathbf{s}}_h)(\vec{\mathbf{t}} - \vec{\mathbf{t}}_h, \vec{\mathbf{r}}) = [\mathbf{P}(\vec{\mathbf{t}}) - \mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}] \qquad \forall\, \vec{\mathbf{r}} \in \mathbb{X} \times \mathbb{M}\,.$$

Then, using that $[\mathbf{P}(\vec{\mathbf{t}}), \vec{\mathbf{r}}] = [\mathcal{F}, \vec{\mathbf{r}}]$ (cf. (3.17)), and adding and subtracting $[\mathbf{P}_h(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}]$, it readily follows from the foregoing identity that

$$\mathcal{D}\mathbf{P}(\vec{\mathbf{s}}_h)(\vec{\mathbf{t}} - \vec{\mathbf{t}}_h, \vec{\mathbf{r}}) \,=\, \mathrm{R}(\vec{\mathbf{r}}) \,+\, [\mathbf{P}_h(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}] \qquad \forall\, \vec{\mathbf{r}} \in \mathbb{X} \times \mathbb{M}\,. \tag{3.23}$$

In turn, applying (3.19) with $\vec{\mathbf{s}} = \vec{\mathbf{s}}_h$ and $\vec{\boldsymbol{\zeta}} \,=\, \vec{\mathbf{t}} - \vec{\mathbf{t}}_h$, and employing (3.23), we deduce after minor algebraic manipulations that

$$C_{\mathbf{P}} \, \|\vec{\mathbf{t}} \,-\, \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \|\mathrm{R}\|_{\left(\mathbb{X}\times\mathbb{M}\right)'} \,+\, \sup_{\substack{\vec{\mathbf{r}}\in\mathbb{X}\times\mathbb{M} \\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{[\mathbf{P}_h(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}]}{\|\vec{\mathbf{r}}\|_{\mathbb{X}\times\mathbb{M}}}\,. \tag{3.24}$$

Next, according to the definitions of $\mathbf{P}$ and $\mathbf{P}_h$ (cf. (3.15) - (3.16)), and using the estimates (2.12) and (2.19), and the definition of r_0 (cf. (2.13)), we obtain

$$\begin{aligned}
\Big| [\mathbf{P}_h(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}] \Big| \,&=\, \Big| [a_2(\mathbf{u}_{\mathrm{S},h} - \mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}}_h), \underline{\mathbf{r}}] \Big| \\
&\leq\, c_2(\Omega_{\mathrm{S}})\, (\kappa_1^2 + 1)^{1/2} \, \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \, \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \, \|\underline{\mathbf{r}}\|_{\mathbf{X}} \\
&\leq\, c_2(\Omega_{\mathrm{S}})\, (\kappa_1^2 + 1)^{1/2} \, \|\vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \, \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \, \|\underline{\mathbf{r}}\|_{\mathbf{X}} \\
&\leq\, \frac{\alpha_0(\Omega)}{2 r_0} \, \tilde{c}_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\} \, \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \, \|\underline{\mathbf{r}}\|_{\mathbf{X}}\,,
\end{aligned}$$

which, thanks to the assumption (3.20) and the fact that $\dfrac{r}{r_0} \le 1$, yields

$$\left|[\mathbf{P}_h(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}]\right| \;\le\; \frac{C_{\mathbf{P}}}{2}\,\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}}\,\|\underline{\mathbf{r}}\|_{\mathbf{X}}\,.$$

Thus, replacing this estimate back into (3.24) we arrive at (3.21). Finally, the fact that R vanishes in $\mathbb{X}_h \times \mathbb{M}_h$, that is (3.22), follows straightforwardly from (3.18). □

According to the upper bound (3.21) provided by the previous lemma, it only remains now to estimate $\|\mathrm{R}\|_{\left(\mathbb{X}\times\mathbb{M}\right)'}$. To this end, we first observe that the functional R can be decomposed as

$$\mathrm{R}(\vec{\mathbf{r}}) := \mathrm{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) + \mathrm{R}_2(\mathbf{v}_{\mathrm{D}}) + \mathrm{R}_3(\mathbf{v}_{\mathrm{S}}) + \mathrm{R}_4(\boldsymbol{\eta}_{\mathrm{S}}) + \mathrm{R}_5(\mathbf{r}_{\mathrm{S}}) + \mathrm{R}_6(q_{\mathrm{D}}) + \mathrm{R}_7(\boldsymbol{\psi}) + \mathrm{R}_8(\xi)$$

for all $\vec{\mathbf{r}} = ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), q_{\mathrm{D}}) \in \mathbb{X}\times\mathbb{M}$, where

$$\begin{aligned}
\mathrm{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) :=&\quad -\kappa_2(\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - \kappa_1\big(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S},h} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\big)_{\mathrm{S}}\\
&\quad -(\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} - (\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\rho}_{\mathrm{S},h})_{\mathrm{S}} - (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S},h})_{\mathrm{S}} - \langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma\,,\\
\mathrm{R}_2(\mathbf{v}_{\mathrm{S}}) :=&\quad (\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - \kappa_3(\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) - \mathbf{t}_{\mathrm{S},h}, \mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}},\\
\mathrm{R}_3(\boldsymbol{\eta}_{\mathrm{S}}) :=&\quad (\boldsymbol{\sigma}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} - \kappa_4\left(\boldsymbol{\rho}_{\mathrm{S},h} - \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S},h} - (\nabla\mathbf{u}_{\mathrm{S},h})^{\mathrm{t}}\right), \boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}},\\
\mathrm{R}_4(\mathbf{r}_{\mathrm{S}}) :=&\quad \big(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S},h} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}, \mathbf{r}_{\mathrm{S}}\big)_{\mathrm{S}},\\
\mathrm{R}_5(\mathbf{v}_{\mathrm{D}}) :=&\quad -(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, p_{\mathrm{D},h})_{\mathrm{D}} + \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,,\\
\mathrm{R}_6(q_{\mathrm{D}}) :=&\quad -(f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}, q_{\mathrm{D}})_{\mathrm{D}},\\
\mathrm{R}_7(\boldsymbol{\psi}) :=&\quad -\langle\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \sum_{l=1}^{2}\omega_l^{-1}\langle\boldsymbol{\varphi}\cdot\mathbf{t}_l, \boldsymbol{\psi}\cdot\mathbf{t}_l\rangle_\Sigma - \langle\boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,,\\
\mathrm{R}_8(\xi) :=&\quad \langle\boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma\,.
\end{aligned}$$

In this way, it follows that

$$\begin{aligned}
\|\mathrm{R}\|_{\left(\mathbb{X}\times\mathbb{M}\right)'} \;\le\; \Big\{&\|\mathrm{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'} \;+\; \|\mathrm{R}_2\|_{\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})'} \;+\; \|\mathrm{R}_3\|_{\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})'} \;+\; \|\mathrm{R}_4\|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}})'}\\
&+ \;+\; \|\mathrm{R}_5\|_{\mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})'} \;+\; \|\mathrm{R}_6\|_{\mathrm{L}^2_0(\Omega_{\mathrm{D}})'} \;+\; \|\mathrm{R}_7\|_{\mathbf{H}^{-1/2}_{00}(\Sigma)} \;+\; \|\mathrm{R}_8\|_{\mathrm{H}^{-1/2}(\Sigma)}\Big\}\,,
\end{aligned} \tag{3.25}$$

and hence our next purpose is to derive suitable upper bounds for each one of the terms on the right hand side of (3.25). We start with the following lemma, which is a direct consequence of the Cauchy–Schwarz inequality.

Lemma 3.6 *There exist C_2, $C_3 > 0$, independent of the meshsizes, such that*

$$\|\mathrm{R}_2\|_{\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})'} \;\le\; C_2\left\{\sum_{T\in\mathcal{T}^{\mathrm{S}}_h}\|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} + \|\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) - \mathbf{t}_{\mathrm{S},h}\|^2_{0,T}\right\}^{1/2}$$

and

$$\|\mathrm{R}_3\|_{\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})'} \;\le\; C_3\left\{\sum_{T\in\mathcal{T}^{\mathrm{S}}_h}\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}^{\mathrm{t}}_{\mathrm{S},h}\|^2_{0,T} + \left\|\boldsymbol{\rho}_{\mathrm{S},h} - \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S},h} - (\nabla\mathbf{u}_{\mathrm{S},h})^{\mathrm{t}}\right)\right\|^2_{0,T}\right\}^{1/2}.$$

In addition, there holds

$$\|\mathrm{R}_4\|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}})'} \,\le\, \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \left\|\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S},h} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}\right\|^2_{0,T}\right\}^{1/2}$$

and

$$\|\mathrm{R}_6\|_{\mathrm{L}^2_0(\Omega_{\mathrm{D}})'} \,\le\, \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|^2_{0,T}\right\}^{1/2}.$$

Next, we derive the upper bounds for R_7 and R_8, the functionals acting on the interface Σ.

Lemma 3.7 *There exist C_7, $C_8>0$, independent of the meshsizes, such that*

$$\|\mathrm{R}_7\|_{\mathbf{H}^{-1/2}_{00}(\Sigma)} \,\le\, C_7\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l + \lambda_h\mathbf{n}\right\|^2_{0,e}\right\}^{1/2}, \tag{3.26}$$

and

$$\|\mathrm{R}_8\|_{\mathrm{H}^{-1/2}(\Sigma)} \,\le\, C_8\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\left\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\right\|^2_{0,e}\right\}^{1/2}. \tag{3.27}$$

Proof. It is clear from the definition of R_7 that

$$\mathrm{R}_7(\boldsymbol{\psi}) \,=\, -\left\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} \,-\, \sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l \,+\, \lambda_h\mathbf{n},\boldsymbol{\psi}\right\rangle_\Sigma \quad \forall\,\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)\,,$$

which certainly yields

$$\|\mathrm{R}_7\|_{\mathbf{H}^{-1/2}_{00}(\Sigma)} \,=\, \left\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l + \lambda_h\mathbf{n}\right\|_{-1/2,00,\Sigma}. \tag{3.28}$$

Then, taking $\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)$ and then $(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h) = (\mathbf{0},(\boldsymbol{\psi}_h,0))\in\mathbb{X}_h$ in the first equation of (2.17), we deduce that

$$\left\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} \,-\, \sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l \,+\, \lambda_h\mathbf{n},\boldsymbol{\psi}_h\right\rangle_\Sigma \,=\, 0 \qquad \forall\,\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)\,,$$

which says that each component of $\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l + \lambda_h\mathbf{n}$ belongs to $\Lambda^{\mathrm{S},\perp}_h(\Sigma)$ (cf. (3.5)). In this way, (3.26) follows from (3.28) and a direct component-wise application of (3.7) (cf. Lemma 3.4). In turn, the proof of (3.27) proceeds analogously by noting now that $\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}\in\Lambda^{\mathrm{D},\perp}_h(\Sigma)$ (cf. (3.6)), and then by applying (3.8) (cf. Lemma 3.4). $\square$

Our next goal is to derive the upper bound for R_1, for which, given $\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, we consider its Helmholtz decomposition provided by part a) of Lemma 3.3. More precisely, we let $\boldsymbol{\eta}\in\mathbf{H}^2(\Omega_{\mathrm{S}})$ and $\boldsymbol{\chi}\in\mathbb{H}^1(\Omega_{\mathrm{S}})$ be such that $\boldsymbol{\tau}_{\mathrm{S}} \,=\, \nabla\boldsymbol{\eta} + \underline{\mathbf{curl}}\,\boldsymbol{\chi}$ in Ω_{S}, and

$$\|\boldsymbol{\eta}\|_{2,\Omega_{\mathrm{S}}} + \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \,\le\, C_{\mathrm{S}}\,\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}\,. \tag{3.29}$$

Then, defining $\boldsymbol{\tau}_{\mathrm{S},h} := \boldsymbol{\Pi}_h^{\mathrm{S}}(\nabla\boldsymbol{\eta}) + \underline{\mathbf{curl}}\,(\mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) \in \mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$ (cf. Section 3.1), which can be seen as a discrete Helmholtz decomposition of $\boldsymbol{\tau}_{\mathrm{S},h}$, and applying from (3.22) that $\mathrm{R}_1(\boldsymbol{\tau}_{\mathrm{S},h}) = 0$, we can write

$$\mathrm{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) \,=\, \mathrm{R}_1(\boldsymbol{\tau}_{\mathrm{S}} - \boldsymbol{\tau}_{\mathrm{S},h}) \,=\, \mathrm{R}_1(\nabla\boldsymbol{\eta} - \boldsymbol{\Pi}_h^{\mathrm{S}}(\nabla\boldsymbol{\eta})) \,+\, \mathrm{R}_1(\underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}))\,.$$

Consequently, we now require to bound the expressions on the right hand side of the foregoing equation, which is provided by the following two lemmas.

Lemma 3.8 *There exists $C > 0$, independent of the meshsizes, such that for each $\boldsymbol{\eta} \in \mathbf{H}^2(\Omega_{\mathrm{S}})$ there holds*

$$|\mathrm{R}_1(\nabla\boldsymbol{\eta} - \boldsymbol{\Pi}_h^{\mathrm{S}}(\nabla\boldsymbol{\eta}))| \,\le\, C\left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \widehat{\Theta}_{1,T}^2\right\}^{1/2} \|\boldsymbol{\eta}\|_{2,\Omega}\,, \tag{3.30}$$

where

$$\begin{aligned}\widehat{\Theta}_{1,T}^2 \,=\, & h_T^2\left\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} \,-\, \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} \,+\, (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}\right\|_{0,T}^2 \,+\, \|\mathbf{f}_{\mathrm{S}} - \mathbf{P}_h^{\mathrm{S}}(\mathbf{f}_{\mathrm{S}})\|_{0,T}^2 \\ &+\; h_T^2\,\left\|\nabla\mathbf{u}_{\mathrm{S},h} - (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h})\right\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_h + \mathbf{u}_{\mathrm{S},h}\|_{0,e}^2\,.\end{aligned} \tag{3.31}$$

Proof. It follows almost straightforwardly from a slight modification of the proof of [39, Lemma 3.10] (see also [37, Lemma 3.6]). We omit further details. □

Lemma 3.9 *There exists $C > 0$, independent of the meshsizes, such that for each $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ there holds*

$$|\mathrm{R}_1(\underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}))| \,\le\, C\left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \widehat{\Theta}_{2,T}^2\right\}^{1/2} \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}}\,, \tag{3.32}$$

where

$$\begin{aligned}\widehat{\Theta}_{2,T}^2 \,=\, & \left\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}\right\|_{0,T}^2 \,+\, h_T^2\,\left\|\underline{\mathbf{curl}}\,(\mathbf{t}_{\mathrm{S},h} \,+\, \boldsymbol{\rho}_{\mathrm{S},h})\right\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\,\left\|[\![(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h})\times\mathbf{n}]\!]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\,\left\|(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h})\times\mathbf{n}\right\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\left\|(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h})\times\mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h\right\|_{0,e}^2\,.\end{aligned} \tag{3.33}$$

Proof. Given $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$, we first notice from the definition of R_1 that there holds

$$\mathrm{R}_1(\underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})) \,=\, \widetilde{\mathrm{T}}_1(\boldsymbol{\chi}) \,+\, \widehat{\mathrm{T}}_1(\boldsymbol{\chi})\,,$$

where

$$\widetilde{\mathrm{T}}_1(\boldsymbol{\chi}) \,:=\, -\,\kappa_1\big(\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h} \,+\, (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}, \underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})\big)_{\mathrm{S}}\,,$$

and, denoting $\boldsymbol{\zeta}_h := \mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}$,

$$\widehat{\mathrm{T}}_1(\boldsymbol{\chi}) \,:=\, -\,(\boldsymbol{\zeta}_h, \underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}))_{\mathrm{S}} - \big\langle\underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})\mathbf{n}, \boldsymbol{\varphi}_h\big\rangle_{\Sigma}\,.$$

For estimating $\widetilde{\mathrm{T}}_1(\boldsymbol{\chi})$ we proceed as in the proof of [39, Lemma 3.9] and apply the boundedness of $\mathbf{I}_h^{\mathrm{S}} : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{H}^1(\Omega_{\mathrm{S}})$ ([24, Lemma 1.127, pag. 69]), as well as the Cauchy-Schwarz and triangle inequalities, to obtain

$$|\widetilde{\mathrm{T}}_1(\boldsymbol{\chi})| \le C \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \left\| \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}} \right\|_{0,T}^2 \right\}^{1/2} \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} . \tag{3.34}$$

Next, for $\widehat{\mathrm{T}}_1(\boldsymbol{\chi})$ we first apply the identities from [41, Chapter I, eq. (2.17) and Theorem 2.11] to deduce that

$$\left\langle \underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})\mathbf{n}, \boldsymbol{\varphi}_h \right\rangle_\Sigma = \left\langle \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h, \boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi} \right\rangle_\Sigma = \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h : (\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) . \tag{3.35}$$

Then, analogously to the proof of [39, Lemma 3.9], we integrate by parts $\left(\boldsymbol{\zeta}_h, \underline{\mathbf{curl}}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})\right)_{\mathrm{S}}$ on each $T\in\mathcal{T}_h^{\mathrm{S}}$, and add (3.35) to the resulting expression, to obtain

$$\begin{aligned} \widehat{\mathrm{T}}_1(\boldsymbol{\chi}) \,=\, & - \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \int_T \underline{\mathbf{curl}}\,\boldsymbol{\zeta}_h : (\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) \,-\, \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{S}})} \int_e [\![\boldsymbol{\zeta}_h\times\mathbf{n}]\!] : (\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) \\ & - \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} \int_e \boldsymbol{\zeta}_h\times\mathbf{n} : (\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) \,-\, \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\boldsymbol{\zeta}_h\times\mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h) : (\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}) . \end{aligned} \tag{3.36}$$

In this way, applying the Cauchy–Schwarz inequality, the approximation properties of the Clément interpolator $\mathbf{I}_h^{\mathrm{S}}$ (cf. Lemma 3.2) and the fact that the number of triangles of the macro-elements $\Delta_{\mathrm{S}}(T)$ and $\Delta_{\mathrm{S}}(e)$ are uniformly bounded, we deduce from (3.36) that

$$\begin{aligned} |\widehat{\mathrm{T}}_1(\boldsymbol{\chi})| \,\le\, & \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Bigg\{ h_T^2 \,\|\underline{\mathbf{curl}}\,(\boldsymbol{\zeta}_h)\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e \,\|[\![\boldsymbol{\zeta}_h\times\mathbf{n}]\!]\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \,\|\boldsymbol{\zeta}_h\times\mathbf{n}\|_{0,e}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \,\|\boldsymbol{\zeta}_h\times\mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}}\boldsymbol{\varphi}_h\|_{0,e}^2 \Bigg\}^{1/2} \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} , \end{aligned} \tag{3.37}$$

which together with (3.34) implies (3.32) and concludes the proof. □

As a direct consequence of Lemmas 3.8 and 3.9, and the stability estimate (3.29) for the Helmholtz decomposition, we obtain the following upper bound for $\|\mathrm{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'}$.

Lemma 3.10 *There exists $C_1 > 0$, independent of the meshsizes, such that*

$$\|\mathrm{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'} \,\le\, C_1 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \widehat{\Theta}_{\mathrm{S},T}^2 \right\}^{1/2} ,$$

where

$$\widehat{\Theta}_{\mathrm{S},T}^2 \,=\, \widehat{\Theta}_{1,T}^2 \,+\, \widehat{\Theta}_{2,T}^2 \,-\, h_T^2 \left\| \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} \,-\, \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} \,+\, (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}} \right\|_{0,T}^2 ,$$

that is

$$\widehat{\Theta}_{S,T}^2 := \left\|\mathbf{f}_S - \mathbf{P}_h^S(\mathbf{f}_S)\right\|_{0,T}^2 + \left\|\boldsymbol{\sigma}_{S,h}^d - \mu(|\mathbf{t}_{S,h}|)\mathbf{t}_{S,h} + (\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})^d\right\|_{0,T}^2$$
$$+ h_T^2 \left\|\nabla\mathbf{u}_{S,h} - \big(\mathbf{t}_{S,h} + \boldsymbol{\rho}_{S,h}\big)\right\|_{0,T}^2 + h_T^2 \left\|\underline{\mathbf{curl}}\,(\mathbf{t}_{S,h} + \boldsymbol{\rho}_{S,h})\right\|_{0,T}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e \left\|[\![(\mathbf{t}_{S,h} + \boldsymbol{\rho}_{S,h})\times\mathbf{n}]\!]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e \left\|(\mathbf{t}_{S,h} + \boldsymbol{\rho}_{S,h})\times\mathbf{n}\right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e \left\|\big(\mathbf{t}_{S,h} + \boldsymbol{\rho}_{S,h}\big)\times\mathbf{n} + \underline{\mathbf{curl}}_{\mathbf{s}}\boldsymbol{\varphi}_h\right\|_{0,e}^2 + h_e \left\|\boldsymbol{\varphi}_h + \mathbf{u}_{S,h}\right\|_{0,e}^2 \right\}$$

Proof. It suffices to see that the first term defining $\widehat{\Theta}_{1,T}^2$ (cf. (3.31) in Lemma 3.8) is dominated by the first term defining $\widehat{\Theta}_{2,T}^2$ (cf. (3.33) in Lemma 3.9), which explains the substraction of the former in the original definition of $\widehat{\Theta}_{S,T}^2$. $\square$

Finally, the corresponding estimate for R_5 is given by the following lemma.

Lemma 3.11 *Assume that there exists a convex domain Ξ such that $\Omega_D \subseteq \Xi$ and $\Gamma_D \subseteq \partial\Xi$. Then there exists $C_5 > 0$, independent of the meshsizes, such that*

$$\|\mathrm{R}_5\|_{\mathbf{H}_0(\mathrm{div}\,;\Omega_D)'} \,\le\, C_5 \left\{ \sum_{T\in\mathcal{T}_h^D} \widehat{\Theta}_{D,T}^2 \right\}^{1/2},$$

where

$$\widehat{\Theta}_{D,T}^2 := h_T^2 \left\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\right\|_{0,T}^2 + h_T^2 \left\|\mathrm{curl}\,(\mathbf{K}^{-1}\mathbf{u}_{D,h})\right\|_{0,T}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)} h_e \left\|[\![\mathbf{K}^{-1}\mathbf{u}_{D,h}\times\mathbf{n}]\!]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_D)} h_e \left\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\times\mathbf{n}\right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e \left\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\times\mathbf{n} + \mathbf{curl}_{\mathbf{s}}\lambda_h\right\|_{0,e}^2 + h_e \left\|p_{D,h} - \lambda_h\right\|_{0,e}^2 \right\}.$$

Proof. The result follows analogously to the proof of Lemmas 3.8, 3.9, and 3.10, taking into account now the Helmholtz decomposition provided by part b) of Lemma 3.3, the fact that $\mathrm{R}_5(\mathbf{v}_{D,h}) = 0$ $\forall\,\mathbf{v}_{D,h} \in \mathbf{H}_{h,0}(\Omega_D)$ (which also follows from (3.22)), and the analogue of the integration by parts formula (3.35), which here becomes

$$\langle \mathrm{curl}\,\phi\cdot\mathbf{n}, \lambda_h\rangle_\Sigma \,=\, \langle \mathbf{curl}_{\mathbf{s}}\lambda_h, \phi\rangle_\Sigma \qquad \forall\,\phi\in \mathrm{H}^1(\Omega_D),$$

where $\mathbf{curl}_{\mathbf{s}}$ is the operator defined in Section 3.1. Additionally we refer to [37, Lemma 3.9] for the proof of the 2D version of this lemma. We omit further details. $\square$

We end this section by concluding that the reliability of Θ, that is the upper bound in (3.14), is a straightforward consequence of Lemmas 3.6, 3.7, 3.10, and 3.11.

3.4 Efficiency of Θ

We now aim to establish the lower bound in (3.14). For this purpose, we will make extensive use of the original system of equations given by (2.4)–(2.5)–(2.6), which is recovered from the augmented-mixed continuous formulation (2.7) by choosing suitable test functions and integrating by parts backwardly the corresponding equations.

We begin the derivation of the efficiency estimates with the following result.

Lemma 3.12 *There hold*

$$\begin{aligned}\big\|\mathbf{f}_{\mathrm{S}}-\mathbf{P}_h^{\mathrm{S}}(\mathbf{f}_{\mathrm{S}})\big\|_{0,T} \leq & \; 2\|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},T} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}},\\ \|\mathbf{f}_{\mathrm{S}}+\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \leq & \; \|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},T} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}},\\ \|f_{\mathrm{D}}-\operatorname{div}\mathbf{u}_{\mathrm{D},h}\|_{0,T} \leq & \; \|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|_{\operatorname{div},T} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{D}},\end{aligned}$$

and there exist constants $c_i>0$, $i\in\{1,\ldots,4\}$, independent of the meshsizes, such that

$$\begin{aligned}\|\boldsymbol{\sigma}_{\mathrm{S},h}-\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{t}}\|_{0,T} \leq & \; c_1\,\|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}},\\ \|\mathbf{e}(\mathbf{u}_{\mathrm{S},h})-\mathbf{t}_{\mathrm{S},h}\|_{0,T} \leq & \; c_2\Big\{\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{1,T}+\|\mathbf{t}_{\mathrm{S}}-\mathbf{t}_{\mathrm{S},h}\|_{0,T}\Big\} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}},\\ \left\|\boldsymbol{\rho}_{\mathrm{S},h}-\frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S},h}-(\nabla\mathbf{u}_{\mathrm{S},h})^{\mathrm{t}}\right)\right\|_{0,T} \leq & \; c_3\Big\{\|\boldsymbol{\rho}_{\mathrm{S}}-\boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T}+\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{1,T}\Big\} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}},\end{aligned}$$

and

$$\Big\|\nabla\mathbf{u}_{\mathrm{S},h}-\big(\mathbf{t}_{\mathrm{S},h}+\boldsymbol{\rho}_{\mathrm{S},h}\big)\Big\|_{0,T} \leq c_4\Big\{\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{1,T}+\|\mathbf{t}_{\mathrm{S}}-\mathbf{t}_{\mathrm{S},h}\|_{0,T}+\|\boldsymbol{\rho}_{\mathrm{S}}-\boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T}\Big\} \quad \forall\, T\in\mathcal{T}_h^{\mathrm{S}}.$$

Proof. It suffices to recall that $\mathbf{f}_{\mathrm{S}}=-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}$, $f_{\mathrm{D}}=\operatorname{div}\mathbf{u}_{\mathrm{D}}$, $\mathbf{t}_{\mathrm{S}}=\mathbf{e}(\mathbf{u}_{\mathrm{S}})$, $\boldsymbol{\rho}_{\mathrm{S}}=\frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S}}-(\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\right)$, and $\boldsymbol{\sigma}_{\mathrm{S}}=\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{t}}$. In particular, for the first estimate we refer to [39, Lemma 3.13]. Further details are omitted. $\square$

Now we turn to provide the corresponding estimates for the rest of terms defining Θ_{S} and Θ_{D}. To do that, we proceed similarly as in [37], [39], and [28] and apply some known results based on inverse inequalities (see [16]) and the localization technique (see [48]) based on tetrahedron-bubble and face-bubble functions. In particular, the following lemma provides local efficiency estimates for several terms on Σ.

Lemma 3.13 *There exist constants $c_i>0$, $i\in\{5,6,7,8\}$, independent of the meshsizes, such that*

a) $h_e\|p_{\mathrm{D},h}-\lambda_h\|_{0,e}^2 \,\leq\, c_5\left\{\|p_{\mathrm{D}}-p_{\mathrm{D},h}\|_{0,T_e}^2+h_T^2\|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2+h_e\|\lambda-\lambda_h\|_{0,e}^2\right\}$,
for all $e\in\mathcal{E}_h(\Sigma)$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{D}}$ having e as a face,

b) $h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \,\leq\, c_6\left\{\|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2+h_T^2\|\operatorname{div}(\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h})\|_{0,T_e}^2+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2\right\}$,
for all $e\in\mathcal{E}_h(\Sigma)$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{D}}$ having e as a face,

c) $$\begin{aligned}& h_e\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n}-\sum_{l=1}^{2}\omega_l^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}_l)\mathbf{t}_l+\lambda_h\mathbf{n}\right\|_{0,e}^2\\ & \leq c_7\left\{\|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2+h_T^2\,\|\mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T_e}^2+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2+h_e\|\lambda-\lambda_h\|_{0,e}^2\right\},\end{aligned}$$

for all $e\in\mathcal{E}_h(\Sigma)$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{S}}$ having e as a face,

d) $h_e \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,\le\, c_8 \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T_e}^2 + h_T^2 |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T_e}^2 + h_e \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \Big\}$,
for all $e \in \mathcal{E}_h(\Sigma)$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{S}}$ having e as a face.

Proof. We notice that all the estimates here can be easily obtained by adapting the proofs of their two-dimensional counterparts. In fact, the estimate in a) can be easily obtained after a slight modification of [2, Lemma 4.12], whereas the proofs of b), c), and d) readily follow from [37, Lemmas 3.15, 3.16 and 3.17], respectively. □

The sixth residual expression defining $\Theta_{\mathrm{S},T}^2$ (cf. (3.12)), that is the one containing the nonlinear operator and the convective term, as well as the rest of terms acting on Σ, are estimated now.

Lemma 3.14 *There exist $c_i > 0$, $i \in \{9, 10, 11\}$, independent of the meshsizes, such that*

a)
$$\Big\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}\Big\|_{0,\Omega_{\mathrm{S}}} \le\, c_9 \Big\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \Big\}.$$

b)
$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \left\| \big(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}\big) \times \mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}} \boldsymbol{\varphi}_h \right\|_{0,e}^2 \le\, c_{10} \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \Big(\|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T_e}^2 + \|\boldsymbol{\rho}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T_e}^2 \Big) \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2 \right\},$$

and

c)
$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \left\| \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \times \mathbf{n} + \mathbf{curl}_{\,\mathbf{s}} \lambda_h \right\|_{0,e}^2 \,\le\, c_{11} \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 + \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{D}}$ having e as a face.

Proof. The efficiency estimate a) follows exactly as in the first part of the proof of [39, Theorem 3.12]. Indeed, after introducing the identity $\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} - \mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} = 0$, the rest of the proof reduces to employ the Lipschitz-continuity of the nonlinear operator induced by μ (cf. [33, Lemma 2.1]), the compact imbedding $\mathbf{i}_c : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{L}^4(\Omega_{\mathrm{S}})$, and the fact that $\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}$ and $\|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}$ are both bounded by r, thus obtaining

$$\|\mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} \,\le\, L_\mu \,\|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}$$

and

$$\begin{aligned}\big\|\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}\big\|_{0,\Omega_{\mathrm{S}}} &\le\, \big\|\big(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\big) \otimes \mathbf{u}_{\mathrm{S}}\big\|_{0,\Omega_{\mathrm{S}}} \,+\, \big\|\mathbf{u}_{\mathrm{S},h} \otimes \big(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\big)\big\|_{0,\Omega_{\mathrm{S}}} \\ &\le\, \|\mathbf{i}_c\|^2 \Big\{ \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \Big\} \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \,\le\, 2\|\mathbf{i}_c\|^2\, r \,\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}\,.\end{aligned}$$

Further details are omitted. In turn, the proofs of b) and c) follow after a straightforward adaptation of that of [29, Lemma 20], and recalling from [6, Proposition 3.6] that the operators $\underline{\mathbf{curl}}_{\,\mathbf{s}}$ and $\mathbf{curl}_{\,\mathbf{s}}$ are bounded. □

We observe here that b) and c) are the only non-local efficiency bounds obtained so far. However, the following lemma shows that local estimates can still be derived for these terms under additional regularity assumptions on $\boldsymbol{\varphi}$ and λ.

Lemma 3.15 *Assume that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ and $\lambda|_e \in \mathrm{H}^1(e)$, for each $e \in \mathcal{E}_h(\Sigma)$. Then there exist c_{12}, $c_{13} > 0$, independent of the meshsizes, such that for each $e \in \mathcal{E}_h(\Sigma)$ there hold*

$$
\begin{aligned}
& h_e \left\| (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}) \times \mathbf{n} + \underline{\mathbf{curl}}_{\,\mathbf{s}} \boldsymbol{\varphi}_h \right\|_{0,e}^2 \\
& \quad \leq \; c_{12} \left\{ \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\rho}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T}^2 \; + \; h_e \, \|\underline{\mathbf{curl}}_{\,\mathbf{s}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{0,e}^2 \right\}
\end{aligned}
\tag{3.38}
$$

and

$$
h_e \left\| \mathbf{K}^{-1} \mathbf{u}_{\mathrm{D},h} \times \mathbf{n} + \mathbf{curl}_{\,\mathbf{s}} \lambda_h \right\|_{0,e}^2 \; \leq \; c_{13} \, \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 + h_e \, \|\mathbf{curl}_{\,\mathbf{s}}(\lambda - \lambda_h)\|_{0,e}^2 \right\}, \tag{3.39}
$$

where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{S}}$ (respectively $\mathcal{T}_h^{\mathrm{D}}$) having e as a face.

Proof. The proof of both estimates follow exactly as in the proof of [29, Lemma 21]. We omit further details. $\square$

Finally, the following lemma provides the corresponding upper bounds for the remaining terms defining $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$. In particular, in order to deal with those involving $\mathbf{K}^{-1}$, we assume from now on that $\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}$ is polynomial on each $T \in \mathcal{T}_h^{\mathrm{D}}$. Otherwise, assuming suitable regularity hypotheses and proceeding similarly as in [11, Section 6.2], higher order terms are obtained, which explains the expression h.o.t. in the lower bound of (3.14).

Lemma 3.16 *There exist positive constants c_i, $i \in \{14, \ldots, 20\}$, independent of the meshsizes, such that*

a) $h_T^2 \|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \; \leq \; c_{14} \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \right\} \quad \forall T \in \mathcal{T}_h^{\mathrm{D}}$,

b) $h_T^2 \left\| \mathrm{curl}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}) \right\|_{0,T}^2 \; \leq \; c_{15} \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \quad \forall T \in \mathcal{T}_h^{\mathrm{D}}$,

c) $h_e \left\| [\![\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \times \mathbf{n}]\!] \right\|_{0,e}^2 \; \leq \; c_{16} \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,\omega_e}^2$
for all $e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$, where the set ω_e is given by $\omega_e := \cup\big\{T' \in \mathcal{T}_h^{\mathrm{D}} : e \in \mathcal{E}(T')\big\}$,

d) $h_e \left\| \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \times \mathbf{n} \right\|_{0,e}^2 \; \leq \; c_{17} \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2$
for all $e \in \mathcal{E}_h(\Gamma_{\mathrm{D}})$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{D}}$ having e as a face,

e) $h_T^2 \left\| \underline{\mathbf{curl}}\,(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}) \right\|_{0,T}^2 \; \leq \; c_{18} \left\{ \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\rho}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T}^2 \right\} \quad \forall T \in \mathcal{T}_h^{\mathrm{S}}$,

f) $h_e \left\| [\![(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}) \times \mathbf{n}]\!] \right\|_{0,e}^2 \; \leq \; c_{19} \left\{ \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,\omega_e}^2 + \|\boldsymbol{\rho}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,\omega_e}^2 \right\}$
for all $e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$, where the set ω_e is given by $\omega_e := \cup\big\{T' \in \mathcal{T}_h^{\mathrm{S}} : e \in \mathcal{E}(T')\big\}$,

g) $h_e \left\| (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\rho}_{\mathrm{S},h}) \times \mathbf{n} \right\|_{0,e}^2 \; \leq \; c_{20} \left\{ \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T_e}^2 + \|\boldsymbol{\rho}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,T_e}^2 \right\}$
for all $e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$, where T_e is the tetrahedron of $\mathcal{T}_h^{\mathrm{S}}$ having e as a face.

Proof. For a) we refer to [10, Lemma 6.3] or alternatively [4, Lemma 4.3] (see also [32, Lemma 4.9]). In turn, noting that

$$
\mathrm{curl}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}) = -\mathrm{curl}\,(\nabla p_{\mathrm{D}}) = \mathbf{0} \quad \text{and} \quad \underline{\mathbf{curl}}\,(\mathbf{t}_{\mathrm{S}} + \boldsymbol{\rho}_{\mathrm{S}}) = \underline{\mathbf{curl}}\,(\nabla \mathbf{u}_{\mathrm{S}}) = \mathbf{0}\,,
$$

we find that the proofs of b) and e) are direct consequences of [28, Lemma 4.9]. Similarly, the proofs of c), d), f) and g) follow after a straightforward application of [28, Lemma 4.10] (see also [10, Lemma 6.2] and [4, Lemma 4.4]). $\square$

We end this section by observing that the required efficiency of the *a posteriori* error estimator Θ (cf. lower bound in (3.14)) is a direct consequence of Lemmas 3.12, 3.14, 3.16, and 3.13. In particular, the terms $h_e\|\lambda-\lambda_h\|_{0,e}^2$ and $h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2$ appearing in Lemma 3.13 (items a) – d)), are bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\lambda-\lambda_h\|_{0,e}^2 \,\le\, h\,\|\lambda-\lambda_h\|_{0,\Sigma}^2 \,\le\, C\,h\,\|\lambda-\lambda_h\|_{1/2,\Sigma}^2\,,$$

and

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2 \,\le\, h\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \,\le\, C\,h\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}^2\,.$$

4 Numerical results

We now turn to the implementation of some numerical tests that confirm the predicted reliability and efficiency of the proposed a posteriori error estimator. For the sake of simplicity, here we restrict ourselves to the two-dimensional case. To do that we remark that the 2D version of the a posteriori error indicators Θ_{S} and Θ_{D} described in (3.12) and (3.13) are defined exactly as their 3D counterparts, considering where appropriate (with $\mathbf{v}:=(v_1,v_2)^{\mathrm{t}}$ and $\boldsymbol{\tau}:=(\tau_{ij})_{2\times2}$), $\mathbf{v}\cdot\mathbf{t}$ and $\boldsymbol{\tau}\mathbf{t}$ instead of $\mathbf{v}\times\mathbf{n}$ and $\boldsymbol{\tau}\times\mathbf{n}$,

$$\mathrm{rot}\,\mathbf{v} := \frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}, \qquad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} := \left(\frac{\partial\tau_{12}}{\partial x_1}-\frac{\partial_{11}}{\partial x_2},\frac{\partial\tau_{22}}{\partial x_1}-\frac{\partial_{21}}{\partial x_2}\right)^{\mathrm{t}},$$

instead of $\mathrm{curl}\,\mathbf{v}$ and $\underline{\mathbf{curl}}\,\boldsymbol{\tau}$, and $\frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}$ and $\frac{d\lambda_h}{d\mathbf{s}}$ instead of $\underline{\mathbf{curl}}_{\mathbf{s}}\boldsymbol{\varphi}_h$ and $\mathbf{curl}_{\mathbf{s}}\lambda_h$, respectively, where $\frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}$ and $\frac{d\lambda_h}{d\mathbf{s}}$ stand for the tangential derivatives of $\boldsymbol{\varphi}_h$ and λ_h, respectively, along Σ.

Our implementation is based on a *FreeFem++* code (see [43]), in conjunction with the direct linear solver UMFPACK (see [20]). Regarding the implementation of the Newton iterative method, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1}-\mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \le tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in R^N, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{L}^2_{\mathrm{tr},h}(\Omega_{\mathrm{S}}), \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}), \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \mathbb{L}^2_{\mathrm{skew},h}(\Omega_{\mathrm{S}}), \mathbf{H}_{h,0}(\Omega_{\mathrm{D}}), \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)$, $\Lambda^{\mathrm{D}}_h(\Sigma)$, and $\mathrm{L}^2_{h,0}(\Omega_{\mathrm{D}})$, and *tol* is a fixed tolerance to be specified later. As usual, the individual errors are denoted by:

$$\begin{gathered}
\mathrm{e}(\mathbf{t}_{\mathrm{S}}) := \|\mathbf{t}_{\mathrm{S}}-\mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \quad \mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}, \quad \mathrm{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}},\\
\mathrm{e}(\boldsymbol{\rho}_{\mathrm{S}}) := \|\boldsymbol{\rho}_{\mathrm{S}}-\boldsymbol{\rho}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \quad \mathrm{e}(p_{\mathrm{S}}) := \|p_{\mathrm{S}}-p_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \quad \mathrm{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}},\\
\mathrm{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}}-p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}, \quad \mathrm{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}, \quad \mathrm{e}(\lambda) := \|\lambda-\lambda_h\|_{1/2,\Sigma},
\end{gathered}$$

where $p_{\mathrm{S},h}$ is the postprocessed pressure given by

$$p_{\mathrm{S},h} := -\frac{1}{2}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S},h}+(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})) \quad \text{in} \quad \Omega_{\mathrm{S}}.$$

In turn, the global error is computed as

$$\mathrm{e}(\vec{\mathbf{t}}) := \Big\{\mathrm{e}(\mathbf{t}_{\mathrm{S}})^2 + \mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}})^2 + \mathrm{e}(\mathbf{u}_{\mathrm{S}})^2 + \mathrm{e}(\boldsymbol{\rho}_{\mathrm{S}})^2 + \mathrm{e}(\mathbf{u}_{\mathrm{D}})^2 + \mathrm{e}(p_{\mathrm{D}})^2 + \mathrm{e}(\boldsymbol{\varphi})^2 + \mathrm{e}(\lambda)^2\Big\}^{1/2},$$

whereas the effectivity index with respect to Θ is given by

$$\mathbf{eff}(\Theta) := \frac{\mathrm{e}(\vec{\mathbf{t}})}{\Theta}.$$

In addition, we define the experimental rates of convergence

$$r(\%) := \frac{\log(\mathrm{e}(\%)/\mathrm{e}'(\%))}{\log(h/h')} \qquad \text{for each } \% \in \{\mathbf{t}_{\mathrm{S}}, \boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\rho}_{\mathrm{S}}, p_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda, \vec{\mathbf{t}}\},$$

where e and e' denote errors computed on two consecutive meshes of sizes h and h', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose $\mathbf{K} = \mathbb{I}$, $\omega_1 = 1$, and according to [12, eq. (3.26) in Section 3.2], the stabilization parameters are taken as $\kappa_1 = \mu_1/L_\mu^2$, with $L_\mu := \max\{\mu_2, 2\mu_2 - \mu_1\}$, $\kappa_2 = \kappa_1$, $\kappa_3 = \mu_1/2$, and $\kappa_4 = C_{\mathrm{Ko}}\mu_1/4$. Since the Korn inequality constant is not known when considering mixed boundary conditions, C_{Ko} is taken here heuristically as 0.5 (see [12, Section 7] for details). In addition, the tolerance *tol* is taken as $1E-6$ in all the examples.

Example 1 is used to corroborate the reliability and efficiency of the a posteriori error estimator Θ, whereas Examples 2 and 3 are utilized to illustrate the behaviour of the associated adaptive algorithm, which applies the following procedure from [47]:

(1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$.

(2) Solve the discrete problem (2.17) for the current mesh $\mathcal{T}_h$.

(3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T \in \mathcal{T}_h^{\star}$, $\star \in \{\mathrm{S},\mathrm{D}\}$.

(4) Check the stopping criterion and decide whether to finish or go to next step.

(5) Use blue-green refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2}\max_{T\in\mathcal{T}_h}\Big\{\Theta_T \,:\, T \in \mathcal{T}_h\Big\}$$

(6) Define resulting meshes as current meshes $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$, and go to step 2.

In Example 1 we consider the regions $\Omega_{\mathrm{S}} := \Big\{(x_1,x_2) \,:\, (x_1-0.5)^2 + (x_2-1)^2 < 0.25,\ x_2 > 1\Big\}$ and $\Omega_{\mathrm{D}} := (0,1)^2$. In this case, we set the nonlinear viscosity to

$$\mu(s) := 2 + \frac{1}{1+s} \qquad \text{for } s \geq 0.$$

The data $\mathbf{f}_{\mathrm{S}}$ and f_{D} are chosen so that the exact solution in the tombstone-shaped domain Ω is given by the smooth functions

$$p_{\mathrm{S}}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2), \quad \mathbf{u}_{\mathrm{S}}(\mathbf{x}) = -\mathbf{curl}\,(\sin(\pi x_1)\sin(\pi x_2)),$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{S}$, and

$$p_\mathrm{D}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2) \quad \forall \mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{D},$$

where $\mathrm{curl}\,(q) := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^\mathrm{t}$ for any sufficiently smooth function q. Notice that this solution satisfies $\mathbf{u}_\mathrm{S}\cdot\mathbf{n} = \mathbf{u}_\mathrm{D}\cdot\mathbf{n}$ on Σ and the boundary condition $\mathbf{u}_\mathrm{D}\cdot\mathbf{n} = 0$ on Γ_D. However, the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_S is non-homogeneous. Then, we need to modify accordingly the functional $\mathbf{F}$ (cf. (2.11)), as follows

$$[\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := -\kappa_2(\mathbf{f}_\mathrm{S}, \mathbf{div}\,\boldsymbol{\tau}_\mathrm{S})_\mathrm{S} + (\mathbf{f}_\mathrm{S}, \mathbf{v}_\mathrm{S})_\mathrm{S} + \langle \boldsymbol{\tau}_\mathrm{S}\mathbf{n}, \mathbf{g}\rangle_{\Gamma_\mathrm{S}} \quad \forall\, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X},$$

where $\mathbf{g} := \mathbf{u}_\mathrm{S}|_{\Gamma_\mathrm{S}} \in \mathbf{H}^{1/2}(\Gamma_\mathrm{S})$.

In Example 2 we consider the inverted L-shaped domain $\Omega = \Omega_\mathrm{S} \cup \Omega_\mathrm{D}$, where $\Omega_\mathrm{S} = (0,1)^2$ and $\Omega_\mathrm{D} := (-1,1)\times(-1,0)$, representing a fluid channel on top of a porous basin. The viscosity follows a Carreau law with $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta = 1.5$, that is

$$\mu(s) := 0.5 + 0.5(1+s^2)^{-1/4} \quad \text{for } s \geq 0,$$

and the data $\mathbf{f}_\mathrm{S}$ and f_D are chosen so that the exact solution is given by

$$p_\mathrm{S}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2), \quad \mathbf{u}_\mathrm{S}(\mathbf{x}) = \mathbf{curl}\,\left(x_1^2(x_1-1)^2 x_2^2(x_2-1)^2\right),$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{S}$, and

$$p_\mathrm{D}(\mathbf{x}) = \frac{(x_1^2-1)^2 x_2^2 (x_2+1)^2}{(x_1+0.01)^2 + (y-0.01)^2} \quad \forall \mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{D}.$$

Notice that the Darcy velocity and pressure exhibit high gradients near the origin.

Finally, in Example 3 we consider $\Omega_\mathrm{D} := (-1,0)^2$ and let Ω_S be the L-shaped domain given by $(-1,1)^2 \setminus \overline{\Omega}_\mathrm{D}$, which yields a porous medium partially surrounded by a fluid. The viscosity follows again a Carreau law (cf. (2.3)) with $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta = 1$, that is

$$\mu(s) := 0.5 + 0.5(1+s^2)^{-1/2} \quad \text{for } s \geq 0,$$

and the data $\mathbf{f}_\mathrm{S}$ and f_D are chosen so that the exact solution is given by

$$p_\mathrm{S}(\mathbf{x}) = \frac{1}{100(x_1^2+x_2^2)+0.01}, \quad \mathbf{u}_\mathrm{S}(\mathbf{x}) = \mathrm{curl}\,\left(0.1(x_2^2-1)^2\sin^2(\pi x_1)\right),$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{S}$, and

$$p_\mathrm{D}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2) \quad \forall \mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{D}.$$

Note that the fluid pressure p_S has high gradients around the origin.

In Table 4.1 we summarize the convergence history of the fully-mixed finite element method (2.17), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain, considering the finite element spaces introduced in Section 2.3, and solving the nonlinear problem with around five Newton iterations. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 2.2 (here $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index $\mathbf{eff}(\Theta)$ remains always in a neighbourhood of 0.98, which illustrates the reliability and efficiency of Θ in the case of a regular solution.

Next, in Tables 4.2, 4.3, 4.4, and 4.5, we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3, solving the nonlinear problem with around three and six Newton iterations, respectively. We observe that the errors of the adaptive procedure decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figures 4.1 and 4.3 where we display the total errors $\mathrm{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p_{\mathrm{D}})$ vs. the number of degrees of freedom N for both refinements. As shown by the values of $r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p_{\mathrm{D}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $O(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figures 4.2 and 4.4. Note that the method is able to recognize the region with high gradients in Examples 2 and 3.

dof	h_{S}	$\mathrm{e}(\mathbf{t}_{\mathrm{S}})$	$r(\mathbf{t}_{\mathrm{S}})$	$\mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(\boldsymbol{\rho}_{\mathrm{S}})$	$r(\boldsymbol{\rho}_{\mathrm{S}})$
854	0.1905	0.5866	–	4.6754	–	0.9306	–	1.7056	–
3195	0.0911	0.2909	1.0633	2.4721	0.9660	0.4707	1.0332	0.9970	0.8139
12543	0.0486	0.1460	1.0085	1.2793	0.9634	0.2381	0.9968	0.5015	1.0050
50188	0.0242	0.0679	1.1031	0.6398	0.9995	0.1142	1.0593	0.2371	1.0804
198838	0.0129	0.0352	0.9553	0.3493	0.8791	0.0580	0.9843	0.1256	0.9231
783886	0.0068	0.0179	0.9822	0.1742	1.0143	0.0294	0.9912	0.0639	0.9862

dof	h_{S}	h_{D}	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$
854	0.1905	0.1901	0.6240	–	1.2480	–	0.0619	–
3195	0.0911	0.0966	0.3409	0.9165	0.6004	1.1092	0.0296	1.1159
12543	0.0486	0.0573	0.1470	1.2302	0.3035	0.9975	0.0150	0.9962
50188	0.0242	0.0259	0.0686	1.0987	0.1516	1.0018	0.0075	1.0023
198838	0.0129	0.0135	0.0364	0.9227	0.0756	1.0106	0.0037	1.0105
783886	0.0068	0.0070	0.0183	1.0003	0.0382	0.9945	0.0019	0.9935

dof	$\widehat{h}$	$\mathrm{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathrm{e}(\lambda)$	$r(\lambda)$	$\mathrm{e}(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathbf{eff}(\Theta)$	iter
854	1/4	1.0668	–	0.2038	–	5.3590	–	5.5271	0.9696	5
3195	1/8	0.5573	0.9844	0.0980	1.1090	2.8448	0.9600	2.9156	0.9757	5
12543	1/16	0.2710	1.0545	0.0479	1.0485	1.4609	0.9746	1.4804	0.9868	5
50188	1/32	0.1345	1.0104	0.0243	0.9767	0.7245	1.0116	0.7312	0.9908	5
198838	1/64	0.0675	1.0025	0.0119	1.0336	0.3909	0.8963	0.3938	0.9926	5
783886	1/128	0.0336	1.0168	0.0064	0.9157	0.1956	1.0098	0.1967	0.9940	5

Table 4.1: EXAMPLE 1, quasi-uniform scheme.

dof	h_S	h_D	$e(\mathbf{t}_S)$	$e(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$e(\boldsymbol{\rho}_S)$	$e(p_S)$	$e(\mathbf{u}_D)$	$e(p_D)$
588	0.2926	0.3297	0.2022	0.5672	0.1893	0.3651	0.1754	39.9583	0.4761
1931	0.1964	0.1901	0.1811	0.3955	0.2030	0.3665	0.1092	73.9004	0.4069
7317	0.0997	0.1000	0.0724	0.1796	0.0814	0.1344	0.0572	59.6887	0.1433
28860	0.0487	0.0534	0.0172	0.0726	0.0068	0.0425	0.0172	76.9741	0.0140
115506	0.0250	0.0263	0.0084	0.0363	0.0034	0.0206	0.0082	66.9770	0.0055
459154	0.0136	0.0147	0.0042	0.0181	0.0015	0.0106	0.0041	54.0296	0.0028

dof	$\widehat{h}$	$e(\boldsymbol{\varphi})$	$e(\lambda)$	$e(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathbf{eff}(\Theta)$	iter
588	1/2	0.2161	0.8872	39.9782	–	40.2227	0.9939	4
1931	1/4	0.2601	1.2136	73.9144	–	74.0108	0.9987	4
7317	1/8	0.0931	0.6470	59.6930	0.3208	59.7396	0.9992	4
28860	1/16	0.0092	0.1010	76.9742	–	76.9937	0.9997	3
115506	1/32	0.0057	0.0485	66.9770	0.2006	66.9905	0.9998	3
459154	1/64	0.0029	0.0320	54.0296	0.3113	54.0370	0.9999	3

Table 4.2: Example 2, quasi-uniform scheme.

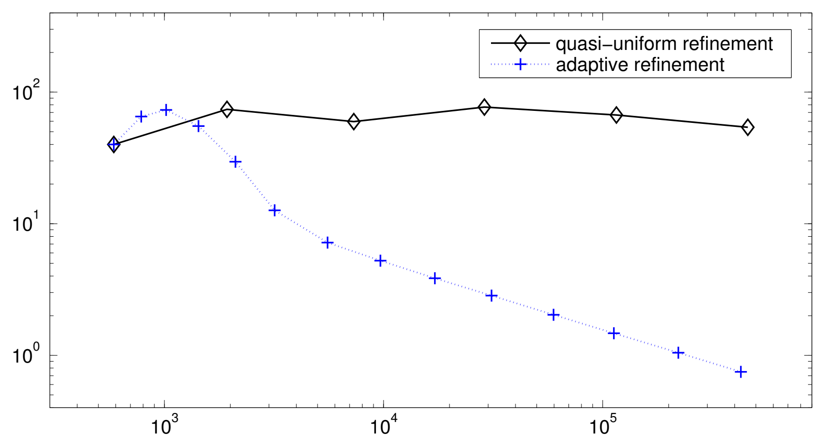

Figure 4.1: Example 2, $e(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p_D)$ vs. N for quasi-uniform/adaptive schemes.

dof	e($\mathbf{t}_S$)	e($\boldsymbol{\sigma}_S$)	e($\mathbf{u}_S$)	e($\boldsymbol{\rho}_S$)	e(p_S)	e($\mathbf{u}_D$)	e(p_D)
588	0.2022	0.5672	0.1893	0.3651	0.1754	39.9583	0.4761
784	0.1505	0.5215	0.1347	0.2404	0.1385	65.1904	0.1957
1019	0.1031	0.4988	0.0802	0.1297	0.1271	73.1960	0.0295
1431	0.0996	0.4973	0.0889	0.0982	0.1536	55.1764	0.0284
2111	0.0991	0.4995	0.0891	0.0863	0.1419	29.5771	0.0283
3185	0.0994	0.5011	0.0890	0.0805	0.1364	12.6187	0.0282
5555	0.0999	0.5028	0.0893	0.0777	0.1493	7.1633	0.0280
9680	0.0996	0.5023	0.0887	0.0848	0.1481	5.2107	0.0208
17147	0.0916	0.3984	0.0628	0.1351	0.1290	3.8253	0.0152
31110	0.0691	0.3124	0.0398	0.1078	0.0935	2.8216	0.0124
59678	0.0490	0.1961	0.0197	0.0866	0.0536	2.0190	0.0076
112409	0.0394	0.1672	0.0165	0.0653	0.0484	1.4593	0.0063
221370	0.0245	0.1003	0.0084	0.0389	0.0271	1.0402	0.0038
427000	0.0206	0.0870	0.0068	0.0327	0.0226	0.7425	0.0032

dof	e($\boldsymbol{\varphi}$)	e(λ)	e($\vec{\mathbf{t}}$)	$r(\vec{\mathbf{t}})$	Θ	$\mathbf{eff}(\Theta)$	iter
588	0.2161	0.8872	39.9782	–	40.2227	0.9939	4
784	0.1439	0.8114	65.1987	–	65.2262	0.9996	4
1019	0.0634	0.1014	73.1980	–	73.2245	0.9996	3
1431	0.0504	0.0475	55.1790	1.6645	55.2056	0.9995	3
2111	0.0444	0.0201	29.5818	3.2070	29.6221	0.9986	3
3185	0.0428	0.0155	12.6297	4.1387	12.7177	0.9931	3
5555	0.0416	0.0152	7.1828	2.0292	7.3110	0.9825	3
9680	0.0459	0.0130	5.2375	1.1374	5.3244	0.9837	3
17147	0.0479	0.0118	3.8503	1.0763	3.9225	0.9816	3
31110	0.0255	0.0060	2.8422	1.0192	2.8947	0.9819	3
59678	0.0217	0.0039	2.0312	1.0314	2.0692	0.9816	3
112409	0.0136	0.0021	1.4710	1.0192	1.4982	0.9818	3
221370	0.0085	0.0012	1.0461	1.0059	1.0651	0.9822	3
427000	0.0072	0.0006	0.7487	1.0184	0.7625	0.9819	3

Table 4.3: Example 2, adaptive scheme.

dof	h_S	h_D	e($\mathbf{t}_S$)	e($\boldsymbol{\sigma}_S$)	e($\mathbf{u}_S$)	e($\boldsymbol{\rho}_S$)	e(p_S)	e($\mathbf{u}_D$)	e(p_D)
1037	0.3529	0.3019	1.2001	19.6901	2.6722	2.8717	0.6257	2.1914	0.1139
3664	0.1947	0.1964	2.1724	58.3515	4.5117	3.3283	1.1404	1.2395	0.0657
13956	0.0960	0.1025	2.4894	114.3013	5.8122	4.1176	1.0524	0.6207	0.0314
55663	0.0520	0.0495	1.7729	94.6633	3.2125	2.7450	0.8685	0.3093	0.0153
220100	0.0293	0.0260	0.8835	51.4952	1.0053	1.4521	0.5310	0.1513	0.0075
879198	0.0145	0.0143	0.5353	33.8233	0.3295	1.2531	0.3642	0.0759	0.0037

dof	$\widehat{h}$	e($\boldsymbol{\varphi}$)	e(λ)	e($\vec{\mathbf{t}}$)	$r(\vec{\mathbf{t}})$	Θ	$\mathbf{eff}(\Theta)$	iter
1037	1/2	1.3033	0.9173	20.2949	–	20.6086	0.9848	5
3664	1/4	2.0423	0.6667	58.7129	–	58.7574	0.9992	5
13956	1/8	2.4754	0.4053	114.5792	–	114.5746	1.0000	5
55663	1/16	1.8369	0.2100	94.7926	0.2741	94.7881	1.0000	5
220100	1/32	0.7827	0.0855	51.5392	0.8865	51.5419	0.9999	5
879198	1/64	0.5891	0.0577	33.8576	0.6068	33.8559	1.0000	5

Table 4.4: Example 3, quasi-uniform scheme.

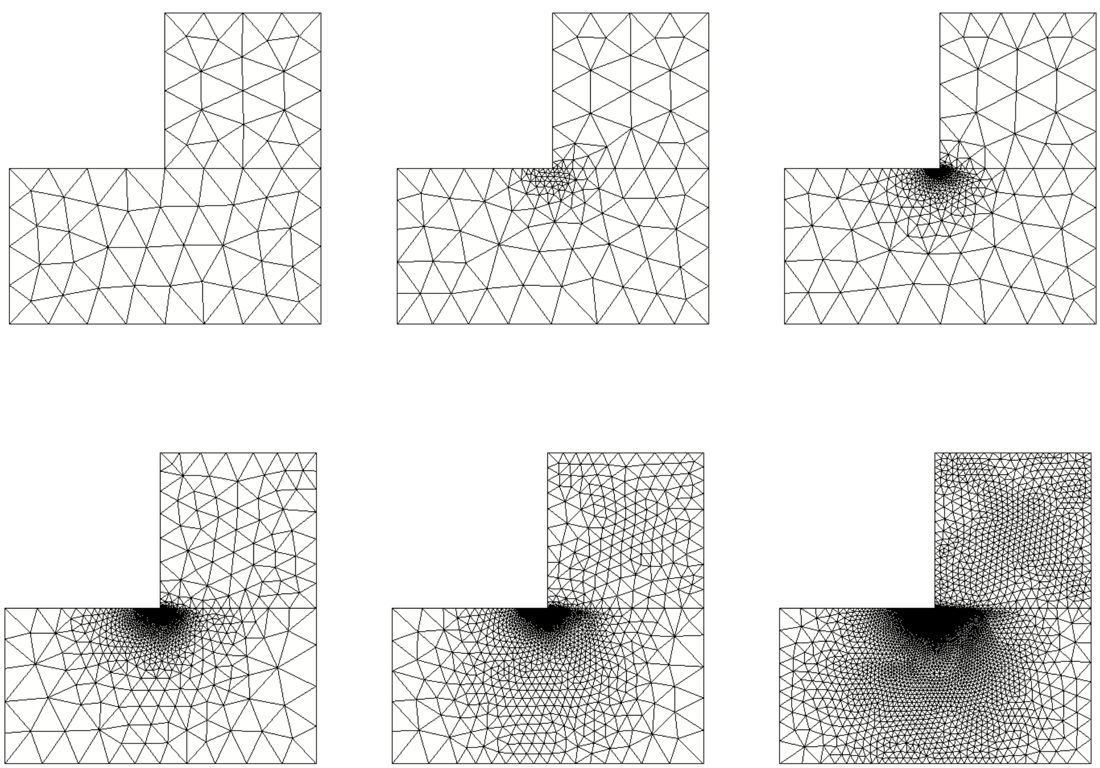

Figure 4.2: Example 2, adapted meshes with 588, 1019, 9680, 31110, 112409, and 447000 degrees of freedom.

dof	$\mathrm{e}(\mathbf{t}_\mathrm{S})$	$\mathrm{e}(\boldsymbol{\sigma}_\mathrm{S})$	$\mathrm{e}(\mathbf{u}_\mathrm{S})$	$\mathrm{e}(\boldsymbol{\rho}_\mathrm{S})$	$\mathrm{e}(p_\mathrm{S})$	$\mathrm{e}(\mathbf{u}_\mathrm{D})$	$\mathrm{e}(p_\mathrm{D})$
1037	1.2001	19.6901	2.6722	2.8717	0.6257	2.1914	0.1139
1579	3.0393	98.1488	6.1149	4.8488	1.2574	1.9106	0.0933
2092	2.0017	84.5704	2.9416	2.9343	1.0256	1.8860	0.0929
2766	0.7046	42.4930	0.8777	1.4751	0.3717	1.8982	0.0936
4748	0.5235	16.4829	0.7695	1.4233	0.2375	1.9012	0.0938
9936	0.4928	8.2112	0.7622	1.4148	0.2143	1.5623	0.0779
18993	0.3580	5.6827	0.5430	1.0980	0.1401	1.0019	0.0495
33974	0.2681	4.2302	0.3972	0.9634	0.1096	0.7945	0.0395
64472	0.1785	3.0476	0.2581	0.7001	0.0752	0.5697	0.0282
122011	0.1438	2.2169	0.2111	0.5401	0.0643	0.4319	0.0215
237874	0.0928	1.5864	0.1340	0.3908	0.0407	0.3042	0.0151
460024	0.0708	1.1390	0.1038	0.3028	0.0301	0.2232	0.0111
915408	0.0456	0.8086	0.0667	0.2074	0.0201	0.1567	0.0078

dof	$\mathrm{e}(\boldsymbol{\varphi})$	$\mathrm{e}(\lambda)$	$\mathrm{e}(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathbf{eff}(\Theta)$	iter
1037	1.3033	0.9173	20.2949	–	20.6086	0.9848	5
1579	5.6381	1.0470	98.6908	–	98.8258	0.9986	5
2092	3.5592	0.5382	84.7936	1.0790	84.8825	0.9990	5
2766	0.6262	0.4790	42.5832	4.9324	42.6743	0.9979	5
4748	0.4162	0.4813	16.6915	3.4667	16.9205	0.9865	5
9936	0.3749	0.4612	8.5469	1.8128	8.7791	0.9736	5
18993	0.3330	0.2994	5.9270	1.1300	6.1023	0.9713	5
33974	0.2066	0.2053	4.4464	0.9885	4.5652	0.9740	5
64472	0.1455	0.1665	3.2017	1.0252	3.3074	0.9680	5
122011	0.1153	0.1202	2.3423	0.9799	2.4071	0.9731	5
237874	0.0839	0.0852	1.6742	1.0060	1.7299	0.9678	5
460024	0.0552	0.0664	1.2092	0.9865	1.2446	0.9716	5
915408	0.0436	0.0463	0.8556	1.0055	0.8850	0.9668	5

Table 4.5: Example 3, adaptive scheme.

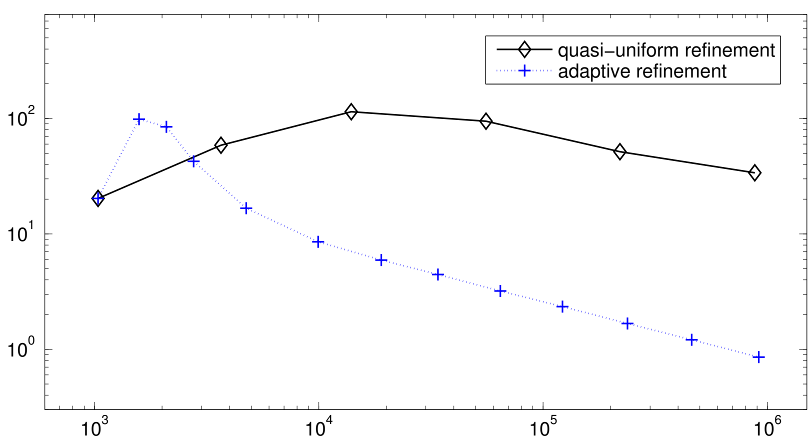

Figure 4.3: Example 3, $\mathrm{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p_\mathrm{D})$ vs. N for quasi-uniform/adaptive schemes.

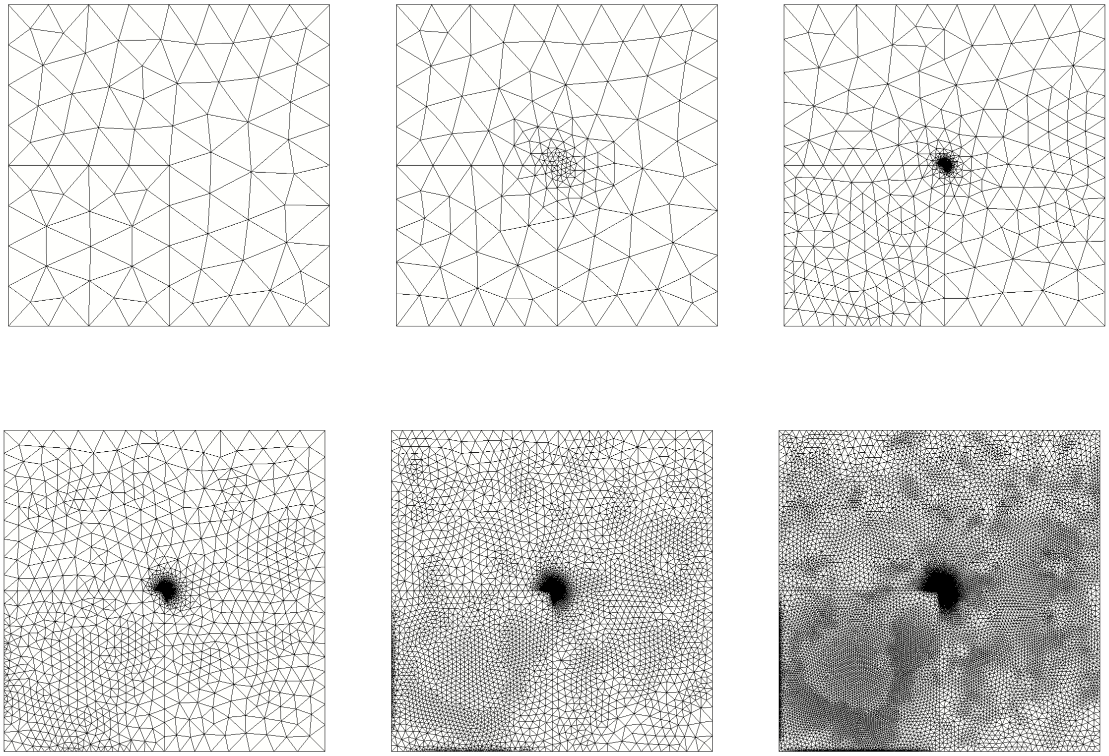

Figure 4.4: Example 3, adapted meshes with 1037, 2092, 18993, 64472, 237874, and 915408 degrees of freedom.

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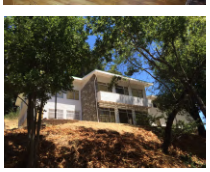

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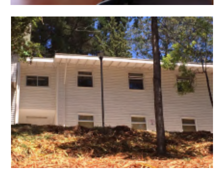

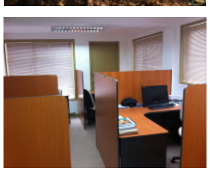

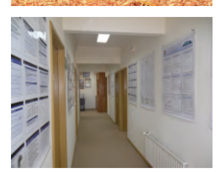

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